MULTIPLE SOLUTIONS FOR POSSIBLY DEGENERATE EQUATIONS IN DIVERGENCE FORM

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Abstract. We establish via variational methods the existence of at least two distinct weak solutions for the Dirichlet problem associated to a possibly degenerate equation in divergence form.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $1 \leq m \leq n$ and $X = (X_1, \ldots, X_m)$ be a family of locally Lipschitz vector fields in $\mathbb{R}^n$. In this paper we prove a multiplicity result for the following problem:

$$\begin{cases} \mathrm{div}_X a(x, Xu) = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ are continuous maps satisfying suitable growth assumptions and $\lambda \in \mathbb{R}$. We denote by $\mathrm{div}_X u = -\sum_{i=1}^m X_i^* u_i$ the so-called $X-$divergence where X_i^* denotes the operator formally adjoint to X_i, that is the operator for which:

$$\int_{\mathbb{R}^n} \psi X_i \varphi dx = \int_{\mathbb{R}^n} \varphi X_i^* \psi dx \quad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n).$$

In this paper we prove that, under suitable assumptions, there is an explicit interval of values for λ for which (1) has at least two distinct weak solutions. This type of problem has been deeply studied when $m = n$ and X is the standard Euclidean gradient, we refer, for instance, to [2, 3, 8, 9, 10, 20, 21, 33] and references therein. If $m < n$ and X is a general family of locally Lipschitz vector fields then equations of type (1) have been studied from several perspectives, see for example [4, 7, 6, 11, 12, 18, 23, 27, 30, 31]. For multiplicity results in the particular case $1 \leq m < n$ we refer to [26, 17, 27] and references therein. In particular, in [27] the authors proved a multiplicity result for the Kohn Laplacian in general Carnot groups using only variational techniques and under the assumption that the nonlinear term f satisfies the Ambrosetti-Rabinowitz condition (see $(F2)$ below). In the present paper we prove that using the approach developed in [28] and under suitable assumptions, the result proved in [27] can be generalized to more general equations and more general settings, see Section 5.
Our main assumptions are:

Assumptions on a:

(A1) $a(x, \xi) = \nabla_\xi \mathcal{A}(x, \xi)$ for some continuous $\mathcal{A} : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ with continuous gradient.

(A2) $\mathcal{A}(x, 0) = 0$ for all $x \in \Omega$.

(A3) There exist $c > 0$ and $p > 1$ such that $|a(x,\xi)| \le c(1+|\xi|^{p-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^m$.

(A4) $\mathcal{A}$ is uniformly convex, i.e. there is $k > 0$ such that

$$\mathcal{A}\Big(x, \frac{\xi+\eta}{2}\Big) \le \frac{1}{2}\mathcal{A}(x,\xi) + \frac{1}{2}\mathcal{A}(x,\eta) - k|\xi - \eta|^p$$

for all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^m$.

(A5) $0 \le a(x,\xi)\xi \le p\mathcal{A}(x,\xi)$ for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^m$.

(A6) There are $C_1, C_2 > 0$ such that

$$C_1|\xi|^p \le \mathcal{A}(x,\xi) \le C_2|\xi|^p$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^m$.

Assumptions on X:

(X1) The control distance d (see [4, Definition 5.2.2]) associated to the family X is defined, moreover $(\mathbb{R}^n, d)$ is complete and the topology generated by d is equivalent to the one generated by the Euclidean distance. For every compact set K of $\mathbb{R}^n$, there exists $C > 1$ and $R_0 > 0$ such that denoted by $B_r(x)$ the $d-$ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ the following condition holds:

$$0 < |B_{2r}(x)| \le C|B_r(x)| \quad \forall x \in K, 0 < r \le R_0,$$

where $|E|$ denotes the $n-$Lebesgue measure of $E \subseteq \mathbb{R}^n$.

(X2) For each compact set $K \subset \mathbb{R}^n$ there are $\theta, \nu > 0$ such that

$$\frac{1}{|B_r|}\int_{B_r(x)} |u - u_r| dx \le \frac{\theta r}{|B_{\nu r}|}\int_{B_{\nu r}(x)} |Xu| dx \quad \forall u \in C^1(\overline{\Omega})$$

for every $x \in K$ and $0 < r \le R_0$. As usual, $u_r := \frac{1}{|B_r|}\int_{B_r} u dx$.

(X3) There exist $p^* = p^*(\Omega) > p$ and $S_p > 0$ such that

$$\|u\|_{L^{p^*}(\Omega)} \le S_p \|Xu\|_{L^p(\Omega)} \quad \forall u \in C^1_0(\Omega). \tag{2}$$

Assumptions on f:

(F1) There are $a_1, a_2 > 0$ and $q \in (p, p^*)$ such that $|f(x,t)| \le a_1 + a_2|t|^{q-1}$ for every $x \in \Omega$ and $t \in \mathbb{R}$.

(F2) There are $\alpha > \frac{C_2}{C_1}p$ and $r_0 > 0$ such that $0 < \alpha\int_0^t f(x,\tau)d\tau \le tf(x,t)$ for every $x \in \overline{\Omega}$ and $|t| \ge r_0$. Here C_1 and C_2 are as in $[A6]$.

By (2), the function $\|u\|_{\mathcal{X}} := \|Xu\|_{L^p(\Omega)}$ is a norm in $C^1_0(\Omega)$. Consequently, we define

$$W^{1,p}_0(\Omega; X) := \overline{C^1_0(\Omega)}^{\|\cdot\|_{\mathcal{X}}}. \tag{3}$$

As pointed out in [23], if $u \in W^{1,p}_0(\Omega; X)$ then $X_j u$ exists in the sense of distributions and $X_j u \in L^p(\Omega)$ for $j = 1, \dots, m$. Consequently, the gradient Xu is well-defined for any $u \in W^{1,p}_0(\Omega; X)$. If follows from (2) that for every $1 \le q \le p^*$, there exits $c_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \le c_q \|u\|_{\mathcal{X}} \quad \forall u \in W^{1,p}_0(\Omega; X). \tag{4}$$

We are now in position to state our main result.

Theorem 1. *Assume* $(A1)-(A6)$, $(X1)-(X3)$ *and* $(F1),(F2)$ *are satisfied. Let* Ω *be an open bounded subset of* $\mathbb{R}^n$. *Then for every* $\rho>0$ *and each*

$$0<\lambda<\Lambda(\rho):=\left(a_1c_1C_1^{-\frac{1}{p}}\rho^{\frac{1}{p}-1}+a_2c_q^qq^{-1}C_1^{-\frac{q}{p}}\rho^{\frac{q}{p}-1}\right)^{-1},$$

problem (1) *has at least two weak solutions one of which has the following property*

$$\|u\|_{\mathcal{X}}^p<\frac{\rho}{C_1}.$$

The plan of the paper is the following. In Section 2 we introduce and describe our variational framework. In Section 3 we collect some results that we will use in the proof of Theorem 1. Section 4 is entirely devoted to the proof of Theorem 1. Finally, in Section 5 we provide some interesting examples satisfying our assumptions.

2. Variational framework

In this Section we describe our variational framework.
The following important theorem has been proved in [18, 16], see also [1, 13, 22].

Theorem 2. *Assume* $(X1)$, $(X2)$ *and* $(X3)$. *If* $\Omega\subset\mathbb{R}^n$ *is a bounded open set and* $p>1$ *then* $W_0^{1,p}(\Omega;X)$ *is reflexive and the embedding*

$$W_0^{1,p}(\Omega;X)\hookrightarrow L^q(\Omega)$$

is compact for every $1\le q<p^*$.

In the sequel we will use the following interesting result, see Theorem 6 in [32] for the proof.

Theorem 3. *Let* Y *be a reflexive real Banach space, and let* $\Phi,\Psi:Y\to\mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semicontinuous and coercive. Further, assume that* Ψ *is sequentially weakly continuous. In addition, assume that, for each* $\mu>0$, *the functional*

$$J_\mu:=\mu\Phi-\Psi$$

satisfies the Palais-Smale condition. Then, for every $\rho>\inf_Y\Phi$ *and every*

$$\mu>\inf_{u\in\Phi^{-1}((-\infty,\rho))}\frac{\sup_{v\in\Phi^{-1}((-\infty,\rho))}\Psi(v)-\Psi(u)}{\rho-\Phi(u)},$$

the following holds: either the functional J_μ *has a strict global minimum in* $\Phi^{-1}((-\infty,\rho))$, *or* J_μ *has at least two critical points one of which lies in* $\Phi^{-1}((-\infty,\rho))$.

For the sake of completeness, we recall that given a Banach space Y with topological dual Y^*, a C^1-functional $\mathcal{J}:Y\to\mathbb{R}$ is said to satisfy the Palais-Smale condition if for every $\eta\in\mathbb{R}$, every sequence $\{x_n\}_{n\in\mathbb{N}}\subset Y$ such that

$$\mathcal{J}(x_n)\to\eta,\quad \|\mathcal{J}'(x_n)\|_{Y^*}\to 0\quad\text{as } n\to\infty$$

admits a convergent subsequence in Y. As usual,

$$\|\mathcal{J}'(u)\|_{Y^*}:=\sup\left\{\left|\mathcal{J}'(u)[\varphi]\right|,\ \varphi\in Y,\|\varphi\|_Y=1\right\}.\tag{5}$$

Let us define the functional $\mathcal{J}_\lambda\colon W_0^{1,p}(\Omega;X)\to\mathbb{R}$ by

$$\mathcal{J}_\lambda(u):=\frac{1}{\lambda}\Phi(u)-\Psi(u),$$

where

$$\Phi(u) := \int_\Omega \mathcal{A}(x, Xu(x))dx \quad \text{and} \quad \Psi(u) := \int_\Omega F(x, u(x))dx$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $F(x,t) := \int_0^t f(x,\tau)d\tau$. It is easy to see that $\mathcal{I}_\lambda \in C^1(W_0^{1,p}(\Omega; X), \mathbb{R})$ and

$$\mathcal{I}_\lambda'(u)[\varphi] = \frac{1}{\lambda}\int_\Omega \langle a(x, Xu), X\varphi\rangle\, dx - \int_\Omega f(x,u)\varphi dx \quad \varphi \in W_0^{1,p}(\Omega; X).$$

We say that $u \in W_0^{1,p}(\Omega; X)$ is a weak solution of (1) if

$$\mathcal{I}_\lambda'(u)[\varphi] = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega; X).$$

3. Structural properties

In this section we collect some interesting consequences of $(A1)-(A6)$, $(F1)$, $(F2)$ and $(X1) - (X3)$.

The following Lemma is [27, Remark 3.2].

Lemma 4. *If f satisfies $(F2)$ (also known as Ambrosetti-Rabinowitz condition) then F satisfies*

$$F(x,t) \geq F(x,v)t^\alpha$$

for every $x \in \overline{\Omega}$ and every $(t,v) \in \mathbb{R}^2$ with $t \geq 1$ and $|v| \geq r_0$.

Let Y be a Banach space and Y^* its dual. We recall [5, 9] that an operator $a : Y \to Y^*$ verifies the $(S_+)-$condition if for any sequence $(x_n)_{n\in\mathbb{N}} \subset Y$, $x_n \rightharpoonup x$ and

$$\limsup_{n\to\infty} \langle a(x_n), x_n - x\rangle \leq 0$$

it holds $x_n \to x$ strongly in Y. We also recall [9] that a convex functional $A : Y \to \mathbb{R}$ is uniformly convex if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$A\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \delta$$

for all $x, y \in Y$ with $\|x - y\| > \varepsilon$. If A is uniformly convex on every ball, A is called locally uniformly convex. The following is [9, Proposition 2.1].

Proposition 5. *Suppose $A : Y \to \mathbb{R}$ is a C^1 locally uniformly convex functional that is locally bounded. Then $a = DA : Y \to Y^*$ verifies the $(S+)-$condition.*

By [28, Remark 3.3], the functional $\Phi(u) = \int_\Omega \mathcal{A}(x, Xu)dx$ is locally bounded and locally uniformly convex. Proposition 5 gives that,

$$\Phi'(u)[\varphi] = \int_\Omega \langle a(x, Xu), X\varphi\rangle\, dx$$

satisfies the $(S_+)-$condition.
We conclude this section with some comments about assumptions $(X1) - (X3)$. In [16], it is proved that $(X1)$ and $(X2)$ implies $(X3)$ for every Ω with sufficiently small diameter, $\overline{\Omega} \subset K^\circ$ and $p^* = pQ/(Q-p)$ with $Q = \log_2(C)$. Moreover, as pointed out in [23], if the family X has the following additional property:

(X4) Let $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ and $R > 0$ we define the map $\delta_R\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$\delta_R(x) = \Big(R^{\alpha_1}x_1, \dots, R^{\alpha_n}x_n\Big).$$

Then

$$X_j(\delta_R u)(x) = R(X_j u)(\delta_R x) \quad \forall u \in C^\infty(\mathbb{R}^n)$$

where $\delta_R u(x) = u(\delta_R(x))$

then (2) holds for every open bounded subset of $\mathbb{R}^n$. As proved in [24, Remark 9] the same conclusion holds if $(X1)$ and $(X2)$ are satisfied for every $r > 0$. We also point out that in general $(X3)$ does not imply $(X1)$, [23, Section 6.2].

4. Main Theorem

In this Section we prove Theorem 1. We start with two preliminary lemmas:

Lemma 6. *Every Palais-Smale sequence $\{u_i\}_{i\in\mathbb{N}} \subset W_0^{1,p}(\Omega; X)$ for $\mathcal{I}_\lambda$ is bounded.*

Proof. We proceed by contradiction. Possibly passing to a subsequence we can assume $\|u_i\|_{\mathcal{X}} \to \infty$ as $i \to \infty$. Let $\alpha > p\frac{C_2}{C_1}$, by definition

$$\begin{aligned}(6)\qquad \mathcal{I}_\lambda(u_i) - \frac{\mathcal{I}'_\lambda(u_i)[u_i]}{\alpha} =& \frac{1}{\lambda}\int_\Omega \mathcal{A}(x, Xu_i)dx - \frac{1}{\alpha\lambda}\int_\Omega \langle a(x, Xu_i), Xu_i\rangle\, dx \\ &+ \int_\Omega \frac{f(x, u_i(x))u_i(x)}{\alpha} - F(x, u_i(x))dx.\end{aligned}$$

Recalling that f is continuous and denoting by

$$\upsilon := \sup\left\{\left|\frac{f(\xi,t)t}{\alpha} - F(\xi,t)\right| \mid \xi \in \overline{\Omega},\ |t| \le r_0\right\} < \infty,$$

we get

$$(7)\qquad \int_{|u_i(x)|\le r_0} \frac{f(x, u_i(x))u_i(x)}{\alpha} - F(x, u_i(x))dx \ge -|\Omega|\upsilon$$

and by $[F2]$,

$$(8)\qquad \int_{|u_i(x)|> r_0} \frac{f(x, u_i(x))u_i(x)}{\alpha} - F(x, u_i(x))dx \ge 0.$$

Assumptions $(A5)$ and $(A6)$ give

$$(9)\qquad \frac{1}{\lambda}\Big(C_1 - \frac{C_2 p}{\alpha}\Big)\|u\|^p_{\mathcal{X}} \le \frac{1}{\lambda}\int_\Omega \mathcal{A}(x, Xu_i)dx - \frac{1}{\alpha\lambda}\int_\Omega \langle a(x, Xu_i), Xu_i\rangle\, dx$$

and using (6), (7), (8) and (9)

$$\frac{1}{\lambda}\Big(C_1 - \frac{C_2 p}{\alpha}\Big)\|u\|^p_{\mathcal{X}} \le \mathcal{I}_\lambda(u_i) - \frac{\mathcal{I}'_\lambda(u_i)[u_i]}{\alpha} + |\Omega|\upsilon,$$

note that $\alpha > p\frac{C_2}{C_1}$ implies $C_1 - \frac{C_2 p}{\alpha} > 0$. Therefore,

$$\frac{1}{\lambda}\Big(C_1 - \frac{C_2 p}{\alpha}\Big)\|u\|^p_{\mathcal{X}} \le \mathcal{I}_\lambda(u_i) + \frac{\|\mathcal{I}'_\lambda(u_i)\|_{\mathcal{X}^{-1}}\|u_i\|_{\mathcal{X}}}{\alpha} + |\Omega|\upsilon.$$

Let $i_0 \in \mathbb{N}$ be such that $\|u_i\|_{\mathcal{X}} \ge 1$ for every $i \ge i_0$. Since $p > 1$ then for every $i \ge i_0$

$$(10)\qquad 0 < \frac{1}{\lambda}\Big(C_1 - \frac{C_2 p}{\alpha}\Big) \le \frac{\mathcal{I}_\lambda(u_i)}{\|u_i\|_{\mathcal{X}}} + \frac{\|\mathcal{I}'_\lambda(u_i)\|_{\mathcal{X}^{-1}}}{\alpha} + \frac{|\Omega|\upsilon}{\|u_i\|_{\mathcal{X}}}.$$

Letting $i \to \infty$ in (10) and recalling that $\{u_i\}_{i\in\mathbb{N}}$ is a Palais-Smale sequence we get a contradiction. $\square$

Lemma 7. *The functional $\mathcal{I}_\lambda$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_i\}_{i\in\mathbb{N}} \subset W_0^{1,p}(\Omega;X)$ be a Palais-Smale sequence for $\mathcal{I}_\lambda$. By Lemma 6, $\{u_i\}_{i\in\mathbb{N}}$ is bounded and since $W_0^{1,p}(\Omega;X)$ is reflexive, there is a subsequence, that we still denote by $\{u_i\}_{i\in\mathbb{N}}$, and $\hat{u} \in W_0^{1,p}(\Omega;X)$ such that $u_i \rightharpoonup \hat{u}$ in $W_0^{1,p}(\Omega;X)$. We have to prove that $u_i \to \hat{u}$ in $W_0^{1,p}(\Omega;X)$. By definition,

$$\begin{aligned}(11)\qquad &\int_\Omega \langle a(x, Xu_i), X(u_i(x) - \hat{u}(x))\rangle\, dx \\ &= \lambda \mathcal{I}_\lambda'(u_i)[u_i - \hat{u}] + \lambda \int_\Omega f(x, u_i(x))(u_i(x) - \hat{u}(x))dx.\end{aligned}$$

Since $\|\mathcal{I}_\lambda'(u_i)\|_{\mathcal{X}^{-1}} \to 0$ and $\{u_i - \hat{u}\}_{i\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega;X)$ and recalling that

$$\left|\mathcal{I}_\lambda'(u_i)[u_i - \hat{u}]\right| \le \|\mathcal{I}_\lambda'(u_i)\|_{\mathcal{X}^{-1}} \|u_i - \hat{u}\|_{\mathcal{X}}$$

we get

$$(12)\qquad \mathcal{I}_\lambda'(u_i)[u_i - \hat{u}] \to 0 \quad \text{as} \quad i \to \infty.$$

By Theorem 2, $u_i \to \hat{u}$ in $L^q(\Omega)$. By $(F1)$

$$\begin{aligned}0 &< \int_\Omega f(x, u_i(x))(u_i(x) - \hat{u}(x))dx \\ &\le a_1 \int_\Omega |u_i(x) - \hat{u}(x)|dx + a_2 \int_\Omega |u_i(x)|^{q-1}|u_i(x) - \hat{u}(x)|dx \\ &\le Ca_1\|u - u_i\|_{L^q(\Omega)} + a_2\|u_i\|_{L^q(\Omega)}\|u - u_i\|_{L^q(\Omega)}\end{aligned}$$

hence

$$(13)\qquad \int_\Omega f(x, u_i(x))(u_i(x) - \hat{u}(x))dx \to 0 \quad \text{as} \quad i \to \infty.$$

Putting together (11), (12) and (13) we conclude

$$(14)\qquad \Phi'(u_i)[u_i - \hat{u}] = \int_\Omega \langle a(x, Xu_i), X(u_i(x) - \hat{u}(x))\rangle\, dx \to 0 \quad \text{as} \quad i \to \infty.$$

Since Φ' has the $(S_+)-$property then $u_i \to \hat{u}$ in $W_0^{1,p}(\Omega;X)$. $\square$

Proof of Theorem 1: By Lemma 7, $\mathcal{I}_\lambda$ satisfies the Palais-Smale condition and by $(A4)$ and $(A6)$, Φ is coercive and sequentially weakly lower semicontinuous. Since f is continuous and using Theorem 2 then Ψ is sequentially weakly continuous. We claim that for every $\rho > 0$ and $0 < \lambda < \Lambda(\rho)$

$$\frac{1}{\lambda} > \Theta(\rho) := \inf_{u\in\Phi^{-1}((-\infty,\rho))} \frac{\sup_{v\in\Phi^{-1}((-\infty,\rho))} \Psi(v) - \Psi(u)}{\rho - \Phi(u)}.$$

By $(A2)$ we get $\Phi(0) = 0$ and by definition $\Psi(0) = 0$. Hence,

$$\Theta(\rho) \le \frac{\sup_{v\in\Phi^{-1}((-\infty,\rho))} \Psi(v)}{\rho}.$$

By $(A6)$,

$$(15)\qquad \Phi^{-1}((-\infty,\rho)) \subseteq \left\{v \in X \mid \|v\|_{\mathcal{X}} \le C_1^{-\frac{1}{p}} \rho^{\frac{1}{p}}\right\},$$

therefore

$$\Theta(\rho) \leq \frac{\sup_{\{v\in X \ | \ \|v\|_X \leq C_1^{-\frac{1}{p}}\rho^{\frac{1}{p}}\}} \Psi(v)}{\rho}.$$

Using $[F1]$, (15) and (4) we easily get

$$\Theta(\rho) \leq \frac{a_1 c_1}{C_1^{\frac{1}{p}}}\rho^{\frac{1}{p}-1} + \frac{a_2 c_q^q}{q C_1^{\frac{q}{p}}}\rho^{\frac{q}{p}-1}$$

and the conclusion follows. Now we prove that J_λ cannot have a strict global minimum in $\Phi^{-1}((-\infty,\rho))$. By Lemma 4 and $(A6)$ it follows

$$\begin{aligned} J_\lambda(tu_0) &= \frac{1}{\lambda}\Phi(tu_0) - \Psi(tu_0) \\ &\leq \frac{C_2}{\lambda}t^p \int_\Omega |Xu_0|^p dx - t^\alpha \int_{\{\xi\in\Omega \ | \ |u_0(x)|\geq r_0\}} F(x,u_0(x))dx + \nu|\Omega|, \end{aligned}$$

for every $u_0 \in W_0^{1,p}(\Omega;X)$, where $\nu = \sup\{|F(x,t)|, \ x \in \overline{\Omega}, |t| \leq r_0\}$. Choosing u_0 such that $|\{x \in \Omega \ | \ |u_0(x)| \geq r_0\}| > 0$, recalling that $\alpha > \frac{C_2}{C_1}p > p$ and $F(x,t) > 0$ for $|t| \geq r_0$ we get

$$\lim_{t\to+\infty} J_\lambda(tu_0) = -\infty.$$

Applying Theorem 3 we conclude the proof. □

5. Examples

In this section we collect some interesting examples of vector fields satisfying $(X1)$, $(X2)$ and $(X3)$.

5.1. **Euclidean space.** Let $m = n$ and X be the standard Euclidean gradient. It is well known that $(X1)$ and $(X2)$ are satisfied for every $r > 0$, therefore also $(X3)$ holds. In this case a result similar to Theorem 1 has been proved in [28]. We invite the reader to have also a look at [3] where the case $\mathcal{A}(x,\xi) = |\xi|^2$ is investigated and [9, 8, 10, 20, 33] for the case $\mathcal{A}(x,\xi) = |\xi|^p$.

5.2. **Carnot Groups.** We recall that a Carnot group $\mathbb{G}$ is a connected Lie groups whose Lie algebra $\mathcal{G}$ is finite dimensional and stratified of step $s \in \mathbb{N}$. Precisely, there exist linear subspaces $V_1, ..., V_s$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_s$$

with

$$[V_1, V_{i-1}] = V_i \text{ if } 2 \leq i \leq s, \text{ and } [V_1, V_s] = \{0\}.$$

Here $[V_1, V_i] := \text{span}\{[a,b] : a \in V_1, \ b \in V_i\}$. Since $\mathcal{G}$ is stratified then every element in $\mathcal{G}$ is the linear combination of commutators of elements V_1. We refer to [4] for a complete introduction to the subject. Let $\dim(V_1) = m$ and $X = (X_1, \ldots, X_m)$ be a basis of V_1. In [18], it is proved that $(X1)$ and $(X2)$ are satisfied for every $r > 0$ therefore also $(X3)$ holds. We point out that when $p = 2$ and $\mathcal{A}(x,\xi) = |\xi|^2$, Theorem 1 boils down to Theorem 3.1 in [27].

5.3. **Hörmander vector fields.** Let $X = (X_1, \dots, X_m)$ be a family of smooth vector fields in $\mathbb{R}^n$. We say that X satisfies the Hörmander condition if

$$\text{rank}\Big(\text{Lie}\{X_1, \dots, X_m\}\Big)(x) = n \quad \forall x \in \mathbb{R}^n$$

where $\text{Lie}\{X_1, \dots, X_m\}$ denotes the Lie algebra generated by X. Clearly, Carnot groups satisfy the Hörmander condition, on the other hand there are plenty of examples of vector fields satisfying the Hörmander condition whose generated Lie algebra is not stratified. For instance, we can consider in $\mathbb{R}^2$ the family $X = (X_1, X_2)$ where

$$X_1 = \partial_x, \quad X_2 = x^2 \partial_y$$

then $\text{rank}\Big(\text{Lie}\{X_1, X_2\}\Big)(x, y) = 2$ for every $(x, y) \in \mathbb{R}^2$ and $\text{Lie}\{X_1, X_2\}$ is not stratified. In [29] and [19] it is proved that $(X1)$ and $(X2)$ hold respectively with $R_0 > 0$.

5.4. **Vector fields not satisfying the Hörmander condition but satisfying** $(X1)$ **and** $(X2)$**.** The following example is contained in [23]. Let us consider the family $X = (X_1, X_2)$ in $\mathbb{R}^3$ where

$$X_1 = \partial_x, \quad X_2 = |x|^m \partial_y + \partial_z, \ m \in [1, \infty),$$

in [23] is it proved that X satisfies $(X1)$ and $(X2)$. The family X satisfies $(X4)$ with $\delta_R(x, y, z) = (Rx, R^{m+1}y, Rz)$, therefore it also satisfies $(X3)$.
The following family of vector fields has been studied in [14, 15],

$$X_j = \lambda_j \partial_{x_j} \quad j = 1, \dots, m$$

where each λ_j is a real-valued function and the family (λ_j) satisfies suitable conditions, see [14, 15]. As proved in [14, 15], these conditions ensure the validity of $(X1)$ and $(X2)$.

We conclude with an explicit application of Theorem 1 to Carnot groups, note that the following result generalizes Theorem 3.1 in [27]. Let $X = (X_1, \dots, X_m)$ be a basis of V_1 and $p > 1$. We define

$$\Delta_p u = \text{div}_X(|Xu|^{p-2} Xu).$$

Theorem 8. *Let $\mathbb{G}$ be a Carnot group, Ω be a bounded open subset of $\mathbb{G}$ and $p \geq 2$. If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies $(F1)$ and $(F2)$ for some $q \in (p, p^*)$ then, for every $\rho > 0$ and each*

$$0 < \lambda < \Lambda(\rho) := \left(a_1 c_1 \rho^{\frac{1}{p}-1} + a_2 c_q^q q^{-1} \rho^{\frac{q}{p}-1} \right)^{-1}$$

the problem

$$\begin{cases} \Delta_p u = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{16}$$

has at least two distinct weak solutions in $W_0^{1,p}(\Omega; X)$ one of which is such that $\|u\|_{\mathcal{X}} \leq \rho$.

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