A MINIMAL PARTITION PROBLEM WITH TRACE CONSTRAINT IN THE GRUSHIN PLANE.

VALENTINA FRANCESCHI

Abstract. We study a variational problem for the perimeter associated with the Grushin plane, called minimal partition problem with trace constraint. This consists in studying how to enclose three prescribed areas in the Grushin plane, using the least amount of perimeter, under an additional "one-dimensional" constraint on the intersections of their boundaries. We prove existence of regular solutions for this problem, and we characterize them in terms of isoperimetric sets, showing differences with the Euclidean case. The problem arises from the study of quantitative isoperimetric inequalities and has connections with the theory of minimal clusters.

1. Introduction

Carnot-Carathéodory spaces are metric spaces in which the distance is defined in association with a family of vector fields $\mathcal{X} = \{X_1, \dots, X_r\}$ on a n-dimensional manifold, $n \geq r$. The theory of perimeters in such spaces has been developed starting from the 1990s in [3], [12], [18], and *isoperimetric inequalities* in Carnot-Carathéodory spaces are a current object of investigation in Calculus of Variations and Analysis on Metric spaces, see [25], [18], [9], [10], [26]. In particular, Pansu's conjecture on the sharp isoperimetric inequality in the Heisenberg groups is still an open problem (see [21], [29], [27], [23]). The *Grushin plane* is an example of Carnot-Carathéodory space, introduced in the context of hypoelliptic operators by Franchi and Lanconelli in [11].

In this paper, we study a variational problem for the perimeter in the Grushin plane, that we call minimal partition problem with trace constraint. We were led to this problem studying quantitative isoperimetric inequalities, as we will explain at the end of the introduction. The minimal partition problem with trace constraint has connections with the theory of minimal clusters.

Let $\alpha \geq 0$. The Grushin plane is defined endowing $\mathbb{R}^2$ with the family of vector fields $\mathcal{X}_\alpha = \{\partial_x, |x|^\alpha \partial_y\}$, where (x, y) denotes a point in $\mathbb{R}^2$ and ∂_x, ∂_y respectively denote the partial derivative with respect to the first and to the second coordinate. Given a Lebesgue measurable set $E \subset \mathbb{R}^2$, the α-*perimeter* of E is defined as

$$P_\alpha(E) = \sup \Big\{ \int_E (\partial_x \varphi_1 + |x|^\alpha \partial_y \varphi_2)\, dxdy : \varphi_1, \varphi_2 \in C^1_c(\mathbb{R}^2),\ \max_{\mathbb{R}^2} \big(\varphi_1^2 + \varphi_2^2\big)^{\frac{1}{2}} \leq 1 \Big\}. \tag{1.1}$$

Notice that when $\alpha = 0$ the α-perimeter is the standard Euclidean perimeter, that we denote by P. If $E \subset \mathbb{R}^2$ is a bounded set with Lipschitz boundary, we have

$$P_\alpha(E) = \int_{\partial E} \sqrt{(N^E_x(x, y))^2 + |x|^{2\alpha}(N^E_y(x, y))^2}\, d\mathcal{H}^1(x, y), \tag{1.2}$$

where $N^E = (N^E_x, N^E_y)$ is the outer unit normal to E and $\mathcal{H}^1$ is the one dimensional Hausdorff measure, see [22, Theorem 2.1], [14, Proposition 2.1]. By the representation formula (1.2), it is

2010 *Mathematics Subject Classification.* 49Q20, 53C17.

Key words and phrases. Minimal clusters, Grushin plane, Quantitative Isoperimetric Inequality.

clear that x-translations modify the α-perimeter with $\alpha > 0$, i.e., there exists $E \subset \mathbb{R}^2$ such that $P_\alpha(E) = P_\alpha(E + (\bar{x}, 0))$, if and only if $|\bar{x}| = 0$. This is an essential difference with the classical perimeter, or with the one defined in *Carnot groups*. In fact, the Grushin plane for $\alpha > 0$ is not isometrically homogeneous (i.e., there exist $p, q \in \mathbb{R}^2$ such that no distance preserving homeomorphism connects p to q), and it doesn't admit any group structure (see [19]).

An important feature of the Grushin plane is that isoperimetric sets are completely characterized. Given $v > 0$, the *isoperimetric problem* for the α-perimeter is

$$\min\{P_\alpha(E) : E \subset \mathbb{R}^2,\ \mathcal{L}^2(E) = v\}, \tag{1.3}$$

where $\mathcal{L}^2$ denotes the two-dimensional Lebesgue measure. Solutions to (1.3) have been studied in [22], and in [14] in Grushin structures of dimension $n \geq 2$: up to a vertical translation $\tau_t(x, y) = (x, y + t)$ and an anisotropic dilation $\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$, the unique solution to problem (1.3), called *isoperimetric set*, is

$$E^\alpha_{\text{isop}} = \{(x, y) \in \mathbb{R}^2 : |y| < \varphi_\alpha(|x|),\ |x| < 1\}, \quad \varphi_\alpha(r) = \int_{\arcsin r}^{\frac{\pi}{2}} (\sin t)^{\alpha+1}\, dt,\ r > 0. \tag{1.4}$$

The one parameter family of dilations δ_λ is such that $P_\alpha(\delta_\lambda E) = \lambda^{Q-1} P_\alpha(E)$, $\mathcal{L}^2(\delta_\lambda E) = \lambda^Q \mathcal{L}^2(E)$, where $Q = \alpha + 2$ is called *homogeneous dimension*. In particular, (1.4) implies the validity of the following *sharp isoperimetric inequality* for any measurable set $E \subset \mathbb{R}^2$ with finite measure:

$$\mathcal{L}^2(E) \leq c(\alpha) P_\alpha(E)^{\frac{Q}{Q-1}}, \quad c(\alpha) = \frac{\alpha+1}{\alpha+2}\Big(2\int_0^\pi \sin^\alpha(t)\, dt\Big)^{-\frac{1}{\alpha+1}}.$$

When $\alpha = 1$, the profile function φ_α corresponds to the conjectured isoperimetric profile function of the Heisenberg groups.

The *minimal partition problem with trace constraint* consists in studying how to enclose three prescribed areas in the Grushin plane, using the least amount of perimeter, under an additional "one-dimensional" constraint on the intersections of their boundaries.

We say that a set $E \subset \mathbb{R}^2$ is *x-symmetric* (resp. *y-symmetric*) if $(x, y) \in E$ implies $(-x, y) \in E$ (resp. if $(x, y) \in E$ implies $(x, -y) \in E$). We say that E is *y-convex* if the section $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is an interval for every $x \in \mathbb{R}$; finally we say that E is *y-Schwarz symmetric* if it is y-symmetric and y-convex. We denote by $\mathscr{S}_x$ the class of $\mathcal{L}^2$-measurable, x-symmetric sets in $\mathbb{R}^2$ and by $\mathscr{S}^*_y$ the class of $\mathcal{L}^2$-measurable and y-Schwarz symmetric sets in $\mathbb{R}^2$.

Given $v_1, v_2, h_1, h_2 \geq 0$, we define the class $\mathcal{A} = \mathcal{A}(v_1, v_2, h_1, h_2)$ of all sets $E \in \mathscr{S}_x \cap \mathscr{S}^*_y$ such that for some $x_0 > 0$, called *partitioning point* of E, the sets

$$E^l = \{(x, y) \in E : x < -x_0\}, \quad E^c = \{(x, y) \in E : |x| < x_0\}, \quad E^r = \{(x, y) \in E : x > x_0\}$$

satisfy

$$\mathcal{L}^2(E^c) = v_1, \quad \mathcal{L}^2(E^l) = \mathcal{L}^2(E^r) = v_2/2, \tag{1.5a}$$

$$[-h_1, h_1] \subset \mathrm{tr}^x_{x_0-} E,\ [-h_2, h_2] \subset \mathrm{tr}^x_{x_0+} E, \tag{1.5b}$$

where $\mathrm{tr}^x_{x_0\pm} E$ denote the *left* and *right traces* of the set E at the point x_0, introduced in Definition A.2. Choosing $h_1 = h_2 = h > 0$, the *trace constraint* (1.5b) is a relaxed version of the trace equality

$$E_{x_0} = E_{-x_0} = [-h, h]. \tag{1.6}$$

In other words, a set $E \in \mathcal{A}$ is such that E^l, E^c, E^r have prescribed areas and E^c touches E^r and E^l in segments of a prescribed length, see Figure 1.

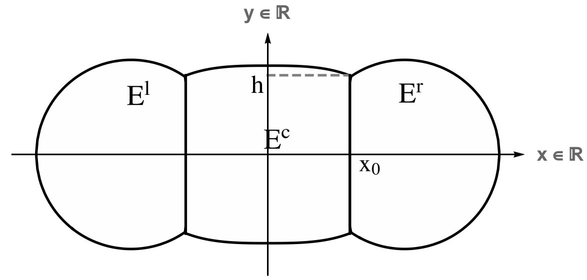

FIGURE 1. A set in the class $\mathcal{A}$, for $h_1 = h_2 = h$.

Given $v_1, v_2, h_1, h_2 \geq 0$, we study existence of regular solutions of

$$\inf\{\mathscr{P}_\alpha(E) : E \in \mathcal{A}(v_1, v_2, h_1, h_2)\}, \tag{1.7}$$

where

$$\mathscr{P}_\alpha(E) = P_\alpha(E^l) + P_\alpha(E^c) + P_\alpha(E^r) - 4h_1 - 4h_2 \tag{1.8}$$

measures the total amount of α-perimeter of the partition $\{E^l, E^c, E^r\}$. The quantity $4h_1 + 4h_2$ is the total perimeter of the common parts counted with multiplicity.

The first solution to the minimal partition problem with no trace constraint and with two prescribed volumes in the Euclidean plane dates back to the 1993 paper [7], where it is proved that the unique minimizers are the so called *double bubbles*. For general dimensions, several open questions about minimal clusters are still open (see [24]).

Our interest in Problem (1.7) comes from the study of the stability of the isoperimetric inequality. In the seminal paper [16], the authors present a symmetrization technique in $\mathbb{R}^n$, $n \in \mathbb{N}$, to prove existence of a dimensional constant $C(n) > 0$ such that any Lebesgue measurable set $E \subset \mathbb{R}^n$ satisfies

$$P(E) - P(B(0, r_E)) \geq C(n)\Big(\min_{x \in \mathbb{R}^n} \mathcal{L}^n(E \triangle B(x, r_E))\Big)^2. \tag{1.9}$$

Here, $B(0, r) = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n : p_1^2 + \cdots + p_n^2 < r^2\}$, and the quantity $r_E \geq 0$ is chosen in order to have $\mathcal{L}^n(E) = \mathcal{L}^n(B(0, r_E))$. Such inequality is known as the sharp *quantitative isoperimetric inequality* in $\mathbb{R}^n$, see also [5], [6], [8], [20]. The minimal partition problem (1.7) is used in [16] to prove (1.9) in a class of symmetric sets in $\mathbb{R}^n$. In [16, Lemma 4.3] the authors implicitly use the solution E to problem (1.7) with trace constraint given by (1.6), in the Euclidean setting ($\alpha = 0$). The solution is, for some x_0, $r_0 > 0$

$$E = E^l \cup E^c \cup E^r, \quad E^c = B(0, r_0) \cap \{(x, y) \in \mathbb{R}^2 : |x| < x_0\}. \tag{1.10}$$

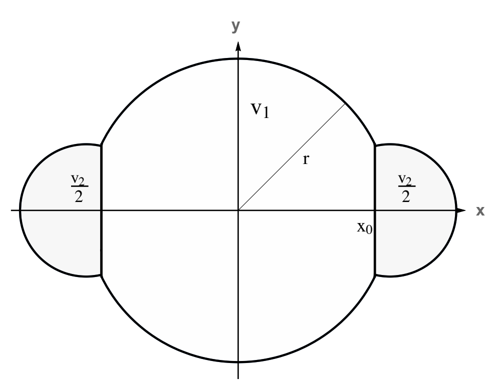

FIGURE 2. The solution of the minimal partition problem in the Euclidean setting.

Notice that the central part E^c is the portion of an isoperimetric set lying in a vertical stripe, see Figure 2. This is due to the invariance under translations of the standard perimeter P.

Quantitative isoperimetric inequalities have been studied in Riemannian manifolds providing results in the Gauss space (see [4]), in the n-Sphere (see [1]) and in the Hyperbolic space (see [2]). Quantitative isoperimetric inequalities in a subRiemannian setting are presented in [15] in the case of the Heisenberg group.

An interesting task would be to study quantitative isoperimetric inequalities in spaces with less isometries, such as the Grushin plane. However, in this paper we show some unexpected obstacles that prevent an adaptation of the techniques in [16] to the Grushin plane. In particular, we will show that, when $\alpha > 0$, solutions to the minimal partition problem (1.7) are not obtained in their central part as portions of isoperimetric sets lying in a vertical stripe.

In the first part of this paper we establish existence of solutions.

Theorem 1.1. *Let $\alpha \geq 0$, $v_1, v_2, h_1, h_2 \geq 0$. There exists a solution $E = E^l \cup E^c \cup E^r \in \mathcal{A}(v_1, v_2, h_1, h_2)$ to the minimal partition problem with trace constraint* (1.7) *such that E^c is a convex set and E^l, E^r have Lipschitz boundaries.*

Minimizers as in Theorem 1.1 are called *regular*. In Proposition 3.3 we show, under a technical assumption, that any regular minimizer $E \in \mathcal{A}$ assumes the least possible traces at the partitioning point $x_0 > 0$, i.e., $\mathrm{tr}^x_{x_0-} E = [-h_1, h_1]$, and $\mathrm{tr}^x_{x_0+} E = [-h_2, h_2]$.

When $\alpha \in \{0, 1\}$ and $v_2 = h_2 = 0$, the geometry of regular solutions can be described in a more precise way.

Theorem 1.2. *Let $\alpha \in \{0, 1\}$. Given $v_1, h_1 > 0$, let $E = E^l \cup E^c \cup E^r \in \mathcal{A}(v_1, 0, h_1, 0)$ be a regular solution for Problem* (1.7) *such that $\mathrm{tr}^-_{x_0} E = [-h_1, h_1]$. Then $E^c = \{(x, y) \in \mathbb{R}^2 : |y| < f(x),\ |x| < x_0\}$, where the function f is given by*

$$f(r) = \lambda^{\alpha+1} \varphi_\alpha\Big(\frac{r}{\lambda}\Big) + y, \tag{1.11}$$

for some $\lambda = \lambda(\alpha, v_1, h_1) > 0$ and $y = y(\alpha, v_1, h_1) \leq 0$ such that $y = 0$ if and only if $\alpha = 0$.

Due to the presence of the vertical translation y in (1.11), we deduce by Theorem 1.2 that a regular solution of the minimal partition problem is not obtained as the portion of an isoperimetric set $\delta_\lambda(E^\alpha_{\mathrm{isop}})$ lying in a vertical stripe, unless $\alpha = 0$. This result shows a delicate point where the techniques of [16] fail in the case of the Grushin geometry.

The paper is organized as follows. In Section 2, we prove existence of regular solutions of the minimal partition problem. The argument is divided into several steps. Lemma 2.1 is an *approximation theorem*, that generalizes the classical results in [12]. In Lemma 2.2 we show how to modify a set in the class $\mathcal{A}$ in order to decrease the perimeter $\mathscr{P}_\alpha$ and gain some *regularity properties*. Finally, in Theorem 2.4 we combine *lower semicontinuity* of the α-perimeter together with a *compactness theorem* for sets of finite α-perimeter to prove existence of minimizers.

In Section 3, we characterize regular solutions of the minimal partition problem. In Propostion 3.1 we find differential equations for the profile function f of a regular minimizer $E = \{(x, y) \in \mathbb{R}^2 : |y| < f(|x|)\}$. In Section 3.2 we prove, under a technical assumption, that regular solutions of the minimal partition problem (1.7) satisfy the trace equality (1.6). We conclude showing formula (1.11) in Proposition 3.5, which is proved under the assumptions $\alpha \in \{0, 1\}$, $v_1, h_1 > 0$ and $v_2 = h_2 = 0$. To this purpose, we use the differential equations of Proposition 3.1 to write the parameters λ and y in terms of α and of the given constraints.

Appendix A is dedicated to the notion of trace of a y-Schwarz symmetric set.

2. Existence of minimizers

In this section we prove existence of solutions to problem (1.7). The proof is divided into several steps that we present in Lemmas 2.1 and 2.2.

We first introduce some notation.

We say that a set $E \subset \mathbb{R}^2$ is *locally Lipschitz (resp. locally C^∞)* if its boundary ∂E is a locally Lipschitz (resp. locally C^∞) curve, i.e., for any $(x, y) \in \partial E$ there exists $r > 0$ such that $\partial E \cap B((x, y), r)$ is a Lipschitz (resp. C^∞) curve, where $B((x, y), r) = \{p \in \mathbb{R}^2 : |p-(x, y)| < r\}$.

For any set $E \subset \mathbb{R}^2$ and $t > 0$ we let

$$E^x_{t-} = \{(x, y) \in E : |x| < t\} \quad \text{and} \quad E^x_t = \{(x, y) \in E : |x| = t\}$$

In the following, we use the short notation $\{|x| < t\}$ for $\{(x, y) \in \mathbb{R}^2 : |x| < t\}$.

We call the *profile function* of a set $E \in \mathscr{S}_x \cap \mathscr{S}^*_y$, the measurable function $f : [0, \infty) \to [0, \infty)$ such that

$$E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\},$$

that exists by definition of x-symmetry and y-Schwarz symmetry.

In the next lemma we show that it is possible to approximate the sets in $\mathcal{A}$ by smooth sets in such a way that the symmetries are maintained and the trace constraint is preserved in the limit. This result is a refinement of the well known approximation theorem for sets with finite α-perimeter, see [12, Theorem 2.2.2].

Lemma 2.1 (Approximation by smooth sets). *Given $v_1, v_2, h_1, h_2 > 0$, let $E \in \mathcal{A}(v_1, v_2, h_1, h_2)$ be a set with finite α-perimeter. Let $x_0 > 0$ be the partitioning point for E and $y_0^\pm$ be such that $\mathrm{tr}^x_{x_0\pm} E = [-y_0^\pm, y_0^\pm]$. Then there exists a sequence of locally C^∞ sets $\mathcal{E}_j \in \mathscr{S}_x \cap \mathscr{S}^*_y$, $j \in \mathbb{N}$ such that*

- $\lim_{j\to\infty} P_\alpha((\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E^x_{x_0-})$ *and* $\lim_{j\to\infty} P_\alpha(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E \setminus E^x_{x_0-})$; (2.1a)
- $\lim_{j\to\infty} \mathcal{L}^2((\mathcal{E}_j)^x_{x_0-}) = \mathcal{L}^2(E^x_{x_0-})$ *and* $\lim_{j\to\infty} \mathcal{L}^2(\mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-}) = \mathcal{L}^2(E \setminus E^x_{x_0-})$; (2.1b)
- *if* $\mathrm{tr}^x_{x_0\pm}\mathcal{E}_j = [-q_j^\pm, q_j^\pm]$, *for some* $q_j^\pm \geq 0$, *we have* $q_j^\pm \to y_0^\pm$ *as* $j \to \infty$. (2.1c)

Proof. To construct the sequence $(\mathcal{E}_j)_{j\in\mathbb{N}}$ we introduce a positive symmetric mollifier $J \in C^\infty(\mathbb{R}^2)$, i.e., $J \in C^\infty_c(B(0, 1))$, with $B(0, 1) = \{p \in \mathbb{R}^2 : |p| < 1\}$, $J \geq 0$, $\int_{\mathbb{R}^2} J(p)\, dp = 1$, and $J(p) = J(q)$, for $p, q \in \mathbb{R}^2$, $|p| = |q|$. For any $\varepsilon > 0$, let $J_\varepsilon(p) = \frac{1}{\varepsilon^2} J(|p|/\varepsilon)$, $p \in \mathbb{R}^2$ and define the mollified function $h_\varepsilon = J_\varepsilon * \chi_E$. For any $t \in (0, 1)$, let $E_{\varepsilon t} = \{(x, y) \in \mathbb{R}^2 : h_\varepsilon(x, y) > t\}$. Consider a sequence $\varepsilon_j \to 0$ as $j \to \infty$. Following [17, Theorem 1.24], we can choose $t \in (0, 1)$ such that the set $\mathcal{E}_j = E_{\varepsilon_j t}$ is a locally smooth set satisfying (2.1b), and, in addition

$$\lim_{j\to\infty} P_\alpha(\mathcal{E}_j; (\mathcal{E}_j)^x_{x_0-}) = P_\alpha(E; E^x_{x_0-}), \ \lim_{j\to\infty} P_\alpha(\mathcal{E}_j; \mathcal{E}_j \setminus (\mathcal{E}_j)^x_{x_0-}\}) = P_\alpha(E; E \setminus E^x_{x_0-}). \tag{2.2}$$

Observe that the sets $\mathcal{E}_j$, $j \in \mathbb{N}$ are y-Schwarz symmetric, i.e., for any $(\bar{x}, \bar{y}) \in \mathcal{E}_j$, $(\bar{x}, y) \in \mathcal{E}_j$ if $|y| < \bar{y}$. In fact, since E is y-Schwarz symmetric, $\chi_E(\bar{x} - x', \bar{y} - y') \leq \chi_E(\bar{x} - x', y - y')$ for every $(x', y') \in \mathbb{R}^2$ and $|y| < |\bar{y}|$. Hence

$$\begin{aligned} t < & h_{\varepsilon_j}(\bar{x}, \bar{y}) = \int_{B_{\varepsilon_j}(0)} J_\varepsilon(x', y')\chi_E(\bar{x} - x', \bar{y} - y')\, dx'dy' \\ & \leq \int_{B_{\varepsilon_j}(0)} J_\varepsilon(x', y')\chi_E(\bar{x} - x', y - y')\, dx'dy' = h_{\varepsilon_j}(\bar{x}, y), \end{aligned}$$

which implies $(\bar{x}, y) \in \mathcal{E}_j$. Moreover, by symmetry of the mollifier J, for every $j \in \mathbb{N}$, $\mathcal{E}_j$ is also x-symmetric. Hence the left and right traces of $\mathcal{E}_j$ are well defined. Let ϕ_j denote the profile function of $\mathcal{E}_j$, i.e., $\mathcal{E}_j = \{(x, y) \in \mathbb{R}^2 : |y| < \phi_j(|x|)\}$, and define

$$q_j^- = \lim_{x \to x_0^-} \phi_j(x_0), \quad q_j^+ = \lim_{x \to x_0^+} \phi_j(x_0).$$

By Remark A.3, $\mathrm{tr}^x_{x_0 \pm} \mathcal{E}_j = [-q_j^\pm, q_j^\pm]$. We prove that $q_j^+ \to y_0^+$ as $j \to \infty$. The same argument applies to prove $q_j^- \to y_0^-$, $j \to \infty$, and (2.1c) follows. Let $0 < \sigma < y_0^+$, by (2.12) there exists $\delta = \delta(\sigma) > 0$ such that

$$|f(x) - y_0^+| < \sigma \quad \text{for} \quad x_0 < x < x_0 + \delta. \tag{2.3}$$

Choose $\bar{j}(\sigma) \in \mathbb{N}$ to have $\varepsilon_j < \min\{\sigma, \ \delta(\sigma)/4\}$ for $j \geq \bar{j}(\sigma)$. We first claim that for any $j \geq \bar{j}(\sigma)$, if $x \in (x_0 + \varepsilon_j, x_0 + \frac{\delta}{2})$ and $y \in (0, y_0^+ - 2\sigma)$, then

$$(x - \xi, y - \eta) \in E, \quad \text{for} \quad (\xi, \eta) \in B(0, \varepsilon_j). \tag{2.4}$$

In fact, the following estimates holds true for $j \geq \bar{j}(\sigma)$, $x \in (x_0 + \varepsilon_j, x_0 + \frac{\delta}{2})$, $-\varepsilon_j < \xi < \varepsilon_j$:

$$x_0 < x - \xi < x_0 + \frac{\delta}{2} - \xi < x_0 + \frac{\delta}{2} + \varepsilon_j < x_0 + \delta$$

hence, by (2.3), for $y \in (0, y_0^+ - 2\sigma)$ and $-\varepsilon_j < \eta < \varepsilon_j$,

$$y - \eta < y + \varepsilon_j < y_0^+ - 2\sigma + \sigma < f(x - \xi).$$

We now deduce from (2.4) that

$$A_\sigma = \big(x_0 + \varepsilon_j, x_0 + \frac{\delta}{2}\big) \times (0, y_0 - 2\sigma) \subset \mathcal{E}_j, \quad \text{for } j \geq j(\sigma). \tag{2.5}$$

This follows applying the definition of the set $\mathcal{E}_j$, since, for any $j \geq \bar{j}(\sigma)$, if $(x, y) \in A_\sigma$ we have

$$h_{\varepsilon_j}(x, y) = \int_{B(0,\varepsilon_j)} J_{\varepsilon_j}(\xi, \eta) \chi_E(x - \xi, y - \eta) \, d\xi d\eta = \int_{B(0,\varepsilon_j)} J_{\varepsilon_j}(\xi, \eta) \, d\xi d\eta = 1 > t.$$

In particular, (2.5) implies

$$(-y_0^+ + 2\sigma, y_0^+ - 2\sigma) \subset \mathrm{tr}^x_{(x_0+\varepsilon_j)+} \mathcal{E}_j \quad \text{for every} \quad j > \bar{j}(\sigma). \tag{2.6}$$

Similarly, we can choose $\bar{\bar{j}}(\sigma) \in \mathbb{N}$ such that

$$\mathrm{tr}_{(x_0+\varepsilon_j)+} \mathcal{E}_j \subset (-y_0^+ - 2\sigma, y_0 + 2\sigma) \quad \text{for} \quad j \geq \bar{\bar{j}}(\sigma). \tag{2.7}$$

We deduce (2.1c) from (2.6) and (2.7). Statement (2.1a) follows from (2.1c) and (2.2). □

Lemma 2.2 (Regularization). *Let $v_1, v_2, h_1, h_2 \geq 0$ and $E \in \mathcal{A}(v_1, v_2, h_1, h_2)$ be a locally C^∞-set with finite α-perimeter. Then, there exists a set $\tilde{E} \in \mathcal{A}(v_1, v_2, h_1, h_2)$ such that, if $\tilde{x}_0$ is the partitioning point for $\tilde{E}$, there holds:*

(1) $\tilde{E}^x_{\tilde{x}_0-}$ is convex and $\tilde{E} \setminus \tilde{E}^x_{\tilde{x}_0-}$ has locally Lipschitz boundary;
(2) $\mathscr{P}_\alpha(\tilde{E}) \leq \mathscr{P}_\alpha(E)$, in particular $P_\alpha(\tilde{E}^x_{\tilde{x}_0-}) \leq P_\alpha(E^x_{x_0-})$ and $P_\alpha(\tilde{E} \setminus \tilde{E}^x_{\tilde{x}_0-}) \leq P_\alpha(E \setminus E^x_{x_0-})$ where x_0 is the partitioning point for E.

Proof. Let $\mathrm{tr}^x_{x_0-} E = [-q^-, q^-]$ and $\mathrm{tr}^x_{x_0+} E = [-q^+, q^+]$. We divide the proof into the following steps, corresponding to operations performed on the set E.

Step 1. (Gluing around the y-axis). Starting from E, we construct a set $\hat{E} \in \mathscr{S}_x \cap \mathscr{S}^*_y$ such that there exist $0 < \hat{x}_0 \leq x_0$ satisfying:

(1) the Euclidean outer unit normal to $\hat{E}$ exists outside a set of $\mathcal{H}^1$-measure zero;

(2) if $\hat{\phi} : [0,\infty) \to [0,\infty)$ denotes the profile function of $\hat{E}$ and $D = \inf\{d \geq 0 : \phi(x) = 0 \text{ for } x \geq d\}$, then $\mathcal{H}^1(\{x \in [0,D] : \hat{\phi}(x) = 0\}) = 0$;
(3) $P_\alpha(\hat{E}^x_{\hat{x}_0-}) \leq P_\alpha(E^x_{x_0-})$ and $P_\alpha(\hat{E} \setminus \hat{E}^x_{\hat{x}_0-}) \leq P_\alpha(E \setminus E^x_{x_0-})$;
(4) $\mathcal{L}^2(\hat{E}^x_{\hat{x}_0-}) = \mathcal{L}^2(E^x_{x_0-})$ and $\mathcal{L}^2(\hat{E} \setminus \hat{E}^x_{\hat{x}_0-}) = \mathcal{L}^2(E \setminus E^x_{x_0-})$;
(5) $\mathrm{tr}^x_{\hat{x}_0-}\hat{E} = \mathrm{tr}^-_{x_0}E$ and $\mathrm{tr}^x_{\hat{x}_0+}\hat{E} = \mathrm{tr}^+_{x_0}E$.

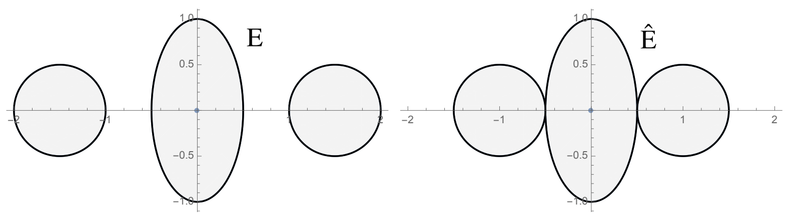

FIGURE 3. From E to $\hat{E}$.

Let $\phi : [0,\infty) \to [0,\infty)$ be the profile function of E. Define the set $Z := \{x \in \mathbb{R} : \phi(x) = 0\}$ and write $Z = Z^1 \cup Z^2$ with

$$Z^1 = \{x \in [0,D] : \phi(x) = 0 \text{ and } \phi(\xi) \neq 0 \text{ for } \xi \in (x-\delta, x+\delta) \setminus \{x\} \text{ for some } \delta > 0\},$$
$$Z^2 = \{x \in [0,D] : \exists \delta > 0 : \phi(\xi) = 0 \text{ for } \xi \in (x-\delta, x] \text{ or } \xi \in [x, x+\delta)\}.$$

By symmetry and smoothness of E, we have $Z^1 = \emptyset$. In fact, suppose by contradiction that there exists $x \in Z^1$, and let $p = (x,0) \in \partial E$. Since ∂E is smooth, there exists the outer unit normal ν at p. By y-symmetry of E, $\nu = (\pm 1, 0)$. Moreover, there exists a smooth function $\theta : B_r(p) \to \mathbb{R}$, defined on a Euclidean ball $B_r(p) = \{q \in \mathbb{R}^2 : |q-p| < r\}$ for some radius $r > 0$, such that

$$\begin{aligned} \theta(q) = 0 &\iff q \in \partial E \cap B_r(p), \\ \theta(q) > 0 &\iff q \in E \cap B_r(p), \\ \theta(q) < 0 &\iff q \in (\mathbb{R}^2 \setminus \overline{E}) \cap B_r(p). \end{aligned}$$

We deduce that $p + \tau\nu \notin E$ for $0 < \tau < \min\{r, \delta\}$, which contradicts $\phi(x \pm \tau) \neq 0$.

On the other hand, the set Z^2 is the complement in $\mathbb{R}$ of the set $\{x \in \mathbb{R} : (x,0) \in \overline{E}\}$, therefore it is open in the $\mathbb{R}$-topology. Hence, Z^2 is the union of at most countably many open intervals. We diversify the notation for the intervals in $Z^2 \cap \{x \in \mathbb{R} : |x| < x_0\}$ and in $Z^2 \cap \{x \in \mathbb{R} : |x| > x_0\}$: there exists a sequence of points $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots \leq x_0 < c_1 < d_1 < c_2 < d_2 < \cdots \leq D$, such that

$$Z^2 = \bigcup_{k \in \mathfrak{I}} (a_k, b_k) \cup \bigcup_{k \in \mathfrak{J}} (c_k, d_k) \cup \bigcup_{k \in \mathfrak{I}} (-b_k, -a_k) \cup \bigcup_{k \in \mathfrak{J}} (-d_k, -c_k),$$

where $\mathfrak{I}, \mathfrak{J} \subset \mathbb{N}$. We rearrange E in at most countably many steps, each one corresponding to an interval (a_k, b_k) for $k \in \mathfrak{I}$.

Base step. We define the set

$$\begin{aligned} E_1 = (E)^x_{a_1-} &\cup \{(x + a_1 - b_1, y) : (x,y) \in E,\ x > b_1\} \\ &\cup \{(x + b_1 - a_1, y) : (x,y) \in E,\ x < -b_1\} \end{aligned}$$

which is x-symmetric and y-Schwarz symmetric. Let $x_1 = x_0 + a_1 - b_1 < x_0$. Since $E \cap \big((a_1,b_1) \times \mathbb{R}\big) = \emptyset$, we have

$$\mathcal{L}^2((E_1)^x_{x_1-}) = \mathcal{L}^2(E^x_{x_0-}), \quad \mathcal{L}^2(E_1 \setminus (E_1)^x_{x_1-}) = \mathcal{L}^2(E \setminus E^x_{x_0-}).$$

Moreover $\mathrm{tr}^x_{x_1\pm}E_1 = \mathrm{tr}^x_{x_0\pm}E$. We prove that $P_\alpha((E_1)^x_{x_1-}) \le P_\alpha(E^x_{x_0-})$. By smoothness of ∂E_1 outside $\{(x,y)\in\mathbb{R}^2 : \ |x| = a_1\}$, we let $N_1(p) = (N_{1x}(p), N_{1y}(p))$ be the Euclidean outer unit normal to E_1 at $p = (x,y)\in\mathbb{R}^2$, for $|x|\ne a_1$. If $N = (N_x, N_y)$ is the Euclidean outer unit normal to ∂E, for $(x,y)\in\partial E$, we have

$$\begin{aligned}
N_1(x,y) &= N(x-a_1+b_1, y) \quad \text{if} \quad x > a_1;\\
N_1(x,y) &= N(x-b_1+a_1, y) \quad \text{if} \quad x < -a_1.
\end{aligned}$$

Hence, by the representation formula (1.2) we get

$$\begin{aligned}
&P_\alpha((E_1)^x_{x_1-}) \le P_\alpha((E_1)^x_{a_1-}) + P_\alpha((E_1)^x_{x_1-}\setminus(E_1)^x_{a_1-})\\
&= \int_{\partial(E_1)^x_{a_1-}} \big(N_{1x}(x,y)^2 + |x|^{2\alpha}N_{1y}(x,y)^2\big)^{\frac12}\, d\mathcal{H}^1(x,y)\\
&\quad + \int_{\partial\left((E_1)^x_{x_1-}\setminus(E_1)^x_{a_1-}\right)} \big(N_x(x-a_1+b_1,y)^2 + |x|^{2\alpha}N_y(x-a_1+b_1,y)^2\big)^{\frac12}\, d\mathcal{H}^1(x,y)\\
&= \int_{\partial(E)^x_{a_1-}} ({N_x}^2 + |x|^{2\alpha}{N_y}^2)^{\frac12}\, d\mathcal{H}^1 + \int_{\partial\left(E^x_{x_0-}\setminus(E)^x_{b_1-}\right)} (N_x^2 + |x-b_1+a_1|^{2\alpha}N_y^2)^{\frac12}\, d\mathcal{H}^1(x,y)\\
&\le \int_{\partial(E)^x_{a_1-}} ({N_x}^2 + |x|^{2\alpha}{N_y}^2)^{\frac12}\, d\mathcal{H}^1 + \int_{\partial\left(E^x_{x_0-}\setminus(E)^x_{b_1-}\right)} ({N_x}^2 + |x|^{2\alpha}{N_y}^2)^{\frac12}\, d\mathcal{H}^1 = P_\alpha(E^x_{x_0-}).
\end{aligned}$$

In the same way it follows

$$P_\alpha(E_1\setminus(E_1)^x_{x_1-}) \le P_\alpha(E\setminus E^x_{x_0-}).$$

Second step. Let E_2 be the x-symmetric and y-Schwarz symmetric set

$$\begin{aligned}
E_2 =& \{(x,y)\in E_1 : |x| \le a_2 - (b_1-a_1)\}\\
&\cup \Big\{\Big(x - \sum_{i=1}^2 (b_i-a_i), y\Big) : (x,y)\in E_1,\ x > b_2 - (b_1-a_1)\Big\}\\
&\cup \Big\{\Big(x + \sum_{i=1}^2 (b_i-a_i), y\Big) : (x,y)\in E_1,\ x < -b_2 + (b_1-a_1)\Big\},
\end{aligned}$$

and $x_2 = x_0 - \sum_{i=1}^2 (b_i-a_i)$. Then,

$$\mathcal{L}^2((E_2)^x_{x_2-}) = \mathcal{L}^2(E^x_{x_0-}) \quad \text{and} \quad \mathcal{L}^2(E_2\setminus(E_2)^x_{x_2-}) = \mathcal{L}^2(E\setminus E^x_{x_0-}),$$

and $\mathrm{tr}^x_{x_2\pm}E_2 = \mathrm{tr}^x_{x_1\pm}E_1 = \mathrm{tr}^x_{x_0\pm}E$. Moreover, as in the previous step, ∂E_2 is locally smooth outside the set $\{(x,y)\in\mathbb{R}^2 : \ |x| = a_1,\ |x| = a_2-(b_1-a_1)\}$, hence, $P_\alpha((E_2)^x_{x_2-}) \le P_\alpha(E^x_{x_0-})$ and $P_\alpha(E_2\setminus(E_2)^x_{x_2-}) \le P_\alpha(E\setminus E^x_{x_0-})$.

Inductive step. Let E_k be the x-symmetric and y-Schwarz symmetric set

$$\begin{aligned}
E_k =& \Big\{(x,y)\in E_{k-1} : |x| \le a_k - \sum_{i=1}^{k-1}(b_i-a_i)\Big\}\\
&\cup \Big\{\Big(x - \sum_{i=1}^k (b_i-a_i), y\Big) : (x,y)\in E_{k-1},\ x > b_k - \sum_{i=1}^{k-1}(b_i-a_i)\Big\}\\
&\cup \Big\{\Big(x + \sum_{i=1}^k (b_i-a_i), y\Big) : (x,y)\in E_{k-1},\ x < -b_k + \sum_{i=1}^{k-1}(b_i-a_i)\Big\},
\end{aligned}$$

and define

$$x_k = x_0 - \sum_{i=1}^{k} (b_i - a_i) < x_0.$$

Then

$$\mathcal{L}^2((E_k)^x_{x_k-}) = \mathcal{L}^2(E^x_{x_0-}) \quad \text{and} \quad \mathcal{L}^2(E_k \setminus (E_k)^x_{x_k-}) = \mathcal{L}^2(E \setminus E^x_{x_0-}),$$

$P_\alpha((E_k)^x_{x_k-}) \le P_\alpha(E^x_{x_0-})$, $P_\alpha(E_k \setminus (E_k)^x_{x_k-}) \le P_\alpha(E \setminus E^x_{x_0-})$, and ∂E_k is locally smooth outside $\{(x,y) \in \mathbb{R}^2 : |x| = a_1 - \sum_{i=2}^{\ell}(b_i - a_i) \text{ for } \ell = 2, \dots, k\}$. Iterating this procedure at most countably many times, we obtain a x-symmetric and y-Schwarz symmetric set $\hat{E}$ satisfying claims 3, 4 and 5 for

$$\hat{x}_0 = x_0 - \sum_{i \in \mathcal{I}} (b_i - a_i). \tag{2.8}$$

Repeating this argument for the intervals (c_k, d_k), $k \in \mathfrak{J}$, we obtain a set, which we still call $\hat{E}$, that satisfies also claims 1 and 2. In fact, let

$$\hat{Z} = \Big\{a_k - \sum_{i=1}^{k-1}(b_i - a_i) : k \in \mathfrak{I}\Big\} \cup \{c_k - \sum_{i=1}^{k-1}(d_i - c_i) : k \in \mathfrak{J}\}$$

which is at most countable, and denote by $\hat{\phi}$ be the profile function of $\hat{E}$. Then the outer unit normal to $\hat{E}$ exists outside the set $\{(x,y) \in \mathbb{R}^2 :\ |x| \in \hat{Z}\}$, and $\{x \in \mathbb{R} : \hat{\phi}(x) = 0\} \subset \hat{Z}$.

Step 2. (Reflection in the vertical direction) We rearrange the set $\hat{E}$ into a x-symmetric and y-Schwarz symmetric set $\hat{\hat{E}}$ with profile function $\hat{\hat{\phi}} : [0,\infty) \to [0,\infty)$ such that

(1) The Euclidean outer unit normal to $\hat{\hat{E}}$ exists outside a set of $\mathcal{H}^1$-measure zero;
(2) $\hat{\hat{\phi}}(|x|) \ge q^-$ for $x \in \mathbb{R}$, $|x| < \hat{x}_0$;
(3) $P_\alpha((\hat{\hat{E}})^x_{\hat{x}_0-}) \le P_\alpha(\hat{E}^x_{\hat{x}_0-})$ and $P_\alpha(\hat{\hat{E}} \setminus (\hat{\hat{E}})^x_{\hat{x}_0-}) = P_\alpha(\hat{E} \setminus (\hat{E})^x_{\hat{x}_0-})$;
(4) $\mathcal{L}^2((\hat{\hat{E}})^x_{\hat{x}_0-}) \ge \mathcal{L}^2(\hat{E}^x_{\hat{x}_0-})$ and $\mathcal{L}^2(\hat{\hat{E}} \setminus (\hat{\hat{E}})^x_{\hat{x}_0-}) = \mathcal{L}^2(\hat{E} \setminus (\hat{E})^x_{\hat{x}_0-})$;
(5) $\mathrm{tr}^x_{\hat{x}_0-}\hat{\hat{E}} = \mathrm{tr}^x_{\hat{x}_0-}\hat{E}$ and $\mathrm{tr}^x_{\hat{x}_0+}\hat{\hat{E}} = \mathrm{tr}^x_{\hat{x}_0+}\hat{E}$.

We define the rearranged function $\hat{\hat{\phi}} : [0,\infty) \to [0,\infty)$,

$$\hat{\hat{\phi}}(x) = \begin{cases} |\hat{\phi}(x) - q^-| + q^- = \begin{cases} \hat{\phi}(x) & \text{if } \hat{\phi}(x) \ge q^- \\ 2q^- - \hat{\phi}(x) & \text{if } \hat{\phi}(x) < q^- \end{cases} & \text{if } |x| < \hat{x}_0, \\ \hat{\phi}(x) & \text{if } |x| > \hat{x}_0. \end{cases}$$

Let $\hat{\hat{E}}$ be the x- and y-symmetric set generated by $\hat{\hat{\phi}}$ (see Figure 4). Clearly $\hat{\hat{E}} \setminus (\hat{\hat{E}})^x_{\hat{x}_0-} = \hat{E} \setminus (\hat{E})^x_{\hat{x}_0-}$ and claims 2 and 5 are satisfied. Claim 4 follows, observing that $\hat{\hat{\phi}} \in L^1(\mathbb{R})$ and $\hat{\hat{\phi}} \ge \hat{\phi}$, thus $\mathcal{L}^2(\hat{\hat{E}}) \ge \mathcal{L}^2(\hat{E})$.

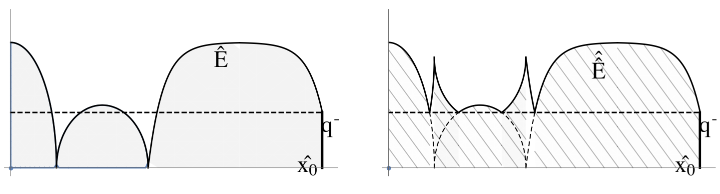

FIGURE 4. The set $\hat{E}$ and the rearranged $\hat{\hat{E}}$.

Denote by $\mathcal{K}$, the set of points $(x,y)\in\partial\hat{E}$ with $\hat{\phi}(x)=q^-$ and for which there exists $\delta>0$ such that $\hat{\phi}(\xi)\neq q^-$ for $\xi\in(x,x+\delta)$ or $\xi\in(x-\delta,x)$. Then, $\mathcal{H}^1(\mathcal{K})=0$ for any $j\in\mathbb{N}$ since $\mathcal{K}$ is countable.

By construction, the outer unit normal to $\hat{\hat{E}}$ exists outside $\mathcal{K}\cup\hat{Z}$, which has $\mathcal{H}^1$-measure zero, and we denote it by $\hat{\hat{N}}=(\hat{\hat{N}}_x,\hat{\hat{N}}_y)$. Moreover, if $\hat{N}=(\hat{N}_x,\hat{N}_y)$ is the outer unit normal to $\partial\hat{E}$, we have for any $(x,y)\in\partial\hat{E}\setminus\big(\mathcal{K}\cup\hat{Z}\big)$, $|x|<\hat{x}_0$,

$$\hat{\hat{N}}(x,|y-q^-|+q^-)=\big(\hat{N}_x(x,y),\operatorname{sgn}(y-q^-)\hat{N}_y(x,y)\big),$$

hence

$$\begin{aligned}
P_\alpha\big((\hat{\hat{E}})^x_{\hat{x}_0-}\big)&=\int_{\partial(\hat{\hat{E}})^x_{\hat{x}_0-}}\sqrt{\hat{\hat{N}}_x^2+|x|^{2\alpha}\hat{\hat{N}}_y^2}\,d\mathcal{H}^1\\
&=\int_{\{p\in\partial\hat{E}^x_{\hat{x}_0-}:\ \hat{N}_y(p)\neq0\}}\sqrt{\hat{N}_x^2+|x|^{2\alpha}\hat{N}_y^2}\,d\mathcal{H}^1+\mathcal{H}^1\big(\{p\in\partial(\hat{\hat{E}})^x_{\hat{x}_0-}:\ \hat{\hat{N}}_y(p)=0\}\big)\\
&\leq\int_{\{p\in\partial\hat{E}^x_{\hat{x}_0-}:\ \hat{N}_y(p)\neq0\}}\sqrt{\hat{N}_x^2+|x|^{2\alpha}\hat{N}_y^2}\,d\mathcal{H}^1+\mathcal{H}^1\big(\{p\in\partial\hat{E}^x_{\hat{x}_0-}:\ \hat{N}_y(p)=0\}\big)\\
&=P_\alpha(\hat{E}).
\end{aligned}$$

Step 3. (Convexification and regularization) We finally rearrange the set $\hat{\hat{E}}$ into a x-symmetric and y-Schwarz symmetric set $\tilde{E}$, such that there exists $0<\tilde{x}_0<x_0$ satisfying:

(1) $(\tilde{E})^x_{\tilde{x}_0-}$ is convex, and $\tilde{E}\setminus\tilde{E}^x_{\tilde{x}_0-}$ is locally lipschitz;
(2) $P_\alpha((\tilde{E})^x_{\tilde{x}_0-})\leq P_\alpha(E^x_{x_0-})$ and $P_\alpha(\tilde{E}\setminus(\tilde{E})^x_{\tilde{x}_0-})\leq P_\alpha(E\setminus E^x_{x_0-})$;
(3) $\mathcal{L}^2((\tilde{E})^x_{\tilde{x}_0-})=\mathcal{L}^2(E^x_{x_0-})$ and $\mathcal{L}^2(\tilde{E}\setminus(\tilde{E})^x_{\tilde{x}_0-})=\mathcal{L}^2(E\setminus E^x_{x_0-})$;
(4) $\mathrm{tr}^x_{\tilde{x}_0-}\tilde{E}\supset[-q^-,q^-]$ and $\mathrm{tr}^x_{\tilde{x}_0+}\tilde{E}\supset[-q^+,q^+]$.

This will conclude the proof.

We introduce the function

$$\Psi:\mathbb{R}^2\to\mathbb{R}^2,\ \Psi(x,y)=\Big(\operatorname{sgn}(x)\frac{|x|^{\alpha+1}}{\alpha+1},y\Big),$$

which is a homeomorphism with inverse

$$\Phi:\mathbb{R}^2\to\mathbb{R}^2,\ \Phi(\xi,\eta)=\Big(\operatorname{sgn}(\xi)|(\alpha+1)\xi|^{\frac{1}{\alpha+1}},\eta\Big).$$

As shown in [22, Proposition 2.3], for any measurable set $F\subset\mathbb{R}^2$, we have

$$P_\alpha(F)=P(\Psi(F))\quad\text{and}\quad\mathcal{L}^2(F)=\mu(\Psi(F)),$$

where P denotes the Euclidean perimeter and μ is a Borel measure on $\mathbb{R}^2$ defined on Borel sets as follows:

$$\mu(A)=\int_A|(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}\,d\xi d\eta,\quad A\subset\mathbb{R}^2\text{ Borel}.$$

Let $F^{\mathrm{c}}=\Psi((\hat{\hat{E}})^x_{\hat{x}_0-})\subset\mathbb{R}^2$ and consider its convex envelope in $\mathbb{R}^2$, $\mathrm{co}(F^{\mathrm{c}})$. First of all, notice that the maps Φ, Ψ preserve the symmetries, namely, since $\mathrm{co}(F^{\mathrm{c}})$ is x- and y-Schwarz symmetric, also F^{c} has such symmetries. We show that it is also a convex set. In fact, let

$t \in (0,1)$, for any $q_1, q_2 \in \Phi(\mathrm{co}(F^{\mathrm{c}}))$ there exist $p_1 = (\xi_1, \eta_1), p_2 = (\xi_2, \eta_2) \in \mathrm{co}(F^{\mathrm{c}})$. Since $\mathrm{co}(F^{\mathrm{c}})$ is convex, we have

$$\begin{aligned}&\Big(\mathrm{sgn}\big(t\xi_1 + (1-t)\xi_2\big)(\alpha+1)|t\xi_1 + (1-t)\xi_2|^{\frac{1}{\alpha+1}}, t\eta_1 + (1-t)\eta_2\Big)\\ &\qquad = \Phi(tp_1 + (1-t)p_2) \in \Phi(\mathrm{co}(F^{\mathrm{c}}))\end{aligned} \tag{2.9}$$

On the other hand, by the concavity inequality

$$|t\xi_1 + (1-t)\xi_2|^{\frac{1}{\alpha+1}} \geq t|\xi_1|^{\frac{1}{\alpha+1}} + (1-t)|\xi_2|^{\frac{1}{\alpha+1}}, \quad t \in (0,1), \quad \xi_1, \xi_2 \geq 0$$

and by x- and y-Schwarz symmetry of $\Phi(\mathrm{co}(F^{\mathrm{c}}))$, we get from (2.9)

$$tq_1 + (1-t)q_2 = \Big((\alpha+1)\big\{\mathrm{sgn}(t\xi_1)|t\xi_1|^{\frac{1}{\alpha+1}} + \mathrm{sgn}((1-t)\xi_2)|(1-t)\xi_2|^{\frac{1}{\alpha+1}}\big\}, t\eta_1 + (1-t)\eta_2\Big) \in \Phi(\mathrm{co}(F^{\mathrm{c}})),$$

which proves that F^{c} is convex. The set F^{c} satisfies

$$\mathcal{L}^2(F^{\mathrm{c}}) = \mu(\mathrm{co}(F^{\mathrm{c}})) \geq \mu(F^{\mathrm{c}}) = \mathcal{L}^2((\hat{\hat{E}})^x_{\hat{x}_0-}) \geq \mathcal{L}^2(\hat{E}^x_{\hat{x}_0-}) \geq \mathcal{L}^2(E^x_{x_0-}), \tag{2.10a}$$

$$P_\alpha(F^{\mathrm{c}}) = P(\mathrm{co}(F^{\mathrm{c}})) \leq P(F^{\mathrm{c}}) = P_\alpha((\hat{\hat{E}})^x_{\hat{x}_0-}), \tag{2.10b}$$

$$\mathrm{tr}^x_{\hat{x}_0-} F^{\mathrm{c}} = [-q^-, q^-]. \tag{2.10c}$$

By (2.10a), we define $\tilde{x}_0 \in [0, \hat{x}_0]$ such that $\mathcal{L}^2\big((F^{\mathrm{c}})^x_{\tilde{x}_0-}\big) = \mathcal{L}^2\big(E^x_{x_0-}\big)$. Notice that $P_\alpha((F^{\mathrm{c}})^x_{\tilde{x}_0-}) \leq P_\alpha(F^{\mathrm{c}})$: this follows using the same calibration argument as in [14, Proposition 4.2]. Moreover, we deduce

$$[-q^-, q^-] \subset \mathrm{tr}^x_{\tilde{x}_0-} F^{\mathrm{c}}$$

by x- and y-Schwarz symmetry of F^c, that implies decreasing monotonicity of its profile function.

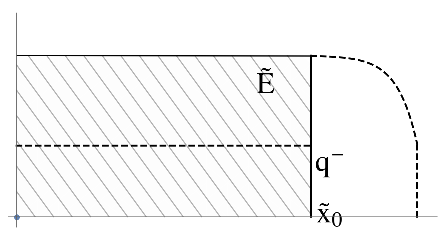

FIGURE 5. The convexified set F^c, cut at $\tilde{x}_0$.

Define the set

$$\begin{aligned}F = (F^{\mathrm{c}})^x_{\tilde{x}_0-} &\cup \{(x - \hat{x}_0 + \tilde{x}_0, y) \in \mathbb{R}^2 : (x,y) \in \hat{\hat{E}},\ x > \hat{x}_0\}\\ &\cup \{(x + \hat{x}_0 - \tilde{x}_0, y) \in \mathbb{R}^2 : (x,y) \in \hat{\hat{E}},\ x < -\hat{x}_0\}.\end{aligned}$$

Arguing as in Step 2, we have $P_\alpha(F \setminus (F)^x_{\tilde{x}_0-}) \leq P_\alpha(\hat{\hat{E}} \setminus (\hat{\hat{E}})^x_{\hat{x}_0-})$, $\mathcal{L}^2(F \setminus (F)^x_{\tilde{x}_0-}) = \mathcal{L}^2(\hat{\hat{E}} \setminus (\hat{\hat{E}})^x_{\hat{x}_0-})$ and $\mathrm{tr}^x_{\tilde{x}_0+} F = \mathrm{tr}^x_{\hat{x}_0+} \hat{\hat{E}}$.

Now, the same argument used to prove (2.10a)-(2.10c), shows that the sets $F^r = \Phi(\mathrm{co}(\Psi(F \cap \{x > \tilde{x}_0\})))$ and $F^l = \{(-x, y) : (x,y) \in F^r\}$, obtained combining the changes of variables Φ and Ψ with a convexification in the plane $\mathbb{R}^2_{(\xi,\eta)}$, satisfy

$$\mathcal{L}^2(F^r \cup F^l) \geq \mathcal{L}^2(F \setminus (F)^x_{\tilde{x}_0-}), \quad \mathrm{tr}_{\tilde{x}_0+} F^r = \mathrm{tr}_{\tilde{x}_0+} F, \quad P_\alpha(F^l \cup F^r) \leq P_\alpha(F \setminus (F)^x_{\tilde{x}_0-}).$$

Moreover, since $\operatorname{co}(\Psi(F\cap\{x>\tilde{x}_0\}))$ is a convex set and Φ is C^∞-smooth outside 0, the set F^r is locally Lipschitz. Let $r\geq \tilde{x}_0$ be such that $\mathcal{L}^2((F^r\cup F^l)^x_{r-})=\mathcal{L}^2(E\setminus E^x_{x_0-})$. The set

$$\tilde{E}=(F^c)^x_{\tilde{x}_0-}\cup(F^r\cup F^l)^x_{r-} \tag{2.11}$$

satisfies all the claims of this Step. □

Remark 2.3. Given a set $F\in\mathcal{A}(v_1,v_2,h_1,h_2)$ with partitioning point $x_0>0$ it is always possible to construct a set $\tilde{E}\in\mathcal{A}(v_1,v_2,h_1,h_2)$ such that $\mathscr{P}_\alpha(\tilde{E})\leq\mathscr{P}_\alpha(F)$, $\tilde{E}^x_{x_0-}=F^x_{x_0-}$, and $\tilde{E}\setminus\tilde{E}^x_{x_0-}$ is locally Lipschitz.

This follows combining the changes of variables $\Psi:\mathbb{R}^2_{(x,y)}\to\mathbb{R}^2_{(\xi,\eta)}$ and $\Phi:\mathbb{R}^2_{(\xi,\eta)}\to\mathbb{R}^2_{(x,y)}$ with a convexification in the plane $\mathbb{R}^2_{(\xi,\eta)}$ as we did to construct the set $\tilde{E}$ defined in (2.11).

We prove that Problem (1.7) admits solutions satisfying suitable regularity properties. Given $v_1,v_2,h_1,h_2\geq 0$, we define the constant

$$C_{MP}=\inf\{\mathscr{P}_\alpha(E):E\in\mathcal{A}(v_1,v_2,h_1,h_2)\}.$$

Notice that, for any $E\in\mathcal{A}$, $\mathscr{P}_\alpha(E)\geq P(E)$. This follows from the formulas

$$P_\alpha(E^x_{t-})=P_\alpha(E;E^x_{t-})+\mathcal{H}^1(E^x_t)\quad\text{and}\quad P_\alpha(E\setminus E^x_{t-})=P_\alpha(E;E\setminus E^x_{t-})+\mathcal{H}^1(E^x_t)$$

holding for a.e. $t>0$ and any set E with finite measure and finite α-perimeter, see [14, Proposition 4.1]. Moreover, if $h_1=h_2=h$, for any set $E\in\mathcal{A}(v_1,v_2,h,h)$ such that $\operatorname{tr}^-_{x_0}E=\operatorname{tr}^+_{x_0}E=[-h,h]$, the functional $\mathscr{P}_\alpha$ corresponds to the α-perimeter:

$$\mathscr{P}_\alpha(E)=P_\alpha(E;E^x_{x_0-})+4h+P_\alpha(E;E\setminus E^x_{x_0-})+4h-8h=P_\alpha(E).$$

We hence deduce that the constant C_{MP} is positive thanks to the validity of the following isoperimetric inequality: for any $\mathcal{L}^2$-measurable set $E\subset\mathbb{R}^2$ with finite measure

$$P_\alpha(E)\geq C\mathcal{L}^2(E)^{\frac{\alpha+1}{\alpha+2}}$$

for some geometric constant $C>0$, see [9], [10], [18] (see also [13, Proposition 1.3.4]).

Theorem 2.4. *Let $v_1,v_2,h_1,h_2\geq 0$. There exists a bounded set $E\in\mathcal{A}$ with partitioning point $x_0\geq 0$ realizing the infimum in* (1.7) *and such that $E^x_{x_0-}$ is convex, and $E\setminus E^x_{x_0-}$ is locally Lipschitz.*

Proof. Let $(E_m)_{m\in\mathbb{N}}$ be a minimizing sequence for the infimum in Problem (1.7), namely

$$E_m\in\mathcal{A}\quad\mathscr{P}_\alpha(E_m)\leq C_{MP}\Big(1+\frac{1}{m}\Big)\quad m\in\mathbb{N}.$$

Let $x_m>0$ be the partitioning point for E_m and $f_m:[0,\infty)\to[0,\infty)$ be its profile function. Moreover, let $y^-_m,y^+_m\geq 0$ be such that $\operatorname{tr}^x_{x_m\pm}E_m=[-y^\pm_m,y^\pm_m]$. By Remark A.3,

$$\lim_{x\to x_m^+}f_m(x)=y^+_m,\qquad\lim_{x\to x_m^-}f_m(x)=y^-_m,\quad\text{with}\quad y^+_m\geq h_2,\ y^-_m\geq h_1. \tag{2.12}$$

Let $m\in\mathbb{N}$. By Lemma 2.1, let $\mathcal{E}^m_j\in\mathscr{S}_x\cap\mathscr{S}^*_y$, $j\in\mathbb{N}$ be a sequence of smooth sets approximating E_m, i.e., satisfying (2.1a)-(2.1c). Define $q^\pm_{jm}\geq 0$ such that $\operatorname{tr}^x_{x_m\pm}\mathcal{E}^m_j=[-q^\pm_{jm},q^\pm_{jm}]$. For any $j\in\mathbb{N}$, apply Lemma 2.2 to $\mathcal{E}^m_j$, obtaining a x-symmetric and y-Schwarz symmetric set $\tilde{\mathcal{E}}^m_j$, such that there exists $0<\tilde{x}^m_j<x_m$ satisfying:

(1) $(\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-}$ is convex, and $\tilde{\mathcal{E}}^m_j\setminus(\tilde{\mathcal{E}^m_j})^x_{\tilde{x}^m_j-}$ is locally lipschitz;
(2) $P_\alpha((\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-})\leq P_\alpha((\mathcal{E}^m_j)^x_{x_m-})$ and $P_\alpha(\tilde{\mathcal{E}}^m_j\setminus(\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-})\leq P_\alpha(\mathcal{E}_j\setminus(\mathcal{E}^m_j)^x_{x_m-})$;
(3) $\mathcal{L}^2((\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-})=\mathcal{L}^2((\mathcal{E}_j)^x_{x_0-})$ and $\mathcal{L}^2(\tilde{\mathcal{E}}_j\setminus(\tilde{\mathcal{E}}_j)^x_{\tilde{x}_j-})=\mathcal{L}^2(\mathcal{E}_j\setminus(\mathcal{E}_j)^x_{x_0-})$;

(4) $\mathrm{tr}^x_{\tilde{x}^m_j-}\tilde{\mathcal{E}}^m_j \supset [-q^-_j, q^-_j]$ and $\mathrm{tr}^x_{\tilde{x}_j+}\tilde{\mathcal{E}}_j \supset [-q^+_j, q^+_j]$.

By (2.1c), for any $m \in \mathbb{N}$ there exists $J(m) \in \mathbb{N}$ such that for $j \geq J(m)$, we have

$$|q^{\pm}_{jm} - y^{\pm}_m| \leq \frac{1}{m} \tag{2.13}$$

and

$$\begin{aligned} &\Big|P_\alpha\big((\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-}\big) - P_\alpha((E_m)^x_{x_m-}))\Big| \leq \frac{1}{m}, \quad \Big|P_\alpha\big(\tilde{\mathcal{E}}^m_j \setminus (\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-}\big) - P_\alpha\big(E_m \setminus (E_m)^x_{x_m-}\big)\Big| \leq \frac{1}{m}, \\ &\Big|\mathcal{L}^2\big((\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-}\big) - \mathcal{L}^2((E_m)^x_{x_m-}))\Big| \leq \frac{1}{m}, \quad \Big|\mathcal{L}^2\big(\tilde{\mathcal{E}}^m_j \setminus (\tilde{\mathcal{E}}^m_j)^x_{\tilde{x}^m_j-}\big) - \mathcal{L}^2(E_m \setminus (E_m)^x_{x_m-}))\Big| \leq \frac{1}{m}. \end{aligned} \tag{2.14}$$

Let $(j_m)_{m\in\mathbb{N}}$ be an increasing sequence of integer numbers such that $j_m \geq J(m)$ for any $m \in \mathbb{N}$. We choose the diagonal sequence $\tilde{E}_m = \tilde{\mathcal{E}}^m_{j_m}$, $m \in \mathbb{N}$ and prove that there exists $\ell > 0$ such that

$$\tilde{E}_m \subset [-\ell, \ell] \times [-\ell, \ell], \quad \text{for any} \quad m \in \mathbb{N}. \tag{2.15}$$

First of all, letting $\tilde{x}_m = \tilde{x}^m_{j_m}$, notice that

$$\sup\{P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) : m \in \mathbb{N}\} < \infty, \quad \text{and} \quad \sup\{P_\alpha(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-}) : m \in \mathbb{N}\} < \infty. \tag{2.16}$$

In fact:

$$\begin{aligned} &\max\{P_\alpha\big((\tilde{E}_m)^x_{\tilde{x}_m-}\big), P_\alpha\big(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-}\big)\} \leq P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) + P_\alpha(\tilde{E}_m \setminus (\tilde{E}_m)^x_{\tilde{x}_m-}) \\ &\quad \leq \mathscr{P}_\alpha(E_m) + 4h_1 + 4h_2 + \frac{2}{m} \leq 2C_{MP} + 4h_1 + 4h_2 + 2. \end{aligned}$$

We prove that the sequence $\tilde{x}_m$ is bounded. Let $\tilde{\phi}_m$ be the profile function of $\tilde{E}_m$ and assume by contradiction that $x_m \to \infty$ as $m \to \infty$. In this case, by the representation formula (1.2) we have:

$$P_\alpha((\tilde{E}_m)^x_{\tilde{x}_m-}) = \int_0^{\tilde{x}_m} \sqrt{\tilde{\phi}_m(x)^2 + |x|^{2\alpha}}\, dx \geq \int_0^{\tilde{x}_m} |x|^\alpha = \frac{\tilde{x}_m^{\alpha+1}}{\alpha+1} \to \infty, \quad m \to \infty$$

which is in contradiction with (2.16). In the same way we can see that, if r_m is such that $\tilde{E}_m \subset (\tilde{E}_m)^x_{r_m-}$, the sequence $(r_m)_{m\in\mathbb{N}}$ is bounded.

Now, we show boundedness in the vertical direction, namely we show that there exists $L \geq 0$ such that $\tilde{E}_m \subset (\tilde{E}_m)^y_{L-}$. Suppose by contradiction that for any $L \geq 0$, there exists $m = m(L) \in \mathbb{N}$ such that $(\tilde{E}_m)^x_{\tilde{x}_m-} \setminus (\tilde{E}_m)^y_{L-} \neq \emptyset$, then by convexity of $(\tilde{E}_m)^x_{\tilde{x}_m-}$, we can equivalently assume that for any $L \geq 0$ there exists $j(L) \geq 0$ such that

$$\tilde{\phi}_m(0) > L \text{ for } m \geq m(L), \tag{2.17}$$

We write for $x \in (0, \tilde{x}_m)$

$$\tilde{\phi}_m(x) = -\int_x^{\tilde{x}_m} \tilde{\phi}'_m(\xi)\, d\xi = \int_x^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi,$$

then

$$\tilde{\phi}_m(0) = \lim_{x\to 0} \int_x^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi = \int_0^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi,$$

which implies, by (2.17)

$$\lim_{m\to\infty} \int_0^{\tilde{x}_m} |\tilde{\phi}'_m(\xi)|\, d\xi = \lim_{m\to\infty} \tilde{\phi}_m(0) = \infty.$$

Therefore

$$P_\alpha((\tilde E_m)^x_{\tilde x_m-}) = 4\int_0^{\tilde x_m}\sqrt{(\tilde\phi_m'(x))^2+x^{2\alpha}}\,dx \geq \int_0^{\tilde x_m}|\tilde\phi_m'(x)|\,dx \to \infty \text{ as } m\to\infty,$$

which is in contradiction with (2.16). Similarly, we exclude the case that for any $L>0$ $\big(\tilde E_m \setminus (\tilde E_m)^x_{\tilde x_m-}\big)\setminus(\tilde E_m)^y_{L-}\neq\emptyset$.

We conclude the proof showing existence of a solution to Problem (1.7). Thanks to (2.16), by the compactness theorem for BV_α functions (see [18, Theorem 1.28]), there exists a set E_∞ which is the L^1_{loc}-limit of $\tilde E_m$ as $m\to\infty$. By (2.15), convergence $\chi_{\tilde E_m}\to\chi_{E_\infty}$ is in $L^1(\mathbb{R}^2)$. Moreover, since the sequence $(\tilde x_m)_{m\in\mathbb{N}}$ is bounded, we let $x_\infty\geq 0$ be the limit up to subsequences of $\tilde x_m$ as $m\to\infty$. We have, by (2.14),

$$\mathcal{L}^2((E_\infty)^x_{\tilde x_\infty-}) = \lim_{m\to\infty}\mathcal{L}^2((\tilde E_m)^x_{\tilde x_m-}) = \lim_{m\to\infty}\mathcal{L}^2((E_m)^x_{x_m-}) = v_1,$$

and

$$\mathcal{L}^2(E_\infty\setminus(E_\infty)^x_{\tilde x_\infty-}) = \lim_{m\to\infty}\mathcal{L}^2(\tilde E_m\setminus(\tilde E_m)^x_{\tilde x_m-}) = \lim_{m\to\infty}\mathcal{L}^2(E_m\setminus(E_m)^x_{x_m-}) = v_2.$$

Now, since $(\tilde E_m)^x_{\tilde x_m-}$, is convex, we can choose a representative for E_∞ such that $(E_\infty)^x_{x_\infty-}$, is convex. By boundedness of the sequence $\tilde E_m$, let $y^\pm_\infty\geq 0$ be such that $\mathrm{tr}^x_{x_\infty\pm}E_\infty = [-y^\pm_\infty, y^\pm_\infty]$. Then, by (2.13) and claim 4 at Step 4, we have

$$y^-_\infty \geq \lim_{m\to\infty}\tilde q^-_m \geq \lim_{m\to\infty} q^-_m \geq \lim_{m\to\infty} y^-_m - \frac{1}{m}\geq h_1,$$

equivalently $y^+_\infty\geq h_2$. Hence $E_\infty\in\mathcal{A}$.

By the lower semi-continuity of the α-perimeter together with (2.14), we have

$$\begin{aligned}
\mathscr{P}_\alpha(E_\infty) &= P_\alpha((E_\infty)^x_{x_\infty-}) + P_\alpha(E_\infty\setminus(E_\infty)^x_{x_\infty-}) - 4h_1 - 4h_2\\
&\leq \liminf_{m\to\infty} P_\alpha((\tilde E_m)^x_{\tilde x_m-}) + \liminf_{m\to\infty} P_\alpha(\tilde E_m\setminus(\tilde E_m)^x_{\tilde x_m-}) - 4h_1 - 4h_2\\
&\leq \liminf_{m\to\infty} P_\alpha((E_m)^x_{x_m-}) + \liminf_{m\to\infty} P_\alpha(E_m\setminus(E_m)^x_{x_m-}) + \frac{2}{m} - 4h_1 - 4h_2\\
&= \liminf_{m\to\infty}\mathscr{F}_\alpha(E_m) + \frac{2}{m}\leq C_{MP}.
\end{aligned}\tag{2.18}$$

By Remark 2.3 applied to $F=E_\infty$, there exists a set $\tilde E_\infty\in\mathcal{A}(v_1,v_2,h_1,h_2)$ such that $\mathscr{P}_\alpha(\tilde E_\infty)\leq\mathscr{P}_\alpha(E_\infty)$, $\tilde E_\infty\setminus(\tilde E_\infty)^x_{x_\infty-}$ is locally Lipschitz and $(\tilde E_\infty)^x_{x_\infty-} = (E_\infty)^x_{x_\infty-}$. In conclusion the set $\tilde E_\infty\subset\mathbb{R}^2$ is such that

$$\mathscr{P}_\alpha(\tilde E_\infty) = \inf\{\mathscr{P}_\alpha(E), E\in\mathcal{A}\}$$

with $\tilde E_\infty\in\mathcal{A}$. It is therefore a bounded minimizer for (1.7) such that $(\tilde E_\infty)^x_{x_\infty-}$ is convex and $\tilde E_\infty\setminus(\tilde E_\infty)^x_{x_\infty-}$ is locally Lipschitz. □

3. Profile of regular solutions

In this section we describe the solutions to the minimal partition problem (1.7) found in Theorem 2.4. If $E\in\mathcal{A}(v_1,v_2,h_1,h_2)$ with partitioning point $x_0>0$ is a bounded solution to Problem (1.7) such that $E^x_{x_0-}$ is convex and $E\setminus E^x_{x_0-}$ is locally Lipschitz, we say that E is a *regular solution of the minimal partition problem.* In this case, writing $E=\{(x,y)\in\mathbb{R}^2 : |y|<f(|x|)\}$, the profile function $f:[0,\infty)\to[0,\infty)$ is *decreasing* in $[0,x_0)$ and *locally Lipschitz* in $[x_0,\infty)$. With a little abuse of notation, in the following we sometimes consider f to be extended to an even function defined on the whole $\mathbb{R}$, and we still call it the profile function of E.

3.1. **Differential equations for the profile function.** In the following proposition we deduce differential equations for regular solutions of the minimal partition problem.

Proposition 3.1. *Let $v_1, v_2, h_1, h_2 \geq 0$ and $E \subset \mathbb{R}^2$ be a regular solution of the minimal partition problem (1.7) in the class $\mathcal{A} = \mathcal{A}(v_1, v_2, h_1, h_2)$, with partitioning point $x_0 \geq 0$. Then, writing*

$$E = \{(x, y) \in \mathbb{R}^2 : |y| < f(x)\},$$

the even function f satisfies

$$f'(x) = -\frac{\operatorname{sgn} x \; c|x|^{\alpha+1}}{\sqrt{1 - cx^2}} \quad \text{if} \quad |x| < x_0, \tag{3.1a}$$

$$f'(x) = \frac{(kx + d)\, x^\alpha}{\sqrt{1 - (kx + d)^2}} \quad \text{if} \quad x > x_0. \tag{3.1b}$$

$$f'(x) = \frac{(kx - d)\, |x|^\alpha}{\sqrt{1 - (kx - d)^2}} \quad \text{if} \quad x < -x_0 \tag{3.1c}$$

for some constants $c \geq 0$, k, $d \in \mathbb{R}$.

Proof. By boundedness of the regular minimizer E, let $r_0 = \inf\{r > 0 : E \subset E^x_{r-}\} < \infty$.

We first prove equation (3.1a). For $\psi_1 \in C^\infty_c(0, x_0)$ with $\int \psi_1 = 0$, and $\varepsilon \in \mathbb{R}$, consider the function $x \mapsto f(|x|) + \varepsilon\psi_1(|x|)$, $x \in \mathbb{R}$, and define the set

$$E_\varepsilon = \{(x, y) \in \mathbb{R}^2 : |y| < f(|x|) + \varepsilon\psi_1(|x|)\} \in \mathcal{A}.$$

By the Representation formula for the α-perimeter (1.2), let

$$p_1(\varepsilon) = P_\alpha((E_\varepsilon)^x_{x_0-}) = 4\Big\{ \int_0^{x_0} \sqrt{(f' + \varepsilon\psi')^2 + |x|^{2\alpha}}\, dr + \lim_{x \to x_0^-} f(x)\Big\}.$$

By minimality of E, we then have

$$\begin{aligned} 0 = p_1'(\varepsilon)\big|_{\varepsilon=0} &= 4 \int_0^{x_0} \frac{d}{d\varepsilon}\big(\sqrt{(f' + \varepsilon\psi_1')^2 + x^{2\alpha}}\big)\Big|_{\varepsilon=0} dx \\ &= 4 \int_0^{x_0} \frac{f'(x)\psi_1'(x)}{\sqrt{f'^2(x) + x^{2\alpha}}}\, dx = -4 \int_0^{x_0} \frac{d}{dx}\left(\frac{f'}{\sqrt{f'^2 + x^{2\alpha}}}\right) \psi_1(x)\, dx. \end{aligned}$$

By arbitrariness of ψ_1, we deduce the following second order ordinary differential equation, holding for some $C \in \mathbb{R}$

$$\frac{d}{dx}\left(\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}}\right) = C \quad \text{for a.e. } 0 < x < x_0. \tag{3.2}$$

The normal form of (3.2) is

$$f''(x) = \frac{\alpha f'(x)}{x} + \frac{C}{x^{2\alpha}}(f'(x)^2 + x^{2\alpha})^{\frac{3}{2}}. \tag{3.3}$$

Hence, since f is even, f' is odd and f'' is even, f satisfies (3.3) for any $x \in \mathbb{R}$, $|x| < x_0$ and we extend (3.2) to $|x| < x_0$. Integrating (3.2) around 0, we obtain existence of a constant $d \in \mathbb{R}$ such that for some $\delta > 0$,

$$\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = Cx + d \quad \text{for } |x| < \delta.$$

Since f' is odd we deduce that $d = 0$, in fact for $|x| < \delta$

$$Cx + d = \frac{f'(x)}{\sqrt{f'^2(x) - x^{2\alpha}}} = -\frac{f'(-x)}{\sqrt{f'^2(-x) + (-x)^2\alpha}} = -(C(-x) + d) = Cx - d.$$

Hence (3.2) reads

$$\frac{f'(x)}{\sqrt{f'^2(x) + x^{2\alpha}}} = Cx \text{ for } |x| < \delta,$$

which implies, by monotonicity of f, that $C < 0$. Letting $c = -C > 0$, we hence get the following ordinary differential equation for f:

$$f'(x) = -\mathrm{sgn}(x)\frac{c|x|^{\alpha+1}}{\sqrt{1 - c^2x^2}} \quad \text{for } |x| < \delta.$$

A solution to the latter equation can be extended up to $(-1/c, 1/c)$. This implies $0 < x_0 \leq 1/c$ and (3.1a) is proved.

To prove (3.1b) and (3.1c), we proceed in the same way, considering a function $\psi_2 \in C_c^\infty(x_0, r_0)$, with $\int \psi_2 = 0$ and the associated perturbation $f + \eta\psi_2$ for $\eta \in \mathbb{R}$. The set $E_\eta = \{(x, y) \in \mathbb{R}^2 : |y| < f(|x|) + \varepsilon\psi_2(|x|)\}$ is inside the class $\mathcal{A}$, hence, as in the previous case, minimality of E leads to

$$\frac{d}{d\eta}\mathscr{P}_\alpha(E_\eta)\Big|_{\eta=0} = \frac{d}{d\eta}P_\alpha(E_\eta \setminus (E_\eta)^x_{x_0-})\Big|_{\eta=0} = 0$$

and we obtain existence of a constant $k \in \mathbb{R}$ such that

$$\frac{d}{dx}\left(\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}}\right) = k \quad \text{for} \quad x_0 < |x| < r_0. \tag{3.4}$$

Let $x_0 < x < r_0$. An integration between x_0 and x shows that, letting

$$d = \lim_{x \to x_0^+} \frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} - kx_0,$$

we have

$$\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = kx + d \quad \text{for} \quad x_0 < x < r_0$$

which is equivalent to (3.1b). In particular, $|kx + d| < 1$ for $x_0 < x < r_0$.

Analogously, for any $x \in (-r_0, -x_0)$, an integration between x and $-x_0$ shows that

$$\frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}} = kx - d \quad \text{for} \quad -r_0 < x < -x_0,$$

which leads to (3.1c) and $|kx - d| < 1$ for $-r_0 < x < -x_0$. □

Remark 3.2. Let E be a regular solution of the minimal partition problem with partitioning point $x_0 > 0$. Then, its profile function f is defined on some bounded interval $[0, r_0]$ and it is a locally Lipschitz function satisfying in a weak sense the ordinary differential equations (3.1a)-(3.1c). By an elementary argument, that is omitted, it follows that $f \in C^2([0, x_0)) \cap C^2(x_0, r_0) \cap C^2(-r_0, -x_0)$.

Notice that equation (3.1a) is scale invariant, i.e., given $c_1, c_2 \geq 0$ and a solution g to (3.1a) for $c = c_1$, the function

$$g_\lambda(x) = \lambda^{\alpha+1}g\left(\frac{x}{\lambda}\right), \quad \lambda = \frac{c_1}{c_2}$$

is a solution to (3.1a) for $c = c_2$. In this sense, in [22, Theorem 3.2], the authors show that the unique solution to equation (3.1a) is the function

$$\varphi_\alpha(x) = \int_{\arcsin|x|}^{\frac{\pi}{2}} \sin^{\alpha+1}(t)\, dt, \quad x \in [-1,1], \tag{3.5}$$

obtained integrating (3.1a) for $c = 1$. In other words, there exist $\lambda > 0$ and $y \in \mathbb{R}$ such that

$$f(x) = \lambda^{\alpha+1}\varphi_\alpha\big(\frac{x}{\lambda}\big) + y, \quad |x| < x_0. \tag{3.6}$$

In particular

$$\lim_{x\to 0^+} \frac{f'(x)}{|x|^{\alpha+1}} = 0.$$

Moreover, by (3.1b),

$$f(r_0) = 0, \quad \lim_{x\to r_0^+} f'(x) = -\infty,$$

and r_0 is characterized by the following equality

$$-1 = \lim_{x\to r_0^-} \frac{f'(x)}{\sqrt{f'(x)^2 + x^{2\alpha}}}. = kr_0 + d,$$

Namely,

$$r_0 = -\frac{1+d}{k}. \tag{3.7}$$

3.2. **Traces of regular solutions.** In this section, we study traces of minimizers. What we expect is that if $E \in \mathcal{A}(v_1, v_2, h_1, h_2)$ is a regular solution of the minimal partition problem with partitioning point x_0, then

$$\mathrm{tr}^-_{x_0} E = [-h_1, h_1] \quad \text{and} \quad \mathrm{tr}^+_{x_0} E = [-h_2, h_2].$$

In Proposition 3.3 we prove the claim for the left trace, under the additional assumption that the profile function of E does not have infinite derivative at x_0. The case of the right trace is equivalent.

Proposition 3.3. *Given $v_1, v_2, h_1, h_2 \geq 0$, let $E \in \mathcal{A}(v_1, v_2, h_1, h_2)$ be a regular solution of Problem* (1.7) *with partitioning point $x_0 > 0$, and let $f : [0,\infty) \to [0,\infty)$ be its profile function. If*

$$\lim_{x\to x_0^-} f'(x) > -\infty,$$

then

$$\mathrm{tr}^-_{x_0} E = [-h_1, h_1].$$

Proof. Assume by contradiction that $f(x_0^-) > h_1$, where

$$f(x_0^-) = \lim_{x\to x_0^-} f(x).$$

We show that in this case, there exists a set $F \in \mathcal{A}$ such that $P_\alpha(F^x_{x_0-}) < P_\alpha(E^x_{x_0-})$, $P_\alpha(F \setminus F^x_{x_0-}) = P_\alpha(E \setminus E^x_{x_0-})$, hence $\mathscr{P}_\alpha(F) < \mathscr{P}_\alpha(E)$, which is in contradiction with the minimality of E.

For a small parameter $\varepsilon > 0$, let $f_\varepsilon : [0, x_0) \to [0,\infty)$ be the function defined by

$$f_\varepsilon(x) = \begin{cases} f(x), & \text{if } 0 < x < x_0 - \varepsilon \\ r_\varepsilon(x), & \text{if } x_0 - \varepsilon < x < x_0 \end{cases}$$

where r_ε is the segment connecting the points $\big(x_0-\varepsilon, f(x_0-\varepsilon)\big)$ and $(x_0, f(x_0^-)-\varepsilon)$, i.e.,

$$r_\varepsilon(x) = m(\varepsilon)(x-x_0) + f(x_0^-) - \varepsilon, \quad m(\varepsilon) = \frac{1}{\varepsilon}(f(x_0^-) - \varepsilon - f(x_0-\varepsilon)) < 0.$$

By convexity of $E^x_{x_0-}$, $f(x) \geq r_\varepsilon(x)$ for $x_0-\varepsilon < x < x_0$. We define the set $E_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |y| < f_\varepsilon(|x|)\}$.

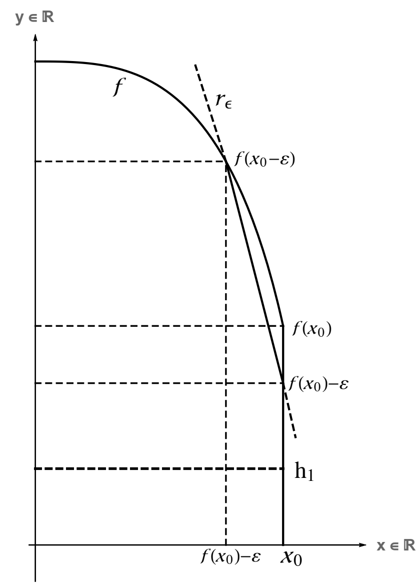

FIGURE 6. Construction of the set E_ε.

We compute the difference $P_\alpha(E) - P_\alpha(E_\varepsilon)$, using the Representation formula for the α-perimeter (1.2). Since $\partial E^x_{x_0} = \partial E \cap \{|x| = x_0\}$ is a vertical segment, the outer unit normal to E is constant on $\partial E^x_{x_0}$, $N_E = (1,0)$. In the same way the outer unit normal to E_ε is constant on $\partial(E_\varepsilon)^x_{x_0}$, $N_{E_\varepsilon} = (1,0)$. Then, we have

$$\begin{aligned} P_\alpha(E^x_{x_0-}) - P_\alpha(E_\varepsilon) &= 4\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + \int_{\partial E^x_{x_0}} d\mathcal{H}^1 - \int_{\partial(E_\varepsilon)^x_{x_0}} d\mathcal{H}^1 \\ &= 4\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + 4\big(f(x_0^-) - (f(x_0^-) - \varepsilon)\big) \\ &= 4\Big\{\int_{x_0-\varepsilon}^{x_0} \sqrt{f'(x)^2 + x^{2\alpha}} - \sqrt{m(\varepsilon)^2 + x^{2\alpha}}\, dx + \varepsilon\Big\}. \end{aligned}$$

Let $A(\varepsilon) = (P_\alpha(E^x_{x_0-}) - P_\alpha(E_\varepsilon))/4$. On the other hand,

$$\mathcal{L}^2(E^x_{x_0-}) - \mathcal{L}^2(E_\varepsilon) = 4\int_{x_0-\varepsilon}^{x_0} f(x) - r_\varepsilon(x)\, dx,$$

and we define $B(\varepsilon) = (\mathcal{L}^2(E^x_{x_0-}) - \mathcal{L}^2(E_\varepsilon))/4$. For any $\varepsilon > 0$, let $y_\varepsilon = B(\varepsilon)/x_0$. We claim that for $\varepsilon > 0$ small enough the set

$$F_\varepsilon = (E_\varepsilon + (0, y_\varepsilon)) \cup \big([-x_0, x_0] \times [-y_\varepsilon, y_\varepsilon]\big),$$

obtained by translating E_ε in the vertical direction of the quantity y_ε, satisfies

$$P_\alpha(F_\varepsilon) < P_\alpha(E). \tag{3.8}$$

It follows that the set $F = F_\varepsilon \cup (E \setminus E^x_{x_0-})$ satisfies $\mathscr{P}_\alpha(F) < \mathscr{P}_\alpha(E)$. Moreover $F \in \mathcal{A}$, since $\mathcal{L}^2(F^x_{x_0-}) = \mathcal{L}^2(E_\varepsilon) + 4x_0 y_\varepsilon = \mathcal{L}^2(E^x_{x_0-})$, and (1.5a) and (1.5b)are clear by construction. By invariance under vertical translations of the α-perimeter, we have

$$P_\alpha(F_\varepsilon) = P_\alpha(E_\varepsilon) + 4y_\varepsilon = P_\alpha(E^x_{x_0-}) - 4\Big(A(\varepsilon) - \frac{B(\varepsilon)}{x_0}\Big)$$

To prove (3.8), it is therefore sufficient to show that for $\varepsilon > 0$ small enough

$$x_0 A(\varepsilon) > B(\varepsilon). \tag{3.9}$$

First of all, notice that, by Lebesgue dominated convergence Theorem,

$$\lim_{\varepsilon\to 0^+} A(\varepsilon) = \lim_{\varepsilon\to 0^+} B(\varepsilon) = 0. \tag{3.10}$$

Let $f'(x_0^-) = \lim_{x\to x_0^-} f'(x)$. By convexity of $E^x_{x_0-}$, we have $f'(x_0^-) \leq 0$ $f''(x_0^-) = \lim_{x\to x_0^-} f''(x) \leq 0$. Moreover, the following limit exists

$$\lim_{\varepsilon\to 0^+} m(\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{f(x_0^-) - f(x_0-\varepsilon)}{\varepsilon} - 1 = f'(x_0^-) - 1.$$

Now, since $f'(x_0^-) > -\infty$, we also have $f''(x_0^-) > -\infty$, hence the following limit exists

$$\begin{aligned} m'(0) &= \lim_{\varepsilon\to 0^+} m'(\varepsilon) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big\{f'(x_0-\varepsilon) - \frac{1}{\varepsilon}\big[f(x_0^-) - f(x_0-\varepsilon)\big]\Big\} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big\{f'(x_0^-) - f''(x_0^-)\varepsilon - \frac{1}{\varepsilon}\Big[f(x_0^-) - \Big(f(x_0^-) - f'(x_0^-)\varepsilon + \frac{f''(x_0^-)}{2}\varepsilon^2\Big)\Big] + o(\varepsilon)\Big\} \\ &= -\frac{f''(x_0^-)}{2}. \end{aligned}$$

On the other hand, by the chain rule

$$\begin{aligned} A'(\varepsilon) &= 1 + \sqrt{f'^2(x_0-\varepsilon) + (x_0-\varepsilon)^{2\alpha}} - \sqrt{m(\varepsilon)^2 + (x_0-\varepsilon)^{2\alpha}} - \int_{x_0-\varepsilon}^{x_0} \frac{m(\varepsilon)m'(\varepsilon)}{\sqrt{m(\varepsilon^2) + x^{2\alpha}}}\,dx \\ &\geq \sqrt{f'^2(x_0-\varepsilon) + (x_0-\varepsilon)^{2\alpha}} - \sqrt{m(\varepsilon)^2 + (x_0-\varepsilon)^{2\alpha}} \end{aligned}$$

that gives

$$A'(0) = \lim_{\varepsilon\to 0} A'(\varepsilon) \geq 1 + \sqrt{f'^2(x_0^-)^2 + x_0^{2\alpha}} - \sqrt{(f'(x_0^-) - 1)^2 + x_0^{2\alpha}} > 0, \tag{3.11}$$

where the last inequality is justified by the following: for any $c < a < 0$, and $b \in \mathbb{R}$,

$$\sqrt{a_2 + b_2} - \sqrt{c_2 + b_2} > c - a.$$

We conclude observing that $B'(0) = \lim_{\varepsilon\to 0^+} B'(\varepsilon) = 0$, in fact:

$$\begin{aligned} B'(\varepsilon) &= f(x_0-\varepsilon) - r_\varepsilon(x_0-\varepsilon) + \int_{x_0-\varepsilon}^{x_0} \frac{d}{d\varepsilon}\Big\{f(x) - m(\varepsilon)(x-x_0) - f(x_0^-) + \varepsilon\Big\}\,dx \\ &= f(x_0-\varepsilon) + \varepsilon m(\varepsilon) - f(x_0^-) + \varepsilon - \int_{x_0-\varepsilon}^{x_0} 1 - m'(\varepsilon)(x-x_0)\,dx \xrightarrow[\varepsilon\to 0]{} 0 \end{aligned}$$

Then, (3.9) follows by (3.11) and (3.10). □

Remark 3.4. Assume that the profile function of a minimizer as in Theorem 2.4 satisfies

$$f'(x_0^-) = \lim_{x\to x_0^-} f'(x) = -\infty.$$

Using the notation of Proposition 3.3, there holds, for $\varepsilon > 0$ small enough:

$$x_0 A(\varepsilon) - B(\varepsilon) = -\frac{x_0^{\alpha+2}}{6\sqrt{2}}\Big(\frac{\varepsilon}{x_0}\Big)^{\frac{3}{2}} + o(\varepsilon^{3/2}) < 0 \quad \text{for } \varepsilon < \varepsilon_0. \tag{3.12}$$

Hence, the construction proposed in the latter proposition does not apply to this case.

3.3. **Center of regular solutions.** In this section, we show that we cannot in general expect a regular minimizer for the minimal partition problem to be obtained in its central part as a dilation of the isoperimetric set $E^\alpha_{\rm isop}$. In fact, we show that, for particular choices of $v_1, v_2, h_1, h_2 > 0$ and $\alpha \geq 0$, a regular minimizer $E \in \mathcal{A}$ with partitioning point $x_0 > 0$ does not satisfy for any $\lambda > 0$

$$E^x_{x_0-} = (\delta^\alpha_\lambda E^\alpha_{\rm isop})^x_{x_0-}. \tag{3.13}$$

Let f be the profile function of a regular solution of the minimal partition problem E. As observed in Remark 3.2, from the study of the differential equations in Proposition 3.1, it follows

$$f(x) = \lambda^{\alpha+1}\varphi_\alpha\big(\frac{x}{\lambda}\big) + y, \quad |x| < x_0$$

for some $\lambda > 0$, and $y \in \mathbb{R}$, where φ_α is the profile function of the isoperimetric set $E^\alpha_{\rm isop}$ defined in (3.5). We characterize the parameters λ and $y \in \mathbb{R}$ in terms of the data $h_1 = h$, $v_1 = v$ in the case of regular solutions of Problem (1.7) with $h_2 = v_2 = 0$ and $\alpha = 0, 1$, see Proposition 3.5. We deduce that, if $\alpha = 1$, the translation y is strictly negative (see Remark 3.6), hence (3.13) does not hold. On the other hand, in the case when $\alpha = 0$, the α-perimeter corresponds to the Euclidean perimeter and we prove that regular solutions of the minimal partition problem satisfy (3.13).

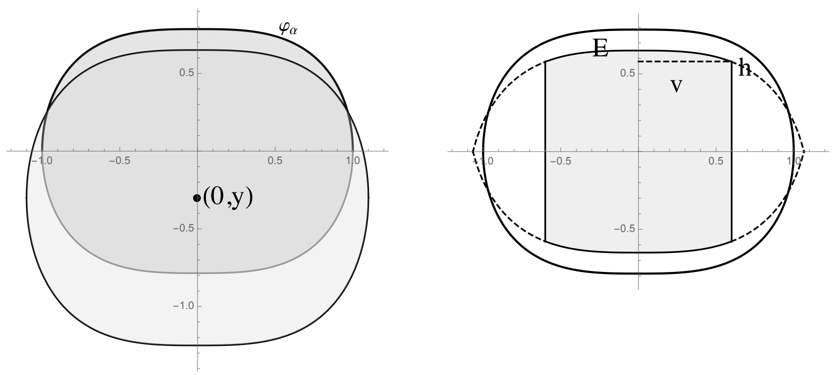

FIGURE 7. For $\alpha > 0$, a regular solution for the minimal partition problem is not obtained as a dilation of the isoperimetric set $E^\alpha_{\rm isop}$ in its central part. Its profile function is in fact the profile of an isoperimetric set vertically translated of a negative quantity y.

Proposition 3.5. *Given $\alpha \in \{0, 1\}$, $h \geq 0$, and $v > 0$, let $E \in \mathcal{A}(v, 0, h, 0)$ be a regular solution of Problem* (1.7) *with $v_1 = v, h_1 = h, v_2 = h_2 = 0$, and partitioning point $x_0 > 0$, satisfying $\mathrm{tr}^-_{x_0} E = [-h, h]$. Let $f : [0, x_0] \to [0, \infty)$ be its profile function. Then there exists $d \in [-1, 1]$ such that*

$$f(t) = \lambda^{\alpha+1}\varphi_\alpha\big(\frac{x}{\lambda}\big) + y \tag{3.14}$$

with

$$\lambda = \frac{x_0}{d}, \quad y = h\Big\{1 - \frac{\varphi_\alpha(d)}{d^\alpha\sqrt{1-d^2}}\Big\}. \tag{3.15}$$

Proof. If $E \in \mathcal{A}$ is as in the statement, it is a convex set such that $E \subset E^-_{x_0}$, and $\mathscr{P}_\alpha(E) = P_\alpha(E) \leq \mathscr{P}_\alpha(F)$ for any $F \in \mathcal{A}$. By Propostion 3.1, the profile function f satisfies for a constant

$c > 0$

$$f'(x) = -\frac{\operatorname{sgn} x\ c|x|^{\alpha+1}}{\sqrt{1-c^2x^2}},$$

hence, as observed in Remark 3.2, there exist $\lambda > 0$, $y \in \mathbb{R}$ such that

$$f(x) = \lambda^{\alpha+1}\varphi_\alpha\big(\frac{x}{\lambda}\big) + y, \quad |x| < x_0.$$

In particular $\lambda = 1/c$. Let $\beta = f(0) > 0$, and define

$$p(\beta, c, x_0) = P_\alpha(E) = \int_0^{x_0} \sqrt{f'^2(t) + t^{2\alpha}}\, dt = \int_0^{x_0} \frac{t^\alpha}{\sqrt{1-(ct)^2}}\, dt = \frac{1}{c^{\alpha+1}} \int_0^{\arcsin cx_0} \sin^\alpha \vartheta\, d\vartheta.$$

Notice that p can be thought of as a functional depending on β, c, x_0 where the triple (β, c, x_0) identify a unique minimizer $E \in \mathcal{A}$ as in the statement. In particular, p is independent of β, as well as P_α is independent of vertical translations. Let $d = cx_0$, that leads to $\lambda = x_0/d$. With a slight abuse of notation, we write p in terms of d and x_0 as

$$p(d, x_0) = x_0^{\alpha+1} g_\alpha(d), \text{ with } g_\alpha(d) = \frac{1}{d^{\alpha+1}} \int_0^{\arcsin d} \sin^\alpha \vartheta\, d\vartheta. \tag{3.16}$$

We write the volume and trace constraints satisfied by the minimizer E in terms of the parameters d and x_0. For any $t \in (0, x_0)$, there holds

$$f(t) = \beta + \int_0^t f'(s)\, ds = \beta - \int_0^t \frac{cs^{\alpha+1}}{\sqrt{1-(cs)^2}}\, ds = \beta - \frac{1}{c^{\alpha+1}} \int_0^{\arcsin ct} \sin^{\alpha+1}\vartheta\, d\vartheta.$$

Hence the trace constraint $f(x_0) = h$ is equivalent to

$$\begin{aligned} &\beta = \beta(d, x_0) = h + x_0^{\alpha+1}\sigma_\alpha(d), \text{ with} \\ &\sigma_\alpha(d) = \frac{1}{d^{\alpha+1}} \int_0^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta > 0 \text{ for } d \in (0,1). \end{aligned} \tag{3.17}$$

Plugging $\beta = \beta(d, x_0)$ in the expression for f we get

$$f(t) = h + x_0^{\alpha+1}\sigma_\alpha(d) - x_0^{\alpha+1}\frac{1}{d^{\alpha+1}} \int_0^{\arcsin(\frac{d}{x_0}t)} \sin^{\alpha+1}\vartheta\, d\vartheta, \tag{3.18}$$

that implies

$$\begin{aligned} y = f(\lambda) &= h + x_0^{\alpha+1} b_\alpha(d), \text{ with} \\ b_\alpha(d) &= \frac{1}{d^{\alpha+1}}\Big\{\int_0^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta - \int_0^{\pi/2} \sin^{\alpha+1}\vartheta\, d\vartheta\Big\} \\ &= -\frac{1}{d^{\alpha+1}} \int_{\arcsin d}^{\pi/2} \sin^{\alpha+1}\vartheta\, d\vartheta = -\frac{\varphi_\alpha(d)}{d^{\alpha+1}} < 0 \text{ for } d \in (0,1). \end{aligned} \tag{3.19}$$

Using (3.17) and the expression for f, the volume constraint $\int f = v$ reads

$$\begin{aligned} v &= \int_0^{x_0} \Big(\beta - \frac{1}{c^{\alpha+1}} \int_0^{\arcsin ct} \sin^{\alpha+1}\vartheta\, d\vartheta\Big)\, dt = \beta x_0 - \frac{1}{c^{\alpha+2}} \int_0^{cx_0} \int_0^{\arcsin r} \sin^{\alpha+1}\vartheta\, d\vartheta\, dr \\ &= \Big(h + x_0^{\alpha+1}\sigma_\alpha(d)\Big)x_0 - x_0^{\alpha+2}\frac{1}{d^{\alpha+2}} \int_0^d \int_0^{\arcsin t} \sin^{\alpha+1}\vartheta\, d\vartheta\, dt, \end{aligned}$$

hence

$$
\begin{aligned}
&v = hx_0 + x_0^{\alpha+2} G_\alpha(d), \text{ with} \\
G_\alpha(d) &= \frac{1}{d^{\alpha+2}}\Big(d\int_0^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta - \int_0^d \int_0^{\arcsin t} \sin^{\alpha+1}\vartheta\, d\vartheta\, dt\Big) \\
&= \frac{1}{d^{\alpha+2}} \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta\, dt > 0 \text{ for } d\in(0,1)
\end{aligned}
\tag{3.20}
$$

The functional

$$
F(d, x_0) = x_0^{\alpha+2} G_\alpha(d) + hx_0 - v.
$$

defines implicitly the constraints of the problem. Existence of a minimizer together with the Lagrange Multipliers theorem imply that there exists $\mu \in \mathbb{R}$ such that $\nabla p(d, x_0) = \mu \nabla F(d, x_0)$, namely

$$
\begin{cases} \partial_d p = \mu \partial_d F \\ \partial_{x_0} p = \mu \partial_{x_0} F \\ F(d, x_0) = 0 \end{cases} \iff \begin{cases} g_\alpha'(d) x_0^{\alpha+1} = \mu x_0^{\alpha+2} G_\alpha'(d) \\ (\alpha+1) x_0^\alpha g_\alpha(d) = \mu\big((\alpha+2)x_0^{\alpha+1} G_\alpha(d) + h\big) \\ x_0^{\alpha+2} G_\alpha(d) + hx_0 - v = 0 \end{cases}
\tag{3.21}
$$

Recalling the definitions of g_α and G_α in (3.16) and (3.20), we write the expressions for the derivatives

$$
\begin{aligned}
g_\alpha'(d) &= -\frac{\alpha+1}{d^{\alpha+2}} \int_0^{\arcsin d} \sin^\alpha \vartheta\, d\vartheta + \frac{1}{d\sqrt{1-d^2}} \\
G_\alpha'(d) &= -\frac{\alpha+2}{d^{\alpha+3}} \int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta + \frac{1}{\sqrt{1-d^2}}
\end{aligned}
$$

it easy to see that when $\alpha = 0, 1$ we have

$$
g_\alpha'(d) = d G_\alpha'(d)
\tag{3.22}
$$

We check it for $\alpha = 1$. In this case $g_\alpha'(d) = -\frac{2}{d^3}[\sqrt{1-d^2} - 1] + \frac{1}{d\sqrt{1-d^2}} = \frac{2-d^2-2\sqrt{1-d^2}}{d^3\sqrt{1-d^2}}$, and

$$
\begin{aligned}
G_\alpha'(d) &= -\frac{3}{d^4}\int_0^d \frac{1}{2}\big[\vartheta - \frac{\sin(2\vartheta)}{2}\big]_{\arcsin t}^{\arcsin d} dt + \frac{1}{\sqrt{1-d^2}} \\
&= \frac{3}{2d^4}\int_0^d \big(\arcsin t - t\sqrt{1-t^2} - \arcsin d + d\sqrt{1-d^2}\big) + \frac{1}{\sqrt{1-d^2}} \\
&= \frac{3}{2d^4}\Big\{\big[t \arcsin t + \sqrt{1-t^2} + \frac{1}{3}(1-t^2)^{3/2}\big]_0^d - d\arcsin d + d^2\sqrt{1-d^2}\Big\} \\
&= \frac{3}{2d^4}\Big\{d\arcsin d + \sqrt{1-d^2} + \frac{1}{3}(1-d^2)^{3/2} - 1 - \frac{1}{3} \\
&\quad - d\arcsin d + d^2\sqrt{1-d^2}\Big\} \\
&= \frac{1}{d^4}\{(2+d^2)\sqrt{1-d^2} - 2\} + \sqrt{1}\sqrt{1-d^2} = \frac{2-d^2-2\sqrt{1-d^2}}{d^4\sqrt{1-d^2}}.
\end{aligned}
$$

Notice that (3.22) is equivalent to

$$
\begin{aligned}
0 &= \frac{1}{d^{\alpha+2}}\Big\{(\alpha+2)\int_0^d \int_{\arcsin t}^{\arcsin d} \sin^{\alpha+1}\vartheta\, d\vartheta\, dt - (\alpha+1)\int_0^{\arcsin d} \sin^\alpha\vartheta\, d\vartheta\Big\} + \frac{1-d^2}{d\sqrt{1-d^2}} \\
&= (\alpha+2)G_\alpha(d) - \frac{1}{d}(\alpha+1)g_\alpha(d) + \frac{1}{d}\sqrt{1-d^2}.
\end{aligned}
\tag{3.23}
$$

From (3.22), the first equation in system (3.21) gives

$$\mu = \frac{g'_\alpha(d)}{x_0 G'_\alpha(d)} = \frac{d}{x_0} = \frac{1}{\lambda}.$$

Plugging μ into the second equation of (3.21) we obtain

$$(\alpha+1)x_0^{\alpha+1} g_\alpha(d) = d\{(\alpha+2)x_0^{\alpha+1} G_\alpha(d) + h\},$$

hence, using (3.23),

$$x_0^{\alpha+1} = \frac{dh}{(\alpha+1)g_\alpha(d) - d(\alpha+2)G_\alpha(d)} = \frac{dh}{\sqrt{1-d^2}} \tag{3.24}$$

We are left to compute $y = f(\lambda) = f(x_0/d)$ with $x_0 = x_0(h,d)$ given by (3.24). Expression (3.19) for y, combined with (3.24) gives

$$\begin{aligned} y = f\Big(\frac{x_0}{d}\Big) &= h + x_0^{\alpha+1} b_\alpha(d) = h - \frac{dh}{\sqrt{1-d^2}} \frac{\varphi_\alpha(d)}{d^{\alpha+1}} \\ &= -\frac{h}{d^\alpha \sqrt{1-d^2}} \Big\{\varphi_\alpha(d) - d^\alpha \sqrt{1-d^2}\Big\} \end{aligned}$$

which concludes the proof. □

Remark 3.6. Let $\alpha \geq 0$ and $E = \{(x,y) \in \mathbb{R}^2 : |y| < f(|x|)\}$ be a regular minimizer of Problem (1.7) with f as in (3.14). We deduce by (3.15) that $y = 0$ if and only if $\alpha = 0$. In fact, for any $\alpha \geq 0$ the function $d \mapsto \varphi_\alpha(d) - d^\alpha\sqrt{1-d^2}$ is 0 at $d = 1$ and it if $\alpha > 0$ it is strictly monotone decreasing since

$$(\varphi_\alpha(d) - d^\alpha\sqrt{1-d^2})' = -\frac{d^{\alpha+1}}{\sqrt{1-d^2}} - \alpha d^{\alpha-1}\sqrt{1-d^2} + \frac{d^{\alpha+1}}{\sqrt{1-d^2}} < 0.$$

Hence if $\alpha > 0$, $\varphi_\alpha(d) - d^\alpha\sqrt{1-d^2} > 0$ for $0 < d < 1$. In particular, $y < 0$. On the other hand, if $\alpha = 0$, $\varphi_\alpha(d) = \sqrt{1-d^2}$, that leads to $y = 0$.

This implies that the central part of Euclidean solutions of Problem (1.7) are portions of isoperimetric sets lying in some stripe $\{|x| < x_0\}$, while this property fails to hold in the Grushin plane with $\alpha = 1$.

Appendix A. Traces of Schwarz symmetric sets

For a set $E \in \mathscr{S}_y^*$ and a point $x_0 \in \mathbb{R}$, the notion of *trace of E at x_0* can be defined thanks to the following Lemma.

Lemma A.1. *Let $E \in \mathscr{S}_y^*$ and let $x_0 \in \mathbb{R}$. Then there exist $y^+, y^- \geq 0$ such that if $T^+ = [-y^+, y^+]$ and $T^- = [-y^-, y^-]$, there holds*

$$\lim_{x \to x_0^\pm} \int_{\mathbb{R}} |\chi_E(x,y) - \chi_{T^\pm}(y)|\, dy = 0.$$

Proof. We prove the statement for the limit as $x \to x_0^-$. Let $u \in C^1(\mathbb{R}^2)$ and $x_1, x_2 \in (-\infty, x_0)$. Consider the α-gradient of u, $D_\alpha u = (\partial_x u, |x|^\alpha \partial_y u)$. We have

$$\begin{aligned} \int_{\mathbb{R}} \big(u(x_2,y) - u(x_1,y)\big) dy &= \int_{\mathbb{R}} \int_{x_1}^{x_2} \partial_x u(\xi, y)\, d\xi dy \\ &\leq \int_{(x_1,x_2)\times\mathbb{R}} |\partial_x u|(\xi,y)\, d\xi dy \leq |D_\alpha u|((x_1,x_2)\times\mathbb{R}). \end{aligned}$$

By the approximation theorem for BV_α-functions, see [12], the last inequality can be extended to $u \in BV_\alpha(\mathbb{R}^2)$ and for $u = \chi_E$ we get

$$\int_{\mathbb{R}} \big(\chi_E(x_2, y) - \chi_E(x_1, y)\big) dy \le P_\alpha(E; (x_1, x_2) \times \mathbb{R}),$$

It hence follows that for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\|\chi_E(x_2, \cdot) - \chi_E(x_1, \cdot)\|_{L^1(\mathbb{R})} \le \varepsilon \text{ for } x_0 - \delta < x_1 < x_2 < x_0, \tag{A.1}$$

which is a Cauchy condition in the complete space $L^1(\mathbb{R})$. We deduce existence of a function $u \in L^1(\mathbb{R})$ which is the limit of $\chi_E(x, \cdot)$ as $x \to x_0^-$. Moreover, since for any $x \in \mathbb{R}$, the section $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is a real interval centered at zero, $u = \chi_{T^-}$, for a symmetric interval $T^- = [-y^-, y^-]$ for some $y^- > 0$. □

Definition A.2 (Traces of Schwarz symmetric sets)**.** Let $E \in \mathscr{S}_y^*$ be a set with finite α-perimeter and let $x_0 \in \mathbb{R}$. The interval T^- (resp. T^+) defined in Lemma A.1 is called the *left (resp. right) trace of E at x_0* and it is denoted by $\mathrm{tr}_{x_0}^-$ (resp. $\mathrm{tr}_{x_0}^+$). If

$$\mathrm{tr}^x_{x_0-} E = \mathrm{tr}^x_{x_0+} E = [-y_0, y_0],$$

we set $\mathrm{tr}^x_{x_0} E = [-y_0, y_0]$ and we call it the *trace of E at x_0 in the x-direction*. In this case we say that the set E *has trace at x_0 in the x-direction*.

Remark A.3. If $E \in \mathscr{S}_x \cap \mathscr{S}_y^*$ has profile function $f : [0, \infty) \to [0, \infty)$, i.e., $E = \{(x, y) \in \mathbb{R}^2 : |y| < f(|x|)\}$, then left and right traces at $x_0 > 0$ can be computed as follows:

$$\mathrm{tr}^x_{x_0\pm} E = [-y_0^\pm, y_0^\pm] \quad \text{with} \quad \lim_{x \to x_0^\pm} f(x) = y_0^\pm. \tag{A.2}$$

In fact, by definition of left and right traces, we have

$$0 = \lim_{x \to x_0^\pm} \int_{\mathbb{R}} |\chi_E(x, y) - \chi_{[-y_0^\pm, y_0^\pm]}(y)|\, dy = \lim_{x \to x_0^\pm} \mathcal{L}^1((E)_x \triangle [-y_0^\pm, y_0^\pm]) = 2 \lim_{x \to x_0^\pm} |f(x) - y_0^\pm|.$$

Remark A.4. If $E \subset \mathbb{R}^2$ is x-spherically symmetric and x-convex, i.e., the section $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$ is an interval for every $y \in \mathbb{R}$, we can define *left and right traces at $y_0 \in \mathbb{R}$* in the y-direction through the formula $\|\chi_E(\cdot, y_2) - \chi_E(\cdot, y_1)\|_{L^1(\mathbb{R})} \le \varepsilon$ for $y_0 - \delta < y_1 < y_2 < y_0$ (see (A.1)). In this paper we are interested in studying traces in the x-direction.

Acknowledgments

The author would like to thank Prof. Roberto Monti from the University of Padova for suggesting her the problem and for the precious help. The author acknowledges the support of the G.N.A.M.P.A. projects *Problemi isoperimetrici e teoria geometrica della misura in spazi metrici*, and *Variational Problems and Geometric Measure Theory in Metric Spaces*.

References

[1] V. Bögelein, F. Duzaar, N. Fusco, A sharp quantitative isoperimetric inequality in higher codimension. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **26** (2015), no. [3], 309–362.

[2] V. Bögelein, F. Duzaar, C. Scheven, A sharp quantitative isoperimetric inequality in hyperbolic n-space. *Calc. Var. Partial Differential Equations* **54** (2015), no. [4], 3967–4017.

[3] L. Capogna, D. Danielli, N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality. *Comm. Anal. Geom.* **2** (1994), no. [2], 203–215.

[4] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, On the isoperimetric deficit in Gauss space. *Amer. J. Math.*, **133** (2011), no. [1], 131–186.

[5] M. Cicalese, G. P. Leonardi, A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.* **206** (2012), no. [4], 617–643.

[6] M. Cicalese, G. P. Leonardi, Best constants for the isoperimetric inequality in quantitative form. *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. [3], 1101–1129.

[7] J. Foisy, M. Alfaro, J. Brock, N. Hodges, J. Zimba, The standard double soap bubble in $\mathbf{R}^2$ uniquely minimizes perimeter. *Pacific J. Math.*, **159** (1993), no. [1], 47–59.

[8] A. Figalli, F. Maggi, A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. *Inv. Math.*, **182** (2010) no. [1], 167–211.

[9] B. Franchi, S. Gallot, R. L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics. *Math. Ann.* **300** (1994), 557–571.

[10] B. Franchi, C. E. Gutiérrez, R. L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators., *Comm. Partial Differential Equations* **19** (1994), no. [3-4], 523–604.

[11] B. Franchi, E. Lanconelli, Une métrique associée à une classe d'opérateurs elliptiques dégénérés. *Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983*, Special Issue, 105Ð114 (1984).

[12] B. Franchi, R. Serapioni, F. Serra Cassano, Meyers-Serrin Type Theorems and Relaxation of Variational Integrals Depending Vector Fields. *Houston Journal of Mathematics* **22** (1996), 859–889.

[13] V. Franceschi, *Sharp and quantitative isoperimetric inequalities in Carnot-Carathéodory spaces*, Ph.D. Thesis, (2016).

[14] V. Franceschi, R. Monti, Isoperimetric Problem in H-type groups and Grushin spaces. *Rev. Mat. Iberoam.*, to appear.

[15] V. Franceschi, G. P. Leonardi, R. Monti, Quantitative isoperimetric inequalities in $\mathbb{H}^n$. *Calc. Var. Partial Differential Equations*, **54** (2015), no. [3], 3229–3239.

[16] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality, *Ann. of Math.*, **168** (2008), no. [3], 941–980.

[17] E. Giusti, *Minimal surfaces and functions of bounded variation.* Monographs in Mathematics, **80**. Birkhäuser Verlag, Basel, 1984.

[18] N. Garofalo, D.-M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.* **49** (1996), 1081–1144.

[19] E. Le Donne, A metric characterization of Carnot groups. *Proc. Amer. Math. Soc.* **143** (2015), no. [2], 845–849.

[20] F. Maggi, Some methods for studying stability in isoperimetric type problems. *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), no. [3], 367–408.

[21] R. Monti, Heisenberg isoperimetric problem. The axial case. *Adv. Calc. Var.* **1** (2008), no. 1, 93–121.

[22] R. Monti, D. Morbidelli, Isoperimetric Inequality in the Grushin Plane. *J. Geom. Anal.*, **14** (2004), no. [2], 355–368.

[23] R. Monti, M. Rickly, Convex isoperimetric sets in the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (2009), no. [2], 391–415.

[24] F. Morgan, *Geometric measure theory. A beginner's guide. Fourth edition.* Elsevier/Academic Press, Amsterdam, 2009. viii+249 pp.

[25] P. Pansu, An isoperimetric inequality on the Heisenberg group. *Conference on differential geometry on homogeneous spaces (Turin, 1983). Rend. Sem. Mat. Univ. Politec. Torino*, Special Issue (1983), 159–174.

[26] D. Prandi, L. Rizzi, M. Seri, A sub-Riemannian Santaló formula with applications to isoperimetric inequalities and Dirichlet spectral gap of hypoelliptic operators, Preprint.

[27] M. Ritoré, A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$. *Calc. Var. Partial Differential Equations*, **44** (2012), no. [1-2], 47–60.

[28] M. Ritoré, C. Rosales, Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$. *J. Geom. Anal.*, **16** (2006), no. [4], 703–720.

[29] M. Ritoré, C. Rosales, Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. *Adv. Math.* **219** (2008), no. [2], 633–671.

E-mail address: `valentina.franceschi@math.unipd.it`

Università di Padova, Dipartimento di Matematica, via Trieste 63, 35121 Padova, Italy