A phase-field damage model based on evolving microstructure

Hauke Hanke, Dorothee Knees *

May 25, 2016

Abstract

In this paper we discuss a damage model that is based on microstructure evolution. In the context of evolutionary Γ-convergence we derive a corresponding effective macroscopic model. In this model, the damage state of a given material point is related to a unit cell problem incorporating a specific microscopic defect. The size and shape of this underlying microscopic defect is determined by the evolution. According to the small intrinsic length scale inherent to the original models a numerical simulation of damage progression in a device of realistic size is hopeless. Due to the scale separation in the effective model, its numerical treatment seems promising.

Key words: Two-scale convergence, folding and unfolding operator, Γ-convergence, discrete gradient, state dependent coefficient, damage model
MCS (2010) 74Q15, 35B27, 35R05, 74A45, 49J40, 74C05, 74R05.

1 Introduction

In many cases of fatal rupture of a macroscopic device the damage progression is initiated on the microscopic scale. There, the loading of the device results in the creation of microscopic cracks which in the long run might coalesce and, thus, cause the complete failure of the device. Since in the beginning of the damage process the size of the microscopic defects is very small, the number of the emerging defects has to grow to notice a significant decrease of the device's robustness. But this combination, namely, the occurrence of a huge number of very small objects, makes the mathematical (and especially the numerical) treatment of such problems very challenging. Therefore, we are interested in providing an effective description of the initial problem, simplifying the occurring microstructure (e.g., the union of all microscopic cracks) to enable numerical simulation but preserving the damage behavior of the original device. For the sake of simplifying the notation as well as the mathematical analysis of the models we are going to consider the device to grow inclusions of material having a very low robustness compared to their surrounding material instead of small cracks. For an extension to damage progression via the growth of microscopic voids or cracks we refer to [12], see also Remark 2.7.

*University of Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany. E-Mail: dknees@mathematik.uni-kassel.de

In this paper, the heterogeneity of the material occupied body $\Omega \subset \mathbb{R}^d$ under consideration is denoted as microstructure. Even in the simplified case of microstructure consisting of only two phases, the appearing geometries being related to their possible distributions might be very complicated. One very common kind of microstructure approximation is a periodically distribution of the two considered phases. Since we are interested in the modeling of damage progression we like to account for local changes of the microstructure in dependence of external influences. Therefore, the assumption of a global periodical response to external forces is too restrictive. For a fixed parameter $\varepsilon > 0$, being associated to the intrinsic length scale of the appearing microstructure, the time-dependent occurrence of the two material phases is captured by a finite number of (time-dependent) parameters. These parameters for instance describe the radii of the damaged subregions and give rise to a piecewise constant function in the sense described below.

The considered body Ω is decomposed in small cells $\varepsilon(\lambda+Y) \subset \Omega$, where $\lambda \in \Lambda$ with Λ being a given periodic lattice and with Y denoting the unit cell. Considering a specific cell $\varepsilon(\lambda+Y) \subset \Omega$ the distribution of the two phases (modeled by the constant tensors $\mathbb{C}_{\text{strong}}$ and $\mathbb{C}_{\text{weak}}$) is given by m geometric parameters $z^{\varepsilon\lambda} \in [0,1]^m$. Hence, the material distribution of the whole body Ω is associated to a piecewise constant function $z_\varepsilon : \Omega \to [0,1]^m$, where $z_\varepsilon|_{\varepsilon(\lambda+Y)\subset\Omega} \equiv z^{\varepsilon\lambda}$. That means, the material properties of the body Ω are modeled by the state-dependent tensor

$$\mathbb{C}_\varepsilon(z_\varepsilon) = \mathbb{1}_{\Omega\setminus\Omega_\varepsilon^D(z_\varepsilon)}\mathbb{C}_{\text{strong}} + \mathbb{1}_{\Omega_\varepsilon^D(z_\varepsilon)}\mathbb{C}_{\text{weak}},$$

where $\mathbb{1}_\mathcal{O} : \mathbb{R}^d \to \{0,1\}$ denotes the characteristic function of the set $\mathcal{O} \subset \mathbb{R}^d$ and $\Omega_\varepsilon^D(z_\varepsilon)$ is the subset of Ω occupied by the material modeled by $\mathbb{C}_{\text{weak}}$. For instance, if $m = 1$, $z^{\varepsilon\lambda}$ may stand for the radius of the soft inclusion. For the detailed relation between the damage variable z_ε and the set $\Omega_\varepsilon^D(z_\varepsilon)$ we refer to Section 2.1. Starting with these types of admissible microstructures for fixed $\varepsilon > 0$ an evolution model is considered accounting for the uni-directionality of damage progression, i.e., material that once is damaged cannot regain stiffness during the whole process. The damage progression is modeled in the framework of the energetic formulation for rate-independent processes developed in [17, 18]. For a suitable state space $\mathcal{Q}_\varepsilon(\Omega) = \mathcal{U}_\varepsilon \times \mathcal{Z}_\varepsilon$ this energetic formulation is based on an energy functional $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ depending on the displacement field u_ε as well as the damage variable z_ε, and a dissipation distance $\mathcal{D}_\varepsilon : \mathcal{Q}_\varepsilon(\Omega)\times\mathcal{Q}_\varepsilon(\Omega) \to [0,\infty]$ depending only on the damage variable. We introduce the energy functional via

$$\mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) = \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \mathcal{G}_\varepsilon(z_\varepsilon) - \langle \ell(t), u_\varepsilon\rangle,$$

where ℓ is a given time-dependent loading, $\mathbf{e}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the linearized strain tensor, and $\mathcal{G}_\varepsilon(z_\varepsilon)$ is a regularization term; see Section 2.1 for details. We are interested in an effective description as $\varepsilon \to 0$ of the damage process described by the energetic formulation. To perform the limit passage $\varepsilon \to 0$ rigorously, the regularization term $\mathcal{G}_\varepsilon(z_\varepsilon)$ is added. This term improves the regularity of the appearing microstructures which enables us to identify an effective limit damage model in the context of Sobolev-spaces. The regularization term is motivated by the theory for broken Sobolev functions and can be interpreted as a discrete gradient, see e.g. [3, 13].

The dissipated energy is proportional to the growth of the weak material and is modeled by the dissipation distance $\mathcal{D}_\varepsilon : \mathcal{Q}_\varepsilon(\Omega)\times\mathcal{Q}_\varepsilon(\Omega) \to [0,\infty]$ given by

$$\mathcal{D}_\varepsilon(z_1,z_2) = \begin{cases} \displaystyle\int_\Omega \gamma |z_1(x) - z_2(x)|_m \mathrm{d}x & \text{if } z_1 \geq z_2 \text{ (component-wise)},\\ \infty & \text{otherwise.} \end{cases}$$

The quantity $\gamma > 0$ is a material dependent constant and plays the role of an averaged fracture toughness. Observe that the dissipation distance ensures the uni-directionality of the damage, meaning that the damaged region of Ω is only allowed to grow with respect to increasing time.
Based on these two functionals the evolution is described by the energetic formulation for rate-independent processes which consists of a stability condition (S^ε) and an energy balance (E^ε); see Section 2.1 for the precise definition. As already mentioned before, the system (S^ε) and (E^ε) models a damage process showing up very fine structures of material distribution. The smaller the intrinsic length scale $\varepsilon > 0$ is chosen the more complicated the material distribution might get. Numerically this leads to an unmanageable large amount of degree of freedom. For this reason, we are interested in an effective description of this damage process which captures the evolution of the microstructure but enables numerical simulations. This is done by performing the limit passage $\varepsilon \to 0$ rigorously. For the limit function space $\mathcal{Q}_0(\Omega)$ and $p > 1$ the limit energy functional $\mathcal{E}_0 : [0,T]\times\mathcal{Q}_0(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}_0(t,u_0,z_0) = \tfrac{1}{2}\langle \mathbb{C}_{\mathrm{eff}}(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{d\times d}} - \langle \ell(t), u_0\rangle,$$

where material properties for $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $x \in \Omega$ are modeled by the effective tensor

$$\langle \mathbb{C}_{\mathrm{eff}}(z_0)(x)\xi,\xi\rangle_{d\times d} = \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} \int_Y \langle \mathbb{C}_0(z_0(x))(y)(\xi + \mathbf{e}_y v(y)), \xi + \mathbf{e}_y v(y)\rangle_{d\times d} \mathrm{d}y. \tag{1.1}$$

Here,

$$\mathbb{C}_0(z_0(x)) = \mathbb{1}_{Y\setminus Y^D(z_0(x))}\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{Y^D(z_0(x))}\mathbb{C}_{\mathrm{weak}},$$

where $Y^D(z_0(x))$ denotes the subset of Y occupied by the material modeled by $\mathbb{C}_{\mathrm{weak}}$. In (1.1) the minimum is taken with respect to all functions $v \in \mathrm{H}^1(Y)^d$, which can be periodically extended (in $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)^d$) and have mean value zero, i.e., it holds $\int_Y v(y)\mathrm{d}y = 0$. Moreover, for $\gamma > 0$ the dissipated energy is modeled by the dissipation distance $\mathcal{D}_0 : \mathcal{Q}_0(\Omega)\times\mathcal{Q}_0(\Omega) \to [0,\infty]$ given by

$$\mathcal{D}_0(z_1,z_2) = \begin{cases} \displaystyle\int_\Omega \gamma |z_1(x) - z_2(x)|_m \mathrm{d}x & \text{if } z_1 \geq z_2 \text{ (component-wise)},\\ \infty & \text{otherwise.} \end{cases}$$

Applying the methods of evolutionary Γ-convergence from [16], to the sequence of evolution systems ((S^ε) and (E^ε))$_{\varepsilon>0}$ defined by $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$, we show that the associated sequence of solutions $((u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega))_{\varepsilon>0}$ converges (in some sense, see Theorem 5.7 for

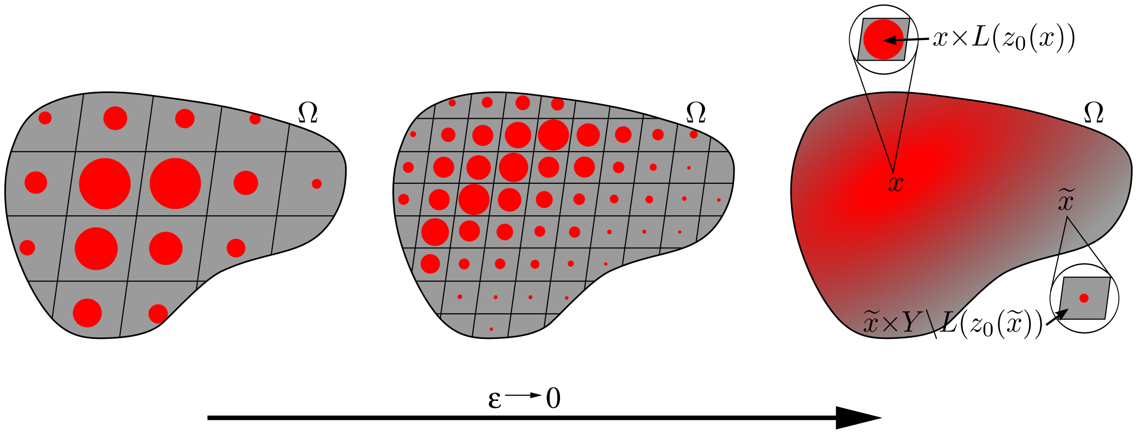

Figure 1: Schematic representation of the limit passage of the microscopic model (S^ε) and (E^ε) to the effective limit model (S^0) and (E^0) for a fixed time t. In this example the microscopic inclusions are assumed to be balls; see Section 2 for the notation.

details) to a function $(u_0, z_0) : [0, T] \to \mathcal{Q}_0(\Omega)$ which is a solution of the energetic formulation (S^0) and (E^0) associated to the limit functionals $\mathcal{E}_0 : [0, T]\times\mathcal{Q}_0(\Omega) \to \mathbb{R}$ and $\mathcal{D}_0 : \mathcal{Q}_0(\Omega)\times\mathcal{Q}_0(\Omega) \to [0, \infty]$.
Comparison with other approaches: The limit model described by $\mathcal{D}_0$ and $\mathcal{E}_0$ with the effective elasticity tensor $\mathbb{C}_{\text{eff}}$ from (1.1) belongs to the class of phase-field damage models, see for instance [9]. In such models, the dependence of the elasticity tensor on the (typically scalar) damage variable z in general is based on phenomenological considerations. The approach discussed in our paper allows for a more detailed modelling of the processes on the micro-scale and also for the modeling of anisotropic effects. Neglecting the gradient regularization term $\|\nabla z_0\|^p_{L^p(\Omega)}$ in $\mathcal{E}_0$ and the discrete gradients $\mathcal{G}_\varepsilon(z_\varepsilon)$ in $\mathcal{E}_\varepsilon$ leads to a class of models that were studied in the papers [8, 7, 10]. There, the authors assume that in each macroscopic material point the material either is undamaged (encoded by $\mathbb{C}_{\text{strong}}$) or maximally damaged (encoded by $\mathbb{C}_{\text{weak}}$). During the evolution a displacement field $u(t)$ and non decreasing sets $D(t) \subset \Omega$ have to be determined such that the total energy

$$\tfrac{1}{2}\langle \mathbb{C}(D(t))\mathbf{e}(u(t)), \mathbf{e}(u(t))\rangle_{L^2(\Omega)} - \langle \ell(t), u(t)\rangle + \kappa|D(t)|$$

with $\mathbb{C}(D(t)) = \mathbb{1}_{\Omega\setminus D(t)}\mathbb{C}_{\text{strong}} + \mathbb{1}_{D(t)}\mathbb{C}_{\text{weak}}$ is minimal. Since this problem is not well-posed, the authors introduce a suitable relaxed problem with effective material tensors belonging to the G-closure of the pair $\mathbb{C}_{\text{weak}}, \mathbb{C}_{\text{strong}}$ with respect to certain time dependent volume fractions. Compared to our approach, this allows for a much higher flexibility in generating effective elasticity tensors. However, information on the specific underlying micro-pattern is not available any more.

2 Damage progression via the growth of inclusions

Let $d \in \mathbb{N}$ denote the space dimension. From now on we are going to assume that the material occupied set $\Omega \subset \mathbb{R}^d$ satisfies the following condition:

$$\begin{array}{l}\text{The set } \Omega \subset \mathbb{R}^d \text{ is assumed to be open, connected, bounded, and}\\ \text{has a locally Lipschitz boundary } \partial\Omega;\ \text{see Definition 2.1 below.}\\ \text{Moreover, } \Gamma_{\mathrm{Dir}} \subset \partial\Omega \text{ is a closed subset of positive measure.}\end{array} \tag{2.1}$$

Definition 2.1 (Locally Lipschitz boundary)**.** A bounded set $\mathcal{O} \subset \mathbb{R}^d$ has a locally Lipschitz boundary, if for each point $x \in \partial\mathcal{O}$ there exists a neighborhood N_x such that $N_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function (with respect to an appropriately rotated system of coordinates) and $\Omega \cap N_x$ is below the graph.

2.1 Microscopic inclusions of weak material causing damage progression

We start by defining the state space $\mathcal{Q}_\varepsilon(\Omega)$ for the microscopic models that describe damage progression by the growth of inclusions of damaged material in an undamaged bulk. As indicated in Section 1 the damage process under investigation is modeled with the help of two variables, namely, the displacement field u_ε and the damage variable z_ε. Consequently, the state space

$$\mathcal{Q}_\varepsilon(\Omega) = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \tag{2.2}$$

is the product of

$$\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega) = \big\{u \in \mathrm{H}^1(\Omega)\,\big|\,\text{the trace } u|_{\Gamma_{\mathrm{Dir}}} \text{ satisfies } u|_{\Gamma_{\mathrm{Dir}}} = 0\big\} \tag{2.3}$$

and the space of piecewise constant functions $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ that is defines as follows: Let $\{b_1, b_2, \ldots, b_d\}$ be an arbitrary basis of $\mathbb{R}^d$, with no need of orthonormality. Furthermore, let

$$\Lambda = \Big\{\lambda \in \mathbb{R}^d\ :\ \lambda = \sum_{i=1}^d k_i b_i,\ k_i \in \mathbb{Z}\Big\} \tag{2.4}$$

be a periodic lattice and

$$Y = \Big\{x \in \mathbb{R}^d\ :\ x = \sum_{i=1}^d l_i b_i,\ l_i \in [-\tfrac{1}{2}, \tfrac{1}{2})\Big\}$$

the associated unit cell. In particular, the unit cell Y is the d-parallelotope whose axis are the basis vectors $\{b_1, b_2, \ldots, b_d\}$. The only restriction on the basis $\{b_1, b_2, \ldots, b_d\}$ is that

$$\mu_d(Y) = 1$$

is satisfied to make the following statements valid without any normalization coefficients. Due to this definition, there is only one vertex contained in $\varepsilon(\lambda+Y)$ such that each of these cells is uniquely determined by $\varepsilon > 0$ and the associated vertex $\varepsilon\lambda$. Moreover, we define

$$\Lambda_\varepsilon^- = \{\lambda \in \Lambda : \varepsilon(\lambda+\overline{Y}) \subset \Omega\}, \quad \Lambda_\varepsilon^+ = \{\lambda \in \Lambda : \varepsilon(\lambda+Y) \cap \Omega \neq \emptyset\}, \quad \Omega_\varepsilon^\pm = \bigcup_{\lambda\in\Lambda_\varepsilon^\pm} \varepsilon(\lambda+Y) \tag{2.5}$$

Finally, for an open set $\Omega \subset \mathbb{R}^d$ the set of piecewise constant functions is given by

$$\mathrm{K}_{\varepsilon\Lambda}(\Omega) = \{v \in \mathrm{L}^1(\Omega) \,|\, \exists\, \widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) : \widetilde{v}|_\Omega = v\},$$

where

$$\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) = \{\widetilde{v} \in \mathrm{L}^1(\mathbb{R}^d) \,|\, \forall\, \lambda \in \Lambda : \widetilde{v}|_{\varepsilon(\lambda+Y)} = const\}.$$

Given a global damage state $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, the set $\Omega_\varepsilon^D(z_\varepsilon) \subset \Omega$ characterizes the distribution of the inclusions of damaged material in the following way: Let $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ be a set valued mapping (with $\mathcal{L}_{\text{Leb}}(Y)$ denoting the Lebesgue measurable subsets of Y). We assume that L satisfies

- $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ is a non-increasing function, i.e., for all (2.6a)
  $\widehat{z}_1 \leq \widehat{z}_2 \in [0,1]^m$ (component-wise) it holds $L(\widehat{z}_2) \subset L(\widehat{z}_1)$.
- For all $\widehat{z} \in [0,1]^m$ with $\widehat{z} \neq 1$ (component-wise) it holds $\mu_d(L(\widehat{z})) > 0$. (2.6b)
- For all $\widehat{z} \in [0,1]^m$ the set $L(\widehat{z})$ is a closed subset of $\overline{Y}$. (2.6c)

For any given $\widehat{z} \in [0,1]^m$ and every $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\widehat{z}_\delta \to \widehat{z}$ in $\mathbb{R}^m$ it holds

- $\mu_d(L(\widehat{z})\backslash L(\widehat{z}_\delta)) + \mu_d(L(\widehat{z}_\delta)\backslash L(\widehat{z})) \to 0$ for $\delta \to 0$ and (2.6d)
- $\forall \Delta > 0\, \exists\, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds $L(\widehat{z}_\delta) \subset \text{neigh}_\Delta(L(\widehat{z}))$. (2.6e)

Here, $\text{neigh}_\Delta(\mathcal{O})$ denotes the Δ-neighborhood of the set $\mathcal{O} \subset \mathbb{R}^d$. For a given damage state $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ we define

$$\Omega_\varepsilon^D(z_\varepsilon) = \bigcup_{\lambda\in\Lambda_\varepsilon^-} \varepsilon(\lambda + L(z^{\varepsilon\lambda})), \tag{2.7}$$

which is the set of damaged material. Assuming that

the tensors $\mathbb{C}_{\text{strong}}, \mathbb{C}_{\text{weak}} \in \text{Lin}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ are positive definite and symmetric, (2.8)

the elasticity tensor for $x \in \Omega$ is modeled by

$$\mathbb{C}_\varepsilon(z_\varepsilon)(x) = \mathbb{1}_{\Omega\backslash\Omega_\varepsilon^D(z_\varepsilon)}(x)\mathbb{C}_{\text{strong}} + \mathbb{1}_{\Omega_\varepsilon^D(z_\varepsilon)}(x)\mathbb{C}_{\text{weak}}, \tag{2.9a}$$

Observe that for small values of ε the set $\Omega_\varepsilon^D(z_\varepsilon)$ may have a very irregular structure on a very small length scale, which can be very challenging from a numerical point of view. Therefore, we are interested in the derivation of an effective macroscopic model preserving the microscopic behavior but enabling a numerical treatment, for instance.

Remark 2.2. The definition of $\Omega_\varepsilon^D(z_\varepsilon)$ (closed set) is chosen in such a way that the inclusions $\Omega_\varepsilon^D(z_\varepsilon)$ (closed set) are contained in the open set Ω and have an empty intersection with $\partial\Omega$. This seems to be a rather technical assumption. But note that in the case of modeling voids ($\mathbb{C}_{\text{weak}} \equiv 0$), condition (2.7) guarantees for any $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])$ that the boundary of Ω is contained in the boundary of the material occupied set $\Omega \backslash \Omega_\varepsilon^D(z_\varepsilon)$. In this way the presumed boundary conditions (see (2.3), for instance) are always well defined.

With $\mathcal{Q}_\varepsilon(\Omega)$ from (2.2) and a given load $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$, the energy functional $\mathcal{E}_\varepsilon : [0,T] \times \mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) = \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \langle \ell(t), u_\varepsilon\rangle, \tag{2.10}$$

where the regularization term $\mathcal{G}_\varepsilon(z_\varepsilon) := \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}$ with $p > 1$ will be specified in Section 4. The last ingredient of the energetic formulation, namely, the dissipation distance $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \times \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$, does only depend on the damage variable and for $\gamma > 0$ is given by

$$\mathcal{D}_\varepsilon(z_1, z_2) = \begin{cases} \displaystyle\int_\Omega \gamma |z_1(x) - z_2(x)|_m \mathrm{d}x & \text{if } z_1 \geq z_2 \text{ (component-wise)}, \\ \infty & \text{otherwise.} \end{cases}$$

Based on $\mathcal{E}_\varepsilon : [0,T] \times \mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ and $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \times \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ we are interested in global energetic solutions $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$, which for all $t \in [0,T]$ are assumed to fulfill the stability condition (S^ε) and the energy balance (E^ε):

$$(\mathrm{S}^\varepsilon)\quad \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_\varepsilon(z_\varepsilon(t), \widetilde{z}) \qquad \text{for all } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_\varepsilon(\Omega),$$

$$(\mathrm{E}^\varepsilon)\quad \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) = \mathcal{E}_\varepsilon(0, u_\varepsilon(0), z_\varepsilon(0)) + \int_0^t \partial_t \mathcal{E}_\varepsilon(s, u_\varepsilon(s), z_\varepsilon(s)) \mathrm{d}s,$$

with $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z; [0,t]) = \sup \sum_{j=1}^N \mathcal{D}_\varepsilon(z(s_{j-1}), z(s_j))$, where $N \in \mathbb{N}$ and the supremum is taken with respect to all finite partitions of $[0,t]$. Moreover, for given initial values (u^0, z^0) the initial condition $(u_\varepsilon(0), z_\varepsilon(0)) = (u^0, z^0)$ has to be satisfied.

Introducing the set of stable states $\mathcal{S}_\varepsilon(\widetilde{t})$ at time $\widetilde{t} \in [0,T]$ via

$$\mathcal{S}_\varepsilon(\widetilde{t}) = \{(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega) \text{ satisfying } (\mathrm{S}^\varepsilon) \text{ for } t = \widetilde{t}\}$$

the stability condition (S^ε) is equivalently written as $(u_\varepsilon(t), z_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t)$ for all $t \in [0,T]$. Adopting the abstract existence result for rate-independent processes modeled by the energetic formulation given in [15], we are able to state the following existence result; see [12, Section 6.5] for the proof:

Proposition 2.3 (Existence of solutions). *Let the material tensors* $\mathbb{C}_{\text{strong}}$ *and* $\mathbb{C}_{\text{weak}}$ *be positive definite. Moreover, assume that the conditions* (2.6) *hold. Then for* $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{S}_\varepsilon(0)$*, there exists an energetic solution* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ *of the rate-independent system* $(\mathcal{Q}_\varepsilon(\Omega), \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ *satisfying* $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ *and*

$$u_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d),$$
$$z_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_\varepsilon}([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)),$$

where $\mathrm{BV}_{\mathcal{D}_\varepsilon}([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)) = \{\, z : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \,|\, \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z; [0,T]) < \infty\}$.

2.2 Effective damage model based on the growth of inclusions of weak material

We will now introduce the macroscopic limit model. For $p > 1$ the limit state space $\mathcal{Q}_0(\Omega)$ is defined via

$$\mathcal{Q}_0(\Omega) = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega; [0,1]^m).$$

For a given damage variable $z_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ the modeling of the material is based on the tensor $\mathbb{C}_0(z_0(x)) \in \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\})$ which for almost every $(x, y) \in \Omega \times Y$ is given by

$$\mathbb{C}_0(z_0(x))(y) = \mathbb{1}_{Y \setminus L(z_0(x))}(y)\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{L(z_0(x))}(y)\mathbb{C}_{\mathrm{weak}}.$$

Here, $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ denotes the set valued mapping chosen in Subsection 2.1; see also condition (2.6).

Lemma 2.4. *Let $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ satisfy the conditions* (2.6a)*,* (2.6c)*, and* (2.6e)*. Then, for any measurable function $z : \mathbb{R}^d \to [0,1]^m$ the mapping*

$$\mathbb{C}_0(z(\cdot))(\cdot) : \begin{cases} \mathbb{R}^d \times Y \to \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\} \\ (x, y) \mapsto \mathbb{C}_0(z(x))(y) \end{cases} \quad \textit{is measurable on } \mathbb{R}^d \times Y. \tag{2.11}$$

Proof. To verify condition (2.11) let $z : \mathbb{R}^d \to [0,1]^m$ be an arbitrary but fixed measurable function. Due to its definition the mapping $\mathbb{C}_0(z(\cdot))(\cdot) : \mathbb{R}^d \times Y \to \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\}$ is constant on the two sets $M(z) = \bigcup_{x \in \mathbb{R}^d} \{x\} \times L(z(x))$ and $(\mathbb{R}^d \times Y) \setminus M(z)$. Hence, (2.11) is proven by showing that $M(z)$ is a measurable subset of $\mathbb{R}^d \times Y$.
For this purpose, we choose a countable sequence $(z_\delta)_{(\delta>0)}$ of simple functions approximating the measurable mapping $z : \mathbb{R}^d \to [0,1]^m$ from below, i.e., $z_\delta(x) \nearrow z(x)$ (component wise) for all $x \in \mathbb{R}^d$. Here, the term *simple function* means, that there is a finite number of disjoint, measurable sets $A_1^\delta, A_2^\delta, \ldots, A_{n_\delta}^\delta \subset \mathbb{R}^d$ and constant vectors $z_1^\delta, z_2^\delta, \ldots, z_{n_\delta}^\delta \in [0,1]^m$ such that $\bigcup_{k=1}^{n_\delta} A_k = \mathbb{R}^d$ and $z_\delta = \sum_{k=1}^{n_\delta} \mathbb{1}_{A_k^\delta} z_k^\delta$. Thus, we now consider the sequence $(M(z_\delta))_{\delta>0}$ of $M(z)$ approximating sets. For $\delta > 0$ the measurability of $M(z_\delta)$ is a consequence of the fact that it can be written as a finite union of measurable sets in the following way:

$$M(z_\delta) = \bigcup_{k=1}^{n_\delta} \Big(\bigcup_{x \in A_k^\delta} \{x\} \times L(z_\delta(x)) \Big) = \bigcup_{k=1}^{n_\delta} (A_k^\delta \times L(z_k^\delta)).$$

Note that for fixed $\delta > 0$ the measurability of the set $L(z_k^\delta)$ for all $k \in \{1, 2, \ldots, n_\delta\}$ is ensured by assumption (2.6c). Due to the relation $z_\delta \le z$ on $\mathbb{R}^d$ and condition (2.6a) we have $M(z) \subset M(z_\delta)$ for every $\delta > 0$ by definition. Moreover, $\bigcap_{\delta>0} M(z_\delta) \subset M(z)$ is shown by the following contradiction argument:
Let $(x^*, y^*) \in \bigcap_{\delta>0} M(z_\delta)$ but $(x^*, y^*) \notin M(z)$. Then for all $\delta > 0$

$$y^* \in L(z_\delta(x^*)) \tag{2.12}$$

but

$$\mathrm{dist}(y^*, L(z(x^*))) = 2\Delta > 0 \tag{2.13}$$

since $L(z(x^*))$ was assumed to be closed; see (2.6c). Condition (2.13) implies

$$y^* \notin \mathrm{neigh}_\Delta(L(z(x^*))). \tag{2.14}$$

Since $z_\delta(x^*) \to z(x^*)$ by assumption, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds

$$y^* \overset{(2.12)}{\in} L(z_\delta(x^*)) \overset{(2.6e)}{\subset} \mathrm{neigh}_\Delta(L(z(x^*)))$$

which is a contradiction to (2.14).
All together we proved $M(z) = \bigcap_{\delta>0} M(z_\delta)$. Since $M(z)$ can be written as the countable intersection of measurable sets, this shows its measurability and hence condition (2.11) is verified. □

Remark 2.5. Let the mapping $f : Y \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \to \mathbb{R}$ be defined by

$$f(y, \xi, \widehat{z}) = \langle \mathbb{C}_0(\widehat{z})(y)\xi, \xi\rangle_{d\times d} = \begin{cases} \langle \mathbb{C}_{\mathrm{strong}}\xi, \xi\rangle_{d\times d} & \text{if } y \in Y\backslash L(\widehat{z}), \\ \langle \mathbb{C}_{\mathrm{weak}}\xi, \xi\rangle_{d\times d} & \text{if } y \in L(\widehat{z}). \end{cases}$$

Then for fixed $y \in Y$ the mapping $f(y, \cdot, \cdot) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \to \mathbb{R}$ is not continuous on $\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m$ as for fixed $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ it only takes the values $\langle \mathbb{C}_{\mathrm{strong}}\xi, \xi\rangle_{d\times d}$ and $\langle \mathbb{C}_{\mathrm{weak}}\xi, \xi\rangle_{d\times d}$. Hence, $f : Y \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \to \mathbb{R}$ does not satisfy the Carathéodory condition. However, as follows from the previous lemma, for every measurable function $z : \mathbb{R}^d \to [0, 1]^m$ the mapping $\widehat{f}_z : \mathbb{R}^d \times Y \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ with $\widehat{f}_z(x, y, \xi) = \langle \mathbb{C}_0(z(x))(y)\xi, \xi\rangle_{d\times d}$ is a Carathéodory function, since the mapping $\xi \mapsto \widehat{f}_z(x, y, \xi)$ for all $(x, y) \in \mathbb{R}^d \times Y$ is continuous and since $(x, y) \mapsto \widehat{f}_z(x, y, \xi)$ for any $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is measurable.

Let $z_0 : \Omega \to [0, 1]^m$ be a measurable function. We define the tensor $\mathbb{C}_{\mathrm{eff}}(z_0)(x) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ via the the following unit cell problem: $\forall \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\langle \mathbb{C}_{\mathrm{eff}}(z_0)(x)\xi, \xi\rangle_{d\times d} = \min_{v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} \int_Y \langle \mathbb{C}_0(z_0(x))(y)(\xi + \mathbf{e}_y(v)(y)), \xi + \mathbf{e}_y(v)(y)\rangle_{d\times d} \mathrm{d}y, \tag{2.15}$$

where $\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ denotes the set of periodically extendable functions (in $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)^d$) having mean value zero. In [13, Proposition 3.3] we showed, that the right hand side of (2.15) is indeed a quadratic expression with respect to $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.
Now, for $p > 1$ the energy functional $\mathcal{E}_0 : [0, T] \times \mathcal{Q}_0(\Omega) \to \mathbb{R}$ is defined in the following way:

$$\mathcal{E}_0(t, u_0, z_0) = \tfrac{1}{2}\langle \mathbb{C}_{\mathrm{eff}}(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t), u_0\rangle.$$

Finally, the limit dissipation distance $\mathcal{D}_0 : \mathrm{W}^{1,p}(\Omega; [0, 1]^m) \times \mathrm{W}^{1,p}(\Omega; [0, 1]^m) \to [0, \infty]$ for $\gamma > 0$ is given by

$$\mathcal{D}_0(z_1, z_2) = \begin{cases} \displaystyle\int_\Omega \gamma |z_2(x) - z_1(x)|_m \mathrm{d}x & \text{if } z_1 \geq z_2 \text{ (component-wise)}, \\ \infty & \text{otherwise}. \end{cases}$$

The proof of the following existence result is carried out in Section 5.2 by showing that subsequences of global energetic solutions of $((\mathrm{S}_\varepsilon)$ & $(\mathrm{E}_\varepsilon))$ converge in a suitable sense to solutions of $((\mathrm{S}_0)$ & $(\mathrm{E}_0))$.

Theorem 2.6 (Existence of solutions). *Let the material tensors $\mathbb{C}_{\mathrm{strong}}$ and $\mathbb{C}_{\mathrm{weak}}$ satisfy* (2.8) *and assume that the conditions* (2.6) *hold. Let $(u_0^0, z_0^0) \in \mathcal{Q}_0(\Omega) \cap \mathcal{S}_0(0)$ and assume that for $\varepsilon > 0$ there exist initial values $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{Q}_\varepsilon(\Omega) \cap \mathcal{S}_\varepsilon(0)$ with $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) = \mathcal{E}_0(0, u_0^0, z_0^0)$. Then there exists an energetic solution $(u_0, z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ of the rate-independent system $(\mathcal{Q}_0(\Omega), \mathcal{E}_0, \mathcal{D}_0)$ with initial condition (u_0^0, z_0^0) satisfying*

$$
\begin{aligned}
u_0 &\in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d),\\
z_0 &\in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_0}([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)),
\end{aligned}
$$

i.e., for all $t \in [0,T]$ it holds

$$(\mathrm{S}^0)\quad \mathcal{E}_0(t, u_0(t), z_0(t)) \le \mathcal{E}_0(t, \tilde{u}, \tilde{z}) + \mathcal{D}_0(z_0(t), \tilde{z}) \qquad \textit{for all } (\tilde{u}, \tilde{z}) \in \mathcal{Q}_0(\Omega),$$

$$(\mathrm{E}^0)\quad \mathcal{E}_0(t, u_0(t), z_0(t)) + \mathrm{Diss}_{\mathcal{D}_0}(z_0; [0,t]) = \mathcal{E}_0(0, u_0(0), z_0(0)) + \int_0^t \partial_t \mathcal{E}_0(s, u_0(s), z_0(s))\mathrm{d}s,$$

with $\mathrm{Diss}_{\mathcal{D}_0}(z;[0,t]) = \sup \sum_{j=1}^N \mathcal{D}_0(z(s_{j-1}), z(s_j))$, where $N \in \mathbb{N}$ and the supremum is taken with respect to all finite partitions of $[0,t]$.

In contrast to the microscopic models introduced in Subsection 2.1, the rate-independent system $(\mathcal{Q}_0(\Omega), \mathcal{E}_0, \mathcal{D}_0)$ shows up a diffuse material distribution. In any point $x \in \Omega$ the material is a mixture (see (2.15)) of the two initial materials modeled by the tensors $\mathbb{C}_{\mathrm{strong}}$ and $\mathbb{C}_{\mathrm{weak}}$. Since by $L(z_0(x))$ the distribution of these initial materials is uniquely determined, the structure of the microscopic models is preserved in some sense. But due to (2.15) the very fine microstructures of the microscopic models $(\mathcal{Q}_\varepsilon(\Omega), \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ are replaced by shifting the occurring inclusions to a second scale. In this way in the effective model $(\mathcal{Q}_0(\Omega), \mathcal{E}_0, \mathcal{D}_0)$ the numerical treatment of the inclusions is independent of the actual microstructure, whereas in $(\mathcal{Q}_\varepsilon(\Omega), \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ it heavily depends on the intrinsic length scale $\varepsilon > 0$, for instance.

Remark 2.7. In [12, Section 8] a similar result is obtained for a model, where damage is described by the growth of microscopic voids, i.e., there the material tensor $\mathbb{C}_{\mathrm{weak}}$ is set to zero. This obviously causes some mathematical issues. First of all, for prescribing the same boundary values independently of the chosen scale $\varepsilon > 0$, the micro-voids (see the definition of $\Omega_\varepsilon^D(z_\varepsilon)$; (2.7)) are not allowed to intersect the boundary $\partial\Omega$. Moreover, to gain a priori estimates independent of $\varepsilon > 0$, uniform coercivity of the energy functionals needs to be shown. In [12] this is done by constructing suitable continuation operators, extending an H^1-function on $\Omega\backslash\Omega_\varepsilon^D(z_\varepsilon)$ to Ω such that its norm can be estimated independently of $\varepsilon > 0$ and z_ε.

3 Two-scale convergence

One of the crucial techniques exploited to derive Theorem 2.6 is the theory of two-scale convergence. This section introduces everything needed in the following sections concerning

the notation and the theory of folding/unfolding and two-scale convergence and does not claim completeness. Note that this is just a rough overview which we already stated in [13] in almost the same way. For further details we recommend to [1, 4, 5].
Before defining the two-scale convergence with the help of the periodic unfolding operator we start by introducing the mappings $[\cdot]_\Lambda$ and $\{\cdot\}_Y$ on $\mathbb{R}^d$.

$$[\cdot]_\Lambda : \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y : \mathbb{R}^d \to Y, \quad \text{and} \quad x = [x]_\Lambda + \{x\}_Y \quad \text{for all } x \in \mathbb{R}^d$$

Let $\lambda \in \Lambda$ and let $x \in \mathbb{R}^d$ be in the cell $\lambda + Y$, then $[x]_\Lambda = \lambda$ and $\{x\}_Y$ is determinable as $\{x\}_Y = x - [x]_\Lambda$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have the following decomposition:

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon\mathcal{V}_\varepsilon(x), \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon\left[\frac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{V}_\varepsilon(x) = \left\{\frac{x}{\varepsilon}\right\}_Y, \tag{3.1}$$

where $\mathcal{N}_\varepsilon(x)$ denotes the macroscopic center of the cell $\mathcal{N}_\varepsilon(x) + \varepsilon Y$ that contains x and $\mathcal{V}_\varepsilon(x)$ is the microscopic part of x in $\mathcal{N}_\varepsilon(x) + \varepsilon Y$. At last, we want to distinguish the unit cell Y from the periodicity cell $\mathcal{Y} = \mathbb{R}^d/_\Lambda$. Following Ref. [22], we introduce the mappings $\mathcal{J}_\varepsilon$ and $\mathcal{S}_\varepsilon$ as follows:

$$\mathcal{J}_\varepsilon : \begin{cases} \mathbb{R}^d & \to \quad \mathbb{R}^d \times \mathcal{Y}, \\ x & \mapsto \quad (\mathcal{N}_\varepsilon(x), \mathcal{V}_\varepsilon(x)), \end{cases} \qquad \mathcal{S}_\varepsilon : \begin{cases} \mathbb{R}^d \times \mathcal{Y} & \to \quad \mathbb{R}^d, \\ (x,y) & \mapsto \quad \mathcal{N}_\varepsilon(x) + \varepsilon y, \end{cases}$$

where in the last sum $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$.
For $q \geq 1$ two-scale convergence is linked to a suitable two-scale embedding of $\mathrm{L}^q(\Omega)$ in the two-scale space $\mathrm{L}^q(\mathbb{R}^d \times Y)$. Such an embedding is called periodic unfolding operator. The following definition of a periodic unfolding operator was given in Ref. [4].

Definition 3.1. (Ref. [4]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $q \in [1, \infty]$. Then the periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined via:

$$\mathcal{T}_\varepsilon : \mathrm{L}^q(\Omega) \to \mathrm{L}^q(\mathbb{R}^d \times Y); \; v \mapsto v^{\mathrm{ex}} \circ \mathcal{S}_\varepsilon,$$

where $v^{\mathrm{ex}} \in \mathrm{L}^q(\mathbb{R}^d)$ is the extension of the function v by 0 to all of $\mathbb{R}^d$.

With this definition the following product rule is valid: Let $q, q', r \in [1, \infty]$ such that $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r}$. Then

$$v_1 \in \mathrm{L}^q(\Omega), v_2 \in \mathrm{L}^{q'}(\Omega) \Longrightarrow \mathcal{T}_\varepsilon(v_1 v_2) = (\mathcal{T}_\varepsilon v_1)(\mathcal{T}_\varepsilon v_2) \in \mathrm{L}^r(\mathbb{R}^d \times Y).$$

Note that $\overline{[\Omega \times Y]_\varepsilon} = \overline{\mathcal{S}_\varepsilon^{-1}(\Omega)} = \overline{\{(x,y) | \mathcal{S}_\varepsilon(x,y) \in \Omega\}}$ is the support of $\mathcal{T}_\varepsilon v$, and this is not contained in $\Omega \times Y$, in general.
Following the lines in Ref. [20] we now will use this periodic unfolding operator to introduce the kind of two-scale convergence, which is used here; the strong and weak two-scale convergence, respectively. But before that, we define the folding operator $\mathcal{F}_\varepsilon$. For details see [20].

Definition 3.2. (Ref. [20]) Let $\Omega \subset \mathbb{R}^d$ be open, $\varepsilon > 0$ and $q \in [1, \infty)$. Then the folding operator $\mathcal{F}_\varepsilon$ is defined via:

$$\mathcal{F}_\varepsilon : \mathrm{L}^q(\mathbb{R}^d \times Y) \to \mathrm{L}^q(\Omega);\ V \mapsto \left(\mathcal{P}_\varepsilon(\mathbb{1}_{[\Omega\times Y]_\varepsilon} V)\right) \circ \mathcal{J}_\varepsilon|_\Omega,$$

where $(\mathcal{P}_\varepsilon V)(x, y) = \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} V(\zeta, y)\mathrm{d}\zeta$.

Definition 3.3. (Ref. [20]) Let $q \in (1, \infty)$ and let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $\mathrm{L}^q(\Omega)$. Then

(a) v_ε converges strongly two-scale to $V \in \mathrm{L}^q(\Omega\times Y)$ in $\mathrm{L}^q(\Omega\times Y)$, $v_\varepsilon \overset{s}{\to} V$ in $\mathrm{L}^q(\Omega\times Y)$, if $\mathcal{T}_\varepsilon v_\varepsilon \to V^{\mathrm{ex}}$ in $\mathrm{L}^q(\mathbb{R}^d\times Y)$.

(b) v_ε converges weakly two-scale to $V \in \mathrm{L}^q(\Omega\times Y)$ in $\mathrm{L}^q(\Omega\times Y)$, $v_\varepsilon \overset{w}{\rightharpoonup} V$ in $\mathrm{L}^q(\Omega\times Y)$, if $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V^{\mathrm{ex}}$ in $\mathrm{L}^q(\mathbb{R}^d\times Y)$.

Referring to (2.5) we have that for all $\varepsilon > 0$ the support of the function $\mathcal{T}_\varepsilon v_\varepsilon$ is contained in $\overline{[\Omega\times Y]_\varepsilon} \subset \overline{\Omega}_\varepsilon^+ \times Y$ which results in the fact that the support of a possible accumulation point U of the sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ has to be in $\overline{\Omega}\times Y$, since $\mu_d(\Omega_\varepsilon^+ \setminus \Omega) \to 0$. Due to $\mu_d(\partial\Omega) = 0$ we also have $\mathrm{L}^q(\Omega\times Y) = \mathrm{L}^q(\overline{\Omega}\times Y)$ and so every accumulation point of $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ can be uniquely identified with an element of $\mathrm{L}^q(\Omega\times Y)$. But notice that it is important to determine the convergence in $\mathrm{L}^q(\mathbb{R}^d\times Y)$ and not in $\mathrm{L}^q(\Omega\times Y)$. We refer to Ref. [20], where it is shown in Example 2.3 that convergence in $\mathrm{L}^q(\Omega\times Y)$ is not sufficient.
Note, that according to the definition of the two-scale convergence in $\mathrm{L}^q(\Omega\times Y)$ via the convergence of the unfolded sequence in $\mathrm{L}^q(\mathbb{R}^d\times Y)$ all convergence properties known for L^q-convergence are transmitted. For a summary of those properties we refer to Proposition 2.4 in [20]. For the convenience of the reader we state here only those properties used in the following.

Proposition 3.4 ([20])**.** *Let $q \in (1, \infty)$ and set $q' = \frac{q}{q-1}$. Furthermore, let $V_0 \in \mathrm{L}^q(\Omega\times Y)$, $W_0 \in \mathrm{L}^{q'}(\Omega\times Y)$ and $M_0 \in \mathrm{L}^1(\Omega\times Y)$ be given. Then for sequences $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^q(\Omega)$ and $(w_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^{q'}(\Omega)$ the following conditions hold.*

(a) *If $v_\varepsilon \overset{w}{\rightharpoonup} V_0$ in $\mathrm{L}^q(\Omega\times Y)$ and $w_\varepsilon \overset{s}{\to} W_0$ in $\mathrm{L}^{q'}(\Omega\times Y)$ then $\langle v_\varepsilon, w_\varepsilon\rangle_{\mathrm{L}^2(\Omega)} \to \langle V_0, W_0\rangle_{\mathrm{L}^2(\Omega\times Y)}$.*

(b) *If $v_\varepsilon \to v_0$ in $\mathrm{L}^q(\Omega)$ then $v_\varepsilon \overset{s}{\to} Ev_0$ in $\mathrm{L}^q(\Omega\times Y)$, where $E : \mathrm{L}^q(\Omega) \to \mathrm{L}^q(\Omega\times Y)$ for $v \in \mathrm{L}^q(\Omega)$ and $(x, y) \in \Omega\times Y$ is defined via $Ev(x, y) = v(x)$.*

(c) *If $v_\varepsilon \overset{s}{\to} V_0$ in $\mathrm{L}^q(\Omega\times Y)$ and if $(m_\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $\mathrm{L}^\infty(\Omega)$ such that $\mathcal{T}_\varepsilon m_\varepsilon(x, y) \to M_0(x, y)$ for almost every $(x, y) \in \Omega\times Y$. Then $m_\varepsilon v_\varepsilon \overset{s}{\to} M_0 V_0$ in $\mathrm{L}^q(\Omega\times Y)$.*

The following corollary extends property (c) of Proposition 3.4 to a special case appearing when applying the two-scale theory to the energy functional in (2.10). The proof is done via a standard contradiction argument.

Corollary 3.5. *For $q \in (1, \infty)$ let $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^q(\Omega)$ and $V_0 \in \mathrm{L}^q(\Omega\times Y)$ be given such that $v_\varepsilon \overset{s}{\to} V_0$ in $\mathrm{L}^q(\Omega\times Y)$. Moreover, let $(m_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{L}^\infty(\Omega)$ satisfying $m_\varepsilon \overset{s}{\to} M_0$ of $\mathrm{L}^1(\Omega\times Y)$ for some function $M_0 \in \mathrm{L}^1(\Omega\times Y)$. Then $m_\varepsilon v_\varepsilon \overset{s}{\to} M_0 V_0$ in $\mathrm{L}^q(\Omega\times Y)$.*

In Section 5, we are going to prove a Γ-convergence result for the energy functionals given by (2.10). There, the following integral identity for $v \in \mathrm{L}^1(\Omega)$ will be central.

$$\int_\Omega v(x)\mathrm{d}x = \int_{[\Omega\times Y]_\varepsilon} \mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x \tag{3.2}$$

Observe that this identity immediately gives us the norm-preservation of the periodic unfolding operator $\mathcal{T}_\varepsilon$. It is proved by decomposing $\mathbb{R}^d$ into cells $\varepsilon(\lambda+Y)$ for $\lambda \in \Lambda$. In preparation for performing the limit passage $\varepsilon \to 0$ in the models of Subsection 2.1, we are now going to state two-scale convergence results for two particular types of sequences of functions. Due to the linearized strain tensor appearing in the energy functional $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ we first of all have to investigate the asymptotic behavior of bounded sequences in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$. In this context we need the function space

$$\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) = \left\{ v \in \mathrm{H}^1_{\mathrm{per}}(\overline{Y}) \,\middle|\, \int_Y v(y)\mathrm{d}y = 0 \right\}.$$

To describe the weak two-scale convergence of gradients we introduce the function space $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$, which is the space of functions $V \in \mathrm{L}^2(\Omega\times Y) = \mathrm{L}^2(\Omega;\mathrm{L}^2(Y))$, having the same traces on opposite faces of Y and satisfying $\int_Y V(x,y)\mathrm{d}y = 0$ for almost every $x \in \Omega$ as well as $\nabla_y V \in \mathrm{L}^2(\Omega\times Y)^d$ in the sense of distributions. We equip this space with the norm $\|V\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))} = \|\nabla_y V\|_{\mathrm{L}^2(\Omega\times Y)^d}$. With this, we have the following compactness result which we will exploit for converging sequences of the displacement components of the microscopic models of Subsection 2.1, cf. [21, Theorem 3.1.4]:

Proposition 3.6. *Let $(v_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{H}^1(\Omega)$. Then there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'>0}$ of $(v_\varepsilon)_{\varepsilon>0}$ and functions $(v_0, V_1) \in \mathrm{H}^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ such that:*

$$\begin{aligned} v_{\varepsilon'} &\rightharpoonup v_0 && \text{in } \mathrm{H}^1(\Omega),\\ v_{\varepsilon'} &\overset{s}{\to} Ev_0 && \text{in } \mathrm{L}^2(\Omega\times Y),\\ \nabla v_{\varepsilon'} &\overset{w}{\rightharpoonup} \nabla_x Ev_0 + \nabla_y V_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^d, \end{aligned}$$

where $E : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega\times Y)$ is defined via $Ev(x,y) = v(x)$.

For the construction of the displacement component of the recovery sequence the following density result is important, cf. [11, Proposition 2.11]:

Proposition 3.7. *Let $(w_0, W_1) \in \mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ be given. Moreover, for every $\varepsilon > 0$ let $w_\varepsilon \in \mathrm{H}^1_0(\Omega)$ be the solution of the following elliptic problem:*

$$\int_\Omega ((w_\varepsilon - \mathcal{F}_\varepsilon(Ew_0)^{\mathrm{ex}})w + \langle \nabla w_\varepsilon - \mathcal{F}_\varepsilon(\nabla_x Ew_0 + \nabla_y W_1)^{\mathrm{ex}}, \nabla v\rangle_d)\mathrm{d}x = 0 \quad \forall\, v \in \mathrm{H}^1_0(\Omega).$$

Then

$$\begin{aligned} w_\varepsilon &\rightharpoonup w_0 && \text{in } \mathrm{H}^1_0(\Omega),\\ w_\varepsilon &\overset{s}{\to} Ew_0 && \text{in } \mathrm{L}^2(\Omega\times Y),\\ \nabla w_\varepsilon &\overset{s}{\to} \nabla_x Ew_0 + \nabla_y W_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^d. \end{aligned}$$

In the context of deriving the effective model (S^0) and (E^0) by performing the limit passage $\varepsilon \to 0$, we have to concern with the two-scale asymptotic behavior of sequences like $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$. Here, for a sequence $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the tensor $\mathbb{C}_\varepsilon(z_\varepsilon) \in \mathrm{L}^\infty(\Omega;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$ is given by (2.9). Moreover, for $p>1$ according to available a priori estimates (see Section 5) it is reasonable to consider the existence of a function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ such that $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$. Starting with these assumptions the two-scale limit of $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ is identified in the following way:

Theorem 3.8 (Two-scale limit of $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$)**.** *Let $L:[0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ satisfy the conditions* (2.6)*. If $(z_\varepsilon)_{\varepsilon>0}$ denotes a sequence of functions satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^1(\Omega;[0,1])^m$, then*

$$\mathbb{C}_\varepsilon(z_\varepsilon) \xrightarrow{s} \mathbb{C}_0(z_0(\cdot))(\cdot) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}),$$

where $\mathbb{C}_\varepsilon(z_\varepsilon)$ is defined by (2.9) *and $\mathbb{C}_0(z_0(\cdot))(\cdot)$ for almost every $(x,y)\in\Omega\times Y$ is given by*

$$\mathbb{C}_0(z_0(x))(y) = \mathbb{1}_{Y\backslash L(z_0(x))}(y)\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{L(z_0(x))}(y)\mathbb{C}_{\mathrm{weak}}. \tag{3.3}$$

Proof. Let the sequence $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^1(\Omega;[0,1])^m$. We start by rewriting the two-scale function $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon) \in \mathrm{L}^\infty(\mathbb{R}^d\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}},0\})$ to gain a preferably simple description to work with.

The case $x\in\mathbb{R}^d\backslash\overline{\Omega}$: For fixed $x\in\mathbb{R}^d\backslash\overline{\Omega}$ there exists $\varepsilon_0>0$ such that $x\in\mathbb{R}^d\backslash\Omega_\varepsilon^+$ for all $\varepsilon\in(0,\varepsilon_0)$. Hence, $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)(x,\cdot)\equiv 0$ on Y for all $\varepsilon\in(0,\varepsilon_0)$; see Definition 3.1. Moreover, the extension $\mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)$ trivially fulfills $\mathbb{C}_0^{\mathrm{ex}}(z_0(x))(\cdot)\equiv 0$ for all $x\in\mathbb{R}^d\backslash\overline{\Omega}$ by definition. Altogether this shows for all $x\in\mathbb{R}^d\backslash\overline{\Omega}$

$$\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)(x,\cdot) \to \mathbb{C}_0^{\mathrm{ex}}(z_0(x))(\cdot) \quad \text{in } \mathrm{L}^1(Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}},0\}). \tag{3.4}$$

The case $x\in\Omega$: Let $x\in\Omega$ be fixed. Since Ω is assumed to be open there exists $\varepsilon_0>0$ such that $x\in\Omega_\varepsilon^-$ for all $\varepsilon\in(0,\varepsilon_0)$. Note that for $(x,y)\in\Omega_\varepsilon^-\times Y$ we have (i) $z_\varepsilon(x)=z_\varepsilon(\mathcal{N}_\varepsilon(x))$, (ii) $\mathcal{N}_\varepsilon(\mathcal{N}_\varepsilon(x)+\varepsilon y)=\mathcal{N}_\varepsilon(x)$, and (iii) $\{\frac{\mathcal{N}_\varepsilon(x)+\varepsilon y}{\varepsilon}\}_Y = y$. Keeping these observations in mind, applying $\mathcal{T}_\varepsilon$ to the tensor $\mathbb{C}_\varepsilon(z_\varepsilon)$ given by (2.9) results in

$$\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)(x,y) = \mathbb{C}_0(z_\varepsilon(x))(y) \quad \text{for all } (x,y)\in\Omega_\varepsilon^-\times Y. \tag{3.5}$$

According to $z_\varepsilon\to z_0$ in $\mathrm{L}^1(\Omega)^m$ there exists a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that

$$z_{\varepsilon'}(x) \to z_0(x) \quad \text{for almost every } x\in\Omega. \tag{3.6}$$

Now, condition (2.6d) together with (3.6) enables us to pass to the limit in relation (3.5) (at least for the subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$), i.e., for almost every $x\in\Omega$ we have

$$\mathcal{T}_{\varepsilon'}\mathbb{C}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) \to \mathbb{C}_0(z_0(x))(\cdot) \quad \text{in } \mathrm{L}^1(Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}). \tag{3.7}$$

Define $f_{\varepsilon'}:\mathbb{R}^d\to[0,\infty)$ by $f_{\varepsilon'}(x)=\|\mathcal{T}_{\varepsilon'}\mathbb{C}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot)-\mathbb{C}_0^{\mathrm{ex}}(z_0)(x,\cdot)\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))}$. Then, by combining (3.4) and (3.7) and exploiting $\mu_d(\partial\Omega)=0$ (see (2.1)) we finally showed

$$f_{\varepsilon'} \to 0 \quad \text{almost every in } \mathbb{R}^d.$$

Note that the sequence $(f_{\varepsilon'})_{\varepsilon'>0}$ is uniformly bounded and that the support of $f_{\varepsilon'} : \mathbb{R}^d \to [0,\infty)$ is contained in $\Omega^+_{\varepsilon_0}$ for all $\varepsilon' \in (0,\varepsilon_0)$. Hence, the theorem of dominated convergence yields

$$\lim_{\varepsilon'\to 0} \|f_{\varepsilon'}\|_{\mathrm{L}^1(\mathbb{R}^d)} = \lim_{\varepsilon'\to 0} \int_{\mathbb{R}^d} |f_{\varepsilon'}(x)| \mathrm{d}x = 0,$$

which proves

$$\mathbb{C}_{\varepsilon'}(z_{\varepsilon'}) \xrightarrow{s} \mathbb{C}_0(z_0(\cdot))(\cdot) \quad \text{in } \mathrm{L}^1(\Omega\times Y; \{\mathbb{C}_{\text{strong}}, \mathbb{C}_{\text{weak}}\}).$$

By a standard contradiction argument it follows that this convergence holds for the whole sequence $(\varepsilon)_{\varepsilon>0}$. □

4 Discrete gradients of piecewise constant functions

This section is devoted to the definition of the regularization term $\mathcal{G}_\varepsilon(z_\varepsilon) = \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$ $(p > 1)$ appearing in the microscopic energy functional $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$. As already mentioned in Section 1, to identify the limit energy by performing the limit passage $\varepsilon \to 0$, we need to improve the a priori regularity of the admissible microstructures. In particular, for the sequence of solutions $((u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega))_{\varepsilon>0}$ of $((\mathrm{S}^\varepsilon)$ and $(\mathrm{E}^\varepsilon))_{\varepsilon>0}$ we need to enforce the strong convergence in $\mathrm{L}^p(\Omega)^m$ with respect to the damage variable. Obviously, when neglecting the regularization term we would only expect weak$*$ convergence in $\mathrm{L}^\infty(\Omega)^m$ of the sequence $(z_\varepsilon)_{\varepsilon>0}$. Models, where the regularization terms are neglected, are discussed in [7, 8, 10], where there is no restriction on the geometry of the occurring microstructure consisting of the two phases modeled by $\mathbb{C}_{\text{strong}}$ and $\mathbb{C}_{\text{weak}}$. But observe that due to the absence of a regularization in [10] some information on the microstructure is lost in the limit model. There, the limit material tensor is an element of the non-single valued G-closure of the tensors $\mathbb{C}_{\text{strong}}$ and $\mathbb{C}_{\text{weak}}$.
Coming back to our models, we are interested in the definition of a discrete gradient for piecewise constant functions on a lattice in such a way that only an overall constant function has gradient zero. Furthermore an in some sense bounded sequence of those piecewise constant functions, where the spacing of the lattice tends to zero, should lead to a limit belonging to a Sobolev space $\mathrm{W}^{1,p}$. Roughly spoken we want to introduce a penalty term, extracting those sequences of BV-functions that converge strongly in L^p to a Sobolev function, such that the discrete gradient of these sequences converge weakly in L^p to the gradient of this Sobolev function.
The definition of the discrete gradient is based on the extension operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)$ extending a piecewise constant function $v \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ for every $\lambda \in \Lambda^+_\varepsilon\backslash\Lambda^-_\varepsilon$ on $\varepsilon(\lambda+Y)\backslash\Omega$ constantly by the (constant) value of v on $\varepsilon(\lambda+Y)\cap\Omega$.

$$\begin{aligned} &\text{For } z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \text{ the function } V_\varepsilon z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m \text{ for every} \\ &\lambda \in \Lambda^+_\varepsilon \text{ and } z^{\varepsilon\lambda} :\equiv z|_{\varepsilon(\lambda+Y)\cap\Omega} \text{ is defined via } V_\varepsilon z|_{\varepsilon(\lambda+Y)} :\equiv z^{\varepsilon\lambda}. \end{aligned} \tag{4.1}$$

Definition 4.1 (Discrete gradient)**.** For $\{b_1, b_2, \ldots, b_d\}$ being the basis of $\mathbb{R}^d$ chosen in Section 2.1, let $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ be defined via $R_{\frac{\varepsilon}{2}}(z) = \sum_{i=1}^d \widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(V_\varepsilon z)$, where for $i = 1, 2, \ldots, d$ the mapping $\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ for $\overline{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ reads as follows:

$$\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(\overline{z})(x) = \begin{cases} \frac{1}{\varepsilon|b_i|}\left(\overline{z}(x+\frac{\varepsilon}{2}b_i) - \overline{z}(x-\frac{\varepsilon}{2}b_i)\right) \otimes n_i & \text{if } x+\frac{\varepsilon}{2}b_i \in \Omega^+_\varepsilon \text{ and } x-\frac{\varepsilon}{2}b_i \in \Omega^+_\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

with $n_i \in \mathbb{R}^d$ given by

$$n_i \in \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_d\}^\perp, \quad |n_i|_d = 1, \quad \text{and} \quad \langle n_i, b_i\rangle_d > 0. \tag{4.2}$$

The function $R_{\frac{\varepsilon}{2}}(z) \in \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ is called discrete gradient of $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$.

This construction of the discrete Gradient is inspired by the lifting operator introduced by A. Buffa and C. Ortner in [3]. For a detailed discussion about the differences of these two approaches we refer to [13]. The following theorem states that the discrete gradient can be used to filter out sequences of piecewise constant functions converging to elements of $\mathrm{W}^{1,p}(\Omega)^m$.

Theorem 4.2 (Compactness result)**.** *For $p \in (1, \infty)$ and every sequence $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1])^m$ which satisfies*

$$\sup_{\varepsilon>0} \left(\|z_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\right) \le C < \infty \tag{4.3}$$

there exist a function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and a sub-sequence $(z_{\varepsilon'})_{\varepsilon'>0}$ of $(z_\varepsilon)_{\varepsilon>0}$ with

$$z_{\varepsilon'} \to z_0 \text{ in } \mathrm{L}^q(\Omega)^m \quad \text{and} \quad R_{\frac{\varepsilon}{2}}(z_{\varepsilon'}) \rightharpoonup \nabla z_0 \text{ in } \mathrm{L}^p(\Omega)^{m\times d},$$

where $1 \le q < p^*$, and p^* denotes the Sobolev conjugate of p.

For the proof of this and the following approximation theorem we refer to [13].

Theorem 4.3 (Approximation result)**.** *For every function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1])^m$ such that*

$$\lim_{\varepsilon\to 0} \left(\|z_0 - z_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|(\nabla z_0)^{\mathrm{ex}} - R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\right) = 0. \tag{4.4}$$

Remark 4.4. For a given function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ one might construct the sequence $(z_\varepsilon)_{\varepsilon>0}$ of Theorem 4.3 explicitly in the following way: For $x \in \mathbb{R}^d$ let the projector $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ to piecewise constant functions be defined via

$$P_\varepsilon v(x) = \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} v(\widehat{x})\mathrm{d}\widehat{x},$$

where $\fint_{\mathcal{O}} g(a)\mathrm{d}a = \frac{1}{\mu_d(\mathcal{O})}\int_{\mathcal{O}} g(a)\mathrm{d}a$ denotes the average of the function g over the set $\mathcal{O}$ with $\mu_d(\mathcal{O}) > 0$ and where $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ is defined by (3.1). Choose $\Delta > 0$ arbitrary but fixed. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $\Omega^+_\varepsilon \subset \mathrm{neigh}_\Delta(\Omega)$, where $\mathrm{neigh}_\Delta(\Omega)$ denotes the Δ-neighborhood of Ω. Moreover, for given $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists an extension $\overline{z}_0 \in \mathrm{W}^{1,p}_0(\mathrm{neigh}_\Delta(\Omega))^m$ with $\overline{z}_0|_\Omega = z_0$ according to Theorem A 6.12 in [2]. Then for $\varepsilon \in (0, \varepsilon_0)$ the sequence $(z_\varepsilon)_{\varepsilon>0}$ defined by $z_\varepsilon = (P_\varepsilon \overline{z}^{\mathrm{ex}}_0)|_\Omega \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ satisfies condition (4.4), see [13, Section 4]. Note that here the application of P_ε has to be understood component-wise.

5 Proof of Theorem 2.6

Since the sequence of material tensors $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ does provide better convergence properties with respect to the two-scale topology, the identification of the limit energy functional $\mathcal{E}_0 : [0,T]\times\mathcal{Q}_0(\Omega) \to \mathbb{R}$ is based on a two-scale translation of the sequence of microscopic energy functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$. For this purpose, for $p > 1$ we introduce the two-scale limit energy $\mathbf{E}_0 : [0,T]\times\mathcal{Q}_0(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ in the following way:

$$\begin{aligned}\mathbf{E}_0(t,u_0,z_0,U_1) = \tfrac{1}{2}\langle\mathbb{C}_0(z_0(\cdot))(\cdot)(\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)),\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}\\ + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle\ell(t),u_0\rangle.\end{aligned} \tag{5.1}$$

According to [13, Theorem 3.1], for all $(u,z) \in \mathcal{Q}_0(\Omega)$ it holds

$$\mathcal{E}_0(t,u,z) = \min\{\mathbf{E}_0(t,u,z,U)\,|\,U \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\}. \tag{5.2}$$

5.1 Mutual recovery sequence

This subsection is in preparation for proving the convergence of the microscopic models introduced in Subsection 2.1 to the effective model of Subsection 2.2. For this purpose, we are going to apply the evolutionary Γ-convergence method which is presented in [16] in an abstract setting. There, the authors pointed out that the crucial issue in performing the limit passage is to guarantee the stability of the limit when starting with a *stable sequence*. Hence, one of the main concerns of [16] is the provision of various sufficient conditions ensuring this stability. The existence of a *mutual recovery sequence* is requested and we are going to focus on one suitable definition and refer to [16] for the general theory.
The state spaces and functionals underlying the following definitions and theorems are those introduced in Section 2. Summarizing, this subsection contains the proof that there are subsequences of solutions of the microscopic models (S^ε) and (E^ε) which converge to a function satisfying the limit stability condition (S^0) for all $t \in [0,T]$ (see Theorem 2.6). We start with the following definitions:

Definition 5.1 (Stable sequence with respect to $t \in [0,T]$)**.** A sequence $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ satisfying $(u_\varepsilon,z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for every $\varepsilon > 0$ is called stable sequence with respect to the time $t \in [0,T]$ if the conditions (a) and (b) hold:

(a) There exists a function $(u_0,z_0) \in \mathcal{Q}_0(\Omega)$ such that:

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \qquad z_\varepsilon \to z_0 \quad \text{in } \mathrm{L}^p(\Omega)^m, \qquad R_{\frac{\varepsilon}{2}}(z_\varepsilon)|_\Omega \rightharpoonup \nabla z_0 \quad \text{in } \mathrm{L}^p(\Omega)^{m\times d},$$

(b) $(u_\varepsilon,z_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ for every $\varepsilon > 0$.

Definition 5.2 (Mutual recovery condition and mutual recovery sequence)**.** A sequences of functionals $(\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)_{\varepsilon\geq 0}$ fulfills the mutual recovery condition, if for every function $(\widetilde{u}_0,\widetilde{z}_0) \in \mathcal{Q}_0(\Omega)$ and for every stable sequence $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ with respect to $t \in [0,T]$ the following holds: There exists a sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ such that

$$\limsup_{\varepsilon\to 0}\mathcal{D}_\varepsilon(z_\varepsilon,\widetilde{z}_\varepsilon) \leq \mathcal{D}_0(z_0,\widetilde{z}_0) \tag{5.3}$$

and

$$\limsup_{\varepsilon\to 0} \left(\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon)\right) \leq \mathcal{E}_0(t,\widetilde{u}_0,\widetilde{z}_0) - \mathcal{E}_0(t,u_0,z_0). \tag{5.4}$$

Such a sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ is called mutual recovery sequence.

Remark 5.3. Observe that Definition 5.2 does not ask the mutual recovery sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ to converge to $(\widetilde{u}_0,\widetilde{z}_0)\in\mathcal{Q}_0(\Omega)$ in any sense.

Theorem 5.4 (Mutual recovery sequence)**.** *Assume that the conditions* (2.6) *hold. If* $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ *is a stable sequence with respect to some* $t\in[0,T]$ *with limit* $(u_0,z_0)\in\mathcal{Q}_0(\Omega)$*, then:*

(a) *For every* $(\widetilde{u}_0,\widetilde{z}_0)\in\mathcal{Q}_0(\Omega)$ *there exists a mutual recovery sequence* $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$.

(b) *The function* (u_0,z_0) *satisfies the stability condition* (S^0) *for* t.

The construction of the u-component of the mutual recovery sequence is based on the two-scale density result concerning Sobolev functions stated in Proposition 3.7. Starting with a given stable sequence $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ the z-component $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is explicitly constructed out of $z_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ in the proof of the following theorem.

Theorem 5.5 (z-component of the mutual recovery sequence)**.** *Let* $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ *be a stable sequence with respect to* $t\in[0,T]$ *with limit* $(u_0,z_0)\in\mathcal{Q}_0(\Omega)$*.*
Then for every $\widetilde{z}_0\in\mathrm{W}^{1,p}(\Omega;[0,1])^m$ *with* $\widetilde{z}_0\leq z_0$ *there exists a sequence* $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ *satisfying* $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$*,* $\widetilde{z}_\varepsilon\leq z_\varepsilon$ *component-wise,* $\widetilde{z}_\varepsilon\to\widetilde{z}_0$ *in* $\mathrm{L}^p(\Omega)^m$*,* $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega\rightharpoonup\nabla\widetilde{z}_0$ *in* $\mathrm{L}^p(\Omega)^{m\times d}$*, and*

$$\limsup_{\varepsilon\to 0}\left(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\right) \leq \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}. \tag{5.5}$$

The construction of the z-component of the mutual recovery sequence generalizes the construction in [19] to the discrete setting. In [19], the authors constructed a mutual recovery sequence for scalar Sobolev functions. The main steps of our proof stay the same but due to the discrete setting on the ε-level and the vectorial case, some new technicalities come into play. The main difficulties arise due to the irreversibility condition.

Proof. 1. Let $z_0,\widetilde{z}_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and $(z_\varepsilon)_{\varepsilon>0}$ be given as assumed in Theorem 5.5. Choose $\Delta>0$ arbitrary but fixed. Then there exists $\varepsilon_0>0$ such that $\Omega^+_\varepsilon\subset\mathrm{neigh}_\Delta(\Omega)$ for all $\varepsilon\in(0,\varepsilon_0)$. Moreover, there exists an extension $\overline{z}_0\in\mathrm{W}^{1,p}_0(\mathrm{neigh}_\Delta(\Omega);[0,1]^m)$ of $\widetilde{z}_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ satisfying $\overline{z}_0|_\Omega=\widetilde{z}_0$ according to Theorem A 6.12 in [2]. Let $P_\varepsilon:\mathrm{L}^p(\mathbb{R}^d)\to\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ denote the projector to piecewise constant functions introduced in Remark 4.4. Then $\overline{z}_\varepsilon=(P_\varepsilon(\overline{z}_0^{\mathrm{ex}}))|_\Omega$ satisfies

$$\lim_{\varepsilon\to 0}\left(\|\widetilde{z}_0-\overline{z}_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|(\nabla\widetilde{z}_0)^{\mathrm{ex}}-R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\right)=0, \tag{5.6}$$

as mentioned in Remark 4.4. Observe that the application of the projector P_ε to the function $\overline{z}_0^{\mathrm{ex}}\in\mathrm{L}^p(\mathbb{R}^d)^m$ has to be understood component-wise. Following the proof in

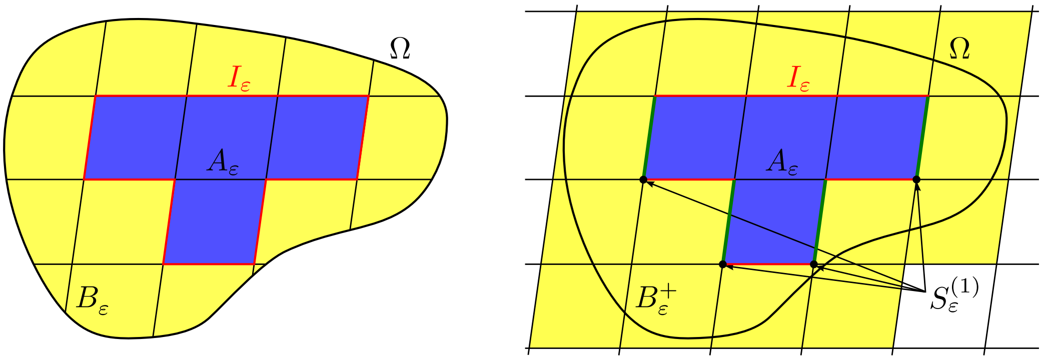

Figure 2: Decomposition of Ω into the subsets A_ε and B_ε.

[19] we introduce the function $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, decomposed for every component $\widetilde{z}_\varepsilon^{(j)}$, $j \in \{1,2,\ldots,m\}$, in the following way:

$$\widetilde{z}_\varepsilon^{(j)}(x) = \begin{cases} \max\{0, \overline{z}_\varepsilon^{(j)}(x) - \delta_\varepsilon^{(j)}\} & \text{if } x \in A_\varepsilon^{(j)} = \Omega_\varepsilon^- \backslash B_\varepsilon^{(j)}, \\ z_\varepsilon^{(j)}(x) & \text{if } x \in B_\varepsilon^{(j)} \cup (\Omega \backslash \Omega_\varepsilon^-), \end{cases}$$

where $B_\varepsilon^{(j)} = \{x \in \Omega_\varepsilon^- : z_\varepsilon^{(j)}(x) < \max\{0, \overline{z}_\varepsilon^{(j)}(x) - \delta_\varepsilon^{(j)}\}\}$. For $j \in \{1,2,\ldots,m\}$ the positive constants δ_ε^j will later be chosen in such a way that $\delta_\varepsilon^j \to 0$ for $\varepsilon \to 0$. This definition immediately results in $0 \le \widetilde{z}_\varepsilon \le z_\varepsilon$.

2. Now, we prove that $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$. Since $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$ is equivalent to $\widetilde{z}_\varepsilon^{(j)} \to \widetilde{z}_0^{(j)}$ in $\mathrm{L}^p(\Omega)$ for every $j \in \{1,2,\ldots,m\}$ we will restrict ourselves to the case $m=1$. Hence, let $A_\varepsilon = A_\varepsilon^{(1)}$, $B_\varepsilon = B_\varepsilon^{(1)}$, and $\delta_\varepsilon = \delta_\varepsilon^{(1)}$ to shorten notation. According to $|z_\varepsilon(x) - \widetilde{z}_0(x)| \le 1$, especially on B_ε, we find

$$\|\widetilde{z}_\varepsilon - \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)} \le \|\max\{0, \overline{z}_\varepsilon - \delta_\varepsilon\} - \widetilde{z}_0\|^p_{\mathrm{L}^p(A_\varepsilon)} + \mu_d(B_\varepsilon). \tag{5.7}$$

By increasing the domain of integration from A_ε to Ω, adding zero $(-\overline{z}_\varepsilon + \overline{z}_\varepsilon)$ and applying the triangle inequality, the first term of (5.7) is bounded by the expression $2^{p-1}\|\max\{0, \overline{z}_\varepsilon - \delta_\varepsilon\} - \overline{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)} + 2^{p-1}\|\overline{z}_\varepsilon - \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)}$. Hence, due to (5.6) the right hand side of (5.7) converges to zero if the sequence $(\delta_\varepsilon)_{\varepsilon>0}$ can be chosen such that $\delta_\varepsilon \to 0$ and $\mu_d(B_\varepsilon) \to 0$.

3. Choice of $\delta_\varepsilon > 0$: As before let $m=1$. Since $\widetilde{z}_0 = \overline{z}_0$ on Ω_ε^- by definition the identity $\overline{z}_\varepsilon = P_\varepsilon \widetilde{z}_0^{\mathrm{ex}}$ on Ω_ε^- holds. Combining this identity with the assumption $\widetilde{z}_0 \le z_0$ results in $\overline{z}_\varepsilon \le P_\varepsilon z_0^{\mathrm{ex}}$ on Ω_ε^-. Due to this estimate

$$B_\varepsilon \subset \{x \in \Omega_\varepsilon^- \mid z_\varepsilon(x) < \max\{0, P_\varepsilon z_0^{\mathrm{ex}}(x) - \delta_\varepsilon\}\} \subset \{x \in \Omega_\varepsilon^- \mid \delta_\varepsilon < |P_\varepsilon z_0^{\mathrm{ex}}(x) - z_\varepsilon(x)|\} = \widehat{B}_\varepsilon$$

such that Markov's inequality (**M**) can be exploited in the following way:

$$\mu_d(B_\varepsilon) \le \mu_d(\widehat{B}_\varepsilon) \overset{(\mathbf{M})}{\le} \tfrac{1}{\delta_\varepsilon^p} \int_{\Omega_\varepsilon^-} |P_\varepsilon z_0^{\mathrm{ex}}(x) - z_\varepsilon(x)|^p \mathrm{d}x.$$

By choosing $\delta_\varepsilon^p = \|P_\varepsilon z_0^{\mathrm{ex}} - z_\varepsilon\|_{\mathrm{L}^p(\Omega_\varepsilon^-)} \le \|P_\varepsilon z_0^{\mathrm{ex}} - z_0\|_{\mathrm{L}^p(\Omega)} + \|z_0 - z_\varepsilon\|_{\mathrm{L}^p(\Omega)}$, for instance, the assumed convergence $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)$ yields $\delta_\varepsilon \to 0$ and $\mu_d(B_\varepsilon) \to 0$ as $\varepsilon \to 0$. As already

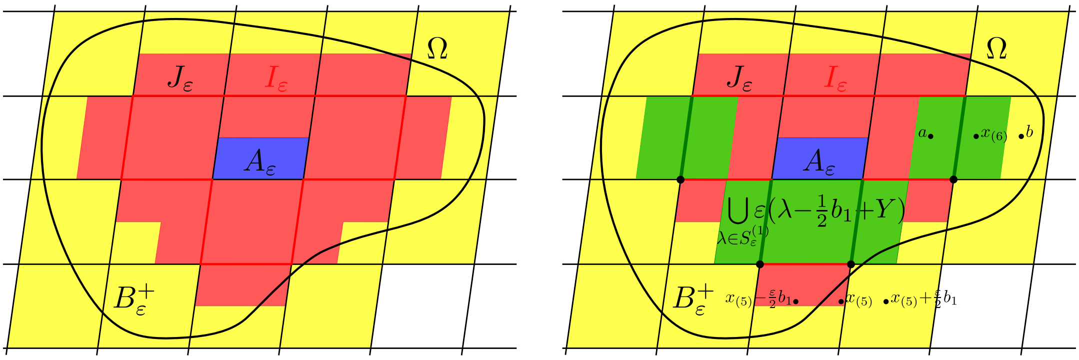

Figure 3: Here, $x_{(5)}$ and $x_{(6)}$ denote points considered in step 5 and 6, respectively.

mentioned in [19], $\delta_\varepsilon > 0$ is necessary to apply Markov's inequality. However, in the case of $\delta_\varepsilon = 0$ the assumed convergence $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)$ implies $(P_\varepsilon z_0^{\mathrm{ex}})|_\Omega - z_\varepsilon \to 0$ in $\mathrm{L}^p(\Omega)$ such that $\lim_{\varepsilon\to 0} \mu_d(\widehat{B}_\varepsilon) = 0$ results immediately.
4. To show: $\limsup_{\varepsilon\to 0} \big(\|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} - \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\big) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d}$:
Roughly spoken, the fact $\mu_d(B_\varepsilon) \to 0$ for $\varepsilon \to 0$ means that in the case of a sequence of Sobolev functions ($z_\varepsilon \in \mathrm{W}^{1,p}(\Omega)$) it is sufficient to prove (5.5) for A_ε instead of Ω_ε^+ on the left hand side. However, since we are interested in the case of piecewise constant functions we have to pay some special attention to the region around the interface $I_\varepsilon = \partial A_\varepsilon \cap \partial B_\varepsilon^+$, where $B_\varepsilon^+ = B_\varepsilon \cup (\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-)$. Note that due to the definition of A_ε and B_ε there are disjoint subsets $\Lambda_{A_\varepsilon}, \Lambda_{B_\varepsilon} \subset \Lambda_\varepsilon^-$ such that $A_\varepsilon = \bigcup_{\lambda\in\Lambda_{A_\varepsilon}} \varepsilon(\lambda+Y)$ and $B_\varepsilon = \bigcup_{\lambda\in\Lambda_{B_\varepsilon}} \varepsilon(\lambda+Y)$. Hence, for $\Lambda_{B_\varepsilon^+} = \Lambda_{B_\varepsilon} \cup (\Lambda_\varepsilon^+ \backslash \Lambda_\varepsilon^-)$ we have $B_\varepsilon^+ = \bigcup_{\lambda\in\Lambda_{B_\varepsilon^+}} \varepsilon(\lambda+Y)$.
For $i \in \{1, 2, \ldots, d\}$ let $n_i \in \mathbb{R}^d$ be given by condition (4.2) and let $F_{n_i}(\varepsilon\lambda)$ denote the face of $\varepsilon(\lambda+Y)$ orthogonal to $n_i \in \mathbb{R}^d$ which is contained in $\varepsilon(\lambda+Y)$. Then, the interface I_ε can be uniquely represented by $I_\varepsilon = \bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \overline{F_{n_i}(\varepsilon\lambda)}$, where $S_\varepsilon^{(i)} \subset \Lambda$ is a suitable finite subset and $\bigcup_{\lambda\in S_\varepsilon^{(i)}} \overline{F_{n_i}(\varepsilon\lambda)}$ are all faces of the interface I_ε that are orthogonal to $n_i \in \mathbb{R}^d$. Observe that $|S_\varepsilon^{(i)}| \le |\Lambda_{B_\varepsilon^+}|$ since the number of faces in $S_\varepsilon^{(i)}$ is bounded by the number of all cells contained in B_ε^+.
Taking the union of all cells

$$J_\varepsilon = \bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \varepsilon(\lambda-\tfrac{1}{2}b_i+Y)$$

containing the face $F_{n_i}(\varepsilon\lambda)$ in the middle (see Figure 3) we have $I_\varepsilon \subset \overline{J_\varepsilon}$ and

$$\mu_d(J_\varepsilon) \le \sum_{i=1}^d \sum_{\lambda\in S_\varepsilon^{(i)}} \varepsilon^d = \sum_{i=1}^d |S_\varepsilon^{(i)}|\varepsilon^d \le \sum_{i=1}^d |\Lambda_{B_\varepsilon^+}|\varepsilon^d = d\mu_d(B_\varepsilon^+). \tag{5.8}$$

The set J_ε has been constructed in such a way that $x \in A_\varepsilon \backslash J_\varepsilon$ implies $x+\frac{\varepsilon}{2}b_i \in A_\varepsilon$ and $x-\frac{\varepsilon}{2}b_i \in A_\varepsilon$ and the analog statement is valid on $B_\varepsilon^+ \backslash J_\varepsilon$. Hence, by exploiting the structure of $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ given by Definition 4.1 we have

$$R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon = \begin{cases} R_{\frac{\varepsilon}{2}}(\max\{0,\overline{z}_\varepsilon-\delta_\varepsilon\}) & \text{in } A_\varepsilon\backslash J_\varepsilon, \\ R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon & \text{in } J_\varepsilon, \\ R_{\frac{\varepsilon}{2}}z_\varepsilon & \text{in } B_\varepsilon^+\backslash J_\varepsilon. \end{cases} \tag{5.9}$$

Keeping (5.5) in mind, we want to estimate $|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|^p$ from above by terms depending only on z_ε and $\overline{z}_\varepsilon$. Due to (5.9) we only have to care about the case $x\in J_\varepsilon$. Therefore, we consider every component $(R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon(x))b_i$ separately.

5. The case $x\in J_\varepsilon\backslash\bigcup_{\lambda\in S_\varepsilon^{(i)}}\varepsilon(\lambda-\frac{1}{2}b_i+Y)$ for $i\in\{1,\dots,d\}$ fixed:
In this case either $x+\frac{\varepsilon}{2}b_i\in A_\varepsilon$ and $x-\frac{\varepsilon}{2}b_i\in A_\varepsilon$ or $x+\frac{\varepsilon}{2}b_i\in B_\varepsilon^+$ and $x-\frac{\varepsilon}{2}b_i\in B_\varepsilon^+$. Combining this result with the definition of the function $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and the structure of the discrete gradient yields the desired estimate

$$\big|(R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon(x))b_i\big| \le \max\big\{\big|(R_{\frac{\varepsilon}{2}}(\max\{0,\overline{z}_\varepsilon(x)-\delta_\varepsilon\}))b_i\big|,\big|(R_{\frac{\varepsilon}{2}}z_\varepsilon(x))b_i\big|\big\}. \tag{5.10}$$

6. The case $x\in\bigcup_{\lambda\in S_\varepsilon^{(i)}}\varepsilon(\lambda-\frac{1}{2}b_i+Y)$ for $i\in\{1,\dots,d\}$ fixed:
In this case either $x+\frac{\varepsilon}{2}b_i\in A_\varepsilon$ and $x-\frac{\varepsilon}{2}b_i\in B_\varepsilon^+$ or $x+\frac{\varepsilon}{2}b_i\in B_\varepsilon^+$ and $x-\frac{\varepsilon}{2}b_i\in A_\varepsilon$ according to the definition of $S_\varepsilon^{(i)}$. Without loss of generality set $a=x+\frac{\varepsilon}{2}b_i\in A_\varepsilon$ and $b=x-\frac{\varepsilon}{2}b_i\in B_\varepsilon^+$. Then due to the definitions of A_ε and B_ε^+ we have

$$1\ge z_\varepsilon(a)\ge\widetilde{z}_\varepsilon(a)=\max\{0,\overline{z}_\varepsilon(a)-\delta_\varepsilon\}\ge 0, \tag{5.11a}$$
$$1\ge\max\{0,\overline{z}_\varepsilon(b)-\delta_\varepsilon\}>\widetilde{z}_\varepsilon(b)=z_\varepsilon(b)\ge 0. \tag{5.11b}$$

Since $b\in B_\varepsilon^+\backslash B_\varepsilon=\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-$ is possible, in relation (5.11b) and in the following table every function has to be understood as its extension with respect to the continuation operator $V_\varepsilon:\mathrm{K}_{\varepsilon\Lambda}(\Omega)\to\mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)$ given by (4.1). Keeping this remark in mind the following estimates are valid.

	if $\widetilde{z}_\varepsilon(a)\ge\widetilde{z}_\varepsilon(b)$	if $\widetilde{z}_\varepsilon(a)<\widetilde{z}_\varepsilon(b)$
$\lvert\widetilde{z}_\varepsilon(a)-\widetilde{z}_\varepsilon(b)\rvert$	$=\widetilde{z}_\varepsilon(a)-\widetilde{z}_\varepsilon(b)$ $\overset{(5.11a)}{\le} z_\varepsilon(a)-\widetilde{z}_\varepsilon(b)$ $\overset{(5.11b)}{=} z_\varepsilon(a)-z_\varepsilon(b)$	$=\widetilde{z}_\varepsilon(b)-\widetilde{z}_\varepsilon(a)$ $\overset{(5.11b)}{<}\max\{0,\overline{z}_\varepsilon(b)-\delta_\varepsilon\}-\widetilde{z}_\varepsilon(a)$ $\overset{(5.11a)}{=}\max\{0,\overline{z}_\varepsilon(b)-\delta_\varepsilon\}-\max\{0,\overline{z}_\varepsilon(a)-\delta_\varepsilon\}$

Hence, we also find

$$\big|(R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon(x))b_i\big| \le \max\big\{\big|(R_{\frac{\varepsilon}{2}}(\max\{0,\overline{z}_\varepsilon(x)-\delta_\varepsilon\}))b_i\big|,\big|(R_{\frac{\varepsilon}{2}}z_\varepsilon(x))b_i\big|\big\}, \tag{5.12}$$

for all $x\in\bigcup_{\lambda\in S_\varepsilon^{(i)}}\varepsilon(\lambda-\frac{1}{2}b_i+Y)$.

7. Summary of the case $x\in J_\varepsilon$: Combining (5.10) and (5.12) these inequalities hold for every $x\in J_\varepsilon$, which finally results in

$$|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|^p \le \begin{cases} |R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon|^p & \text{in } A_\varepsilon\backslash J_\varepsilon, \\ |R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon|^p+|R_{\frac{\varepsilon}{2}}z_\varepsilon|^p & \text{in } J_\varepsilon, \\ |R_{\frac{\varepsilon}{2}}z_\varepsilon|^p & \text{in } B_\varepsilon^+\backslash J_\varepsilon, \end{cases} \tag{5.13}$$

by recalling (5.9), since $|\max\{C_1, C_2\}|^p \leq |C_1|^p + |C_2|^p$ and since

$$|R_{\frac{\varepsilon}{2}} \max\{0, \overline{z}_\varepsilon(x) - \delta_\varepsilon\}| \leq |R_{\frac{\varepsilon}{2}}(\overline{z}_\varepsilon(x) - \delta_\varepsilon)| = |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|.$$

Exploiting (5.13) we conclude in the case $m = 1$ that

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \big(\|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} - \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\big)& \\
\leq \limsup_{\varepsilon \to 0} \bigg(\int_{A_\varepsilon \setminus J_\varepsilon} & |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \\
&+ \int_{B_\varepsilon^+ \setminus J_\varepsilon} |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \\
&+ \int_{J_\varepsilon} |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p + |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \bigg) \\
= \limsup_{\varepsilon \to 0} \bigg(\int_{A_\varepsilon \cup J_\varepsilon} & |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p \mathrm{d}x - \int_{A_\varepsilon \setminus J_\varepsilon} |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \bigg) \\
\leq \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon^+} & |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p \mathrm{d}x - \liminf_{\varepsilon \to 0} \int_\Omega |\mathbb{1}_{A_\varepsilon \setminus J_\varepsilon}(x) R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \\
= \|\nabla \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d} & - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d},
\end{aligned}$$

where in the second last line the first term converges to $\|\nabla \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d}$ according to (5.6). Moreover, weak lower semi-continuity of the norm together with the weak convergence $\mathbb{1}_{A_\varepsilon \setminus J_\varepsilon} R_{\frac{\varepsilon}{2}} z_\varepsilon \rightharpoonup \nabla z_0$ in $\mathrm{L}^p(\Omega)^d$ is exploited for the second one. Note that due to estimate (5.8) we have $\mathbb{1}_{A_\varepsilon \setminus J_\varepsilon} \to \mathbb{1}_\Omega$ in $\mathrm{L}^q(\Omega)$ for every $q \in [1, \infty)$, since $\lim_{\varepsilon \to 0} \mu_d(B_\varepsilon) = 0$ implies $\lim_{\varepsilon \to 0} \mu_d(B_\varepsilon^+) = 0$.

8. The general case $m > 1$: Up to now, in the case $m > 1$ it holds ($j \in \{1, 2, \dots, m\}$)

$$\limsup_{\varepsilon \to 0} \big(\|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon^{(j)}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} - \|R_{\frac{\varepsilon}{2}} v_\varepsilon^{(j)}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\big) \leq \|\nabla \widetilde{z}_0^{(j)}\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla v_0^{(j)}\|^p_{\mathrm{L}^p(\Omega)^d}$$

for every component $\widetilde{z}_\varepsilon^{(j)}, v_\varepsilon^{(j)}, \widetilde{z}_0^{(j)}, v_0^{(j)}$ of the functions $\widetilde{z}_\varepsilon, z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ and $\widetilde{z}_0, z_0 \in \mathrm{W}^{1,p}(\Omega; [0,1])^m$. Summing up these inequalities for all $j = 1, 2, \dots, m$ we finally have

$$\limsup_{\varepsilon \to 0} \big(\|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\big) \leq \|\nabla \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}.$$

9. $R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$: According to step 8, Theorem 4.2 can be applied for the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$. Moreover, due to step 2 the limit-function of Theorem 4.2 is identified as $\widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ which altogether yields $R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ for a subsequence (not relabeled). □

Now, Theorem 5.5 enables us to construct the mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$.

Proof of Theorem 5.4. Part (a): Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a the stable sequence with respect to $t \in [0,T]$ converging to the limit $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$; see Definition 5.1. Then, for a given function $(\widetilde{u}_0, \widetilde{z}_0) \in \mathcal{Q}_0(\Omega)$ we start by constructing the mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$.
1. First, the z-component $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is constructed and (5.3) is verified. Observe that in the case of $\mathcal{D}_0(z_0,\widetilde{z}_0)=\infty$, the lim sup-inequality (5.3) is trivially fulfilled for the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ mentioned in Remark 4.4. Hence, without loss of generality we assume $\widetilde{z}_0 \le z_0$ (component-wise) from now on. According to Theorem 5.5 there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ satisfying $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, $\widetilde{z}_\varepsilon \le z_\varepsilon$, $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, and

$$\limsup_{\varepsilon\to0}\big(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\big) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}.$$

Recalling the structure of the involved functionals results in $\lim_{\varepsilon\to0}\mathcal{D}_\varepsilon(z_\varepsilon,\widetilde{z}_\varepsilon) = \mathcal{D}_0(z_0,\widetilde{z}_0)$ and (5.3) is shown.
2. Now, the u-component $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is constructed. Since $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ by assumption, according to Proposition 3.6 there exists a function $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ such that $\nabla u_\varepsilon \xrightharpoonup{w} \nabla_x E u_0 + \nabla_y U_1$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ at least for a subsequence (not relabeled). For $(u,z)=(\widetilde{u}_0,\widetilde{z}_0)$ let $\widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ be the unique solution of (5.2). Therefore,

$$\mathcal{E}_0(t,\widetilde{u}_0,\widetilde{z}_0) = \mathbf{E}_0(t,\widetilde{u}_0,\widetilde{z}_0,\widetilde{U}_1) \tag{5.14}$$

by definition. Adopting the notation of Proposition 3.7 let $w_\varepsilon \in \mathrm{H}^1_0(\Omega)^d$ be the solution of the elliptic problem stated there with $w_0 :\equiv 0 \in \mathrm{H}^1_0(\Omega)^d$ and $W_1 = \widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Then according to Proposition 3.7 we have $w_\varepsilon \rightharpoonup 0$ in $\mathrm{H}^1_0(\Omega)^d$, $w_\varepsilon \xrightarrow{s} 0$ in $\mathrm{L}^2(\Omega\times Y)^d$, and $\nabla w_\varepsilon \xrightarrow{s} \nabla_y\widetilde{U}_1$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$. Thus, the u-component of the mutual recovery sequence is defined via

$$\widetilde{u}_\varepsilon = \widetilde{u}_0 + w_\varepsilon.$$

Using property (b) of Proposition 3.4 and the convergence results for $(w_\varepsilon)_{\varepsilon>0}$ we find

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d,\\ \nabla\widetilde{u}_\varepsilon &\xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y\widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

3. Now we are in the position to prove the lim sup-inequality stated in (5.4). According to the assumption and step 2 we have $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ which implies

$$\lim_{\varepsilon\to0}\big(\langle \ell(t),u_\varepsilon\rangle - \langle \ell(t),\widetilde{u}_\varepsilon\rangle\big) = \langle \ell(t),u_0\rangle - \langle \ell(t),\widetilde{u}_0\rangle.$$

4. Next we prove that

$$\begin{aligned} \limsup_{\varepsilon\to0}\big(&\langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon),\mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} - \langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon),\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\big)\\ &\le \langle \mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)(\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1)),\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}\\ &\quad - \langle \mathbb{C}_0(z_0(\cdot))(\cdot)(\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)),\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \end{aligned} \tag{5.15}$$

Combining this with the convergence results of step 1 and 3 implies the lim sup-inequality (5.4). To show relation (5.15) we are going to prove

$$\lim_{\varepsilon\to 0}\langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon),\mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}=\langle \mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)(\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1)),\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} \tag{5.16}$$

and

$$\liminf_{\varepsilon\to 0}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon),\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\geq\langle \mathbb{C}_0(z_0(\cdot))(\cdot)(\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)),\mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \tag{5.17}$$

Ad (5.16): Since $\widetilde{z}_\varepsilon\to\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$ according to Theorem 3.8 we have $\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\overset{s}{\to}\mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)$ in $\mathrm{L}^1(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$. Adopting the notation of Corollary 3.5 let $m_\varepsilon=\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)$, $M_0=\mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)$, and $v_\varepsilon=\mathbf{e}(\widetilde{u}_\varepsilon)$, $V_0=\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1)$. Then Corollary 3.5 together with the convergence results for $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ give $w_\varepsilon=\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon)\overset{s}{\to}\mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)(\mathbf{e}_x(\widetilde{u}_0)+\mathbf{e}_y(\widetilde{U}_1))=W_0$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$. With this, Proposition 3.4(a) yields (5.16).
Ad (5.17): We start with the following integral identity valid according to identity (3.2) and the product rule for the unfolding operator $\mathcal{T}_\varepsilon$:

$$\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon),\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}=\langle \mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon),\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}. \tag{5.18}$$

Since $z_\varepsilon\to z_0$ in $\mathrm{L}^p(\Omega)^m$ according to Theorem 3.8 we have $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\to\mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)$ in $\mathrm{L}^1(\mathbb{R}^d\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}},0\})$. Moreover, due to the definition of two-scale convergence it holds $\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rightharpoonup\mathbf{e}_x^{\mathrm{ex}}(u_0)+\mathbf{e}_y^{\mathrm{ex}}(U_1)$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}$, which enables us to apply Theorem 3.23 of [6] yielding the following inequality:

$$\begin{aligned}\liminf_{\varepsilon'\to 0}&\langle \mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon),\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}\\&\geq\langle \mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)(\mathbf{e}_x^{\mathrm{ex}}(u_0)+\mathbf{e}_y^{\mathrm{ex}}(U_1)),\mathbf{e}_x^{\mathrm{ex}}(u_0)+\mathbf{e}_y^{\mathrm{ex}}(U_1)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}.\end{aligned}$$

Taking into account that $\mathrm{supp}(\mathbb{C}_0^{\mathrm{ex}}(z_0))\subset\Omega\times Y$ this inequality together with (5.18) gives (5.17). Combining the convergence results of step 1, step 3, and (5.15) with the equality (5.14) we showed

$$\begin{aligned}\limsup_{\varepsilon\to 0}(\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)-\mathcal{E}_\varepsilon(t,u_\varepsilon,z_\varepsilon))&\leq\mathcal{E}_0(t,\widetilde{u}_0,\widetilde{z}_0)-\mathbf{E}_0(t,u_0,z_0,U_1)\\&\leq\mathcal{E}_0(t,\widetilde{u}_0,\widetilde{z}_0)-\mathcal{E}_0(t,u_0,z_0),\end{aligned}$$

where we minimized the right hand side with respect to all functions of $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. With this, the proof of point (a) in Theorem 5.4 is done.
Part (b) is a consequence of point (a): Let $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ be a stable sequence with respect to $t\in[0,T]$ converging to the limit $(u_0,z_0)\in\mathcal{Q}_0(\Omega)$; see Definition 5.1. Then, for an arbitrary function $(\widetilde{u}_0,\widetilde{z}_0)\in\mathcal{Q}_0(\Omega)$ with $\widetilde{z}_0\leq z_0$ choose $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ as constructed in the steps 1 and 2. Note that in the case $\widetilde{z}_0\not\leq z_0$ according to $\mathcal{D}_0(\widetilde{z}_0,z_0)=\infty$ the stability

condition (S^0) is trivially fulfilled. Due to the stability of $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ at time $t \in [0,T]$ we have

$$0 \leq \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) + \mathcal{D}_\varepsilon(z_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon).$$

Applying the limsup with respect to the sequence $(\varepsilon)_{\varepsilon>0}$ to the right hand side according to (5.3) and (5.4) results in

$$0 \leq \mathcal{E}_0(t, \widetilde{u}_0, \widetilde{z}_0) + \mathcal{D}_0(z_0, \widetilde{z}_0) - \mathcal{E}_0(t, u_0, z_0),$$

which is nothing else than the stability condition (S^0) of $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$ at time $t \in [0,T]$ for the arbitrarily chosen test-function $(\widetilde{u}_0, \widetilde{z}_0) \in \mathcal{Q}_0(\Omega)$. ☐

5.2 Convergence result

This subsection provides the main result of this paper, saying that the model of Subsection 2.2 is the limit of the microscopic models introduced in Subsection 2.1. However, before that we show that $\mathcal{E}_0 : [0,T]\times\mathcal{Q}_0(\Omega) \to \mathbb{R}$ is the Γ-limit of the sequence $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ of functionals $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ with respect to our special topology.

Theorem 5.6 ($\mathcal{E}_\varepsilon \stackrel{\Gamma}{\rightharpoonup} \mathcal{E}_0$)**.** *Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a sequence satisfying $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ and*

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \qquad z_\varepsilon \to z_0 \quad \text{in } \mathrm{L}^p(\Omega)^m, \qquad R_{\frac{\varepsilon}{2}}(z_\varepsilon)|_\Omega \rightharpoonup \nabla z_0 \quad \text{in } \mathrm{L}^p(\Omega)^{m\times d}.$$

Then for every $t \in [0,T]$ it holds $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathcal{E}_0(t, u_0, z_0)$. Moreover, for every function $(\widetilde{u}_0, \widetilde{z}_0) \in \mathcal{Q}_0(\Omega)$ there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for every $\varepsilon > 0$, with

$$\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0 \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \qquad \widetilde{z}_\varepsilon \to \widetilde{z}_0 \quad \text{in } \mathrm{L}^p(\Omega)^m, \qquad R_{\frac{\varepsilon}{2}}(\widetilde{z}_\varepsilon)|_\Omega \to \nabla \widetilde{z}_0 \quad \text{in } \mathrm{L}^p(\Omega)^{m\times d},$$

and with $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) = \mathcal{E}_0(t, \widetilde{u}_0, \widetilde{z}_0)$.

Proof. Ad lim inf-inequality: Due to the assumptions of Theorem 5.6 we already have $\lim_{\varepsilon\to 0}\langle \ell(t), u_\varepsilon\rangle = \langle \ell(t), u_0\rangle$ and $\liminf_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(z_\varepsilon)\|_{\mathrm{L}^p(\Omega)^{m\times d}} \geq \|\nabla z_0\|_{\mathrm{L}^p(\Omega)^{m\times d}}$. Moreover, Theorem 3.8 states $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon) \to \mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)$ in $\mathrm{L}^1(\mathbb{R}^d\times Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}, 0\})$. According to Proposition 3.6 there exists a function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ such that $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla_x u_0^{\mathrm{ex}} + \nabla_y U_1^{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}$ at least for a subsequence. Thus, we are in the position to apply Theorem 3.23 of [6] which yields the following inequality:

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\langle \mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon), \mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}\\ &\geq \langle \mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)(\mathbf{e}_x^{\mathrm{ex}}(u_0)+\mathbf{e}_y^{\mathrm{ex}}(U_1)), \mathbf{e}_x^{\mathrm{ex}}(u_0)+\mathbf{e}_y^{\mathrm{ex}}(U_1)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}.\end{aligned}$$

Recalling the definition of $\mathbf{E}_0 : [0,T]\times\mathcal{Q}_0(\Omega)\times\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ (see (5.1)) we proved $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathbf{E}_0(t, u_0, z_0, U_1)$ for every $t \in [0,T]$, by taking the integral identity

(3.2) and $\mathrm{supp}(\mathbb{C}_0^{\mathrm{ex}}(z_0(\cdot))(\cdot)) \subset \Omega\times Y$ into account. This immediately gives us the estimate $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathcal{E}_0(t, u_0, z_0)$ due to (5.2).

Ad lim(sup)-(in)equality: For a given function $(\widetilde{u}_0, \widetilde{z}_0) \in \mathcal{Q}_0(\Omega)$ and $(u, z) = (\widetilde{u}_0, \widetilde{z}_0)$ let $\widetilde{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ be the minimizer of (5.2). For $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ chosen as in step 2 of the proof of Theorem 5.4 it holds

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \\ \nabla \widetilde{u}_\varepsilon &\overset{s}{\to} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

According to Theorem 4.3 for $\widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ such that $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, and $R_{\frac{\varepsilon}{2}}(\widetilde{z}_\varepsilon)|_\Omega \to \nabla \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$. Moreover, condition (4.4) implies

$$\lim_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(\widetilde{z}_\varepsilon)\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} = \|\nabla \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}. \tag{5.19}$$

Finally, Theorem 3.8 yields $\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon) \overset{s}{\to} \mathbb{C}_0(\widetilde{z}_0)$ in $\mathrm{L}^1(\Omega\times Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\})$. By adopting the notation of Corollary 3.5, with $m_\varepsilon = \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)$, $M_0 = \mathbb{C}_0(\widetilde{z}_0)$, $v_\varepsilon = \mathbf{e}(\widetilde{u}_\varepsilon)$, and $V_0 = \mathbf{e}_x(\widetilde{u}_0) + \mathbf{e}_y(\widetilde{U}_1)$ we have $w_\varepsilon = \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon) \overset{s}{\to} \mathbb{C}_0(\widetilde{z}_0)(\mathbf{e}_x(\widetilde{u}_0) + \mathbf{e}_y(\widetilde{U}_1)) = W_0$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$. Additionally exploiting Proposition 3.4(a) results in

$$\begin{aligned} &\lim_{\varepsilon\to 0} \langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \\ &\qquad = \langle \mathbb{C}_0(\widetilde{z}_0(\cdot))(\cdot)(\mathbf{e}_x(\widetilde{u}_0) + \mathbf{e}_y(\widetilde{U}_1)), \mathbf{e}_x(\widetilde{u}_0) + \mathbf{e}_y(\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \end{aligned} \tag{5.20}$$

Combining (5.19), (5.20), and $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon\rangle = \langle \ell(t), \widetilde{u}_0\rangle$ concludes the proof. □

Now we are in the position to state the final result of this paper, saying that the sequence of solutions of the microscopic models (S^ε) and (E^ε) introduced in Subsection 2.1 converges to a solution of the effective limit model (S^0) and (E^0) introduced in Subsection 2.2.

Theorem 5.7 (Convergence result ensuring the existence of solutions to (S^0) and (E^0))**.** *Let the material tensors $\mathbb{C}_{\mathrm{strong}}$ as well as $\mathbb{C}_{\mathrm{weak}}$ be positive definite and assume that the conditions* (2.6) *hold. If for every $\varepsilon > 0$ the function $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ is an energetic solution of* (S^ε) *and* (E^ε) *with $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ and if there exists a tuple $(u_0^0, z_0^0) \in \mathcal{Q}_0(\Omega)$ of initial values of* (S^0) *and* (E^0) *such that*

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) = \mathcal{E}_0(0, u_0^0, z_0^0)$$

then there exists a function $(u_0, z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ with

$$\begin{aligned} &u_0 \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d), \\ &z_0 \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega; [0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_0}([0,T]; \mathrm{W}^{1,p}(\Omega; [0,1]^m)) \end{aligned}$$

and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) satisfying for all $t \in [0,T]$

$$\begin{aligned} u_\varepsilon(t) \rightharpoonup u_0(t) \quad &\text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \qquad z_\varepsilon(t) \to z_0(t) \quad \text{in } \mathrm{L}^p(\Omega)^m, \\ &R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))|_\Omega \rightharpoonup \nabla z_0(t) \ \text{in } \mathrm{L}^p(\Omega)^{m\times d}. \end{aligned}$$

Furthermore, $(u_0, z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ *is an energetic solution to* (S^0) *and* (E^0) *with* $(u_0(0), z_0(0)) = (u_0^0, z_0^0)$*. Additionally, for all* $t \in [0,T]$ *it holds*

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) = \mathcal{E}_0(t, u_0(t), z_0(t)),$$
$$\lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon\varepsilon}(z_\varepsilon; [0,t]) = \mathrm{Diss}_{\mathcal{D}_0}(z_0; [0,t]).$$

Note that since $(u_0^0, z_0^0) \in \mathcal{Q}_0(\Omega)$ are assumed to be initial values of (S^0) and (E^0) the tuple (u_0^0, z_0^0) has to satisfy the stability condition (S^0) at time $t = 0$.

Proof. 1. Let $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ be an energetic solution of (S^ε) and (E^ε) with $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$. We start by proving a priori estimates. Due to Korn's inequality, for $C_\ell = \|\ell\|_{\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)} < \infty$ inequality (5.21) below is obtained and is further estimated by exploiting the non-negativity of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$ in the energy balance (E^ε).

$$\begin{aligned} C_{\mathrm{Korn}} \|u_\varepsilon(t)\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} &\leq \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \\ &\overset{(\mathrm{E}^\varepsilon)}{\leq} \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \end{aligned} \tag{5.21}$$

According to the assumptions on $(u_\varepsilon^0, z_\varepsilon^0)_{\varepsilon>0}$ there exists a constant $C_0 > 0$ such that $\mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) \leq C_0$ for all $\varepsilon > 0$. Applying the scaled version of Young's estimate to the product $C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}$ on the right hand side of (5.21) and taking the supremum with respect to $t \in [0,T]$ on both sides afterwards, yields the uniform estimate

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq c, \tag{5.22}$$

where $c > 0$ only depends on $C_0 > 0$, $T > 0$, and $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. This estimate implies that the energy balance's right hand side is uniformly bounded which results in a uniform bound for the total dissipation $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$ on its left hand side. Hence, $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ is a (component-wise) non-increasing function. Estimating $\|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}$ in the same way as in (5.21) gives

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} \leq C_0 + cC_\ell(T+1),$$

where we already exploited (5.22). Moreover, $\|z_\varepsilon(t)\|^p_{\mathrm{L}^p(\Omega)^m} \leq m\mu_d(\Omega)$ for every $\varepsilon > 0$ and all $t \in [0,T]$ since $0 \leq z_\varepsilon(t) \leq 1$ by definition. Combining all estimates results in the following uniform bound of the solution $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$: There exists a constant $C > 0$ depending only on $C_0 > 0$, $T > 0$, and $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ such that for all $\varepsilon > 0$ it holds:

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \big(\|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} + \|z_\varepsilon(t)\|^p_{\mathrm{L}^p(\Omega)^m} + \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\big) \leq C. \tag{5.23}$$

2. Now we are going to construct a function $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ and choose a subsequence $(\widetilde{\varepsilon})_{\widetilde{\varepsilon}>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for any $t \in [0,T]$ the sequence $(z_{\widetilde{\varepsilon}}(t))_{\widetilde{\varepsilon}>0}$ converges to

$z_0(t)$ with respect to the strong L^1-topology. Similarly to the proceeding in [14, Section 3], we start by constructing the function $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega;[0,1]^m)$. This construction is based on the limit of the sequence $(F_\varepsilon)_{\varepsilon>0}$ of functions $F_\varepsilon : [0,T] \to \mathbb{R}$ defined via

$$F_\varepsilon(t) = \|z_\varepsilon(t)\|_{\mathrm{L}^1_1(\Omega)^m}, \tag{5.24}$$

where the subscript *1* denotes that the space $\mathrm{L}^1(\Omega)^m$ for $v \in \mathrm{L}^1(\Omega)^m$ is equipped with the norm $\|v\|_{\mathrm{L}^1_1(\Omega)^m} = \sum_{j=1}^m \|v_j\|_{\mathrm{L}^1(\Omega)}$. As already mentioned in step 1, $F_\varepsilon : [0,T] \to \mathbb{R}$ is monotonously decreasing and uniformly bounded by $m\mu_d(\Omega)$. Therefore, the Helly selection principle is applicable saying that there exists a monotonously decreasing function $F_0 \in \mathrm{BV}([0,T];\mathbb{R})$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for all $t \in [0,T]$ it holds

$$F_{\varepsilon'}(t) \overset{\varepsilon'\to 0}{\longrightarrow} F_0(t). \tag{5.25}$$

Let $J_0 \subset [0,T]$ be the jump set of F_0, which is at most countable since $F_0 \in \mathrm{BV}([0,T];\mathbb{R})$ is monotone. Furthermore, let $K_T \subset [0,T]\backslash J_0$ be a dense and countable subset and choose $(t_n)_{n\in\mathbb{N}}$ such that $(t_n)_{n\in\mathbb{N}} = K_T \cup J_0$. For arbitrary but fixed $n \in \mathbb{N}$ according to the uniform bound (5.23) the assumptions of Theorem 4.2 and Theorem 3.8 for the sequence $(z_{\varepsilon'}(t_n))_{\varepsilon'>0}$ are satisfied. Hence, there exists a function $z_0^{(t_n)} \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and a subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ satisfying for $\varepsilon \to 0$

$$z_{\varepsilon''}(t_n) \to z_0^{(t_n)} \quad \text{in } \mathrm{L}^p(\Omega)^m, \tag{5.26a}$$

$$R_{\frac{\varepsilon''}{2}}(z_{\varepsilon''}(t_n))|_\Omega \rightharpoonup \nabla z_0^{(t_n)} \quad \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \tag{5.26b}$$

$$\mathbb{C}_{\varepsilon''}(z_{\varepsilon''}(t_n)) \overset{s}{\to} \mathbb{C}_0(z_0^{(t_n)}) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}). \tag{5.26c}$$

Let $(z_0^{(t_n)})_{n\in\mathbb{N}} \subset \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ denote the set of all limit functions. Since $(t_n)_{n\in\mathbb{N}}$ is a countable set, by a diagonalization argument we are able to construct a (possibly different but not relabeled) subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ satisfying (5.26) for all $n \in \mathbb{N}$.
Due to (5.26a) for all $n \in \mathbb{N}$ we have $F_{\varepsilon''}(t_n) = \|z_{\varepsilon''}(t_n)\|_{\mathrm{L}^1_1(\Omega)^m} \overset{\varepsilon''\to 0}{\longrightarrow} \|z_0^{(t_n)}\|_{\mathrm{L}^1_1(\Omega)^m}$ which results in $F_0(t_n) = \|z_0^{(t_n)}\|_{\mathrm{L}^1_1(\Omega)^m}$ by keeping (5.25) in mind. Moreover, the monotonicity of $z_{\varepsilon''} : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ together with (5.26a) results in $z_0^{(t_l)} \le z_0^{(t_k)}$ for all $t_k < t_l \in K_T$. According to this relation of $z_0^{(t_k)}$ and $z_0^{(t_l)}$ for $t_k < t_l \in K_T$ we find

$$C_m\|z_0^{(t_k)}-z_0^{(t_l)}\|_{\mathrm{L}^1(\Omega)^m} \le \|z_0^{(t_k)}-z_0^{(t_l)}\|_{\mathrm{L}^1_1(\Omega)^m} = \|z_0^{(t_k)}\|_{\mathrm{L}^1_1(\Omega)^m}-\|z_0^{(t_l)}\|_{\mathrm{L}^1_1(\Omega)^m} = F_0(t_k)-F_0(t_l)$$

which due to the continuity of F_0 on $[0,T]\backslash J_0 \supset K_T$ converges to 0 for $t_k \nearrow t_l$ or $t_l \searrow t_k$. Here, $C_m > 0$ is the constant resulting from the utilization of the norm equivalence in dimension m. Hence, the function $\zeta_0 : K_T \to \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ for all $t_k \in K_T$ defined by $\zeta_0(t_k) = z_0^{(t_k)}$ is continuous with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)^m}$. This function enables us to construct the limit function $z_0 : [0,T] \to \mathrm{L}^1(\Omega)^m$ in the following way:

(a) $z_0(t_n) = z_0^{(t_n)}$ for all $n \in \mathbb{N}$,

(b) $z_0|_{[0,T]\backslash J_0}$ is the continuous extension of ζ_0 with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)^m}$.

Observe that according to $J_0 \subset (t_n)_{n\in\mathbb{N}}$ and the density of $K_T \subset [0,T]\backslash J_0$ the function $z_0 : [0,T] \to \mathrm{L}^1(\Omega)^m$ is defined everywhere on $[0,T]$.
3. Now we show that the sequence $(z_{\varepsilon''}(t))_{\varepsilon''>0}$ for all $t \in [0,T]$ converges to the function $z_0(t)$ in the sense of (5.26). Since the monotonicity of $z_{\varepsilon''} : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ has to be understood as $z_{\varepsilon''}(\widetilde{t}) \le z_{\varepsilon''}(t)$ (component-wise) for all $t < \widetilde{t} \in [0,T]$ it holds

$$\|z_{\varepsilon''}(t)-z_{\varepsilon''}(\widetilde{t})\|_{\mathrm{L}^1_1(\Omega)^m} = \|z_{\varepsilon''}(t)\|_{\mathrm{L}^1_1(\Omega)^m} - \|z_{\varepsilon''}(\widetilde{t})\|_{\mathrm{L}^1_1(\Omega)^m} \overset{(5.24)}{=} F_{\varepsilon''}(t)-F_{\varepsilon''}(\widetilde{t}).$$

Exploiting this relation in the following calculation yields $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$. For $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}} \subset [0,T]\backslash J_0$ we choose $t_m \in K_T$ such that $t < t_m$. Then

$$\begin{aligned}
\lim_{\varepsilon''\to 0} \|z_{\varepsilon''}(t)-z_0(t)\|_{\mathrm{L}^1(\Omega)^m} &\le \lim_{\varepsilon''\to 0} \big(\|z_{\varepsilon''}(t)-z_{\varepsilon''}(t_m)\|_{\mathrm{L}^1(\Omega)^m} + \|z_{\varepsilon''}(t_m)-z_0(t_m)\|_{\mathrm{L}^1(\Omega)^m}\big) \\
&\quad + \|z_0(t_m)-z_0(t)\|_{\mathrm{L}^1(\Omega)^m} \\
&\overset{(5.26a)}{\le} \lim_{\varepsilon''\to 0} C_m^{-1}\big(F_{\varepsilon''}(t)-F_{\varepsilon''}(t_m)\big) + \|z_0(t_m)-z_0(t)\|_{\mathrm{L}^1(\Omega)^m} \\
&\overset{(5.25)}{=} C_m^{-1}\big(F_0(t)-F_0(t_m)\big) + \|z_0(t_m)-z_0(t)\|_{\mathrm{L}^1(\Omega)^m}. \qquad (5.27)
\end{aligned}$$

Since F_0 and z_0 are continuous on $[0,T]\backslash J_0$, $t_m \in K_T$ with $t < t_m$ can be chosen such that (5.27) gets arbitrarily small, which proves $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$ for every $t \in [0,T]$.
On the other hand, according to estimate (5.23) we are able to apply Theorem 4.2 and Theorem 3.8 again such that for arbitrary but fixed $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}}$ there exists a function $z^{(t)} \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and a subsequence $(\varepsilon''')_{\varepsilon'''>0}$ of $(\varepsilon'')_{\varepsilon''>0}$ satisfying

$$\begin{aligned}
z_{\varepsilon'''}(t) &\to z^{(t)} && \text{in } \mathrm{L}^p(\Omega)^m, && (5.28a)\\
R_{\frac{\varepsilon'''}{2}}(z_{\varepsilon'''}(t))|_\Omega &\rightharpoonup \nabla z^{(t)} && \text{in } \mathrm{L}^p(\Omega)^{m\times d}, && (5.28b)\\
\mathbb{C}_{\varepsilon'''}(z_{\varepsilon'''}(t)) &\overset{s}{\to} \mathbb{C}_0(z^{(t)}) && \text{in } \mathrm{L}^1(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}). && (5.28c)
\end{aligned}$$

Since $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}}$ was chosen arbitrarily and we already proved $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$ for all $t \in [0,T]$, this convergence result first of all gives $z_0(t) \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ for every $t \in [0,T]$. Observe that the validity of this statement for all $t \in (t_n)_{n\in\mathbb{N}}$ is already guaranteed by (5.26). Secondly, with $z^{(t)} = z_0(t)$ the convergence result (5.28) is valid for all converging subsequences of $(\varepsilon'')_{\varepsilon''>0}$ such that we conclude that (5.28) holds for the whole sequence $(\varepsilon'')_{\varepsilon''>0}$.
Recapitulating all results proven in step 2 and 3 there exists a piecewise continuous, monotone function $z_0 \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m))$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) such that the following is valid for all $t \in [0,T]$ if $\varepsilon \to 0$:

$$\begin{aligned}
z_\varepsilon(t) &\to z_0(t) && \text{in } \mathrm{L}^p(\Omega)^m, && (5.29a)\\
R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))|_\Omega &\rightharpoonup \nabla z_0(t) && \text{in } \mathrm{L}^p(\Omega)^{m\times d}, && (5.29b)\\
\mathbb{C}_\varepsilon(z_\varepsilon(t)) &\overset{s}{\to} \mathbb{C}_0(z_0(t)) && \text{in } \mathrm{L}^1(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}). && (5.29c)
\end{aligned}$$

4. Now for every $t \in [0,T]$ we prove the displacement field's convergence for the same subsequence constructed in step 2 and 3. For this purpose, let $u_0 : [0,T] \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ be uniquely defined by

$$u_0(t) \in \mathrm{Argmin}\{\mathcal{E}_0(t,u,z_0(t)) \,|\, u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\}, \tag{5.30}$$

where $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega)$ is the function defined in step 2.
On the other hand for fixed $t \in [0,T]$ we have $(u_\varepsilon(t), z_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t)$ by assumption. Due to (5.23) and Proposition 3.6 there exist $u_0^{(t)} \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of the sequence $(\varepsilon)_{\varepsilon>0}$ considered in (5.29) such that

$$u_{\varepsilon'}(t) \rightharpoonup u_0^{(t)} \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d. \tag{5.31}$$

Thus, we verified the applicability of Theorem 5.4 which states that $(u_0^{(t)}, z_0(t)) \in \mathcal{Q}_0(\Omega)$ satisfies the stability condition (S^0) at $t \in [0,T]$. By choosing $\widetilde{z} = z_0(t)$ in the stability condition (S^0) we find

$$u_0^{(t)} \in \mathrm{Argmin}\{\mathcal{E}_0(t,u,z_0(t)) \,|\, u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\}. \tag{5.32}$$

Comparing (5.30) and (5.32) we obtain $u_0^{(t)} = u_0(t)$. This identification shows

$$u_\varepsilon(t) \rightharpoonup u_0(t) \quad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \tag{5.33}$$

where the validity for the whole sequence $(\varepsilon)_{\varepsilon>0}$ considered in (5.29) is proven via a standard contradiction argument.
Note that in this step we already proved that $(u_0(t), z_0(t)) \in \mathcal{Q}_0(\Omega)$ satisfies the limit stability condition (S^0) for all $t \in [0,T]$, which includes $(u_0^0, z_0^0) \in \mathcal{Q}_0(\Omega)$. Since the pointwise limit of a sequence of measurable functions is measurable again, according to the uniform estimate (5.23) we have $u_0 \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)$.
5. For proving that $(u_0, z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ satisfies the limit energy balance (E^0) we pass in (E^ε) to the limit $\varepsilon \to 0$. We start with the right hand side. Due to the uniform bound (5.23) we have $|\langle \dot{\ell}(s), u_\varepsilon(s)\rangle| \le C_\ell C$ for every $\varepsilon > 0$ and all $s \in [0,T]$ such that

$$\lim_{\varepsilon\to 0} \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \lim_{\varepsilon\to 0} \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s$$

by applying the theorem of dominated convergence and making use of $u_\varepsilon(s) \rightharpoonup u_0(s)$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ for all $s \in [0,t]$. According to the assumptions we already have $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon^0, z_\varepsilon^0) = \mathcal{E}_0(t, u_0^0, z_0^0)$.
6. Left hand side of (E^ε): According to the convergence results (5.29) and (5.33) all assumptions of Theorem 5.6 are fulfilled, such that for all $t \in [0,T]$ we have

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \ge \mathcal{E}_0(t, u_0(t), z_0(t)). \tag{5.34}$$

For $N \in \mathbb{N}$ let $\pi_N = \{0 = t_0 < t_1 < \ldots < t_N = t\}$ be an arbitrary partition of the interval $[0,t]$. Then, by exploiting the definition of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])$ and the convergence result (5.29a) the following estimate holds:

$$\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \geq \lim_{\varepsilon\to 0} \sum_{j=1}^{N} \mathcal{D}_\varepsilon(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j)) = \sum_{j=1}^{N} \mathcal{D}_0(z_0(t_{j-1}), z_0(t_j)).$$

By taking the supremum with respect to all finite partition π_N of the interval $[0,t]$ on the right hand side this inequality yields

$$\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \geq \mathrm{Diss}_{\mathcal{D}_0}(z_0;[0,t]). \tag{5.35}$$

Since $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])$ is uniformly bounded with respect to $\varepsilon > 0$ and $t \in [0,T]$, relation (5.35) implies $z_0 \in \mathrm{BV}_{\mathcal{D}_0}([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m))$. Adding (5.34) and (5.35) and combing this with the convergence results of step 5 for all $t \in [0,T]$ we have

$$(\mathrm{E}^0_\mathrm{l}) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) + \liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \leq \lim_{\varepsilon\to 0}(\mathrm{E}^\varepsilon_\mathrm{r}) \overset{\text{step5}}{=} (\mathrm{E}^0_\mathrm{r}), \tag{5.36}$$

where the index l and r denote the left and right hand side of the respective energy balance. Due to the stability $(u_0(t), z_0(t)) \in \mathcal{Q}_0(\Omega)$ proved in step 4 we immediately obtain the opposite inequality $(\mathrm{E}^0_\mathrm{l}) \geq (\mathrm{E}^0_\mathrm{r})$ according to Proposition 2.4 of [16], such that finally $(u_0, z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ satisfies for all $t \in [0,T]$ the energy balance

$$\mathcal{E}_0(t,u_0(t),U_1(t),z_0(t)) + \mathrm{Diss}_{\mathcal{D}_0}(z_0;[0,t]) = \mathcal{E}_0(t,u_0(0),U_1(0),z_0(0)) - \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s.$$

Due to the validity of the energy balance (E^0) actually all inequalities in (5.36) are equalities. This implies that (5.34) and (5.35) also have to be equalities and that their limits exist. Hence, it holds

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) = \mathcal{E}_0(t,u_0(t),z_0(t)), \\ &\lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) = \mathrm{Diss}_{\mathcal{D}_0}(z_0;[0,t]) \end{aligned} \tag{5.37}$$

and the proof is concluded. □

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