Preface

Our aim in organizing this CIME Course was to present to young students and researchers the impressive recent achievements in differential geometry and topology obtained by means of techniques based on Ricci flow. We then invited some of the leading researchers in the field of geometric analysis and low-dimensional geometry/topology to introduce some of the central ideas in their work. Here is the list of speakers together with the titles of their lectures:

- Gérard Besson (Grenoble) – *The differentiable sphere theorem (after S. Brendle and R. Schoen)*.
- Michel Boileau (Toulouse) – *Thick/thin decomposition of three-manifolds and the geometrization conjecture*.
- Carlo Sinestrari (Roma "Tor Vergata") – *Singularities of three-dimensional Ricci flows*.
- Gang Tian (Princeton) – *Kähler–Ricci flow and geometric applications*.

The summer school had around 50 international attendees (mostly PhD students and PostDocs). Even though they were sometimes technically heavy, the lectures were followed by all the students with interest. The participants were very satisfied by the high quality of the course. The not-so-intense scheduling of the lectures gave the students many opportunities to interact with the speakers, who were always very friendly and available for discussion. It should be mentioned that the wonderful location and the careful CIME organization were also greatly appreciated.

We think that the fast-growing field of geometric flows and more generally of geometric analysis, which has always received great attention in the international community, but which is still relatively "young" in Italy, will benefit from its diffusion by this CIME Course.

We briefly describe the contents of the lectures collected in this volume.

Gérard Besson presented the impact of the Ricci flow technique on the theory of positively curved manifolds, the central result being the *differentiable 1/4-pinched sphere theorem*, proved by Brendle and Schoen. It says that a complete, simply connected Riemannian manifold whose sectional curvature varies in $(1/4, 1]$ is diffeomorphic to the standard sphere.

The problem was first proposed by H. Hopf, then in 1951 H.E. Rauch showed that a complete Riemannian manifold whose sectional curvature is positive and varies between two numbers whose ratio is close to 1 has a universal cover homeomorphic to a sphere. In the 1960s M. Berger and W. Klingenberg obtained the optimal result: a simply connected Riemannian manifold which is *strictly* $1/4$-pinched is *homeomorphic* to the sphere. The analogous *diffeomorphic* conclusion remained open until S. Brendle and R. Schoen proved the following.

Theorem (S. Brendle–R. Schoen, 2008) *Let M be a pointwise strictly $1/4$-pinched Riemannian manifold of positive sectional curvature. Then M carries a metric of constant sectional curvature. Hence, it is diffeomorphic to the quotient of a sphere by a finite subgroup of $O(n)$.*

The proof relies on the use of the Ricci flow introduced by R. Hamilton and culminating in the work of G. Perelman. The idea is to construct a deformation of the Riemannian metric, evolving it by means of Ricci flow toward a constant curvature metric. We recall that this was the method that R. Hamilton used in his seminal paper, proving the following theorem.

Theorem (R. Hamilton, 1982) *Let M be a closed 3-dimensional Riemannian manifold which carries a metric of positive Ricci curvature, then it also carries a metric of positive constant curvature.*

The lectures also focus on the extension to higher dimensions of the following result, due to C. Böhm and B. Wilking. Recall that a curvature operator is 2-positive if the sum of its two smallest eigenvalues is positive.

Theorem (C. Böhm–B. Wilking, 2008) *Let M be a closed Riemannian manifold whose curvature operator is 2-positive, then M carries a constant curvature metric.*

In the lectures the connection between this method and the algebraic properties of the Riemann curvature operator is stressed, the main focus being the identification of those properties of the curvature operator which are preserved under the Ricci flow.

In his lectures, Michel Boileau gave an introduction to the geometrization of 3-manifolds. Sections 1 and 2 cover Thurston's classification of the eight 3-dimensional geometries and the characterization of *geometric* (and *Seifert*) closed 3-manifolds in terms of basic topological properties. This follows by combining Thurston's hyperbolization theorems (in particular the characterization of hyperbolic 3-manifolds that are fibered over $\mathbb{S}^1$), Perelman's general geometrization theorem, and Agol's recent (2013) proof of a deep conjecture of Thurston that closed hyperbolic 3-manifolds are "virtually fibered".

Section 3 discusses: (1) A central result of classical 3-dimensional geometric topology, that is, the *canonical decomposition of a 3-manifold* by splitting it along spheres and tori; (2) *Thurston's geometrization conjecture*. This roughly says that every piece of a canonical decomposition is geometric together with a prediction on the geometry carried by the piece in terms of basic topological properties. It includes as a particular case the celebrated *Poincaré conjecture*; (3) Thurston's fundamental *hyperbolization theorem for Haken manifolds*.

Perelman's proof of the general *geometrization theorem* deals with all of these topics and also allows us to recover, as a by-product, the canonical decomposition itself. This is done by completing the program based on *Ricci flow with surgeries*, first proposed by R. Hamilton. This is the subject of Boileau's notes from Section 4.

Since the appearance of Perelman's three celebrated preprints, several simplifications and variants of the original proofs have been developed by various authors. At the end of the day, we can say that the *Poincaré conjecture* (that is, the case when the Ricci flow with surgery becomes extinct in finite time) is in a sense the "simplest" case. The general case (when the Ricci flow with surgery exists at all times, which includes the complete hyperbolization theorem) requires non-trivial extra arguments, in particular to obtain a key *non-collapsing Theorem*. In Perelman's original work these come from the theory of *Alexandrov spaces*. Bessières, Besson, Boileau, Maillot, and Porti developed instead an alternative approach where the basic tools are Thurston's hyperbolization theorem for Haken manifolds and some well established properties of Gromov's *simplicial volume*, allowing one to bypass the need for the (somewhat more exotic) theory of Alexandrov spaces. Boileau's notes are largely based on the monograph

L. Bessières, G. Besson, M. Boileau, S. Maillot and J. Porti, *Geometrisation of 3-manifolds*, EMS Tracts in Mathematics **13**, 2010.

In this tract, the authors developed a slightly different notion of surgery by defining the so-called *Ricci flow with bubbling-off*. Actually, one might roughly say that the Ricci flow with bubbling-off reduces the general hyperbolization theorem to Thurston's hyperbolization theorem for Haken manifolds.

Carlo Sinestrari provided an extensive introduction to Ricci flow by first giving a survey of the basic results and examples, then concentrating on the analysis of the singularities of the flow in the three-dimensional case, which is needed in Hamilton and Perelman's surgery construction. After reviewing the properties of the Ricci flow and the fundamental estimates of the theory, such as Hamilton's Harnack differential inequality, the Hamilton–Ivey pinching estimate and Perelman's no collapsing result, he presented Perelman's analysis of kappa-solutions and the canonical neighborhood property which gives a full description of the singular behavior of the solutions in dimension 3. All these results are central to the proof of the Poincaré and geometrization conjectures.

The exposition is quite accessible to non-experts. Indeed, the presentation is often informal and the proofs are omitted except in some simple and significant cases, focusing more on the description of the results and their applications and consequences. A final detailed bibliographical section gives to the interested reader all the references needed for an advanced study of these topics.

Gang Tian's expository notes, based on his lectures, discuss some aspects of the Analytic Minimal Model Program through Kähler–Ricci flow, developed in collaboration with other authors, particularly, J. Song and Z. Zhang. Very stimulating open problems and conjectures are also presented.

Section 2 contains a detailed account of the sharp version of the Hamilton–DeTurck local existence theorem, which holds in the Kähler case: the maximal time $T_{max} \in (0, +\infty]$ such that the flow exists on the interval $[0, T_{max})$ is precisely determined in terms of a cohomological property of the initial Kähler metric. As a corollary, one deduces that $T_{max} = +\infty$ for every initial metric on a compact Kähler manifold with a numerically positive canonical bundle.

In Section 3 the limit singularities that can arise when $t \to T_{max} < +\infty$ are analyzed. After having established a general convergence theorem (Theorem 4.3.1), one faces questions concerning regularity and the geometric properties of the limit. A combination of partial results (particularly in the case of projective varieties and when one can apply deep results of algebraic geometry) and well motivated conjectures outlines a pregnant scenario.

Sections 4 and 5 discuss the construction of a *Kähler–Ricci flow with surgery* (assuming the truth of a conjecture stated in Section 3) and its asymptotic behavior. Numerous conjectures arise throughout this discussion such as: the characterization (up to birational isomorphism) of "Fano-like" manifolds as those whose flow becomes extinct at a finite time; the characterization of uniruled manifolds (up to birational isomorphism) as those whose flow collapses in finite time; and the existence of a flow with surgery globally defined in time and with only finitely many surgery times.

In Section 6 algebraic surfaces are considered, showing how most of the program is carried out in this case.

We are pleased to express our thanks to the speakers for their excellent lectures and to the participants for contributing with their enthusiasm to the success of the Summer School.

The speakers, the participants and the CIME organizers collectively created a stimulating, rich, pleasant and friendly atmosphere at Cetraro. For this reason we would finally like to thank the Scientific Committee of CIME and, in particular, Pietro Zecca and Elvira Mascolo.

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