Almost Periodic Methods in the Theory of Homogenization

Andrea BRAIDES

Dipartimento di Automazione Industriale, Università di Brescia, via Valotti 9, I-25060 BRESCIA

Introduction

The study of the asymptotic behaviour of oscillating structures has been carried on successfully under the hypothesis of periodicity (see for example Bensoussan, J.L. Lions & Papanicolaou [7], Marcellini [35], Müller [38], etc.). However, the periodical setting is, under certain aspects, at times unsatisfactory; *e.g.*,

– the family of periodic functions is not stable under simple algebraic operations (sum, product) or superposition, unless we restrict to functions with the same period;

– in some cases we have to consider the restriction of periodic functions to linear subspaces (which in general is not periodic);

– we would like our homogenization results to be stable under "small" perturbations (for example compact support perturbations) of the functionals considered.

The almost periodic framework provides a suitable answer to these requirements.

In this paper we shall present a homogenization theorem (Theorem 2.4) for functionals of the calculus of variations of the form

$$\int_\Omega f\Big(x, \frac{x}{\varepsilon}, u(x), \frac{u(x)}{\varepsilon}, Du(x)\Big) dx,$$

with very mild conditions of almost periodicity on the oscillating variables, and standard growth hypotheses on the integrand f. Particular case of problems of this type have already been treated by Buttazzo & Dal Maso [15], Acerbi & Buttazzo [1] and E [23], under periodicity assumptions (see also Boccardo & Murat [9]).

We use the techniques of De Giorgi's Γ-convergence, which has proven particularly suited for the treatment of asymptotic problems. Almost-periodic problems have already been studied by Braides [10], [11], Ambrosio & Braides [4], De Arcangelis and Serra Cassano [21], De Arcangelis [20], and, in the framework of the G-convergence, by Kozlov [28], [29], Oleinik & Zhikov [39] (for the linear case), and recently by Braides, Chiadò Piat & Defranceschi [13], and Braides [12] (for quasilinear equations).

Section 1 contains the main definitions of almost periodicity and Γ-convergence. In Section 2 we recall some important results concerning (weakly lower semicontinuous) integral functionals on Sobolev spaces, and state the main results of the paper. Section 3 is devoted to the proof of the homogenization theorem under stronger hypotheses of uniform almost periodicity, which already cover the case of oscillating Riemmanian metrics. The simplification allows us to underline in the proof the main feature of uniform almost periodic functions here utilized: the keeping of their properties while passing to lower dimensional linear subspaces. Let us remark that even in the periodic case (and no non-oscillating variables), we find ourselves with a non periodic problem when considering the superposition of f and a linear function: $f(x, \xi x, \xi)$. This would lead to some technical problems, if we wanted to use only periodic techniques (see E [23], Section 4 and 5, Acerbi & Buttazzo [1] Section 3). In Section 4 we briefly recall a closure theorem for the homogenization (Theorem 4.2) proven in [11], and adapt it to the present situation. This permits the extension to the general almost periodic case. Finally, in Section 5 we deduce from Theorem 2.4 a homogenization result for viscosity solution of Hamilton-Jacobi equations.

We have given detailed proofs of the new results, when the almost periodicity plays an important role, while, for the proofs involving routine procedures of Γ-convergence, we refer to well-known results (see for example Buttazzo & Dal Maso [16], Dal Maso & Modica [19], Fusco [25]), and in particular to representation theorems for variational functionals (see Buttazzo [14], Alberti [3]).

Acknowledgements

This problem was pointed out to me by Gianni Dal Maso in one out of countless fruitful discussions. My thanks extend to Giovanni Alberti for valuable bibliographical help on integral representation problems.

1. Notation and definitions

Notation

By $M^{m\times n}$ we will denote the space of $m \times n$ real matrices, if $\xi \in M^{m\times n}$ and $x \in \mathbf{R}^n$, $\xi x \in \mathbf{R}^m$ will be defined by the usual product between matrices and vectors; if $x, y \in \mathbf{R}^n$, $(x, y) \in \mathbf{R}$ will denote their scalar product. $\mathrm{H}^{1,p}(\Omega) = \mathrm{H}^{1,p}(\Omega; \mathbf{R}^m)$ will be the usual Sobolev space (of $\mathbf{R}^m$-valued functions). $\mathcal{A}_n$ will be the family of all bounded open subsets of $\mathbf{R}^n$; if $\Omega \in \mathcal{A}_n$, $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ will be the families of open and Borel subsets of Ω, respectively.

Almost-periodic functions

Definition 1.1. (see for example Levitan & Zhikov [30] Def. 3 Ch. 1, Besicovitch [8] Def. 2 Ch. 1 Parag. 12) Let $(X, \|\ \|)$ be a complex Banach space. We say that a measurable function $v : \mathbf{R}^m \to X$ is *uniformly almost periodic* (u.a.p. for short), and we write $v \in UAP(\mathbf{R}^m, X)$, if it is the uniform limit of a sequence of trigonometric polynomials on X; *i.e.,*

$$\lim_{h\to\infty} \|P_h(\cdot) - v(\cdot)\|_\infty = 0$$

for some

$$P_h(y) = \sum_{j=1}^{r_h} \mathbf{x}_j^h \ e^{i(\lambda_j^h, y)},$$

with $\mathbf{x}_j^h \in X$, $\lambda_j^h \in \mathbf{R}^m$, and $r_h \in \mathbf{N}$. The definition easily extends to real Banach spaces. If $X = \mathbf{R}$, this definition is the usual definition of uniformly almost periodic functions (in the sense of Bohr, see [8]), and we will write UAP_m for $UAP(\mathbf{R}^m, \mathbf{R})$. A special class of u.a.p. functions are *quasiperiodic* functions, *i.e.,* diagonal functions of continuous periodic functions of a larger number of variables: $v(x) = V(x, \dots, x)$ (see [30] Chapter 3).

Remark 1.2. If $u : \mathbf{R}^N \to \mathbf{R}$ is a L^1_{loc} function, we define the *mean value* of u (over $\mathbf{R}^n$) as

$$\fint u dx = \limsup_{T\to\infty} \frac{1}{(2T)^n} \int_{[-T,T]^n} u(x)dx.$$

It is easy to see that the mean value of u.a.p. functions is finite.

Let us first recall a characterization of uniformly almost periodic functions, which will be an essential tool in the proof of the homogenization result (see for example, [8] Ch. 1 Parag. 12 Theorem 10 and Theorem 3).

Theorem 1.3. *Let $g : \mathbf{R}^N \to \mathbf{R}$; then the following statements are equivalent:*

i) g is uniformly almost periodic;

ii) g is a continuous function, and for every $\eta > 0$ the set

$$T_\eta = \{ \ \tau \in \mathbf{R}^N \ : \ |g(x+\tau) - g(x)| < \eta \text{ for every } x \in \mathbf{R}^N \ \}$$

is relatively dense in $\mathbf{R}^N$(*);

iii) g is a continuous function and for every $y \in \mathbf{R}^N$ and $\eta > 0$ the set

$$T^y_\eta = \{\tau \in \mathbf{R} \ : \ |g(x+\tau y) - g(x)| < \eta \text{ for every } x \in \mathbf{R}^N\}$$

is relatively dense in $\mathbf{R}$.

Following [8] and [30], we can define a more general class of almost periodic functions, which includes u.a.p. functions, periodic measurable functions, and their perturbations with $L^1(\mathbf{R}^n)$ functions (more in general with L^1_{loc} functions with mean value zero on $\mathbf{R}^n$).

Definition 1.4. Let $(X, \| \ \|)$ be a complex Banach space. We say that a measurable function $v : \mathbf{R}^n \to X$ is (Besicovitch) *almost periodic*, and we write $v \in AP(\mathbf{R}^n, X)$, if it is the limit in mean value of a sequence of trigonometric polynomials on X; *i.e.,*

$$\lim_{h\to\infty} \fint \|P_h(y) - v(y)\| dy = 0 \tag{1.1}$$

for some

$$P_h(y) = \sum_{j=1}^{r_h} \mathbf{x}^h_j \ e^{i(\lambda^h_j, y)}, \tag{1.2}$$

(*) We say that a set $T \subset \mathbf{R}^N$ is *relatively dense* in $\mathbf{R}^N$ if there exists an *inclusion length* $L > 0$ such that $T + [0, L]^N = \mathbf{R}^N$; *i.e.,* for every $z \in \mathbf{R}^N$ there exists $\tau \in z + [0, L]^N$.

with $\mathbf{x}_j^h \in X$, $\lambda_j^h \in \mathbf{R}^n$, and $r_h \in \mathbf{R}$. The definition easily extends to real Banach spaces. In particular we can take $X = UAP_m$ (which is a Banach space if equipped with the uniform norm). If $X = \mathbf{R}$, this definition is the usual definition of Besicovitch almost periodic functions (see [8]).

Γ-*convergence*

Definition 1.5. (De Giorgi & Franzoni [22]) Let (X, τ) be a metric space (throughout the paper we shall consider always L^p spaces), and let $(F_i)_{i\in I}$ be a family of real functions defined on X, $I \subset]0, +\infty[$ with $0 \in \overline{I}$. Then for $x_0 \in X$, we define

$$\Gamma(\tau)\text{-}\liminf_{i\to 0} F_i(x_0) = \inf\{\liminf_{i\to 0} F_i(x_i) \ : \ x_i \stackrel{\tau}{\to} x_0\},$$

and

$$\Gamma(\tau)\text{-}\limsup_{i\to 0} F_i(x_0) = \inf\{\limsup_{i\to 0} F_i(x_i) \ : \ x_i \stackrel{\tau}{\to} x_0\};$$

if these two quantities coincide their common value will be called the Γ-*limit* of the sequence (F_i) in x_0, and will be denoted by

$$\Gamma(\tau)\text{-}\lim_{i\to 0} F_i(x_0).$$

It is easy to check that the following statements are equivalent:

i) $l = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(x_0)$;

ii) for every sequence of positive numbers (ε_h) converging to 0 there exists a subsequence $(\varepsilon_{h'})$ for which we have

$$l = \Gamma(\tau)\text{-}\lim_{h'\to 0} F_{\varepsilon_{h'}}(x_0);$$

iii) for every sequence of positive numbers (ε_h) converging to 0 we have

a) for every sequence (x_h) converging to x_0 we have

$$l \le \liminf_{h\to\infty} F_{\varepsilon_h}(x_h);$$

b) there exists a sequence (x_h) converging to x_0 such that

$$l \ge \limsup_{h\to\infty} F_{\varepsilon_h}(x_h).$$

Remark 1.6. The Γ-upper and lower limits defined above are τ-lower semicontinuous functions.

The value of the Γ-limit $\Gamma(\tau)\text{-}\lim_{i\to 0} F_i(x)$ does not change if we substitute F_i with its lower τ-semicontinuous envelope; *i.e.,* the greatest lower τ-semicontinuous function less than or equal to F_i.

The importance of the Γ-convergence in the calculus of variations is clearly described by the following theorem.

Theorem 1.7. (De Giorgi & Franzoni [22]) *Let us suppose that the* Γ*-limit*

$$\Gamma(\tau)\text{-}\lim_{i\to 0} F_i(x) = F_0(x)$$

exists for every $x \in X$*, and that* G *is a* τ*-continuous function defined on* X*. Then if the functions* $F_i + G$ *are equicoercive on* X*, we have*

$$\lim_{i\to 0} \inf_X (F_i + G) = \min_X (F_0 + G).$$

2. Preliminaries and statement of the main results

In the following we will consider functionals of the type

$$F(u,\Omega) = \int_\Omega f(x,u,Du)dx, \tag{2.1}$$

where $\Omega \in \mathcal{A}_n$ and $u \in \mathrm{H}^{1,p}(\Omega)$, and the integrand $f : \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n} \to \mathbf{R}$ is a Carathéodory function and satisfies the growth condition

$$|\xi|^p \le f(x,u,\xi) \le C(1+|\xi|^p) \tag{2.2}$$

for all $(x,u,\xi) \in \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n}$.

We will be concerned with Γ-limits of functionals of the form (2.1) in the strong topology of $\mathrm{L}^p(\Omega)$. Let us remark that, thanks to (2.2) this is equivalent to taking the Γ-limits in the weak topology of $\mathrm{H}^{1,p}(\Omega)$, and that in this topology these functionals are equicoercive. Moreover, we can specialize Theorem 1.7, taking into account boundary data as follows (see for example Fusco [25] Lemma 2.1).

Remark 2.1. If $(F_\varepsilon(\cdot,\Omega))$ is a sequence of functionals of the form (2.1) that Γ-converge to a functional $F(\cdot,\Omega)$ as $\varepsilon \to 0$, then we have

$$\min\{F(u,\Omega) \ : \ u-\varphi \in \mathrm{H}_0^{1,p}(\Omega)\} = \lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u,\Omega) \ : \ u-\varphi \in \mathrm{H}_0^{1,p}(\Omega)\}$$

for any $\varphi \in \mathrm{H}^{1,p}(\Omega)$.

Lower semicontinuous integral functionals on $\mathrm{H}^{1,p}(\Omega)$ of the form (2.1) are completely described by the following definition and the subsequent semicontinuity theorem.

Definition 2.2. (Morrey [37], Ball & Murat [6]) We will say that a continuous function $f : M^{m\times n} \to \mathbf{R}$ is *quasiconvex* if for every $\xi \in M^{m\times n}$, $\Omega \in \mathcal{A}_n$, $u \in \mathcal{C}_0^1(\Omega)$, we have

$$|\Omega| f(\xi) \le \int_\Omega f(\xi + Du)dx. \tag{2.3}$$

If f satisfies growth conditions as in (2.2), then (2.3) is satisfied for all $u \in \mathrm{H}_0^{1,p}(\Omega)$.

An equivalent definition of quasiconvexity is given by Ball, Currie & Olver [5] using u.a.p. functions: f is quasiconvex if

$$f\left(\fint Du dx\right) \le \fint f(Du)dx$$

for any $u \in \mathcal{C}^1(\mathbf{R}^n;\mathbf{R}^m)$ such that $Du \in UAP_{mn}$.

Theorem 2.3. (see Morrey [37], Acerbi & Fusco [2], Dacorogna [18]) *Let us consider a Carathéodory function* $f : \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n} \to \mathbf{R}$ *satisfying (2.2). A necessary and sufficient condition for the functional* $F(\cdot,\Omega)$ *in (2.1) to be sequentially lower semicontinuous in the weak topology of* $\mathrm{H}^{1,p}(\Omega)$ *is the function* f *to be quasiconvex in the last variable. More precisely, the relaxed functional of* F*, i.e., the greatest sequentially lower semicontinuous (integral) functional less than or equal to* F*, in the weak topology of* $\mathrm{H}^{1,p}(\Omega)$*, is*

$$\overline{F}(u,\Omega) = \int_\Omega Qf(x,u,Du)dx, \tag{2.4}$$

where the Carathéodory function Qf *is the quasiconvex envelope of* f*, given by*

$$Qf(x,u,\xi) = \inf\Big\{\frac{1}{|A|}\int_A f(x,u,Dw(y)+\xi)dy \ : \ w \in \mathrm{H}^{1,p}_0(A)\Big\}. \tag{2.5}$$

This formula is independent of $A \in \mathcal{A}_n$*. Let us remark that thanks to the lower semicontinuity and coerciveness of* $\overline{F}$*, we have*

$$\inf\{F(u,\Omega) \ : \ u-\varphi \in \mathrm{H}^{1,p}_0(\Omega)\} = \min\{\overline{F}(u,\Omega) \ : \ u-\varphi \in \mathrm{H}^{1,p}_0(\Omega)\} \tag{2.6}$$

for all $\varphi \in \mathrm{H}^{1,p}(\Omega)$*.*

We can state now the main result of the paper.

Theorem 2.4. *Let* $f : \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n} \to \mathbf{R}$ *satisfy:*

(i) for every $(x,u,\xi) \in \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n}$

$$|\xi|^p \le f(x,u,\xi) \le C(1+|\xi|^p);$$

(ii) for every $\xi \in M^{m\times n}$ *the function* $x \mapsto f(x,\cdot,\xi)$ *belongs to* $AP(\mathbf{R}^n, UAP_m)$ *(as in Definition 1.4);*

(iii) for every $(x,u) \in \mathbf{R}^n \times \mathbf{R}^m$ *the function* $f(x,u,\cdot)$ *is quasiconvex.*

Then there exists a quasiconvex function $\overline{f} : M^{m\times n} \to \mathbf{R}$ *such that for every bounded open subset* Ω *of* $\mathbf{R}^n$ *and every* $u \in \mathrm{H}^{1,p}(\Omega)$ *the limit*

$$\Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{\varepsilon\to 0}\int_\Omega f\Big(\frac{x}{\varepsilon},\frac{u(x)}{\varepsilon},Du(x)\Big)dx = \int_\Omega \overline{f}(Du(x))dx \tag{2.7}$$

exists, and the function $\overline{f}$ *satisfies*

$$\overline{f}(\xi) = \lim_{t\to\infty}\inf\Big\{\frac{1}{t^n}\int_{]0,t[^n} f(x,u(x)+\xi x,Du(x)+\xi)dx \ : \ u \in \mathrm{H}^{1,p}_0(]0,t[^n)\Big\}. \tag{2.8}$$

Remark 2.5. 1) Theorem 2.4 can be easily extended, with standard techniques, to the limit of functionals of the type

$$\int_\Omega f(x, \frac{x}{\varepsilon}, u, \frac{u}{\varepsilon}, Du)dx,$$

under suitable continuity conditions on the non oscillating variables. We refer to Buttazzo & Dal Maso [16], Fusco [25], or to E [23] Corollary 6.1 for a proof.

2) The function $\overline{f}$ may turn out to be less regular than f, as seen by Buttazzo & Dal Maso [15] Section 4 (see also E [23]), where it is considered the case $f(u,\xi) = |\xi|^\alpha + g(u)$. Acerbi & Buttazzo in [1] Chapter 4, show that even if $f(u,\xi)$ is a quadratic form in ξ, $\overline{f}$ is in general not quadratic (while Boccardo & Murat [9] prove that the G-limit of operators of the type $-\mathrm{div}\big(b(\frac{u}{\varepsilon})Du\big)$ is still of the form $-\tilde{b}\triangle u$).

Remark 2.6. The condition (ii) is easily verified in some important cases:

a) if $x \mapsto f(x,\cdot,\xi)$ is periodic with values in UAP_m;

b) if $(x,u) \mapsto f(x,u,\xi)$ is a function of UAP_{n+m};

c) if $f(x,u,\xi) = a(x)g(u,\xi)$ or $f(x,u,\xi) = b(u)h(x,\xi)$, with the functions a and $h(\cdot,\xi)$ in $AP(\mathbf{R}^n,\mathbf{R})$, and b and $g(\cdot,\xi)$ in UAP_m, for every ξ.

The following corollary to Theorem 2.4 allows us to drop the quasiconvexity assumption in two important cases.

Proposition 2.7. *Let f satisfy condition (i) of Theorem 2.4 and one of the following assumptions:*

(iia) $f(\cdot,\cdot,\xi)$ *is periodic, with periods independent of* ξ*;*

(iib) $f(\cdot,\cdot,\xi)$ *is quasiperiodic; i.e.,* $f(x,u,\xi) = \mathcal{F}(x,\dots,x,u,\dots,u,\xi)$, *where* $\mathcal{F}$ *il a continuous periodic function of a larger number of variables, with periods independent of* ξ.

Then the thesis of Theorem 2.4 still holds.

Proof. It suffices to remark that the operation of quasiconvexification (2.5) mantains both periodicity and quasiperiodicity, and apply Theorem 2.4 to the quasiconvex envelope Qf of f. The formula for $\overline{f}$ does not change by Remark 1.6 and Theorem 2.3. □

Remark 2.8. Formula (2.8) is a straightforward consequence of (2.7). In fact, by Remark 2.1 we obtain, taking $\Omega =]0,1[^n$ and $\varphi(x) = \xi x$,

$$\min\Big\{ \int_{]0,1[^n} \overline{f}(\xi + Du)dx \ : \ u \in \mathrm{H}_0^{1,p}(]0,1[^n) \Big\}$$

$$= \lim_{\varepsilon\to 0} \inf\Big\{ \int_{]0,1[^n} f\Big(\frac{x}{\varepsilon}, \frac{\xi x + u}{\varepsilon}, \xi + Du\Big)dx \ : \ u \in \mathrm{H}_0^{1,p}(]0,1[^n) \Big\}.$$

By the quasiconvexity of $\overline{f}$, the left-hand side is equal to $\overline{f}(\xi)$, while, making a change of variables in the right-hand side, we obtain

$$\overline{f}(\xi) = \lim_{\varepsilon\to 0} \inf\Big\{\varepsilon^n \int_{]0,\frac{1}{\varepsilon}[^n} f(x, \xi x + u, \xi + Du)dx \ : \ u \in \mathrm{H}^{1,p}_0(]0, \frac{1}{\varepsilon}[^n)\Big\},$$

and hence (2.8).

We cannot expect to obtain in general for the function $\overline{f}$ a homogenization formula involving a single minimization problem even when $f = f(x, \xi)$ is purely periodic in the first variables, as shown by a counterexample of S.Müller [38]. This is possible however in some special case; *e.g.*, in the scalar case $m = 1$ and $f = f(x, \xi)$ (see for example Marcellini [35]), or when $m = 1$ and $f = f(u, |\xi|)$ (see Buttazzo & Dal Maso [15]).

We end this section with two technical results that will be needed in Section 4.

Remark 2.9. If $F(\cdot, \Omega)$ is defined as in (2.1), we can define, for $\lambda > 0$, the λ-*Moreau-Yosida transform* of F, as

$$T_\lambda F(u, \Omega) = \inf\{F(u + v, \Omega) + \lambda \|v\|^p_{\mathrm{L}^p(\Omega)} \ : \ v \in \mathrm{H}^{1,p}_0(\Omega)\}.$$

Then if $(F_i)_{i\in I}$ is a family of functionals of the type (2.1) (I as in Definition 1.5), we have (see [22])

$$\Gamma(\mathrm{L}^p(\Omega))\text{-}\liminf_{i\to 0} F_i(u, \Omega) = \lim_{\lambda\to\infty} \liminf_{i\to 0} T_\lambda F_i(u, \Omega),$$

$$\Gamma(\mathrm{L}^p(\Omega))\text{-}\limsup_{i\to 0} F_i(u, \Omega) = \lim_{\lambda\to\infty} \limsup_{i\to 0} T_\lambda F_i(u, \Omega).$$

We will need also the following Meyers type theorem for minima of integral functionals (Giaquinta & Giusti [26]).

Theorem 2.10. *Let $f : \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n} \to \mathbf{R}$ be a Carathéodory function satisfying (2.2), quasiconvex in the last variable, $\Omega \in \mathcal{A}_n$ and $\overline{u} \in \mathcal{C}^1(\overline{\Omega})$. Then there exists $\eta > 0$ depending only on Ω, $\overline{u}$, p and the constant C in (2.2), such that for every $\lambda > 0$ every minimum point u_λ of the problem*

$$\min\Big\{\int_\Omega (f(x, \overline{u} + v, D\overline{u} + Dv) + \lambda|v|^p)dx \ : \ v \in \mathrm{H}^{1,p}_0(\Omega)\Big\}$$

is in $H^{1,p+\eta}(\Omega)$. Moreover, there exists a constant $C(\lambda)$ depending only on λ, Ω, $\overline{u}$, p and the constant C in (2.2) such that

$$\|u_\lambda\|_{\mathrm{H}^{1,p+\eta}(\Omega)} \le C(\lambda).$$

3. Proof of the homogenization theorem

In this section we prove the homogenization theorem with a hypothesis of uniform almost periodicity on the function f, which already covers some interesting cases; for example

$$f(u, Du) = \sum_{i,j=1}^{n} \sum_{\alpha\beta}^{m} a_{ij}^{\alpha\beta}(u) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j},$$

with $a_{ij}^{\alpha\beta}$ u.a.p. and satisfying (2.2). The next section will deal with the extension to a more general class of integrands.

In view of the characterization of uniformly almost periodic functions in Theorem 1.3 and the growth hypothesis (2.2), we will make the following assumption on the function f: for every $\zeta \in M^{m\times n}$ and $\eta > 0$ the sets

$$\text{(3.1)} \qquad \begin{aligned} T_\eta^\zeta &= \{\tau \in \mathbf{R}^n \ : |f(x+\tau, u+\zeta\tau, \xi) - f(x,u,\xi)| < \eta(1+|\xi|^p) \\ &\qquad \text{for every } (x,u,\xi) \in \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n}\} \\ T_\eta^0 &= \{\tau \in \mathbf{R}^m \ : |f(x, u+\tau, \xi) - f(x,u,\xi)| < \eta(1+|\xi|^p) \\ &\qquad \text{for every } (x,u,\xi) \in \mathbf{R}^n \times \mathbf{R}^m \times M^{m\times n}\} \end{aligned}$$

are relatively dense in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively.

We can now proceed in the proof of the homogenization theorem for function f satisfying (2.2) and (3.1) (no hypothesis of quasiconvexity is needed in this case). Following a procedure which is typical of Γ-convergence, we first prove a compactness result.

Proposition 3.1. *For every sequence (ε_h) of positive real numbers converging to 0, there exists a subsequence $(\varepsilon_{h'})$, and a quasiconvex function $\varphi : M^{m\times n} \to \mathbf{R}$ such that for every $\Omega \in \mathcal{A}_n$ and every $u \in \mathrm{H}^{1,p}(\Omega)$, there exists the Γ-limit*

$$\text{(3.2)} \qquad \Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u(x)}{\varepsilon_{h'}}, Du(x)\Big) dx = \int_\Omega \varphi(Du(x)) dx.$$

Proof. By a standard compactness argument (see Buttazzo & Dal Maso [16], Fusco [25] Lemma 2.1, or E [23] Theorem 3.1), we obtain the existence of a subsequence $(\varepsilon_{h'})$ such that the limit

$$F(u, \Omega) = \Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u(x)}{\varepsilon_{h'}}, Du(x)\Big) dx$$

exists for every $\Omega \in \mathcal{A}_n$ and every $u \in \mathrm{H}^{1,p}(\Omega)$, and satisfies:

(i) for every $\Omega \in \mathcal{A}_n$ and every $u \in \mathrm{H}^{1,p}(\Omega)$

$$\int_\Omega |Du|^p dx \le F(u, \Omega) \le C \int_\Omega (1 + |Du|^p) dx.$$

(ii) F is local; *i.e.,* $F(u, \Omega) = F(v, \Omega)$, whenever $\Omega \in \mathcal{A}_n$ and $u = v$ a.e. on Ω;

(iii) for every fixed $u \in \mathrm{H}^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$ the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}_n$.

Moreover:

(iv) for every $\Omega \in \mathcal{A}_n$, $u \in \mathrm{H}^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$, $a \in \mathbf{R}^m$,

$$F(u+a,\Omega) = F(u,\Omega).$$

In fact, let $u_{h'} \rightharpoonup u$ be a sequence in $\mathrm{H}^{1,p}(\Omega)$ such that

$$F(u,\Omega) = \lim_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u_{h'}(x)}{\varepsilon_{h'}}, Du(x)\Big)dx.$$

Fixed $\eta > 0$, let $a_{h'} \in \mathbf{R}^m$ such that $a_{h'} \to a$ and $\tau_{h'} = a_{h'}/\varepsilon_{h'} \in T^0_\eta$; *i.e.*,

$$|f(x, u+\tau_{h'}, \xi) - f(x,u,\xi)| \le \eta(1+|\xi|^p)$$

for every $x \in \mathbf{R}^n, u \in \mathbf{R}^m, \xi \in M^{m\times n}$. Then we have

$$\begin{aligned}
F(u+a,\Omega) \le\ & \liminf_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u_{h'}(x)+a_{h'}}{\varepsilon_{h'}}, Du(x)\Big)dx \\
=\ & \liminf_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u_{h'}(x)}{\varepsilon_{h'}} + \tau_{h'}, Du(x)\Big)dx \\
=\ & \lim_{h'\to\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_{h'}}, \frac{u_{h'}(x)}{\varepsilon_{h'}}, Du(x)\Big)dx + \eta \liminf_{h'\to\infty}(|\Omega| + \|Du_{h'}\|^p_{\mathrm{L}^p(\Omega)}) \\
\le\ & F(u,\Omega) + \eta(|\Omega| + \sup_{h'} \|Du_{h'}\|^p_{\mathrm{L}^p(\Omega)}).
\end{aligned}$$

By the arbitrariness of η, we have $F(u+a,\Omega) \le F(u,\Omega)$. In the same way we prove the opposite inequality.

By (i)–(iv) and the lower semicontinuity of the Γ-limit, we can apply well-known integral representation theorems (for example Theorem 4.3.2 of Buttazzo [14]), to obtain the existence of a quasiconvex function $\varphi : \mathbf{R}^n \times M^{m\times n} \to \mathbf{R}$ such that

$$F(u,\Omega) = \int_\Omega \varphi(x, Du(x))dx,$$

for every $\Omega \in \mathcal{A}_n$ and $u \in \mathrm{H}^{1,p}(\Omega)$. In order to complete the proof we have to show that the function φ can be chosen independent of the variable x. It suffices to prove that for every open ball B in $\mathbf{R}^n$, $\xi \in M^{m\times n}$, and $a \in \mathbf{R}^n$, we have $F(\xi x, B) = F(\xi x, B+a)$. This can be proven similarly to (iv) above (see also Braides [10] Proposition 5.1). □

Arguing as in Remark 2.8, we obtain for the function φ in (3.2) the formula

$$\varphi(\xi) = \lim_{h'\to\infty} \inf\{\varepsilon^n_{h'} \int_{]0,1/\varepsilon_{h'}[^n} f(x, u(x)+\xi x, Du(x)+\xi)dx \ :\ u \in \mathrm{H}^{1,p}_0(]0,1/\varepsilon_{h'}[^n)\}.$$

The proof of Theorem 2.4, for f satisfying (3.1) and (2.2), will be then completed by ii) of Definition 1.5 and by the following proposition, which shows that the function φ is independent of the subsequence $(\varepsilon_{h'})$.

Proposition 3.2. *For every $\xi \in M^{m\times n}$ there exists the limit*

$$\overline{f}(\xi) = \lim_{t\to\infty} \inf\Big\{\frac{1}{t^n}\int\limits_{]0,t[^n} f(x, u(x)+\xi x, Du(x)+\xi)dx \;:\; u \in \mathrm{H}^{1,p}_0(]0,t[^n)\Big\}. \tag{3.3}$$

Proof. The matrix $\zeta \in M^{m\times n}$ will remain fixed throughout the proof. Let us define, for every $t > 0$, the quantity

$$g_t = g_t(\zeta) = \inf\Big\{\frac{1}{t^n}\int\limits_{]0,t[^n} f(x, u(x)+\zeta x, Du(x)+\zeta)dx \;:\; u \in \mathrm{H}^{1,p}_0(]0,t[^n)\Big\}.$$

Fixed $t > 0$, let $u_t \in \mathrm{H}^{1,p}_0(]0,t[^n)$ be such that

$$\frac{1}{t^n}\int\limits_{]0,t[^n} f(x, u_t(x)+\zeta x, Du_t(x)+\zeta)dx \le g_t + \frac{1}{t}.$$

We want to estimate g_s, for $s > t$, in terms of g_t. To such purpose, we will construct $u_s \in \mathrm{H}^{1,p}_0(]0,s[^n)$, by a patchwork procedure, exploiting the uniform almost periodicity of the function f.

Fixed $\eta > 0$, let T^ζ_η be as in (3.1), and let L_η be the inclusion length related to T^ζ_η. If $s \ge t + L_\eta$, let I_s be the set of all $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{Z}^n$ such that

$$0 \le z_j \le \frac{s}{t+L_\eta} - 1 \qquad j = 1, \ldots, n,$$

and, for every $\mathbf{z} \in I_s$, let

$$\tau_{\mathbf{z}} \in \big((t+L_\eta)\mathbf{z} + [0, L_\eta]^n\big) \cap T^\zeta_\eta;$$

when $s < t + L_\eta$ we set $I_s = 0$ and we choose $\tau_{\mathbf{z}} = 0$.

If $s > t$, let us define

$$u_s(x) = \begin{cases} u_t(x-\tau_{\mathbf{z}}) & \text{if } x \in \tau_{\mathbf{z}}+]0,t[^n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_s =]0,s[^n \setminus \bigcup_{\mathbf{z}\in I_s} (\tau_{\mathbf{z}}+]0,t[^n).$$

If $s \ge t + L_\eta$, then we have $|Q_s| = s^n - \Big([\frac{s}{t+L_\eta}] - 1\Big)^n t^n < s^n\Big(1 - \big(\frac{t}{t+L_\eta}\big)^n\Big)$(*).

(*) If $a \in \mathbf{R}$, then $[a] \in \mathbf{Z}$ is the *integral part* of a.

We can now estimate g_s $(s \geq t + L_\eta)$:

$$
\begin{aligned}
g_s \leq & \frac{1}{s^n} \int\limits_{]0,s[^n} f(x, u_s(x) + \zeta x, Du_s(x) + \zeta) dx \\
= & \frac{1}{s^n} \Bigg(\sum_{\mathbf{z} \in I_s} \int\limits_{\tau_{\mathbf{z}} +]0,t[^n} f(x, u_t(x - \tau_{\mathbf{z}}) + \zeta x, Du_t(x - \tau_{\mathbf{z}}) + \zeta) dx + \int\limits_{Q_s} f(x, \zeta x, \zeta) dx \Bigg) \\
= & \frac{1}{s^n} \Bigg(\sum_{\mathbf{z} \in I_s} \int\limits_{]0,t[^n} f(x, u_t(x) + \zeta x + \zeta \tau_{\mathbf{z}}, Du_t(x) + \zeta) dx + \int\limits_{Q_s} f(x, \zeta x, \zeta) dx \Bigg) \\
= & \frac{1}{s^n} \Bigg(\sum_{\mathbf{z} \in I_s} \int\limits_{]0,t[^n} \big(f(x, u_t(x) + \zeta x + \zeta \tau_{\mathbf{z}}, Du_t(x) + \zeta) - f(x, u_t(x) + \zeta x, Du_t(x) + \zeta) \big) dx \\
& \quad + \Big([\frac{s}{t + L_\eta}] - 1 \Big)^n \int\limits_{]0,t[^n} f(x, u_t(x) + \zeta x, Du_t(x) + \zeta) dx + \int\limits_{Q_s} f(x, \zeta x, \zeta) dx \Bigg) \\
\leq & \frac{1}{s^n} \Bigg(\sum_{\mathbf{z} \in I_s} \eta \int\limits_{]0,t[^n} \big(1 + |Du_t|^p\big) dx + \Big([\frac{s}{t + L_\eta}] - 1 \Big)^n t^n (g_t + \frac{1}{t}) + C |Q_s| (1 + |\zeta|^p) \Bigg) \\
\leq & \eta \big(\frac{t}{t + L_\eta} \big)^n (1 + g_t + \frac{1}{t}) + \big(\frac{t}{t + L_\eta} \big)^n (g_t + \frac{1}{t}) + C \Big(1 - \big(\frac{t}{t + L_\eta} \big)^n \Big) (1 + |\zeta|^p).
\end{aligned}
$$

We have thus

$$
g_s \leq g_t \big(\frac{t}{t + L_\eta} \big)^n (1 + \eta) + \big(\frac{t}{t + L_\eta} \big)^n \big(\eta (1 + \frac{1}{t}) + \frac{1}{t} \big) + C \Big(1 - \big(\frac{t}{t + L_\eta} \big)^n \Big) (1 + |\zeta|^p).
$$

Taking the limit, first as $s \to \infty$ and then as $t \to \infty$, we obtain

$$
\limsup_{s \to \infty} g_s \leq (1 + \eta) \liminf_{t \to \infty} g_t + \eta.
$$

By the arbitrariness of η we have the desired result. □

4. Generalization to almost periodic functionals

The proof of Theorem 2.4 follows an approximation argument, already used by Braides [11], in view of the following closure lemma for the homogenization.

Lemma 4.1. *Let f and g_h $(h \in \mathbf{N})$ satisfy (2.2), and for every $R \geq 0$*

$$
\lim_{h \to \infty} ⨍ \sup_{|\xi| \leq R} \| f(x, \cdot, \xi) - g_h(x, \cdot, \xi) \|_\infty dx = 0. \tag{4.1}
$$

If for every bounded open subset Ω of $\mathbf{R}^n$ and every $u \in \mathrm{H}^{1,p}(\Omega)$ the limit

$$
G_h(u, \Omega) = \Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{\varepsilon \to 0} \int\limits_{\Omega} g_h \Big(\frac{x}{\varepsilon}, \frac{u(x)}{\varepsilon}, Du(x) \Big) dx \tag{4.2}
$$

exists, then there exists also the limit

$$F(u,\Omega) = \Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{\varepsilon\to 0}\int_\Omega f\Big(\frac{x}{\varepsilon},\frac{u(x)}{\varepsilon},Du(x)\Big)dx, \tag{4.3}$$

and we have

$$F(u,\Omega) = \Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{h\to\infty} G_h(u,\Omega) \tag{4.4}$$

for every bounded open subset Ω *of* $\mathbf{R}^n$*, and every* $u\in\mathcal{C}^1(\overline{\Omega})$*.*

Remark. Taking $g = g_h$, this lemma gives a "stability" result for the homogenization, under "small" perturbation, in the sense that, under the growth hypothesis (2.2), if $f(x,u,\xi)$ is homogenizable, then so is also the function $f(x,u,\xi)+g(x,u,\xi)$ whenever we have $⨍\|g(x,\cdot,\xi)\|_\infty dx = 0$ for all $\xi\in M^{m\times n}$ (for example when g is of compact support in x, or $x\mapsto\|g(x,\cdot,\xi)\|_\infty$ belongs to $\mathrm{L}^1(\mathbf{R}^n)$.

Proof of Lemma 4.1. We just give a sketch of the proof, since it follows closely the proof of Lemma 3.4 in [11].

Step 1: we observe that, fixed $\lambda>0$ and $u\in\mathcal{C}^1(\overline{\Omega})$, by Theorem 2.10 we have uniform Meyers estimates for the minimizers of the problems

$$T_\lambda G^h_\varepsilon(u,\Omega) = \min\{\int_\Omega g_h(\frac{x}{\varepsilon},\frac{u+w}{\varepsilon},Du+Dw)dx + \lambda\int_\Omega |w|^p dx : w\in\mathrm{H}^{1,p}_0(\Omega)\},$$

$$T_\lambda F_\varepsilon(u,\Omega) = \min\{\int_\Omega f(\frac{x}{\varepsilon},\frac{u+w}{\varepsilon},Du+Dw)dx + \lambda\int_\Omega |w|^p dx : w\in\mathrm{H}^{1,p}_0(\Omega)\},$$

thanks to the estimate (2.2). The functionals $T_\lambda G^h_\varepsilon$ and $T_\lambda F_\varepsilon$ are the Moreau-Yosida transforms of the functionals $\int_\Omega g_h(\frac{x}{\varepsilon},\frac{u}{\varepsilon},Du)dx$ and $\int_\Omega f(\frac{x}{\varepsilon},\frac{u}{\varepsilon},Du)dx$ as in 2.9.

Step 2: using Step 1 and the condition (4.1), we can pass to the limit, obtaining

$$\lim_{\lambda\to\infty}\liminf_{h\to\infty}\limsup_{\varepsilon\to 0} T_\lambda G^h_\varepsilon(u,\Omega) = \lim_{\lambda\to\infty}\limsup_{\varepsilon\to 0} T_\lambda F_\varepsilon(u,\Omega)$$

and

$$\lim_{\lambda\to\infty}\limsup_{h\to\infty}\liminf_{\varepsilon\to 0} T_\lambda G^h_\varepsilon(u,\Omega) = \lim_{\lambda\to\infty}\liminf_{\varepsilon\to 0} T_\lambda F_\varepsilon(u,\Omega).$$

The proof repeats word by word the argument of [11], replacing the condition

$$\lim_{h\to\infty}⨍\sup_{|\xi|\le R}|f(x,\xi)-g_h(x,\xi)|dx = 0$$

with (4.1).

Step 3: by Theorem 1.7, we can pass to the limit on the right-hand sides as $\varepsilon\to 0$ obtaining

$$\lim_{\lambda\to\infty}\liminf_{h\to\infty} T_\lambda G^h(u,\Omega) = \lim_{\lambda\to\infty}\limsup_{\varepsilon\to 0} T_\lambda F_\varepsilon(u,\Omega)$$

and

$$\lim_{\lambda\to\infty}\limsup_{h\to\infty} T_\lambda G^h(u,\Omega) = \lim_{\lambda\to\infty}\liminf_{\varepsilon\to 0} T_\lambda F_\varepsilon(u,\Omega).$$

By Remark 2.9 this implies (4.4) passing to the limit. □

Proposition 4.2. *Let f and (g_h) satisfy (4.1) and (4.2). Then there exists a (quasiconvex) function $\overline{f}: M^{m\times n}\to\mathbf{R}$ such that for every bounded open subset Ω of $\mathbf{R}^n$ and every $u\in \mathrm{H}^{1,p}(\Omega)$ the limit*

$$\Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{\varepsilon\to 0}\int_\Omega f\Big(\frac{x}{\varepsilon},\frac{u(x)}{\varepsilon},Du(x)\Big)dx=\int_\Omega \overline{f}(Du(x))dx \tag{4.5}$$

exists, and the function $\overline{f}$ satisfies

$$\overline{f}(\xi)=\lim_{t\to\infty}\inf\Big\{\frac{1}{t^n}\int_{]0,t[^n} f(x,u(x)+\xi x,Du(x)+\xi)dx\ :\ u\in \mathrm{H}^{1,p}_0(]0,t[^n)\Big\}. \tag{4.6}$$

Proof. As in the proof of Proposition 3.1, for every sequence (ε_h) converging to 0, there exists a subsequence $(\varepsilon_{h'})$ such that the limit

$$F(u,\Omega)=\Gamma(\mathrm{L}^p(\Omega))\text{-}\lim_{h'\to\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_{h'}},\frac{u(x)}{\varepsilon_{h'}},Du(x)\Big)dx$$

exists for every $\Omega\in\mathcal{A}_n$ and every $u\in \mathrm{H}^{1,p}(\Omega)$, and, fixed $\Omega\in\mathcal{A}_n$, we have:

(i) for every $B\in\mathcal{B}(\Omega)$ and every $u\in \mathrm{H}^{1,p}(\Omega)$

$$\int_B |Du|^p dx\le F(u,B)\le C\int_B (1+|Du|^p)dx;$$

(ii) F is local on $\mathcal{A}(\Omega)$;

(iii) for every fixed $u\in \mathrm{H}^{1,p}(\Omega)$ the set function $F(u,\cdot)$ is the restriction of a Borel measure to $\mathcal{A}_n$;

(iv) for every $A\in\mathcal{A}(\Omega)$, $F(\cdot,A)$ is weakly lower semicontinuous in $\mathrm{H}^{1,p}(\Omega)$.

Then F is also local on $\mathcal{B}(\Omega)$ (see Buttazzo & Dal Maso [17]); a well-known result (see Liu [34] or Federer [24] Theorem 3.1.16) implies then that F is determined by its behaviour on $\mathcal{C}^1(\overline{\Omega})$. Hence, by (4.3), we have that the Γ-limit is independent of the subsequence $(\varepsilon_{h'})$. From (4.4) we deduce also that

$$F(u,\Omega)=F(u+a,\Omega)\qquad\text{for every } a\in\mathbf{R}^m,$$

for all $u\in \mathrm{H}^{1,p}(\Omega)$, and

$$F(\xi x,\Omega)=F(\xi x,\Omega+b)\qquad\text{for every } b\in\mathbf{R}^n,$$

for all $\xi\in M^{m\times n}$.

Hence we can apply the representation result 4.3.2 of Buttazzo [14], and obtain (4.5). The formula (4.6) follows from Remark 2.8. □

The proof of Theorem 2.4 will be then completed if we exhibit a sequence (g_h) satisfying (4.1) and the hypotheses (3.1) of Section 3, in order to obtain (4.2).

We follow the construction of [11]. Let us first remark that by Definition 1.4 and the definition of UAP_m, we have that for every $\xi \in M^{m\times n}$ there exists a sequence of trigonometric polynomials P_h^ξ on $\mathbf{R}^n \times \mathbf{R}^m$ such that

$$\fint \|P_h^\xi(x,\cdot) - f(x,\cdot,\xi)\|_\infty dx \to 0,$$

as $h \to \infty$.

Let (ξ_h) be a dense sequence in $M^{m\times n}$. For every $h \in \mathbf{N}$ let us choose trigonometric polynomials $P_1^h, \dots, P_h^h$ such that

$$\fint \|P_j^h(x,\cdot) - f(x,\cdot,\xi_j)\|_\infty dx \le \frac{1}{h^2}. \tag{4.7}$$

Define then

$$f_h(x,u,\xi) = \begin{cases} f(x,u,\xi) & \xi = \xi_j \quad j = 1,\dots,h \\ C(1+|\xi|^p) & \text{otherwise,} \end{cases}$$

(4.8)

$$\varphi_h(x,u,\xi) = \begin{cases} C(1+|\xi_j|^p) \wedge (P_j^h(x,u) \vee |\xi_j|^p) & \xi = \xi_j \quad j = 1,\dots,h \\ C(1+|\xi|^p) & \text{otherwise,} \end{cases}$$

and Qf_h, $Q\varphi_h$ using formula (2.5) We obtain then easily the following proposition.

Proposition 4.3. *The functions Qf_h and $Q\varphi_h$ are quasiconvex and satisfy (3.1). Moreover*

$$Qf_h(x,u,\xi_j) = f(x,u,\xi_j) \tag{4.9}$$

for $j = 1,\dots,h$.

Proof. The proof follows directly from (2.5) as in Proposition 4.3, Remark 4.6 and Remark 4.7 of [11]. □

Proposition 4.4. *The functions $g_h = Q\varphi_h$ satisfy the hypotheses of Proposition 4.2.*

Proof. By Proposition 4.3, we have only to prove that (4.1) holds. Let us first show that

$$\lim_{h\to\infty} \fint \sup_{|\xi|\le R} \|f(x,\cdot,\xi) - Qf_h(x,\cdot,\xi)\|_\infty dx = 0. \tag{4.10}$$

Now, since quasiconvex functions satisfying (2.2) are locally uniformly equi-Lipschitz in the last variable (see for example Fusco [25] Lemma 1.2, or Marcellini [36]), for every $R > 0$ there exists $L > 0$, depending only on the constant C in (2.2) and p, such that

$$\begin{gathered} |f(x,u,\xi) - f(x,u,\xi')| \le L|\xi - \xi'| \\ |Qf_h(x,u,\xi) - Qf_h(x,u,\xi')| \le L|\xi - \xi'| \end{gathered} \tag{4.11}$$

for all $(x,u) \in \mathbf{R}^n \times \mathbf{R}^m$ and $\xi, \xi' \in M^{m\times n}$ with $|\xi|, |\xi'| \le R$, and for every $h \in \mathbf{N}$. Using (4.11), and (4.9) we get

$$\begin{aligned}|f(x,u,\xi) - Qf_h(x,u,\xi)| &= |f(x,u,\xi) - f(x,u,\xi_j) + Qf_h(x,u,\xi_j) - Qf_h(x,u,\xi)| \\ &\le 2L|\xi - \xi_j|,\end{aligned}$$

where $j \in 1,\dots,h$ and $|\xi_j| \le R$. Hence

$$\sup_{|\xi|\le R} \|f(\cdot,\cdot,\xi) - Qf_h(\cdot,\cdot,\xi)\|_\infty \le 2L \sup_{|\xi|\le R} \Big(\inf\{|\xi - \xi_j| \ : \ j \in 1,\dots,h, \ |\xi_j| \le R\}\Big).$$

By the density of (ξ_h) the right-hand side tends to 0 as $h \to \infty$, and we obtain (4.10).

Let us now prove that

$$\lim_{h\to\infty} ⨍ \sup_{|\xi|\le R} \|Q\varphi_h(x,\cdot,\xi) - Qf_h(x,\cdot,\xi)\|_\infty dx = 0. \tag{4.12}$$

Using (2.5) it is easy to see that we have

$$\begin{aligned}|Q\varphi_h(x,u,\xi) - Qf_h(x,u,\xi)| &\le \sum_{j=1}^h |\varphi_h(x,u,\xi_j) - f_h(x,u,\xi_j)| \\ &= \sum_{j=1}^h |P_j^h(x,u) - f_h(x,u,\xi_j)|.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&⨍ \sup_{|\xi|\le R} \|Q\varphi_h(x,\cdot,\xi) - Qf_h(x,\cdot,\xi)\|_\infty dx \\ &\le \sum_{j=1}^h ⨍ \|P_j^h(x,\cdot) - f_h(x,\cdot,\xi_j)\|_\infty dx,\end{aligned}$$

and (4.12), by (4.7).

Eventually, (4.1) follows from (4.10) and (4.12). □

5. Homogenization of Hamilton-Jacobi equations

As pointed out by P.L.Lions [32], P.L.Lions, Papanicolaou & Varadhan [33], and E [23], we can derive from Theorem 2.4 a homogenization result for Hamilton-Jacobi equations, via Γ-convergence of functionals related to the Legendre transforms of the Hamiltonian. More precisely, let us suppose that $H : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is a continuous function satisfying:

i) $H(t,x,\cdot)$ is convex for every (t,x);

ii) there exists $1 < p < +\infty$, and $C > 0$ such that

$$|\xi|^p \le H(t,x,\xi) \le C(1 + |\xi|^p)$$

for every (t,x,ξ);

iii) the function $t \mapsto H(t, \cdot, \xi)$ belongs to $AP(\mathbf{R}, UAP_n)$.
We shall study the limiting behaviour of the viscosity solutions of the Cauchy problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + H(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, Du_\varepsilon) = 0 & \text{in } \mathbf{R}^n \times [0, +\infty[\\ u_\varepsilon(x, 0) = \varphi(x) & \text{in } \mathbf{R}^n, \end{cases} \tag{5.1}$$

where φ is a given bounded and uniformly continuous function in $\mathbf{R}^n$.

Let us define the Legendre transform of H:

$$L(t, u, \xi) = \sup_{\xi' \in \mathbf{R}^n} \{(\xi, \xi') - H(t, u, \xi')\},$$

for every (t, u, ξ). Then we have the following proposition.

Proposition 5.1. *For every $\xi \in \mathbf{R}^n$ there exists the limit*

$$\overline{L}(\xi) = \lim_{T \to \infty} \frac{1}{T} \inf \left\{ \int_0^T L(\tau, u(\tau) + \xi\tau, u'(\tau) + \xi) d\tau \ : \ u \in \mathrm{H}_0^{1,q}(0, T) \right\}. \tag{5.2}$$

Proof. By the growth hypothesis *ii*), if we set

$$C_1 = (p-1)\frac{p^{-q}}{C^{\frac{p}{q}}} \qquad C_2 = \frac{p-1}{p^q},$$

we have that for every (t, u, ξ)

$$C_1|\xi|^q - C \leq L(t, u, \xi) \leq C_2|\xi|^q.$$

The existence of the limit in (5.2) follows then by (2.8) in Theorem 2.4, where $f = L(t, u, \xi) + C$. □

We can intoduce now the *effective Hamiltonian* $\overline{H}$ as

$$\overline{H}(\xi) = \sup_{\xi' \in \mathbf{R}^n} \{(\xi, \xi') - \overline{L}(\xi')\},$$

and state the convergence result as follows.

Theorem 5.2. *Let φ be a given bounded and uniformly continuous function in $\mathbf{R}^n$, and let u_ε be the unique viscosity solution of (5.1). Then as $\varepsilon \to 0$, u_ε converges uniformly on compact sets to the unique viscosity solution of the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} + \overline{H}(Du) = 0 & \text{in } \mathbf{R}^n \times [0, +\infty[\\ u(x, 0) = \varphi(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{5.3}$$

Proof. Following P.L.Lions [31] Theorem 11.1, and Lax [27], we can define for $x, y \in \mathbf{R}^n$ and $0 \le s < t$

$$S_\varepsilon(x,t;y,s) = \inf\left\{\int_s^t L(\frac{\tau}{\varepsilon}, \frac{u(\tau)}{\varepsilon}, u'(\tau))d\tau \ : \ u(s)=y,\ u(t)=x,\ u \in \mathrm{H}^{1,\infty}(s,t)\right\}$$

$$= \inf\left\{\int_s^t L(\frac{\tau}{\varepsilon}, \frac{u(\tau)}{\varepsilon}, u'(\tau))d\tau \ : \ u(\tau) - \big(\frac{y-x}{s-t}(\tau - s) + y\big) \in \mathrm{H}^{1,q}_0(s,t)\right\}.$$

Then the unique viscosity solution to problem (5.1) is given by the Lax formula:

$$u_\varepsilon(x,t) = \inf\{\varphi(y) + S_\varepsilon(x,t;y,s) \ : \ y \in \mathbf{R}^n, 0 \le s < t\}.$$

By Theorem 2.4, we have that for every $x, y \in \mathbf{R}^n$ and $0 \le s < t$

$$S_\varepsilon(x,t;y,s) \to \min\left\{\int_s^t \overline{L}(u'(\tau))d\tau \ : \ u(\tau) - \big(\frac{y-x}{s-t}(\tau - s) + y\big) \in \mathrm{H}^{1,q}_0(s,t)\right\}$$
$$= (t-s)\overline{L}(\frac{y-x}{s-t}),$$

the last equality following by the convexity of $\overline{L}$ and Jensen's inequality. By the growth hypothesis on L we obtain that the functions $S_\varepsilon(x,t;\cdot,\cdot)$ are equicontinuous in $\{y \in \mathbf{R}^n, 0 \le s \le t-\eta\}$, and then

$$u_\varepsilon(x,t) \to u(x,t)$$

pointwise, where

$$u(x,t) = \inf\{\varphi(y) + (t-s)\overline{L}\big(\frac{y-x}{s-t}\big) \ : \ y \in \mathbf{R}^n, 0 \le s < t\}.$$

Since the functions u_ε are equicontinuous on compact sets, the convergence is uniform on bounded sets. Again by the Lax formula in [31] Theorem 11.1, and by the definition of $\overline{H}$, u is the unique viscosity solution of (5.3). □

An Example. Let $H = H(x,\xi)$ be uniformly almost periodic in the first variable. As shown in [33] and [23], we can give an alternative definition of $\overline{H}$: for every $\xi \in \mathbf{R}^n$, $\overline{H}$ is the unique constant such that the stationary problem

$$H(x, \xi + Du(x)) = \overline{H}(\xi) \tag{5.4}$$

has a uniformly almost periodic solution. We can apply this remark to obtain an explicit formula for $\overline{H}$, when $n = 1$,

$$H(x,\xi) = |\xi|^2 - V(x),$$

V is u.a.p. and

$$\inf V = 0.$$

When $H(\xi) > 0$, from equation (5.4) we have

$$|u'(x) + \xi|^2 = V(x) + \overline{H}(\xi) > 0 \quad ,$$

hence, by the requirement that u' is continuous,

$$u'(x) = -\xi + \sqrt{V(x) + \overline{H}(\xi)} \quad \text{or} \quad u'(x) = -\xi - \sqrt{V(x) + \overline{H}(\xi)}.$$

The function u is then u.a.p. if and only if the mean value of u' is zero, *i.e.,*

$$|\xi| = ⨍ \sqrt{V(x) + \overline{H}(\xi)}\, dx.$$

Since $\overline{H}$ is positive and convex, we obtain the formula

$$\overline{H}(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq ⨍\sqrt{V(x)}\, dx \\ \alpha & \text{if } |\xi| = ⨍\sqrt{V(x) + \alpha}\, dx. \end{cases}$$

The flat piece in the graph of $\overline{H}$ corresponds to the lack of differentiability of $\overline{L}$ in 0, as already observed by Buttazzo & Dal Maso [15] Section 4a.

References

[1] E. ACERBI & G. BUTTAZZO - On the limits of periodic Riemmanian metrics. *J. Analyse Math.* **43** (1984), 183–201.

[2] E. ACERBI & N. FUSCO - Semicontinuity problems in the calculus of variations; *Arch. Rational Mech. Anal.* **86** (1986), 125–145

[3] G. ALBERTI - Integral Representation of Local Functionals. *Ann. Mat. Pura Appl.* **165** (1993), 49–86.

[4] L. AMBROSIO & A. BRAIDES - Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization; *J. Math. Pures Appl.* **69** (1990) 307–333.

[5] J. M. BALL, J. C. CURRIE & P. J. OLVER - Null Lagrangians, Weak Continuity, and Variational Problems of Arbitrary Order. *J. Functional Anal.* **41** (1981), 135–174.

[6] J. M. BALL & F. MURAT - $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. *J. Functional Anal.* **58** (1984), 225–253.

[7] A. BENSOUSSAN, J. L. LIONS & G. PAPANICOLAOU - *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978

[8] A. BESICOVITCH - *Almost Periodic Functions.* Cambridge UP, 1932

[9] L. BOCCARDO & F. MURAT - Remarques sur l'homogénéisation de certaines problèmes quasi-linéaires. *Portugaliae Math.* **41** (1982), 535–562.

[10] A. BRAIDES - Homogenization of Some Almost Periodic Functionals. *Rend. Accad. Naz. Sci. XL* **103** (1985), 313–322.

[11] A. BRAIDES - A Homogenization Theorem for Weakly Almost Periodic Functionals. *Rend. Accad. Naz. Sci. XL* **104** (1986), 261–281.

[12] A. BRAIDES - Correctors for the homogenization of almost periodic monotone operators. *Asymptotic Analysis* **5** (1991), 47–74.

[13] A. BRAIDES, V. CHIADÒ PIAT & A. DEFRANCESCHI - Homogenization of almost periodic monotone operators. *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **9** (1992), 399–432.

[14] G. BUTTAZZO - *Semicontinuity, relaxation and integral representation in the calculus of variations.* Pitman, London, 1989.

[15] G. BUTTAZZO & G. DAL MASO - Γ-limit of a sequence of non-convex and non-equi-Lipschitz integral functionals. *Ricerche Mat.* **27** (1978) 235–251.

[16] G. BUTTAZZO & G. DAL MASO - Γ-limits of integral functionals. *J. Analyse Math.* **37** (1980), 145–185.

[17] G. BUTTAZZO & G. DAL MASO - A characterization of nonlinear functionals on Sobolev Spaces which admit an integral representation with a Carathéodory integrand. *J. Math. Pures Appl.* **64** (1985), 337–361.

[18] B. DACOROGNA - *Weak continuity and weak lower semicontinuity of nonlinear functionals.* Lecture Notes in Math. 922, Springer, Berlin, 1980.

[19] G. DAL MASO & L. MODICA - A General Theory of Variational Functionals, "Topics in Functional Analysis 1980-81" Quaderno della Scuola Normale Superiore di Pisa, 1981, 149–221.

[20] R. DE ARCANGELIS - Homogenization of Besicovitch type almost periodic functionals. *Preprint Univ. Salerno* (1989).

[21] R. DE ARCANGELIS & F. SERRA CASSANO - Convergence of minima for non equicoercive Bohr like almost periodic functionals. *Asymptotic Anal.* **3** (1990), 145–171.

[22] E. DE GIORGI & T. FRANZONI- Su un tipo di convergenza variazionale; *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.* (8) **58** (1975), 842–850

[23] W. E - A Class of Homogenization Problems in the Calculus of Variations. *Comm. Pure Appl. Math.* **44** (1991), 733–759.

[24] H. FEDERER - *Geometric Measure Theory.* Springer, Berlin, 1969.

[25] N. FUSCO - On the convergence of integral functionals depending on vector-valued functions; *Ricerche Mat.* **32** (1983), 321–339.

[26] M. GIAQUINTA & E. GIUSTI - Quasi-minima. *Ann. Inst. H. Poincarè* **1** (1984), 79–107.

[27] P. D. LAX - Hyperbolic systems of conservation laws. *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[28] S. M. KOZLOV - Averaging differential operators with almost-periodic rapidly oscillating coefficients. *Math. USSR-Sb.* **35** (1979), 481–498.

[29] S. M. KOZLOV - Reducibility of quasiperiodic differential operators and averaging. *Trans. Moscow Math. Soc.* **2** (1984), 101–126.

[30] B. M. LEVITAN & V. V. ZHIKOV - Almost periodic functions and differential equations. Cambridge Univ. Press, Cambridge, 1982.

[31] P. L. LIONS - *Generalized Solutions of Hamilton-Jacobi Equations.* Pitman, London, 1982.

[32] P. L. LIONS - Equations de Hamilton-Jacobi et solutions de viscosité. In *Ennio De Giorgi Colloquium*, ed. P. Kree, pp. 83–97, Pitman, London, 1985.

[33] P. L. LIONS, G. PAPANICOLAOU & S. R. VARADHAN - Homogenization of Hamilton-Jacobi equations. Preprint 1987.

[34] F. C. LIU - A Luzin Type Property of Sobolev Functions. *Indiana Univ. Math. J.* **26** (1977), 645–651.

[35] P. MARCELLINI - Periodic solutions and homogenization of nonlinear variational problems. *Ann. Mat. Pura Appl.* **117** (1978), 139-152.

[36] P. MARCELLINI - Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. *Manuscripta Math.* **51** (1985), 1–28.

[37] C. B. MORREY - Quasiconvexity and the semicontinuity of multiple integrals. *Pacific J. Math.* **2** (1952), 25–53.

[38] S. MÜLLER - Homogenization of nonconvex integral functionals and cellular elastic materials; *Arch. Rational Mech. Anal.* **99** (1987), 189–212.

[39] O. A. OLEINIK & V. V. ZHIKOV - On the homogenization of elliptic operators with almost-periodic coefficients. *Rend. Sem. Mat. Fis. Milano* **52** (1982), 149–166.