Density of polyhedral partitions

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Abstract

We prove the density of polyhedral partitions in the set of finite Caccioppoli partitions. Precisely, we consider a decomposition u of a bounded Lipschitz set $\Omega \subset \mathbb{R}^n$ into finitely many subsets of finite perimeter, which can be identified with a function in $SBV_{\rm loc}(\Omega;\mathcal{Z})$ with $\mathcal{Z} \subset \mathbb{R}^N$ a finite set of parameters. For all $\varepsilon > 0$ we prove that such a u is ε-close to a small deformation of a polyhedral decomposition v_ε, in the sense that there is a C^1 diffeomorphism $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ which is ε-close to the identity and such that $u \circ f_\varepsilon - v_\varepsilon$ is ε-small in the strong BV norm. This implies that the energy of u is close to that of v_ε for a large class of energies defined on partitions. Such type of approximations are very useful in order to simplify computations in the estimates of Γ-limits.

1 Introduction

Besides their theoretical interest, approximation results have a great technical importance in the treatment of variational problems; in particular, in the computation of Γ-limits for varying energies. The density of piecewise-affine maps in Sobolev spaces, for example, often allows computations for integral energies to be performed only in the simplified setting of maps with constant gradient. Similarly, the approximation of sets of finite perimeter by polyhedral sets, which sometimes is taken as the definition of sets of finite perimeter itself, allows to reduce problems involving surface energies to the case of a planar interface. The use of approximation theorems for the computation of Γ-limits is not strictly necessary, since more abstract integral-representation theorems can be used, whose application though is often quite technical. The computation is simpler if representation formulas are available such as relaxation or homogenization formulas. Indeed, in that case it is easier to prove a lower bound for a Γ-limit by the blow-up technique elaborated by Fonseca and Müller [FM93]. Approximation results are crucial to reduce the proof of the upper bound to simpler functions for which recovery sequences are suggested by the representation formulas themselves.

In multi-phase problems, i.e., for interfacial problems when more than two sets are involved, the proper variational setting is that of partitions into sets of finite perimeter, or Caccioppoli partitions, for which a theory of relaxation and Γ-convergence has been first developed by Ambrosio and Braides [AB90a, AB90b]. The study of Caccioppoli partitions is also a fundamental step in the analysis of free-discontinuity problems defined on (special) functions of bounded variation, since lower-semicontinuity conditions and representation formulas for the latter can be often deduced from those for partitions. In that spirit, integral-representation theorems for partitions have been proved by Braides and Chiadò Piat [BCP96] and Bouchitté et al. [BFLM02].

The scope of this paper is to fill a gap that seemingly exists in the treatment of problems on Caccioppoli partitions, namely the existence of approximations by polyhedral sets. This is a widely expected result, so much expected that sometimes it is mistakenly referred to as proved in some reference text. Conversely, its non-availability makes it more complicated to obtain homogenization results even when formulas are available (see e.g. the recent work by Braides and Cicalese [BC15]). A "dual" result for systems of rectifiable curves has recently been proved by Conti et al. [CGM15] and used to show convergence of linear elasticity to a dislocation model [CGO15]. In a two-dimensional setting one can use that approximation result to obtain polyhedral partitions by considering boundaries of sets as rectifiable curves, and as such it has been recently used to the study of systems of chiral molecules [BGP16].

It must be noted that the method usually followed to obtain approximating sets for a single Caccioppoli set cannot be used for partitions. Indeed, for a single set of finite perimeter $E \subset \mathbb{R}^n$ we can use that the characteristic function $u := \chi_E$ is by definition a function with bounded variation; hence, by a mollification argument it can be approximated by smooth functions u_ρ and this mollification process does not increase the corresponding variation. Approximating sets are then obtained by taking super-level sets of the form $E_\rho := \{x : u_\rho(x) > c_\rho\}$. By Sard's theorem the set E_ρ is smooth for almost all values of c_ρ, and by the coarea formula c_ρ can be chosen so that the boundary of E_ρ is not larger than the boundary of E. Finally, polyhedral sets are obtained by triangulation using the smoothness of E_ρ. Such a simple argument cannot be repeated if we have a partition. Indeed, identify such a partition $(E_1, \dots, E_N)$ in $\mathbb{R}^n$ with a BV-function by setting $u := \sum_j a_j \chi_{E_j}$, for suitable labelling parameters a_j. If we choose a_j real numbers, the process outlined above will require the choice of more superlevel sets $\{u_\rho > c_\rho^j\}$, which will introduce artificial interfaces. To picture this situation, think of having a partition into three sets of finite perimeter and choose as labels the numbers $a_j := j$. Then in the process above we will have two approximating sets $E_\rho^1 := \{x : c_\rho^1 < u_\rho(x) \le c_\rho^2\}$ and $E_\rho^2 := \{x : u_\rho(x) > c_\rho^2\}$ with $1 < c_\rho^1 < 2 < c_\rho^2 < 3$ and the interface between the set E_1 and E_3 will be approximated by a double interface: one between E_ρ^1 and E_ρ^2 and another one between E_ρ^2 and E_ρ^3. Although these approximations weakly converge to the original partition, the total length of the surface has doubled and the energy of the partitions will not converge. If otherwise we label the sets with a_j in some higher-dimensional $\mathbb{R}^m$ then the use of the coarea formula is not possible. It is then necessary, as is done in the proof of integral-representation results, to make a finer use of the structure of boundaries of sets of finite perimeter.

In our construction we use the fact that essential boundaries between sets of finite perimeter are contained in C^1 hypersurfaces that can be locally deformed onto portions of hyperplanes. By a covering argument we can thus transform most of the interfaces with a small deformation into open subsets of a finite system of hyperplanes, which can in turn be approximated by polyhedral sets. We finally introduce a decomposition of the ambient space into a system of small polyhedra whose boundaries contain the above-mentioned lower-dimensional polyhedral sets, and define a Caccioppoli partition by choosing the majority phase (i.e., the label corresponding to the set with the largest measure) on each of the small polyhedra. This finally gives the desired approximating sets.

A scalar version of this result is proven in [Fed69, Th. 4.2.20] and then refined in [ADC05] and [QdG08]. The vector-valued approximation, however, does not follow from the scalar one working componentwise since approximation of the energy requires to choose a single deformation f for all components. Approximation of vector-valued SBV^p functions was studied in [CT99, KR16], but the case of partitions does not seem to follow directly from the arguments therein, which introduce large gradients in small regions. The vectorial case of SBD^p functions was addressed for $p = 2$ in [Cha04, Cha05, Iur14].

2 The density result and its proof

We will consider partitions of an open set $\Omega \subset \mathbb{R}^n$ into N sets of finite perimeter $(E_1, \dots, E_N)$ for some $N \ge 1$. We say that a set $\Sigma \subset \Omega$ is *polyhedral* if there is a finite number of $n-1$-dimensional simplexes $T_1, \dots, T_M \subset \mathbb{R}^n$ such that Σ coincides, up to $\mathcal{H}^{n-1}$-null sets, with $\bigcup_{j=1}^M T_j \cap \Omega$. We are interested in showing that for a general partition $(E_1, \dots, E_N)$ there exist polyhedral approximations; i.e., partitions $(E_1^j, \dots, E_N^j)$ of Ω into sets whose boundaries are polyhedral, such that for all $k \in \{1, \dots, N\}$ we have $|E_k^j \triangle E_k| \to 0$ and $\mathcal{H}^{n-1}(\partial E_k^j) \to \mathcal{H}^{n-1}(\partial E_k)$ as $j \to +\infty$ (where in the last formula ∂E_k denotes the reduced boundary of E_k), and the normal to ∂E_k^j converges in a suitable sense to the normal to ∂E_k (see Corollary 2.5).

It will be handy to use a finite set $\mathcal{Z} := \{z_1, \dots, z_N\} \subset \mathbb{R}^N$ as a set of labels for the different phases, and identify each partition with the function $u : \Omega \to \mathbb{R}^N$ given by $u(x) = z_k$ on E_k. In this way the set of partitions into N sets of finite perimeter is identified with a subset of the space $SBV_{\text{loc}}(\Omega; \mathcal{Z})$ (see [AB90a, AB90b]). Note that our results will be independent of the labelling, but the latter allows to state the convergence of boundaries of sets as a convergence of the derivatives of functions.

We recall that a function $u \in SBV_{\text{loc}}(\Omega; \mathcal{Z})$, for $\Omega \subset \mathbb{R}^n$ open, has the property that its distributional derivative is a bounded measure of the form $Du = [u] \otimes \nu \mathcal{H}^{n-1} \llcorner J_u$. Here $J_u \subset \Omega$ is a $n-1$-rectifiable set, called the jump set of u, the unit vector $\nu : J_u \to S^{n-1}$ is the normal to J_u, and $[u] := (u^+ - u^-)$ is the jump of u, where u^+ and $u^- : J_u \to \mathcal{Z}$ are the traces of u on the two sides of J_u, which are $\mathcal{H}^{n-1} \llcorner J_u$-measurable. We use the notation $\mu \llcorner E$ for the restriction of a measure μ to a μ-measurable set E, defined by $(\mu \llcorner E)(A) := \mu(E \cap A)$.

The main result of this paper is the following approximation statement.

Theorem 2.1. *Let $\mathcal{Z} \subset \mathbb{R}^N$ be finite, let $u \in SBV_{\text{loc}}(\Omega; \mathcal{Z})$ with $|Du|(\Omega) < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz set with $\partial\Omega$ compact. Then there is a sequence $u_j \in SBV_{\text{loc}}(\Omega; \mathcal{Z})$ such that J_{u_j} is polyhedral, $u_j \to u$ in $L^1_{\text{loc}}(\Omega; \mathcal{Z})$ and $Du_j \rightharpoonup Du$ as measures, and there are bijective maps $f_j \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, with inverse also in C^1, which converge strongly in $W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$ to the identity map such that $|D(u \circ f_j) - Du_j|(\Omega) \to 0$.*

We remark that $u \circ f_j$ is defined on the set $f_j^{-1}(\Omega)$, and so is the measure $D(u \circ f_j)$, which is then implicitly extended by zero to the rest of $\mathbb{R}^n$. In particular,

$$|D(u \circ f_j) - Du_j|(\Omega) = |D(u \circ f_j) - Du_j|(\Omega \cap f_j^{-1}(\Omega)) + |Du_j|(\Omega \setminus f_j^{-1}(\Omega)).$$

The rest of this paper contains the proof of Theorem 2.1. We shall first (Theorem 2.2) prove the analogous statement for functions defined on $\mathbb{R}^n$, and then (Lemma 2.7) give an extension argument to deal with general domains. We remark that the assumption that $\partial\Omega$ is compact is only used in constructing the extension, so that our result can be extended immediately to some other unbounded sets, such as, for example, the half space.

Theorem 2.2. *Let $\mathcal{Z} \subset \mathbb{R}^N$ be finite, and let $u \in SBV_{\text{loc}}(\mathbb{R}^n; \mathcal{Z})$ with $|Du|(\mathbb{R}^n) < \infty$. Then there is a sequence $u_j \in SBV_{\text{loc}}(\mathbb{R}^n; \mathcal{Z})$ such that J_{u_j} is polyhedral, $u_j \to u$ in $L^1_{\text{loc}}(\mathbb{R}^n; \mathcal{Z})$, $Du_j \rightharpoonup Du$ as measures, and there are bijective maps $f_j \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, with inverse also in C^1, which converge strongly in $W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$ to the identity map such that $|D(u \circ f_j) - Du_j|(\mathbb{R}^n) \to 0$.*

The proof of Theorem 2.2 relies on a deformation argument allowed by the rectifiability of J_u. We recall that the latter means that J_u coincides, up to an $\mathcal{H}^{n-1}$-null set, with a Borel subset of the union of countably many C^1 surfaces [AFP00, Sect. 2.9]. Furthermore, in this characterization one also has that, for $\mathcal{H}^{n-1}$-almost all points $y \in J_u$, denoting by M_y the C^1 surface containing y, the vector $\nu(y)$ is the normal in y to the surface M_y and

$$\lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \mathcal{H}^{n-1}((J_u \triangle M_y) \cap B_\rho(y)) = 0, \tag{2.1}$$

where $B_\rho(y)$ is the open ball of radius ρ centered in y. The measurability of the traces $u^\pm(y)$ and the finiteness of $\mathcal{Z}$ imply, via the Lebesgue point theorem, that the traces are locally approximately constant, in the sense that

$$\lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \mathcal{H}^{n-1}(\{x \in J_u \cap B_\rho(y) : u^+(x) \neq u^+(y)\}) = 0 \tag{2.2}$$

for $\mathcal{H}^{n-1}$-almost every $y \in J_u$. We refer to [AFP00] for a more detailed treatment of these concepts. The idea of the proof is to cover most of the jump set of u by disjoint balls, such that in each of them the jump set is an (almost flat) C^1 graph (see Step 2). In each of the balls the jump set can then be explicitly deformed into a plane, up to an interpolation region (see Step 1).

Proof. Step 1. *We perform a local construction around $\mathcal{H}^{n-1}$-almost all points of the jump set.*

Fix $\varepsilon \in (0, 1)$, whose value will be chosen below. Assume that $y \in J_u$ has the following properties: there are $g = g_y \in C^1(\mathbb{R}^{n-1})$, $r = r_y > 0$ and an affine isometry $I_y : \mathbb{R}^n \to \mathbb{R}^n$, $I_y(x) = Q_y x + b_y$, satisfying $g(0) = 0$, $Dg(0) = 0$,

$$\mathcal{H}^{n-1}((I_y J_u) \triangle \{(x', g(x')) : x' \in B'_r\}) < \varepsilon r^{n-1}, \tag{2.3}$$

where B'_r denotes the $n-1$-dimensional ball of radius r centered in 0,

$$\mathcal{H}^{n-1}(\{J_u \cap B_r(y) : u^+(x) \neq u^+(y)\}) < \varepsilon r^{n-1} \tag{2.4}$$

and the same for u^-. Since we chose the isometry I_y to make $Dg(0) = 0$, choosing r sufficiently small we can ensure that additionally $|Dg| \le \varepsilon^2$ in B'_r, which in turn implies $|g| \le \varepsilon^2 r$ in B'_r. By (2.1) and (2.2) $\mathcal{H}^{n-1}$-almost every $y \in J_u$ has the properties above.

We fix $\psi \in C^1_c(B_r(y); [0, 1])$ such that $\psi = 1$ on $B_{(1-\varepsilon)r}(y)$ and $\|D\psi\|_\infty \le 2/(\varepsilon r)$ and define $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x) := x - \psi(x) g(\Pi\, I_y x) \nu_y\,,$$

where $\nu_y := Q_y^{-1} e_n$ is the normal to J_u at y, and $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ components. We compute

$$Df = \mathrm{Id} - g \nu_y \otimes D\psi - \psi \nu_y \otimes (D' g\, \Pi\, Q_y).$$

Here we use the notation $D'g$ in place of Dg to highlight the derivation in $\mathbb{R}^{n-1}$. The bounds on g and ψ imply that $|Df - \mathrm{Id}| < 3\varepsilon$ everywhere. In particular, f is a diffeomorphism, which is the identity outside $B_r(y)$.

Let $\mu := Du \llcorner B_r(y) - [u](y) \otimes \nu \mathcal{H}^{n-1} \llcorner \{I_y^{-1}(x', g(x')) : x' \in B'_r\}$, where ν is the normal to the last set. By (2.3) and (2.4), we obtain $|\mu|(\mathbb{R}^n) \leq c\varepsilon r^{n-1}$.

We choose a closed $n-1$-dimensional polyhedron $\hat{P}$ contained in $B'_{(1-\varepsilon)r}$ and such that

$$\mathcal{H}^{n-1}(B'_{(1-\varepsilon)r} \setminus \hat{P}) \leq \varepsilon r^{n-1},$$

and define $P_y := I_y^{-1}(\hat{P} \times \{0\})$ and

$$\hat{\mu} := D(u \circ f) \llcorner B_r(y) - [u](y) \otimes \nu_y \mathcal{H}^{n-1} \llcorner P_y\,. \tag{2.5}$$

By the change-of-variable formula for BV functions, the bounds on f and the estimate in μ we obtain $|\hat{\mu}|(\mathbb{R}^n) \leq c\varepsilon r^{n-1} \leq c\varepsilon |Du|(B_r(y))$. All constants may depend only on n and $\mathcal{Z}$.

Step 2. *By a covering argument we conclude the construction.*

Using Vitali's covering theorem, we choose finitely many points $x_1, \dots, x_M \in \mathbb{R}^n$ and radii $r_i \in (0,1)$ with the properties stated in Step 1, such that

$$|Du| \left(\mathbb{R}^n \setminus \bigcup_{i=1}^{M} B_{r_i}(x_i) \right) < \varepsilon$$

and the balls $B_{r_i}(x_i)$ are disjoint. Let $f_1, \dots, f_M \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and $P_1, \dots, P_M \subset \mathbb{R}^n$ be the corresponding deformations and polyhedra, respectively, and let $u_i^{\pm} \in \mathcal{Z}$, $\nu_i \in S^{n-1}$ be the corresponding traces and normals. Let

$$f := f_1 \circ f_2 \circ \cdots \circ f_M \in C^1(\mathbb{R}^n; \mathbb{R}^n).$$

Since $f_i(x) = x$ outside $B_{r_i}(x_i)$ we have

$$|Df(x) - \mathrm{Id}| + |f(x) - x| \leq 6\varepsilon$$

for all $x \in \mathbb{R}^n$. We define $v := u \circ f$. Then, letting

$$\mu^* := \sum_{j=1}^{M} (u_i^+ - u_i^-) \otimes \nu_i \mathcal{H}^{n-1} \llcorner P_i$$

be the polyhedral measure we have constructed in Step 1, we obtain

$$\begin{aligned} |Dv - \mu^*|(\mathbb{R}^n) &\leq \sum_{i=1}^{M} |\hat{\mu}_i|(\mathbb{R}^n) + |Du| \left(\mathbb{R}^n \setminus \bigcup_{i=1}^{M} B_{r_i}(x_i) \right) \\ &\leq c\varepsilon |Du|(\mathbb{R}^n) + \varepsilon\,. \end{aligned} \tag{2.6}$$

Here $\hat{\mu}_i$ denotes the analog for the ball $B_{r_i}(x_i)$ of the remainder $\hat{\mu}$ obtained in (2.5).

Step 3. *We construct a piecewise-constant SBV function with polyhedral jump set whose gradient is close to* μ^*.

To that end, we will use the results of Lemma 2.6 separately stated and proved below. We consider c_* and the polyhedral decomposition into the cells $\{V_q\}_{q\in G}$ of $\mathbb{R}^n$ obtained from Lemma 2.6 taking in its hypothesis the polyhedra $P_1, \dots, P_M$, with a spacing $\delta > 0$ such that $2c_*\delta < \operatorname{dist}(P_i, P_l)$ for all $i \neq l$.

For any $q \in G$, we choose a value $z_q \in \mathcal{Z}$ such that $|V_q \cap v^{-1}(z_q)| = \max_{z'\in\mathcal{Z}} |V_q \cap v^{-1}(z')|$. We define $w : \mathbb{R}^n \to \mathbb{R}^N$ by setting $w(x) = z_q$ if $x \in V_q$. By the geometric properties of the cells V_q described in Lemma 2.6, using Poincaré's inequality and the trace theorem we obtain

$$\|v - z_q\|_{L^1(V_q)} \le c\delta|Dv|(V_q) \text{ and } \|v - z_q\|_{L^1(\partial V_q)} \le c|Dv|(V_q)\,, \tag{2.7}$$

where c may depend only on n, $\mathcal{Z}$ and on c_*. To see this, we observe that since $B_{\delta/c_*}(x_q) \subset V_q$ there is $m_q \in \mathbb{R}^N$ such that $\|v - m_q\|_{L^1(B_{\delta/c_*}(x_q))} \le c\delta|Dv|(V_q)$. The estimate is then extended to V_q passing to polar coordinates centered in x_q and using the one-dimensional Poincaré inequality in the radial direction; note that since $V_q \subset B_{c_*\delta}(x_q)$ the Jacobian determinant is bounded. Finally one replaces m_q by z_q using the fact that the volume of the set $V_q \cap \{v = z_q\}$ is at least $|V_q|/\#\mathcal{Z}$. The trace estimate, in turn, follows by using the one-dimensional trace estimate on each segment connecting a point on ∂V_q with x_q, and estimating again the Jacobian determinant using $B_{\delta/c_*}(x_q) \subset V_q \subset B_{c_*\delta}(x_q)$.

It remains to check that the map w has the desired properties. Since w takes finitely many values, and is piecewise constant on each of the polyhedra V_q which cover $\mathbb{R}^n$, we see that $w \in SBV_{\rm loc}(\mathbb{R}^n; \mathcal{Z})$ and that $J_w \subset \bigcup_{q\in G} \partial V_q$ is polyhedral.

To estimate Dw, we consider two indices $q \neq q' \in G$ such that $\mathcal{H}^{n-1}(\partial V_q \cap \partial V_{q'}) > 0$. Denoting by $T_q v$ and $T_{q'} v$ the inner traces of v on the boundaries of V_q and $V_{q'}$ respectively, we obtain, using a triangular inequality and (2.7),

$$\begin{aligned}
|Dw|(\partial V_q \cap \partial V_{q'}) =& |z_q - z_{q'}|\mathcal{H}^{n-1}(\partial V_q \cap \partial V_{q'}) \\
\le& \|T_q v - T_{q'} v\|_{L^1(\partial V_q \cap \partial V_{q'})} \\
&+ \|T_q v - z_q\|_{L^1(\partial V_q)} + \|T_{q'} v - z_{q'}\|_{L^1(\partial V_{q'})} \\
\le& |Dv|(\partial V_q \cap \partial V_{q'}) + c|Dv|(V_q) + c|Dv|(V_{q'}) \\
\le& c|Dv|(V_q \cup V_{q'} \cup (\partial V_q \cap \partial V_{q'})).
\end{aligned}$$

If $|\mu^*|(\partial V_q \cap \partial V_{q'}) = 0$, this estimate suffices. Otherwise, there is exactly one j such that $\mathcal{H}^{n-1}(P_i \cap \partial V_q \cap \partial V_{q'}) > 0$. Assuming that ν_i is oriented from $V_{q'}$ to V_q, a computation similar to the one above gives

$$\begin{aligned}|Dw-\mu^*|(\partial V_q\cap\partial V_{q'}) =&\|z_q-z_{q'}-(u_i^+-u_i^-)\chi_{P_i}\|_{L^1(\partial V_q\cap\partial V_{q'})}\\ \le&\|T_q v-T_{q'}v-(u_i^+-u_i^-)\chi_{P_i}\|_{L^1(\partial V_q\cap\partial V_{q'})}\\ &+\|T_q v-z_q\|_{L^1(\partial V_q)}+\|T_{q'}v-z_{q'}\|_{L^1(\partial V_{q'})}\\ \le&|Dv-\mu^*|(\partial V_q\cap\partial V_{q'})+c|Dv|(V_q)+c|Dv|(V_{q'})\\ \le&c|Dv-\mu^*|(V_q\cup V_{q'}\cup(\partial V_q\cap\partial V_{q'})).\end{aligned}$$

We finally sum over all pairs. Since the number of faces of the polyhedra is uniformly bounded, each V_q is included only in the estimates for a uniformly bounded number of faces and therefore

$$|Dw-\mu^*|(\mathbb{R}^n)\le c|Dv-\mu^*|(\mathbb{R}^n)\le c\varepsilon|Du|(\mathbb{R}^n)+\varepsilon\,.$$

Since $\varepsilon>0$ was arbitrary, this concludes the proof of Step 3.

The proof of the result then follows by choosing $\varepsilon=1/j$ and defining f_j, with a slight abuse of notation, as the corresponding function f in Step 2. □

Remark 2.3. Since in Step 1 of the proof of Theorem 2.1 we may assume $r_y<\varepsilon$, the construction above additionally gives that $\operatorname{dist}(x,\operatorname{supp}Du)<1/j$ for all x such that $f_j(x)\ne x$.

Corollary 2.4. *In the setting of Theorem 2.1, if $\psi:S^{n-1}\times\mathcal{Z}\times\mathcal{Z}\to[0,\infty)$ is continuous and symmetric and $E[u]:=\int_{J_u\cap\Omega}\psi(\nu,u^+,u^-)d\mathcal{H}^{n-1}$, then $E[u_j]\to E[u]$.*

Proof. Since $|D(u\circ f_j)-Du_j|(\Omega)\to 0$ we have

$$\mathcal{H}^{n-1}(\Omega\cap(J_{u\circ f_j}\triangle J_{u_j}))+\mathcal{H}^{n-1}(\{x\in J_{u\circ f_j}\cap J_{u_j}\cap\Omega: u_j^\pm\ne(u\circ f_j)^\pm\})=o(1).$$

Since $\mathcal{Z}$ is finite and S^{n-1} compact the function ψ is bounded. Therefore the previous estimate implies that

$$E[u_j]=\int_{J_{u\circ f_j}\cap\Omega}\psi(\nu_{u\circ f_j},(u\circ f_j)^+,(u\circ f_j)^-)d\mathcal{H}^{n-1}+o(1).$$

We remark that $u\circ f_j$ is defined on $f_j^{-1}(\Omega)$, and denote by $\nu_{u\circ f_j}$ the normal to its jump set $J_{u\circ f_j}=f_j^{-1}(J_u)\subset f_j^{-1}(\Omega)$. One easily checks that $\nu_{u\circ f_j}(x)=Df_j^T(x)\nu(f_j(x))/|Df_j^T(x)\nu(f_j(x))|$.

By the change-of-variables formula, see [AFP00, Th. 2.91], we have

$$\begin{aligned}&\int_{J_{u\circ f_j}\cap\Omega}\psi(\nu_{u\circ f_j},(u\circ f_j)^+,(u\circ f_j)^-)d\mathcal{H}^{n-1}\\ &\qquad=\int_{J_u\cap f_j(\Omega)}\psi(\nu_j,u^+,u^-)J_{n-1}d^{J_u}f_j^{-1}d\mathcal{H}^{n-1},\end{aligned}$$

where $\nu_j := \nu_{u\circ f_j} \circ f_j^{-1}$ is the normal to $J_{u\circ f_j}$ transported by f_j, which converges uniformly to ν as $j \to \infty$, and $J_{n-1}d^{J_u} f_j^{-1}$ is the Jacobian of the tangential differential of f_j^{-1}. The claim then follows by dominated convergence using continuity of $\nu \mapsto \psi(\nu, \alpha, \beta)$, that ∇f_j tends to the identity, and the fact that $\mathcal{H}^{n-1}(J_u \setminus f_j(\Omega)) \to 0$. □

Corollary 2.5. *In the setting of Theorem 2.1, we obtain that for all $z, z' \in \mathcal{Z}$ the polyhedral sets $A_j^z := \{x \in \Omega : u_j(x) = z\}$ are such that $\mathcal{H}^{n-1}(\partial A_j^z \cap \partial A_j^{z'} \cap \Omega) \to \mathcal{H}^{n-1}(\partial A^z \cap \partial A^{z'} \cap \Omega)$, where $A^z := \{x \in \Omega : u(x) = z\}$ and ∂ denotes the reduced boundary.*

Proof. It follows from the previous Corollary choosing $\psi(\nu, \alpha, \beta) = 1$ if $\{\alpha, \beta\} = \{z, z'\}$ and $\psi(\nu, \alpha, \beta) = 0$ otherwise. □

We finally state and prove the lemma used in the proof of Step 3 above.

Lemma 2.6. *There is $c_* > 0$, depending only on n, such that the following holds: Let $P_1, \dots, P_M$ be $n-1$-dimensional disjoint closed polyhedra in $\mathbb{R}^n$. Then for $\delta > 0$ sufficiently small there are countably many pairwise disjoint open convex n-dimensional polyhedra $V_q \subset \mathbb{R}^n$, $q \in G$, such that $|\mathbb{R}^n \setminus \bigcup_q V_q| = 0$ and $P_j \cap V_q = \emptyset$ for all $j \in \{1, \dots, M\}$ and $q \in G$. For any $q \in G$ there is $x_q \in \mathbb{R}^n$ such that $B_{\delta/c_*}(x_q) \subset V_q \subset B_{c_*\delta}(x_q)$. Each polyhedron V_q has at most c_* faces.*

The idea of the proof is to define G as a set of points in $\mathbb{R}^n$ with a spacing of order δ; and then to construct $(V_q)_{q\in G}$ as the corresponding Voronoi tessellation. In order for the polyhedral P_j to be contained in the boundaries between the V_q, in a neighbourhood of each P_j, we use a grid oriented as P_j. The remaining difficulty is to interpolate between grids of different orientation. This is done superimposing the grids and removing, in an intermediate layer, some points so that the remaining ones have approximately distance δ from each other.

Proof. We set $\delta_0 := \frac{1}{5n} \min_{i,j} \operatorname{dist}(P_i, P_j)$. For any j we define the t-neighbourhood of P_j by $(P_j)_t := \{x \in \mathbb{R}^n : \operatorname{dist}(x, P_j) < t\}$ and fix an affine isometry $I_j : \mathbb{R}^n \to \mathbb{R}^n$ such that $P_j \subset I_j(\mathbb{R}^{n-1} \times \{0\})$.

For $\delta \in (0, \delta_0)$ we set

$$\hat{G}_0 := \delta\mathbb{Z}^n \setminus \bigcup_{j=1}^{M} (P_j)_{2n\delta_0}$$

and, for $j = 1, \dots, M$,

$$\hat{G}_j := I_j\Big(\delta\mathbb{Z}^n + \frac{1}{2}\delta e_n\Big) \cap (P_j)_{3n\delta_0}\,.$$

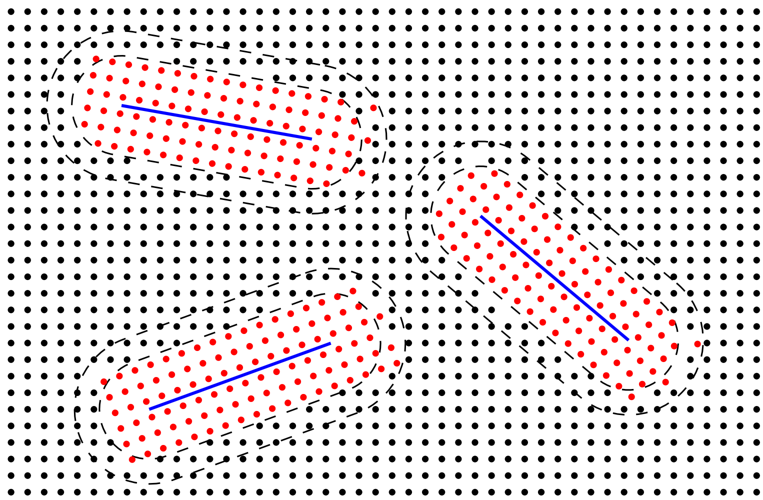

Figure 1: Sketch of the grid construction in the proof of Lemma 2.6.

The set $\hat{G} := \bigcup_{j=0}^{M} \hat{G}_j$ is a discrete set with the property that any $x \in \mathbb{R}^n$ has distance at most $\sqrt{n}\delta$ from $\hat{G}$, see Figure 1 for an illustration. Inside each of the disjoint sets $(P_j)_{2n\delta_0}$ the set $\hat{G}$ coincides with $I_j(\delta\mathbb{Z}^n + \frac{1}{2}\delta e_n)$. We define $G \subset \hat{G}$ as a maximal subset with the property that any two points of G have a distance of at least δ/n. By maximality, for any $z \in \hat{G} \setminus G$ there is $q \in G$ with $|q - z| < \delta/n$; hence for any $x \in \mathbb{R}^n$ there is a point $q \in G$ with $|x - q| \leq (\sqrt{n} + 1/n)\delta \leq n\delta$. Further, $G \cap (P_j)_{n\delta_0} = \hat{G} \cap (P_j)_{n\delta_0}$.

For $q \in G$, let $V_q := \{x \in \mathbb{R}^n : |x - q| < |x - z| \text{ for all } z \in G, z \neq q\}$. The family of all such V_q is the Voronoi tessellation of $\mathbb{R}^n$ induced by G. The V_q are open, disjoint, convex polyhedra which cover $\mathbb{R}^n$ up to a null set. This concludes the construction.

It remains to prove the stated properties. Since the distance of two points in G is at least δ/n, we have $B_{\delta/(2n)}(q) \subset V_q$. Since any point in $\mathbb{R}^n$ is at distance smaller than $n\delta$ from a point of G, we have $V_q \subset B_{n\delta}(q)$. In particular, $\overline{V}_q \cap \overline{V}_{q'} \neq \emptyset$ implies $|q - q'| \leq 2n\delta$. Since the balls $B_{\delta/(2n)}(q)$, with $q \in G$, are disjoint, given $q \in G$ there are at most $(4n^2)^n$ points $q' \in G$ such that $|q - q'| \leq 2n\delta$. It follows that V_q is a polyhedron with at most $(4n^2)^n$ faces.

We finally show that the polyhedra P_j are cointained in the union of the boundaries of the V_q. To do this, fix one $j \in \{1, \ldots, M\}$. Set $G_j := I_j(\delta\mathbb{Z}^n + \frac{1}{2}\delta e_n) \cap (P_j)_{n\delta_0}$. By construction, $G \cap (P_j)_{n\delta_0} = G_j$. In particular, $P_j \subset \bigcup_{q \in G_j} \overline{V}_q$. At the same time, since G_j is symmetric with respect to the hyperplane which contains P_j, each point of P_j is equidistant from at least two of its points, and therefore $P_j \subset \bigcup_{q \in G_j} \partial V_q$. This concludes the proof. $\square$

We finally turn to the extension argument which is needed for the derivation of Theorem 2.1 from Theorem 2.2.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz set with $\partial\Omega$ bounded, let $\mathcal{Z} \subset \mathbb{R}^N$ be a finite set, and let $u \in SBV_{\mathrm{loc}}(\Omega; \mathcal{Z})$. Then, there is an extension $\tilde{u} \in SBV_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{Z})$ with $\tilde{u} = u$ in Ω, $|D\tilde{u}|(\partial\Omega) = 0$, $|D\tilde{u}|(\mathbb{R}^n) < c|Du|(\Omega)$.*

Proof. To construct the extension, we fix $\eta \in (0,1)$ and $\hat{\nu} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ a smoothing of the outer normal ν to $\partial\Omega$, such that $|\hat{\nu}| = 1$ and $\hat{\nu} \cdot \nu > \eta$ $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$. The map $\hat{\nu}$ is constructed by considering a covering of $\partial\Omega$ by balls in which Ω is a Lipschitz subgraph, in the sense that $\Omega \cap B_r(x) = \{y \in B_r(x) : (Q_x y)_n < \psi_x(\Pi Q_x y)\}$, with $x \in \partial\Omega$, $Q_x \in O(n)$, $\psi_x : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz, and $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection onto the first $n-1$ components. This implies $Q_x^T e_n \cdot \nu \geq \eta_x := 1/\sqrt{1 + (\mathrm{Lip}(\psi_x))^2}$ on $B_r(x) \cap \partial\Omega$. By compactness, $\partial\Omega$ is covered by finitely many such balls $\{B_{r_j}(x_j)\}_{j=1,\dots,J}$. We fix a partition of unity $g_j \in C_c^\infty(B_{r_j}(x_j))$ with $\sum_j g_j = 1$ on $\partial\Omega$ and define $\hat{\nu}_* := \sum_j g_j Q_{x_j}^T e_n$, $\eta := \min_j \eta_{x_j}$. It remains only to rescale so that $|\hat{\nu}| = 1$ on $\partial\Omega$. Since we already know that $|\hat{\nu}_*| \geq \hat{\nu}_* \cdot \nu \geq \eta$ on $\partial\Omega$ this can be done setting $\hat{\nu} := \varphi(\hat{\nu}_*)$, where $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ coincides with the projection onto the unit sphere outside $B_\eta(0)$.

Having constructed $\hat{\nu}$ and η, we observe that there is $\rho > 0$ such that $(x,t) \mapsto \Phi(x,t) := x + t\hat{\nu}(x)$ is a bilipschitz map from $\partial\Omega \times (-\rho, \rho)$ to a tubular neighbourhood of $\partial\Omega$. To see this, one first uses the implicit function theorem on the map $\mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto (\Phi(x,t), t) \in \mathbb{R}^n \times \mathbb{R}$ to see that it is a diffeomorphism in a neighbourhood of any $(x,t) \in \partial\Omega \times \{0\}$, then the compactness of $\partial\Omega$ to show that it is covered by a finite number of such sets, and finally one restricts to $x \in \partial\Omega$.

We define $\tilde{u}(x + t\hat{\nu}(x)) = u(x - t\hat{\nu}(x))$ for $x \in \partial\Omega$ and $t \in (0, \rho)$, or equivalently $\tilde{u}(x) = u(\Phi(P_t \Phi^{-1}(x))$ for $x \in \Phi(\partial\Omega \times (0,t))$, where P_t is the linear map that flips the sign of the last argument. We further set $\tilde{u} = u$ in Ω, and $\tilde{u}$ equal to a constant arbitrary element z_0 of $\mathcal{Z}$ on the rest of $\mathbb{R}^n$. Then $\tilde{u} : \mathbb{R}^n \to \mathcal{Z}$. By the chain rule for SBV functions, $\tilde{u} \in SBV_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{Z})$. By the construction $\tilde{u}$ has the same trace on both sides of $\partial\Omega$, hence $|D\tilde{u}|(\partial\Omega) = 0$. □

Proof of Theorem 2.1. It suffices to apply Theorem 2.2 to the extension $\tilde{u}$ of u constructed in Lemma 2.7. □

Remark 2.8. In the statement of Theorem 2.2 we can replace the Lipschitz and boundedness assumption on Ω by the requirement that an extension as in Lemma 2.7 exists. Such an assumption is satisfied for example if Ω is a half space, taking the extension by reflection.

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