REGULARITY FOR A FRACTIONAL p-LAPLACE EQUATION

ARMIN SCHIKORRA, TIEN-TSAN SHIEH, AND DANIEL SPECTOR

ABSTRACT. In this note we consider regularity theory for a fractional p-Laplace operator which arises in the complex interpolation of the Sobolev spaces, the $H^{s,p}$-Laplacian. We obtain the natural analogue to the classical p-Laplacian situation, namely $C^{s+\alpha}_{loc}$-regularity for the homogeneous equation.

1. INTRODUCTION AND MAIN RESULT

In recent years equations involving what we will call the distributional $W^{s,p}$-Laplacian, defined for test functions φ as

$$(-\Delta)^s_p u[\varphi] := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{d+sp}}\,dy\,dx,$$

have received a lot of attention, e.g. [3, 6, 7, 12, 14, 15, 19]. The $W^{s,p}$-Laplacian $(-\Delta)^s_p$ appears when one computes the first variation of certain energies involving the $W^{s,p}$ semi-norm

$$[u]_{W^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

which was introduced by Gagliardo [10] and independently by Slobodeckij [25] to describe the trace spaces of Sobolev maps.

Regularity theory for equations involving this fractional p-Laplace operator is a very challenging open problem and only partial results are known: $C^{0,\alpha}_{loc}$-regularity for suitable right-hand-side data was obtained by Di Castro, Kuusi and Palatucci [6, 7]; A generalization of the Gehring lemma was proven by Kuusi, Mingione and Sire [14, 15];

A.S. is supported by DFG-grant SCHI-1257-3-1 and the DFG-Heisenberg fellowship. D.S. is supported by the Taiwan Ministry of Science and Technology under research grants 103-2115-M-009-016-MY2 and 105-2115-M-009-004-MY2. Part of this work was written while A.S. was visiting NCTU with support from the Taiwan Ministry of Science and Technology through the Mathematics Research Promotion Center.

A stability theorem similar to the Iwaniec stability result for the p-Laplacian was established by the first-named author [21]. The current state-of-the art with respect to regularity theory is higher Sobolev-regularity by Brasco and Lindgren [3].

Aside from their origins as trace spaces, the fractional Sobolev spaces

$$W^{s,p}(\mathbb{R}^d) := \left\{ u \in L^p(\mathbb{R}^d) : [u]_{W^{s,p}(\mathbb{R}^d)} < +\infty \right\}$$

also arise in the real interpolation of L^p and $\dot{W}^{1,p}$. If one alternatively considers the complex interpolation method, one is naturally led to another kind of fractional Sobolev space $H^{s,p}(\mathbb{R}^d)$, where taking the place of the differential energy (1.1) one can utilize the semi-norm

$$[u]_{H^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |D^s u|^p \right)^{\frac{1}{p}}. \tag{1.2}$$

Here $D^s = (\frac{\partial^s}{\partial x_1^s}, \ldots, \frac{\partial^s}{\partial x_d^s})$ is the fractional gradient for

$$\frac{\partial^s u}{\partial x_i^s}(x) := c_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+s}} \frac{x_i - y_i}{|x-y|} \, dy, \quad i = 1, \ldots, d.$$

Composition formulae for the fractional gradient have been studied in the classical work [11], while more recently they have been considered by a number of authors [1, 4, 5, 20, 22, 24]. While it is common in the literature (for example in [17]) to see $H^{s,p}(\mathbb{R}^d)$ equipped with the L^p-norm of the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ (see Section 2 for a definition), we here utilize (1.2) because it preserves the structural properties of the spaces for $s \in (0,1)$ more appropriately. In particular, for $s = 1$ we have $D^1 = D$ (the constant $c_{d,s}$ tends to zero as s tends to one), while for $s \in (0,1)$ the fractional Sobolev spaces defined this way support a fractional Sobolev inequality in the case $p = 1$ (see [23]). Let us also remark that for $p = 2$ these spaces are the same, $W^{s,2} = H^{s,2}$, but for $p \neq 2$ this is not the case.

Returning to the question of a fractional p-Laplacian, in the context of $H^{s,p}(\mathbb{R}^d)$ computing the first variation of energies involving the $H^{s,p}$ semi-norms (1.2) yields an alternative fractional version of a p-Laplacian, we shall call it the $H^{s,p}$-Laplacian

$$\operatorname{div}_s(|D^s u|^{p-2} D^s u) = \sum_{i=1}^{d} \frac{\partial^s}{\partial x_i^s} (|D^s u|^{p-2} \frac{\partial^s u}{\partial x_i^s}).$$

Somewhat surprisingly while the regularity theory for the homogeneous equation of the $W^{s,p}$-Laplacian

$$(-\Delta)_p^s u = 0$$

is far from being understood, the regularity for the $H^{s,p}$-Laplacian

$$\operatorname{div}_s(|D^s u|^{p-2}\, D^s u) = 0 \tag{1.3}$$

actually follows the classical theory, which is the main result we prove in this note:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be open, $p \in (2 - \frac{1}{d}, \infty)$ and $s \in (0,1]$. Suppose $u \in H^{s,p}(\mathbb{R}^d)$ is a distributional solution to (1.3), that is*

$$\int_{\mathbb{R}^d} |D^s u|^{p-2} D^s u \cdot D^s \varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.4}$$

Then $u \in C^{s+\alpha}_{loc}(\Omega)$ for some $\alpha > 0$ only depending on p.

The key observation for Theorem 1.1 is that $v := I_{1-s}u$, where I_{1-s} denotes the Riesz potential, actually solves an inhomogeneous classical p-Laplacian equation with good right-hand side.

Proposition 1.2. *Let u be as in Theorem 1.1. Then $v := I_{1-s}u$ satisfies*

$$-\operatorname{div}(|Dv|^{p-2} Dv) \in L^\infty_{loc}(\Omega).$$

Therefore, Theorem 1.1 follows from the regularity theory of the classical p-Laplacian: By Proposition 1.1, v is a distributional solution to

$$\operatorname{div}(|Dv|^{p-2}\, Dv) = \mu$$

and μ is sufficiently integrable whence $v \in C^{1,\alpha}_{loc}(\Omega)$ [8, 9, 26] (see also the excellent survey paper by Mingione [18]). In particular, one can apply the potential estimates by Kuusi and Mingione [13, Theorem 1.4, Theorem 1.6] to deduce that $Dv \in C^{0,\alpha}_{loc}(\Omega)$, which implies that $u \in C^{s+\alpha}_{loc}(\Omega)$.

Let us also remark, that the reduction argument used for Proposition 1.2 extends the class of fractional partial differential equations introduced in [24], which will be treated in a forthcoming work.

2. Proof of Proposition 1.2

With $(-\Delta)^{\frac{\sigma}{2}}$ we denote the fractional Laplacian

$$(-\Delta)^{\frac{\sigma}{2}} f(x) := \tilde{c}_{d,\sigma} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\sigma}}\, dy,$$

and with I_σ its inverse, the Riesz potential. Let $v := I_{1-s}u$ where u satisfies (1.4), so that

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.1}$$

Now let $\Omega_1 \Subset \Omega$ be an arbitrary open set compactly contained in Ω, and let ϕ be a test function supported in Ω_1. Pick an open set Ω_2 so that $\Omega_1 \Subset \Omega_2 \Subset \Omega$ and a cutoff function η, supported in Ω and constantly one in Ω_2. Then in particular one can take

$$\varphi := \eta(-\Delta)^{\frac{1-s}{2}}\phi$$

as a test function in (2.1) to obtain

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s(\eta(-\Delta)^{\frac{1-s}{2}}\phi) = 0.$$

That is,

$$\int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D\phi = \int_{\mathbb{R}^d} |Dv|^{p-2} Dv \cdot D^s(\eta^c(-\Delta)^{\frac{1-s}{2}}\phi).$$

where $\eta^c := (1-\eta)$. We set

$$T(\phi) := D^s(\eta^c(-\Delta)^{\frac{1-s}{2}}\phi)$$

Now we show that by the disjoint support of η^c and ϕ we have

$$\|T(\phi)\|_{L^p(\mathbb{R}^d)} \le C_{\Omega_1,\Omega_2,d,s,p}\, \|\phi\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

Once we have this, the claim is proven as Hölder's inequality and realizing the L^∞ norm via duality implies

$$-\operatorname{div}(|Dv|^{p-2}Dv) \in L^\infty_{loc}(\Omega).$$

To see (2.2), we use the disjoint support arguments as in [2, Lemma A.1] [16, Lemma 3.6.]: First we see that since $\eta^c(x)\phi(x) \equiv 0$,

$$T(\phi) = \tilde{c}_{d,1-s}\, D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\phi(y)}{|x-y|^{N+1-s}}\, dy.$$

Now taking a cutoff-function ζ supported in Ω_2, $\zeta \equiv 1$ on Ω_1 we have

$$T(\phi) = \tilde{c}_{d,1-s}\, D^s \int_{\mathbb{R}^d} \frac{\eta^c(x)\zeta(y)\phi(y)}{|x-y|^{N+1-s}}\, dy = \tilde{c}_{d,1-s} \int_{\mathbb{R}^d} k(x,y)\,\phi(y)\, dy,$$

where
$$k(x,y) := D^s_x \kappa(x,y) := D^s_x \frac{\eta^c(x)\,\zeta(y)}{|x-y|^{N+1-s}}.$$
The positive distance between the supports of η^c and ζ implies that these kernels k, κ are a smooth, bounded, integrable (both, in x and in y), and thus by a Young-type convolution argument we obtain (2.2). One can also argue by interpolation,
$$\left\| \int_{\mathbb{R}^d} \kappa(x,y)\,\phi(y) \right\|_{L^p(\mathbb{R}^d)} \leq \|\phi\|_{L^1(\mathbb{R}^d)},$$
as well as
$$\left\| \int_{\mathbb{R}^d} D_x\kappa(x,y)\,\phi(y)\,dy \right\|_{L^p(\mathbb{R}^d)} \leq \|\phi\|_{L^1(\mathbb{R}^d)}.$$
Interpolating this implies the desired result that
$$\left\| \int_{\mathbb{R}^d} D^s_x\kappa(x,y)\,\phi(y)\,dy \right\|_{L^p(\mathbb{R}^d)} \leq \|\phi\|_{L^1(\mathbb{R}^d)}.$$
Thus (2.2) is established and the proof of Proposition 1.2 is finished. □

References

1. P. Biler, C. Imbert, and G. Karch, *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 497–529. MR 3294409
2. S. Blatt, Ph. Reiter, and A. Schikorra, *Harmonic analysis meets critical knots. Critical points of the Möbius energy are smooth*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6391–6438. MR 3461038
3. L. Brasco and E. Lindgren, *Higher sobolev regularity for the fractional p-laplace equation in the superquadratic case*, Adv.Math. (2015).
4. L. Caffarelli, F. Soria, and J.-L. Vázquez, *Regularity of solutions of the fractional porous medium flow*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 5, 1701–1746. MR 3082241
5. L. Caffarelli and J. L. Vazquez, *Nonlinear porous medium flow with fractional potential pressure*, Arch. Ration. Mech. Anal. **202** (2011), no. 2, 537–565. MR 2847534
6. A. Di Castro, T. Kuusi, and G. Palatucci, *Local behaviour of fractional p-minimizers*, preprint (2014).
7. ______, *Nonlocal harnack inequalities*, J. Funct. Anal. **267** (2014), 1807–1836.
8. E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.
9. L. C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e*, J. Differential Equations **45** (1982), no. 3, 356–373.

10. E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova **27** (1957), 284–305. MR 0102739 (21 #1525)
11. J. Horváth, *On some composition formulas*, Proc. Amer. Math. Soc. **10** (1959), 433–437. MR 0107788
12. J. Korvenpää, T. Kuusi, and G. Palatucci, *The obstacle problem for nonlinear integro-differential operators*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 63, 29. MR 3503212
13. T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. **262** (2012), no. 10, 4205–4269.
14. T. Kuusi, G. Mingione, and Y. Sire, *A fractional Gehring lemma, with applications to nonlocal equations*, Rend. Lincei - Mat. Appl. **25** (2014), 345–358.
15. ———, *Nonlocal self-improving properties*, Analysis & PDE **8** (2015), 57–114.
16. L. Martinazzi, A. Maalaoui, and A. Schikorra, *Blow-up behaviour of a fractional adams-moser-trudinger type inequality in odd dimension*, Comm.P.D.E (accepted) (2015).
17. V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011. MR 2777530
18. G. Mingione, *Recent advances in nonlinear potential theory*, pp. 277–292, Springer International Publishing, Cham, 2014.
19. A. Schikorra, *Integro-differential harmonic maps into spheres*, Comm. Partial Differential Equations **40** (2015), no. 3, 506–539. MR 3285243
20. ———, *L^p-gradient harmonic maps into spheres and $SO(N)$*, Differential Integral Equations **28** (2015), no. 3-4, 383–408. MR 3306569
21. ———, *Nonlinear commutators for the fractional p-laplacian and applications*, Mathematische Annalen (2015), 1–26.
22. ———, *ε-regularity for systems involving non-local, antisymmetric operators*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 3531–3570. MR 3426086
23. A. Schikorra, D. Spector, and J. Van Schaftingen, *An L^1-type estimate for Riesz potentials*, Rev. Mat. Iberoamer. (accepted) (2014).
24. T.-T. Shieh and D. Spector, *On a new class of fractional partial differential equations*, Adv. Calc. Var. **8** (2015), no. 4, 321–336. MR 3403430
25. L. N. Slobodeckiĭ, *S. L. Sobolev's spaces of fractional order and their application to boundary problems for partial differential equations*, Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 243–246. MR 0106325
26. N. N. Ural′ceva, *Degenerate quasilinear elliptic systems*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 184–222.

Armin Schikorra, Mathematisches Institut, Abt. für Reine Mathematik, Albert-Ludwigs-Universität, Eckerstrasse 1, 79104 Freiburg im Breisgau, Germany armin.schikorra@math.uni-freiburg.de

Tien-Tsan Shieh, National Center for Theoretical Sciences, 2F Astronomy-Mathematics Building, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei City 106, Taiwan, R.O.C. ttshieh@ncts.ntu.edu.tw

Daniel Spector, National Chiao Tung University, Department of Applied Mathematics, 1001 Ta Hsueh Rd, 30010 Hsinchu, Taiwan, R.O.C. dspector@math.nctu.edu.tw