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Regularity for non-autonomous functionals with almost linear growth

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Abstract. We consider non-autonomous functionals $\mathcal{F}(u;\Omega) = \int_\Omega f(x, Du)\, dx$, where the density $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ has almost linear growth, i.e.,

$$f(x, \xi) \approx |\xi| \log(1 + |\xi|).$$

We prove partial $C^{1,\gamma}$-regularity for minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ under the assumption that $D_\xi f(x, \xi)$ is Hölder continuous with respect to the x-variable. If the x-dependence is C^1 we can improve this to full regularity provided additional structure conditions are satisfied.

Keywords. Variational integrals; Non-standard growth conditions; Partial regularity.

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1 Introduction

This paper is concerned with variational functionals of the form

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, Du)\, dx \tag{1.1}$$

for a mapping $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $n \geq 2, N \geq 1$ and Ω a bounded open set in $\mathbb{R}^n$. Here the integrand $f : (x, \xi) \in \Omega \times \mathbb{R}^{n\times N} \to [0, +\infty)$ is strictly convex with respect to the variable $\xi \in \mathbb{R}^{n\times N}$ and therefore the existence of minimizers is established by the direct methods of the calculus of variations.

The study of $C^{1,\gamma}$-partial regularity for minimizers of the functional (1.1) has been achieved when the integrand grows as a power function $|\xi|^p$ for some $p > 1$ (see [21] for an exhaustive treatment) or in case it satisfies the so called non standard growth conditions, i.e.

$$c|\xi|^p \leq f(x, \xi) \leq C(1 + |\xi|^q)$$

for some $1 < p \leq q < +\infty$ and positive constants c, C(see [2, 4, 6, 15, 24, 28] and [25] for a nice survey).

In this paper we will not be concerned with such cases in any essential way. In fact, we will focus our attention on integrands which are not too far from being linear in $|\xi|$, that is

$$\lim_{|\xi|\to+\infty}\frac{|f(x,\xi)|}{|\xi|}=+\infty,\qquad \lim_{|\xi|\to+\infty}\frac{|f(x,\xi)|}{|\xi|^p}=0\qquad \forall p>1. \tag{1.2}$$

It is worth mentioning that many regularity results have been established for integrals with nearly linear growth in case they do not depend on the x variable.

The earliest paper on this subject is due to Greco, Iwaniec and Sbordone (see [22]), in which the higher integrability of the minimizers has been proved in the scale of Orlicz spaces for a large class of autonomous functionals satisfying (1.2).

After that, Fuchs and Seregin in [20] proved the $C^{1,\gamma}$ partial regularity for minimizers of

$$J(u)=\int_\Omega |Du|\log(1+|Du|)\,dx$$

under the assumption $n\le 4$. Such result has been extended to any dimension n by Esposito and Mingione in [17] and later on the full $C^{1,\gamma}$-regularity has been established in [18, 27]. All the quoted papers concern the autonomous case.

Actually, variational functionals whose integrand depend on x arise in problems of mathematical physics and engineering and they attracted great interest.

Regularity results for minimizers of non-autonomous functionals satisfying non standard growth conditions have been established in [6, 7, 12, 13, 16].

Note that functionals with nearly linear growth have features in common with ones satisfying non standard growth since, by virtue of (1.2), we have that

$$c|\xi|\le f(x,\xi)\le C(1+|\xi|^p),\qquad \forall p>1.$$

The aim of this paper is to establish $C^{1,\gamma}$-partial regularity of minimizers of (1.1) with an integrand f satisfying the assumption

$$c_0\mathcal{A}(|\xi|)-c_1\le f(x,\xi)\le c_2\mathcal{A}(|\xi|)+c_3 \tag{F1}$$

where c_i are positive constants, $\xi\in\mathbb{R}^{nN}$ and

$$\mathcal{A}(t)=t\log(1+t),$$

with $t\ge 0$.

Here we shall assume that there exist constants $c_4,c_5,\nu>0$ and an exponent $\alpha\in(0,1)$ such that f is a function fulfilling (F1) and whose derivatives satisfy the following assumptions:

$$|D_\xi f(x,\xi)|\le c_4(1+\log(1+|\xi|)); \tag{F2}$$

$$|D_\xi f(x_1,\xi)-D_\xi f(x_2,\xi)|\le c_5|x_1-x_2|^\alpha\log(1+|\xi|); \tag{F3}$$

$$\nu(1+|\xi_1|+|\xi_2|)^{-1}|\xi_1-\xi_2|^2 \le \left\langle D_\xi f(x,\xi_1)-D_\xi f(x,\xi_2),\xi_1-\xi_2\right\rangle; \qquad \text{(F4)}$$

for any $\xi,\xi_1,\xi_2 \in \mathbb{R}^{nN}$ and for any $x,x_1,x_2 \in \Omega$. Moreover to perform the blow up procedure we shall need $D_{\xi\xi}f \in C^0(\Omega\times\mathbb{R}^{nN})$ and satisfying the following assumption

$$\nu(1+|\xi|)^{-1}|\zeta|^2 \le \left\langle D_{\xi\xi}f(x,\xi)\zeta,\zeta\right\rangle \le c_6\frac{\log(1+|\xi|)}{|\xi|}|\zeta|^2, \qquad \text{(F5)}$$

with a positive constant c_6. Note that (F1) and the convexity assumption (F4) imply (F2).

The first result of this paper is the following higher integrability property of minimizers of the functional $\mathcal{F}$. This result will be useful to prove regularity and it is also of interest by itself. It will be proved under weaker assumptions than the ones needed to prove $C^{1,\gamma}$ regularity.

Theorem 1.1. Let $u \in W^{1,\mathcal{A}}_{loc}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (F1) – (F4). Then we have

$$Du \in L^s_{loc}(\Omega), \qquad \forall\, s < \frac{n}{n-\alpha},$$

and

$$||\,(V_1(Du))^2\,||_{L^{\frac{n}{n-2b}}(B_\rho)} \le c\int_{B_{2R}}|Du|\log(1+|Du|)\,dx + c\int_{B_{2R}}|V_1(Du)|^2\,dx,$$

for every pair of concentric balls $B_\rho \subset B_{2R} \Subset \Omega$ and for every $b \in (0,\frac{\alpha}{2})$. Here α is the exponent appearing in (F3) and we denoted by $V_1(\xi)=(1+|\xi|^2)^{-\frac{1}{4}}\xi$.

Corollary 1.2. Under the same assumptions of Theorem 1.1, if $u \in W^{1,\mathcal{A}}_{loc}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then we have

$$Du \in W^{k,p}_{loc}(\Omega,\mathbb{R}^{nN}), \qquad (1.3)$$

for every $k \in (0,\frac{\alpha}{2})$ and for every $1 < p < \frac{n}{n-\frac{\alpha}{2}}$.

The higher integrability of Theorem 1.1 allows us to prove a $C^{1,\gamma}$-partial regularity result which is formulated in the following

Theorem 1.3. Let f be a $C^2(\Omega,\mathbb{R}^{nN})$-integrand satisfying the assumptions (F1) and (F3) – (F5). If $u \in W^{1,\mathcal{A}}_{loc}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then there exists an open subset Ω_0 of Ω such that

$$\text{meas}(\Omega\setminus\Omega_0)=0$$

and

$$u \in C^{1,\gamma}_{loc}(\Omega_0, \mathbb{R}^N) \qquad \text{for every} \qquad \gamma < \frac{\alpha}{2},$$

where α is the exponent appearing in (F3).

Our proof is based on a blow up argument aimed to establish a decay estimate for the excess function of the minimizer. The proof has features in common with [17], since we use the higher integrability Theorem 1.1 in order to define the excess function as

$$E(x,r) = \fint_{B_r(x)} |V_p(Du) - V_p((Du)_r)|^2 + r^\beta$$

with

$$V_p(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}} \xi.$$

The main difference with [17] is that, in order to perform the blow up procedure, we use a Caccioppoli type inequality for minimizers of a suitable perturbation of the rescaled functional, as done in [12].

The main difficulty in order to prove the Caccioppoli type inequality is the proof of a uniform higher integrability result for the minimizers of the rescaled functionals. We have to combine the difference quotient method with properties of Orlicz-Sobolev classes generated by an Orlicz function which grows almost linearly. We also use the properties of the function $V_p(\xi)$ which is an useful tool to deal with subquadratic setting.

In order to improve this to everywhere regularity, additional assumptions are necessary. The first is the modulus dependence, i.e.,

$$f(x,\xi) = \widehat{f}(x, |\xi|) \tag{F6}$$

for a function $\widehat{f} : \Omega \times [0, \infty) \to \mathbb{R}$ which is strictly increasing in the real variable. According to counterexamples of De Giorgi (see [10]), when dealing with vectorial minimizers, i.e. $N > 1$, it is well-known that without this assumption there is no hope for full regularity. On the other hand we need a Caccioppoli-type inequality in order to apply De Giorgi arguments, hence we assume for every $s \in \{1, ..., n\}$

$$\partial_s D_\xi f \in C^0(\Omega \times \mathbb{R}^{nN}, \mathbb{R}^{nN}) \quad \text{and} \quad |\partial_s D_\xi f(x,\xi)| \leq c(1 + |\xi|)^{p-1} \tag{F7}$$

for an exponent $1 < p < \frac{n - \frac{\alpha}{2}}{n - \alpha}$. Finally we suppose that

$$|D^2_{\xi\xi}(x, \xi_1) - D^2_{\xi\xi}(x, \xi_2)| \leq c(1 + |\xi_1| + |\xi_2|)^{p-2-\mu} |\xi_1 - \xi_2|^\mu \tag{F8}$$

for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^{nN}$ and for an exponent $\mu \in (0, 1)$. Of course (F7) and (F8) are true in the autonomous case for $f(x,\xi) = |\xi| \log^\theta(1 + |\xi|)$, $\theta > 0$, for every choice of $p > 1$. The full regularity result of this paper is the following

Theorem 1.4. Let $u \in W^{1,\mathcal{A}}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (F1) and (F3) – (F8). Then we have

$$u \in C^{1,\gamma}_{loc}(\Omega, \mathbb{R}^N), \quad \text{for all } \gamma < 1.$$

Thanks to Theorem 1.3 we have a nonempty set of regular points for every minimizer of the functional $\mathcal{F}$ with a general integrand function f. Therefore Corollary 1.2 allows us to apply Lemma 2.16 (stated in the next section) to give an estimate of the Hausdorff dimension of the singular set of minimizers of $\mathcal{F}$.

Corollary 1.5. If f is a C^2 function satisfying the assumptions $(F1)$ and $(F3) - (F5)$ and the function $u \in W^{1,\mathcal{A}}(\Omega; \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ in Ω, then for the Hausdorff dimension of the singular set Σ of the function u the following estimate hold

$$dim_{\mathcal{H}}(\Sigma) \leq n - \frac{\alpha}{2} q$$

where $q = \frac{n}{n - \frac{\alpha}{2}}$.

See also [11].

2 Notations and preliminaries

In this section we recall some standard definitions and collect several Lemmas that we shall need to establish our main results.

We shall indicate with $B_R(x_0)$ the ball centered at the point $x_0 \in \mathbb{R}^n$ and having radius $R > 0$. We shall omit the center of the ball when no confusion arises. All the balls considered will be concentric unless differently specified.

As usual $\{e_s\}_{1 \leq s \leq n}$ is the standard basis in $\mathbb{R}^n$ and if $u, v \in \mathbb{R}^k$ the tensor product $u \otimes v \in \mathbb{R}^{k^2}$ of u and v is defined by $(u \otimes v)_{i,j} := v_i w_j$.

In the estimates c will denote a constant, depending on the data of the problem, that may change from line to line.

Now we recall the definition of the Orlicz-Sobolev space (for more details on this topic we refer to [3])

Definition 2.1. a) A function $\varphi : [0, \infty) \to [0, \infty)$ is called a Young function, if φ is strictly increasing, convex and satisfies

$$\lim_{t \to 0} \frac{\varphi(t)}{t} = \lim_{t \to \infty} \frac{t}{\varphi(t)} = 0.$$

b) If φ satisfies in addition a global (Δ_2)-condition, i.e.,

$$\varphi(2t) \leq c\varphi(t) \quad \text{for all } t \geq 0,$$

then we define

$$L_\varphi(\Omega,\mathbb{R}^N) := \left\{ u \in L_1(\Omega,\mathbb{R}^N) : \int_\Omega \varphi(|u|)\,dx < \infty \right\},$$

which is a Banach space endowed with the Luxemburg norm

$$\|u\|_\varphi := \inf \left\{ k \geq 0 : \int_\Omega \varphi\left(\frac{|u|}{k}\right)\,dx \leq 1 \right\}.$$

c) A function $u : \Omega \to \mathbb{R}^N$ belongs to the space $W^{1,\varphi}(\Omega,\mathbb{R}^N)$ if $u \in L_\varphi(\Omega,\mathbb{R}^N)$ and its distributional gradient $Du \in L_\varphi(\Omega,\mathbb{R}^{nN})$. $W^{1,\varphi}(\Omega,\mathbb{R}^N)$ is a Banach-space together with the norm

$$\|u\|_{1,\varphi} := \|u\|_\varphi + \|Du\|_\varphi.$$

d) We define $W^{1,\varphi}_0(\Omega,\mathbb{R}^N)$ as the closure of $C^\infty_0(\Omega,\mathbb{R}^N)$ with respect to the $W^{1,\varphi}(\Omega,\mathbb{R}^N)$-norm.

Now we can give the definition of a local minimizer, that in our case takes place:

Definition 2.2. A function $u \in W^{1,\mathcal{A}}_{loc}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if

$$\int_{supp\,\varphi} f(x,Du)\,dx \leq \int_{supp\,\varphi} f(x,Du+D\varphi)\,dx,$$

for any $\varphi \in W^{1,\mathcal{A}}_{loc}(\Omega,\mathbb{R}^N)$ with $supp\,\varphi \subset \Omega$.

As usual, in order to prove the higher integrability of the local minimizers, we shall need the machinery of fractional order Sobolev spaces. These spaces are defined as follows.

Definition 2.3. If A is a smooth, bounded open subset of $\mathbb{R}^n$ and $\theta \in (0,1)$, $1 \leq p < +\infty$ a function u belongs to the fractional order Sobolev space $W^{\theta,p}(A;\mathbb{R}^n)$ if and only if

$$||u||_{W^{\theta,p}} := \left(\int_A |u(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_A\int_A \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\theta}}\,dx\,dy\right)^{\frac{1}{p}} < \infty.$$

This quantity is a norm making $W^{\theta,p}(A;\mathbb{R}^n)$ a Banach space.

In the context of fractional order Sobolev spaces we have to use fractional difference quotients. Therefore we recall the finite difference operator.

Definition 2.4. For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by

$$\tau_{s,h}F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \dots, n\}$.

The difference quotient is defined for $h \in \mathbb{R} \setminus \{0\}$ as

$$\Delta_{s,h}F(x) = \frac{\tau_{s,h}F(x)}{h}.$$

The following proposition describes some elementary properties of the finite difference operator and can be found, for example, in [21].

Proposition 2.5. Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

$(d1)$ $\tau_{s,h}F \in W^{1,p}(\Omega_{|h|})$ and

$$D_i(\tau_{s,h}F) = \tau_{s,h}(D_iF).$$

$(d2)$ If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then

$$\int_\Omega F\, \tau_{s,h}G\, dx = -\int_\Omega G\, \tau_{s,-h}F\, dx.$$

$(d3)$ We have

$$\tau_{s,h}(FG)(x) = F(x + he_s)\tau_{s,h}G(x) + G(x)\tau_{s,h}F(x).$$

Next Lemma was proved in [1] (See Lemma 2.2).

Lemma 2.6. For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have

$$(2\gamma + 1)|\xi - \eta| \leq \frac{|(\mu^2 + |\xi|^2)^\gamma \xi - (\mu^2 + |\eta|^2)^\gamma \eta|}{(\mu^2 + |\xi|^2 + |\eta|^2)^\gamma} \leq \frac{c(k)}{2\gamma + 1}|\xi - \eta|$$

for every $\xi, \eta \in \mathbb{R}^k$.

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.7. If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1, \dots, n\}$ and $F, D_s F \in L^p(B_R)$ then

$$\int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le |h|^p \int_{B_R} |D_s F(x)|^p \, dx.$$

Moreover

$$\int_{B_\rho} |F(x + he_s)|^p \, dx \le c(n,p) \int_{B_R} |F(x)|^p \, dx.$$

Now we recall the fundamental embedding properties for fractional order Sobolev spaces. (For the proof we refer to [3]).

Lemma 2.8. If $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^2(B_R)$ and for some $\rho \in (0, R)$, $\beta \in (0, 1]$, $M > 0$,

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^2 \, dx \le M^2 |h|^{2\beta}$$

for every h with $|h| < \frac{R-\rho}{2}$, then $F \in W^{k,2}(B_\rho; \mathbb{R}^N) \cap L^{\frac{2n}{n-2k}}(B_\rho; \mathbb{R}^N)$ for every $k \in (0, \beta)$ and

$$||F||_{L^{\frac{2n}{n-2k}}(B_\rho)} \le c \left(M + ||F||_{L^2(B_R)} \right),$$

with $c \equiv c(n, N, R, \rho, \beta, k)$.

Previous Lemma can be reformulated as follows

Lemma 2.9. If $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1 < p < +\infty$ and for some $\rho \in (0, R)$, $\beta \in (0, 1]$, $M > 0$,

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^p \, dx \le M^p |h|^{p\beta}$$

for every h with $|h| < \frac{R-\rho}{2}$, then $F \in W^{k,p}(B_\rho; \mathbb{R}^N) \cap L^{\frac{np}{n-kp}}(B_\rho; \mathbb{R}^N)$ for every $k \in (0, \beta)$ and

$$||F||_{L^{\frac{np}{n-kp}}(B_\rho)} \le c \left(M + ||F||_{L^p(B_R)} \right),$$

with $c \equiv c(n, N, R, \rho, \beta, k)$.

Next Lemma finds an important application in the so called hole-filling method. Its proof can be found in [21] (See Lemma 6.1).

Lemma 2.10. Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $0 \le A$, $0 \le B$ and $0 < \beta$. Assume that

$$h(r) \le \frac{A}{(d-r)^\beta} + B + \theta h(d)$$

for $\rho \leq r < d \leq R_0$. Then

$$h(\rho) \leq \frac{cA}{(R_0-\rho)^\beta} + B,$$

where $c = c(\theta, \beta) > 0$.

We shall need the following Poincaré-Sobolev inequality, whose proof can be found in [14] (for other versions of this inequality we refer to [8, 9]).

Lemma 2.11. Assume $1 < p < 2$ and let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then there exists a positive constant $c \equiv c(n, N, p)$ such that

$$\left(\fint_{B_\rho(x_0)} \left| V_p\left(\frac{u-(u)_\rho}{\rho} \right) \right|^{\frac{2n}{n-p}} dx \right)^{\frac{n-p}{2n}} \leq c \left(\fint_{B_\rho(x_0)} |V(Du)|^2 \, dx \right)^{\frac{1}{2}}.$$

Next result is a simple consequence of the a priori estimates for solutions to linear elliptic systems with constant coefficients.

Proposition 2.12. Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, $p \geq 1$ be such that

$$\int_\Omega A^{ij}_{\alpha\beta} D_\alpha u^i D_\beta \varphi^j \, dx = 0$$

for every $\varphi \in C^\infty_0(\Omega; \mathbb{R}^N)$, where $A^{ij}_{\alpha\beta}$ is a constant matrix satisfying the strong Legendre Hadamard condition

$$A^{ij}_{\alpha\beta}\lambda^i\lambda^j\mu_\alpha\mu_\beta \geq \nu|\lambda|^2|\mu|^2 \qquad \forall\lambda \in \mathbb{R}^N,\ \mu \in \mathbb{R}^n.$$

Then $u \in C^\infty$ and for any ball $B_R(x_0) \Subset \Omega$ we have

$$\sup_{B_{\frac{R}{2}}(x_0)} |Du| \leq \frac{c}{R^n} \int_{B_R} |Du| \, dx$$

For the proof see [8].

We shall use the following auxiliary function defined for $\xi \in \mathbb{R}^k$

$$V_\beta(\xi) = (1 + |\xi|^2)^{\frac{\beta-2}{4}} \xi,$$

for any exponent $\beta \geq 1$. Recall that for $\beta > 1$

$$|V_\beta(\xi)| \text{ is a non-decreasing function of } |\xi|; \tag{2.1}$$

$$|V_\beta(\xi+\eta)| \leq c(\beta)(|V_\beta(\xi)| + |V_\beta(\eta)|); \tag{2.2}$$

$$\min\{t^2, t^\beta\}|V_\beta(\xi)|^2 \leq |V_\beta(t\xi)|^2 \leq \max\{t^2, t^\beta\}|V_\beta(\xi)|^2; \tag{2.3}$$

$$|V(\xi)-V(\eta)|\le c(\beta)|V(\xi-\eta)|\le c(\beta,|\eta|)|V(\xi)-V(\eta)| \text{ if } 1<\beta<2; \quad (2.4)$$

$$(1+|\xi|^2+|\eta|^2)^{\frac{\beta}{2}}\le 1+(1+|\xi|^2+|\eta|^2)^{\frac{\beta-2}{2}}(|\xi|^2+|\eta|^2) \qquad \text{if} \beta\le 2; \quad (2.5)$$

$$c(\beta)(|\xi|^2+|\xi|^\beta)\le |V_\beta(\xi)|^2\le C(\beta)(|\xi|^2+|\xi|^\beta) \qquad \text{if} \quad \beta\ge 2; \quad (2.6)$$

$$|V_\beta(\xi)|^2 \qquad \text{is convex if} \qquad 1<\beta<2. \quad (2.7)$$

Many of the previous properties of the function V_β can be easily checked and they have been successfully employed in the study of the regularity of minimizers of convex and quasiconvex integrals under subquadratic growth conditions ([1, 8, 9, 29]). In our context, the following elementary inequality will also be useful.

Lemma 2.13. Set

$$V_p(\xi)=(1+|\xi|^2)^{\frac{p-2}{4}}\xi.$$

Then for every $\rho>0$ and function v with the suitable integrability degree, we have

$$\int_{B_\rho}|V_p(Dv)|^2\,dx\le c(p)\int_{B_\rho}|V_1(Dv)|^2\,dx+c(p)\int_{B_\rho}|V_1(Dv)|^{2p}\,dx,$$

for a constant c depending only on p.

Proof. We start by noting that

$$(1+|\xi|^2)^{\frac{1}{2}}\le 2[1+(1+|\xi|^2)^{-\frac{1}{2}}|\xi|^2]. \quad (2.8)$$

Indeed if $|\xi|\le 1$ we have

$$(1+|\xi|^2)^{\frac{1}{2}}\le\sqrt{2},$$

while, if $|\xi|>1$ we have

$$(1+|\xi|^2)^{\frac{1}{2}}=\frac{1+|\xi|^2}{(1+|\xi|^2)^{\frac{1}{2}}}\le\frac{2|\xi|^2}{(1+|\xi|^2)^{\frac{1}{2}}}.$$

Hence, recalling that $p>1$, we can conclude that

$$\begin{aligned}
\int_{B_\rho}|V_p(Dv)|^2\,dx&=\int_{B_\rho}|Dv|^2(1+|Dv|^2)^{\frac{p-2}{2}}\,dx\\
&=\int_{B_\rho}|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}(1+|Dv|^2)^{\frac{p-1}{2}}\,dx\\
&\le 2\int_{B_\rho}|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\left[1+|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\right]^{p-1}\,dx\\
&\le c(p)\int_{B_\rho}|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\,dx+c(p)\int_{B_\rho}\left(|Dv|^2(1+|Dv|^2)^{-\frac{1}{2}}\right)^p\,dx,
\end{aligned}$$

where we also used (2.8). □

We shall also need the following elementary inequality.

Lemma 2.14. For every $x \geq 0$ and $1 < p < 2$ we have

$$\log(1+x) \leq cx(1+x^2)^{\frac{p-2}{2}}.$$

for a constant $c = c(p)$.

Proof. The function

$$\varphi(x) = \frac{\log(1+x)}{x}(1+x^2)^{\frac{2-p}{2}}$$

is nonnegative for every $x > 0$ and

$$\lim_{x \to 0^+} \varphi(x) = 1.$$

Moreover, since $p < 2$, we have

$$\lim_{x \to +\infty} \varphi(x) = 0.$$

Since φ is continuous, there exists $c = c(p) \geq 0$ such that $\varphi(x) \leq c$ for every $x \in [0, +\infty]$. Hence the conclusion follows. □

In the linearization procedure we shall use the rescaled functional of $\mathcal{F}$ on the unit ball $B \equiv B_1(0)$

$$\mathcal{I}(v) := \int_B g(y, Dv)\, dy$$

defined by setting

$$g(y,\xi) = \lambda^{-2}[f(x_0 + r_0 y, A + \lambda\xi) - f(x_0 + r_0 y, A) - D_\xi f(x_0 + r_0 y, A)\lambda\xi], \quad (2.9)$$

where A is a matrix such that $|A|$ is uniformly bounded by a positive constant M. Next Lemma contains the growth conditions on g.

Lemma 2.15. Let $f \in C^2(\Omega \times \mathbb{R}^{n\times N})$ be a function satisfying the assumptions (F1) and (F3)-(F5) and let $g(y,\xi)$ be the function defined by (2.9). Then we have

$$\widetilde{\nu}\frac{|\xi|^2}{1+|\lambda\xi|} \leq |g(y,\xi)| \leq c\frac{\log(1+|\lambda\xi|)}{|\lambda\xi|}|\xi|^2; \quad \text{(I1)}$$

$$|D_\xi g(y,\xi)| \leq c\frac{\log(e+|\lambda\xi|)}{\lambda}; \quad \text{(I2)}$$

$$|D_\xi g(y_1,\xi) - D_\xi g(y_2,\xi)| \le \frac{c r_0^\alpha}{\lambda} |y_1 - y_2|^\alpha \left(\log(e + |\xi|)\right); \tag{I3}$$

$$\widetilde{\nu} \frac{|\zeta|^2}{1+|\lambda\xi|} \le \left\langle D_{\xi\xi} g(y,\xi)\zeta, \zeta \right\rangle \tag{I4}$$

where the constant c depends on M in all statements.

Proof. (I2), (I3) and (I4) can be proven as in [12] (Lemma 2.9) using the growth conditions of f. The lower bound in (I1) is a consequence of the representation

$$g(y,\xi) = \int_0^1 \int_0^t D_{\xi\xi} f(x_0 + r_0 y, A + s\lambda\xi)(\xi,\xi)\, ds\, dt$$

since we have by (F4)

$$\begin{aligned} D_{\xi\xi} f(x_0 + r_0 y, A + s\lambda\xi)(\xi,\xi) &\ge \mu \frac{|\xi|^2}{1 + |A + s\lambda\xi|} \\ &\ge \widetilde{\nu} \frac{|\zeta|^2}{1+|\lambda\xi|}. \end{aligned}$$

The upper bound is an immediate consequence of (F5). □

Now let us recall that the singular set Σ of a local minimizer u of the functional $\mathcal{F}$ is included in the set of non-Lebesgue points of Du. Therefore the estimate for the Hausdorff dimension of Σ is an immediate corollary of the regularity Theorem 1.1 through the application of the following proposition that can be found, for example, in [23] (see also Section 4 in [26] for a simple proof).

Lemma 2.16. Let $v \in W^{\theta,p}(\Omega,\mathbb{R}^N)$ where $\theta \in (0,1)$, $p > 1$ and set

$$A := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \fint_{B(x,\rho)} |v(y) - (v)_{x,\rho}|^p \, dy > 0 \right\},$$

$$B := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} |(v)_{x,\rho}| = +\infty \right\}.$$

Then

$$dim_{\mathcal{H}}(A) \le n - \theta p \quad \text{and} \quad dim_{\mathcal{H}}(B) \le n - \theta p.$$

3 Higher integrability

This section is devoted to the proof of the higher integrability result stated in Theorem 1.1.

Proof of Theorem 1.1. Let $u \in W^{1,\mathcal{A}}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with an integrand function f satisfying (F1) – (F4). Then u satysfies the Euler system related to the functional $\mathcal{F}$:

$$\int_\Omega D_\xi f(x, Du) D\varphi \, dx = 0 \tag{3.1}$$

for every $\varphi \in W^{1,\mathcal{A}}_0(\Omega)$ with compact support. Fix a ball $B_{2R} \Subset \Omega$ and let η be a cut-off function in $C^1_0(B_{3R/2})$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_R and $|D\eta| < c/R$. Let us consider the function $\varphi = \tau_{s,-h}(\eta^2(x)\tau_{s,h}u)$ with s fixed in $\{1, \ldots, n\}$ (which from now on we shall omit for the sake of simplicity) and $|h| < R/10$. Substituting in (3.1) the function φ and using $(d2)$ of Proposition 2.5 we get

$$\int_{B_{2R}} \tau_h(D_\xi f(x, Du)) \, D(\eta^2 \tau_h u) \, dx = 0.$$

This equality can be written as

$$\begin{aligned}
I = &\int_{B_{2R}} \eta^2 \left[D_\xi f(x + he_s, Du(x + he_s)) - D_\xi f(x + he_s, Du(x)) \right] \tau_h Du \, dx \\
= &- \int_{B_{2R}} \eta^2 [D_\xi f(x + he_s, Du(x)) - D_\xi f(x, Du(x))] \, \tau_h Du \, dx \\
&- 2 \int_{B_{2R}} \eta \, [D_\xi f(x + he_s, Du(x + he_s)) - D_\xi f(x, Du)] D\eta \otimes \tau_h u \, dx \\
= &- II - III
\end{aligned} \tag{3.2}$$

where we used $(d1)$ of Proposition 2.5. Assumption $(F4)$ yields that

$$\nu \int_{B_{2R}} \eta^2 \, (1 + |Du(x + he_s)| + |Du(x)|)^{-1} \, |\tau_h Du|^2 \, dx \leq I. \tag{3.3}$$

Using assumption $(F3)$ we obtain:

$$|II| \leq c|h|^\alpha \int_{B_{3R/2}} \log(1 + |Du|) |\tau_h Du| \, dx$$

and hence, by Young's Inequality for Young functions and properties of η, it follows that

$$|II| \leq c|h|^\alpha \left(\int_{B_{3R/2}} |Du| \log(1 + |Du|) \, dx + \int_{B_{3R/2}} |\tau_h Du| \log(1 + |\tau_h Du|) \, dx \right)$$

$$\leq c|h|^\alpha \int_{B_{2R}} |Du| \log(1+|Du|)\, dx. \tag{3.4}$$

To estimate III we use assumption $(F2)$ and Young's Inequality as follows

$$\begin{aligned}
|III| \leq & c|h| \int_{B_{2R}} \eta |D\eta| (1+\log(1+|Du(x+he_s)|)) |\Delta_h u|\, dx \\
& + c|h| \int_{B_{2R}} \eta |D\eta| (1+\log(1+|Du(x)|)) |\Delta_h u|\, dx \\
\leq & c|h| \int_{B_{3R/2}} \log(1+|Du(x+he_s)|) |Du(x+he_s)|\, dx \\
& + c|h| \int_{B_{3R/2}} \log(1+|\Delta_h u|) |\Delta_h u|\, dx + c|h| \int_{B_{2R}} \log(1+|Du|)|Du|\, dx \\
& + c|h| \int_{B_{2R}} (1+|Du|)\, dx \\
\leq & c|h|^\alpha \int_{B_{2R}} \log(1+|Du|)|Du|\, dx + c|h|^\alpha \int_{B_{2R}} (1+|Du|)\, dx. \qquad (3.5)
\end{aligned}$$

In order to estimate the $\Delta_h u$ integral in the last step, we used the following inequality which is valid for each convex function φ according to Jensen's Inequality:

$$\begin{aligned}
\int_{B_{3R/2}} \varphi(|\Delta_h u|)\, dx &= \int_{B_{3R/2}} \varphi\left(\left| \int_0^1 \frac{du}{ds}(x+the_s)\, dt \right| \right) dx \\
&\leq \int_{B_{3R/2}} \int_0^1 \varphi\left(\left| \frac{du}{ds}(x+the_s) \right| \right) dt\, dx \\
&\leq \int_{2R} \varphi\left(|Du|\right)\, dx. \qquad (3.6)
\end{aligned}$$

Inserting estimates (3.3), (3.4) and (3.5) into (3.2) we get

$$\begin{aligned}
&\nu \int_{B_{2R}} \eta^2 \left(1+|Du(x+he_s)|+|Du(x)|\right)^{-1} |\tau_h Du|^2\, dx \\
&\leq c|h|^\alpha \int_{B_{2R}} \log(1+|Du|)|Du|\, dx + c|h|^\alpha \int_{B_{2R}} (1+|Du|)\, dx. \qquad (3.7)
\end{aligned}$$

The left hand side of (3.7) can be controlled from below as follows

$$\begin{aligned}
\nu \int_{B_{2R}} \eta^2 \frac{|\tau_h Du|^2}{1+|Du(x+he_s)|+|Du(x)|} dx &\geq c \int_{B_{2R}} \eta^2 \frac{|\tau_h Du|^2}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{2}}}\, dx \\
&= c \int_{B_{2R}} \eta^2 \left(\frac{|Du(x+he_s)-Du(x)|}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{4}}} \right)^2 dx.
\end{aligned}$$

Lemma 2.6 applied for $\gamma = -\frac{1}{4}$ implies that

$$\begin{aligned}&\frac{|Du(x+he_s)-Du(x)|}{(1+|Du(x+he_s)|^2+|Du(x)|^2)^{\frac{1}{4}}}\\&\geq c|(1+|Du(x+he_s)|^2)^{-\frac{1}{4}}Du(x+he_s)-(1+|Du(x)|^2)^{-\frac{1}{4}}Du(x)|\\&=c|\tau_{s,h}V_1(Du(x))|.\end{aligned}$$

Hence

$$\nu\int_{B_{2R}}\eta^2\frac{|\tau_h Du|^2}{1+|Du(x+he_s)|+|Du(x)|}dx \geq c\int_{B_{2R}}\eta^2|\tau_{s,h}(V_1(Du))|^2\,dx.$$

Plugging this estimate in (3.7) we get

$$\int_{B_{2R}}\eta^2|\tau_{s,h}(V_1(Du))|^2\,dx \leq c|h|^\alpha\int_{B_{2R}}(1+|Du|\log(1+|Du|))\,dx. \tag{3.8}$$

Lemma 2.8 implies that

$$V_1(Du)\in W^{b,2}\cap L^{\frac{2n}{n-2b}}\qquad \forall\, b\in\left(0,\frac{\alpha}{2}\right),$$

and

$$||V_1(Du)||_{L^{\frac{2n}{n-2b}}(B_\rho)}\leq c\left(\int_{B_{2R}}(1+|Du|\log(1+|Du|))\,dx\right)^{\frac{1}{2}}+c\left(\int_{B_{2R}}|V_1(Du)|^2\,dx\right)^{\frac{1}{2}},$$

for every $\rho<2R$. Hence we get the claim and the final estimate:

$$||\left(V_1(Du)\right)^2||_{L^{\frac{n}{n-2b}}(B_\rho)}\leq c\int_{B_{2R}}(1+|Du|\log(1+|Du|))\,dx+c\int_{B_{2R}}|V_1(Du)|^2\,dx,$$

for every $\rho<2R$. □

The proof of Corollary 1.3 can be immediately obtained by applying Young's inequality with exponents $2/p$ and $2/(2-p)$ to the right hand side of the following equality

$$\begin{aligned}&\int_\Omega\eta^p|\tau_{h,s}Du|^p\,dx\\&=\int_\Omega[h^{-\chi}\eta^p(1+|Du(x+he_s)|+|Du(x)|)^{-\frac{p}{2}}|\tau_{h,s}Du|^p\\&\qquad\cdot h^\chi(1+|Du(x+he_s)|+|Du(x)|)^{\frac{p}{2}}]\,dx,\end{aligned}$$

where η is a suitable cut-off function and

$$\chi = \left(\frac{2}{2-p} + \frac{2}{p}\right)^{-1} \alpha = \frac{p(2-p)}{4}\alpha.$$

It follows

$$\begin{aligned}
&\int_\Omega \eta^p |\tau_{h,s} Du|^p \, dx \\
&\quad = |h|^{-\chi\frac{2}{p}} \int_\Omega \eta^2 (1 + |Du(x+he_s)| + |Du(x)|)^{-1} |\tau_{h,s} Du|^2 \, dx \\
&\quad + |h|^{\chi\frac{2}{2-p}} \int_\Omega \eta^{\frac{p}{2-p}} (1 + |Du(x+he_s)| + |Du(x)|)^{\frac{p}{2-p}} \, dx \\
&\quad \le c|h|^{\alpha-\chi\frac{2}{p}} + c|h|^{\chi\frac{2}{2-p}} = c|h|^{p\frac{\alpha}{2}},
\end{aligned}$$

by (3.8), the choice of χ and Theorem 1.1 provided $\frac{p}{2-p} < \frac{n}{n-\alpha}$ which is equivalent to $p < \frac{n}{n-\frac{\alpha}{2}}$. Hence we obtain the claim by Lemma 2.9.

4 Decay estimate

Define the excess function in accordance to [17] as

$$E(x,r) = ⨍_{B_r(x)} |V_p(Du) - V_p((Du)_r)|^2 + r^\beta \tag{4.1}$$

with $\beta < \alpha$ and $p < \frac{n}{n-\alpha}$. We remark that the higher integrability stated in Theorem 1.1 together with Lemma 2.13 allows us to give sense to $E(x,r)$ when $p < \frac{n}{n-\alpha}$ and therefore we may use a blow-up technique similar to the one used for functionals with p-growth, when $p < 2$.

The blow-up argument needed to prove Theorem 1.3 is contained in the following

Proposition 4.1. Fix $M > 0$. There exists a constant $C(M) > 0$ such that, for every $0 < \tau < \frac{1}{4}$, there exists $\epsilon = \epsilon(\tau, M)$ such that, if

$$|(Du)_{x_0,r}| \le M \qquad \text{and} \qquad E(x_0,r) \le \epsilon,$$

then

$$E(x_0, \tau r) \le C(M)\, \tau^\beta\, E(x_0, r),$$

where β is the exponent appearing in (4.1).

Proof. Step 1. Blow up

Fix $M > 0$. Assume by contradiction that there exists a sequence of balls $B_{r_j}(x_j) \Subset \Omega$ such that

$$|(Du)_{x_j,r_j}| \leq M \qquad \text{and} \qquad \lambda_j^2 = E(x_j, r_j) \to 0 \tag{4.2}$$

but

$$\frac{E(x_j, \tau r_j)}{\lambda_j^2} > \tilde{C}(M)\tau^\beta \tag{4.3}$$

where $\tilde{C}(M)$ will be determined later. Setting $A_j = (Du)_{x_j,r_j}$, $a_j = (u)_{x_j,r_j}$ and

$$v_j(y) = \frac{u(x_j + r_j y) - a_j - r_j A_j y}{\lambda_j r_j} \tag{4.4}$$

for all $y \in B_1(0)$, one can easily check that $(Dv_j)_{0,1} = 0$ and $(v_j)_{0,1} = 0$. By the definition of λ_j it follows that

$$\fint_{B_1(0)} \frac{|V_p(\lambda_j Du_j)|^2}{\lambda_j^2}\, dy + \frac{r_j^\beta}{\lambda_j^2} = 1. \tag{4.5}$$

Therefore passing possibly to not relabeled sequences (note that we obtain by (4.5) uniform L^p-bounds on Du_j)

$$\begin{aligned}
&v_j \rightharpoonup v && \text{weakly in } W^{1,p}(B_1(0); \mathbb{R}^N)\\
&\lambda_j v_j \to 0 && \text{strongly in } W^{1,p}(B_1(0); \mathbb{R}^N)\\
&v_j \to v && \text{strongly in } L^p(B_1(0); \mathbb{R}^N)\\
&A_j \to A && \\
&r_j \to 0 && \frac{r_j^\vartheta}{\lambda_j^2} \to 0, \qquad \vartheta > \beta.
\end{aligned} \tag{4.6}$$

Step 2. Minimality of v_j

We normalize f around A_j as follows

$$f_j(y, \xi) = \frac{f(x_j + r_j y, A_j + \lambda_j \xi) - f(x_j + r_j y, A_j) - D_\xi f(x_j + r_j y, A_j)\lambda_j \xi}{\lambda_j^2} \tag{4.7}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_j(w) = \int_{B_1(0)} [f_j(y, Dw)] dy. \tag{4.8}$$

The minimality of u and a simple change of variable yield that

$$\int_{B_1(0)} f(x_j+r_jy, Du(x_j+r_jy))\, dy \le \int_{B_1(0)} f(x_j+r_jy, Du(x_j+r_jy)+D\varphi(y))\, dy$$

for every $\varphi \in W_0^{1,h}(B_1(0);\mathbb{R}^N)$, that is

$$\int_{B_1(0)} f(x_j+r_jy, A_j+\lambda_j Dv_j(y))\, dy \le \int_{B_1(0)} f(x_j+r_jy, A_j+\lambda_j Dv_j(y)+D\varphi(x_j+r_jy))\, dy,$$

for every $\varphi \in W_0^{1,h}(B_{r_j}(x_j);\mathbb{R}^N)$. Thus, by the definition of the rescaled functionals, we have

$$\mathcal{I}_j(v_j) \le \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{D_\xi f(x_j+r_jy, A_j)D\varphi}{\lambda_j}\, dy. \tag{4.9}$$

Using (F3) we conclude that

$$\begin{aligned} \mathcal{I}_j(v_j) &\le \mathcal{I}_j(v_j+\varphi) + \int_{B_1(0)} \frac{[D_\xi f(x_j+r_jy, A_j) - D_\xi f(x_j, A_j)]D\varphi}{\lambda_j}\, dy \\ &\le \mathcal{I}_j(v_j+\varphi) + c(M)\frac{r_j^\alpha}{\lambda_j}\int_{B_1(0)} |D\varphi|\, dy. \end{aligned} \tag{4.10}$$

Step 3. v solves a linear system

Using that v_j satisfies inequality (4.10), we have that

$$0 \le \mathcal{I}_j(v_j+s\varphi) - \mathcal{I}_j(v_j) + c(M)\frac{r_j^\alpha}{\lambda_j}\int_{B_1(0)} |sD\varphi|\, dy, \tag{4.11}$$

for every $\varphi \in C_0^1(B)$ and for every $s \in (0,1)$. Now, using again the definition of the rescaled functionals, we observe that

$$\begin{aligned} \mathcal{I}_j(v_j+s\varphi) - \mathcal{I}_j(v_j) &= \int_{B_1(0)}\int_0^1 [D_\xi f_j(x_j+r_jy, A_j+\lambda_j(Dv_j+tsD\varphi))]sD\varphi\, dt\, dy \\ = \frac{1}{\lambda_j}\int_{B_1(0)}\int_0^1 &[D_\xi f(x_j+r_jy, A_j+\lambda_j(Dv_j+tsD\varphi)) - D_\xi f(x_j+r_jy, A_j)]sD\varphi\, dt\, dy. \end{aligned} \tag{4.12}$$

Inserting (4.12) in (4.11), dividing by s and taking the limit as $s \to 0$, we conclude that

$$0 \le \frac{1}{\lambda_j}\int_{B_1(0)} [D_\xi f(x_j+r_jy, A_j+\lambda_j Dv_j) - D_\xi f(x_j+r_jy, A_j)]D\varphi\, dy$$

$$+ \quad \frac{c(M) r_j^{\alpha}}{\lambda_j} \int_{B_1(0)} |D\varphi| \, dy. \tag{4.13}$$

Let us split

$$B_1(0) = E_j^+ \cup E_j^- = \{y \in B_1 : \ \lambda_j |Dv_j| > 1\} \cup \{y \in B_1 : \ \lambda_j |Dv_j| \leq 1\}.$$

Using (4.5) we get

$$|E_j^+| \leq \int_{E_j^+} \lambda_j^p |Dv_j|^p \, dy \leq \lambda_j^p \int_{E_j^+} |Dv_j|^p \, dy \leq c \lambda_j^p. \tag{4.14}$$

Using (F2), the elementary inequality $\log(1+t) \leq ct^p$ and (4.5), we obtain

$$\frac{1}{\lambda_j} \left| \int_{E_j^+} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \right| \tag{4.15}$$

$$\leq \frac{1}{\lambda_j} \int_{E_j^+} (1 + \log(1 + |A_j + \lambda_j Dv_j|) + \log(1 + |A_j|)) \, dy \tag{4.16}$$

$$\leq c(M) \frac{|E_j^+|}{\lambda_j} + \frac{1}{\lambda_j} \int_{E_j^+} |\lambda_j Dv_j|^p \, dy \tag{4.17}$$

$$\leq c(M) \lambda_j^{p-1}. \tag{4.18}$$

Hence, we infer that

$$\lim_{j \to \infty} \frac{c}{\lambda_j} \left| \int_{E_j^+} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy \right| = 0. \tag{4.19}$$

On E_j^- we have

$$\frac{1}{\lambda_j} \int_{E_j^-} [D_\xi f(x_j + r_j y, A_j + \lambda_j Dv_j) - D_\xi f(x_j + r_j y, A_j)] D\varphi \, dy$$
$$= \int_{E_j^-} \int_0^1 D_{\xi\xi} f(x_j + r_j y, A_j + t\lambda_j Dv_j) \, dt Dv_j D\varphi \, dy. \tag{4.20}$$

Note that (4.14) yields that $\chi_{E_j^-} \to \chi_{B_1}$ in L^r, for every $r < \infty$. Moreover by (4.6) we have, at least for subsequences, that

$$\lambda_j Dv_j \to 0 \qquad \text{a.e. in } B_1$$

$$r_j \to 0$$

and

$$x_j \to x_0.$$

Hence the uniform continuity of $D_{\xi\xi}f$ on bounded sets implies

$$\begin{aligned}&\lim_j \frac{1}{\lambda_j}\int_{E_j^-}[D_\xi f(x_j+r_jy,A_j+\lambda_jDv_j)-D_\xi f(x_j+r_jy,A_j)]D\varphi\,dy\\ =&\int_{B_1}D_{\xi\xi}f(x_0,A)DvD\varphi\,dy. \qquad (4.21)\end{aligned}$$

Since $\beta<\alpha$, by (4.6) we deduce that

$$\lim_j \frac{r_j^\alpha}{\lambda_j^2}=0. \qquad (4.22)$$

By estimates (4.19), (4.21) and (4.22), passing to the limit as $j\to\infty$ in (4.13) yields

$$0\le\int_{B_1}D_{\xi\xi}f(x_0,A)DvD\varphi\,dy$$

Changing φ in $-\varphi$ we finally get

$$\int_{B_1}D_{\xi\xi}f(x_0,A)DvD\varphi\,dy=0$$

that is v solves a linear system which is uniformly elliptic thanks to the uniform convexity of f. The regularity result stated in Proposition 2.12 implies that $v\in C^\infty(B_1)$ and for any $0<\tau<1$

$$⨍_{B_\tau}|Dv-(Dv)_\tau|^2\,dy\le c\tau^2⨍_{B_1}|Dv-(Dv)_1|^2\,dy\le c\tau^2, \qquad (4.23)$$

for a constant c depending on M.

Step 4. Higher integrability of v_j

In this step we will prove a higher integrability result for Dv_j which is uniform with respect to the rescaling procedure. We will drop the index j for simplicity.

Lemma 4.2. Let g be a function satisfying (I1)-(I4) and $v\in W^{1,\mathcal{A}}(B;\mathbb{R}^N)$ a solution of

$$\mathcal{I}(v)\quad\le\quad\mathcal{I}(v+\varphi)+c(M)\frac{r_0^\alpha}{\lambda}\int_{B_1(0)}D_\xi f(x_0+r_0y,A)D\varphi\,dy$$

for every $\varphi\in W_0^{1,\mathcal{A}}(B_1(0);\mathbb{R}^N)$. Then we have

$$\left(⨍_{B_{\frac{\rho}{2}}}\left|\lambda^{-1}V_1(\lambda Dv)\right|^{\frac{2n}{n-2k}}\,dy\right)^{\frac{n-2k}{2n}}\le\frac{c}{\lambda}\left(⨍_{B_\rho}|V_p(\lambda Dv)|^2\,dy\right)^{\frac{1}{2}}$$

$$+ c\frac{r_0^{\frac{\alpha}{2}}}{\lambda}\left(\int_{B_\rho}\{1+|\lambda Dv|+\log(1+|\lambda Dv|)|\lambda Dv|\}\,dy\right)^{\frac{1}{2}}+\left(\int_{B_\rho}\left|\lambda^{-1}V_1(\lambda Dv)\right|^2\,dy\right)^{\frac{1}{2}}$$

for every $k < \frac{\alpha}{2}$ and for every ball $B_\rho \Subset B_1$. Here c does not depend on r_0, λ and v.

Proof. Let us fix two radii $\frac{\rho}{2} < r < s < \rho$ and a cut-off function $\eta \in C_0^\infty(B_s)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and $|\nabla\eta| \le \frac{c}{s-r}$. As in [12], using $\varphi = \tau_{s,-h}(\eta^2\tau_{s,h}v)$, we obtain

$$\begin{aligned}&\int_{B_\rho}\int_0^1 \eta^2 D_{\xi\xi}g(y, Dv+t\tau_h Dv)(\tau_h Dv,\tau_h Dv)\,dt\,dy\\ &\le -\int_{B_\rho}\eta^2[D_\xi g(y+he_s, Dv(y+he_s))-D_\xi g(y, Dv(y+he_s))]\tau_h Dv\,dy\\ &-2\int_{B_\rho}\eta\tau_h\{D_\xi g(y,Dv)\}\,D\eta\otimes\tau_h v\,dy + c\frac{r_0^\alpha}{\lambda}|h|^\alpha\int_B|D(\eta^2\tau_h v)|\,dy. \qquad (4.24)\end{aligned}$$

By the definition of g, we can write the second integral in previous inequality as follows

$$\begin{aligned}-2 &\int_{B_\rho}\eta\tau_h\{D_\xi g(y,Dv)\}\,D\eta\otimes\tau_h v\,dy =\\ = &-\frac{2}{\lambda}\int_{B_\rho}\eta\tau_h\{D_\xi f(x_0+r_0y, A+\lambda Dv(y))-D_\xi f(x_0+r_0y,A)\}\,D\eta\otimes\tau_h v\,dy\\ = &-\frac{2}{\lambda}\int_{B_\rho}\eta\{D_\xi f(x_0+r_0(y+he_s), A+\lambda Dv(y+he_s))-D_\xi f(x_0+r_0(y+he_s),A)\\ &\quad -D_\xi f(x_0+r_0y, A+\lambda Dv(y))+D_\xi f(x_0+r_0y,A)\}D\eta\otimes\tau_h v\,dy\\ = &-\frac{2}{\lambda}\int_{B_\rho}\eta\Big\{D_\xi f(x_0+r_0(y+he_s), A+\lambda Dv(y+he_s))-\\ &\quad -D_\xi f(x_0+r_0y, A+\lambda Dv(y+he_s))\\ &\quad +D_\xi f(x_0+r_0y, A+\lambda Dv(y+he_s))-D_\xi f(x_0+r_0y, A+\lambda Dv(y))\\ &\quad -D_\xi f(x_0+r_0(y+he_s),A)+D_\xi f(x_0+r_0y,A)\Big\}D\eta\otimes\tau_h v\,dy. \qquad (4.25)\end{aligned}$$

By (I4) and the argumentation at the end of the previous section the l.h.s. in (4.24) is bounded from below by

$$c\int_{B_\rho}\eta^2(1+|\lambda Dv|+|\lambda Dv(y+he_s)|)^{-1}|\tau_h Dv|^2\,dy \ge c\int_{B_\rho}\eta^2\left|\tau_h\left\{\lambda^{-1}V_1(\lambda Dv)\right\}\right|^2\,dy. \qquad (4.26)$$

Whereas on the r.h.s. of (4.24), taking into account (4.25), using (I3) and (F3) we are led to

$$\begin{aligned}
T_1 &= c\frac{r_0^\alpha}{\lambda}|h|^\alpha \int_{B_\rho} \eta^2(1+\log(1+|\lambda Dv(y+he_s)|))|\tau_h Dv|\,dy;\\
T_2 &= c\frac{r_0^\alpha}{\lambda}|h|^\alpha \int_{B_\rho} \eta|D\eta|\log(1+|A|+|\lambda Dv(y+he_s)|)|\tau_h v|\,dy\\
&\quad + \frac{c}{\lambda}\int_{B_\rho}\eta|D\eta|\left|\int_0^1 D_{\xi\xi}f(x_0+r_0y, A+s\lambda\tau_h(Dv)))ds\right||\lambda\tau_h(Dv)||\tau_h v|\,dy;\\
&= T_{2,1}+T_{2,2}\\
T_3 &= c\frac{r_0^\alpha}{\lambda}|h|^\alpha \int_{B_\rho}|D(\eta^2\tau_h v)|\,dy.
\end{aligned}$$

Using Young's inequality for $\mathcal{A}(t) = t\log(1+t)$ and choosing $h \ll 1$ we get

$$\begin{aligned}
T_1 &\le c\frac{r_0^\alpha}{\lambda^2}|h|^\alpha \int_{B_\rho}\{1+|\lambda Dv|+\log(1+|\lambda Dv|)|\lambda Dv|\}\,dy;\\
T_{2,1} &\le c\frac{r_0^\alpha}{\lambda^2}|h|^\alpha \int_{B_\rho}\{1+|\lambda Dv|+\log(1+|\lambda Dv|)|\lambda Dv|\}\,dy;\\
T_3 &\le c\frac{r_0^\alpha}{\lambda^2}|h|^\alpha \int_{B_\rho}|\lambda Dv|\,dy,
\end{aligned}$$

In order to estimate the integral $T_{2,2}$ we use (F5) and Young's Inequality as follows

$$\begin{aligned}
\left|\int_0^1 D_{\xi\xi}f(x_0+r_0y, A+s\lambda\tau_h(Dv)))ds\right| &\le c\int_0^1 \frac{\log(1+|A+s\lambda\tau_h(Dv)|)}{|A+s\lambda\tau_h(Dv)|}\,ds\\
&\le c\int_0^1 (1+|A+s\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}\,ds\\
&\le c(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}},
\end{aligned}$$

where we used Lemma 2.14 and Lemma 2.1 of [1]. Hence

$$\begin{aligned}
T_{2,2} &\le \frac{c}{\lambda}\int_{B_s}(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}|\lambda\tau_h(Dv)||\tau_h v|\\
&= \frac{c|h|}{\lambda^2}\int_{B_s}(1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}}|\lambda\tau_h(Dv)||\lambda\Delta_h v|.
\end{aligned}$$

We observe that for the Young function $\varphi(t) := (1+t^2)^{\frac{p-2}{2}}t^2$ we have

$$\varphi'(t) \approx (1+t^2)^{\frac{p-2}{2}}t; \quad \varphi^*(\varphi'(t)) \approx \varphi(t). \tag{4.27}$$

Here φ^* denotes the conjugate Young function. The second statement in (4.27) is a consequence of

$$\varphi^*(\varphi'(t)) = \int_0^{\varphi'(t)} (\varphi')^{-1}(s)\, ds = \int_0^t s\varphi''(s)\, ds \approx \int_0^t \varphi'(s)\, ds = \varphi(t).$$

Hence we obtain with the help of Young's Inequality for Young functions, (3.6) and Lemma 2.14

$$\begin{aligned} T_{2,2} &\le \frac{c|h|}{\lambda^2} \left\{ \int_{B_s} \varphi^* \left((1+|\lambda\tau_h(Dv)|^2)^{\frac{p-2}{2}} |\lambda\tau_h(Dv)| \right) dy + \int_{B_s} \varphi(|\lambda\Delta_h v|)\, dy \right\} \\ &\le \frac{c|h|}{\lambda^2} \left\{ \int_{B_s} \varphi\left(|\lambda\tau_h(Dv)|\right)\right)\, dy + \int_{B_s} \varphi(|\lambda\Delta_h v|)\, dy \right\} \\ &\le c\frac{c|h|}{\lambda^2} \int_{B_\rho} |V_p(\lambda Dv)|^2\, dy. \end{aligned}$$

Inserting the estimates for T_i in (4.24) and using (4.26), we finally get

$$\begin{aligned} &\int_{B_\rho} \eta^2 \left|\tau_h \left\{\lambda^{-1} V_1(\lambda Dv)\right\}\right|^2\, dy \\ \le\ & c\frac{r_0^\alpha}{\lambda^2}|h|^\alpha \int_{B_\rho} \{1+|\lambda Dv| + \log(1+|\lambda Dv|)|\lambda Dv|\}\, dy \\ +\ & \frac{c|h|}{\lambda^2} \int_{B_\rho} |V_p(\lambda Dv)|^2\, dy \end{aligned} \tag{4.28}$$

The conclusion follows applying Lemma 2.8. □

Step 5. A Caccioppoli type inequality

The higher integrability of the previous step allows us to prove a Caccioppoli type inequality for minimizers of the rescaled functional, which is contained in the following

Proposition 4.3. Let g be a function satisfying (I1)-(I4) and $v \in W^{1,h}(B;\mathbb{R}^N)$ a solution of

$$\mathcal{I}(v) \ \le\ \mathcal{I}(v+\varphi) + c(M)\frac{r_0^\alpha}{\lambda} \int_{B_1(0)} |D\varphi|\, dy \tag{4.29}$$

for every $\varphi \in W_0^{1,h}(B_1(0);\mathbb{R}^N)$. Then we have

$$\int_{B_{\frac{\tau}{2}}} \left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 \le \frac{c}{\lambda^2} \int_{B_\tau} \left| V_p\left(\lambda \frac{|v-v_\tau|}{\tau}\right)\right|^2\, dy$$

$$
\begin{aligned}
&+c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_p(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p + c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_1(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p \\
&+c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}} 1+|\lambda Dv|\,dy\right)^p \\
&+c\frac{r_0^{\alpha}}{\lambda^2}\int_{B_\tau} \lambda|Dv|\,dy. \qquad (4.30)
\end{aligned}
$$

Proof. Let us fix two radii $\frac{\tau}{2} < r < s < \tau$ and a cut-off function $\eta \in C_0^\infty(B_s)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and $|\nabla\eta| \le \frac{c}{s-r}$. Using $\varphi = \eta(v_\tau - v)$ as a test function in (4.29), by virtue of the left inequality at (I1), we get

$$
\begin{aligned}
\int_{B_r}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 &\le \int_{B_1} g(y,Dv)\,dy \\
&\le \int_{B_1} g(y,Dv+D\varphi) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_1(0)}|D\varphi| \\
&= \int_{B_s\setminus B_r} g(y,Dv+D(\eta(v_\tau-v))) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_s}|D(\eta(v_\tau-v))| \\
&= \int_{B_s\setminus B_r} g(y,(1-\eta)Dv+D\eta(v_\tau-v)) + c(M)\frac{r_0^\alpha}{\lambda}\int_{B_s}|Dv| \\
&+ \ c(M)\frac{r_0^\alpha}{\lambda(s-r)}\int_{B_s}|v-v_\tau|. \qquad (4.31)
\end{aligned}
$$

The first integral in the right hand side of (4.31) can be estimated by the right inequality at (I1) and the properties of η as follows

$$
\begin{aligned}
&\int_{B_s\setminus B_r} g(y,(1-\eta)Dv+D\eta(v_\tau-v)) \\
\le\ &\frac{c}{\lambda}\int_{B_s\setminus B_r} \log(1+\lambda|Dv|+\lambda|D\eta||v-v_\tau|)(|Dv|+|D\eta||v-v_\tau|) \\
\le\ &\frac{c}{\lambda}\int_{B_s\setminus B_r} \log\left(1+\lambda|Dv|+\lambda\frac{|v-v_\tau|}{s-r}\right)\left(|Dv|+\frac{|v-v_\tau|}{s-r}\right). \qquad (4.32)
\end{aligned}
$$

By (I1), Lemma 2.14 and Lemma 2.13 we obtain

$$
\begin{aligned}
&\int_{B_s\setminus B_r} g(y,(1-\eta)Dv+D\eta(v_\tau-v)) \\
\le\ &\frac{c}{\lambda^2}\int_{B_s\setminus B_r}\left|V_p\left(\lambda|Dv|+\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy \\
\le\ &\frac{c}{\lambda^2}\int_{B_s\setminus B_r}|V_p(\lambda|Dv|)|^2\,dy + \frac{c}{\lambda^2}\int_{B_s\setminus B_r}\left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy
\end{aligned}
$$

$$\begin{aligned} &\le \frac{c}{\lambda^2}\int_{B_s\setminus B_r}|V_1(\lambda|Dv|)|^2\,dy+\frac{c}{\lambda^2}\int_{B_s\setminus B_r}|V_1(\lambda|Dv|)|^{2p}\,dy\\ &+\frac{c}{\lambda^2}\int_{B_s\setminus B_r}\left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy. \end{aligned} \tag{4.33}$$

Inserting (4.33) in (4.31), we get

$$\begin{aligned} c\int_{B_r}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 &\le \frac{c}{\lambda^2}\int_{B_s\setminus B_r}|V_1(\lambda|Dv|)|^2\,dy\\ &+\frac{c}{\lambda^2}\int_{B_s\setminus B_r}|V_1(\lambda|Dv|)|^{2p}\,dy\\ &+\frac{c}{\lambda^2}\int_{B_s\setminus B_r}\left|V_p\left(\lambda\frac{|v-v_\tau|}{s-r}\right)\right|^2 dy\\ &+c\frac{r_0^\alpha}{\lambda}\int_{B_s}|Dv|\\ &+c\frac{r_0^\alpha\tau}{\lambda(s-r)}\int_{B_\tau}|Dv|, \end{aligned} \tag{4.34}$$

where we also used Poincaré's Inequality. Now we fill the hole by adding to both sides of (4.34) the quantity

$$\int_{B_r}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2$$

and use the iteration Lemma 2.10 to obtain

$$\begin{aligned} \int_{B_{\frac{\tau}{2}}}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2 &\le \frac{c}{\lambda^2}\int_{B_\tau}|V_1(\lambda|Dv|)|^{2p}\,dy\\ &+\frac{c}{\lambda^2}\int_{B_\tau}\left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy+c\frac{r_0^\alpha}{\lambda}\int_{B_\tau}|Dv|. \end{aligned} \tag{4.35}$$

Now we apply to the first integral in the right hand side of (4.35) the estimate of Lemma 4.2 with $p=\frac{n}{n-2k}$, thus having

$$\begin{aligned} &\int_{B_\tau}|V_1(\lambda Dv)|^{2p}\,dy\le c\left(\int_{B_{2\tau}}|V_p(\lambda Dv)|^2\,dy\right)^p\\ &+cr_0^{\alpha p}\left(\int_{B_{2\tau}}\{1+|\lambda Dv|+\log(1+|\lambda Dv|)|\lambda Dv|\}\,dy\right)^p+\left(\int_{B_{2\tau}}|V_1(\lambda Dv)|^2\,dy\right)^p. \end{aligned} \tag{4.36}$$

Inserting (4.36) in (4.35) and using again Lemma 2.14, we have

$$\int_{B_{\frac{\tau}{2}}}\left|\frac{V_1(\lambda Dv)}{\lambda}\right|^2\le\frac{c}{\lambda^2}\int_{B_\tau}\left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy$$

$$
\begin{aligned}
&+ \quad \frac{c}{\lambda^2}\left(\int_{B_{2\tau}} |V_p(\lambda Dv)|^2\,dy\right)^p + c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}} \{1+|\lambda Dv|+\log(1+|\lambda Dv|)|\lambda Dv|\}\ dy\right)^p \\
&+ \quad \frac{c}{\lambda^2}\left(\int_{B_{2\tau}} |V_1(\lambda Dv)|^2\ dy\right)^p + c\frac{r_0^{\alpha}}{\lambda^2}\int_{B_\tau} \lambda|Dv| \\
&\le \quad \frac{c}{\lambda^2}\int_{B_\tau} \left|V_p\left(\lambda\frac{|v-v_\tau|}{\tau}\right)\right|^2 dy + c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_p(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p \\
&+ \quad c\lambda^{2p-2}\left(\int_{B_{2\tau}} \frac{|V_1(\lambda Dv)|^2}{\lambda^2}\,dy\right)^p + c\frac{r_0^{\alpha p}}{\lambda^2}\left(\int_{B_{2\tau}} (1+|\lambda Dv|)\,dy\right)^p \\
&+ \quad c\frac{r_0^{\alpha}}{\lambda^2}\int_{B_\tau} \lambda|Dv|\,dy \qquad (4.37)
\end{aligned}
$$

which is the conclusion. □

Step 6. Conclusion

Fix $\tau \in (0, \frac{1}{4})$, set $b_j = (v_j)_{B_{2\tau}}$, $B_j = (Dv_j)_{B_\tau}$ and define

$$w_j(y) = v_j(y) - b_j - B_j y.$$

After rescaling, we note that $\lambda_j w_j$ satisfies the following integral inequality

$$\int_{B_1(0)} g_j(y, \lambda_j Dw_j)\,dy \le \int_{B_1(0)} g_j(y, \lambda_j Dw_j + D\varphi)\,dy + c\frac{r_j^\alpha}{\lambda_j}\int_{B_1(0)} |D\varphi|\,dy,$$

for every $\varphi \in W_0^{1,h}(B_1(0))$ where ($z_j := x_j + r_j y$)

$$g_j(y,\xi) = \frac{f(z_j, A_j+\lambda_j B_j+\xi) - f(z_j, A_j+\lambda_j B_j) - D_\xi f(z_j, A_j+\lambda_j B_j)\xi}{\lambda_j^2}.$$

It is easy to check that Lemma 2.15 applies to each g_j, for some constants that could depend on τ through $|\lambda_j B_j|$. But, given τ, we may always choose j large enough to have $|\lambda_j B_j| \le c\frac{\lambda_j}{\tau^{\frac{n}{p}}} < 1$ (remember (4.6)). Hence we can apply Proposition 4.3 to each $\lambda_j w_j$ obtaining for (compare Lemma 2.13 and (2.4))

$$
\begin{aligned}
\lim_j \frac{E(x_j,\tau r_j)}{\lambda_j^2} &\le \lim_j \frac{c}{\lambda_j^2}\fint_{B_{\tau r_j}(x)} |V_p(Du-(Du)_{\tau r_j})|^2\,dy + \lim_j \frac{\tau^\beta r_j^\beta}{\lambda_j^2} \\
&\le \lim_j \frac{c}{\lambda_j^2}\fint_{B_\tau} |V_p(\lambda_j Dw_j)|^2\,dy + \tau^\beta \\
&\le \lim_j \frac{c}{\lambda_j^2}\fint_{B_\tau} |V_p(\lambda_j Dw_j)|^2\,dy + \tau^\beta
\end{aligned}
$$

$$\leq \lim_j \frac{c}{\lambda_j^2} \fint_{B_\tau} |V_1(\lambda_j Dw_j)|^2\, dy + \lim_j \frac{c}{\lambda_j^2} \fint_{B_\tau} |V_1(\lambda_j Dw_j)|^{2p}\, dy + \tau^\beta$$

the estimation (note that the second term on the r.h.s. can also be estimated by the r.h.s. of Proposition 4.3, see the calculations after (4.36))

$$\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j (w_j - (w_j)_{2\tau})}{\tau}\right)\right|^2 dy \\
&\quad + c \lim_j \lambda_j^{2p-2} \left(\fint_{B_{2\tau}} \frac{|V_p(\lambda_j Dw_j)|^2}{\lambda_j^2}\, dy\right)^p \\
&\quad + c \lim_j \lambda_j^{2p-2} \left(\fint_{B_{2\tau}} \frac{|V_1(\lambda_j Dw_j)|^2}{\lambda_j^2}\, dy\right)^p \\
&\quad + c \lim_j \frac{r_j^{\alpha p}}{\lambda_j^2} \left(\fint_{B_\tau} \lambda_j |Dw_j|\, dy\right)^p + c \lim_j \frac{r_j^{\alpha}}{\lambda_j^2} \left(\fint_{B_\tau} (1 + \lambda_j |Dw_j|)\, dy\right) + \tau^\beta \\
&\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j (w_j - (w_j)_{2\tau})}{\tau}\right)\right|^2 dy + \tau^\beta,
\end{aligned}$$

since $\lim_j \lambda_j^{2p-2} = 0$, $\lim_j \frac{r_j^\alpha}{\lambda_j^2} = 0$, $\lim_j \frac{r_j^{\alpha p}}{\lambda_j^2} = 0$ and the integrals appearing as their factors are bounded as $j \to \infty$. Now, since $v_j \to v$ strongly in $L^p(B_1(0))$, using the Sobolev-Poincaré inequality stated in Lemma 2.11, one can easily check that

$$\lim_{j \to +\infty} \int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\, dy = 0. \tag{4.38}$$

In fact, for every $\vartheta \in (0, \frac{p}{2})$ we can use Hölder's inequality of exponents $\frac{p}{2\vartheta}$ and $\frac{p}{p-2\vartheta}$ as follows

$$\begin{aligned}
&\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\, dy = \int_{B_{\frac{1}{2}}} |v_j - v|^2 (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p-2}{2}}\, dy \\
&\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)}{4}} dy \right)^{\frac{2\vartheta}{p}} \\
&\qquad \times \left(\int_{B_{\frac{1}{2}}} |v_j - v|^{\frac{2p(1-\vartheta)}{p-2\vartheta}} (1 + \lambda_j^2 |v_j - v|^2)^{\frac{p(p-2)(1-\vartheta)}{2(p-2\vartheta)}}\, dy \right)^{\frac{p-2\vartheta}{p}} \\
&\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p\, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \left(\frac{|V_p(\lambda_j (v_j - v))|^2}{\lambda_j^2}\right)^{\frac{p(1-\vartheta)}{p-2\vartheta}} dy \right)^{\frac{p-2\vartheta}{p}}
\end{aligned}$$

$$\leq \left(\int_{B_{\frac{1}{2}}} |v_j - v|^p \, dy \right)^{\frac{2\vartheta}{p}} \left(\int_{B_{\frac{1}{2}}} \frac{|V_p(\lambda_j(Dv_j - Dv))|^2}{\lambda_j^2} \, dy \right)^{1-\vartheta}.$$

Last inequality is obtained applying Lemma 2.11 to the second integral, choosing $\vartheta \in (0, \frac{p}{2})$ such that $\frac{p(1-\vartheta)}{p-2\vartheta} = \frac{n}{n-p}$. Hence (4.38) follows noticing that the first integral vanishes as j goes to infinity and second one stays bounded thanks to (4.5), since $v \in C^\infty(B_1(0))$.
Since $b_j \to (v)_{2\tau}$ and $B_j \to (Dv)_\tau$, using (4.38) and the definition of w_j we get

$$\begin{aligned}
\lim_j \frac{E(x_j, \tau r_j)}{\lambda_j^2} &\leq c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j(w_j - v + v)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&= c \lim_j \fint_{B_{2\tau}} \frac{1}{\lambda_j^2} \left| V_p\left(\frac{\lambda_j(v_j - v + v - b_j - B_j y)}{\tau} \right) \right|^2 \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + c \fint_{B_{2\tau}} \frac{|(Dv)_\tau y - (Dv)_{2\tau} y|^2}{\tau^2} \, dy + \tau^\beta \\
&\leq c \fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \, dy + c|(Dv)_\tau - (Dv)_{2\tau}|^2 + \tau^\beta \\
&\leq c\tau^2 + c\tau^\beta \leq c_M^\star \tau^\beta.
\end{aligned}$$

The contradiction follows by choosing $c_M^\star > \tilde{C}(M)$. □

5 Full regularity

In this section we will prove that the minimizer u belongs to the space $C^{1,\gamma}(\Omega, \mathbb{R}^N)$ for every $\gamma < 1$ if we assume (F1) and (F3)-(F8). We follow the lines of [7] (section 4) and use the fact that the range of anisotropy in the almost linear growth situation is arbitrary small. Note that in [7] Breit studies (p,q)-elliptic integrands. We just clarify the main differences. The first step is to regularize the problem. Here we consider the standard regularization (compare, for example, [5] and the references therein): u_δ is defined as the unique minimizer of

$$\mathcal{F}_\delta(u, B) := \int_B \left\{ f(x, Du) + \delta(1 + |Du|^2)^{\frac{q}{2}} \right\} dx$$

in $(u)_\epsilon + W_0^{1,q}(B)$ for $B \Subset \Omega$ and $1 < p < q < \frac{n - \frac{\alpha}{2}}{n - \alpha}$ (p is defined in (F7)). Thereby $(u)_\epsilon$ is the mollification of u with parameter ϵ and

$$\delta = \delta(\epsilon) := \frac{1}{1 + \epsilon^{-1} + \|D(u)_\epsilon\|_{L^q(B)}^{2q}}.$$

For u_δ we obtain:

Lemma 5.1. • As $\epsilon \to 0$ we have: $u_\delta \rightharpoonup u$ in $W^{1,1}(B,\mathbb{R}^N)$,

$$\delta \int_B \left(1+|Du_\delta|^2\right)^{\frac{q}{2}} dx \to 0; \quad \int_B F(Du_\delta)dx \to \int_B F(\nabla u)dx;$$

• $Du_\delta \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega,\mathbb{R}^{nN})$.

For the last statement we can refer to [6] (Lemma 2.7), since u_δ is the minimizer of a isotropic problem and the second derivatives $D_{\xi\xi} f_\delta$ fulfills a Hölder-condition by (F8) ($f_\delta(x,\xi) := f(x,\xi)+\delta(1+|\xi|^2)^{\frac{q}{2}}$). The rest can be quoted from [6], Lemma 2.1. Only the week convergence needs a comment: Following the ideas of [6] one easily sees that Du_δ in bounded in $L_h(B)$. According to the Poincaré-inequality in Orlicz spaces (see [19]) and the uniform boundedness of u_ϵ in $W^{1,h}_{loc}(\Omega)$ (remember $u \in W^{1,h}(\Omega)$) we obtain $\sup_\delta \|u_\delta\|_{W^{1,h}(B)} < \infty$. By the De La Valée Poussin Lemma we can select a subsequence such that

$$u_\delta \rightharpoonup: v \in W^{1,1}(B), \quad v = u \quad \text{on } \partial B$$

and v minimizes $\mathcal{F}(\cdot, B)$ with respect to boundary data u which means $v = u$.

Next we prove higher integrability with respect to the parameter δ, i.e.,

$$Du_\delta \in L^t_{loc}(B) \quad \text{uniformly in } \delta \text{ for all } t < \frac{n}{n-\alpha}. \tag{5.1}$$

Here we proceed exactly as in section 3, observing that our bounds are now independent of δ. We only have to calculate the additional integral ($F_(Z) := (1+|Z|^2)^{\frac{q}{2}}$)

$$\begin{aligned}
&\delta \int_B D_\xi F_0(Du_\delta) D\tau_{-h}(\eta^2 \tau_h u_\delta)\, dx = -\delta \int_B \tau_h D_\xi F_0(Du_\delta) D(\eta^2 \tau_h u_\delta)\, dx \\
&= -\delta \int_B \eta^2 \int_0^1 D_{\xi\xi} F_0(Du_\delta + t\tau_h Du_\delta)(\tau_h Du_\delta, \tau_h Du_\delta)\, dx \\
&- 2\delta \int_B \eta \tau_h D_\xi F_0(Du_\delta) D\eta \otimes \tau_h u_\delta\, dx
\end{aligned}$$

on the r.h.s. Here the first integral on the last calculation is nonnegative, so we can drop it. The last one can be estimated by (using Lemma 5.1)

$$c(\eta) h \int_B \left(1+|Du_\delta|^2\right)^{\frac{q}{2}} dx \le c(\eta) h.$$

Hence we obtain (5.1) if we apply the arguments of section 3 (remember the uniform $W^{1,h}(B)$-bounds on u_δ).

In order to prove Lipschitz-regularity of the solution u we have to show a growth condition for the function

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\delta^{q-\frac{1}{2}} (\omega_\delta - k)^2 \, dx$$

where we abbreviated $\Gamma_\delta := 1 + |Du_\delta|^2$, $\omega_\delta := \log \Gamma_\delta$ and $A(k,r) := B_r \cap [|Du_\delta| > k]$. We want to show

$$\tau(h,r) \leq \frac{c}{(\widehat{r}-r)^\kappa (h-k)^\Theta} \tau(k,\widehat{r})^\mu \tag{5.2}$$

for $0 < h < k$, $0 < r < \widehat{r} < R_0$ with exponents $\kappa, \Theta > 0$ and $\mu > 1$. From (5.2) we arrive at uniform L^∞_{loc}-bounds on Du_δ using Stampacchia's Lemma ([30], Lemma 5.1, p. 219), details are given in [4]. Note that uniform bounds for τ (which are necessary) follows from (5.1) and

$$q < \frac{n - \frac{\alpha}{2}}{n - \alpha}.$$

Hence we have $u_\delta \in W^{1,\infty}_{loc}(B)$ uniformly in δ (remember Lemma 5.1). It follows with the help of Arzela -Ascoli's Theorem that $u \in W^{1,\infty}_{loc}(B)$ and since B is arbitrary $u \in W^{1,\infty}_{loc}(\Omega)$. This means that

$$\int f(x, Du) \, dx \longrightarrow \min$$

is a problem with quadratic growth (at least locally, compare (F5)) and the claim follows from [6], Lemma 2.7.
In order to prove (5.2) we have to notice that the integrand satisfies the growth conditions

$$\begin{aligned} \nu(1+|\xi|^2)^{-\frac{1}{2}}|Z|^2 \leq D^2_{\xi\xi} f(x,\xi)(Z,Z) &\leq \Lambda(1+|\xi|^2)^{\frac{q-2}{2}}|Z|^2, \\ |\partial_s D_\xi f(x,\xi)| &\leq \Lambda(1+|\xi|^2)^{\frac{q-1}{2}}. \end{aligned}$$

Since the exponent from above ($p = 1$) and below are close enough, we can exactly argue as in [7] (section 4) and obtain (5.2). Note that in this part of [7] the condition $p > 1$ is not used.

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