# $BV$ functions in abstract Wiener spaces

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June 12, 2009

**Abstract**

Functions of bounded variation in an abstract Wiener space, i.e., an infinite dimensional Banach space endowed with a Gaussian measure and a related differential structure, have been introduced by M. Fukushima and M. Hino using Dirichlet forms, and their properties have been studied with tools from analysis and stochastics. In this paper we reformulate, in an integralgeometric vein and with purely analytical tools, the definition and the main properties of $BV$ functions, and investigate further properties.

## 1 Introduction

The spaces $BV$ of functions of bounded variation in Euclidean spaces are by now a classical setting where several problems, mainly (but not exclusively) of variational nature, find their natural framework. Recently, generalizations in metric measure spaces have been studied, but mostly under the hypothesis that the measure is *doubling*, which is not the case when dealing with probability measures in vector spaces, not even locally in infinite dimensions. However, there are several motivations for studying $BV$ functions in this context, i.e., Banach spaces, also of infinite dimensions, endowed with a probability measure. Among them, let us quote isoperimetric inequalities and mass concentration, see [20], [21], infinite dimensional analysis and semigroups (see e.g. [9], [11]). Moreover, the importance of generalizing the classical notion of perimeter and variation has been pointed out in several occasions by E. De Giorgi: we refer to [13], where the infinite dimensional context is explicitly mentioned.

$BV$ functions in an abstract Wiener space, i.e., a Banach space $X$ endowed with a Gaussian measure $\gamma$ and a related differential structure, have been defined by M. Fukushima in [18], M. Fukushima and M. Hino in [19], relying upon Dirichlet forms theory. The starting point of these papers has been a characterization of sets with finite perimeter in finite dimensions in terms of the behaviour of suitable stochastic processes (see [17]), and in fact some arguments in [18], [19] come from stochastics.

In this infinite-dimensional context, the main difficulty arises from the fact that the measures appearing in the integration by parts formula can't be built by a direct approximation by smooth functions, since Riesz theorem is not available and the dual of $C_b(X)$ strictly contains the class of signed measures; so, looking for instance at the approximation of $u$ by

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the Ornstein–Uhlenbeck semigroup $T_t u$ as $t \downarrow 0$, one has to prove tightness of the family of measures

$$\nabla_H T_t u \gamma$$

as $t \downarrow 0$. In the above mentioned papers this difficulty is overcome (after a reduction to nonnegative functions $u$) by looking at the Dirichlet form $\mathcal{E}_u(v,v) = \int_X u \|\nabla_H v\|_H^2 d\gamma$ and applying the capacity theory of Dirichlet forms.

One of the aims of this paper is to study $BV$ functions in abstract Wiener spaces with tools closer to those used in the Euclidean case. We analyse the connections between the distributional notion of (vector-valued) measure gradient and the approximation by smooth functions, as well as the relevant properties of the Ornstein-Uhlenbeck semigroup. Our methods rely basically on an integralgeometric approach, a viewpoint which has already proved to be very useful in the finite-dimensional theory (see also [16] for a recent contribution in the same direction in Wiener spaces). This approach allows to build directly the measure distributional derivative and to obtain the tightness property only as a consequence. Some of the main results of this paper have been presented, together with several open problems, in [2].

The paper is organized as follows: in Section 2 we describe the Wiener space setting and the tools useful to rephrase in this framework the characterizations of total variation that are known in the Euclidean case. A more detailed comparison with the Euclidean case is presented in [2]. In Section 3 we define $BV$ functions in Wiener spaces and discuss their basic properties. In Section 4 we provide a complete characterization of $BV$ functions in terms of integration by parts formulae and approximation by smooth functions.

It is known that neither Sobolev nor $BV$ spaces are compactly embedded in $L^p$ spaces, but our integral-geometric viewpoint provides new natural compactness criteria, both in the Sobolev and $BV$ classes. In Section 5, in the case when the Wiener space is Hilbert, we compare Sobolev and $BV$ classes with those developed by Da Prato and collaborators, proving compactness of these "stronger" Sobolev and $BV$ classes.

## 2 Wiener space setting

In this section we describe our setting: given an (infinite dimensional) separable Banach space $X$, we denote by $\|\cdot\|_X$ its norm and by $B_X(x,r) = \{y \in X : \|y-x\|_X < r\}$ the open ball centred at $x \in X$ and with radius $r > 0$. $X^*$ denotes the topological dual, with duality $\langle \cdot, \cdot \rangle$. Given the elements $x_1^*, \ldots, x_m^*$ in $X^*$, we denote by $\Pi_{x_1^*,\ldots,x_m^*} : X \to \mathbb{R}^m$ the map

$$\Pi_{x_1^*,\ldots,x_m^*} x = (\langle x, x_1^* \rangle, \ldots, \langle x, x_m^* \rangle), \tag{1}$$

also denoted by $\Pi_m : X \to \mathbb{R}^m$ if it is not necessary to specify the elements $x_1^*, \ldots, x_m^*$. The symbol $\mathcal{F}C_b^k(X)$ denotes the space of $k$ times continuously differentiable cylindrical functions with bounded derivatives up to the order $k$, that is, $u \in \mathcal{F}C_b^k(X)$ if $u(x) = v(\Pi_m x)$ for some $v \in C_b^k(\mathbb{R}^m)$.

We divide this section in some subsections; first of all we recall some notion of measure theory, with particular emphasis on the infinite dimensional (i.e., non locally compact) setting, then we pass to the definition and description of abstract Wiener spaces. In the third subsection we discuss the integration by parts formula and recall the definitions of gradient and divergence. Finally, we introduce Sobolev classes and the Ornstein-Uhlenbeck semigroup together with some of their basic properties.

## 2.1 Infinite dimensional measure theory

We denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra; since $X$ is separable, $\mathcal{B}(X)$ is generated by the cylindrical sets, that is by the sets of the form $E = \Pi_m^{-1} B$ with $B \in \mathcal{B}(\mathbb{R}^m)$, see [24, Theorem I.2.2]; this fact remains true even if we fix a sequence $(x_i^*) \subset X^*$ which separates the points in $X$ and use only elements from that sequence to generate the maps $\Pi_m$. We shall make later some special choices of $(x_i^*)$, induced by a Gaussian probability measure $\gamma$ in $X$.

We also denote by $\mathscr{M}(X, Y)$ the set of countably additive measures on $X$ with finite total variation with values in a Hilbert space $Y$, $\mathscr{M}(X)$ if $Y = \mathbb{R}$. We denote by $|\mu|$ the total variation measure of $\mu$, defined by

$$|\mu|(B) := \sup\left\{ \sum_{h=1}^{\infty} \|\mu(B_h)\|_Y : \ B = \bigcup_{h=1}^{\infty} B_h \right\}, \tag{2}$$

for every $B \in \mathcal{B}(X)$, where the supremum runs along all the countable disjoint unions. Notice that, using the polar decomposition, there is a unit $|\mu|$-measurable vector field $\sigma : X \to Y$ such that $\mu = \sigma|\mu|$, and then the equality

$$|\mu|(X) = \sup\left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \phi \in C_b(X, Y^*), \|\phi(x)\|_{Y^*} \le 1 \ \forall x \in X \right\},$$

holds, where $\langle \cdot, \cdot \rangle$ denotes the duality between $Y$ and $Y^*$. Note that, by the Stone-Weierstrass theorem, the algebra $\mathcal{F}C_b^1(X)$ of $C^1$ cylindrical functions is dense in $C(K)$ in sup norm, since it separates points, for all compact sets $K \subset X$. Since $|\mu|$ is tight, it follows that $\mathcal{F}C_b^1(X)$ is dense in $L^1(X, |\mu|)$. Arguing componentwise, it follows that also the space $\mathcal{F}C_b^1(X, Y^*)$ of cylindrical functions with a finite-dimensional range is dense in $L^1(X, |\mu|, Y^*)$. As a consequence $\sigma$ can be approximated in $L^1(X, |\mu|, Y^*)$ by a uniformly bounded sequence of functions in $\mathcal{F}C_b^1(X, Y^*)$, and we may restrict the supremum above to these functions only to get

$$|\mu|(X) = \sup\left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \phi \in \mathcal{F}C_b^1(X, Y^*), \|\phi(x)\|_{Y^*} \le 1 \ \forall x \in X \right\}. \tag{3}$$

We now recall a tightness criterion and we include its proof for the reader's convenience.

**Lemma 2.1.** *Let $(\sigma_n) \subset \mathscr{M}_+(X)$ be a bounded sequence, $\sigma \in \mathscr{M}_+(X)$ and assume that $\limsup_n \sigma_n(X) \le \sigma(X)$, while*

$$\liminf_{n\to\infty} \sigma_n(A) \ge \sigma(A) \qquad \text{for all } A \subset X \text{ open.}$$

*Then $(\sigma_n)$ is tight and $\sigma_n \to \sigma$ in the duality with $C_b(X)$.*

*Proof.* Let $(x_i) \subset X$ a dense sequence and $\varepsilon > 0$ be fixed. We claim that for all $k \ge 1$ there exists $N = N(k)$ such that

$$\sup_n \sigma_n\big(X \setminus \bigcup_{i=1}^{N} \overline{B}_{1/k}(x_i)\big) < \varepsilon 2^{-k}.$$

It this property holds, the totally bounded and closed set $K_\varepsilon := \cap_k \cup_1^{N(k)} \overline{B}_{1/k}(x_i)$ satisfies $\sup_n \sigma_n(X \setminus K_\varepsilon) < \varepsilon$, proving the tightness of $(\sigma_n)$.

We prove the claimed property by contradiction, assuming that for some $k$ there exist $n(\ell)$ such that $\sigma_{n(\ell)}(X \setminus \cup_{i=1}^{\ell} \overline{B}_{1/k}(x_i)) > \varepsilon 2^{-k}$ for all $\ell \geq 1$. Obviously $n(\ell) \to \infty$ as $\ell \to \infty$ and for any $\ell_0$ we have

$$\begin{aligned}
\sigma\Big(\bigcup_{i=1}^{\ell_0} B_{1/k}(x_i)\Big) &\leq \liminf_{\ell\to\infty} \sigma_{n(\ell)}\Big(\bigcup_{i=1}^{\ell_0} B_{1/k}(x_i)\Big) \leq \liminf_{\ell\to\infty} \sigma_{n(\ell)}\Big(\bigcup_{i=1}^{\ell} B_{1/k}(x_i)\Big) \\
&\leq \limsup_{n\to\infty} \sigma_n(X) - \varepsilon 2^{-k} \leq \sigma(X) - \varepsilon 2^{-k}.
\end{aligned}$$

Letting $\ell_0 \to \infty$ gives a contradiction, since $\cup_1^{\ell_0} B_{1/k}(x_i) \uparrow X$.

The last statement is a simple consequence of the Cavalieri formula, taking into account that $\sigma_n(E) \to \sigma(E)$ for all Borel sets $E$ with $\sigma(\partial E) = 0$, and that $\sigma(\{u = t\}) = 0$ with at most countably many exceptions for all $u \in C_b(X)$. □

Finally, let us define the sup of (the total variation of) an arbitrary family of measures $\{\mu_\alpha : \alpha \in I\}$ by setting

$$\bigvee_{\alpha\in I} |\mu_\alpha|(A) = \sup\left\{\sum_{n=1}^{\infty} |\mu_{\alpha_n}|(A_n)\right\},$$

where the supremum runs along all the countable pairwise disjoint partitions $A = \bigcup_n A_n$ and all the choices of the sequence $(\alpha_n) \subset I$.

## 2.2 The abstract Wiener space

Assume that a nondegenerate centred Gaussian measure $\gamma$ is defined on $X$. This means that $\gamma$ is a probability measure and for all $x^* \in X^*$ the law $x^*_{\#}\gamma$ is a centred Gaussian measure on $\mathbb{R}$, that is, the Fourier transform of $\gamma$ is given by

$$\hat{\gamma}(x^*) = \int_X \exp\{-i\langle x, x^*\rangle\} d\gamma(x) = \exp\Big\{-\frac{1}{2}\langle Qx^*, x^*\rangle\Big\}, \qquad \forall x^* \in X^*,$$

where $Q \in \mathcal{L}(X^*, X)$ is the covariance operator. The nondegeneracy hypothesis means that $\gamma$ is not concentrated on a proper subspace of $X$, in terms of $Q$ this means that $\langle Qx^*, x^*\rangle > 0$ for $x^* \neq 0$. The covariance operator is a symmetric and positive operator uniquely determined by the relation

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x), \quad \forall x^*, y^* \in X^*; \tag{4}$$

we also write $\mathscr{N}(0, Q)$ for $\gamma$. The fact that the operator $Q$ defined by (4) is bounded is a consequence of Fernique's Theorem (see e.g. [7, Theorem 2.8.5]), asserting that

$$\int_X \exp\{\alpha\|x\|_X^2\} d\gamma(x) < \infty \tag{5}$$

if and only if

$$\alpha^{-1} > \sigma := \sup\Big\{\langle Qx^*, x^*\rangle^{1/2} : x^* \in X^*, \|x^*\|_{X^*} \leq 1\Big\};$$

as another consequence of this we also get that any $x^* \in X^*$ defines a function $x \mapsto \langle x, x^*\rangle$ that belongs to $L^p(X, \gamma)$ for all $p \geq 1$. In particular, we can think of any $x^* \in X^*$ as a continuous element of $L^2(X,\gamma)$. Let us denote by $R^* : X^* \to L^2(X,\gamma)$ the embedding, $R^*x^*(x) = \langle x, x^*\rangle$. The space $\mathscr{H}$ given by the closure of $R^*X^*$ in $L^2(X,\gamma)$ is called the *reproducing kernel* of the Gaussian measure $\gamma$ and obviously $R^*X^*$ turns out to be dense in it. The above definition is motivated by the fact that if we consider the operator $R : \mathscr{H} \to X$ whose adjoint is $R^*$, then

$$R\hat{h} = \int_X \hat{h}(x) x \, d\gamma(x), \tag{6}$$

where the integral is understood in Bochner's sense. As a consequence $Q = RR^*$:

$$\langle RR^*x^*, y^*\rangle = [R^*x^*, R^*y^*]_{\mathscr{H}} = \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x) = \langle Qx^*, y^*\rangle.$$

It can be easily proved that $R$ is injective. In addition, the operator $R$ is compact and even more, i.e., it belongs to the ideal $\gamma(\mathscr{H}, X)$ of $\gamma$-Radonifying, or Gaussian-Radonifying operators, see e.g. [22]. This remark shows that another presentation is possible: one can start with $R \in \gamma(\mathscr{H}, X)$ for some separable Hilbert space $\mathscr{H}$ and construct a Gaussian measure $\gamma$ whose covariance operator is $Q = RR^*$. In any case, the measure $\gamma$ built with this construction is concentrated on the separable subspace of $X$ defined as the closure of $R\mathscr{H}$ in $X$.

The space $H = R\mathscr{H} \subset X$ is called the Cameron-Martin space; it is a separable Hilbert space with inner product defined by

$$[h_1, h_2]_H = [\hat{h}_1, \hat{h}_2]_{\mathscr{H}}$$

for all $h_1$, $h_2 \in H$, where $h_i = R\hat{h}_i$, $i = 1, 2$. It is a dense subspace of $X$ and the embedding of $(H, \|\cdot\|_H)$ in $(X, \|\cdot\|)$ is compact since $R$ is compact. In addition, $\gamma(H) = 0$ if $X$ is infinite-dimensional [7, Theorem 2.4.7], while $H = X$ if $X$ is finite-dimensional.

With this notation, the Fourier transform of the Gaussian measure $\gamma$ reads

$$\hat{\gamma}(x^*) = \exp\Big\{ -\frac{1}{2}\|\hat{x^*}\|^2_{\mathscr{H}} \Big\}, \qquad \forall x^* \in X^*,$$

where $\hat{x}^* = R^*x^*$.

Using the embedding $R^*X^* \subset \mathscr{H}$, we shall say that a family $\{x_j^*\}$ of elements of $X^*$ is orthonormal if the corresponding family $\{R^*x_j^*\}$ is orthonormal in $\mathscr{H}$. Starting from a sequence $(y_j^*)$ in $X^*$ whose image under $R^*$ is dense in $\mathscr{H}$, we may construct an orthonormal basis $(R^*x_j^*)$ in $\mathscr{H}$ (by the Gram-Schmidt procedure), hence $h_j = Qx_j^* = RR^*x_j^*$ provide an orthonormal basis of $H$. Set also $H_m = \mathrm{span}\{h_1, \dots, h_m\}$, and define $X_m^\perp = \ker \Pi_{x_1^*,\dots,x_m^*}$ and $X_m$ the ($m$-dimensional) complementary space. Since the variables $R^*x_j^*$ are Gaussian and uncorrelated, they are independent, hence the image $\gamma_m$ of $\gamma$ under $\Pi_{x_1^*,\dots,x_m^*}$ is a standard Gaussian in $\mathbb{R}^m$; in addition it can be proved that we have a product decomposition $\gamma = \gamma_m \otimes \gamma_m^\perp$ of the measure $\gamma$, with $\gamma_m^\perp$ Gaussian. Since $R^*X^*$ is dense in $\mathscr{H}$ the following proposition is easily estabilished by approximation:

**Proposition 2.2.** *Let $\hat{h}_1, \dots, \hat{h}_m$ be in $\mathscr{H}$. Then the law of the variable*

$$x \mapsto (\hat{h}_1, \dots, \hat{h}_m)$$

*under $\gamma$ is Gaussian. If $\hat{h}_i$ are orthonormal, the law is the standard Gaussian $\gamma_m$ in $\mathbb{R}^m$.*

One more property of Gaussian measures we shall use is *rotation invariance*, i.e., if $\varrho : X \times X \to X \times X$ is given by $\varrho(x, y) = (\cos\vartheta x + \sin\vartheta y, -\sin\vartheta x + \cos\vartheta y)$ for some $\vartheta \in \mathbb{R}$, then $\varrho_{\#}(\gamma \otimes \gamma) = \gamma \otimes \gamma$. We shall use, in particular, the following equality:

$$\int_X \int_X u(\cos\vartheta x + \sin\vartheta y)d\gamma(x)d\gamma(y) = \int_X u(x)d\gamma(x), \qquad \forall u \in L^1(X, \gamma), \tag{7}$$

which is obtained by the above relation by integrating the function $u \otimes 1$ on $X \times X$.

For every function $u \in L^1(X, \gamma)$ we define its *canonical cylindrical approximations* $\mathbb{E}_m u$ as the conditional expectations relative to the $\sigma$-algebras generated by $\{\langle x, x_1^*\rangle, \dots, \langle x, x_m^*\rangle\}$, denoted by $\mathcal{B}_m(X)$, i.e.,

$$\int_A u d\gamma = \int_A \mathbb{E}_m u d\gamma, \qquad \forall A \in \mathcal{B}_m(X). \tag{8}$$

Then, $\mathbb{E}_m u \to u$ in $L^1(X, \gamma)$ and $\gamma$-a.e. (see e.g. [7, Corollary 3.5.2]). More explicitly, we have

$$\mathbb{E}_m u(x) = \int_X u(\Pi_m x + (I - \Pi_m)y)d\gamma(y) = \int_{X_m^\perp} u(\Pi_m x + y')d\gamma_m^\perp(y'),$$

where $\Pi_m$ is the projection onto $X_m$. Notice that $\mathbb{E}_m u$ is invariant under translations along all the vectors in $X_m^\perp$, hence we may write $\mathbb{E}_m u(x) = v(\Pi_m x)$ for some function $v$, and, with an abuse of notation, we may write $\mathbb{E}_m u(x_m)$ instead of $\mathbb{E}_m u(x)$.

Finally, let us recall the Cameron–Martin Theorem: the shifted measure

$$\gamma_h(B) = \gamma(B - h), \qquad B \in \mathcal{B}(X),$$

also denoted by $\mathscr{N}(h, Q)$, is absolutely continuous with respect to $\gamma$ if and only if $h \in H$ and, in this case, with the usual notation $h = R\hat{h}$, we have, see e.g. [7, Corollary 2.4.3],

$$d\gamma_h(x) = \exp\left\{\hat{h}(x) - \frac{1}{2}\|h\|_H^2\right\}d\gamma(x). \tag{9}$$

It is also important to notice that if we define for any $\lambda \in \mathbb{R}$ the measure

$$\gamma_\lambda(B) = \gamma(\lambda B), \qquad \forall B \in \mathcal{B}(X) \tag{10}$$

then $\gamma_\lambda \ll \gamma_\sigma$ if and only if $|\lambda| = |\sigma|$ (see for instance [7, Example 2.7.4]).

## 2.3 Gradient, divergence and Sobolev spaces

For a given function $f : X \to \mathbb{R}$ and $h \in H$, we define

$$\partial_h f(x) := \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

and

$$\partial_h^* f(x) = \partial_h f(x) - f(x)\hat{h}(x),$$

wherever this makes sense. Here, as usual, $h = R\hat{h}$. We shall use the shorter notation $\partial_j = \partial_{h_j}$, $\partial_j^* = \partial_{h_j}^*$. The gradient $\nabla_H f : X \to H$ of $f$ is defined as

$$\nabla_H f(x) := \sum_{j \in \mathbb{N}} \partial_j f(x) h_j.$$

Notice that if $f(x) = g(\Pi_m x)$ with $g \in C^1(\mathbb{R}^m)$, then

$$\partial_h f(x) = \nabla g(\Pi_m x) \cdot \Pi_m h.$$

The operator $\partial_h^*$ is (up to a change of sign) the adjoint of $\partial_h$ with respect to $L^2(X, \gamma)$, namely

$$\int_X \phi \partial_h f d\gamma = -\int f \partial_h^* \phi d\gamma \qquad \forall \phi,\ f \in \mathcal{F}C_b^1(X). \tag{11}$$

The divergence operator is defined for $\Phi : X \to H$ as

$$\nabla_H^* \Phi(x) := \sum_{j \in \mathbb{N}} \partial_j^* [\Phi(x), h_j]_H.$$

We define the space $\mathcal{F}C_b^1(X, H)$ of cylindrical $H$-valued functions as the vector space spanned by functions $\phi h$, where $\phi$ runs in $\mathcal{F}C_b^1(X)$ and $h$ in $H$. With this notation, the integration by parts formula (11) gives

$$\int_X [\nabla_H f, \Phi]_H d\gamma = -\int_X f \nabla_H^* \Phi \ d\gamma \tag{12}$$

for every $f \in \mathcal{F}C_b^1(X),\ \Phi \in \mathcal{F}C_b^1(X, H)$.

Thanks to (12), the gradient $\nabla_H$ is a closable operator in the topologies $L^p(X, \gamma)$, $L^p(X, \gamma, H)$ for any $p \in [1, \infty)$ and we denote by $\mathbb{D}^{1,p}(X, \gamma)$ the domain of its closure. Notice that the space denoted by $\mathbb{D}^{1,p}(X, \gamma)$ by Fukushima in [18] is denoted by $W^{p,1}(X, \gamma)$ in [7]. Anyway, these spaces coincide, see [7, Section 5.2] and (12) holds for every $f \in \mathbb{D}^{1,p}(X, \gamma),\ \Phi \in \mathcal{F}C_b^1(X, H)$.

Let us recall the *Gaussian isoperimetric inequality*, see [20]. Let $E \subset X$, and set $B_r = \{x \in H : \|x\|_H < r\},\ E_r = E + B_r$; then

$$\Phi^{-1}(\gamma(E_r)) \geq \Phi^{-1}(\gamma(E)) + r, \quad \text{with } \ \Phi(t) = \int_{-\infty}^t \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds. \tag{13}$$

Then, setting

$$\mathscr{U}(t) = (\Phi' \circ \Phi^{-1})(t) \approx t\sqrt{2\log(1/t)}, \quad \text{as } t \to 0; \tag{14}$$

the inequality

$$\gamma(E_r) \geq \gamma(E) + r\mathscr{U}(\gamma(E)) + o(r) \tag{15}$$

follows. The isoperimetric inequality implies also the following Gauss–Sobolev inequality

$$\|\nabla_H f\|_{L^1(X,\gamma)} \geq \int_0^\infty \mathscr{U}\left(\gamma(\{|f| > s\})\right) ds, \tag{16}$$

which implies the continuous embedding of $\mathbb{D}^{1,1}(X, \gamma)$ into the Orlicz space

$$L\log^{1/2} L(X, \gamma) := \Big\{u : X \to \mathbb{R} \text{ measurable} : A_{1/2}(\lambda|u|) \in L^1(X, \gamma) \ \text{ for some } \lambda > 0\Big\},$$

endowed with the Luxenbourg norm

$$\|u\|_{L\log^{1/2}L} := \inf\left\{\lambda > 0 : \int_X A_{1/2}(|u|/\lambda)\,d\gamma \le 1\right\},$$

see [19, Proposition 3.2]. Here $A_{1/2}$ is defined by

$$A_{1/2}(t) := \int_0^t \log^{1/2}(1+s)ds.$$

Analogously, using the continuity of the map $f \mapsto |f|^p$ from $\mathbb{D}^{1,p}(X,\gamma)$ to $\mathbb{D}^{1,1}(X,\gamma)$, one obtains that $\mathbb{D}^{1,p}(X,\gamma)$ embeds continuously into the Orlicz space

$$L^p\log^{1/2}L(X,\gamma) := \Big\{u : X \to \mathbb{R} \text{ meas.} : A_{1/2}(\lambda|u|^p) \in L^1(X,\gamma) \text{ for some } \lambda > 0\Big\}. \tag{17}$$

Finally we recall the Poincaré inequality, see [7, Theorem 5.5.11]: for any $u \in \mathbb{D}^{1,p}(X,\gamma)$, $p \ge 1$

$$\int_X \left|u - \int_X u d\gamma\right|^p d\gamma \le C_p \int_X \|\nabla_H u\|_H^p d\gamma, \tag{18}$$

where $C_p$ depends only on $p$.

## 2.4 The Ornstein–Uhlenbeck semigroup

Let us define the Ornstein–Uhlenbeck semigroup $(T_t)_{t\ge0}$, by Mehler's formula

$$T_t u(x) = \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)d\gamma(y) \tag{19}$$

for all $u \in L^1(X,\gamma)$, $t \ge 0$.

For our purposes, the following properties of the Ornstein–Uhlenbeck semigroup are relevant: $T_t$ is a contraction semigroup in $L^1(X,\gamma)$ and $T_t u \in \mathbb{D}^{1,1}(X,\gamma)$ for any $u \in L\log^{1/2}L(X,\gamma)$, $t > 0$, (see [19, Proposition 3.6]). In addition, a direct consequence of (7) and of Jensen's inequality is

$$\int_X \|T_t u\|_Y d\gamma \le \int_X \|u\|_Y d\gamma, \qquad \text{for any } u \in L^1(X,\gamma,Y), \tag{20}$$

with $Y$ Hilbert (here $T_t$ is defined componentwise, namely $\langle T_t u, y\rangle = T_t\langle u, y\rangle$).

Stronger smoothing properties of $T_t$ hold for bounded functions; in particular if $f \in L^p(X,\gamma)$ for any $p > 1$, then $T_t f \in \mathbb{D}^{k,q}(X,\gamma)$ for any $k \in \mathbb{N}$, $q > 1$ (see [7, Proposition 5.4.8]). Moreover, the following commutation relation holds (again componentwise) for any $u \in \mathbb{D}^{1,1}(X,\gamma)$:

$$\nabla_H T_t u = e^{-t} T_t \nabla_H u, \qquad t > 0. \tag{21}$$

Therefore, we get

$$\nabla_H T_{t+s} u = \nabla_H T_t(T_s u) = e^{-t} T_t \nabla_H T_s u, \quad t \ge 0,\ s > 0$$

for any $u \in L\log^{1/2}L(X,\gamma)$, see [7, Proposition 5.4.8]. As a consequence, we obtain that the limit (possibly infinite)

$$\mathcal{I}(u) := \lim_{t\downarrow 0}\int_X \|\nabla_H T_t u\|_H d\gamma \tag{22}$$

always exists for $u \in L\log^{1/2}L(X,\gamma)$. Indeed, consider the map

$$s \longmapsto \int_X \|\nabla_H T_{t+s}u\|_H \; d\gamma = e^{-t}\int_X \|T_t\nabla_H T_s u\|_H \, d\gamma$$

and observe that

$$\begin{aligned}\int_X \|\nabla_H T_t u\|_H \; d\gamma \le & \liminf_{s\to 0}\int_X \|\nabla_H T_{t+s}u\|_H \; d\gamma = e^{-t}\liminf_{s\to 0}\int_X \|T_t\nabla_H T_s u\|_H \; d\gamma \\ \le & e^{-t}\liminf_{s\to 0}\int_X \|\nabla_H T_s u\|_H \; d\gamma\end{aligned}$$

by (20), which obviously implies that the limit exists.

It also follows from (21) and (12) that

$$\int_X T_t f \nabla_H^* \Phi d\gamma = e^{-t}\int_X f\nabla_H^*(T_t\Phi)d\gamma, \tag{23}$$

for all $f \in L\log^{1/2}L(X,\gamma)$, $\Phi \in \mathcal{F}C_b^1(X,H)$. Indeed, we can assume by a density argument that $f \in \mathbb{D}^{1,1}(X,\gamma)$ to get

$$\begin{aligned}\int_X T_t f\nabla_H^*\Phi d\gamma &= -\int [\nabla_H T_t f, \Phi]_H d\gamma = -e^{-t}\int [\nabla_H f, T_t\Phi]_H d\gamma \\ &= e^{-t}\int_X f\nabla_H^* T_t\Phi d\gamma.\end{aligned}$$

Another important consequence of (21) is that if $u \in \mathbb{D}^{1,1}(X,\gamma)$ then

$$\lim_{t\to 0}\|\nabla_H T_t u - \nabla_H u\|_{L^1(X,\gamma)} = 0. \tag{24}$$

Finally, notice that if $\mathbb{E}_m u$ are the canonical cylindrical approximations of a function $u \in L\log^{1/2}L(X,\gamma)$ defined in (8) then

$$\int_X \|\nabla_H T_t \mathbb{E}_m u\|_H d\gamma \le \int_X \|\nabla_H T_t u\|_H d\gamma, \quad \forall\, t > 0. \tag{25}$$

To prove (25), let us first notice that, by the rotational invariance of $\gamma_m^\perp$,

$$T_t\mathbb{E}_m u = \mathbb{E}_m T_t u.$$

Indeed,

$$\begin{aligned}T_t\mathbb{E}_m u(x) &= \int_X \mathbb{E}_m u(e^{-t}x + \sqrt{1-e^{-2t}}z)d\gamma(z) \\ &= \int_X\int_{X_m^\perp} u(e^{-t}\Pi_m x + \sqrt{1-e^{-2t}}\Pi_m z + y')d\gamma_m^\perp(y')d\gamma(z)\end{aligned}$$

and by apply the rotation invariance of Gaussian measures (7) to $\gamma_m^\perp$ we get

$$\begin{aligned}
\mathbb{E}_m T_t u(x) &= \int_{X_m^\perp} T_t u(\Pi_m x + w') d\gamma_m^\perp(w') \\
&= \int_{X_m^\perp} \int_X u\Big(e^{-t}(\Pi_m x + w') + \sqrt{1-e^{-2t}} z\Big) d\gamma_m^\perp(w') d\gamma(z) \\
&= \int_{X_m} \int_{X_m^\perp} \int_{X_m^\perp} u\Big(e^{-t}\Pi_m x + e^{-t} w' + \sqrt{1-e^{-2t}} z_m \\
&\qquad + \sqrt{1-e^{-2t}} z'\Big) d\gamma_m^\perp(w') d\gamma_m^\perp(z') d\gamma_m(z_m) \\
&= \int_X \int_{X_m^\perp} u\Big(e^{-t}\Pi_m x + \sqrt{1-e^{-2t}} \Pi_m z + y'\Big) d\gamma_m^\perp(y') d\gamma(z).
\end{aligned}$$

From the above commutation relation it follows that the vector $\nabla_H T_t \mathbb{E}_m u = \nabla_H \mathbb{E}_m T_t u$ coincides with its projection $\nabla_m$ on $H_m$, since $\mathbb{E}_m u$ depends only on $x_m \in X_m$. Moreover, by Jensen's inequality we have

$$\begin{aligned}
\|\nabla_H T_t \mathbb{E}_m u(x)\|_H =& \|\mathbb{E}_m(\nabla_m T_t u)(x)\|_H = \Big\| \int_{X_m^\perp} \nabla_m T_t u(\Pi_m x + x') d\gamma_m^\perp(x') \Big\|_H \\
&\leq \int_{X_m^\perp} \|\nabla_m T_t u(\Pi_m x + x')\|_H d\gamma_m^\perp(x') = \mathbb{E}_m \|\nabla_m T_t u(x)\|_H.
\end{aligned}$$

Since $\|\nabla_m T_t u(x)\|_H \leq \|\nabla_H T_t u(x)\|_H$ we can integrate both sides to get (25).

# 3 $BV$ functions in infinite dimensions

We have collected in the preceding section the tools we need in order to discuss $BV$ functions in the Wiener space setting. The $BV(X,\gamma)$ class can be defined as follows.

**Definition 3.1** ($BV$ space)**.** *Let $u \in L\log^{1/2}L(X,\gamma)$. We say that $u \in BV(X,\gamma)$ if there exists a measure $\mu \in \mathscr{M}(X,H)$ such that for any $\phi \in \mathcal{F}C_b^1(X)$ we have*

$$\int_X u(x)\partial_j^*\phi(x) d\gamma(x) = -\int_X \phi(x) d\mu_j(x) \qquad \forall j \in \mathbb{N}, \tag{26}$$

*where $\mu_j = [h_j, \mu]_H$. In particular, if $u = \chi_E$ and $u \in BV(X,\gamma)$, then we say that $E$ has finite perimeter.*

Equivalently, we may require the existence of measures $\mu_j$ as in (26) satisfying

$$\sup_m |(\mu_1, \ldots, \mu_m)|(X) < \infty. \tag{27}$$

Indeed, if $\mu_j = [\mu, h_j]_H$, then the total variation of the $\mathbb{R}^m$ valued measure $(\mu_1, \ldots, \mu_m)$ in $X$ is less than $|\mu|(X)$. Conversely, if (27) holds, then the measure $\mu := \sum_j \mu_j h_j$ is well defined and belongs to $\mathscr{M}(X,H)$ (it suffices to consider the densities $f_i$ of $\mu_i$ with respect to the measure $\sigma := \sup_m |(\mu_1, \ldots, \mu_m)|$ to obtain $\sum_1^m f_i^2 \leq 1$ $\sigma$-a.e., hence $\|(f_i)\|_{\ell_2} \leq 1$ $\sigma$-a.e. and $\mu = \sum f_j h_j \sigma$).

**Remark 3.2.** Notice that in the previous definition we have required that the measure $\mu$ is defined on the whole of $\mathcal{B}(X)$ and is $\sigma$-additive there. Since cylindrical functions generate the Borel $\sigma$-algebra the measure $\mu$ verifying (26) is unique, and will be denoted $D_\gamma u$. Using (3) the total variation of $D_\gamma u$ is given by

$$\begin{aligned} |D_\gamma u|(X) &= \sup\Big\{\langle D_\gamma u, \Phi\rangle;\ \Phi \in C_b(X,H), \|\Phi(x)\|_H \le 1\ \forall\ x \in X\Big\} \\ &= \sup\Big\{\int_X u\nabla_H^*\Phi d\gamma;\ \Phi \in \mathcal{F}C_b^1(X,H), \|\Phi(x)\|_H \le 1\ \forall\ x \in X\Big\}. \end{aligned} \tag{28}$$

Actually it will be convenient to define also an even smaller space $\mathcal{F}C_c^1(X,H)$ of vector fields $\Phi$ representable as follows:

$$\Phi(x) = \sum_{i=1}^m \phi_i(\langle x_1^*, x\rangle, \ldots, \langle x_m^*, x\rangle)h_i$$

with $h_i = Qx_i^*$ and $\phi_i \in C_c^1(\mathbb{R}^m)$. The space $\mathcal{F}C_c^1(X,H)$ has an additional technical advantage: the divergence $\nabla_H^*\Phi$ of $\Phi$ above is $\sum_i \partial_i\phi_i - \phi_i R^* x_i^*$ and is a bounded function because $R^*x_i^*$ is bounded on the support of $\phi_i$. By a further approximation, based on the fact that any $\Phi \in \mathcal{F}C_b^1(X,H)$ is the pointwise limit of a uniformly bounded sequence $(\Phi_n) \subset \mathcal{F}C_c^1(X,H)$, we have also

$$|D_\gamma u|(X) = \sup\Big\{\int_X u\nabla_H^*\Phi d\gamma;\ \Phi \in \mathcal{F}C_c^1(X,H), \|\Phi(x)\|_H \le 1\ \forall x \in X\Big\}. \tag{29}$$

In the case $u = \chi_E$, we write $P_\gamma(E,\cdot)$ for the measure $|D_\gamma\chi_E|$ and we shall also write $P_\gamma(E)$ for $P_\gamma(E,X)$.

**Remark 3.3** (About the $L\log^{1/2}L(X,\gamma)$ assumption)**.** The $L\log^{1/2}L(X,\gamma)$ membership hypothesis we made on $u$ is necessary to give sense to the integral of the product $u\partial_j^*\phi$: indeed, this term is the sum of the function $u\partial_j\phi$ (which makes sense for $u \in L^1(X,\gamma)$ only) and $u\phi\hat{h}_j$, which makes sense by Orlicz duality if $u \in L\log^{1/2}L(X,\gamma)$ and $\exp(c|\hat{h}_j|^2) \in L^1(X,\gamma)$ for some $c > 0$. Since $\hat{h}_j = R^*x_j^*$ for some $x_j^* \in X^*$, by our construction of $h_j$, this exponential integrability property follows by Fernique's theorem (5). Nevertheless, we shall provide different equivalent definitions of $BV$ where this extra integrability property is not needed (as in the finite-dimensional case) but rather derived as a consequence. These equivalent definitions will also show that Definition 3.1 is independent of the choice of the basis $(h_j)$.

Let us see an equivalent way of defining the $BV$ class, with partial derivatives along all directions $h \in H$; in this case, since $\hat{h}$ is in $L^2(X,\gamma)$ and not better, we have to assume $u \in L^2(X,\gamma)$ to give a sense to the integration by parts formula, even when cylindrical test functions are involved.

**Proposition 3.4.** *Let $u \in L^2(X,\gamma)$. Then, $u \in BV(X,\gamma)$ if and only if for every $h \in H$ there is a real measure $\mu_h$ such that*

$$\int_X u(x)\partial_h^*\phi(x)d\gamma(x) = -\int_X \phi(x)d\mu_h(x) \qquad \forall\phi \in \mathcal{F}C_b^1(X), \tag{30}$$

*with $\bigvee_{\|h\|_H=1} |\mu_h|$ finite. In this case, $|D_\gamma u| = \bigvee_{\|h\|_H=1} |\mu_h|$.*

*Proof.* If $u \in BV(X,\gamma)$ then the existence of $\mu_h = [h, D_\gamma u]_H$ for all $h \in H$ follows from the linearity of the $\partial_h$ operator with respect to $h$, and the boundedness of $|\mu_h|$ from the finiteness of $|D_\gamma u|$. In particular, $\bigvee_{\|h\|_H=1} |\mu_h| \le |D_\gamma u|(X)$.

Conversely, define $\mu_j = \mu_{h_j}$ and

$$\mu = \sum_{j=1}^{\infty} \mu_j h_j,$$

so that $\mu_h = [h,\mu]_H$. The integration by parts (26) clearly holds; we have to prove that $\mu$ is a finite measure. If we fix a partition $(B_n)_{n\in\mathbb{N}}$ of $X$, if $\mu(B_n) \neq 0$, we define

$$\alpha_n = \frac{\mu(B_n)}{\|\mu(B_n)\|_H}$$

so that we obtain

$$\begin{aligned}\sum_{n\in\mathbb{N}} \|\mu(B_n)\|_H =& \sum_{n\in\mathbb{N}} \|[\mu(B_n),\alpha_n]_H \alpha_n\|_H = \sum_{n\in\mathbb{N}} |\mu_{\alpha_n}(B_n)| \\ \le& \sum_{n\in\mathbb{N}} |\mu_{\alpha_n}|(B_n) \le \bigvee_{\|h\|_H=1} |\mu_h|,\end{aligned}$$

and then $u \in BV(X,\gamma)$ with $D_\gamma u := \mu$ and $|\mu| \le \bigvee_{\|h\|_H=1} |\mu_h|$. ☐

It is easy to verify that if $u \in \mathbb{D}^{1,1}(X,\gamma)$, then $u \in BV(X,\gamma)$ with $D_\gamma u = \nabla_H u \gamma$.

Now we relate $BV$ functions in $\mathbb{R}^m$ with cylindrical functions in $X$. We denote as before by $\gamma_m$ the standard Gaussian distribution on $\mathbb{R}^m$ and we point out that all directional derivatives $\partial_\nu$ and their adjoints $\partial_\nu^*$ have the same meaning as in the infinite dimensional case, but without restriction on directions, since $H = X$ in this case; we shall try to use as much as possible a consistent notation, valid both for the finite-dimensional and the infinite-dimensional case.

**Proposition 3.5.** *Let $u \in L\log^{1/2}L(X,\gamma)$ be a cylindrical function,*

$$u(x) = v(\Pi_{x_1^*,\ldots,x_m^*} x),$$

*with $R^* x_i^*$ orthonormal. Then $v \in BV(\mathbb{R}^m,\gamma_m)$ if and only if $u \in BV(X,\gamma)$ and*

$$|D_\gamma u|(X) = |D_{\gamma_m} v|(\mathbb{R}^m).$$

*Proof.* Recalling that the law of $\Pi$ under $\gamma$ is $\gamma_m$, we have

$$\begin{aligned}|D_\gamma u|(X) &= \sup\Big\{ \int_X u(x)\nabla^*\Phi(x)d\gamma(x) : \Phi \in \mathcal{F}C_b^1(X,H),\ \|\Phi(x)\|_H \le 1\ \forall x \in X \Big\} \\ &= \sup\Big\{ \int_X u(x)\nabla^*\Phi(x)d\gamma(x) : \Phi \in \mathcal{F}C_b^1(X,H_m),\ \|\Phi(x)\|_H \le 1\ \forall x \in X \Big\} \\ &= \sup\Big\{ \int_{\mathbb{R}^m} v(y)\nabla^*\Psi(y)d\gamma_m(y) : \Psi \in [C_b^1(\mathbb{R}^m)]^m,\ \|\Psi\|_\infty \le 1 \Big\}.\end{aligned}$$

☐

As a particular example of Sobolev functions we can consider Lipschitz functions.

**Definition 3.6** ($H$-Lipschitz functions)**.** *A Borel function $f : X \to \mathbb{R}$ is said to be $H$-Lipschitz if there exists a constant $C$ such that for $\gamma$-a.e. $x$ one has*

$$|f(x+h) - f(x)| \le C\|h\|_H, \quad \forall h \in H. \tag{31}$$

It can be proved that for a $H$-Lipschitz function $f$ there exists a full-measure $\gamma$-measurable set $X_0$ such that $X_0 + H = X_0$ and, for every $x \in X_0$, one has

$$|f(x+h) - f(x)| \le C\|h\|_H, \quad \forall h \in H.$$

In particular, $f$ has a version such that the previous inequality is satisfied for every $x \in X$. By the arguments in [7, Section 5.11], it can be proved that $H$-Lipschitz functions belong to $\mathbb{D}^{1,p}(X,\gamma)$ for every $p \ge 1$, and in particular to $BV(X,\gamma)$.

An important result is the following coarea formula, which can be proved by following *verbatim* the proof of [15, Section 5.5].

**Theorem 3.7.** *If $u \in BV(X,\gamma)$, then for every Borel set $B \subset X$ the following equality holds:*

$$|D_\gamma u|(B) = \int_{\mathbb{R}} P_\gamma(\{u > t\}, B)dt. \tag{32}$$

As a corollary, we have that almost every ball has finite perimeter, since the distance function is $H$–Lipschitz. We do not know whether every ball has finite perimeter, because $P_\gamma(B_r(x))$ is not trivially monotone with respect to $r$ and no homothety argument can be used in view of (10).

We now extend from finite dimensions to infinite dimensions some typical tools of the $BV$ theory. The following definition is a very convenient tool in the theory of $BV$ functions:

**Definition 3.8** (Total variation)**.** *We define total variation of a function $v \in L^1(X,\gamma)$ by*

$$V_\gamma(v) := \sup\Big\{ \int_X v\nabla_H^*\Phi d\gamma : \Phi \in \mathcal{F}C_c^1(X,H),\ \|\Phi(x)\|_H \le 1\ \forall\, x \in X \Big\}. \tag{33}$$

The name is justified by the following observation: if $v \in BV(X,\gamma)$, then (29) shows that $V_\gamma(v) = |D_\gamma v|(X)$; on the other hand, if $X$ *is finite-dimensional* and the supremum in (33) is finite, then a direct application of Riesz theorem provides us with a $X$-valued measure $\mu$, with total variation less than $V_\gamma(v)$, such that

$$\int_X v\nabla_H^*\Phi d\gamma = -\langle \Phi, \mu\rangle \qquad \forall \Phi \in C_b^1(X,H).$$

Hence $\mu = D_\gamma u$ and $V_\gamma(u) = |D_\gamma v|(X)$. This equivalence is much less obvious in the infinite-dimensional case, since Riesz theorem is not available, and it will be discussed in the next section. Notice that $v \mapsto V_\gamma(v)$ is lower semicontinuous with respect to the $L^1(X,\gamma)$ convergence, since it is the supremum of a family of continuous functionals. Since $v \mapsto V_\gamma(v)$ is easily seen to be nonincreasing under cylindrical approximation (with the same argument used in Proposition 3.5), we can combine this property with lower semicontinuity to get

$$V_\gamma(v) = \lim_{m\to\infty} V_\gamma(\mathbb{E}_m v). \tag{34}$$

Analogous definitions can be given for the total variation along a direction $\nu \in H_m$ for some $m$; in this case, in order to have a bounded adjoint derivative $\nabla_\nu^*\phi$, we consider the space $\mathcal{F}^\nu C_c^1(X)$ of cylindrical functions $\phi$ with support contained in a strip $\{a < \langle x, x^*\rangle < b\}$, where $\nu = Qx^*$.

**Definition 3.9** (Directional total variation)**.** *Let $\nu \in \cup_m H_m$ be a unit vector. We define total variation of a function $v \in L^1(X,\gamma)$ along $\nu$ by*

$$V_\gamma^\nu(v) := \sup\Big\{ \int_X v\partial_\nu^* \phi d\gamma : \phi \in \mathcal{F}^\nu C_c^1(X),\ \|\phi(x)\|_H \le 1\ \forall x \in X \Big\}. \tag{35}$$

Again $v \mapsto V_\gamma^\nu(v)$ is lower semicontinuous with respect to the $L^1(X,\gamma)$ convergence and, in finite dimensional spaces, Riesz theorem shows that the quantity is finite if and only if the integration by parts formula

$$\int_X v\partial_\nu^*\phi d\gamma = -\int_X \phi d\mu \qquad \forall \phi \in C_b^1(X) \tag{36}$$

holds for some real-valued measure $\mu$ with finite total variation, that we shall denote by $D_\gamma^\nu v$; if this happens, $|\mu|(X)$ coincides with $V_\gamma^\nu(v)$. Finally,

$$V_\gamma^\nu(v) = \lim_{m\to\infty} V_\gamma^\nu(\mathbb{E}_m v). \tag{37}$$

We can now discuss 1-dimensional sections of Gaussian $BV$ functions in the same spirit as Section 3.11 of [1]. Notice that any $u \in BV(\mathbb{R}^m,\gamma_m)$ is in the classical space $BV_{\rm loc}(\mathbb{R}^m)$, so that we can use all the (local) properties known in Euclidean case, and

$$D_\gamma u = G_m Du \tag{38}$$

where $G_m(x) = (2\pi)^{-m/2}\exp(-|x|^2/2)$ is the standard Gaussian kernel and $Du$ stands for the classical derivative of $u$ in the sense of distributions. Let us fix a unit direction $\nu = Qx^* \in H$, let $\Pi(x) = \langle x, x^*\rangle$ be the induced projection and let us write $x \in X$ as $y + \Pi(x)\nu$. Then, denoting by $K$ the kernel of $\Pi$, $\gamma$ admits a product decomposition $\gamma = \gamma^\perp \otimes \gamma_1$ with $\gamma^\perp$ Gaussian in $K$. For $u : X \to \mathbb{R}$ and $y \in K$ we define the function $u_y : \mathbb{R} \to \mathbb{R}$ by $u_y(t) = u(y + t\nu)$.

**Theorem 3.10.** *Let $u \in L\log^{1/2}L(X,\gamma)$ and let $\nu \in \cup_m H_m$; then*

$$V_\gamma^\nu(u) = \int_K V_{\gamma_1}(u_y)\, d\gamma^\perp(y).$$

*In particular Definition 3.9 is independent of the choice of the basis and makes sense for all $h \in H$.*

*Proof.* Let us fix $\phi \in \mathcal{F}^\nu C_b^1(X)$ with $\|\phi\|_\infty \le 1$. Then

$$\int_X u(x)\partial_\nu^*\phi(x)d\gamma(x) = \int_K \Big(\int_{\mathbb{R}} u_y(t)(\phi_y'(t) - t\phi_y(t)d\gamma_1(t)\Big)d\gamma^\perp(y) \le \int_K V_{\gamma_1}(u_y)d\gamma^\perp(y),$$

whence

$$V_\gamma^\nu(u) \le \int_K V_{\gamma_1}(u_y)d\gamma^\perp(y).$$

For the reverse inequality we can assume that $V_\gamma^\nu(u)$ is finite. First we prove the inequality in the finite-dimensional case $X = \mathbb{R}^m$ and $\gamma = \gamma_m$, and then we consider the general case. If $X = \mathbb{R}^m$ and $\gamma = \gamma_m$ then the measure $\mu = D_{\gamma_m}^\nu u$ in (36) is a real valued measure with total variation in $\mathbb{R}^m$ less than $V_\gamma^\nu(u)$. Let us show that we may find a sequence $(u_n) \subset C^\infty(\mathbb{R}^m) \cap \mathbb{D}^{1,1}(\mathbb{R}^m, \gamma_m)$ such that $u_n \to u$ in $L^1(\mathbb{R}^m, \gamma_m)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} |\partial_\nu u_n(y)| d\gamma_m(y) \leq V_{\gamma_m}^\nu(u).$$

For, set $u_n = T_{1/n} u$ and notice that for every $\phi \in \mathcal{F}^\nu C_b^1(\mathbb{R}^m)$ with $|\phi| \leq 1$

$$\Big| \int_{\mathbb{R}^m} \phi \partial_\nu u_n d\gamma_m \Big| = \Big| \int_{\mathbb{R}^m} \partial_\nu^* \phi u_n d\gamma_m \Big| \to \Big| \int_{\mathbb{R}^m} \partial_\nu^* \phi u d\gamma_m \Big| \leq V_{\gamma_m}^\nu(u),$$

then the sequence $(\partial_\nu u_n)$ is bounded in $\mathscr{M}(\mathbb{R}^m)$, and (up to a subsequence which we do not relabel) weakly$^*$ converges to a measure $\mu$. The above limit relation shows that $\mu = D_{\gamma_m}^\nu$ and that the whole sequence is convergent. By Fubini theorem (possibly passing to a subsequence) for $\gamma^\perp$-a.e. $y \in K$ the sequence $(u_{n,y})$ converges to $u_y$ in $L^1(\mathbb{R}, \gamma_1)$ and then by semicontinuity we get

$$\begin{aligned} V_\gamma^\nu(u) &\geq \liminf_{n\to+\infty} \int_{\mathbb{R}^m} |\partial_\nu u_n(y)| d\gamma(y) = \liminf_{n\to+\infty} \int_K \int_{\mathbb{R}} |u'_{n,y}(t)| d\gamma_1(t) d\gamma^\perp(y) \\ &= \liminf_{n\to+\infty} \int_K V_{\gamma_1}(u_{n,y}) d\gamma^\perp(y) \geq \int_K V_{\gamma_1}(u_y) d\gamma^\perp(y). \end{aligned}$$

In the infinite-dimensional case we consider the cylindrical approximations $v_m := \mathbb{E}_m u$; since they converge in $L^1(X, \gamma)$ we can find a subsequence $(m_i)$ such that $v_{m_i,y} \to u_y$ in $L^1(\mathbb{R}, \gamma_1)$ for $\gamma^\perp$-a.e. $y \in K$; then, lower semicontinuity of $v \mapsto V_{\gamma_1}(v)$, Fatou's lemma and monotonicity give

$$\int_K V_{\gamma_1}(u_y) d\gamma^\perp(y) \leq \liminf_{i\to\infty} \int_K V_{\gamma_1}(u_{m_i y}) d\gamma^\perp(y) \leq \sup_m V_\gamma(v_m) \leq V_\gamma(u).$$

□

**Corollary 3.11.** *Let $h = R\hat{h} \in H$ and $c \in \mathbb{R}$; then the sets $E = \{x \in X : \ \hat{h}(x) \leq c\}$ have finite perimeter with*

$$P_\gamma(E) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{c^2}{2\|h\|_H^2} \right\}. \tag{39}$$

*Proof.* With no loss of generality we can assume that $\|h\|_H = 1$. We prove the identity first in the case when $\hat{h} = R^* x^*$ for some $x^* \in X^*$; without loss of generality, we may assume that $\|Qx^*\|_H = 1$. In this case the assertion simply follows by noticing that $E$ is a cylindrical set of the form

$$E = \Big\{ x \in X : \langle x, x^* \rangle \in B \Big\},$$

with $B = \{s \in \mathbb{R} : s \leq c\}$. This implies that $P_\gamma(E) = P_{\gamma_1}(B) = e^{-c^2/2}/\sqrt{2\pi}$. In the general case, density of $QX^*$ in $H$ and lower semicontinuity of the perimeter provides the inequality $\leq$ in (39); to prove the inequality $\geq$ we fix $\phi \in C_c^\infty(\mathbb{R})$ with $\phi(c) = 1$, $|\phi| \leq 1$, $k = Qk^*$, $\|k\|_H = 1$ and $\|k - h\|_H^2 < 2\varepsilon$ and the field $\Phi_\varepsilon(x) = \phi(\langle x, k^* \rangle) k$; then

$$P_\gamma(E) \geq \int_E \partial_k^* \phi(\langle x, k^* \rangle) d\gamma.$$

By Proposition 2.2 and considering the map $x \mapsto (\hat{h}(x), \langle x, k^* \rangle)$, the right hand side can be represented as $\int_{\{x_1 \le c\}} \partial^* \phi(x_2) d\eta_\varepsilon(x_1, x_2)$, where $\eta_\varepsilon$ are Gaussian in $\mathbb{R}^2$ with $\gamma_1$ as marginals and $\int x_1 x_2 d\eta_\varepsilon(x_1, x_2) > 1 - \varepsilon$; as $\varepsilon \to 0$ these Gaussians converge to the standard Gaussian on the diagonal of $\mathbb{R}^2$, so that

$$P_\gamma(E) \ge \int_{\{z \le c\}} \partial^* \phi(z) d\gamma_1(z) = \frac{\phi(c)}{\sqrt{2\pi}} e^{-c^2/2}.$$

□

In connection with the proof of the previous Corollary, notice that it would be desirable to extend Theorem 3.10 even to the case when $\nu = R\hat{h} \in H$; however, this extension presents some difficulties, since $\hat{h}$ is not really a linear map on $X$.

# 4 Main results

We are now in a position to characterize $BV$ functions in the same way as they are characterized in the classical case. Notice that in the classical case the original definition of $BV$ given by E. De Giorgi in [12] was based on property **(4)** below, with the heat semigroup instead of the Ornstein-Uhlenbeck one.

**Theorem 4.1.** *Given $u \in L^1(X, \gamma)$, the following are equivalent:*

**(1)** *$u$ belongs to $BV(X, \gamma)$;*

**(2)** *the quantity $V_\gamma(u)$ in* (33) *is finite;*

**(3)** $L_\gamma(u) := \inf \left\{ \liminf_{n \to \infty} \int_X \|\nabla_H u_n\|_H d\gamma : u_n \in \mathbb{D}^{1,1}(X, \gamma),\ u_n \stackrel{L^1}{\to} u \right\} < \infty$;

**(4)** *$u \in L \log^{1/2} L(X, \gamma)$ and the quantity $\mathcal{I}(u)$ in* (22) *is finite.*

*Moreover, $|D_\gamma u|(X) = V_\gamma(u) = L_\gamma(u) = \mathcal{I}[u]$ and*

$$\int_X \|\nabla_H T_t u\|_H d\gamma \le e^{-t} |D_\gamma u|(X) \qquad \forall t > 0. \tag{40}$$

*Proof.* **(1) ⇒ (2).** Simply comparing the classes of competitors, we notice that $V_\gamma(u) \le |D_\gamma u|(X)$.
**(2) ⇒ (3).** Let $t_n \downarrow 0$ and $u_n = T_{t_n} u$. Then, for all $\Phi \in \mathcal{F}C_b^1(X, H)$ with $\|\Phi(x)\|_H \le 1$ for every $x \in X$, from (23) we deduce

$$\int_X [\nabla_H u_n, \Phi]_H d\gamma = -e^{-t_n} \int_X u \nabla^* (T_{t_n} \Phi) d\gamma \le V_\gamma(u).$$

Therefore, $\|\nabla_H u_n\|_{L^1(X,\gamma)} \le V_\gamma(u)$. In particular, we have proved that $L_\gamma(u) \le V_\gamma(u)$.
**(3) ⇒ (4).** Let $(u_n)_{n \in \mathbb{N}}$ be such that $u_n \to u$ in $L^1(X, \gamma)$ and $\|\nabla_H u_n\|_{L^1(X,\gamma)} \to L_\gamma(u)$. Then,

$$\int_X \|\nabla_H T_t u\|_H d\gamma \le \liminf_{n \to \infty} \int_X \|\nabla_H T_t u_n\|_H d\gamma = e^{-t} \liminf_{n \to \infty} \int_X \|\nabla_H u_n\|_H d\gamma = L_\gamma(u).$$

In addition, Fatou's lemma and (16) yield

$$\int_0^\infty \mathscr{U}\left(\gamma(\{|f|>s\})\right)ds \le L_\gamma(u) < \infty, \tag{41}$$

so that $u \in L\log^{1/2}L(X,\gamma)$. Observe that in particular we have proved that $\mathcal{I}(u) \le L_\gamma(u)$.
**(4) ⇒ (1).** We first prove that for all $j$ the derivative $\mu_j$ along the direction $h_j$ exists, and then we prove (27) to conclude that $u \in BV(X,\gamma)$.

Since $T_t u \in \mathbb{D}^{1,1}(X,\gamma)$ for $t>0$, we have

$$V_\gamma^{h_j}(T_t u) = |D_\gamma^{h_j} T_t u| = \int_X |\partial_j T_t u| d\gamma.$$

In particular, setting $\nu = h_j$ and adopting the same notation as in Theorem 3.10, we have

$$\int_K V_{\gamma_1}((T_t u)_y) d\gamma^\perp(y) = \int_X |\partial_j T_t u| d\gamma \le \int_X \|\nabla_H T_t u\|_H d\gamma.$$

Now we can find $t_n \to 0$ sufficiently fast in such a way that $(T_{t_n} u)_y$ converge to $u_y$ in $L^1(\mathbb{R},\gamma_1)$ for $\gamma^\perp$-a.e. $y \in K$ and conclude, by the lower semicontinuity of $v \mapsto V_{\gamma_1}(v)$, that

$$\int_K V_{\gamma_1}(u_y) d\gamma^\perp(y) \le \mathcal{I}(u) < \infty.$$

It follows that for $\gamma^\perp$-a.e. $y \in K$ the function $u_y$ has bounded variation in $\mathbb{R}$. By a Fubini argument, based on the factorization $\gamma = \gamma^\perp \otimes \gamma_1$, the 1-dimensional integration by parts formula yields that the measure $\mu_j = D_{\gamma_1} u_y \otimes \gamma^\perp$, i.e.

$$\mu_j(A) = \int_K D_{\gamma_1} u_y(A_y) d\gamma^\perp(y)$$

(where $A_y := \{t :\ y + t h_j \in A\}$ is the $y$-section of a Borel set $A$) provides the derivative of $u$ along $h_j$. Notice that $\mu_j$ is well defined, since we have just proved that $\int_K |D_{\gamma_1} u_y|(\mathbb{R}) d\gamma^\perp$ is finite.

Now, setting $\mu_i = D_\gamma^{h_i} u$, we check (27); by a density argument, it suffices to prove that

$$\sum_{i=1}^m \int \phi_i d\mu_i \le \mathcal{I}(u)$$

for all $\phi_i \in \mathcal{F}C_b^1(X)$ with $\sum_i \phi_i^2 \le 1$; by integration by parts, it suffices to show that

$$\limsup_{t\downarrow 0} \sum_{i=1}^m \int \phi_i d D_\gamma^{h_i} T_t u \le \mathcal{I}(u)$$

or equivalently

$$\limsup_{t\downarrow 0} \sum_{i=1}^m \int \phi_i \partial_i T_t u d\gamma \le \mathcal{I}(u).$$

The latter inequality is trivial, since $|\sum_1^m \phi_i \partial_{h_i} T_t u| \le \|\nabla_H T_t u\|_H$.

Passing to the limit as $s \downarrow 0$ in the inequality

$$\int_X \|\nabla_H T_{t+s} u\|_H d\gamma \le e^{-t} \int_X \|\nabla_H T_s u\|_H d\gamma$$

provides $\int_X \|\nabla_H T_t u\|_H d\gamma \le e^{-t}\mathcal{I}(u)$ and concludes the proof. □

**Remark 4.2.** Arguing as in the proof of the implication **(2)** $\Rightarrow$ **(3)** we see that for all $u \in BV(X,\gamma)$ and all closed subspaces $K \subset H$ the following inequality holds:

$$\limsup_{t\downarrow 0} \int_X \|\pi_K \nabla_H T_t\|_H d\gamma \leq |D_\gamma^K u|(X).$$

Here $\pi_K : H \to K$ is the orthogonal projection and $D_\gamma^K u = \pi_K D_\gamma u$.

**Remark 4.3.** It is worth noticing that sets of finite perimeter $E$ can also be characterized using the functional

$$J_E(t) = \int_X \sqrt{\mathscr{U}(T_t\chi_E)^2 + \|\nabla_H T_t \chi_E\|_H^2} d\gamma,$$

in the spirit of [5]. In fact, it can be proved that

$$\frac{d}{dt} J_E(t) \leq 0, \qquad \lim_{t\to 0} J_E(t) = P_\gamma(E). \tag{42}$$

With minor modifications the proof of Theorem 4.1 allows also to show that, for all $j$, the family of measures $\nabla_j T_t u\gamma$ is tight and the limit as $t \downarrow 0$ is $\langle D_\gamma u, h_j\rangle$, in the duality with $C_b(X)$; in addition, the limit of $|\langle D_\gamma u, h_j\rangle\gamma|$ is $|\langle D_\gamma u, h_j\rangle|$. We give just a sketch of proof, since this result is a consequence, rather than a tool, in our characterization theorem of $BV$ functions. We also notice that similar results could be stated and proved for the measures $\|\pi_K \nabla_H T_t u\|_H \gamma$ and $|\pi_K D_\gamma u|$, with $K \subset H$ closed subspace.

**Theorem 4.4.** *Let $u \in BV(X,\gamma)$ and $j \geq 1$. Then*

$$\lim_{t\downarrow 0} \partial_j T_t u\gamma = \langle D_\gamma u, h_j\rangle, \qquad \lim_{t\downarrow 0} |\partial_j T_t u|\gamma = |\langle D_\gamma u, h_j\rangle|$$

*in the duality with $C_b(X)$.*

*Proof.* Since, by the integration by parts formula, $\nabla_j T_t u\gamma$ weakly converge to $\langle D_\gamma u, h_j\rangle$ in the duality with $\mathcal{F}C_b^1(X)$ as $t \downarrow 0$, in order to show the convergence of $\partial_j T_t u\gamma$ it suffices to show that $|\nabla_j T_t u|\gamma$ is tight. Indeed, this ensures by Prokhorov theorem the compactness in the duality with $C_b(X)$, and the weak limit must be the same as above, by the density of $\mathcal{F}C_b^1(X)$ in $C_b(X)$.

By Remark 4.2 we have

$$\limsup_{t\downarrow 0} \int_X |\partial_j T_t u| d\gamma \leq |\langle D_\gamma u, h_j\rangle|(X),$$

hence tightness is achieved, thanks to Lemma 2.1, by

$$\liminf_{t\downarrow 0} \int_A |\partial_j T_t u| d\gamma \leq |\langle D_\gamma u, h_j\rangle|(A) \qquad \text{for all } A \subset X \text{ open.}$$

This inequality, in turn, can be derived as in the proof of the implication **(3)** $\Rightarrow$ **(4)** in the proof of Theorem 4.1, using the fact that the sections $A_y = \{t \in \mathbb{R} : \ y + th_j \in A\}$ are open, and the lower semicontinuity of

$$v \in BV(\mathbb{R},\gamma_1) \mapsto |D_{\gamma_1} v|(J)$$

with respect to the $L^1(\gamma_1)$ convergence, for all $J \subset \mathbb{R}$ open. $\square$

An immediate consequence of Theorem 4.1 is the extension, by approximation, of many properties of Sobolev functions to $BV$: for instance $BV(X,\gamma)\cap L^\infty(X,\gamma)$ is an algebra (and therefore sets of finite perimeter are stable under union and intersection), $BV(X,\gamma)$ is stable under left composition with Lipschitz maps $f$, and $|D_\gamma f\circ u|(X)\leq \mathrm{Lip}(f)|D_\gamma u|(X)$, etc. We need in particular the inequalities stated in the following proposition, a direct consequence of (16) and (18) for $\mathbb{D}^{1,1}(X,\gamma)$ functions, and of the equality $|D_\gamma u|(X)=L_\gamma(u)$.

**Proposition 4.5** (Sobolev and Poincaré inequalities)**.** *Let $u\in BV(X,\gamma)$. Then*

$$\int_0^{+\infty}\mathscr{U}(\gamma(\{|u|>s\}))ds\leq |D_\gamma u|(X)$$

*and*

$$\int_X\left|u-\int_X u d\gamma\right|d\gamma\leq C_1|D_\gamma u|(X),$$

*where $C_1$ is the constant in the Poincaré inequality* (18) *for $p=1$.*

The following approximation result for sets of finite perimeter is a consequence of the approximation in $BV$ through smooth functions and the coarea formula.

**Proposition 4.6.** *Let $E\subset X$ be a set with finite perimeter; then there exist cylindrical sets $E_j=\Pi_{m_j}^{-1}B_j$, with $B_j\in\mathbb{R}^{m_j}$ smooth sets, such that*

$$\lim_{j\to\infty}\|\chi_{E_j}-\chi_E\|_{L^1(X,\gamma)}=0\quad and\quad \lim_{j\to\infty}P_\gamma(E_j)=P_\gamma(E).$$

*Proof.* According to Theorem 4.1, for every $u\in BV(X,\gamma)$ there is a sequence of $\mathbb{D}^{1,1}(X,\gamma)$ functions such that the $L^1$ norms of their gradients converge to the total variation of $u$. Moreover, smooth cylindrical functions are dense in $\mathbb{D}^{1,1}(X,\gamma)$, hence there exists a sequence $(u_j)$ of smooth cylindrical functions with

$$u_j\to\chi_E \text{ in } L^1(X,\gamma)\quad\text{and}\quad \int_X\|\nabla_H u_j\|_H d\gamma\to P_\gamma(E).$$

Assuming with no loss of generality that $0\leq u_j\leq 1$, the conclusion then follows from the coarea formula by taking smooth levels $B_j$ of $u_j$. ☐

Due to the previous proposition, we say that $E$ is a *smooth set* if $E=\Pi_m^{-1}B$ for some set $B\in\mathbb{R}^m$ with smooth boundary. Denoting by $\mathcal{H}^{m-1}$ the Hausdorff $(m-1)$-dimensional measure in $\mathbb{R}^m$, since by (38) $D_{\gamma_m}\chi_B=G_mD\chi_B$ and $|D\chi_B|(A)=\mathcal{H}^{m-1}(A\cap\partial B)$ for all Borel sets $A$, for the sets $E_j$ of the previous proposition we get

$$P_\gamma(E)=\lim_{j\to+\infty}\int_{\partial B_j}G_{m_j}d\mathcal{H}^{m_j-1}.$$

**Remark 4.7.** Using the results in [5], it is possible to repeat the same argument contained in [8] to show that the isoperimetric inequality and extremality of half–spaces hold in infinite dimension as well. That is, for any $E\subset X$ with finite perimeter,

$$P_\gamma(E)\geq\mathscr{U}(\gamma(E)),$$

with $\mathscr{U}$ defined in (14) with equality if and only if $E$ is an arbitrary half–space. The proof is based on analysis of the quantity $J_E$ introduced in Remark 4.3; in particular, once one has (42), by using the fact that $J_E(t)$ is twice differentiable, (see [7, Proposition 5.4.8]), one shows that if equality holds for a set $E$ in the isoperimetric inequality, then $\chi_E$ is affine (see [8, Lemma 2.1]), that is $E$ is a half–space.

In the next corollary we consider a finite dimensional subspace $K \subset H$. We shall denote by $\Pi_K : X \to K$ the canonical projection (induced by the choice of a basis of $K$) and set $\Pi_K^\perp(x) = x - \Pi_K(x)$; since $\Pi_K \circ \Pi_K = \Pi_K$ and $\Pi_K|_K = \Pi_K$ we may write in a unique way $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in \mathrm{Ker}(\Pi_K)$ and, accordingly, $u_{x_1}(x_2) = u(x_1 + x_2)$. Setting $X_1 = K$ and $X_2 = \mathrm{Ker}(\Pi_K)$ (the closure of $K^\perp$ in $X$), we have also the factorization $d\gamma(x_1, x_2) = d\gamma_1(x_1) \otimes d\gamma_2(x_2)$ with $\gamma_1$ Gaussian in $X_1$ and $\gamma_2$ Gaussian in $X_2$; furthermore, the Cameron-Martin spaces are respectively $K$ and $K^\perp$.

The next proposition is the natural complement of Theorem 3.10: in that theorem the slices are 1-dimensional, and with minor changes the same result could be proved for finite-dimensional slices. Here we consider, instead, slices of finite codimension; for the sake of simplicity we state it assuming a priori that the map $u$ is globally $BV$, hence without using variations, and we consider only one implication.

In the sequel, for a given closed subspace $L \subset H$ and $u \in BV(X, \gamma)$ we set, in accordance with Remark 4.2,

$$D_\gamma^L u := \pi_L D_\gamma u \in \mathscr{M}(X, L),$$

where $\pi_L : H \to L$ is the orthogonal projection.

**Proposition 4.8.** *Let $u \in BV(X, \gamma)$; then $u_{x_1} \in BV(X_2, \gamma_2)$ for $\gamma_1$-a.e. $x_1 \in X_1$ and*

$$|D_\gamma^{K^\perp} u|(X) = \int_{X_1} |D_{\gamma_2} u_{x_1}|(X_2) d\gamma_1(x_1). \tag{43}$$

*Proof.* Let $u_n = T_{t_n} u$, with $t_n \to 0$, and assume with no loss of generality that $(u_n)_{x_1}$ converge to $u_{x_1}$ in $L^1(X_2, \gamma_2)$ for $\gamma_1$-a.e. $x_1$. We have

$$\int_{X_1} \int_{X_2} \|\nabla_{K^\perp} (u_n)_{x_1}(x_2)\|_{K^\perp} d\gamma_2(x_2) d\gamma_1(x_1) = \int_X \|\nabla_{K^\perp} u_n\|_H d\gamma$$

and, passing to the limit as $n \to \infty$, Fatou's lemma and Remark 4.2 give

$$\int_{X_1} L_{\gamma_2}(u_{x_1}) d\gamma_1(x_1) \le |D_\gamma^{K^\perp} u|(X),$$

with $L_{\gamma_2}$ as in **(3)** of Theorem 4.1. From Theorem 4.1 we deduce $u_{x_1} \in BV(X_2, \gamma_2)$ for $\gamma_1$-a.e. $x_1 \in X_1$ and the inequality $\ge$ holds in (43).

Arguing as in the first part of the proof of Theorem 3.10 we see that the factorization $\gamma = \gamma_1 \otimes \gamma_2$ yields

$$\langle h, D_\gamma u \rangle = \int_{X_1} \langle h, D_{\gamma_2} u_{x_1} \rangle d\gamma_1(x_1)$$

for all $h \in K^\perp$ (indeed, both measures satisfy the integration by parts formula in the direction $h$), hence

$$D_\gamma^{K^\perp} u = \int_{X_1} D_{\gamma_2} u_{x_1} d\gamma_1(x_1).$$

This immediately gives

$$|D_\gamma^{K^\perp} u|(X) \le \int_{X_1} |D_{\gamma_2} u_{x_1}|(X_2) d\gamma_1(x_1).$$

□

**Corollary 4.9.** *Let $u \in BV(X,\gamma)$ let $K \subset H$ be finite-dimensional and $\mathbb{E}_K u$ the conditional expectation relative to $K$. If $K^\perp$ is the complementary subspace of $K$, we have*

$$\int_X |u - \mathbb{E}_K u| d\gamma \le C_1 |D_\gamma^{K^\perp} u|(X).$$

*Proof.* We write as in Proposition 4.8 $x = x_1 + x_2$ with $x_1 \in X_1 = K$ and $x_2 \in X_2 = \mathrm{Ker}(\Pi_K)$, and denote by $u_{x_1}(x_2) = u(x_1 + x_2)$; using the Poincaré inequality in $X_K^\perp$ we get

$$\begin{aligned}\int_X |u(x) - \mathbb{E}_K u(x)| d\gamma(x) &= \int_{X_1 \times X_2} |u(x_1 + x_2) - \mathbb{E}_K u(x_1)| d\gamma_1(x_1) d\gamma_2(x_2) \\ &= \int_{X_1} \int_{X_2} \left| u_{x_1}(x_2) - \int_{X_2} u_{x_1}(z) d\gamma_2(z) \right| d\gamma_2(x_2) d\gamma_1(x_1) \\ &\le C_1 \int_{X_1} |D_{\gamma_2}^{K^\perp} u_{x_1}|(X_2) d\gamma_1(x_1) = C_1 |D_\gamma^{K^\perp} u|(X),\end{aligned}$$

where in the last line we have used Proposition 4.8. □

In an analogous way one can prove that

$$\int_X |u - \mathbb{E}_K u|^p d\gamma \le C_p \int_X \|\pi_{K^\perp} \nabla_H u\|_H^p \, d\gamma \qquad \forall u \in \mathbb{D}^{1,p}(X,\gamma). \tag{44}$$

The next Theorem generalises Theorem 5.12.5 of [7], for the part concerning Sobolev spaces. The part concerning $BV$ functions is, to our knowledge, totally new.

**Theorem 4.10.** *The following statements hold.*

**(i)** *For $p > 1$, let $\mathcal{F}$ be a bounded family of functions in $\mathbb{D}^{1,p}(X,\gamma)$; assume that for every $\varepsilon > 0$ there exists a finite dimensional subspace $K$ of $H$ such that,*

$$\int_X \|\nabla_{K^\perp} u\|_H^p d\gamma \le \varepsilon, \qquad \forall u \in \mathcal{F}; \tag{45}$$

*then $\mathcal{F}$ is relatively compact in $L^p(X,\gamma)$.*

**(ii)** *Let $\mathcal{F}$ be a bounded family of functions in $BV(X,\gamma)$; assume that for every $\varepsilon > 0$ there exists a finite dimensional subspace $K$ of $H$ such that,*

$$|D_{K^\perp} u|(X) \le \varepsilon, \qquad \forall u \in \mathcal{F}; \tag{46}$$

*then $\mathcal{F}$ is relatively compact in $L^1(X,\gamma)$.*

*Proof.* We first discuss briefly the finite-dimensional case, $X = \mathbb{R}^m$, $\gamma = \gamma_m$, for $BV$ functions (for Sobolev functions, see [6, Theorem 9.3.19] or adapt the argument below, taking into account the continuous embedding of $\mathbb{D}^{1,p}$ into the space $L^p \log^{1/2} L$ in (17)). Since the family $\mathcal{F}$ is bounded also in $L \log^{1/2} L(\mathbb{R}^m, \gamma_m)$, we obtain that $\mathcal{F}$ is equi-integrable in $L^1(\mathbb{R}^m, \gamma_m)$, hence by a truncation argument we can assume with no loss of generality that $\mathcal{F}$ is uniformly bounded also in $L^\infty(\mathbb{R}^m, \gamma_m)$. Under this assumption relative compactness follows obviously from relative compactness in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$; the latter is a consequence of the classical compact embedding of $BV_{\mathrm{loc}}$ in $L^1_{\mathrm{loc}}$ and of the identity $D_{\gamma_m} u = G_m Du$, showing that $\sup_{u \in \mathcal{F}} |Du|(K)$ is finite for any compact set $K \subset \mathbb{R}^m$.

We shall prove only **(ii)** by showing that $\mathcal{F}$ is totally bounded (the proof of **(i)** is analogous). By Corollary 4.9 we have

$$\int_X |u - \mathbb{E}_K u| d\gamma \le C_1 \varepsilon \qquad \forall\, u \in \mathcal{F}.$$

Since the result holds in finite dimension, the family $\mathcal{F}_K = \{\mathbb{E}_K u : u \in \mathcal{F}\}$ is totally bounded in $L^1(X,\gamma)$ and the thesis follows. □

**Remark 4.11.** Statement **(i)** is not true in $\mathbb{D}^{1,1}(X,\gamma)$ under condition (45) with $p = 1$. In this case the family $\mathcal{F}$ is only pre–compact and the limit is in general only in $BV(X,\gamma)$. Moreover, bounded families in $BV(X,\gamma)$ are not in general pre–compact; as an example it suffices to consider the family $\mathcal{F}$ of characteristic functions of the sets

$$\{x : \langle x, x^*\rangle \le 0\},\ x^* \in X^*,\ \|Qx^*\|_H = 1.$$

Condition (45) is satisfied if there exist a compact operator $\mathcal{K}$ on $H$ such that $\nabla_H u \in \mathcal{K}(H)$ almost everywhere for all $u \in \mathcal{F}$ and $\int_X \|\mathcal{K}^{-1}\nabla_H u\|_H^p d\gamma$ is uniformly bounded on $\mathcal{F}$. Indeed, a simple compactness argument proves that

$$\|h - \Pi_m h\|_H \to 0 \qquad \text{uniformly on } \{h \in \mathcal{K}(H) :\ \|\mathcal{K}^{-1}h\|_H \le 1\},$$

hence we may find $\omega_m \to 0$ such that

$$\int_X \|\nabla_H u - \nabla_{H_m} u\|_H^p d\gamma \le \omega_m \int_X \|\mathcal{K}^{-1}\nabla_H u\|_H^p d\gamma.$$

Analogously, in the $BV$ case we have that (46) is fulfilled if there exists a compact operator $\mathcal{K}$ on $H$ such that the Radon-Nikodym densities $D_\gamma u/|D_\gamma u|$ belong to $\mathcal{K}(H)$ $|D_\gamma u|$-a.e. for all $u \in \mathcal{F}$ and

$$\int_X \left\| \mathcal{K}^{-1} \frac{D_\gamma u}{|D_\gamma u|} \right\|_H d|D_\gamma u|$$

is uniformly bounded on $\mathcal{F}$. Indeed,

$$\begin{aligned} |D_\gamma^{H_m^\perp} u|(X) = |D_\gamma u - D_\gamma^{H_m} u|(X) &= \int_X \left\| \frac{D_\gamma u}{|D_\gamma u|} - \pi_{H_m}\frac{D_\gamma u}{|D_\gamma u|} \right\|_H d|D_\gamma u| \\ &\le \omega_m \int_H \left\| \mathcal{K}^{-1} \frac{D_\gamma u}{|D_\gamma u|} \right\|_H d|D_\gamma u|. \end{aligned}$$

# 5 The case when $X$ is a Hilbert space

The results presented above show that there are strict links between the notion of derivative $\partial_h$, the semigroup $T_t$ and the measure $\gamma$, which turns out to be invariant under $T_t$. Indeed, if $X$ is an Hilbert space, different notions of derivative can be given, related to different semigroups still having $\gamma$ as invariant measure (see Remark 5.2).

In this section we briefly describe another point of view and confine ourselves to deriving the corresponding compact embedding theorem both for Sobolev and $BV$ functions.

In this section we assume that $(X, \langle\cdot,\cdot\rangle_X)$ is a separable Hilbert space; let $\gamma = \mathscr{N}(0,Q)$ be as before. Identifying $X$ and its dual $X^*$ with the inner product, we fix an orthonormal basis $(e_k)$ in $X$ such that

$$Qe_k = \lambda_k e_k, \quad \forall\, k \ge 1,$$

with $\lambda_k > 0$ and $\sum_k \lambda_k < \infty$. If we set $x_k = \langle x, e_k \rangle$, since $R^* e_k = x_k$ and $R x_k = \lambda_k e_k$ it follows that

$$\|\lambda_k e_k\|_H = \|x_k\|_{L^2(\mathcal{X},\gamma)} = \sqrt{\lambda_k}.$$

Consequently an orthonormal basis in $H$ is given by $\varepsilon_k = \lambda_k^{1/2} e_k$, the Cameron-Martin norm is $(\sum_k x_k^2/\lambda_k)^{1/2}$ and

$$H = Q^{1/2} X = \Big\{ x \in X : \ \exists y \in X \text{ with } x = Q^{1/2} y \Big\} = \Big\{ x \in X : \ \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k} < \infty \Big\}.$$

Notice that $Q^{1/2}$ is still a compact operator on $X$.

Setting for all $k \geq 1$

$$D_k f(x) = \partial_{e_k} f(x) = \lim_{t \to 0} \frac{f(x + t e_k) - f(x)}{t},$$

we define by linearity a gradient operator $D : \mathcal{F}C_b^1(X) \to \mathcal{F}C_b(X, X)$. The gradient turns out to be a closable operator with respect to the topologies $L^p(X, \gamma)$ and $L^p(X, \gamma, X)$ for every $p \geq 1$, and we denote by $W^{1,p}(X, \gamma)$ the domain of the closure in $L^p(X, \gamma)$, endowed with the norm

$$\|u\|_{1,p} = \Big( \int_X |u(x)|^p d\gamma + \int_X \Big( \sum_{k=1}^{\infty} |D_k u(x)|^2 \Big)^{p/2} d\gamma \Big)^{1/p},$$

where we keep the notation $D_k$ also for the closure of the partial derivative operator.

As a consequence of the relation $\varepsilon_k = \lambda_k^{1/2} e_k$ we have also

$$\partial_{\varepsilon_k} = \lambda_k^{1/2} D_k, \tag{47}$$

so that $W^{1,p}(X, \gamma) \subset \mathbb{D}^{1,p}(X, \gamma)$, since $\|\nabla_H f\|_H = (\sum_k \lambda_k |D_k f|^2)^{1/2}$.

Since $\hat{e}_k = x_k/\lambda_k$, the integration by parts formula (11) becomes

$$\int_X g(x) D_k f(x) d\gamma(x) = -\int_X f(x) D_k g(x) d\gamma(x) + \frac{1}{\lambda_k} \int_X x_k f(x) g(x) d\gamma(x) \tag{48}$$

for $f$, $g \in \mathcal{F}C_b^1(X)$. However, we point out that even though $D_v f = \sum_i v_i D_i f$ makes sense for $v \in X$, the corresponding integration by parts along $v$ does not, since at least convergence of

$$\sum_k \frac{|v_k|}{\lambda_k} \int |x_k| d\gamma = \frac{1}{\sqrt{2\pi}} \sum_k \frac{|v_k|}{\sqrt{\lambda_k}}$$

is needed (this is ensured for $v \in Q(X) = Q^{1/2}(H)$).

In this context, we may give a corresponding definition of $BV$ functions.

**Definition 5.1.** *A function $u \in L^1(X, \gamma)$ belongs to $BV_X(X, \gamma)$ if there exists $\nu^u \in \mathscr{M}(X, X)$ such that for any $k \in \mathbb{N}$ we have*

$$\int_X u(x) D_k \varphi(x) d\gamma = -\int_X \varphi(x) d\nu_k^u + \frac{1}{\lambda_k} \int_X x_k u(x) \varphi(x) d\gamma, \qquad \varphi \in \mathcal{F}C_b^1(X),$$

*with $\nu_k^u = \langle \nu^u, k \rangle_X$.*

It is immediate to check that $BV_X(X,\gamma)$ is contained in $BV(X,\gamma)$ and that

$$\langle D_\gamma u, \varepsilon_i \rangle = \lambda_i^{1/2} \nu_i^u. \tag{49}$$

**Remark 5.2.** Even though we do not further pursue this point of view here, let us point out that the transition semigroup corresponding to the gradient $D$ is given by

$$R_t f(x) = \int_X f(e^{-tQ^{-1}/2}x + y) d\gamma_t(y), \qquad f \in C_b(X),$$

where $\gamma_t = \mathscr{N}(0, Q_t)$ and

$$Q_t = Q \int_0^t e^{-sQ^{-1}} ds = Q(1 - e^{-tQ^{-1}}).$$

Notice that $\|e^{-tQ^{-1}}\| \le e^{-\omega t}, \quad t \ge 0$, where $\omega = \inf \frac{1}{\lambda_k}$.

Therefore $\mathscr{N}(0, Q_t) \to \mathscr{N}(0, Q) = \gamma$ weakly as $t \to \infty$, so that $\gamma$ is invariant for $R_t$. Moreover $R_t$ maps $L^1(X,\gamma)$ into $W^{1,1}(X,\gamma)$ for every $t > 0$, see e.g. [11, Proposition 10.3.1], and this is coherent with the hypothesis $u \in L^1(X,\gamma)$ in Definition 5.1. In [3] we plan to investigate further relations between $BV_X(X,\gamma)$ and the semigroup $R_t$.

Let us show that both Sobolev and $BV$ spaces in the present context are compactly embedded into the corresponding Lebesgue spaces. The following statement easily follows from Theorem 4.10 and the relation (47).

**Theorem 5.3.** *For every $p \ge 1$, the embedding of $W^{1,p}(X,\gamma)$ into $L^p(X,\gamma)$ is compact. The embedding of $BV_X(X,\gamma)$ into $L^1(X,\gamma)$ is compact.*

*Proof.* Let us prove the statement in the Sobolev case, that of $BV$ is similar and uses (49). It suffices to show that every family $\mathcal{F}$ bounded in $W^{1,p}(X,\gamma)$ is totally bounded in $L^p(X,\gamma)$. If $\|u\|_{1,p} \le C$ for all $u \in \mathcal{F}$, then in particular

$$\int_X \Big( \sum_{k=1}^\infty |D_k u|^2 \Big)^{p/2} d\gamma \le C^p \qquad \forall\, u \in \mathcal{F},$$

whence by (47)

$$\int_X \Big( \sum_{k=1}^\infty \Big| \frac{1}{\lambda_k} \partial_{\varepsilon_k} u \Big|^2 \Big)^{p/2} d\gamma \le C^p \qquad \forall\, u \in \mathcal{F}.$$

By applying Remark 4.11 with $\mathcal{K} = Q^{1/2}$ the thesis follows. □

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