ON THE CONCENTRATION OF ENTROPY FOR SCALAR CONSERVATION LAWS

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Abstract. We prove that the entropy for an L^∞-solution to a scalar conservation laws with continuous initial data is concentrated on a countably 1-rectifiable set. To prove this result we introduce the notion of Lagrangian representation of the solution and give regularity estimates on the solution.

1. Introduction

Let us consider the nonlinear first order PDE

$$u_t + f(u)_x = 0, \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth flux function. It is a textbook example to prove that if f is nonlinear then the solution u develops discontinuities, so that the natural setting where to study existence is using the notion of weak solutions. More precisely, taking advantage of the divergence form of (1.1), a *weak solution* is a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ such that $f(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ and such that for all $\phi \in C^1_c(\mathbb{R}^+ \times \mathbb{R})$ it holds

$$\iint \big(u\phi_t + f(u)\phi_x\big) dxdt = 0.$$

The initial data can be naturally inserted in the weak formulation through the notion of weak trace on $\{t = 0\}$.
Another well known fact is that the weak solutions are not unique. This means that the conservation of u is not a sufficiently strict requirement to select a unique solution.

The theory then proceeds by requiring that u is an *entropic weak solution*. The idea comes from physics, where the time evolution of a thermodynamic system increases the thermodynamics entropy. From a mathematical point of view, a smooth convex function $\eta : \mathbb{R} \to \mathbb{R}$ is an *entropy* of (1.1) with corresponding flux q if

$$q'(u) = f'(u)\eta'(u). \tag{1.2}$$

The above relation is due to the fact that for smooth solutions u, the PDE (1.1) is equivalent to

$$u_t + f'(u)u_x = 0,$$

and multiplying by $\eta'(u)$ we obtain the conservation of entropy

$$\eta(u)_t + q(u)_x = 0.$$

For nonsmooth functions the chain rule $f(u)_x = f'(u)u_x$ is in general meaningless, so that we define u solution to (1.1) *entropic* if for all convex entropies η it holds

$$\eta_t + q_x \leq 0 \tag{1.3}$$

in the sense of distributions. A simple way of getting this inequality is to assume that u is the limit of the vanishing viscosity approximations

$$u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx}. \tag{1.4}$$

This is a natural assumption, based on the idea that the equation (1.1) is the limit of a more complicated physical system with dissipation, dispersions, etc.: from this point of view the solution u of the hyperbolic PDE tries to capture the macroscopic behavior of the system (i.e. large spatial and temporal scales).
Multiplying (1.4) by $\eta'(u^\epsilon)$ and using the definition of entropy flux q we obtain

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon(\eta(u^\epsilon))_{xx} - \epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2 \leq \epsilon(\eta(u^\epsilon))_{xx}.$$

Passing to the limit for $\varepsilon \to 0$, one formally gets (1.3). The fact that

$$t \mapsto \int_{\mathbb{R}} \eta(u(t))dx$$

is decreasing is because in mathematics one considers convex entropies, while the usual thermodynamic entropy is concave.

Since for scalar equations every convex function is an entropy ((1.2) is always solvable by integration), it follows that there are enough entropies to select a unique entropy solution. This result is the celebrated Kruzkhov theorem of uniqueness of L^∞ weak solutions, which is also valid in several space variables. Thus the theory of Cauchy problems in L^∞ is completely settled.

A natural development of the theory is the study of the *structure* of the solution: the question is whether u satisfies additional regularity of being just L^∞, as u solves (1.1).

For solution of bounded variation, the results proved in [Bre00], [BY14], [BC12], [BL99] and [AL04] provide a satisfactory answer, showing that the solution enjoys additional regularity of just being BV, because of the rigidity requirement of being a solution to (1.1).

The analysis of the structure of L^∞ solutions with general flux functions is definitely much more complicated. The best known results indeed rely on some nonlinear assumption on the flux f, and prove that u has approximate jumps on a rectifiable set J, but little is said on the complement $\mathbb{R}^+ \times \mathbb{R} \setminus J$, see [LOW04].

Another result in the literature concerns the structure of the dissipation measure $\mu := \eta_t + q_x$ ($\mu \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R})$, being a distribution with sign). In [LR03] it is shown that if f has only finitely many inflection points, then μ is concentrated on the jump set J.

The more than 10 years-old conjecture we are concerned in this work is the following one:

If $u \in L^\infty$ is an entropy solution to (1.1) and η is a convex entropy, then the measure $\mu = \eta_t + q_x$ is concentrated on a countably $\mathcal{H}^1$-rectifiable set.

From the above discussion, it is clear that this conjecture has a strong mathematical interest, because it requires to introduce new techniques in order to analyze L^∞ functions which satisfy a nonlinear constraint. Note that for BV solutions, by Volpert rule, the dissipation of entropy is concentrated on the countably $\mathcal{H}^1$-rectifiable jump set of u.

It is surprising that the same conjecture arises from a completely different context, namely the speed of convergence of discrete stochastic approximations to the continuum conservation equation. In [BBMN10] an old conjecture of Varadhan on the speed of convergence is related to the distance of a solution u from being entropic. Suppose that $u \in L^\infty$ is a non necessarily entropic solution to

$$u_t + [u(1-u)]_x = 0, \qquad u \in (0,1).$$

Consider the entropy $\eta(u) = u \ln u + (1-u)\ln(1-u)$ and the corresponding flux q satisfying $q'(u) = (1-2u)\eta'(u)$. If

$$\eta_t + q_x = \mu \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}),$$

then the probability of u being the limit of the stochastic system is exponentially small with coefficient $\mu^+(\mathbb{R}^+ \times \mathbb{R})$, i.e. the *positive* part of the dissipation. The fundamental assumption of [BBMN10] on the structure of u is that μ is countably $\mathcal{H}^1$-rectifiable.

We are now ready to state our main result.

Main Theorem. *Assume that u is an entropy solution to (1.1) with continuous initial datum.*

Then there exists a countably 1-rectifiable set J such that u is continuous outside J and $\mathcal{H}^1$-almost every point of J is an approximate jump point. Denote with $(u^-(y), u^+(y), \mathtt{n}(y))$ the left, right limits and the normal to J in $\mathcal{H}^1$-a.e. point $y \in J$.

If η is an entropy with flux q, then $\mu = \eta_t + q_x$ is concentrated on J. More precisely

$$\mu = \int_J \big\langle \big(\eta(u^+(y)) - \eta(u^-(y)),\ q(u^+(y)) - q(u^-(y))\big), \mathtt{n}(y) \big\rangle \mathcal{H}^1(dy).$$

Some comments are in order.

First, the technique we use is completely different from previous approaches. It is in fact based on the so called *Lagrangian representation* of an entropy solution u.

Definition 1.1. A *Lagrangian representation* of u is a pair of functions

$$\mathtt{X} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \quad \mathtt{u} : \mathbb{R} \to \mathbb{R},$$

such that $r \mapsto \mathtt{X}(t,r)$ is increasing for all t, $t \mapsto \mathtt{X}(t,r)$ is Lipschitz for all r, and for all $t \geq 0, \varphi \in C^1_c(\mathbb{R})$ it holds

$$\int_{\mathbb{R}} u(t,x) D_x \varphi(x)\, dx = \int_{\mathbb{R}} \mathtt{u}(r) D_r(\varphi \circ \mathtt{X}(t,r))(dr).$$

The *characteristics equation*, satisfied by $\mathtt{X}$, is the following:

$$D_t \mathtt{X}(t,s) = \tilde{\lambda}(t, \mathtt{X}(t,s)),$$

where $\tilde{\lambda} = \tilde{\lambda}(t,x)$ is given by

$$\tilde{\lambda}(t,x) = \begin{cases} f'(u(t,x)) & \text{if } u(t) \text{ is continuous at } x, \\ \dfrac{f(u^+) - f(u^-)}{u^+ - u^-} & \text{if } u(t) \text{ has a jump at } x. \end{cases}$$

The fundamental part of the definition is that the initial data $\mathtt{u}$ is transported by the flow $\mathtt{X}$, and the equation is hidden in the fact that $t \mapsto \mathtt{X}(t,r)$ is a characteristic. We observe that for L^∞ solutions the definition of the characteristic speed $\tilde{\lambda}$ is a *consequence* of the representation: the representation yields enough regularity in order to give a meaning to the characteristic equation. A fundamental observation here is that $\mathtt{X}$ enjoys a natural compactness, independent on the BV regularity of the solution. This hidden compactness of u (through $\mathtt{X}$) has never been exploited in the literature.

Secondly, the regularity results on the structure of u of the above theorem are new to the literature, and are interesting by themselves. We remark that they are immediately deduced from the Lagrangian representation. Here we want to underline the fact that proving results on the structure of the entropy dissipation is *intrinsically* tied to proving regularity results for the solution: indeed, in order to give a meaning to the jump set and an explicit formula for the dissipation, one has to prove sufficient regularity for u allowing the computation of the left/right limits.

Finally, as we said, the proof uses a completely new approach. Indeed, since no bounds on u are a priori known (but of being L^∞), the convergence of the dissipation for approximations u^n to the dissipation of u is only *weak* in general, so that it is almost hopeless to pass to the limit any structure of the dissipations μ^n of u^n to the dissipation μ of u.
Our approach is first to consider the subclass of entropies $\eta_k = (u-k)^+$, from which we can recover all the others by superposition: more precisely

$$\eta(u) = \eta(a) + \eta'(a)(u-a) + \int_a^b \eta''(k)(u-k)^+\, dk.$$

Then, using again the structure of u of the first part of the theorem, we study the boundary problem for conservation laws in $\{u > 0\}$. These domains, while open as a consequence of the structure proved in the Main Theorem, are much less regular than the standard boundary problems for conservation laws: usually the boundary data are given on one or two disjoint Lipschitz curves. We thus have to extend the boundary problem theory to this case. Some simplifications occur, because we only work with positive solutions and null boundary conditions.
The fact that we are able to solve the boundary problem gives us the possibility to construct an increasing sequence of positive BV solutions u_j in the domain $\{u > 0\}$, with approximate BV initial datum u^0_j such that $0 \leq u^0_j \leq u(0) \vee 0$. Since BV functions satisfy a chain rule, we are able to compute the dissipation μ_j of these approximations u_j, proving that it is a negative measure concentrated on J and bounded by the dissipation of u. The key step here is that no Cantor part appears in μ_j.
The conclusion is then to prove that $u_j \nearrow u^+$: this is done by comparison of the dissipations μ_j and μ, and provides a sort of uniqueness of solutions in Ω which we do not completely exploit here. Using the explicit expression of the dissipation of u_j and of $\mu \llcorner J$, one concludes that the limit u^+ dissipates only on J and the formula of the theorem holds.

The results of this paper are contained in the Master Thesis in Mathematics of the second author, at the University of Trieste (July 2014).

1.1. **Structure of the paper.** The paper is organized as follows.

In Section 2 we recall the fundamental properties of solutions to (1.1) for $x \in \mathbb{R}$ and in the presence of boundaries. Our aim is to obtain the tools which we need in the next sections in order to prove the regularity of solutions and the concentration of entropy: in particular the uniform BV bounds, the maximum principle and a superposition formula for entropies.

In Section 3 we present the first new results of this paper. On one hand we obtain the *Lagrangian representation* of the solution u (Proposition 3.4): this can be thought as the nonlinear counterpart of the Lagrangian representation of solutions to linear transport equations. An immediate corollary is the regularity of the solution u itself: in Theorem 3.5 we show that its structure is completely similar to a BV function of 1 space variable, in particular it has a countably 1-rectifiable set of approximate jump J. This proves the first part of the Main Theorem.

In the last section (Section 4) we prove the second part of the Main Theorem, namely the entropy concentration. The proof is done for the entropy $\eta_0^+(u) = (u)^+ =: \check{u}$, the positive part of u.
The first step is to construct a positive BV solution u_j in the open set $\{u > 0\}$ (Sections 4.1 and 4.2): here the regularity obtained in the previous section plays a major role.
Next, being BV, one can compute explicitly its dissipation μ_j, and prove that it is concentrated on the set of approximate jump J of the original u (Proposition 4.8 of Section 4.3). Proposition 4.9 of Section 4.4 gives the explicit expression of the dissipation μ_j.
Finally, showing on one hand the uniform estimate $\mu \leq \mu_j \leq 0$, μ being the dissipation of $\check{u}$, (Lemma 4.11), and that $u_j \nearrow u$ (Proposition 4.13), one concludes the proof of the second part of the Main Theorem (Theorem 4.15).

2. Entropy solutions to scalar conservation laws with boundary

We recall the essential concepts in the mathematical theory of scalar conservation laws. We give the well-posedness theorem for the Cauchy problem

$$u_t + f(u)_x = 0, \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}. \tag{2.1}$$

We will also consider the same equation in the presence of boundary. In the last part of this section we present a superposition argument for entropies.

All this material is very classical, see [Bre00, Daf10, Ser99].

2.1. **Existence and uniqueness of solutions.** The following theorem is well known [Kru70]: it gives the well posedness of the Cauchy problem (2.2)

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,\cdot) = u_0, \end{cases} \tag{2.2}$$

in the L^∞-class, as well as the maximum principle property.

In order to establish it, we recall the fundamental notion of entropy solution.

Definition 2.1. A continuously differentiable convex function $\eta : \mathbb{R} \to \mathbb{R}$ is called *entropy* for the equation (2.1), with *entropy flux* $q : \mathbb{R} \to \mathbb{R}$, if

$$\eta'(u) f'(u) = q'(u) \qquad \text{for all } u \in \mathbb{R}. \tag{2.3}$$

We refer to (η, q) as entropy-entropy flux pair.

We say that a weak solution u of (2.1) is *entropy admissible* if

$$\eta(u)_t + q(u)_x \leq 0,$$

in the distributional sense, for every entropy-entropy flux (η, q).

The key point in the selection of the "right" solutions is the fact that we require $u(t) = S_t u_0$ to be *entropic*.

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous. Then there exists a continuous semigroup $S : [0,\infty) \times L^1 \to L^1$ with the following properties.*

(i) $S_0(\bar{u}) = \bar{u}, \quad S_s(S_t\bar{u}) = S_{s+t}\bar{u}$.
(ii) $\|S_t\bar{u} - S_t\bar{v}\|_1 \leq \|\bar{u} - \bar{v}\|_1$.

(iii) *For each $u_0 \in L^1 \cap L^\infty$, the trajectory $t \mapsto S_t u_0$ yields the unique, bounded, entropy solution of the corresponding Cauchy problem* (2.2).

(iv) *If $u_0(x) \leq v_0(x)$ for all $x \in \mathbb{R}$, then $S_t(u_0)(x) \leq S_t(v_0)(x)$ for every $x \in \mathbb{R}$, $t \geq 0$.*

In the definition of entropy functions, one can more generally consider locally Lipschitz-continuous maps η, q that satisfy (2.3) almost everywhere. If we work with bounded solutions, for a proof it is sufficient to approximate η uniformly by smooth convex functions.

In the same way we prove Rankine-Hugoniot conditions, we get that at approximate jump points of an entropy admissible solution u it holds

$$\lambda[\eta(u^+) - \eta(u^-)] \geq q(u^+) - q(u^-).$$

In particular we will consider entropy-entropy flux pairs (η_k, q_k) defined by

$$\eta_k(u) = |u - k|, \quad q_k(u) = \operatorname{sign}(u - k)\big(f(u) - f(k)\big), \qquad k \in \mathbb{R},$$

and $\eta_k^\pm = (u - k)^\pm$ with relative fluxes q_k^+, q_k^-:

$$q_k^+(u) = \chi_{[k,+\infty)}(u)[f(u) - f(k)], \qquad q_k^-(u) = \chi_{(-\infty,k]}(u)[f(k) - f(u)]. \tag{2.4}$$

It is fairly easy to see that we can rewrite a generic convex function $\eta \in C^2(\mathbb{R})$ in terms of η_k. Suppose in fact that $a < 0$ sufficiently small and $b > 0$ sufficiently large,

$$a \leq \operatorname{ess\,inf} u_0 \leq \operatorname{ess\,sup} u_0 \leq b.$$

For every $u \in (a, b)$, elementary computations shows that

$$\eta(u) = \eta(a) + \eta'(a)(u - a) + \int_a^b \eta''(w)(u - w)^+ \, dw,$$

$$q(u) = q(a) + \eta'(a)[f(u) - f(a)] + \int_a^b \eta''(w)\chi_{(-\infty,u]}(w)(f(u) - f(w)) \, dw.$$

Let now μ_k be the dissipation of the entropies η_k:

$$\partial_t \eta_k + \partial_x q_k = \mu_k.$$

Then a simple superposition argument yields that if μ is the dissipation of η, then

$$\mu = \int_{-\infty}^{+\infty} \eta''(w)\mu_w \, dw. \tag{2.5}$$

In the following we will work mainly with the entropies η_k.

2.2. **Boundaries.**

The conservation law (2.1) has been also studied in the presence of boundaries ([Ama97, BlRN79, Ott96] and the references therein). We will present only the results which are needed for our purposes.

Let Ω be a domain of this form:

$$\Omega = \big\{(t, x) \in \mathbb{R}^+ \times \mathbb{R} : x \in (\gamma_1(t), \gamma_2(t))\big\}, \tag{2.6}$$

where $\gamma_i : \mathbb{R}^+ \to \mathbb{R}$ are Lipschitz functions and $\gamma_1 \leq \gamma_2$. We consider the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega, \\ u(0, x) = u_0^\nu(x) & x \in (\gamma^-(0), \gamma^+(0)), \\ u(t, \gamma^-(t)) = u_b^-(t) & t > 0, \\ u(t, \gamma^+(t)) = u_b^+(t) & t > 0, \end{cases} \tag{2.7}$$

where $u_0 \in L^\infty(\gamma^-(0), \gamma^+(0))$ and $u_b^-, u_b^+ \in L^\infty(0, +\infty)$.

Definition 2.3. We say that $u \in L^\infty(\Omega) \cap C([0, \infty); L^1(\mathbb{R}))$ is an *entropy solution* of (2.7) if

1. u satisfies the equation in the sense of distributions and it is entropic in Ω;
2. u satisfies the initial condition;

(3) u has a strong trace u_+ at the boundary such that for almost every $t \in \mathbb{R}^+$ and for all $k \in \mathbb{R}$

$$-\dot\gamma(\tilde\eta_k(u_+(t)) - \tilde\eta_k(u_b(t))) + \tilde q_k(u_+(t)) - \tilde q_k(u_b(t)) \leq 0,$$

where

$$\tilde\eta_k(u) = \begin{cases} (u-k)^+ & \text{if } u_+ \geq u_b, \\ (u-k)^- & \text{if } u_+ < u_b, \end{cases}$$

and $\tilde q_k$ denotes his flux (2.4).

The strong trace assumption is clearly satisfied for BV functions, but not for L^∞ solutions in general: however in the class of solutions we are considering Point (3) above holds.

With no additional difficulties with respect to the more classical single-curve boundary case, one can prove the following results (see the references above for a proof).

Theorem 2.4. *Let $u_0 \in \mathrm{BV}(\mathbb{R})$ and $u_{b,1}, u_{b,2} \in \mathrm{BV}(\mathbb{R}^+)$. Then there exists a unique entropy solution to (2.7) which satisfies the following:*

(i) for all $t \geq 0$,

$$\begin{aligned} \mathrm{Tot.Var.}\{u(t);\mathbb{R}\} \leq{}& \mathrm{Tot.Var.}\{u_0;\mathbb{R}\} \\ &+ \mathrm{Tot.Var.}\{u_{b,1};[0,\infty)\} + \mathrm{Tot.Var.}\{u_{b,2};[0,\infty)\} \\ &+ |u_{b,1}(0) - u_0(\gamma^-(0))| + |u_{b,2}(0) - u_0(\gamma^+(0))|; \end{aligned}$$

(ii) there exists a positive constant C such that

$$\|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} + C\big(\|u_b^+ - v_b^+\|_{L^1} + \|u_b^- - v_b^-\|_{L^1}\big);$$

(iii) if $u_0 \leq v_0$ and $u_b^\pm \leq v_b^\pm$, then $u \leq v$.

3. Lagrangian representation

The main goal of this section is to show that the solution to scalar conservation laws with continuous initial data enjoys a representation which is some sense is the continuous version of the classical wavefront tracking construction. This is the *Lagrangian representation* of the solution u.

We first start with the BV case.

Definition 3.1. A *Lagrangian representation* of u is a pair of functions

$$\begin{aligned} &\mathtt{X} : \mathbb{R}^+ \times (0, \mathrm{Tot.Var.}\{u_0\}] \to \mathbb{R}, && \text{the } \textit{position of the wave } s, \\ &\mathtt{u} : (0, \mathrm{Tot.Var.}\{u_0\}] \to \mathbb{R}, && \text{the } \textit{value of the wave } s, \end{aligned}$$

such that $s \mapsto \mathtt{X}(t,s)$ is increasing for all t, $t \mapsto \mathtt{X}(t,s)$ is Lipschitz for all s, and for all $t \geq 0, \varphi \in C^1_c(\mathbb{R})$ it holds

$$-\int_{\mathbb{R}} u(t,x) D_x\varphi(x)\,dx = \int_{(0,\mathrm{Tot.Var.}\{u_0\}]} \varphi(\mathtt{X}(t,s)) D_s\mathtt{u}(s)\,ds.$$

This last equation can be written shortly as

$$D_x u(t) = \mathtt{X}(t)_\sharp\big(D_s\mathtt{u}(t,s)\mathcal{L}^1\big). \tag{3.1}$$

Remark 3.2. Observe that $\mathtt{u}$ is defined up to additive constants. We uniquely determine $\mathtt{u}$ by requiring $\mathtt{u}(\mathrm{Tot.Var.}\{u\}) = u(+\infty)$.

The fact that this representation deserves the *Lagrangian* attribute is to the following fact: for all $s \in (0, \mathrm{Tot.Var.}\{u_0\}]$ the map $t \mapsto \mathtt{X}(t,s)$ are characteristic curves of

$$D_t u + D_x f(u) = 0.$$

Since we are considering BV solutions, we can rewrite the above PDE in the sense of measures:

$$D_t u + \tilde\lambda D_x u = 0, \tag{3.2}$$

where

$$\tilde\lambda(t,x) = \begin{cases} f'(u(t,x)) & \text{if } u(t) \text{ is continuous at } x, \\ \dfrac{f(u^+) - f(u^-)}{u^+ - u^-} & \text{if } u(t) \text{ has a jump at } x. \end{cases}$$

By (3.2) we can deduce the representation formula for $D_t u$:

$$D_t u = -\tilde{\lambda} D_x u = -\tilde{\lambda} \mathcal{L}^1 \otimes D_x u(t) = \mathcal{L}^1 \otimes [-\tilde{\lambda}(t) D_x u(t)].$$

By (3.1), we deduce

$$D_t u(t) = -\tilde{\lambda} \mathtt{X}(t)_\sharp [D_s \mathtt{u} \mathcal{L}^1 \llcorner J] = \mathtt{X}(t)_\sharp [-\hat{\lambda} D_s \mathtt{u} \mathcal{L}^1 \llcorner J], \tag{3.3}$$

where $\hat{\lambda}(t,s) = \tilde{\lambda}(t, \mathtt{X}(t,s))$.

A non difficult analysis shows that on the other hand

$$D_t u(t) = \mathtt{X}(t)_\sharp \big[-\partial_t \mathtt{X}(t,s) \hat{\lambda} D_s \mathtt{u} \mathcal{L}^1 \llcorner J \big],$$

and since the monotonicity of $s \mapsto \mathtt{X}(t,s)$ implies that for a.e. $t \in \mathbb{R}^+$ the time derivative $\partial_t \mathtt{X}(t,s)$ is constant on the intervals where $s \mapsto \mathtt{X}(t,s)$ is constant, we obtain the following proposition.

Proposition 3.3. *The following holds:*

$$D_t \mathtt{X}(t,s) = \tilde{\lambda}(t, \mathtt{X}(t,s)).$$

In order to pass the Lagrangian representation for BV solutions to L^∞-solutions with continuous initial data, first observe that we can rewrite (3.3) as

$$\int_{\mathbb{R}} u(t,x) D_x \varphi(x)\, dx = \int_{(0, \mathrm{Tot.Var.}\{u_0\}]} D_s \varphi(\mathtt{X}(t,s)) \mathtt{u}(s)\, ds. \tag{3.4}$$

This equation is meaningful even if $\mathtt{u}$ is only continuous, and is invariant w.r.t. reparameterizations of s.

Considering thus a sequence of BV initial data u_0^n converging uniformly to u_0 (here the assumption of continuity plays a major role) and passing to the limit of (3.4) (using suitable reparameterizations of the variable s), then L^1-stability of the entropy solutions gives the following result.

Proposition 3.4. *For all $t \geq 0$ and $\varphi \in C_c^1(\mathbb{R})$, it holds*

$$\int_{\mathbb{R}} u(t,x) D_x \varphi(x)\, dx = \int_{\mathbb{R}} \mathtt{u}(r) D_r (\varphi \circ \mathtt{X}(t))(dr),$$

where u is the entropy solution with initial data u_0 and $\mathtt{u}$ is determined by $\mathtt{u}(s) = u_0(\mathtt{X}(0,s))$.

The formula above implies that

$$u(t,x) = \mathtt{u}(\mathtt{X}(t)^{-1}(x))$$

for every x such that $\mathtt{X}(t)^{-1}(x)$ is single valued. In particular, the fact that $\mathtt{u}$ is continuous and $\mathtt{X}(t)$ is monotone, implies that the only discontinuities of $u(t)$ are in the points where $\mathtt{X}^{-1}(t)$ has jumps, which correspond to the intervals where $\mathtt{X}(t)$ is constant. The fact that $t \mapsto \mathtt{X}(t,s)$ is Lipschitz gives that continuity points of $u(t)$ are actually continuity points of u with respect to both variables and that discontinuity points lie on a countably 1-rectifiable set J.

We collect these regularity results here.

Theorem 3.5. *Let u be the unique entropic solution of (2.2) with continuous initial data. Then there exist a representative $\tilde{u}$ and $J \subset \mathbb{R}^+ \times \mathbb{R}$ such that:*

(a) J is contained in the union of countably many curves $t \mapsto (t, \alpha_i(t))$ with α_i uniformly Lipschitz;
(b) $\tilde{u}$ is continuous outside J;
(c) $\mathcal{H}^1$-almost every point in J is an approximate jump point of u;
(d) for every $t \geq 0$ and for every $x \in J_t = \{x \in \mathbb{R} : (t,x) \in J\}$, $\tilde{u}$ admits left and right limits at x.

In particular $\tilde{u}$ is uniquely determined by requiring that for all $t \geq 0$, $\tilde{u}(t)$ is right continuous.

4. Concentration of entropy

Our goal is to prove that, given u solution of (2.2) with continuous initial data u_0, for any entropy-entropy flux pair (η, q) the measure

$$\mu = \eta(u)_t + q(u)_x$$

is concentrated on J, the set of jump points of u: it is sufficient to prove it for entropies $(\eta_k^+)_{k \in \mathbb{R}}$.

The strategy is the following: for any entropy η_k^+ we construct a sequence of BV functions u_j such that μ_j is concentrated on J and converges monotonically to μ. This is not enough because the set J is not closed: indeed we need also to prove that the measures μ_j are controlled by $\mu \llcorner J$.

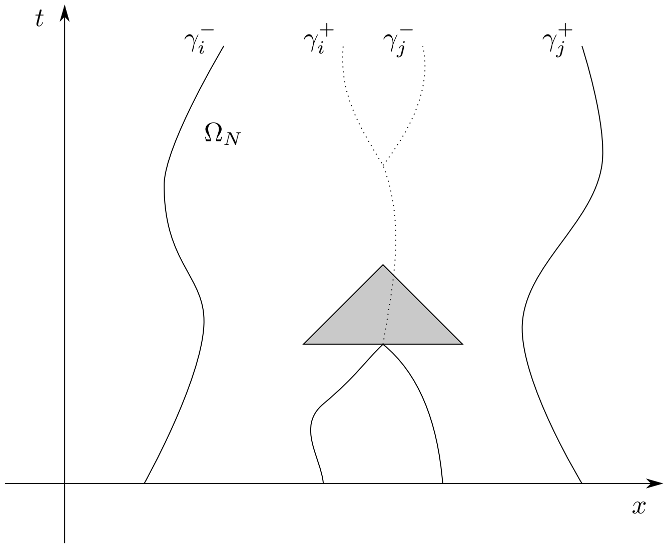

FIGURE 1. Positive triangle property.

Without loss of generality we give the proof for $k = 0$ and in what follows $\check{u}$ will denote positive part of u:

$$\check{u}(t, x) = \eta_0^+(u(t, x)) = (u(t, x))^+.$$

Denote also by $(I_n)_{n\in\mathbb{N}}$ the connected components of $\{u_0 > 0\}$. Since u_0 is continuous, I_n are open intervals. Let $\gamma_n^- = \mathtt{X}(\cdot, \inf I_n)$, $\gamma_n^+ = \mathtt{X}(\cdot, \sup I_n)$ and

$$t_1^N = \min \Big\{ t : \exists\, l, m \le N,\, l \neq m,\, \gamma_l^+(t) = \gamma_m^-(t) \Big\}.$$

Define

$$\Omega_N \cap \{(t, x) : t \in (0, t_1^N]\} = \bigcup_{n=1}^{N} \Big\{ (t, x) : t \in (0, t_1^N], \gamma_n^-(t) < x < \gamma_n^+(t) \Big\}.$$

We describe the restarting procedure which allows the construction of the open set Ω_n after each collision time: our aim is to obtain Proposition 4.2 below.

Suppose to have

$$\gamma_{l_1}^+(t_1^N) = \gamma_{l_2}^-(t_1^N) = \gamma_{l_2}^+(t_1^N) = \ldots = \gamma_{l_k}^-(t_1^N) \neq \gamma_m^\pm(t_1^N),$$

for $m \notin \{l_1, \ldots, l_k\}$.

If there are several collision points at t_1^N, we repeat the following construction at each of them.

We prolong the definition of Ω_N until a time t_2^N which is defined as the minimum time greater than t_1^N such that one of the following happens:

- there exist $l, m \notin \{l_1, \ldots, l_k\}$ different such that $\gamma_l^+(t) = \gamma_m^-(t)$;
- there exists $l \notin \{l_1, \ldots, l_k\}$ such that $\gamma_l^+(t) = \gamma_{l_1}^-(t)$;
- there exists $l \notin \{l_1, \ldots, l_k\}$ such that $\gamma_l^-(t) = \gamma_{l_k}^+(t)$.

So we define

$$\begin{aligned}\Omega_N \cap \{(t, x) : t \in (t_1^N, t_2^N]\} = & \bigcup_{n \notin \{l_1,\ldots,l_k\}} \Big\{ (t, x) : t \in (t_1^N, t_2^N], \gamma_n^-(t) < x < \gamma_n^+(t) \Big\} \\ & \cup \Big\{ (t, x) : t \in (t_1^N, t_2^N], \gamma_{l_1}^-(t) < x < \gamma_{l_k}^+(t) \Big\}.\end{aligned}$$

Observe that each time section of Ω_N has at most N connected components and for $t > t_i^N$ they are up to $N - i$. In particular the procedure has at most $N - 1$ restarts.

(The case $\gamma_l^-(t) = \gamma_l^+(t)$ is treated by removing the couple (γ_l^-, γ_l^+), because by maximum principle the solution will remain ≤ 0 in the interval (γ_l^-, γ_l^+) from that time on.)

In the following we collect some properties of Ω_N.

Definition 4.1. Let $B \subset \mathbb{R}^2$. We say that B enjoys the *positive triangle property* if there exists $L \in \mathbb{R}$ such that the following holds: suppose

$$\big(\bar{t}, (1-\theta)a + \theta b\big) \in B \qquad \text{for all } \theta \in [0,1];$$

then

$$\big(\bar{t}+t, (a+Lt)(1-\theta) + \theta(b-Lt)\big) \in B \qquad \text{for all } \theta \in [0,1], t \in \left[0, \frac{b-a}{2L}\right].$$

Proposition 4.2. *The following hold:*

(1) Ω_N *is open;*
(2) $\Omega_N \subset \Omega_{N+1}$*;*
(3) Ω_N *and* $\overline{\Omega_N}$ *enjoy the positive triangle property.*
(4) $\Omega_N \subset \{u \geq 0\}$.

4.1. **Initial-boundary value problem on Ω_N.** Consider the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \Omega_N, \\ u = u_b & \text{on } \partial\Omega_N, \\ u(0,\cdot) = u_0, \end{cases} \tag{4.1}$$

where $u_0 \in L^\infty(\mathbb{R})$ and $u_b \in L^\infty(\partial\Omega_N)$ are given.

One can assign boundary condition with functions $(u^{\pm}_{b,i})_{i=1}^N \in L^\infty(0,+\infty)$ and require that for all i from 1 to N, u has internal traces along $\gamma_i^{\pm} \cap \partial\Omega_N$ and they satisfy the boundary condition in the sense of Definition 2.3.

Definition 4.3. We say that $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \cap C([0,+\infty); L^1(\mathbb{R}))$ is a *solution* of (4.1) if it satisfies the equation in the sense of distributions in Ω_N, the initial condition is assumed and u has internal traces on $\gamma_i^{\pm} \cap \partial\Omega_N$ which satisfy the boundary conditions $u^{\pm}_{b,i}$ in the sense of Definition 2.3.

We see that we can construct a solution for BV data using solution on domain of the form (2.6) as building block. In fact, by construction, Ω_N is composed by finitely many of such domains.

Up to t_1^N solution is obtained solving separately problems of type (2.6). At t_1^N we join intervals that meet as in construction of Ω_N, we obtain finitely many domains of type (2.6) and we solve the problem with initial data $u(t_1^N)$ on that interval. The procedure can restart because $u(t_1^N)$ is BV: indeed applying Theorem 2.4 on each subdomain bounded by two Lipschitz curves, it follows that

$$\begin{aligned} \mathrm{Tot.Var.}\{u(t_1^N)\} \leq \mathrm{Tot.Var.}\{u_0, \mathbb{R}\} + \sum_{i=1}^N \mathrm{Tot.Var.}\{u^{\pm}_{b,i}; (0, t_1^N]\} \\ + \sum_{i=1}^N |u^{\pm}_{b,i}(0) - u_0(\gamma_i^{\pm}(0))|. \end{aligned}$$

The contractivity property and comparison principle follow from the previous case because it is easy to see that they hold after a restart. Let us state them in this context too.

Proposition 4.4. *Let u, v be solutions of (4.1) in the sense of Definition 4.3 with initial data u_0, v_0 and boundary data u_b, v_b respectively. Then there exists a positive constant C such that*

$$\|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} + C\sum_{i=1}^N \|u_{b,i} - v_{b,i}\|_{L^1},$$

In particular we obtained well-posedness of problem (4.1) endowed with BV initial and boundary data.

Proposition 4.5. *Let u, v be solutions of (4.1) in the sense of Definition 4.3 with initial data u_0, v_0 and boundary data u_b, v_b respectively. Suppose $u_0 \leq v_0$ and $u_b \leq v_b$. Then $u \leq v$.*

4.2. **BV approximations.** Let $(u_{0,j})_{j\in\mathbb{N}}$ be an increasing sequence of non negative BV functions such that $0 \leq u_{0,j} \nearrow (u_0)^+$ in L^1. Define $u_{j,N}$ as the unique solution of problem (4.1) with initial data $u_{0,j}$ and zero boundary conditions.

The following lemma can be proved using the monotonicity of the semigroup, Proposition 4.5.

Lemma 4.6. *For all $j, N \in \mathbb{N}$,*

(1) $0 \leq u_{j,N} \leq \check{u}$;
(2) $u_{j,N} \leq u_{j,N+1}$;
(3) $u_{j,N} \leq u_{j+1,N}$.

By points (1) and (2) in Lemma 4.6 we obtain that $(u_{j,N})_N$ converges in L^1 to some function $u_j \leq \check{u}$. Passing to the limit inequality (3) in Lemma 4.6 we get that the sequence u_j is increasing so it converges to some function $\bar{u} \leq \check{u}$. We will prove that actually $\bar{u} = \check{u}$.

In what follows we will consider the precise representative of BV functions.

4.3. **Concentration of μ_j.** The idea here is that the jump part of the derivative of a BV function is 0 on the set of continuity. We define

$$\Omega = \bigcup_{N=1}^{\infty} \Omega_N, \quad A = \bigcup_{N=1}^{\infty} \overline{\Omega}_N.$$

Since $u_j \leq \check{u}$, then the Lagrangian representation gives the following.

Lemma 4.7. *Let $(\bar{t}, \bar{x}) \in A^c$, then $u_j(\bar{t}, \bar{x}) = 0$ and u_j is continuous in $(\bar{t}, \bar{x})$.*

We introduce the set

$$\Gamma = \bigcup_{n=1}^{\infty} \{(t, \gamma_n^{\pm}(t))\}.$$

We will prove that dissipation is concentrated on Γ.

The first step is that, by the previous lemma, we can easily deduce that μ is concentrated on A, and since u_j is a solution in the set Ω, it follows that

Proposition 4.8. *The dissipation*

$$(u_j)_t + f(u_j)_x =: \mu_j$$

is concentrated on Γ.

Observe that in the divergence computation of the above proposition the function u_j is considered in $\mathbb{R}^+ \times \mathbb{R}$ (by extending it to 0 outside Ω): this would correspond in some sense to the dissipation of $\eta_0^+(u_j)$.

4.4. **Formula for μ_j.** In this section we give an explicit formula for μ_j.

We observe that the function u_j belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$: indeed, u_j is obtained as limit of $u_{j,N}$. By Theorem 2.4 (extended to Ω in a natural way), for all $N \in \mathbb{N}$

$$\mathrm{Tot.Var.}\{u_{j,N}(t), \mathbb{R}\} \leq \mathrm{Tot.Var.}\{u_j(0); \mathbb{R}\},$$

and since total variation is lower semicontinuous,

$$\mathrm{Tot.Var.}\{u_j(t); \mathbb{R}\} \leq \mathrm{Tot.Var.}\{u_j(0); \mathbb{R}\}.$$

In particular $u_j \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$, hence the right (left) trace $u_{j,\Gamma}^+$ $(u_{j,\Gamma}^-)$ of u_j on Γ is well defined.

The fact that Γ is countably 1-rectifiable and u_j BV gives

Proposition 4.9. *It holds*

$$\mu_j = \mu_j \llcorner \Gamma = \big[-\lambda(u_{j,\Gamma}^+ - u_{j,\Gamma}^-) + f(u_{j,\Gamma}^+) - f(u_{j,\Gamma}^-) \big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma. \tag{4.2}$$

4.5. **Comparison with μ.** The aim of this section is to prove that $\mu \le \mu_j \le 0$ for all $j \in \mathbb{N}$. We reach our goal proving two inequalities.

By Lagrangian representation there exist traces of u on Γ. In particular $\check{u}_\Gamma^+$ and $\check{u}_\Gamma^-$, traces of $\check{u}$, are well defined.

Lemma 4.10. *The following inequality holds:*

$$\mu \le \big[-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma. \tag{4.3}$$

In fact, the expression (4.3) is the "jump part" of the measure μ. Since $u_j \le \check{u}$, we can now prove that the measures μ_j are uniformly bounded.

Lemma 4.11. *The following inequality holds:*

$$\big[-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma \le \mu_j. \tag{4.4}$$

Proof. By formula (4.2), it suffices to show that for $\mathcal{H}^1$-almost every point in Γ

$$-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-) \le -\lambda(u_{j,\Gamma}^+ - u_{j,\Gamma}^-) + f(u_{j,\Gamma}^+) - f(u_{j,\Gamma}^-).$$

By Theorem 3.5, we can consider left and right limits u^- and u^+ of $u(t)$ in $(t, \gamma_n(t))$. We need to distinguish three cases:

Case 1: $u^- \le 0$ and $u^+ \le 0$. Since $0 \le u_j \le \check{u}$,

$$-\lambda(\check{u}^+ - \check{u}^-) + f(\check{u}^+) - f(\check{u}^-) = -\lambda(u_j^+ - u_j^-) + f(u_j^+) - f(u_j^-) = 0$$

because all terms vanish.

Case 2: $u^- \le 0$ and $u^+ > 0$. Since u is entropic we have $\lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ and for all $k \in (u^-, u^+)$, it holds $f(k) \ge f(u^+) - \lambda(u^+ - k)$. In particular, since $u_j^+ \le \check{u}^+ = u^+$ and $u_j^- = \check{u}^- = 0$,

$$-\lambda u^+ + f(u^+) - f(0) \le -\lambda u_j^+ + f(u_j^+) - f(0).$$

The case where $u^- > 0$ and $u^+ \le 0$ is analogous.

Case 3: $u^- > 0$ and $u^+ > 0$. We show that such points, except for countably many, are internal to Ω and therefore both quantities vanish thanks to Rankine-Hugoniot conditions. It suffices to prove our claim for a single curve γ_n.

Take a point $(\bar{t}, \gamma_n(\bar{t}))$ with both positive limits and consider $\varepsilon < u^- \wedge u^+$. Since $\mathtt{u}$ is uniformly continuous there exist finitely many k such that $\max_{[\gamma_k^-, \gamma_k^+]} \mathtt{u} > \varepsilon$. Let N be bigger of all these k. We prove that $\bar{t}$ has a right neighborhood of times t' such that $(t', \gamma_n(t')) \in \Omega_N$. Being $\varepsilon < u^- \wedge u^+$, the set $\Omega_N(\bar{t}) = \Omega_N \cap \{t = \bar{t}\}$ has density 1 at $\bar{x}$. Since $\Omega_N(t)$ is made of finitely many open intervals I_i two situations can occur:

- there exists i such that $\bar{x} \in I_i$;
- there exist i, j such that $\inf I_i < \sup I_i = \bar{x} = \inf I_j < \sup I_j$.

In the first case the claim follows from the positive triangle property of Ω_N. One gets the result in the second case using the positive triangle property of $\overline{\Omega_N}$. □

By previous lemmas we get $\mu \le \mu_j$.

Remark 4.12. A completely similar proof shows that for any $j \in \mathbb{N}$,

$$\mu_{j+1} \le \mu_j.$$

4.6. **Convergence of u_j.** Now we can prove that $\bar{u} = \check{u}$. The idea is that the initial "mass" is the same and $\bar{u}$ dissipates less than $\check{u}$, but at every time $\bar{u} \le \check{u}$. The only possibility is that they are equal.

Proposition 4.13. *The sequence of BV approximations u_j converges to $\check{u}$.*

Proof. Consider the map $\Phi : [0, +\infty) \to \mathbb{R}$ defined by

$$\Phi(t) = \int_{\mathbb{R}} (\bar{u}(t) - \check{u}(t))\, dx.$$

This map is continuous, non-positive and $\Phi(0) = 0$. We prove that its distributional derivative is non negative so that it is constant equal to zero. The monotone convergence of u_j and the bound obtained in previous section imply the convergence of dissipations:

$$D_t u_j + D_x f(u_j) = \mu_j \to \bar{\mu} = D_t \bar{u} + D_x f(\bar{u})$$

in the sense of Radon measures. In particular it holds

$$\iint \{\bar{u} D_t \varphi + f(\bar{u}) D_x \varphi\} dxdt \le \iint \{\check{u} D_t \varphi + f(\check{u}) D_x \varphi\} dxdt$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ non negative.

We consider test functions of the form $\varphi(t,x) = \tilde{\varphi}(t)\psi(x)$, where $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on a sufficiently large interval. Then we have

$$\begin{aligned}
0 \ge{}& \iint \{(\bar{u} - \check{u}) D_t \varphi + (f(\bar{u}) - f(\check{u})) D_x \varphi\}\, dxdt \\
={}& \iint (\bar{u} - \check{u}) D_t \tilde{\varphi}\, dxdt + \iint (f(\bar{u}) - f(\check{u})) D_x \varphi\, dxdt \\
={}& \int D_t \tilde{\varphi}(t) \int (\bar{u}(t) - \check{u}(t))\, dxdt + \int \tilde{\varphi}(t) \int (f(\bar{u}) - f(\check{u})) D_x \psi(x)\, dxdt \\
={}& -\Phi'(\tilde{\varphi}).
\end{aligned}$$

Then $\Phi' \le 0$ in the sense of distributions and the claim follows. ☐

4.7. **Formula for μ.** In the following proposition we present the explicit formula for the dissipation μ.

Proposition 4.14. *The dissipation μ is given by the following formula:*

$$\mu = \big[-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma. \tag{4.5}$$

In particular, we obtain that μ is concentrated on the set J of the jumps of u .

Proof. By equations (4.2), (4.3) and (4.4) we get

$$\begin{aligned}
\mu_j ={}& \big[-\lambda(u_{j,\Gamma}^+ - u_{j,\Gamma}^-) + f(u_{j,\Gamma}^+) - f(u_{j,\Gamma}^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma \\
\ge{}& \big[-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma \\
\ge{}& \mu.
\end{aligned}$$

By Proposition 4.13, we get

$$\begin{aligned}
\mu ={}& \lim_j \big[-\lambda(u_{j,\Gamma}^+ - u_{j,\Gamma}^-) + f(u_{j,\Gamma}^+) - f(u_{j,\Gamma}^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma \\
\ge{}& \big[-\lambda(\check{u}_\Gamma^+ - \check{u}_\Gamma^-) + f(\check{u}_\Gamma^+) - f(\check{u}_\Gamma^-)\big] \frac{1}{\sqrt{1+\lambda^2}} \mathcal{H}^1 \llcorner \Gamma \\
\ge{}& \mu
\end{aligned} \tag{4.6}$$

hence we can conclude that all inequalities in (4.6) are in fact equalities and formula 4.5 holds. From that formula we can deduce that μ is concentrated on the set of jumps of $\check{u}$, which is contained in J. ☐

4.8. **Conclusion.** By the superposition formula (2.5), we finally get the following result for entropy solution u of (2.2) with continuous initial data.

Theorem 4.15. *There exists a countably $\mathcal{H}^1$-rectifiable set J such that u is continuous outside J and $\mathcal{H}^1$-almost every point of J is an approximate jump point. Denote with $(u^-(y), u^+(y), \mathtt{n}(y))$ the left, right limits and the normal to J in $\mathcal{H}^1$-a.e. point $y \in J$. If η is an entropy with flux q, then*

$$\eta_t + q_x = \int_J \big(\eta(u^+(y)) - \eta(u^-(y)),\ q(u^+(y)) - q(u^-(y))\big) \cdot \mathtt{n}(y) \mathcal{H}^1(dy).$$

This result, together with Theorem 3.5 gives the Main Theorem stated in the introduction, page 2.

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