

# CONFORMALITY AND $Q$ -HARMONICITY IN SUB-RIEMANNIAN MANIFOLDS

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ABSTRACT. We establish regularity of conformal maps between sub-Riemannian manifolds from regularity of  $Q$ -harmonic functions, and in particular we prove a Liouville-type theorem, i.e., 1-quasiconformal maps are smooth in all contact sub-Riemannian manifolds. Together with the recent results in [CLD17], our work yields a new proof of the smoothness of boundary extensions of biholomorphisms between strictly pseudoconvex smooth domains [Fef74].

**Résumé** On étudie la régularité des applications 1–quasiconformes entre variétés sub-Riemanniennes qui satisfait une hypothèse de régularité pour fonctions  $Q$ –harmonique. En particulier on prouve que toute applications 1–quasiconformes entre variétés sub-Riemanniennes de contact sont des difféomorphismes conformes.

## 1. INTRODUCTION

The focus of this paper is on the interplay between analysis and geometry in the study of conformal maps. Our setting is that of sub-Riemannian manifolds, and our main contribution is to show that one can deduce smoothness of 1-quasiconformal homeomorphisms (see below for the definition) out of certain regularity estimates for weak solutions of a class of quasilinear degenerate elliptic PDE, i.e., the subelliptic  $p$ -Laplacian, see (2.15).

Moreover, we also adapt recent results of Zhong [Zho09] to show that such PDE regularity estimates hold in the important special case of sub-Riemannian contact manifolds, thus fully establishing a Liouville type theorem in this setting. In doing this we provide an extension of a result of Ferrand [LF76, Fer77, LF79] (see also Liimatainen and Salo [LS14]) from the Riemannian to the sub-Riemannian setting.

**Theorem 1.1.** *Every 1-quasiconformal map between sub-Riemannian contact manifolds is conformal.*

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For the proof see Section 6. For some related results in the setting of CR 3-manifolds see [Tan96].

Prior to the present paper, the connection between regularity of quasiconformal maps and the  $p$ -Laplacian, and the equivalence of different definitions of conformality, were only well understood in the Euclidean, Riemannian, and Carnot-group settings. The general sub-Riemannian setting presents genuinely new difficulties, e.g., sub-Riemannian manifolds are not locally bi-Lipschitz equivalent to their tangent cones, Hausdorff measures are not smooth, there is a need to construct adequate coordinate charts that are compatible both with the nonlinear PDE and with the sub-Riemannian structure, although no complete system of harmonic or  $p$ -harmonic coordinates can be constructed. Last but not least, Ferrand's proof of the biLipschitz regularity for 1-quasiconformal maps does not carry through to the sub-Riemannian setting since we do not have yet a sharp isoperimetric inequality.

**Motivations.** One of the motivations that drove our work consists in establishing a connection between the problem of classification of open sets in  $\mathbb{C}^n$  by bi-holomorphisms and the study of quasiconformal maps in sub-Riemannian geometry, in the spirit of Gromov hyperbolicity and Mostow's rigidity: In [BB00], Balogh and Bonk proved that the boundary extension of isometries with respect to the Bergman metric (and so in particular of bi-holomorphisms) between strongly pseudoconvex smooth domains in  $\mathbb{C}^n$  are quasiconformal with respect to the underlying sub-Riemannian metric on the boundaries associated to their Levi form. In [CLD17], two of the authors of the present paper have refined this result and established that such boundary extensions of isometries are in fact 1-quasiconformal with respect to these sub-Riemannian structures. Since the boundaries of smooth strictly pseudoconvex domains are contact manifolds, our main regularity result Theorem 1.1 yields immediately the smoothness of the boundary extension of every biholomorphism between strictly pseudoconvex domains. This alternative proof of Fefferman celebrated result [Fef74] was originally suggested by Michael Cowling.

**Previous results from the literature.** The issue of regularity of 1-quasiconformal homeomorphisms in the Euclidean case was first studied in 1850 in Liouville's work, where the initial regularity of the conformal homeomorphism was assumed to be  $C^3$ . In 1958, the regularity assumption was lowered to  $C^1$  by Hartman [Har58] and then, in conjunction with the proof of the De Giorgi-Nash-Moser Regularity Theorem, further decreased to the Sobolev spaces  $W^{1,n}$ , in the work of Gehring [Geh62] and Rešetnjak [RYGR67]. The role of the De Giorgi-Nash-Moser Theorem in Gehring's proof consists in providing adequate  $C^{1,\alpha}$  estimates for solutions of the Euclidean  $n$ -Laplacian, that are later bootstrapped to  $C^\infty$  estimates by means of elliptic regularity theory.

The regularity of 1-quasiconformal maps in the Riemannian case is considerably more difficult than the Euclidean case. It was finally settled in 1976 by Ferrand [LF76, Fer77, LF79], in occasion of her work on Lichnerowicz's conjecture and was modeled after Rešetnjak's original proof. More recently, inspired by Taylor's regularity proof for isometries via harmonic coordinates, Liimatainen and Salo [LS14] provided a new proof for the regularity of biLipschitz 1-quasiconformal maps between Riemannian manifolds. Their argument is based on the notion of  $n$ -harmonic coordinates, on the morphism property for 1-quasiconformal maps, and on the  $C^{1,\alpha}$  regularity estimates for the  $n$ -Laplacian on manifolds. The proofs in the present paper are modeled on the arguments developed by two of us in [CL16] and on

Taylor's approach, as developed in [LS14] (see also the earlier [CC06] where that strategy was used in the Carnot group case).

The introduction of conformal and quasiconformal maps in the sub-Riemannian setting goes back to the proof of Mostow's Rigidity Theorem [Mos73], where such maps arise as boundary limits of quasi-isometries between certain Gromov hyperbolic spaces. Because the class of spaces that arises as such boundaries in other geometric problems includes sub-Riemannian manifolds that are not Carnot groups, it becomes relevant to study conformality and quasiconformality in this more general environment.

In the sub-Riemannian setting the regularity is currently known only in the special case of 1-quasiconformal maps in Carnot groups, see [Pan89, KR85, Tan96, CC06, CO15, AT17]. Since such groups arise as tangent cones of sub-Riemannian manifolds then the regularity of 1-quasiconformal maps in Carnot groups setting is an analogue of the Euclidean case as studied by Gehring and Rešetnjak. As remarked above, the extension non the non-Carnot setting, even in the special step two case, brings in genuinely new challenges.

**From the regularity theory for the subelliptic  $p$ -Laplacian to the regularity of 1-quasiconformal homeomorphisms.** Theorem 1.1 follows from a more general theorem. In fact, we show that in the class of sub-Riemannian manifolds the Liouville theorem follows from a regularity theory for  $p$ -harmonic functions, with  $p$  corresponding to the conformal dimension of the manifold. This class includes every sub-Riemannian manifold that is locally contactomorphic to a Carnot group of step 2 or, equivalently, every Carnot group of step 2 with a sub-Riemannian metric that is not necessarily left-invariant. We remark that there are examples of step-2 sub-Riemannian manifolds that are not contactomorphic to any Carnot group, see [LOW14]. In order to describe in detail the more general result we introduce the following definition.

**Definition 1.2.** Consider an equiregular<sup>1</sup> sub-Riemannian manifold  $M$  of Hausdorff dimension  $Q$ , with horizontal bundle of dimension  $r$ , endowed with a smooth volume form. We say that  $M$  *supports regularity for  $Q$ -harmonic functions* if the following holds: For every  $g = (g^1, \dots, g^r) \in C^\infty(M, \mathbb{R}^r)$ ,  $U \subset\subset M$  and for every  $\ell > 0$ , there exist constants  $\alpha \in (0, 1)$ ,  $C = C(\ell, g) > 0$  such that for each weak solution  $u$  of the equation  $L_Q u = X_i^* g^i$  on  $M$  with  $\|u\|_{W_{\mathbb{H}}^{1,Q}(U)} < \ell$ , one has

$$\|u\|_{C_{\mathbb{H}}^{1,\alpha}(U)} \leq C.$$

In view of the work of Uraltseva [Ura68] (but see also [Uhl77, Tol84, DiB83]) every Riemannian manifold supports regularity for  $Q$ -harmonic functions. Things are less clear in the sub-Riemannian setting. The Hölder regularity of weak solutions of quasilinear PDE  $\sum_{i=1}^r X_i^* A(x, \nabla_H u) = 0$ , modeled on the subelliptic  $p$ -Laplacian, for  $1 < p < \infty$ , and for their parabolic counterpart, is well known, see [CDG93, ACCN14]. However, in this generality the higher regularity of solutions is still an open problem. The only results in the literature are for the case of left-invariant sub-Riemannian structures on step two Carnot groups. Under these assumptions one has that solutions in the range  $p \geq 2$  have Hölder regular horizontal gradient. This is a formidable achievement in itself, building on contributions by several authors [Cap97, Dom04, DFDFM05, MM07, DM09a, MZGZ09, DM09b, Ric15],

<sup>1</sup>see Definition 2.1

with the final result being established eventually by Zhong in [Zho09]. Beyond the Heisenberg group one has some promising results due to Domokos and Manfredi [Dom08, DM10b, DM10a] in the range of  $p$  near 2. In this paper we build on these previous contributions, particularly on Zhong's work [Zho09] to include the dependence on  $x$  and prove that contact sub-Riemannian manifolds support regularity for  $Q$ -harmonic functions (see Theorem 6.15). The novelty of our approach is that we use a Riemannian approximation scheme to regularize the  $Q$ -Laplacian operator, thus allowing to approximate its solutions with smooth functions. In carrying out this approximation the main difficulty is to show that the regularity estimates do not blow up as the approximating parameter approaches the critical case. Our main result in this context, proved in Section 6, is the following.

**Theorem 1.3.** *sub-Riemannian contact manifolds support regularity for  $p$ -harmonic functions for every  $p \geq 2$ .*

The regularity hypotheses in Definition 1.2 have two important consequences. First, it allows us to construct horizontal  $Q$ -harmonic coordinates. Second, together with the existence of such coordinates, it eventually leads to an initial  $C^{1,\alpha}$  regularity for 1-quasiconformal maps (see Theorem 1.4.(ii)). When this basic regularity is present, one can use classical PDE arguments to derive smoothness without the additional hypothesis of Definition 1.2 (see Theorem 1.4.(i)).

**Theorem 1.4.** *Let  $f : M \rightarrow N$  be a 1-quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ , endowed with smooth volume forms.*

- (i) *If  $f$  is bi-Lipschitz and in  $C_{\text{H,loc}}^{1,\alpha}(M, N) \cap W_{\text{H,loc}}^{2,2}(M, N)$ , then  $f$  is conformal.*
- (ii) *If  $M$  and  $N$  support regularity for  $Q$ -harmonic functions (in the sense of Definition 1.2), then  $f$  is bi-Lipschitz and in  $C_{\text{H,loc}}^{1,\alpha}(M, N) \cap W_{\text{H,loc}}^{2,2}(M, N)$ , and hence conformal.*

The function spaces in Theorem 1.4 are defined componentwise, see Section 5. Theorem 1.4.(i) is proved in Section 5.1. Theorem 1.4.(ii) is proved in Section 5.2.

The above theorem provides the following result.

**Corollary 1.5.** *Let  $f$  be a homeomorphism between two equiregular sub-Riemannian manifolds each supporting the regularity estimates in Definition 1.2. The map  $f$  is conformal if and only if it is 1-quasiconformal.*

The proof of the first part of Theorem 1.4 rests on the morphism property for 1-quasiconformal maps (see Theorem 3.17) and on Schauder's estimates, as developed by Rothschild and Stein [RS76] and Xu [Xu92]. The second part is based on the construction of ad-hoc systems of coordinates, the *horizontal  $Q$ -harmonic coordinates*, that play an analogue role to that of the  $n$ -harmonic coordinates in the work of Liimatainen–Salo [LS14]. However, in contrast to the Riemannian setting, only a subset of the coordinate systems (the *horizontal* components) can be constructed so that they are  $Q$ -harmonic, but not the remaining ones. This yields a potential obstacle, as  $Q$ -harmonicity is the key to the smoothness of the map. We remedy to this potential drawback by producing an argument showing that if an ACC map has suitably regular horizontal components then such regularity is transferred to all the other components (see Proposition 4.14). This method was introduced in [CC06] in the

special setting of Carnot groups, where  $Q$ -harmonic horizontal coordinates arise naturally as the exponential coordinates associated to the first layer of the stratification.

Looking ahead, it seems plausible to conjecture that the Liouville theorem holds in any equiregular sub-Riemannian manifold. Our work shows in fact that this is implied by the regularity theory for  $p$ -Laplacians and the latter is widely expected to hold for general systems of Hörmander vector fields. However the latter remains a challenging open problem.

We conclude this introduction with a comparison between our work and the Carnot group case as studied in [CC06]. In the latter setting one has that all the canonical exponential horizontal coordinates happen to be also smooth  $Q$ -harmonic (in fact they are also harmonic). Moreover, a simple argument based on the existence of dilations and the 1-quasiconformal invariance of the conformal capacity (see [Pan89]) yields the bi-Lipschitz regularity for 1-quasiconformal maps immediately, without having to invoke any PDE result. As a consequence the Liouville theorem in the Carnot group case can be proved relying on a much weaker regularity theory than the one above, i.e., one has just to use the  $C^{1,\alpha}$  estimates for the  $Q$ -Laplacian in the simpler case where the gradient is bounded away from zero and from infinity (established in [Cap99]) in the Carnot group setting. In our more general, non-group setting, there are no canonical  $Q$ -harmonic coordinates, and so one has to invoke the PDE regularity to construct them. Similarly, the lack of dilations makes it necessary to rely on the PDE regularity also to show bi-Lipschitz regularity.

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## 2. PRELIMINARIES

**2.1. Sub-Riemannian geometry.** A *sub-Riemannian manifold* is a connected, smooth manifold  $M$  endowed with a subbundle  $HM$  of the tangent bundle  $TM$  that bracket generates  $TM$  and a smooth section of positive-definite quadratic forms  $g$  on  $HM$ , see [Mon02]. The form  $g$  is locally completely determined by any orthonormal frame  $X_1, \dots, X_r$  of  $HM$ . The bundle  $HM$  is called *horizontal distribution*. The section  $g$  is called *sub-Riemannian metric*.

Analogously to the Riemannian setting, one can endow a sub-Riemannian manifold  $M$  with a metric space structure by defining the *Carnot-Carathéodory* distance: For any pair  $x, y \in M$  set

$$d(x, y) = \inf\{\delta > 0 \text{ such that there exists a curve } \gamma \in C^\infty([0, 1]; M) \text{ with endpoints } x, y \\ \text{such that } \dot{\gamma} \in H_\gamma M \text{ and } |\dot{\gamma}|_g \leq \delta\}.$$

Consider a sub-Riemannian manifold  $M$  with horizontal distribution  $HM$  and denote by  $\Gamma(HM)$  the smooth sections of  $HM$ , i.e., the vector fields tangent to  $HM$ . For all  $k \in \mathbb{N}$ , consider

$$H^k M := \bigcup_{q \in M} \text{span}\{[Y_1, [Y_2, [\dots [Y_{l-1}, Y_l]]]]_q : l \leq k, Y_j \in \Gamma(HM), j = 1, \dots, l\}.$$

The bracket generating condition (also called *Hörmander's finite rank hypothesis*) is expressed by the existence of  $s \in \mathbb{N}$  such that  $H^s M = TM$ .

**Definition 2.1.** A sub-Riemannian manifold  $M$  with horizontal distribution  $HM$  is *equiregular* if, for all  $k \in \mathbb{N}$ , each set  $H^k M$  defines a subbundle of  $TM$ .

Consider the metric space  $(M, d)$  where  $M$  with horizontal distribution  $\Delta$  is an equiregular sub-Riemannian manifold and  $d$  is the corresponding Carnot-Carathéodory distance. As a consequence of Chow-Rashevsky Theorem such a distance is always finite and induces on  $M$  the original topology. As a result of Mitchell [Mit85], the Hausdorff dimension of  $(M, d)$  coincides with the Hausdorff dimension of its tangents spaces.

Let  $X_1, \dots, X_r$  be an orthonormal frame of the horizontal distribution of a sub-Riemannian manifold  $M$ . We define the *horizontal gradient* of a function  $u : M \rightarrow \mathbb{R}$  with respect to  $X_1, \dots, X_r$  as

$$(2.2) \quad \nabla_{\text{H}} u := (X_1 u)X_1 + \dots + (X_r u)X_r.$$

*Remark 2.3.* Let  $X'_1, \dots, X'_r$  be another frame of the same distribution. Let  $B$  be the matrix such that

$$X'_j(p) = \sum_{i=1}^r B_j^i(p) X_i(p).$$

Then the horizontal gradient  $\nabla'_{\text{H}} u$  of  $u$  with respect to  $X'_1, \dots, X'_r$  is

$$\begin{aligned} \nabla'_{\text{H}} u(p) &= \sum_j (X'_j u(p)) X'_j(p) \\ &= \sum_j \left( \sum_i B_j^i(p) X_i(p) u \right) \sum_k B_j^k(p) X_k(p) \\ &= \sum_i \sum_j \sum_k B_j^i(p) B_k^j(p)^T X_i u(p) X_k(p) \\ &= (B(p) B(p)^T)_k^i X_i u(p) X_k(p). \end{aligned}$$

*Remark 2.4.* If  $X_1, \dots, X_r$  and  $X'_1, \dots, X'_r$  are two frames that are orthonormal with respect to a sub-Riemannian structure on the distribution, then  $\nabla'_{\text{H}} u = \nabla_{\text{H}} u$ . Indeed, in this case the matrix  $B(p)$  would be in  $O(r)$  for every  $p$ .

**2.2. PDE preliminaries.** In this section we collect some of the PDE results that will be used later in the paper. Let  $X_1, \dots, X_r$  be an orthonormal frame of the horizontal bundle of a sub-Riemannian manifold  $M$ . For each  $i = 1, \dots, r$  denote by  $X_i^*$  the *adjoint* of  $X_i$  with respect to a smooth volume form  $\text{vol}$ , i.e.,

$$\int_M u X_i \phi \, d \text{vol} = \int_M X_i^* u \phi \, d \text{vol},$$

for every compactly supported  $\phi$  for which the integral is finite. In any system of coordinates, the smooth volume form can be expressed in terms of the Lebesgue measure  $\mathcal{L}$  through a smooth density  $\omega$ , i.e.,  $d \text{vol} = \omega \, d \mathcal{L}$ . If in local coordinates we write  $X_i = \sum_{k=1}^n b_k^i \partial_k$ , then one has

$$(2.5) \quad X_i^* u = -\omega^{-1} (X_i(\omega u)) - u \partial_k b_k^i.$$

Next we define some of the function spaces that will be used in the paper.

**Definition 2.6.** Let  $X_1, \dots, X_r$  be an orthonormal frame of the horizontal bundle of a sub-Riemannian manifold  $M$  and consider an open subset  $\Omega \subset M$ . For any  $k \in \mathbb{N}$ , and  $\alpha \in (0, 1)$  we define the  $C_{\mathbb{H}}^{k, \alpha}$  norm

$$\|u\|_{C_{\mathbb{H}}^{k, \alpha}(\Omega)}^2 := \sup_{\Omega} \left( \sum_{|I| \leq k-1} |X^I u|^2 \right) + \sup_{p, q \in \Omega \text{ and } p \neq q} \frac{\sum_{|I|=k} |X^I u(p) - X^I u(q)|^2}{d(p, q)^{2\alpha}},$$

where, for each  $m = 0, \dots, k$  and each  $m$ -tuple  $I = (i_1, \dots, i_m) \in \{1, \dots, r\}^m$ , we have denoted by  $X^I$  the  $m$ -order operator  $X_{i_1} \cdots X_{i_m}$  and we set  $|I| = m$ . We write

$$C_{\mathbb{H}}^{k, \alpha}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : X^I u \text{ is continuous in } \Omega \text{ for } |I| \leq k \text{ and } \|u\|_{C_{\mathbb{H}}^{k, \alpha}(\Omega)} < \infty \right\}.$$

A function  $u$  is in  $C_{\mathbb{H}, \text{loc}}^{k, \alpha}(\Omega)$ , if for any  $K \subset\subset \Omega$  one has  $\|u\|_{C_{\mathbb{H}}^{k, \alpha}(K)} < \infty$ .

**Definition 2.7.** Let  $X_1, \dots, X_r$  be an orthonormal frame of the horizontal bundle of a sub-Riemannian manifold  $M$  and consider an open subset  $\Omega \subset M$ . For  $k \in \mathbb{N}$  and for any multi-index  $I = (i_1, \dots, i_k) \in \{1, \dots, r\}^k$  we define  $|I| = k$  and  $X^I u = X_{i_1} \cdots X_{i_k} u$ . For  $p \in [1, \infty)$  we define the *horizontal Sobolev space*  $W_{\mathbb{H}}^{k, p}(\Omega)$  to be the space of all  $u \in L^p(\Omega)$  whose distributional derivatives  $X^I u$  are also in  $L^p(\Omega)$  for all multi-indexes  $|I| \leq k$ . This space can also be defined as the closure of the space of  $C^\infty(\Omega)$  functions with respect to the norm

$$(2.8) \quad \|u\|_{W_{\mathbb{H}}^{k, p}}^p := \|u\|_{L^p(\Omega)}^p + \int_{\Omega} \left[ \sum_{|I|=1}^k (X^I u)^2 \right]^{p/2} \text{dvol},$$

see [GN98], [FSS96] and references therein. A function  $u \in L^p(\Omega)$  is in the local Sobolev space  $W_{\mathbb{H}, \text{loc}}^{k, p}(\Omega)$  if, for any  $\phi \in C_c^\infty(\Omega)$ , one has  $u\phi \in W_{\mathbb{H}}^{k, p}(\Omega)$ .

**2.3. Schauder estimates.** Here we discuss Schauder estimates for second order, non-divergence form subelliptic linear operators. Given an orthonormal frame  $X_1, \dots, X_r$  of the horizontal bundle of  $M$ , one defines the *subLaplacian* on  $M$  of a function  $u$  as

$$(2.9) \quad L_2 u := \sum_{i=1}^r X_i^* X_i u.$$

One can check that such an operator does not depend on the choice of the orthonormal frame, but only on the sub-Riemannian structure of  $M$  and the choice of the volume form.

Let  $\Omega$  be an open set of  $M$ . A function  $u : \Omega \rightarrow \mathbb{R}$  is called *2-harmonic* (or, more simply, *harmonic*) if  $L_2 u = 0$  in  $\Omega$ , in the sense of distribution. Hörmander's celebrated Hypocoellipticity Theorem [Hör67] implies that harmonic functions are smooth.

A well known result of Rothschild and Stein [RS76], yields *Schauder estimates* for sub-Laplacians, that is if  $L_2 u \in C_{\mathbb{H}}^\alpha(\Omega)$ , then for any  $K \subset\subset \Omega$ , there exists a constant  $C$  depending on  $K, \alpha$  and the sub-Riemannian structure such that

$$\|u\|_{C_{\mathbb{H}}^{2, \alpha}(K)} \leq C \|L_2 u\|_{C_{\mathbb{H}}^\alpha(\Omega)}.$$

In particular we shall use that

$$(2.10) \quad \|u\|_{C_{\mathbb{H}}^{1,\alpha}(\bar{B}_{\epsilon/2})} \leq C \|L_2 u\|_{C_{\mathbb{H}}^{\alpha}(B_{\epsilon})}.$$

The Schauder estimates have been extended to subelliptic operators with low regularity by a number of authors. For our purposes we will consider operators of the form

$$L_{a(x)}u(x) := \sum_{i,j=1}^r a_{ij}(x)X_iX_ju(x),$$

where  $a_{ij}$  is a symmetric matrix such that for some constants  $\lambda, \Lambda > 0$  one has

$$(2.11) \quad \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for every  $x \in M$  and for all  $\xi \in \mathbb{R}^r$ . We recall a version of the classical Schauder estimates as established in [Xu92]

**Proposition 2.12.** *Let  $u \in C_{\mathbb{H},\text{loc}}^{2,\alpha}(M)$  for some  $\alpha \in (0, 1)$ . Let  $a_{ij} \in C_{\mathbb{H},\text{loc}}^{k,\alpha}(M)$ . If  $L_a u \in C_{\mathbb{H},\text{loc}}^{k,\alpha}(M)$ , then  $u \in C_{\mathbb{H},\text{loc}}^{k+2,\alpha}(M)$  and for every  $U \subset\subset M$  there exists a positive constant  $C = C(U, \alpha, k, X)$  such that*

$$\|u\|_{C^{k+2,\alpha}(U)} \leq C \|Lu\|_{C^{k,\alpha}(M)}.$$

In a similar spirit, the Schauder estimates hold for any operator of the form  $Lu = \sum_{i,j=1}^r a_{ij}(x)X_i^*X_ju$  where  $X_i^*$  denotes the adjoint of  $X_i$  with respect to some fixed smooth volume form.

Next, following an argument originally introduced by Agmon, Douglis and Nirenberg [ADN59, Theorem A.5.1] in the Euclidean setting, we show that one can lift the burden of the a-priori regularity hypothesis from the Schauder estimates.

**Lemma 2.13.** *Let  $\alpha \in (0, 1)$  and assume that  $u \in W_{\mathbb{H},\text{loc}}^{2,2}(M)$  is a function that satisfies for a.e.  $x \in M$*

$$L_{A(x)}u(x) = \sum_{i,j=1}^r a_{ij}(x)X_iX_ju(x) \in C_{\mathbb{H},\text{loc}}^{\alpha}(M).$$

*If  $a_{ij} \in C_{\mathbb{H},\text{loc}}^{\alpha}(M)$ , then  $u$  is in fact a  $C_{\mathbb{H},\text{loc}}^{2,\alpha}(M)$  function.*

*Proof.* The strategy in [ADN59] consists in setting up a bootstrap argument through which the integrability of the weak second order derivatives  $X_iX_ju$  of the solution increases until, in a finite number of steps, one achieves that they are continuous. At this point one invokes a standard extension of a classical result of Hopf [Hop32] or [ADN59, page 723] (for a proof in the subelliptic setting see for instance Bramanti et al., [BBLU10, Theorem 14.4]) which yields the last step in regularity, i.e., if  $X_iX_ju$  are continuous then  $u \in C_{\mathbb{H},\text{loc}}^{2,\alpha}$ .

For a fixed  $p_0 \in M$  consider the *frozen coefficients* operator

$$L_{A(p_0)}w = \sum_{i,j=1}^r a_{ij}(p_0)X_iX_jw.$$



For sake of simplicity we will write  $L_p, L_{p_0}$  for  $L_{A(p)}, L_{A(p_0)}$ . Denote by  $\Gamma_{p_0}(p, q)$  the fundamental solution of  $L_{p_0}$ . For fixed  $r > 0$ , consider a smooth function  $\eta \in C_0^\infty(B(p_0, 2r))$  such that  $\eta = 1$  in  $B(p_0, r)$ . For any  $p \in M$  and any smooth function  $w$  one has

$$\eta(p)w(p) = \int \Gamma_{p_0}(p, q)L_{p_0}(\eta w)(q) \, d \, \text{vol}(q).$$

Differentiating the latter along two horizontal vector fields  $X_i, X_j$   $i, j = 1, \dots, r$  one obtains that for any  $p \in B(p_0, r)$

$$X_{i,p}u(p) = \int \left[ X_{i,p}\Gamma_{p_0}(p, q)L_q(u\eta) + X_{i,p}\Gamma_{p_0}(p, q)(L_{p_0} - L_q)u\eta(q) \right] \, d \, \text{vol}(q),$$

and

$$X_{i,p}X_{j,p}u(p) = \int \left[ X_{i,p}X_{j,p}\Gamma_{p_0}(p, q)L_q(u\eta) + X_{i,p}X_{j,p}\Gamma_{p_0}(p, q)(L_{p_0} - L_q)u\eta(q) \right] \, d \, \text{vol}(q) + C(p_0)L_p(u\eta),$$

where  $X_{i,p}$  denotes differentiation in the variable  $p$  and  $C$  is a Hölder continuous function arising from the principal value of the integral.

Setting  $p = p_0$  one obtains the identity  
(2.14)

$$X_{i,p}X_{j,p}u(p) = \int \left[ X_{i,p}X_{j,p}\Gamma_p(p, q)L_q(u\eta) + X_{i,p}X_{j,p}\Gamma_p(p, q)(L_p - L_q)u\eta(q) \right] \, d \, \text{vol}(q) + C(p_0)L_p(u\eta),$$

where the differentiation in the first term in the integrand is intended in the first set of the argument variables only. The next task is to show that identity (2.14) holds also for functions in  $W_{\mathbb{H}}^{2,2}$ , in the sense that the difference between the two sides has  $L^2$  norm zero. To see this we consider a sequence of smooth approximations  $w_n \rightarrow u \in W_{\mathbb{H}}^{2,2}$  in  $W_{\mathbb{H}}^{2,2}$  norm. To guarantee convergence we observe that in view of the work in [RS76] and [NSW85], the expression  $X_{i,p}X_{j,p}\Gamma_p(p, q)$  is a Calderon-Zygmund kernel. To prove our claim it is then sufficient to invoke the boundedness between Lebesgue spaces of Calderon-Zygmund operators in the setting of homogenous spaces (see [CW71]), and [DH95]).

Our next goal is to show an improvement in the integrability of the second derivatives of the solution  $u \in W_{\mathbb{H}}^{2,2}$ . We write

$$X_{i,p}X_{j,p}u(p) = I_1 + I_2 + I_3 + I_4$$

where

$$I_1(p) = \int X_{i,p}X_{j,p}\Gamma_p(p, q)\eta(q)L_q u(q) \, d \, \text{vol}(q) + C(p)L_p u(p),$$

$$I_2(p) = \int X_{i,p}X_{j,p}\Gamma_p(p, q) \sum_{i,j=1}^r a_{ij}(q)X_i\eta(q)X_j u(q) + u(q) \sum_{i,j=1}^r a_{ij}(q)X_iX_j\eta(q) \, d \, \text{vol}(q),$$

$$I_3(p) = \int X_{i,p}X_{j,p}\Gamma_p(p, q) \sum_{i,j=1}^r (a_{ij}(p) - a_{ij}(q))X_iX_j(\eta u) \, d \, \text{vol}(q).$$

Since  $Lu \in C^\alpha$  and in view of the continuity of singular integral operators in Hölder spaces (see Rothschild and Stein [RS76]) then  $I_1 \in C^\alpha$  and we can disregard this term in our argument.

Next we turn our attention to  $I_2$  and  $I_3$ . Since  $u \in W^{2,2}$  then Sobolev embedding theorem [HK00] yields  $\nabla_{\mathbb{H}} u \in L_{loc}^{\frac{2Q}{Q-2}}$  and as a consequence of the continuity of Calderon-Zygmund operators in homogenous spaces one has  $I_2 \in L^{\frac{2Q}{Q-2}}$ .

In view of the estimates on the fundamental solution for sublaplacians by Nagel, Stein and Wainger [NSW85], one has that

$$|X_i X_j \Gamma_p(p, q)| \sup_{i,j} |a_{ij}(p) - a_{ij}(q)| \leq C(K) d(p, q)^{\alpha-Q},$$

for every  $q \in K \subset\subset M$ . One can then bound  $I_3$  with fractional integral operators

$$\mathcal{I}_\alpha(\psi)(p) := \int d(p, q)^{\alpha-Q} \psi(q) d\text{vol}(q).$$

In the context of homogenous spaces (see for instance [CW71]), these operators are bounded between the Lebesgue spaces  $L^\beta \rightarrow L^\gamma$  with  $\frac{1}{\beta} - \frac{1}{\gamma} = \frac{\alpha}{Q}$ , whenever  $1 < \beta < \frac{\alpha}{Q}$ . When  $1 + \frac{\alpha}{Q} > \beta > \frac{\alpha}{Q}$  one has that  $\mathcal{I}_\alpha$  maps continuously  $L^\beta$  into the Hölder space  $C_{\mathbb{H}}^{\beta - \frac{\alpha}{Q}}$ .

In view of such continuity we infer that  $I_3 \in L^{2\kappa}$  with  $\frac{Q}{Q-2} > \kappa = \frac{Q}{Q-2\alpha} > 1$ .

In conclusion, so far we have showed that if  $u \in W_{\mathbb{H},loc}^{2,2}(M)$  is a solution of  $L_p u(p) \in C_{\mathbb{H}}^\alpha$  then one has the integrability gain  $u \in W_{\mathbb{H},loc}^{2,2\frac{Q}{Q-2\alpha}}(M)$ . Iterating this process for a finite number of steps, in the manner described in [ADN59, page 721-722], one can increase the integrability exponent until it is larger than  $\alpha/Q$  and at that point the fractional integral operators maps into a Hölder space and one finally has that  $X_i X_j u$  are continuous. As described above, to complete the proof one now invokes Bramanti et al., [BBLU10, Theorem 14.4].

□

**2.4. Subelliptic  $Q$ -Laplacian and  $C^\infty$  estimates for non-degeneracy.** Denote by  $Q$  the Hausdorff dimension of  $M$ . For  $u \in W_{\mathbb{H},loc}^{1,Q}(M)$ , define the  $Q$ -Laplacian  $L_Q u$  by means of the following identity

$$(2.15) \quad \int_M L_Q u \phi \, d\text{vol} = \int_M |\nabla_{\mathbb{H}} u|^{Q-2} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \phi \rangle \, d\text{vol}, \quad \text{for any } \phi \in W_{\mathbb{H},0}^{1,Q}(M).$$

If  $|\nabla_{\mathbb{H}} u|^{Q-2} X_i u \in W_{\mathbb{H},loc}^{1,2}(M)$  and  $u \in W_{\mathbb{H},loc}^{1,Q}(M)$  one can then write almost everywhere in  $M$

$$(2.16) \quad L_Q u = X_i^* (|\nabla_{\mathbb{H}} u|^{Q-2} X_i u).$$

**Definition 2.17** ( $Q$ -harmonic function). Let  $M$  be an equiregular sub-Riemannian manifold of Hausdorff dimension  $Q$ . Fixed a measure  $\text{vol}$  on  $M$ , a function  $u \in W_{\mathbb{H},loc}^{1,Q}(M)$  is called  $Q$ -harmonic if

$$\int_M |\nabla_{\mathbb{H}} u|^{Q-2} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \phi \rangle \, d\text{vol} = 0, \quad \forall \phi \in W_{\mathbb{H},0}^{1,Q}(M).$$

**Proposition 2.18.** *Let  $M$  be an equiregular sub-Riemannian manifold endowed with a smooth volume form  $\text{vol}$ . Let  $u \in W_{\text{H,loc}}^{1,Q}(M)$  be a weak solution of  $L_Q u = h$  in  $M$ , with  $h \in C_{\text{H,loc}}^\alpha(M)$  and  $|\nabla_{\text{H}u}|$  not vanishing in  $M$ . If  $u \in C_{\text{H,loc}}^{1,\alpha}(M) \cap W_{\text{H,loc}}^{2,2}(M)$ , then  $u \in C_{\text{H,loc}}^{2,\alpha}(M)$ .*

*Proof.* In coordinates, let  $\omega \in C^\infty$  such that  $d \text{vol} = \omega d\mathcal{L}$ , where  $\mathcal{L}$  is the Lebesgue measure. Since  $|\nabla_{\text{H}u}|$  is continuous and bounded from above, and since  $u \in W_{\text{H,loc}}^{2,2}(M)$ , then, a.e. in  $M$ , the  $Q$ -Laplacian can be expressed in non-divergence form

$$(2.19) \quad (L_Q u)(x) = \alpha_{ij}(x, \nabla_{\text{H}u}) X_i X_j u + g(x, \nabla_{\text{H}u}) = h(x),$$

where

$$\alpha_{ij}(x, \xi) = -|\xi|^{Q-4} (\delta_{ij} + (Q-2)\xi_i \xi_j)$$

and

$$g(x, \xi) = -\omega(x)^{-1} X_i \omega(x) |\xi|^{Q-2} \xi_i + \partial_k b_k^i(x) |\xi|^{Q-2} \xi_i.$$

Set  $a_{ij}(x) = \alpha_{ij}(x, \nabla_{\text{H}u})$ . Since  $u \in C_{\text{H,loc}}^{1,\alpha}(M)$ , we have

$$a_{ij}(\cdot) \text{ and } g(\cdot, \nabla_{\text{H}u}) \in C_{\text{H,loc}}^\alpha(M).$$

In view of the non-vanishing of  $\nabla_{\text{H}u}$ , one can invoke Lemma 2.13, to obtain  $u \in C_{\text{H,loc}}^{2,\alpha}(M)$ .  $\square$

### 3. DEFINITIONS OF 1-QUASICONFORMAL MAPS

In this section we introduce the notions of conformal and quasiconformal maps between sub-Riemannian manifolds.

**Definition 3.1** (Conformal map). A smooth diffeomorphism between two sub-Riemannian manifolds is *conformal* if its differential maps horizontal vectors into horizontal vectors, and its restrictions to the horizontal spaces are similarities<sup>2</sup>.

The notion of quasiconformality can be formulated with minimal regularity assumptions in arbitrary metric spaces.

**Definition 3.2** (Quasiconformal map). A *quasiconformal* map between two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  is a homeomorphism  $f : X \rightarrow Y$  for which there exists a constant  $K \geq 1$  such that for all  $p \in X$

$$H_f(p) := \limsup_{r \rightarrow 0} \frac{\sup\{d_Y(f(p), f(q)) : d_X(p, q) \leq r\}}{\inf\{d_Y(f(p), f(q)) : d_X(p, q) \geq r\}} \leq K.$$

We want to address the case  $K = 1$ , and clarify which one is the correct definition of 1-quasiconformality, since in the literature there are several equivalent definitions of quasiconformality associated to possibly different bounds for different types of distortion (metric, geometric, or analytic).

<sup>2</sup>A map  $F : X \rightarrow Y$  between metric spaces is called a *similarity* if there exists a constant  $\lambda > 0$  such that  $d(F(x), F(x')) = \lambda d(x, x')$ , for all  $x, x' \in X$ .

In order to state our results we need to recall a few basic notions and introduce some notation. We consider the following metric quantities

$$L_f(p) := \limsup_{q \rightarrow p} \frac{d(f(p), f(q))}{d(p, q)} \quad \text{and} \quad \ell_f(p) := \liminf_{q \rightarrow p} \frac{d(f(p), f(q))}{d(p, q)}.$$

The quantity  $L_f(p)$  is sometimes denoted by  $\text{Lip}_f(p)$  and is called the pointwise Lipschitz constant. Given an equiregular sub-Riemannian manifold  $M$ , we denote by  $Q$  its Hausdorff dimension with respect to the Carnot-Carathéodory distance, and we write  $\nabla_{\text{H}}$  for the horizontal gradient, see Section 2.1 for these definitions. We denote by  $\text{vol}_M$  the Popp measure on  $M$  and denote by  $J_f^{\text{Popp}}$  the Jacobian of a map  $f$  between equiregular sub-Riemannian manifolds when these manifolds are equipped with their Popp measures (see Section 3.4). By  $W_{\text{H}}^{1,Q}(M)$  we indicate the space of functions  $u \in L^Q(\text{vol}_M)$  such that  $|\nabla_{\text{H}} u| \in L^Q(\text{vol}_M)$ . We use the standard notation  $\text{Cap}_Q$  and  $\text{Mod}_Q$  for capacity and modulus (see Section 3.7). We also consider the nonlinear pairing

$$I_Q(u, \phi; U) := \int_U |\nabla_{\text{H}} u|^{Q-2} \langle \nabla_{\text{H}} u, \nabla_{\text{H}} \phi \rangle d \text{vol}_M,$$

with  $u, \phi \in W_{\text{H}}^{1,Q}(U)$  and  $U \subset M$  an open subset. For short, we write  $I_Q(u, \phi)$  for  $I_Q(u, \phi; M)$  and denote by  $E_Q(u) = I_Q(u, u; M)$  the  $Q$ -energy of  $u$ . The functional  $I_Q(u, \cdot)$  defines the weak form of the  $Q$ -Laplacian  $L_Q$  when acting on the appropriate function space, see Section 2.4. Given a quasiconformal homeomorphism  $f$  between two equiregular sub-Riemannian manifolds, we denote by  $\mathcal{N}_p(f)$  the Margulis-Mostow differential of  $f$  and by  $(d_{\text{H}} f)_p$  its horizontal differential (see Section 3.2).

Our results rest on the following equivalence theorem, which we prove later in the section.

**Theorem 3.3.** *Let  $f$  be a quasiconformal map between two equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . The following are equivalent:*

$$(3.4) \quad H_f(p) = 1 \text{ for a.e. } p;$$

$$(3.5) \quad H_f^{\equiv}(p) := \limsup_{r \rightarrow 0} \frac{\sup\{d(f(p), f(q)) : d(p, q) = r\}}{\inf\{d(f(p), f(q)) : d(p, q) = r\}} = 1 \text{ for a.e. } p;$$

$$(3.6) \quad (d_{\text{H}} f)_p \text{ is a similarity for a.e. } p;$$

$$(3.7) \quad \mathcal{N}_p(f) \text{ is a similarity for a.e. } p;$$

$$(3.8) \quad \ell_f(p) = L_f(p) \text{ for a.e. } p, \text{ i.e., the limit } \lim_{q \rightarrow p} \frac{d(f(p), f(q))}{d(p, q)} \text{ exists for a.e. } p;$$

$$(3.9) \quad \ell_{\mathcal{N}_p(f)}(e) = L_{\mathcal{N}_p(f)}(e) \text{ for a.e. } p;$$

$$(3.10) \quad J_f^{\text{Popp}}(p) = L_f(p)^Q \text{ for a.e. } p;$$

$$(3.11) \quad \text{The } Q\text{-modulus (w.r.t. Popp measure) is preserved:}$$

$$\text{Mod}_Q(\Gamma) = \text{Mod}_Q(f(\Gamma)), \quad \forall \Gamma \text{ family of curves in } M;$$

$$(3.12) \quad \text{The operators } I_Q \text{ (w.r.t. Popp measure) are preserved:}$$

$$I_Q(v, \phi; V) = I_Q(v \circ f, \phi \circ f; f^{-1}(V)), \quad \forall V \subset N \text{ open}, \forall v, \phi \in W_H^{1,Q}(V).$$

**Definition 3.13** (1-quasiconformal map). We say that a quasiconformal map between two equiregular sub-Riemannian manifolds is *1-quasiconformal* if any of the conditions in Theorem 3.3 holds.

The equivalence of the definitions in Theorem 3.3 have as consequences some invariance properties that are crucial in the proofs of this paper.

**Corollary 3.14.** *Let  $f$  be a 1-quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . Then*

(i) *the  $Q$ -energy (w.r.t. Popp measure) is preserved:*

$$(3.15) \quad E_Q(v) = E_Q(v \circ f), \quad \forall v \in W_H^{1,Q}(N);$$

(ii) *the  $Q$ -capacity (w.r.t. Popp measure) is preserved:*

$$(3.16) \quad \text{Cap}_Q(E, F) = \text{Cap}_Q(f(E), f(F)), \quad \forall E, F \subset M \text{ compact}.$$

The proofs of Theorem 3.3 and Corollary 3.14 will be given later in this section.

While the Hausdorff measure may seem to be the natural volume measure to use in this context, there is a subtle and important reason for choosing the Popp measure rather than the Hausdorff measure. Indeed, the latter may not be smooth, even in equiregular sub-Riemannian manifolds, see [ABB12]. However, we show that for 1-quasiconformal maps the corresponding Jacobians coincide. As a consequence of Theorem 3.3 and Proposition 3.54, we show that if  $f$  is a 1-quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . Then for almost every  $p$

$$\ell_f(p)^Q = L_f(p)^Q = J_f^{\text{Popp}}(p) = J_f^{\text{Haus}}(p).$$

Moreover, the inverse map  $f^{-1}$  is 1-quasiconformal.

Since the Popp measure is smooth, the associated  $Q$ -Laplacian operator  $L_Q$  will involve smooth coefficients and consequently it is plausible to conjecture the existence of a regularity theory of  $Q$ -harmonic functions (see Section 2.4 for the definitions). In fact such a theory exists in the important subclass of contact manifolds (see Section 6.2). The following result is the *morphism property* for 1-quasiconformal maps, and it is proved in Section 3.8. The  $Q$ -Laplacian operator  $L_Q$  is defined in (2.15).

**Corollary 3.17** (Morphism property). *Let  $f : M \rightarrow N$  be a 1-quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$  equipped with their Popp measures. The following hold:*

(i) *The  $Q$ -Laplacian is preserved:*

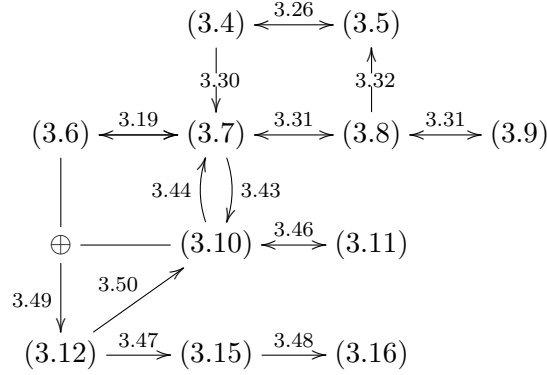
*If  $v \in W_H^{1,Q}(N)$ , then  $L_Q(v \circ f) \circ f^* = L_Q v$ , where  $f^*$  denotes the pull-back operator on functions.*

(ii) *The  $Q$ -harmonicity is preserved:*

*If  $v$  is a  $Q$ -harmonic function on  $N$ , then  $v \circ f$  is a  $Q$ -harmonic function on  $M$ .*

Note that in the Euclidean case the converse is also true: Every map that satisfies the morphism property is 1–quasiconformal. This is a result of Manfredi and Vespi [MV94].

In the rest of the section we prove Theorem 3.3 and the corollaries thereafter. In particular, we show the equivalence of the definitions (3.4) - (3.12) of 1-quasiconformal maps, and show how (3.15) and (3.16) are consequences. To help the reader, we provide the following road map. The nodes of the graph indicate the definitions in Theorem 3.3, the tags on the arrows are the labels of Propositions, Corollaries and Remarks in the present section.



**3.1. Ultratangents of 1-quasiconformal maps.** We refer the reader who is not familiar with the notions of nonprincipal ultrafilters and ultralimits to Chapter 9 of Kapovich’s book [Kap09]. Roughly speaking, taking ultralimits with respect to a nonprincipal ultrafilter is a consistent way of using the axiom of choice to select an accumulation point of any bounded sequence of real numbers. Let  $\omega$  be a nonprincipal ultrafilter. Given a sequence  $X_j$  of metric spaces with base points  $\star_j \in X_j$ , we shall consider the *based ultralimit metric space*

$$(X_\omega, \star_\omega) := (X_j, \star_j)_\omega := \lim_{j \rightarrow \omega} (X_j, \star_j).$$

We recall briefly the construction. Let

$$X_b^\mathbb{N} := \{(p_j)_{j \in \mathbb{N}} : p_j \in X_j, \sup\{d(p_j, \star_j) : j \in \mathbb{N}\} < \infty\}.$$

For all  $(p_j)_j, (q_j)_j \in X_b^\mathbb{N}$ , set

$$d_\omega((p_j)_j, (q_j)_j) := \lim_{j \rightarrow \omega} d_j(p_j, q_j),$$

where  $\lim_{j \rightarrow \omega}$  denotes the  $\omega$ -limit of a sequence indexed by  $j$ . Then  $X_\omega$  is the metric space obtained by taking the quotient of  $(X_b^\mathbb{N}, d_\omega)$  by the semidistance  $d_\omega$ . We denote by  $[p_j]$  the equivalence class of  $(p_j)_j$ . The base point  $\star_\omega$  in  $X_\omega$  is  $[\star_j]$ .

Suppose  $f_j : X_j \rightarrow Y_j$  are maps between metric spaces,  $\star_j \in X_j$  are base points, and we have the property that  $(f_j(p_j))_j \in Y_b^\mathbb{N}$ , for all  $(p_j)_j \in X_b^\mathbb{N}$ . Then the ultrafilter  $\omega$  assigns a limit map  $f_\omega := \lim_{j \rightarrow \omega} f_j : (X_j, \star_j)_\omega \rightarrow (Y_j, f_j(\star_j))_\omega$  as  $f_\omega([p_j]) := [f_j(p_j)]$ .

Let  $X$  be a metric space with distance  $d_X$ . We fix a nonprincipal ultrafilter  $\omega$ , a base point  $\star \in X$ , and a sequence of positive numbers  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ . We define the *ultratangent at  $\star$  of  $X$*  as

$$T_\omega(X, \star) := \lim_{j \rightarrow \omega} (X, \lambda_j d_X, \star).$$

Moreover, given  $f : (X, d_X) \rightarrow (Y, d_Y)$ , we call the *ultratangent map* of  $f$  at  $\star$  the limit, whenever it exists, of the maps  $f : (X, \lambda_j d_X, \star) \rightarrow (Y, \lambda_j d_Y, f(\star))$ , denoted  $T_\omega(f, \star)$ .

**Lemma 3.18.** *Let  $X$  and  $Y$  be geodesic metric spaces and let  $f : X \rightarrow Y$  be a quasi-conformal map satisfying  $H_f(\star) = 1$  at some point  $\star \in X$ . Fix a nonprincipal ultrafilter  $\omega$  and dilations factors  $\lambda_j \rightarrow \infty$ . If the ultratangent map  $f_\omega = T_\omega(f, \star)$  exists, then for  $p, q \in T_\omega(X, \star)$*

$$d(\star_\omega, p) = d(\star_\omega, q) \implies d(f_\omega(\star_\omega), f_\omega(p)) = d(f_\omega(\star_\omega), f_\omega(q)).$$

*Proof.* Take  $p = [p_j], q = [q_j] \in T_\omega(X, \star)$  with  $d(\star_\omega, p) = d(\star_\omega, q) =: R$ . Namely,

$$\lim_{j \rightarrow \omega} \lambda_j d(\star, p_j) = \lim_{j \rightarrow \omega} \lambda_j d(\star, q_j) = R.$$

Set  $r_j := \min\{d(\star, p_j), d(\star, q_j)\}$ . Fix  $j$  and suppose  $r_j = d(\star, p_j)$  so  $r_j \leq d(\star, q_j)$ . Since  $Y$  is geodesic, there exists  $q'_j \in X$  along a geodesic between  $\star$  and  $q_j$  with

$$d(\star, q'_j) = r_j \quad \text{and} \quad d(q_j, q'_j) = d(\star, q_j) - r_j.$$

We claim that  $[q'_j] = [q_j]$ . Indeed,

$$\begin{aligned} d_\omega([q'_j], [q_j]) &= \lim_{j \rightarrow \omega} \lambda_j d(q'_j, q_j) \\ &= \lim_{j \rightarrow \omega} \lambda_j (d(\star, q_j) - r_j) \\ &= \lim_{j \rightarrow \omega} \lambda_j d(\star, q_j) - \lambda_j d(\star, p_j) \\ &= R - R = 0. \end{aligned}$$

Reasoning similarly with  $p_j$ 's, we may conclude that  $p = [p'_j]$  and  $q = [q'_j]$  with  $d(\star, p'_j) = d(\star, q'_j) = r_j$ . Hence, by definition of  $f_\omega$  we have  $f_\omega(p) = f_\omega([p'_j]) = [f_j(p'_j)]$  and  $f_\omega(q) = f_\omega([q'_j]) = [f_j(q'_j)]$ . We then calculate

$$\begin{aligned} \frac{d_\omega(f_\omega(\star_\omega), f_\omega(p))}{d_\omega(f_\omega(\star_\omega), f_\omega(q))} &= \frac{\lim_{j \rightarrow \omega} \lambda_j d(f(\star), f(p'_j))}{\lim_{j \rightarrow \omega} \lambda_j d(f(\star), f(q'_j))} \\ &= \frac{\lim_{j \rightarrow \omega} d(f(\star), f(p'_j))}{\lim_{j \rightarrow \omega} d(f(\star), f(q'_j))} \\ &\leq \lim_{j \rightarrow \omega} \frac{\sup\{d(f(\star), f(a)) : d(\star, a) \leq r_j\}}{\inf\{d(f(\star), f(b)) : d(\star, b) \geq r_j\}} \\ &= 1. \end{aligned}$$

Arguing along the same lines one obtains  $d_\omega(f_\omega(\star_\omega), f_\omega(q)) \leq d_\omega(f_\omega(\star_\omega), f_\omega(p))$  and hence the statement of the lemma follows.  $\square$

**3.2. Tangents of quasiconformal maps in sub-Riemannian geometry.** We recall now some known results due to Mitchell [Mit85] and Margulis, Mostow [MM95], which are needed to show that every 1-quasiconformal map induces at almost every point a 1-quasiconformal isomorphism of the relative ultratangents. For the sake of our argument, we rephrase their results using the convenient language of ultrafilters.

Let  $M$  be an equiregular sub-Riemannian manifold. From [Mit85], for every  $p \in M$  the ultratangent  $T_\omega(M, p)$  is isometric to a Carnot group, denoted  $\mathcal{N}_p(M)$ , also called nilpotent approximation of  $M$  at  $p$ . Each horizontal vector of  $M$  at  $p$  has a natural identification with an horizontal vector of  $\mathcal{N}_p(M)$  at the identity. Such identification is an isometry between the horizontal space  $H_p M$  and the horizontal space of  $\mathcal{N}_p(M)$  at the identity, both equipped with the scalar products given by respective sub-Riemannian structures. Next, consider  $f : M \rightarrow N$  a quasiconformal map between equiregular sub-Riemannian manifolds  $M$  and  $N$ . By the work of Margulis and Mostow [MM95], there exists at almost every  $p \in M$  the ultratangent map  $T_\omega(f, p)$  that is a group isomorphism

$$\mathcal{N}_p(f) : \mathcal{N}_p(M) \rightarrow \mathcal{N}_{f(p)}(N)$$

that commutes with the group dilations, and it is independent on the ultrafilter  $\omega$  and the sequence  $\lambda_j$ . Part of Margulis and Mostow's result is that the map  $f$  is almost everywhere differentiable along horizontal vectors. Hence, for almost every  $p \in M$  and for all horizontal vectors  $v$  at  $p$ , we can consider the push-forwarded vector, which we denote by  $(d_H f)_p(v)$ . We call the map

$$(d_H f)_p : H_p M \rightarrow H_{f(p)} N$$

the *horizontal differential* of  $f$  at  $p$ .

*Remark 3.19.* With the above identification, we have

$$(3.20) \quad (d_H f)_p(v) = \mathcal{N}_p(f)_* v, \quad \forall v \in H_p M,$$

so  $(d_H f)_p$  is a restriction of  $\mathcal{N}_p(f)_*$ . Vice versa,  $(d_H f)_p$  completely determines  $\mathcal{N}_p(f)$ , since  $\mathcal{N}_p(f)$  is a homomorphism and  $H_p M$  generates the Lie algebra of  $\mathcal{N}_p(M)$ . In particular,  $(d_H f)_p$  is a similarity if and only if  $\mathcal{N}_p(f)$  is a similarity with same factor. Hence, Conditions (3.7) and (3.6) are equivalent.

Next we introduce some expressions that can be used to quantify the distortion.

$$\begin{aligned} \underline{\mathcal{L}}_f(p) &:= \liminf_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{r}, \\ \overline{\mathcal{L}}_f(p) &:= \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{r}, \\ \overline{\mathcal{L}}_f^-(p) &:= \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r\}}{r}, \\ \underline{\mathcal{L}}_f^-(p) &:= \liminf_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r\}}{r}, \end{aligned}$$

$$\begin{aligned} \|\mathcal{N}_p(f)\| &:= \max\{d(e, \mathcal{N}_p(f)(y)) : d_{\mathcal{N}_p(M)}(e, y) \leq 1\} \\ &= \max\{d(e, \mathcal{N}_p(f)(y)) : d_{\mathcal{N}_p(M)}(e, y) = 1\}. \end{aligned}$$

*Remark 3.21.* There exists a horizontal vector at  $p$  such that  $\|X\| = 1$  and  $\|f_* X\| = \|\mathcal{N}_p(f)\|$ , which in other words means that  $X$  is in the first layer of the Carnot group  $\mathcal{N}_p(M)$ ,  $d_{\mathcal{N}_p(M)}(e, \exp(X)) = 1$ , and  $d_{\mathcal{N}_{f(p)}(N)}(e, \mathcal{N}_p(f)(\exp(X))) = \|\mathcal{N}_p(f)\|$ .

The following holds.



**Lemma 3.22.** *Let  $M$  and  $N$  be (equiregular) sub-Riemannian manifolds and let  $f : M \rightarrow N$  be a quasiconformal map. Let  $p$  be a point of differentiability for  $f$ . We have*

$$L_f(p) = \|\mathcal{N}_p(f)\| = L_{\mathcal{N}_p(f)}(e) = \bar{\mathcal{L}}_f(p) = \underline{\mathcal{L}}_f(p) = \bar{\mathcal{L}}_f^-(p) = \underline{\mathcal{L}}_f^-(p).$$

*Proof.* Proof of  $L_f(p) \leq \|\mathcal{N}_p(f)\|$ . Let  $p_j \in M$  such that  $p_j \rightarrow p$  and

$$L_f(p) = \lim_{j \rightarrow \infty} \frac{d(f(p), f(p_j))}{d(p, p_j)}.$$

Let  $\lambda_j := 1/d(p, p_j)$ , so  $\lambda_j \rightarrow \infty$ . We fix now any nonprincipal ultrafilter  $\omega$  and consider ultratangents with respect to dilations  $\lambda_j$ . Hence,

$$\begin{aligned} L_f(p) &= \lim_{j \rightarrow \infty} \lambda_j d(f(p), f(p_j)) \\ &= d_\omega([f(p)], [f(p_j)]) \\ &= d_\omega(\mathcal{N}_p f([p]), \mathcal{N}_p f([p_j])) \\ &\leq \|\mathcal{N}_p f\| d_\omega([p], [p_j]) \\ &= \|\mathcal{N}_p f\| \lim_{j \rightarrow \omega} \lambda_j d(p, p_j) \\ &= \|\mathcal{N}_p f\|. \end{aligned}$$

Proof of  $L_f(p) \geq \|\mathcal{N}_p(f)\|$ . Take  $y \in \mathcal{N}_p(M)$  with  $d(e, y) = 1$  that realizes the maximum in  $\|\mathcal{N}_p(f)\|$ . Choose a sequence  $q_j \in M$  such that  $[q_j]$  represents the point  $y$ . Let  $\lambda_j \rightarrow \infty$  be the dilations factors for which we calculate the ultratangent. Since

$$1 = d(e, y) = \lim_{j \rightarrow \omega} \lambda_j d(p, q_j),$$

then, up to passing to a subsequence of indices,  $d(p, q_j) \rightarrow 0$ . Moreover,

$$\begin{aligned} L_f(p) &\geq \limsup_{j \rightarrow \infty} \frac{d(f(p), f(q_j))}{d(p, q_j)} \\ &= \limsup_{j \rightarrow \infty} \lambda_j d(f(p), f(q_j)) \\ &= d_\omega([f(p)], [f(q_j)]) \\ &= d_\omega(e, \mathcal{N}_p f(y)) \\ &= \|\mathcal{N}_p f\|. \end{aligned}$$

Proof of  $\bar{\mathcal{L}}_f(p) \leq \|\mathcal{N}_p(f)\|$ . There exists  $r_j \rightarrow 0$  and  $p_j \in M$  with  $d_M(p, p_j) \leq r_j$  such that

$$\bar{\mathcal{L}}_f(p) = \lim_j \frac{d_N(f(p), f(p_j))}{r_j}.$$

Then, using  $1/r_j$  as scaling for the ultratangent, we have  $d_\omega([p], [p_j]) \leq 1$  and  $\bar{\mathcal{L}}_f(p) = \lim_j \frac{1}{r_j} d_N(f(p), f(p_j)) = d_\omega([f(p)], [f(p_j)]) \leq \|\mathcal{N}_p(f)\|$ .

Proof of  $\|\mathcal{N}_p(f)\| \leq \underline{\mathcal{L}}_f(p)$ . Take  $y \in \mathcal{N}_p(M)$  with  $d_{\mathcal{N}_p(M)}(e, y) \leq 1$  that realizes the maximum in  $\|\mathcal{N}_p(f)\|$ . Choose subsequences  $s_j \rightarrow 0$  that realizes the limit in the definition

of  $\underline{\mathcal{L}}_f(p)$ , i.e., so that

$$\underline{\mathcal{L}}_f(p) = \lim_j \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq s_j\}}{s_j}.$$

We use  $1/s_j$  as scaling factors for the ultratangent space. For any  $\mu \in (0, 1)$  choose a sequence  $q_j \in M$  such that  $[q_j]$  represents the point  $\delta_\mu(y)$ . Therefore, we have that

$$\lim_j \frac{d_M(p, q_j)}{s_j} = d_{\mathcal{N}_p(M)}(e, \delta_\mu(y)) \leq \mu d_{\mathcal{N}_p(M)}(e, y) \leq \mu < 1.$$

For  $j$  big enough we then have  $d_M(p, q_j) < s_j$ . So

$$d_N(f(p), f(q_j)) \leq \sup\{d_N(f(p), f(q)) : d_M(p, q) \leq s_j\},$$

whence, dividing both sides by  $s_j$  and letting  $j \rightarrow \infty$ , we get

$$d_\omega([f(p)], [f(q_j)]) \leq \underline{\mathcal{L}}_f(p),$$

which, in view of the homogeneity of  $\mathcal{N}_p(f)$ , yields

$$\mu \|\mathcal{N}_p(f)\| = d_{\mathcal{N}_p(M)}(e, \mathcal{N}_p(f)(\delta_\mu y)) \leq \underline{\mathcal{L}}_f(p).$$

Since the last inequality holds for all  $\mu \in (0, 1)$ , the conclusion follows.

Proof of  $\underline{\mathcal{L}}_f(p) \geq \overline{\mathcal{L}}_f^{\overline{\overline{}}}(p)$ . Since

$$\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\} \geq \sup\{d_N(f(p), f(q)) : d_M(p, q) = r\},$$

one has

$$\begin{aligned} \underline{\mathcal{L}}_f(p) &= \liminf_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{r} \\ &\geq \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r\}}{r} = \overline{\mathcal{L}}_f^{\overline{\overline{}}}(p). \end{aligned}$$

Proof of  $\underline{\mathcal{L}}_f(p) \leq \overline{\mathcal{L}}_f^{\overline{\overline{}}}(p)$ . Choose a sequence  $r_j \rightarrow 0$  such that

$$\frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r_j\}}{r_j} = \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r_j\}}{r_j},$$

and so in particular

$$\begin{aligned} \underline{\mathcal{L}}_f(p) &= \liminf_j \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r_j\}}{r_j} \\ &\leq \limsup_j \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r_j\}}{r_j} \leq \overline{\mathcal{L}}_f^{\overline{\overline{}}}(p). \end{aligned}$$

Proof of  $\|\mathcal{N}_p(f)\| \leq \overline{\mathcal{L}}_f^{\overline{\overline{}}}(p)$ . Take  $y \in \mathcal{N}_p(M)$  with  $d(e, y) = 1$  that realizes the maximum in  $\|\mathcal{N}_p(f)\|$ . Choose subsequences  $s_j \rightarrow 0$  that realizes the limit in the definition of  $\overline{\mathcal{L}}_f^{\overline{\overline{}}}(p)$ , i.e., so that

$$\overline{\mathcal{L}}_f^{\overline{\overline{}}}(p) = \lim_j \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = s_j\}}{s_j}.$$

We use  $1/s_j$  as scaling factors for the ultratangent space. For any  $\epsilon > 0$  choose a sequence  $q'_j \in M$  such that  $[q'_j]$  represents the point  $\delta_{1+\epsilon}(y)$ . Therefore, we have that

$$1 + \epsilon = d(e, \delta_{1+\epsilon}(y)) = \lim_{j \rightarrow \omega} \frac{d(p, q'_j)}{s_j}.$$

For  $j$  big enough we then have  $d(p, q'_j) \in (s_j, (1 + 2\epsilon)s_j)$ . Since  $M$  is a geodesic space, we consider a point  $q''_j \in M$  such that  $d(p, q''_j) = s_j$  and lies in the geodesic between  $p$  and  $q'_j$ , consequently  $d(q'_j, q''_j) \leq 2\epsilon s_j$ .

Set  $y_\epsilon \in \mathcal{N}_p(M)$  the point being represented by the sequence  $q''_j$ . We have  $d(\delta_{1+\epsilon}y, y_\epsilon) < 2\epsilon$ . From which we get that  $y_\epsilon \rightarrow y$ , as  $\epsilon \rightarrow 0$ . We then bound

$$\underline{\mathcal{L}}_f^{\bar{}}(p) \geq \lim_j \frac{d(f(p), f(q''_j))}{s_j} = d(\mathcal{N}_p(f)(y_\epsilon), e).$$

Since  $d(\mathcal{N}_p(f)(y_\epsilon), e)$  is continuous at  $\epsilon = 0$  and converges to  $\|\mathcal{N}_p(f)\|$ , as  $\epsilon \rightarrow 0$ , we obtain the desired estimate.

To conclude the proof of the proposition, one observes that  $\underline{\mathcal{L}}_f(p) \leq \bar{\mathcal{L}}_f(p)$  and  $\underline{\mathcal{L}}_f^{\bar{}}(p) \leq \bar{\mathcal{L}}_f^{\bar{}}(p)$  are trivial.  $\square$

**Corollary 3.23.** *Let  $M$  and  $N$  be (equiregular) sub-Riemannian manifolds and let  $f : M \rightarrow N$  be a quasiconformal map. Let  $p$  be a point of differentiability for  $f$ . We have*

$$(3.24) \quad L_f(p) = L_{\mathcal{N}_p(f)}(e) \quad \text{and} \quad \ell_f(p) = \ell_{\mathcal{N}_p(f)}(e).$$

*Proof.* The proof follows from Lemma 3.22 applied to  $f$  and  $f^{-1}$ , and by observing that

$$(3.25) \quad \ell_f(p) = 1/L_{f^{-1}}(f(p)), \quad \text{and} \quad \mathcal{N}_p(f)^{-1} = \mathcal{N}_{f(p)}(f^{-1}).$$

$\square$

**Corollary 3.26.** *Let  $M$  and  $N$  be (equiregular) sub-Riemannian manifolds and let  $f : M \rightarrow N$  be a quasiconformal map. Then for almost every  $p \in M$*

$$H_f(p) = H_f^{\bar{}}(p).$$

*Proof.* Note that in every geodesic metric space

$$\inf\{d_N(f(p), f(q)) : d_M(p, q) \geq r\} = \inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}.$$

Hence  $H_f(p) \geq H_f^{\bar{}}(p)$  is immediate.

Regarding the opposite inequality, let  $p$  be a point of differentiability for  $f$ . Consequently,

$$\begin{aligned}
H_f(p) &\stackrel{def}{=} \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{\inf\{d_N(f(p), f(q)) : d_M(p, q) \geq r\}} \\
&= \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \\
&\leq \limsup_{r \rightarrow 0} \frac{r}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) \leq r\}}{r} \\
&= \limsup_{r \rightarrow 0} \frac{r}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \overline{\mathcal{L}}_f(p) \\
&= \limsup_{r \rightarrow 0} \frac{r}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \underline{\mathcal{L}}_f^-(p) \\
&= \limsup_{r \rightarrow 0} \frac{r}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \liminf_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r\}}{r} \\
&\leq \limsup_{r \rightarrow 0} \frac{\sup\{d_N(f(p), f(q)) : d_M(p, q) = r\}}{\inf\{d_N(f(p), f(q)) : d_M(p, q) = r\}} \\
&\stackrel{def}{=} H_{\overline{f}}(p),
\end{aligned}$$

where in the last two steps we have used that  $\overline{\mathcal{L}}_f(p) = \underline{\mathcal{L}}_f^-(p)$  from Lemma 3.22 and the fact that  $\limsup a_j \liminf b_j \leq \limsup(a_j b_j)$ .  $\square$

**Proposition 3.27.** *Let  $f : M \rightarrow N$  be a quasiconformal map between sub-Riemannian manifolds. The function  $p \mapsto \|\mathcal{N}_p(f)\|$  is the minimal upper-gradient of  $f$ .*

*Proof.* The function  $p \mapsto \|\mathcal{N}_p(f)\|$  is an upper-gradient of  $f$  since  $L_f(\cdot)$  is such and  $L_f(p) = \|\mathcal{N}_p(f)\|$  by Lemma 3.22. Regarding the minimality, let  $g$  be a weak upper-gradient of  $f$ . We need to show that

$$(3.28) \quad g(p) \geq \|\mathcal{N}_p(f)\|, \quad \text{for almost all } p.$$

Localizing, we take a unit horizontal vector field  $X$ . For  $p \in M$ , let  $\gamma_p$  be the curve defined by the flow of  $X$ , i.e.,

$$\gamma_p(t) := \Phi_X^t(p),$$

which is defined for  $t$  small enough. We remark that the subfamilies of  $\{\gamma_p\}_{p \in M}$  that have zero  $Q$ -modulus are of the form  $\{\gamma_p\}_{p \in E}$  with  $E \subset M$  of zero  $Q$ -measure. Then, for every unit horizontal vector field  $X$ , there exists a set  $\Omega_X \subseteq M$  of full measure such that for all  $p \in \Omega_X$  we have

$$\int_{\gamma_p|_{[0, \epsilon]}} g \geq d(f(\gamma_p(0)), f(\gamma_p(\epsilon))).$$

Since  $\|X\| \equiv 1$ , then each  $\gamma_p$  is parametrized by arc length. Thus

$$\frac{1}{\epsilon} \int_0^\epsilon g(\gamma_p(t)) dt \geq \frac{1}{\epsilon} d(f(p), f(\Phi_X^\epsilon(p))).$$

Assuming that  $p$  is a Lebesgue point for  $g$ , taking the limit as  $\epsilon \rightarrow 0$ , and considering ultratangents with dilations  $1/\epsilon$ , we have

$$(3.29) \quad \begin{aligned} g(p) &\geq d_\omega(e, \mathcal{N}_p(f)[\Phi_X^\epsilon(p)]), \\ &= d_\omega(e, \mathcal{N}_p(f) \exp(\tilde{X}_p)), \quad \forall p \in \Omega_X, \end{aligned}$$

where  $\tilde{X}_p$  is the vector induced on  $\mathcal{N}_p(M)$  by  $X_p$ .

Set now  $X_1, \dots, X_r$  an orthonormal frame of  $\Delta$  and consider for all  $\theta \in \mathbb{S}^{r-1} \subset \mathbb{R}^r$ , the unit horizontal vector field  $X^\theta := \sum_{i=1}^r \theta_i X_i$ . Fix  $\{\theta_j\}_{j \in \mathbb{N}}$  a countable dense subset of  $\mathbb{S}^{r-1}$  and define  $\Omega := \bigcap_j \Omega_{X^{\theta_j}}$ , which has full measure. Take  $p \in \Omega$  and, recalling Remark 3.21, take  $Y \in \Delta_p$  such that  $\|Y\| = 1$  and

$$d_\omega(e, \mathcal{N}_p(f) \exp(\tilde{Y})) = \|\mathcal{N}_p(f)\|$$

By density, there exists a sequence  $j_k$  of integers such that  $\theta_{j_k}$  converges to some  $\theta$  with the property that  $Y = (X^\theta)_p$ . Therefore, by (3.29) we conclude (3.28).  $\square$

### 3.3. Equivalence of metric definitions.

**Proposition 3.30** (Tangents of 1-QC maps). *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds. Condition (3.4) implies Condition (3.7).*

*Proof.* For almost every  $p \in M$ , the map  $\mathcal{N}_p(f)$  exists and coincides with the ultratangent  $f_\omega$  with respect to any nonprincipal ultrafilter and any sequence of dilations. Hence, we can apply Lemma 3.18 and deduce that spheres about the origin are sent to spheres about the origin. Therefore, the distortion  $H_{\mathcal{N}_p(f)}(e)$  at the origin is 1. Being  $\mathcal{N}_p(f)$  an isomorphism, the distortion is 1 at every point, and in fact  $\mathcal{N}_p(f)$  is a similarity.  $\square$

**Corollary 3.31.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds. Conditions (3.7), (3.8), and (3.9) are equivalent.*

*Proof.* For every point  $p$  of differentiability for  $f$ , we have that  $\mathcal{N}_p(f)$  is a similarity if and only if  $L_{\mathcal{N}_p(f)}(e) = \ell_{\mathcal{N}_p(f)}(e)$ , which by Corollary 3.23 is equivalent to  $\ell_f(p) = L_f(p)$ .  $\square$

**Proposition 3.32.** *Let  $f : M \rightarrow N$  be a quasiconformal map between sub-Riemannian manifolds. At every point  $p \in M$  such that  $L_f(p) = \ell_f(p)$  one has that  $H_f^-(p) = 1$ . Hence, Condition (3.8) implies Conditions (3.5).*

*Proof.* Notice that at every point in which  $L_f(p) = \ell_f(p)$  one has the existence of the limit

$$\lim_{d(p,q) \rightarrow 0} \frac{d(f(p), f(q))}{r}.$$

Consequently, at those points one has

$$H_f^-(p) = \lim_{r \rightarrow 0} \frac{\sup\{d_Y(f(p), f(q)) : d_X(p, q) = r\}}{\inf\{d_Y(f(p), f(q)) : d_X(p, q) = r\}} = \frac{L_f(p)}{\ell_f(p)} = 1. \quad \square$$

Therefore, we proved the equivalence of the metric definitions, i.e., Conditions (3.4), (3.5), (3.7), (3.8), and (3.9).

**3.4. Jacobians and Popp measure.** Let  $(M, \mu_M)$  and  $(N, \mu_N)$  be metric measure spaces and let  $f : M \rightarrow N$  be a homeomorphism. We say that  $J_f : M \rightarrow \mathbb{R}$  is a *Jacobian* for  $f$  with respect to the measures  $\mu_M$  and  $\mu_N$ , if  $f^* \mu_N = J_f \mu_M$ , which is equivalent to the change of variable formula:

$$(3.33) \quad \int_{f(A)} h \, d\mu_N = \int_A (h \circ f) J_f \, d\mu_M,$$

for every  $A \subset M$  measurable and every continuous function  $h : N \rightarrow \mathbb{R}$ .

If  $M$  and  $N$  are equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ , we consider  $\mu_M$  and  $\mu_N$  to be both either the  $Q$ -dimensional spherical Hausdorff measures or the Popp measures. See [Mon02, BR13] for the definition of the Popp measure and Example 3.37 for the case of step-2 Carnot groups. In these cases, we denote the corresponding Jacobians as  $J_f^{\text{Haus}}$  and  $J_f^{\text{Popp}}$ , respectively. If  $f$  is a quasiconformal map, such Jacobians are uniquely determined up to sets of measure zero. In fact, by Theorem [HK98, Theorem 4.9, Theorem 7.11] and [MM95, Theorem 7.1], they can be expressed as volume derivatives. Moreover, by an elementary calculation using just the definition one checks that the Jacobian satisfies the formula

$$(3.34) \quad J_f(p) = 1/J_{f^{-1}}(f(p)).$$

*Remark 3.35.* We have that if  $f : M \rightarrow N$  is quasiconformal and at almost every point  $p$  its differential  $\mathcal{N}_p(f)$  is a similarity, then for almost every  $p \in M$  the Carnot groups  $\mathcal{N}_p(M)$  and  $\mathcal{N}_{f(p)}(N)$  are isometric. Indeed, if  $\lambda_p$  is the dilation factor of  $\mathcal{N}_p(f)$ , then the composition of  $\mathcal{N}_p(f)$  and the group dilation by  $\lambda_p^{-1}$  gives an isometry. As a consequence,  $\mathcal{N}_p(M)$  and  $\mathcal{N}_{f(p)}(N)$  are isomorphic as metric measure spaces when equipped with their Popp measures  $\text{vol}_{\mathcal{N}_p(M)}$  and  $\text{vol}_{\mathcal{N}_{f(p)}(N)}$ , respectively. In particular, for almost every  $p \in M$ , we have

$$(3.36) \quad \text{vol}_{\mathcal{N}_p(M)}(B_{\mathcal{N}_p(M)}(e, 1)) = \text{vol}_{\mathcal{N}_{f(p)}(N)}(B_{\mathcal{N}_{f(p)}(N)}(e, 1)).$$

**Example 3.37.** We recall in a simple case the construction of the Popp measure. Namely, we consider a Carnot group of step 2, that is, the Lie algebra is stratified as  $V_1 \oplus V_2$ . Let  $B \subseteq V_1 \subseteq T_e G$  be the (horizontal) unit ball with respect to a sub-Riemannian metric tensor  $g_1$  at the identity, which is the intersection of the metric unit ball at the identity with  $V_1$ , in exponential coordinates. The set  $[B, B] := \{[X, Y] : X, Y \in B\}$  is the unit ball of a unique scalar product  $g_2$  on  $V_2$ . The formula  $g := \sqrt{g_1^2 + g_2^2}$  defines the unique scalar product on  $V_1 \oplus V_2$  that make  $V_1$  and  $V_2$  orthogonal and extend  $g_1$  and  $g_2$ . Extending the scalar product on  $T_e G$  by left translation, one obtains a Riemannian metric tensor  $\tilde{g}$  on the Lie group  $G$ . For such a Carnot group the Popp measure is by definition the Riemannian volume measure of  $\tilde{g}$ .

*Remark 3.38.* In Carnot groups the Popp measure is strictly monotone as a function of the distance, in the sense that if  $d$  and  $d'$  are two distances on the same Carnot group such that  $d' \leq d$  and  $d' \neq d$ , then  $\text{Popp}_{d'} \leq \text{Popp}_d$  and  $\text{Popp}_{d'} \neq \text{Popp}_d$ . Indeed, this claim follows easily from the construction of the measure. For simplicity of notation, we illustrate the proof for Popp measures in Carnot groups of step 2 as we recalled in Example 3.37. If  $B'$  is a set that strictly contains  $B$  then clearly  $[B, B] \subseteq [B', B']$  and hence the unit ball for  $g$  is strictly contained in the unit ball for  $g'$ . In other words, the vector space  $T_e G$  is equipped

with two different (Euclidean) distances, say  $\rho$  and  $\rho'$ , and by assumption, the identity  $\text{id} : (T_e G, \rho) \rightarrow (T_e G, \rho')$  is 1-Lipschitz. Therefore, the Hausdorff measure with respect to  $\rho$  is greater than the one with respect to  $\rho'$ . At this point we recall that the Hausdorff measure of a Euclidean space equals the Lebesgue measure with respect to orthonormal coordinates. In other words, the Hausdorff measure is equal to the measure induced by the top-dimensional form that takes value 1 on any orthonormal basis, which is by definition the Riemannian volume form. We therefore deduce that the Riemannian volume measure of  $\tilde{g}$  is less than the Riemannian volume measure of  $\tilde{g}'$ . Hence,  $\text{Popp}_{d'} \leq \text{Popp}_d$ . Moreover, the equality holds only if  $\tilde{g} = \tilde{g}'$ , which holds if and only if  $B' = B$ .

**Lemma 3.39.** *Let  $A : G \rightarrow G'$  be an isomorphism of Carnot groups of Hausdorff dimension  $Q$ . If either  $J_A(e) = (L_A(e))^Q$  or  $J_A(e) = (\ell_A(e))^Q$ , then  $A$  is a similarity.*

*Proof.* Up to composing  $A$  with a dilation, we assume that  $L_A(e) = 1$ , i.e.,  $A$  is 1-Lipschitz. Then if  $J_A(e) = (L_A(e))^Q$  we have that  $J_A = 1$ , which means that the push forward via  $A$  of the Popp measure on  $G$  is the Popp measure on  $G'$ . Moreover, identifying the group structures via  $A$ , we assume that we are in the same group  $G$  (algebraically) that is equipped with two different Carnot distances  $d$  and  $d'$  such that  $d' \leq d$ , since the identity  $A = \text{id} : (G, d) \rightarrow (G, d')$  is 1-Lipschitz. If  $d' \neq d$ , then by Remark 3.38  $\text{Popp}_{d'} \neq \text{Popp}_d$ , which contradicts the assumption. We conclude that  $d' = d$ , i.e.,  $A = \text{id}$  is an isometry. The case when  $J_A(e) = (\ell_A(e))^Q$  is similar.  $\square$

**3.5. A remark on tangent volumes.** We prove that the Jacobian of a quasiconformal map coincides with the Jacobian of its tangent map almost everywhere. We begin by recalling the Margulis and Mostow's convergence [MM95]. Fix a point  $p$  in a sub-Riemannian manifold  $M$  and consider privileged coordinates centered at  $p$ , see [MM95, page 418]. Let  $g$  be the sub-Riemannian metric tensor of  $M$ . Let  $\delta_\epsilon$  be the dilations associated to the privileged coordinates. Notice that  $(\delta_\epsilon)_*g$  is isometric via  $\delta_\epsilon$  to  $g$  and  $g_\epsilon := \frac{1}{\epsilon}(\delta_\epsilon)_*g$  is isometric via  $\delta_\epsilon$  to  $\frac{1}{\epsilon}g$ . A key fact is that  $g_\epsilon$  converge to  $g_0$ , as  $\epsilon \rightarrow 0$ , which is a sub-Riemannian metric. (This convergence is the convergence of some orthonormal frames uniformly on compact sets).

Mitchell's theorem [Mit85] can be restated as the fact that  $(\mathbb{R}^n, g_0)$  is the tangent Carnot group  $\mathcal{N}_p(M)$ . Margulis and Mostow actually proved that the maps  $\delta_\epsilon^{-1} \circ f \circ \delta_\epsilon$  converge uniformly, as  $\epsilon \rightarrow 0$ , on compact sets to the map  $\mathcal{N}_p(f)$ . Moreover, by functoriality of the construction of the Popp measure, we have that  $\text{vol}^{g_\epsilon} \rightarrow \text{vol}^{g_0}$ , in the sense that if  $\omega_\epsilon$  is the smooth function such that  $\text{vol}^{g_\epsilon} = \omega_\epsilon \mathcal{L}$ , then  $\omega_\epsilon \rightarrow \omega_0$  uniformly on compact sets.

**Proposition 3.40.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . For almost every  $p \in M$*

$$J_{\mathcal{N}_p(f)}(e) = J_f(p).$$

*Proof.* Denote by  $B_r^{g_\epsilon}$  the ball at 0 of radius  $r$  with respect to the metric  $g_\epsilon$ . We have

$$\begin{aligned}
(3.41) \quad \epsilon^{-Q} \operatorname{vol}^g(f(B_\epsilon^g)) &= \operatorname{vol}^{\frac{1}{\epsilon}g}(f(B_1^{\frac{1}{\epsilon}g})) \\
&= \operatorname{vol}^{\frac{1}{\epsilon}g}(\delta_\epsilon \circ \delta_\epsilon^{-1} \circ f \circ \delta_\epsilon(B_1^{g_\epsilon})) \\
&= \operatorname{vol}^{g_\epsilon}(\delta_\epsilon^{-1} \circ f \circ \delta_\epsilon(B_1^{g_\epsilon})) \\
&\rightarrow \operatorname{vol}^{g^\infty}(\mathcal{N}_p(f)(B_1^{g^0})).
\end{aligned}$$

By [GJ14, Lemma 1 (iii)], for all  $q \in M$  we have the expansion

$$(3.42) \quad \operatorname{vol}_M(B(q, \epsilon)) = \epsilon^Q \operatorname{vol}_{\mathcal{N}_q(M)}(B_{\mathcal{N}_q(M)}(e, 1)) + o(\epsilon^Q).$$

Using (3.41) and the latter, we conclude

$$\begin{aligned}
J_{\mathcal{N}_p(f)}(e) &= \frac{\operatorname{vol}_{\mathcal{N}_q(M)}(N_q(f)(B_{\mathcal{N}_q(M)}(e, 1)))}{\operatorname{vol}_{\mathcal{N}_q(M)}(B_{\mathcal{N}_q(M)}(e, 1))} \\
&= \lim_{\epsilon \rightarrow 0} \frac{\operatorname{vol}_N(f(B(p, \epsilon)))}{\epsilon^Q \operatorname{vol}_{\mathcal{N}_p(M)}(B_{\mathcal{N}_p(M)}(e, 1))} \\
&= \lim_{\epsilon \rightarrow 0} \frac{\operatorname{vol}_N(f(B(p, \epsilon)))}{\operatorname{vol}_M(B(p, \epsilon))} \\
&= J_f(p).
\end{aligned}$$

□

### 3.6. Equivalence of the analytic definition.

**Lemma 3.43.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . If the differential  $\mathcal{N}_p(f)$  of  $f$  is a similarity for almost every  $p \in M$ , then*

$$\ell_f(p)^Q = J_f^{\text{Popp}}(p) = L_f(p)^Q, \quad \text{for almost every } p \in M.$$

*Proof.* Let  $p$  be a point where  $J_f^{\text{Popp}}(p)$  is expressed as volume derivative. By definition, for all  $\epsilon > 0$ , there exists  $\bar{r} > 0$  such that, if  $q \in M$  is such that  $d(q, p) \in (0, \bar{r})$ , then

$$\frac{d(f(q), f(p))}{d(p, q)} \leq L_f(p) + \epsilon.$$

Hence for every  $r \in (0, \bar{r})$ ,

$$f(B(p, r)) \subset B(f(p), r(L_f(p) + \epsilon)).$$

So one has

$$\frac{\operatorname{vol}_N(f(B(p, r)))}{\operatorname{vol}_M(B(p, r))} \leq \frac{\operatorname{vol}_N(B(f(p), r(L_f(p) + \epsilon)))}{\operatorname{vol}_M(B(p, r))}.$$

Letting  $r \rightarrow 0$ , using (3.42) with  $q = p$  and  $q = f(p)$ , and using (3.36), we have

$$J_f^{\text{Popp}}(p) \leq (L_f(p) + \epsilon)^Q.$$

Notice that equation (3.36) requires the assumption of the differential being a similarity. Since  $\epsilon$  is arbitrary,  $J_f^{\text{Popp}}(p) \leq L_f(p)^Q$ . Once we recall that  $J_f^{\text{Popp}}(p) \cdot J_{f^{-1}}^{\text{Popp}}(f(p)) = 1$  and



$\ell_f(p) \cdot L_{f^{-1}}(f(p)) = 1$ , the same argument applied to  $f^{-1}$  yields  $\ell_f(p)^Q \leq J_f^{\text{OPP}}(p)$ . With Corollary 3.31 we conclude.  $\square$

For an arbitrary quasiconformal map we expect the relation

$$\ell_f(p)^Q \leq J_f^{\text{OPP}}(p) \leq L_f(p)^Q$$

to hold. However, our proof of Lemma 3.43 makes a crucial use of equation (3.36), which is not true in general.

**Lemma 3.44.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . If for almost every  $p \in M$  either*

$$\ell_f(p)^Q = J_f^{\text{OPP}}(p)$$

or

$$J_f^{\text{OPP}}(p) = L_f(p)^Q,$$

then  $\mathcal{N}_p(f)$  is a similarity, for almost every  $p$ .

*Proof.* In view of Proposition 3.40 and Corollary 3.23, we have either  $\ell_{\mathcal{N}_p(f)}(e)^Q = J_{\mathcal{N}_p(f)}(e)$  or  $J_{\mathcal{N}_p(f)}(e) = L_{\mathcal{N}_p(f)}(e)$ . Therefore, by Lemma 3.39 we get that  $\mathcal{N}_p(f)$  is a similarity.  $\square$

**3.7. Equivalence of geometric definitions.** We recall the definition of the modulus of a family  $\Gamma$  of curves in a metric measure space  $(M, \text{vol})$ . A Borel function  $\rho : M \rightarrow [0, \infty]$  is said to be *admissible* for  $\Gamma$  if for every rectifiable  $\gamma \in \Gamma$ ,

$$(3.45) \quad \int_{\gamma} \rho \, ds \geq 1.$$

The  $Q$ -modulus of  $\Gamma$  is

$$\text{Mod}_Q(\Gamma) = \inf \left\{ \int_M \rho^Q \, d\text{vol} : \rho \text{ is admissible for } \Gamma \right\}.$$

**Proposition 3.46.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . Then  $\text{Mod}_Q(\Gamma) = \text{Mod}_Q(f(\Gamma))$  for every family  $\Gamma$  of curves in  $M$  if and only if  $L_f^Q(p) = J_f(p)$  for a.e.  $p$ .*

*Proof.* This equivalence is actually a very general fact after the work of Cheeger [Che99] and Williams [Wil12]. Since locally sub-Riemannian manifolds are doubling metric spaces that satisfy a Poincaré inequality, we have that the pointwise Lipschitz constant  $L_f(\cdot)$  is the minimal upper gradient of the map  $f$ , see Proposition 3.27 and Lemma 3.22. We also remark that any quasiconformal map is in  $W_{\text{loc}}^{1,Q}$  and hence in the Newtonian space  $N_{\text{loc}}^{1,Q}$ , see [BKR07]. By a result of Williams [Wil12, Theorem 1.1],  $L_f(p)^Q \leq J_f(p)$ , for almost every  $p$ , if and only if  $\text{Mod}_Q(\Gamma) \leq \text{Mod}_Q(f(\Gamma))$ , for every family  $\Gamma$  of curves in  $M$ . Hence, we get the inequality  $L_f(p)^Q \leq J_f(p)$ .

Now consider the inverse map  $f^{-1}$ . Such a map satisfies the same assumptions of  $f$ . In particular, applying to  $f^{-1}$  Williams's result, we have that  $\text{Mod}_Q(\Gamma) \leq \text{Mod}_Q(f^{-1}(\Gamma))$  for every family  $\Gamma$  of curves in  $N$  if and only if  $L_{f^{-1}}(q)^Q \leq J_{f^{-1}}(q)$ , for almost every  $q \in N$ . Writing  $f^{-1}(\Gamma) = \Gamma'$  and  $q = f(p)$  and using (3.25) and (3.34), we conclude that

$\text{Mod}_Q(f(\Gamma')) \leq \text{Mod}_Q(\Gamma')$  for every family  $\Gamma'$  of curves in  $M$  if and only if  $J_f(p) \leq \ell_f^Q(p) \leq L_f^Q(p)$ , for almost every  $p \in M$

□

Let  $M$  be an equiregular sub-Riemannian manifold of Hausdorff dimension  $Q$ . Let  $\text{vol}_M$  be the Popp measure of  $M$ . For all  $u \in W_{\mathbb{H}}^{1,Q}(M, \text{vol}_M)$ , the  $Q$ -energy of  $u$  is

$$E_Q(u) := \int_M |\nabla_{\mathbb{H}} u|^Q \, d \text{vol}_M.$$

*Remark 3.47.* Since  $E_Q(u) = I_Q(u, u)$ , if the operator  $I_Q$  is preserved, then the  $Q$ -energy is preserved. Namely, Condition (3.12) implies Condition (3.15).

**Proposition 3.48.** *For a quasiconformal map  $f : M \rightarrow N$  between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ , Condition (3.15) implies Condition (3.16).*

*Proof.* Let  $E, F \subset M$  compact sets in  $M$ . We set  $\mathcal{S}(E, F)$  to denote the family of all  $u \in W_{\mathbb{H}}^{1,Q}(M)$  such that  $u|_E = 1$ ,  $u|_F = 0$  and  $0 \leq u \leq 1$ . Recall that the  $Q$ -capacity  $\text{Cap}_Q(E, F)$  is then defined as the infimum of the  $Q$ -energy  $E_Q(u)$  among all competitors  $u \in \mathcal{S}(E, F)$ :

$$\text{Cap}_Q(E, F) = \inf \int_M |\nabla_{\mathbb{H}} u|^Q \, d \text{vol}.$$

Since  $f$  satisfies (3.15), the map  $v \mapsto v \circ f$  is a bijection between  $\mathcal{S}(f(E), f(F))$  and  $\mathcal{S}(E, F)$  that preserves the  $Q$ -energy. Correspondingly, one has that

$$\begin{aligned} \text{Cap}_Q(f(E), f(F)) &= \inf \{E_Q(v) : v \in \mathcal{S}(f(E), f(F))\} \\ &= \inf \{E_Q(v \circ f) : v \in \mathcal{S}(f(E), f(F))\} \\ &= \inf \{E_Q(u) : u \in \mathcal{S}(E, F)\} \\ &= \text{Cap}_Q(E, F), \end{aligned}$$

completing the proof. □

**Proposition 3.49.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . Either of Condition (3.6) and Condition (3.10) implies Condition (3.12).*

*Proof.* Let  $p$  be a point of differentiability of  $f$ . Given an orthonormal basis  $\{X_j\}$  of  $H_p M$ , from (3.6) we have that vectors

$$Y_j := L_f(p)^{-1}(\text{d}_{\mathbb{H}} f)_p X_j$$

form an orthonormal basis of  $H_q N$ , with  $q = f(p)$ . Then, for every open subset  $V \subset N$  and for every  $v \in W_{\mathbb{H}}^{1,Q}(N)$ ,

$$\begin{aligned} X_j(v \circ f)_p &= \text{d}_{\mathbb{H}}(v \circ f)_p(X_j) \\ &= (\text{d}_{\mathbb{H}} v)_q(\text{d}_{\mathbb{H}} f)_p(X_j) \\ &= (\text{d}_{\mathbb{H}} v)_q(L_f(p)Y_j) = L_f(p)(Y_j v)_q. \end{aligned}$$

Therefore, for any  $v, \phi \in W_{\mathbb{H}}^{1,Q}(V)$ ,

$$\begin{aligned} \langle \nabla_{\mathbb{H}}(v \circ f), \nabla_{\mathbb{H}}(\phi \circ f) \rangle_p &= \sum_j X_j(v \circ f)_p X_j(\phi \circ f)_p \\ &= L_f^2(p) \sum_j Y_j(v)_q Y_j(\phi)_q \\ &= L_f^2(p) \langle \nabla_{\mathbb{H}} v, \nabla_{\mathbb{H}} \phi \rangle_q. \end{aligned}$$

In particular

$$|\nabla_{\mathbb{H}}(v \circ f)| = L_f(p) |(\nabla_{\mathbb{H}} v)_{f(\cdot)}|.$$

So, using Condition (3.10) and writing  $U = f^{-1}(V)$ ,

$$\begin{aligned} I_Q(v \circ f, \phi \circ f; U) &= \int_U |\nabla_{\mathbb{H}}(v \circ f)|^{Q-2} \langle \nabla_{\mathbb{H}}(v \circ f), \nabla_{\mathbb{H}}(\phi \circ f) \rangle \, d \text{vol}_M \\ &= \int_U L_f^{Q-2} |(\nabla_{\mathbb{H}} v)_{f(\cdot)}|^{Q-2} L_f^2 \langle \nabla_{\mathbb{H}} v, \nabla_{\mathbb{H}} \phi \rangle_{f(\cdot)} \, d \text{vol}_M \\ &= \int_U J_f |(\nabla_{\mathbb{H}} v)_{f(\cdot)}|^{Q-2} \langle \nabla_{\mathbb{H}} v, \nabla_{\mathbb{H}} \phi \rangle_{f(\cdot)} \, d \text{vol}_M \\ &= \int_V |\nabla_{\mathbb{H}} v|^{Q-2} \langle \nabla_{\mathbb{H}} v, \nabla_{\mathbb{H}} \phi \rangle \, d \text{vol}_N \\ &= I_Q(v, \phi; V), \end{aligned}$$

where we used (3.33). □

**Proposition 3.50.** *Let  $f : M \rightarrow N$  be a quasiconformal map between equiregular sub-Riemannian manifolds of Hausdorff dimension  $Q$ . Then Condition (3.12) implies Condition (3.10).*

*Proof.* We start with the following chain of equalities, where we use (3.12), the chain rule and the change of variable formula (3.33). For every open subset  $U \subset M$ , denote  $V = f(U) \subset N$ . For every  $v, \phi \in W_{\mathbb{H}}^{1,Q}(V)$ ,

$$\begin{aligned} \int_V |\nabla_{\mathbb{H}} v|^{Q-2} \langle \nabla_{\mathbb{H}} v, \nabla_{\mathbb{H}} \phi \rangle \, d \text{vol}_N &= \int_U |\nabla_{\mathbb{H}}(v \circ f)|^{Q-2} \langle \nabla_{\mathbb{H}}(v \circ f), \nabla_{\mathbb{H}}(\phi \circ f) \rangle \, d \text{vol}_M \\ &= \int_U |(\text{d}_{\mathbb{H}} f)_{f(\cdot)}^{\text{T}}(\nabla_{\mathbb{H}} v)_{f(\cdot)}|^{Q-2} \langle (\text{d}_{\mathbb{H}} f)_{f(\cdot)}^{\text{T}}(\nabla_{\mathbb{H}} v)_{f(\cdot)}, (\text{d}_{\mathbb{H}} f)_{f(\cdot)}^{\text{T}}(\nabla_{\mathbb{H}} \phi)_{f(\cdot)} \rangle \, d \text{vol}_M \\ &= \int_V J_{f^{-1}(\cdot)} |(\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}|^{Q-2} \langle (\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}, (\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} \phi)_{\cdot} \rangle \, d \text{vol}_N \\ &= \int_V J_{f^{-1}(\cdot)} |(\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}|^{Q-2} \langle (\text{d}_{\mathbb{H}} f)_{f^{-1}(\cdot)} (\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}, (\nabla_{\mathbb{H}} \phi)_{\cdot} \rangle \, d \text{vol}_N, \end{aligned}$$

where  $(\text{d}_{\mathbb{H}} f)_q^{\text{T}}$  denotes the adjoint of  $(\text{d}_{\mathbb{H}} f)_{f^{-1}(q)}$  with respect to the metrics on  $N$  and  $M$  at  $q$  and  $f^{-1}(q)$  respectively. We then proved that

$$(3.51) \quad \int_V \langle |(\nabla_{\mathbb{H}} v)_{\cdot}|^{Q-2} (\nabla_{\mathbb{H}} v)_{\cdot} - J_{f^{-1}(\cdot)} |(\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}|^{Q-2} (\text{d}_{\mathbb{H}} f)_{f^{-1}(\cdot)} (\text{d}_{\mathbb{H}} f)^{\text{T}}(\nabla_{\mathbb{H}} v)_{\cdot}, (\nabla_{\mathbb{H}} \phi)_{\cdot} \rangle \, d \text{vol}_N = 0$$

for every  $v, \phi \in W_{\mathbb{H}}^{1,Q}(V)$  and for every open subset  $V \subset N$ . Note that (3.51) holds true for every measurable subset  $V \subset N$ . We claim that, for almost every  $q \in N$ ,

$$(3.52) \quad |(\nabla_{\mathbb{H}v})_q|^{Q-2} |(\nabla_{\mathbb{H}v})_q - J_{f^{-1}}(q)| (d_{\mathbb{H}}f)_q^{\mathbb{T}} (\nabla_{\mathbb{H}v})_q|^{Q-2} (d_{\mathbb{H}}f)_{f^{-1}(q)} (d_{\mathbb{H}}f)_q^{\mathbb{T}} (\nabla_{\mathbb{H}v})_q = 0$$

for every  $v \in W_{\mathbb{H}}^{1,Q}(N)$ . Arguing by contradiction, assume that there is a set  $V \subset N$  of positive measure where (3.52) fails for some  $v \in W_{\mathbb{H}}^{1,Q}(N)$ . Choose any smooth frame  $X_1, \dots, X_r$  of  $HN$ , and write the left hand side of (3.52) as  $\sum_{i=1}^r \psi_i X_i$ , with  $\psi_i \in L^Q(N)$  for every  $i = 1, \dots, r$ . Then at least one of the  $\psi_i$  must be different from zero in  $V$ . Without losing generality, say  $\psi_1 \neq 0$  on  $V$ . By possibly taking  $V$  smaller, we may assume that  $\int_V \psi_1 d \text{vol}_N \neq 0$ . Let  $\phi$  be the coordinate function  $x_1$ , that is  $X_j \phi = \delta_1^j$ . Substituting in the left hand side of (3.51), we conclude

$$\int_V \langle (\psi_1, \dots, \psi_r), \nabla_{\mathbb{H}} \phi \rangle d \text{vol}_N = \int_V \psi_1 d \text{vol}_N \neq 0,$$

which contradicts (3.51). This completes the proof of (3.52).

Next, fix  $q \in N$  a point of differentiability where (3.52) holds. For every vector  $\xi \in H_q N$ , consider  $v_{\xi}$  such that  $(\nabla_{\mathbb{H}} v_{\xi})_q = \xi$ . For every  $\xi \in H_q N$  such that  $|\xi| = 1$ , the following holds

$$J_{f^{-1}}(q) |(d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi|^{Q-2} \langle (d_{\mathbb{H}}f)_{f^{-1}(q)} (d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi, \xi \rangle = 1.$$

Using (3.34), the equality above becomes

$$|(d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi|^{Q-2} \langle (d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi, (d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi \rangle = J_f(f^{-1}(q))$$

which is equivalent to

$$|(d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi|^Q = J_f(f^{-1}(q))$$

for every  $\xi$  on  $H_q N$  of norm equal to one. From (3.20) we have  $|(d_{\mathbb{H}}f)_q^{\mathbb{T}} \xi(q)|^Q = |\mathcal{N}_q(f)_*^{\mathbb{T}} \xi(q)|^Q$ . Therefore, at every point  $q \in N$  of differentiability,

$$\|\mathcal{N}_{f^{-1}(q)}(f)_*\|^Q = \max\{|\mathcal{N}_q(f)_*^{\mathbb{T}} \xi|^Q : \xi \in H_q N, |\xi| = 1\} = J_f(f^{-1}(q)).$$

By Lemma 3.22 and writing  $p = f^{-1}(q)$ , we conclude  $L_f(p)^Q = J_f(p)$  for almost every  $p \in M$ , establishing (3.10).  $\square$

### 3.8. The morphism property.

*Proof of Corollary 3.17.* Let  $v \in W_{\mathbb{H}}^{1,Q}(N)$  and  $\phi \in W_{\mathbb{H},0}^{1,Q}(N) \subset W_{\mathbb{H}}^{1,Q}(N)$ , then from (3.12) it follows

$$L_Q(v)(\phi) = I_Q(v, \phi) = I_Q(v \circ f, \phi \circ f) = L_Q(v \circ f) \circ f^*(\phi). \quad \square$$

**3.9. Equivalence of the two Jacobians.** Given  $M$  an equiregular sub-Riemannian manifold of Hausdorff dimension  $Q$ , we prefer to work with the Popp measure  $\text{vol}_M$  rather than the spherical Hausdorff measure  $\mathcal{S}_M^Q$  since  $\text{vol}_M$  is always smooth whereas there are cases in which  $\mathcal{S}_M^Q$  is not (see [ABB12]). However, one has the following formula (see [ABB12, pages 358-359], [GJ14, Section 3.2]).

$$(3.53) \quad d \text{vol}_M = 2^{-Q} \text{vol}_{\mathcal{N}_p(M)}(B_{\mathcal{N}_p(M)}(e, 1)) d\mathcal{S}_M^Q,$$

where we used the fact that the measure induced on  $\mathcal{N}_p(M)$  by  $\text{vol}_M$  is  $\text{vol}_{\mathcal{N}_p(M)}$ .

**Proposition 3.54.** *If  $f : M \rightarrow N$  is a 1-quasiconformal map between equiregular sub-Riemannian manifolds, then for almost every  $p \in M$ ,*

$$J_f^{\text{Popp}}(p) = J_f^{\text{Haus}}(p).$$

*Proof.* Let  $A \subseteq M$  be a measurable set. Since  $f^{-1}$  is <sup>3</sup> also 1-quasiconformal, then we have (3.36) with  $p = f^{-1}(q)$  for almost all  $q \in M$ . Then, using twice (3.53), we have

$$\begin{aligned} 2^Q(f^* \text{vol}_N)(A) &= 2^Q \text{vol}_N(f(A)) \\ &= \int_{f(A)} \text{vol}_{\mathcal{N}_q(N)}(B_{\mathcal{N}_q(N)}(e, 1)) \, d\mathcal{S}_N^Q(q) \\ &= \int_{f(A)} \text{vol}_{\mathcal{N}_{f^{-1}(q)}(M)}(B_{\mathcal{N}_{f^{-1}(q)}(M)}(e, 1)) \, d\mathcal{S}_N^Q(q) \\ &= \int_A \text{vol}_{\mathcal{N}_p(M)}(B_{\mathcal{N}_p(M)}(e, 1)) J_f^{\text{Haus}}(p) \, d\mathcal{S}_M^Q(p) \\ &= 2^Q(J_f^{\text{Haus}} \text{vol}_M)(A). \end{aligned}$$

Thus, we conclude that  $J_f^{\text{Popp}} \text{vol}_M = f^* \text{vol}_N = J_f^{\text{Haus}} \text{vol}_M$ . □

#### 4. COORDINATES IN SUB-RIEMANNIAN MANIFOLDS

Given any system of coordinates near a point of a sub-Riemannian manifolds, we will identify special subsets of these coordinates, that we call *horizontal*. By adapting a method of Liimatainen and Salo [LS14], we show that they can be constructed so that in addition they are also either harmonic or  $Q$ -harmonic (the more general construction of  $p$ -harmonic coordinates follows along the same lines, modifying appropriately the hypothesis). The construction of  $Q$ -harmonic coordinates is based upon a very strong hypothesis, namely that the sub-Riemannian structure supports regularity for  $Q$ -harmonic functions. In contrast, the construction of horizontal harmonic coordinates rests on well known Schauder estimates. The key point of this section, and one of the main contributions of this paper, is that we can prove that the smoothness of maps that preserve in a weak sense the horizontal bundles can be derived by the smoothness of the horizontal components alone.

##### 4.1. Horizontal coordinates.

**Definition 4.1.** Let  $M$  be a sub-Riemannian manifold. Let  $x^1, \dots, x^n$  be a system of coordinates on an open set  $U$  of  $M$  and let  $X_1, \dots, X_r$  be a frame of the horizontal distribution on  $U$ . We say that  $x^1, \dots, x^r$  are *horizontal coordinates* with respect to  $X_1, \dots, X_r$  if the matrix  $(X_i x^j)(p)$ , with  $i, j = 1, \dots, r$ , is invertible, for every  $p \in U$ .

*Remark 4.2.* It is clear that *any* system of coordinates  $x^1, \dots, x^n$  around a point  $p \in M$  can be reordered so that the first  $r$  components become a system of horizontal coordinates.

The next result states that the notion of horizontal coordinate does not depend on the choice of frame.

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<sup>3</sup>Here we need to invoke [MM95, Corollary 6.5] or [HK98]

**Proposition 4.3.** *Assume that  $x^1, \dots, x^n$  are coordinates such that  $x^1, \dots, x^r$  are horizontal with respect to the frame  $X_1, \dots, X_r$ . Then*

- (i)  $\nabla_{\mathbb{H}}x^1, \dots, \nabla_{\mathbb{H}}x^r$  are linearly independent and form a frame of  $\Delta$ .
- (ii) If  $X'_1, \dots, X'_r$  is another frame of  $\Delta$ , then  $x^1, \dots, x^r$  are horizontal coordinates with respect to  $X'_1, \dots, X'_r$ .

*Proof.* Since  $O := (X_i x^j)_{ij}$  is invertible and  $X_1, \dots, X_r$  is a frame, then

$$\nabla_{\mathbb{H}}x^i = \sum_{k=1}^r (X_k x^i) X_k = \sum_k O_i^k X_k$$

and (i) follows. Regarding (ii), let  $B$  be the matrix such that  $X'_i x^j = \sum_{k=1}^r B_i^k X_k x^j = (BO)_{ij}$ . The conclusion follows from the invertibility of  $BO$ .  $\square$

**4.2. Horizontal harmonic coordinates.** Let  $M$  be a sub-Riemannian manifold endowed with a volume form  $\text{vol}$ . Our goal is to construct horizontal coordinates in the neighborhood of any point  $p \in M$ , that are also in the kernel of the subLaplacian  $L_2$ , defined in (2.9), associated to the sub-Riemannian structure and a volume form.

**Theorem 4.4.** *Let  $M$  be an equiregular sub-Riemannian structure endowed with a smooth volume form  $\text{vol}$ . For any point  $p \in M$  there exists a set of horizontal harmonic coordinates defined in a neighborhood of  $p$ .*

To prove this result we start by considering any system of coordinates  $x^1, \dots, x^n$  in a neighborhood of  $p \in M$ . Without loss of generality we can assume that the vectors  $\nabla_{\mathbb{H}}x^1, \dots, \nabla_{\mathbb{H}}x^r$  are linearly independent in a neighborhood of  $p$ , i.e.,  $x^1, \dots, x^r$  are horizontal coordinates. Set  $B_\epsilon := B_\epsilon(p) = \{q \in M \mid d(p, q) < \epsilon\}$ . For  $\epsilon > 0$ , let  $u_\epsilon^1, \dots, u_\epsilon^n$  be the unique weak solution of the Dirichlet problem

$$\begin{cases} L_2 u_\epsilon^i = 0 \text{ in } B_\epsilon, & i = 1, \dots, n \\ u_\epsilon^i = x^i \text{ in } \partial B_\epsilon, & i = 1, \dots, n. \end{cases}$$

We will show that for  $\epsilon > 0$  sufficiently small, the  $n$ -tuple  $u_\epsilon^1, \dots, u_\epsilon^r, x^{r+1}, \dots, x^n$  is a system of coordinates. Note that  $u_\epsilon^1, \dots, u_\epsilon^n$  may fail to be a system of coordinates.

Hörmander's hypoellipticity result [Hör67] yields  $u_\epsilon^i \in C^\infty(B_\epsilon) \cap W_{\mathbb{H}}^{1,2}(B_\epsilon)$ . Consider now

$$w_\epsilon^i := u_\epsilon^i - x^i \in C^\infty(B_\epsilon) \cap W_{\mathbb{H},0}^{1,2}(B_\epsilon).$$

**Lemma 4.5.** *For  $p \in K \subset\subset M$ , the following estimate holds*

$$(4.6) \quad \int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d\text{vol} \leq C' \epsilon^2.$$

for a constant  $C' > 0$  depending only on  $K$ , on the coordinates  $x^1, \dots, x^n$ , the Riemannian structure of  $M$  and the volume form.

*Proof.* For every  $i = 1, \dots, n$ , the function  $w_\epsilon^i$  solves

$$\begin{cases} L_2 w_\epsilon^i = -L_2 x^i =: g_i \\ w_\epsilon^i = 0 \text{ in } \partial B_\epsilon \end{cases}$$

The equation can be interpreted in a weak sense as

$$\int_{B_\epsilon} \nabla_{\mathbb{H}} w_\epsilon^i \nabla_{\mathbb{H}} \phi \, d \text{vol} = \int_{B_\epsilon} g_i \phi \, d \text{vol}$$

for every  $\phi \in W_{\mathbb{H},0}^{1,2}(B_\epsilon)$ . Choosing  $\phi = w_\epsilon^i$  gives

$$\int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d \text{vol} = \int_{B_\epsilon} g_i w_\epsilon^i \, d \text{vol} \leq \left( \int_{B_\epsilon} g_i^2 \, d \text{vol} \right)^{1/2} \left( \int_{B_\epsilon} (w_\epsilon^i)^2 \, d \text{vol} \right)^{1/2}.$$

Poincaré inequality for functions with compact support gives

$$\int_{B_\epsilon} (w_\epsilon^i)^2 \, d \text{vol} \leq C\epsilon^2 \int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d \text{vol},$$

whence

$$\int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d \text{vol} \leq \left( \int_{B_\epsilon} g_i^2 \, d \text{vol} \right)^{1/2} \left( C\epsilon^2 \int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d \text{vol} \right)^{1/2}.$$

We have

$$\left( \int_{B_\epsilon} |\nabla_{\mathbb{H}} w_\epsilon^i|^2 \, d \text{vol} \right)^{1/2} \leq \epsilon C^{1/2} \left( \int_{B_\epsilon} g_i^2 \, d \text{vol} \right)^{1/2} \leq \epsilon C^{1/2} \text{vol}(B_\epsilon)^{1/2} \left( \sup_{B_\epsilon} g_i^2 \right)^{1/2}.$$

This completes the proof of (4.6).  $\square$

Next we need an interpolation inequality that allows us to bridge the  $L^2$  estimates (4.6) and the  $C_{\mathbb{H}}^{1,\alpha}$  estimates from (2.10) to produce  $L^\infty$  bounds. The following is very similar to the analogue interpolation lemma in [LS14].

**Lemma 4.7.** *Let  $p \in K \subset\subset M$  and let  $h$  be a function defined on  $B_\epsilon$ . If there are constants  $\mathcal{A}, \mathcal{B} > 0$  such that for  $\epsilon > 0$  sufficiently small one has*

$$(i) \quad \|h\|_{L^2(B_\epsilon)}^2 \leq \mathcal{A}\epsilon^2 |B_\epsilon|^2,$$

$$(ii) \quad \|h\|_{C_{\mathbb{H}}^\alpha(B_{\epsilon/2})} \leq \mathcal{B},$$

then  $\|h\|_{L^\infty(B_{\epsilon/4})} \leq o(1)$  as  $\epsilon \rightarrow 0$ , uniformly in  $p \in K$ .

*Proof.* Set  $q \in B_{\frac{\epsilon}{4}}(p)$  so that  $B_{\frac{\epsilon}{4}}(q) \subset B_{\frac{3\epsilon}{4}}(p)$ . One has

$$\begin{aligned} \|h\|_{L^2(B_{\frac{\epsilon}{4}}(q))} &\geq \|h(q)\|_{L^2(B_{\frac{\epsilon}{4}}(q))} - \|h - h(q)\|_{L^2(B_{\frac{\epsilon}{4}}(q))} \\ &= |h(q)| \cdot |B_{\frac{\epsilon}{4}}(q)|^{\frac{1}{2}} - \left( \int_{B_{\frac{\epsilon}{4}}(q)} |h(\cdot) - h(q)|^2 \, d \text{vol} \right)^{\frac{1}{2}} \\ &\geq |h(q)| \cdot |B_{\frac{\epsilon}{4}}(q)|^{\frac{1}{2}} - \sup_{B_{\frac{\epsilon}{4}}(q)} \frac{|h(\cdot) - h(q)|}{d(\cdot, q)^\alpha} \left( \int_{B_{\frac{\epsilon}{4}}(q)} d(\cdot, q)^{2\alpha} \, d \text{vol} \right)^{\frac{1}{2}}. \end{aligned}$$

We then obtain that there exists constants  $C_1$  and  $C_2$ , depending only on the sub-Riemannian structure, the exponent  $\alpha$ , and the compact set  $K$ , such that

$$\|h\|_{L^2(B_{\frac{\epsilon}{4}}(q))} \geq C_1 \epsilon^{\frac{Q}{2}} |h(q)| - C_2 \epsilon^{\alpha + \frac{Q}{2}} \|h\|_{C_{\mathbb{H}}^\alpha(B_{\frac{\epsilon}{4}}(q))}.$$

Using the hypotheses (i) and (ii), we conclude for all  $q \in B_{\frac{\epsilon}{2}}(p)$

$$\begin{aligned} |h(q)| &\leq C_1^{-1} \epsilon^{-\frac{Q}{2}} \left( \|h\|_{L^2(B_{\frac{\epsilon}{4}}(q))} + C_2 \epsilon^{\alpha + \frac{Q}{2}} \|h\|_{C_{\mathbb{H}}^{\alpha}(B_{\frac{\epsilon}{4}}(q))} \right) \\ &\leq C_1^{-1} \{ \mathcal{A}^{1/2} \epsilon + \mathcal{B} C_2 \epsilon^{\alpha} \} = o(1) \end{aligned}$$

as  $\epsilon \rightarrow 0$ .  $\square$

In view of (4.6) and (2.10) we can apply the previous lemma to  $h = \nabla_{\mathbb{H}} w_{\epsilon}^i$  and infer

$$\sup_{B_{\frac{\epsilon}{4}}} |\nabla_{\mathbb{H}} u_{\epsilon}^i - \nabla_{\mathbb{H}} x^i| \leq o(1)$$

as  $\epsilon \rightarrow 0$ . Since the matrix  $(X_i x^j)_{ij}$  for  $i, j = 1, \dots, r$  is invertible in a neighborhood of  $p$ , then for  $\epsilon > 0$  sufficiently small the same holds for the matrix  $(X_i u_{\epsilon}^j)_{ij}$ . Consequently, the  $n$ -tuple  $(u_{\epsilon}^1, \dots, u_{\epsilon}^r, x^{r+1}, \dots, x^n)$  yields a system of coordinates in a neighborhood of  $p$  and its first  $r$  components are both horizontal and harmonic. This concludes the proof of Theorem 4.4.  $\square$

**4.3. Horizontal  $Q$ -harmonic coordinates.** Throughout this section we will assume that  $M$  is an equivariant sub-Riemannian structure, endowed with a smooth volume form  $\text{vol}$ , that supports regularity for  $Q$ -harmonic functions, in the sense of Definition 1.2.

We will need an interpolation lemma analogue to Lemma 4.7.

**Lemma 4.8.** *Let  $p \in K \subset\subset M$  and let  $f$  be a function defined on  $B_{\epsilon}$ . If there are constants  $\beta, \mathcal{A}, \mathcal{B} > 0$  and  $\alpha \in (0, 1)$  such that for  $\epsilon > 0$  sufficiently small one has*

$$(i) \quad \|h\|_{L^Q(B_{\epsilon})} \leq \mathcal{A} \epsilon^{1+\beta}$$

$$(ii) \quad \|h\|_{C_{\mathbb{H}}^{\alpha}(B_{\frac{\epsilon}{2}})} \leq \mathcal{B},$$

then  $\|h\|_{L^{\infty}(B_{\epsilon/4})} \leq o(1)$  as  $\epsilon \rightarrow 0$ , uniformly in  $p \in K$ .

*Proof.* Using the notation and the argument in the proof of Lemma 4.7, one concludes that for any  $q \in B_{\frac{\epsilon}{4}}(p)$  one has

$$\|h\|_{L^Q(B_{\frac{\epsilon}{4}}(q))} \geq |h(q)| \cdot |B_{\frac{\epsilon}{4}}(q)|^{\frac{1}{Q}} - \|h\|_{C_{\mathbb{H}}^{\alpha}(B_{\frac{\epsilon}{4}}(q))} \epsilon^{\alpha + \frac{Q}{p}}.$$

The proof follows immediately from the latter and from the hypothesis.  $\square$

**Theorem 4.9.** *Let  $M$  be an equiregular sub-Riemannian structure endowed with a smooth volume form  $\text{vol}$  that supports regularity for  $Q$ -harmonic functions. For any point  $p \in M$  there exists a set of horizontal coordinates defined in a neighborhood of  $p$  that are  $Q$ -harmonic.*

*Proof.* We follow the argument outlined in the special case of Theorem 4.4. For  $p \in K \subset\subset M$  and  $\epsilon > 0$  to be determined later, we consider weak solutions  $u_{\epsilon}^i \in W_{\mathbb{H}}^{1,Q}(B_{\epsilon})$  to the Dirichlet problems

$$\begin{cases} L_Q u_{\epsilon}^i = 0 & \text{in } B_{\epsilon}, \quad i = 1, \dots, n \\ u_{\epsilon}^i = x^i & \text{in } \partial B_{\epsilon}, \quad i = 1, \dots, n, \end{cases}$$



where  $x^1, \dots, x^n$  is an arbitrary set of coordinates near  $p$ . These solutions exist and are unique in view of the convexity of the  $Q$ -energy. The  $C_H^{1,\alpha}$  estimates assumptions guarantee that  $u_\epsilon^i \in C_{H,\text{loc}}^{1,\alpha}(B_\epsilon) \cap W_{H,\text{loc}}^{1,Q}(B_\epsilon)$ . Arguing as in Lemma 4.5, we set

$$w_\epsilon^i := u_\epsilon^i - x^i \in C_{H,\text{loc}}^{1,\alpha}(B_\epsilon) \cap W_{H,0}^{1,Q}(B_\epsilon)$$

and observe that

$$\int_{B_\epsilon} |\nabla_H u_\epsilon^i|^{Q-2} X_k u_\epsilon^i X_k w_\epsilon^i \, d \text{vol} = 0.$$

As a consequence one has

$$\begin{aligned} \int_{B_\epsilon} |\nabla_H w_\epsilon^i|^Q \, d \text{vol} &\leq \int_{B_\epsilon} (|\nabla_H u_\epsilon^i| + |\nabla_\epsilon x^i|)^{Q-2} |\nabla_H w_\epsilon^i|^2 \, d \text{vol} \\ &\leq \int_{B_\epsilon} (|\nabla_H u_\epsilon^i|^{Q-2} X_k u_\epsilon^i - |\nabla_H x^i|^{Q-2} X_k x^i) X_k w_\epsilon^i \, d \text{vol} \\ &= \int_{B_\epsilon} -X_k^* (|\nabla_H x^i|^{Q-2} X_k x^i) w_\epsilon^i \, d \text{vol} \\ &\leq \left( \int_{B_\epsilon} |L_Q x^i|^Q \, d \text{vol} \right)^{\frac{Q-1}{Q}} \left( \int_{B_\epsilon} |w_\epsilon^i|^Q \, d \text{vol} \right)^{\frac{1}{Q}} \\ \text{(applying Poincaré inequality)} &\leq C\epsilon \left( \int_{B_\epsilon} |L_Q x^i|^Q \, d \text{vol} \right)^{\frac{Q-1}{Q}} \left( \int_{B_\epsilon} |\nabla_H w_\epsilon^i|^Q \, d \text{vol} \right)^{\frac{1}{Q}} \\ (4.10) &\leq C'\epsilon^Q \|\nabla_H w^i\|_{L^Q(B_\epsilon)}, \end{aligned}$$

for constants  $C, C' > 0$  depending only on  $Q, K$ , on the coordinates  $x^1, \dots, x^n$ , the sub-Riemannian structure, and the volume form. From the latter it immediately follows that

$$(4.11) \quad \|\nabla_H w_\epsilon^i\|_{L^Q(B_\epsilon)} \leq C'' \epsilon^{1+\frac{1}{Q-1}}$$

Arguing as in Theorem 4.4, and applying the  $C_H^{1,\alpha}$  estimates from the hypothesis that  $M$  supports regularity for  $Q$ -harmonic functions, (4.11) and the interpolation Lemma 4.8, one has that for  $\epsilon > 0$  sufficiently small the matrix  $(X_i u_\epsilon^j)_{ij}$ , for  $i, j = 1, \dots, r$  is invertible in a neighborhood of  $q$ . On the other hand, this implies that for each  $i = 1, \dots, r$  one has that  $|\nabla_H u_\epsilon^i|$  is a  $C_H^\alpha$  function bounded away from zero in a neighborhood of  $p$ , and hence by part (2) of Definition 1.2 and by Proposition 2.18 one has that  $u_\epsilon^1, \dots, u_\epsilon^r, x^{r+1}, \dots, x^n$  is a smooth system of coordinates in a neighborhood of  $p$ , with  $u_\epsilon^1, \dots, u_\epsilon^r$  both horizontal and  $Q$ -harmonic.  $\square$

**4.4. Regularity from horizontal regularity.** Let  $\gamma$  be an horizontal curve in  $M$ . Let  $x^1, \dots, x^n$  be coordinates on  $M$  such that  $x^1, \dots, x^r$  are horizontal coordinates with respect to an horizontal frame  $X_1, \dots, X_r$ . We write

$$\gamma_H = (x^1 \circ \gamma, \dots, x^r \circ \gamma) \quad \text{and} \quad \gamma_V = (x^{r+1} \circ \gamma, \dots, x^n \circ \gamma).$$

Hence  $\gamma = (\gamma_H, \gamma_V)$  and  $\dot{\gamma} = (\dot{\gamma}_H, \dot{\gamma}_V)$ . There are functions  $\beta_1, \dots, \beta_r$  so that

$$\dot{\gamma} = \sum_{j=1}^r \beta_j (X_j \circ \gamma).$$

In coordinates we write  $X_j = \sum_{k=1}^n X_j^k \frac{\partial}{\partial x^k}$ . So

$$\begin{aligned} (\dot{\gamma}_H, \dot{\gamma}_V) &= \sum_{j=1}^r \beta_j (X_j \circ \gamma) \\ &= \sum_{j=1}^r \beta_j \sum_{k=1}^n (X_j^k \circ \gamma) \frac{\partial}{\partial x^k} \\ &= \sum_{k=1}^r \sum_{j=1}^r \beta_j (X_j^k \circ \gamma) \frac{\partial}{\partial x^k} + \sum_{k=r+1}^n \sum_{j=1}^r \beta_j (X_j^k \circ \gamma) \frac{\partial}{\partial x^k}. \end{aligned}$$

Set  $O = (X_j x^i)_{ij} = X_j^i$ . We have  $\dot{\gamma}_H = \sum_{k=1}^r \sum_{j=1}^r \beta_j (O_j^k \circ \gamma) \frac{\partial}{\partial x^k} = O\beta$ , where we denoted  $\beta = (\beta_1, \dots, \beta_r)$ . Since  $O$  is invertible,  $(\beta_1, \dots, \beta_r) = (O^{-1} \circ \gamma) \dot{\gamma}_H$ . Thus

$$(4.12) \quad \dot{\gamma}_V = \sum_{k=r+1}^n \sum_{j=1}^r \left[ (O^{-1} \circ \gamma) \dot{\gamma}_H \right]_j (X_j^k \circ \gamma) \frac{\partial}{\partial x^k}.$$

In particular, the following holds.

**Proposition 4.13.** *Let  $\gamma$  be an absolute continuous curve. If  $\gamma_H$  is smooth, then  $\gamma$  is smooth.*

*Proof.* By hypothesis  $\gamma$  and  $\dot{\gamma}_H$  are absolute continuous. Then by (4.12) also  $\dot{\gamma}_V$  is absolute continuous. Thus  $\dot{\gamma}$  is continuous. A bootstrap argument shows that  $\gamma$  is smooth.  $\square$

In the following, we will consider maps that are *absolutely continuous on curves* ( $ACC_Q$ ). We recall that such maps send almost every (with respect to the  $Q$ -modulus measure) rectifiable curve into a rectifiable curve (see [Sha00] for more details). In the case of a sub-Riemannian manifold  $M$ ,  $ACC$  maps defined on  $M$  have the following property. Let  $X$  be any horizontal vector field in  $M$  and denote by  $\phi_X^t$  the corresponding flow. Then for almost every  $p \in M$  (with respect to Lebesgue measure), one has that  $t \rightarrow f(\phi_X^t(p))$  is a rectifiable curve.

**Proposition 4.14.** *Let  $M$  and  $N$  two sub-Riemannian manifolds. Let  $f : M \rightarrow N$  be an  $ACC$  map. Let  $k \geq 1$ ,  $\alpha \in (0, 1)$ , and  $p \geq 1$ . If  $f^1, \dots, f^r$  are in  $C_{H,loc}^{k,\alpha}(M)$  (resp. in  $W_{H,loc}^{k,p}(M)$ ), then  $f^1, \dots, f^n$  is  $C_{H,loc}^{k,\alpha}(M)$  (resp. in  $W_{H,loc}^{k,p}(M)$ ).*

*Proof.* Let  $X$  be any horizontal vector field in  $M$ . Notice that if  $f^1, \dots, f^r$  are in  $C_{H,loc}^{k,\alpha}(M)$  (resp. in  $W_{H,loc}^{k,p}(M)$ ), then  $Xf^1, \dots, Xf^r$  are in  $C_{H,loc}^{k-1,\alpha}(M)$  (resp. in  $W_{H,loc}^{k-1,p}(M)$ ). For almost every  $p \in M$ , the curve

$$f(\phi_X^t(p)) =: \gamma(p, t) = (\gamma_H(p, t), \gamma_V(p, t)),$$

is an horizontal curve and hence (4.12) holds. Therefore, for almost every  $p$ , we have

$$\begin{aligned} (Xf^{m+1}(p), \dots, Xf^n(p)) &= \frac{d}{dt}\gamma_V(p, t)|_{t=0} \\ &= \sum_{j=1}^m \sum_{k=m+1}^n \left[ (O^{-1} \circ \gamma(p, 0)) \gamma_{\mathbb{H}}(p, 0) \right]_j X_j^k(\gamma(p, 0)) \frac{\partial}{\partial x^k} \\ &= \sum_{j=1}^m \sum_{k=m+1}^n \left[ O^{-1}(f(p)) (Xf^1(p), \dots, Xf^r(p))^T \right]_j (X_j^k \circ f)(p) \frac{\partial}{\partial x^k}. \end{aligned}$$

Since the functions  $X_j^k \circ f$  and  $Xf^1, \dots, Xf^r$  are continuous (resp. in  $L^p$ ), then the functions  $Xf^{m+1}, \dots, Xf^n$  are continuous (resp. in  $L^p$ ), for all horizontal  $X$ . Hence,  $f^1, \dots, f^n \in C_{\mathbb{H}, \text{loc}}^1(M)$  (resp. in  $W_{\mathbb{H}, \text{loc}}^{1,p}(M)$ ) and then  $X_j^k \circ f \in C_{\mathbb{H}, \text{loc}}^1(M)$  (resp. in  $W_{\mathbb{H}, \text{loc}}^{1,p}(M)$ ). Notice that, if  $f^1, \dots, f^n \in C_{\mathbb{H}, \text{loc}}^1(M)$  then on any compact  $K$  the functions  $\nabla_{\mathbb{H}} f^1, \dots, \nabla_{\mathbb{H}} f^n$  are bounded, say by a constant  $C$ , therefore, for all horizontal curve  $\sigma : [0, 1] \rightarrow K$ ,

$$\text{Length}(f(\sigma)) = \int_0^1 \|f_*\sigma'\| ds \leq C \int_0^1 \|\sigma'\| ds = C \text{Length}(\sigma).$$

Hence,  $f^1, \dots, f^n \in C_{\mathbb{H}, \text{loc}}^1(M)$  implies that  $f$  is Lipschitz and therefore its components are in  $C^\alpha$ . Bootstrapping, we conclude that  $f^1, \dots, f^n$  is  $C_{\mathbb{H}, \text{loc}}^{k,\alpha}(M)$  (resp. in  $W_{\mathbb{H}, \text{loc}}^{k,p}(M)$ ).  $\square$

## 5. REGULARITY OF 1-QUASICONFORMAL MAPS

In this section we prove Theorem 1.4. Let us first clarify the definition of the function spaces involved. Given two equiregular sub-Riemannian manifolds  $M, N$ , we say that a homeomorphism  $f$  is in  $C_{\mathbb{H}, \text{loc}}^{1,\alpha}(M, N) \cap W_{\mathbb{H}, \text{loc}}^{2,2}(M, N)$  if, in any (smooth) coordinate system of  $N$ , the components of  $f$  belong to  $C_{\mathbb{H}, \text{loc}}^{1,\alpha}(M) \cap W_{\mathbb{H}, \text{loc}}^{2,2}(M)$ .

**5.1. Every 1-quasiconformal map in  $C_{\mathbb{H}, \text{loc}}^{1,\alpha}(M, N) \cap W_{\mathbb{H}, \text{loc}}^{2,2}(M, N)$  is conformal.** We now show that, assuming that a 1-quasiconformal map has the basic regularity, then the map is smooth. The proof is independent from the results in Section 4. Namely, we do not need to assume any regularity theory for  $Q$ -Laplacian.

*Proof of Theorem 1.4.(i).* Denote by  $\text{vol}_M$  and  $\text{vol}_N$  the Popp measures of  $M$  and  $N$ . For  $p \in M$ , consider any system of smooth coordinates  $y^1, \dots, y^n$  in a neighborhood of  $f(p) \in N$ . Set  $f^i := y^i \circ f$  and  $h^i := L_Q(y^i) \in C^\infty(N)$ . From Corollary 3.17.(i), it follows that for all  $u \in C_0^\infty(M)$

$$\int_M L_Q(f^i) u \, d\text{vol}_M = \int_M h^i \circ f J_f^{\text{Popp}} u \, d\text{vol}_M.$$

For  $i = 1, \dots, n$ , set  $H^i := h^i \circ f J_f^{\text{Popp}}$ . Since the Popp measures are smooth and  $f \in C_{\mathbb{H}, \text{loc}}^{1,\alpha}(M, N)$ , we have that  $J_f^{\text{Popp}} \in C_{\mathbb{H}, \text{loc}}^\alpha(M)$  and therefore  $H^i \in C_{\mathbb{H}, \text{loc}}^\alpha(M)$ . At this point we have that  $L_Q f^i \in C_{\mathbb{H}, \text{loc}}^\alpha(M)$  and that  $f^i \in C_{\mathbb{H}, \text{loc}}^{1,\alpha}(M) \cap W_{\mathbb{H}, \text{loc}}^{2,2}(M)$ . Notice that  $|\nabla_{\mathbb{H}} f^i|$  is bounded away from 0, since  $f$  is bi-Lipschitz. Therefore, Proposition 2.18 applies, yielding that  $f \in C_{\mathbb{H}, \text{loc}}^{2,\alpha}(M, N)$ . The proof follows by bootstrap using the Schauder estimates in Proposition 2.12.  $\square$

**5.2. Regularity of  $Q$ -harmonic functions implies conformality.** We now reduce the smoothness assumption by using horizontal  $Q$ -harmonic coordinates, see Section 4. To ensure their existence and to use them we need to assume that the manifolds support the regularity theory for  $Q$ -Laplacian as defined in Definition 1.2.

Note that a standard argument, see for instance [Cap97], shows that for every  $g = (g^1, \dots, g^r) \in C^\infty(M, \mathbb{R}^r)$ ,  $U \subset\subset M$  and for every  $\ell, \ell' > 0$ , there exists a constant  $C > 0$  such that for each weak solution  $u$  of the equation  $L_Q u = X_i^* g^i$  on  $M$  with  $\|u\|_{W_H^{1,Q}(U)} < \ell$  and  $\frac{1}{\ell'} < |\nabla_H u| < \ell'$  on  $U$ , one has

$$\|u\|_{W_H^{2,2}(U)} \leq C.$$

*Proof of Theorem 1.4.(ii).* We shall use Proposition 4.14. Since sub-Riemannian manifolds are  $Q$ -regular, by [HK98] any quasiconformal map is  $\text{ACC}_Q$  (see also [MM95, Corollary 6.5]) In view of Theorem 4.9, consider  $u_1, \dots, u_n$  a system of local coordinates around a point  $f(p) \in M$  for which the horizontal coordinates  $u^1, \dots, u^r$  are  $Q$ -harmonic.

In view of the morphism property (Corollary 3.17) the pull-backs  $f_i = u_i \circ f$ , for  $i = 1, \dots, r$  are  $Q$ -harmonic functions in a neighborhood of  $p \in M$ . By the  $Q$ -harmonic regularity assumption, both  $u^i$  and  $f^i = u^i \circ f$  are in  $C_{H,\text{loc}}^{1,\alpha}(M)$ , for  $i = 1, \dots, r$ . Apply Proposition 4.14 to  $f$  with  $k = 1$  and get  $f \in C_{H,\text{loc}}^{1,\alpha}(M, N)$ . Since also  $f^{-1}$  is 1-quasiconformal, the same argument shows that  $f^{-1} \in C_{H,\text{loc}}^{1,\alpha}(N, M)$ . In particular, the map  $f$  is bi-Lipschitz and  $f^1, \dots, f^n$  is a local system of bi-Lipschitz coordinates. In particular,  $|\nabla_H f^1|, \dots, |\nabla_H f^n|$  are bounded away from zero. Because of the observation above, we have that  $f^1, \dots, f^r$  are in  $W_{H,\text{loc}}^{2,2}(M)$ . Invoking Proposition 4.14 once more, we have that  $f^1, \dots, f^n$  are in  $W_{H,\text{loc}}^{2,2}(M)$ .  $\square$

We remark that in the setting of Carnot groups both the existence of horizontal  $Q$ -harmonic coordinates and the Lipschitz regularity of 1-quasiconformal can be proven directly without using any PDE argument, see [Pan89].

## 6. LIOUVILLE THEOREM FOR CONTACT SUB-RIEMANNIAN MANIFOLDS

**6.1.  $Q$ -Laplacian with respect to a divergence-free frame.** In this section we intend to write the  $Q$ -Laplacian in a sub-Riemannian manifold using a horizontal frame that is not necessarily orthonormal, but is divergence-free with respect to some other volume form. Recall that a vector field  $X$  is divergence-free with respect to a volume form  $\mu$  if its adjoint with respect to  $\mu$  equals  $-X$ .

Let  $M$  be a sub-Riemannian manifold equipped with a smooth volume form  $\text{vol}$ . Let  $Y_1, \dots, Y_r$  be an orthonormal frame for the horizontal distribution  $HM$  of  $M$ . Recall from (2.16) that the  $Q$ -Laplacian of a twice differentiable function is

$$(6.1) \quad L_Q u = \sum_i Y_i^* \left( \left( \sum_k (Y_k u)^2 \right)^{\frac{Q-2}{2}} Y_i u \right)$$

Assume that there exists another frame  $X_1, \dots, X_r$  of  $HM$  and another smooth volume form  $\mu$  such that each  $X_i$  is divergence-free with respect to  $\mu$ . If  $g$  is the sub-Riemannian metric of  $M$ , let

$$g_{ij} := g(X_i, X_j) \in C^\infty(M).$$

For all  $x \in M$ , let  $g^{ij}(x)$  be the inverse matrix of  $g_{ij}(x)$  and define the family of scalar products on  $\mathbb{R}^r$  as

$$\tilde{g}_x(v, w) := v_i g^{ij}(x) w_j, \quad x \in M, v, w \in \mathbb{R}^r.$$

Then there exists  $a_i^j \in C^\infty(M)$  such that

$$(6.2) \quad Y_i = a_i^j X_j.$$

So  $\delta_{ij} = a_i^k a_j^l g_{kl}$  and  $g^{ij} = a_k^i a_k^j$ .

Let  $\omega$  be the smooth function such that  $\text{vol} = \omega\mu$ . Since  $X_i$  are divergence-free with respect to  $\mu$ , the adjoint vector fields with respect to  $\text{vol}$  of  $Y_i$  are such that

$$Y_i^* u = X_j^*(a_i^j u) = -\omega^{-1} X_j(\omega a_i^j u).$$

We use the notation

$$\nabla_0 u := (X_1 u, \dots, X_r u).$$

Noticing that  $\sum_k (Y_k u)^2 = \tilde{g}(\nabla_0 u, \nabla_0 u)$ , the expression (6.1) becomes

$$(L_Q u)(x) = -\omega(x)^{-1} X_i A_i(x, \nabla_0 u),$$

where

$$(6.3) \quad A_i(x, \xi) := \omega(x) \tilde{g}_x(\xi, \xi)^{\frac{Q-2}{2}} g^{ik}(x) \xi_k, \quad \text{for } \xi \in \mathbb{R}^r, x \in M.$$

The derivatives of such functions are

$$\partial_{x_j} A_i(x, \xi) = \partial_{x_j} \omega \tilde{g}(\xi, \xi)^{\frac{Q-2}{2}} g^{ik} \xi_k + \omega \frac{Q-2}{2} \tilde{g}(\xi, \xi)^{\frac{Q-2}{2}-1} \partial_{x_j} g^{l,l'} \xi_l \xi_{l'} g^{ik} \xi_k + \omega \tilde{g}(\xi, \xi)^{\frac{Q-2}{2}} \partial_{x_j} g^{ik} \xi_k$$

and

$$\partial_{\xi_j} A_i(x, \xi) = \omega \left( (Q-2) \tilde{g}(\xi, \xi)^{\frac{Q-4}{2}} g^{lj} g^{ik} \xi_l \xi_k + \tilde{g}(\xi, \xi)^{\frac{Q-2}{2}} g^{ij} \right).$$

Hence,

$$\partial_{\xi_j} A_i(x, \xi) \eta_i \eta_j = \omega \left( (Q-2) \tilde{g}(\xi, \xi)^{\frac{Q-4}{2}} \tilde{g}(\xi, \eta)^2 + \tilde{g}(\xi, \xi)^{\frac{Q-2}{2}} \tilde{g}(\eta, \eta) \right).$$

Using Cauchy-Schwarz inequality, the equivalence of norms in  $\mathbb{R}^r$ , and the smoothness of the functions  $\omega$  and  $g^{ij}$ 's, the functions  $A_i$  in (6.3) satisfy the following estimates: on each compact set of  $M$ , for some  $\lambda, \Lambda > 0$  depending only on  $Q$ , and for every  $\chi \in \mathbb{R}^r$ ,

$$(6.4) \quad \lambda |\xi|^{Q-2} |\chi|^2 \leq \partial_{\xi_j} A_i(x, \xi) \chi_i \chi_j \leq \Lambda |\xi|^{Q-2} |\chi|^2$$

and

$$(6.5) \quad |\partial_{x_j} A_i(x, \xi)| \leq \Lambda |\xi|^{Q-1}.$$

Summarizing, we have the following.

**Proposition 6.6.** *Let  $M$  be a sub-Riemannian manifold and consider  $\text{vol}$  and  $\mu$  two smooth volume forms on  $M$ . Assume there is a horizontal frame  $X_1, \dots, X_r$  on  $M$  of vector fields that are divergence-free with respect to  $\mu$ . If  $u$  is a function on  $M$  that is  $Q$ -harmonic with respect to  $\text{vol}$ , then  $u$  satisfies*

$$\sum_{i=1}^r X_i A_i(x, \nabla_0 u) = 0,$$

for some  $A_i$  for which (6.4) and (6.5) hold.

*Remark 6.7.* In the above we used two different structures of metric measure space on the same manifold  $M$ . These are  $(M, g, \text{vol})$  and  $(M, g_0, \mu)$ , where  $g_0$  is the metric for which  $X_1, \dots, X_m$  form an orthonormal frame. For each of these structures we may define corresponding Sobolev spaces  $W_H^{p,q}(M, g, \text{vol})$  and  $W_H^{p,q}(M, g_0, \mu)$ . Similarly, we consider spaces  $C_H^{1,\alpha}(M, g)$  and  $C_H^{1,\alpha}(M, g_0)$ . Since the matrix  $(a_i^j)$  in (6.2) and its inverse have locally Lipschitz coefficients, it follows that on compact sets  $\Omega \subset M$  the space  $W_H^{p,2}(\Omega, g, \text{vol})$  is biLipschitz to  $W_H^{p,2}(\Omega, g_0, \mu)$  for  $p = 1, 2$ , and  $C_H^{1,\alpha}(\Omega, g)$  is biLipschitz to  $C_H^{1,\alpha}(\Omega, g_0)$ .

**6.2. Darboux coordinates on contact manifolds.** On every contact manifold, the existence of a frame of divergence-free vector fields with respect to some measure is ensured by Darboux Theorem. More generally, every sub-Riemannian manifold that is contactomorphic to a unimodular (e.g., nilpotent) Lie group equipped with a horizontal left-invariant distribution admits such a frame. The reason is that left-invariant vector fields are divergence-free with respect to the Haar measure of the group. We shall recall now Darboux Theorem and we recall the standard contact structures, which are those of the Heisenberg groups.

Darboux Theorem states, see [Etn03], that every two contact manifolds of the same dimension are locally contactomorphic. In particular, any contact  $2n+1$ -manifold is locally contactomorphic to the standard contact structure on  $\mathbb{R}^{2n+1}$ , a frame of which is given by

$$(6.8) \quad X_i := \partial_{x_i} - \frac{x_{n+i}}{2} \partial_{x_{2n+1}}, \quad X_{n+i} := \partial_{x_{n+i}} + \frac{x_i}{2} \partial_{x_{2n+1}},$$

where  $i = 1, \dots, n$ . For future reference we will also set  $X_{2n+1} = \partial_{x_{2n+1}}$ . This frame is left-invariant for a specific Lie group structure, which we denote by  $\mathbb{H}^n$ : the Heisenberg group.

**Corollary 6.9.** *(of Darboux Theorem) Let  $M$  be a contact sub-Riemannian  $2n+1$ -manifold equipped with a volume form  $\text{vol}$ . There are local coordinates  $x_1, \dots, x_{2n+1}$  in which the horizontal distribution is given by the vector fields in (6.8), which are divergence-free with respect to the Lebesgue measure  $\mathcal{L}$ , and there exists  $\omega \in C^\infty$  such that  $\omega^{-1} \in C^\infty$  and  $d\text{vol} = \omega d\mathcal{L}$ .*

**6.3. Riemannian approximations.** Let us consider a contact  $2n+1$  manifold  $M$ , with sub-Riemannian metric  $g_0$  and volume form  $\text{vol}$ . Let  $Y_1, \dots, Y_{2n}$  denote a  $g_0$ -orthonormal horizontal frame in a neighborhood  $\Omega \subset M$ , and denote by  $Y_{2n+1}$  the Reeb vector field. For every  $\epsilon \in (0, 1)$  we may define a 1-parameter family of Riemannian metrics  $g_\epsilon$  on  $M$  so that the frame  $Y_1, \dots, Y_{2n}, \epsilon Y_{2n+1}$  is orthonormal. Denote by  $Y_1^\epsilon, \dots, Y_{2n+1}^\epsilon$  such  $g_\epsilon$ -orthonormal

frame. For  $\epsilon \geq 0$  and  $\delta \geq 0$  we will consider the family of regularized  $Q$ -Laplacian operators

$$(6.10) \quad L_Q^{\epsilon, \delta} u := \sum_{i=1}^{2n+1} Y_i^{\epsilon*} \left( \left( \delta + \sum_k (Y_k^\epsilon u)^2 \right)^{\frac{Q-2}{2}} Y_i^\epsilon u \right)$$

Invoking Corollary 6.9, and applying the same arguments as in Proposition 6.6, one can see that such  $Q$ -Laplacian operators  $L_Q^{\epsilon, \delta}$ , can be written in the form

$$(6.11) \quad L_Q^{\epsilon, \delta} u = \sum_{i=1}^{2n+1} X_i^\epsilon A_i^{\epsilon, \delta}(x, \nabla_\epsilon u) = 0,$$

where  $X_i^\epsilon = X_i$  for  $i = 1, \dots, 2n$  and  $X_{2n+1}^\epsilon = \epsilon X_{2n+1}$ , with  $X_1, \dots, X_{2n+1}$  as in (6.8). Here we have set  $\nabla_\epsilon f = (X_1^\epsilon f, \dots, X_{2n+1}^\epsilon f)$ . The case  $\epsilon = \delta = 0$  in (6.11) reduces to the subelliptic  $Q$ -Laplacian. The components  $A_i^{\epsilon, \delta}$  in (6.11) are defined as in (6.3), starting with the  $g_\epsilon$  metric, i.e., for every  $\xi \in \mathbb{R}^{2n+1}$  and  $x \in \Omega$ ,

$$(6.12) \quad A_i^{\epsilon, \delta}(x, \xi) := \omega(x) (\delta + \tilde{g}_{\epsilon, x}(\xi, \xi))^{\frac{Q-2}{2}} g_\epsilon^{ik}(x) \xi_k.$$

By the same token as in (6.4), one has that there exists  $\lambda, \Lambda > 0$  depending only on  $Q$ , such that the estimates

$$(6.13) \quad \lambda (\delta + |\xi|^2)^{\frac{Q-2}{2}} |\chi|^2 \leq \sum_{i,j=1}^{2n+1} \partial_{\xi_j} A_i^{\epsilon, \delta}(x, \xi) \chi_i \chi_j \leq \Lambda (\delta + |\xi|^2)^{\frac{Q-2}{2}} |\chi|^2.$$

$$(6.14) \quad |\partial_{x_j} A_i^{\epsilon, \delta}(x, \xi)| \leq \Lambda (\delta + |\xi|^2)^{\frac{Q-1}{2}}.$$

hold for all  $\epsilon \geq 0$  and  $\delta \geq 0$  and for all  $\xi \in \mathbb{R}^{2n+1}$  and  $\chi \in \mathbb{R}^{2n+1}$ .

In the next section we prove that contact sub-Riemannian manifolds support regularity for  $Q$ -harmonic functions. The same arguments also imply regularity for  $p$ -harmonic functions for every  $p \geq 2$ . Hence, together with Theorem 1.4, this result will yield Theorem 1.1.

**6.4.  $C^{1, \alpha}$  estimates after Zhong.** In this section we consider weak solutions  $u \in W_{\mathbb{H}, \text{loc}}^{1, Q}(\Omega)$  of  $L_Q^0 u = 0$ , where  $L_Q^0$  denotes the  $Q$ -Laplacian operator corresponding to a sub-Riemannian metric  $g_0$  (not necessarily left-invariant) in an open set  $\Omega \subset \mathbb{H}^n$ , endowed with its Haar measure, which coincides with the Lebesgue measure in  $\mathbb{R}^{2n+1}$ . We prove the following theorem

**Theorem 6.15.** *For every open  $U \subset\subset \Omega$  and for every  $\ell > 0$ , there exist constants  $\alpha \in (0, 1)$ ,  $C > 0$  such that for each  $u \in W_{\mathbb{H}, \text{loc}}^{1, Q}(\Omega)$  weak solution of  $L_Q^0 u = 0$  with  $\|u\|_{W_{\mathbb{H}}^{1, Q}(U)} < \ell$ , one has*

$$\|u\|_{C_{\mathbb{H}}^{1, \alpha}(U)} \leq C.$$

This result is due to Zhong [Zho09], in the case when  $g_0$  is a left invariant sub-Riemannian metric in  $\mathbb{H}^n$ . A simpler proof, in the case  $p > 4$ , was recently given by Ricciotti in [Ric17].

The proof in [Zho09] breaks down with the additional dependence on  $x$ , in the coefficients of the equation as expressed in Proposition 6.6. In fact, in one of the approximations used in [Zho09], the argument relies on the existence of explicit barrier functions, which one does not have in our setting. To deal with this issue we use a Riemannian approximation scheme to carry out the regularization. Apart from this aspect, the arguments in [Zho09] apply to the present setting as well. Note that the Hölder regularity of the solution  $u$  is considerably simpler (see for instance [CDG93]).

*Remark 6.16.* The proof in [Zho09] applies to any Carnot group of step two, and likewise the conclusion of Theorem 6.15 continues to hold in this more general setting.

*Riemannian approximation.* Throughout the rest of the section we will assume  $\delta > 0$  and let  $u$  denote a solution of  $L_Q^0 u = 0$  in  $\Omega \subset \mathbb{H}^n$ . For  $\epsilon > 0$  we consider  $W_{\epsilon, \text{loc}}^{k,p}$  and  $C_\epsilon^{k,\alpha}$  to be the Sobolev and Hölder spaces corresponding to the frame  $X_1^\epsilon, \dots, X_{2n+1}^\epsilon$ . Observe that by virtue of classical elliptic theory (see for instance [LU68]) for  $\delta > 0$  one has that the weak solutions  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1,Q}(\Omega)$  of (6.11) are in fact smooth in  $\Omega$ . For a fixed ball  $D \subset\subset \Omega$  and for any  $\epsilon \geq 0$ , standard PDE arguments (see for instance [HKM06]) yield the existence and unicity of the solution to the Dirichlet problem

$$(6.17) \quad \begin{cases} L_Q^{\epsilon, \delta} u^\epsilon = 0 \text{ in } D \\ u^\epsilon - u \in W_{\epsilon, 0}^{1,Q}(D). \end{cases}$$

Although the smoothness of  $u^\epsilon$  may degenerate as  $\epsilon \rightarrow 0$  and  $\delta \rightarrow 0$ , we will show that the estimates on the Hölder norm of the gradient do not depend on these parameters and hence will hold uniformly in the limit. Note that in view of the Caccioppoli inequality and of the uniform bounds on the Hölder norm of  $u^\epsilon$  as  $\epsilon \rightarrow 0$  (such bounds depend only on the stability of the Poincaré inequality and on the doubling constants of the Riemannian Heisenberg groups  $(\mathbb{H}^n, g_\epsilon)$  which are stable in view of [CCR13]), one has that for any  $K \subset\subset D$  there exists a constant  $M_{K,Q} > 0$  depending only on  $Q, K$  such that

$$\|\nabla_\epsilon u^\epsilon\|_{L^Q(K)} \leq M_{K,Q}.$$

The next proposition addresses the non trivial uniform bounds.

**Proposition 6.18.** *For every open  $U \subset\subset D$  and for every  $\ell > 0$ , there exist constants  $\alpha \in (0, 1), C > 0$  such that if  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1,Q}(D) \cap C^\infty(D)$  is the unique solution of (6.17) with  $\|u\|_{W_{\mathbb{H}}^{1,Q}(D)} < \ell$ , then one has*

$$\|u^\epsilon\|_{C_\epsilon^{1,\alpha}(U)} \leq C, \quad \forall \epsilon > 0.$$

The main regularity result Theorem 6.15 then follows from Proposition 6.18, by means of Ascoli-Arzelà theorem and the uniqueness of the Dirichlet problem (6.17) when  $\epsilon = 0$ .

The proof of Proposition 6.18 follows very closely the arguments in [Zho09]. For the reader's convenience we reproduce them in the two sections below. For the sake of notation's simplicity, and without any loss of generality, we will just present the proof in the case  $n = 1$ .

*Uniform Lipschitz regularity.* The aim of this section is to establish Lipschitz estimates that are uniform as  $\epsilon \rightarrow 0$ , on a open ball  $B \subset\subset D$ .

**Theorem 6.19.** *Let  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1,Q}(D) \cap C^\infty(D)$  be the unique solution of (6.17). If  $B \subset 2B \subset\subset D$  then there exists  $C > 0$ , depending only on  $Q, \Lambda, \lambda$  of (6.13) and (6.14), such that*

$$\sup_B |\nabla_\epsilon u^\epsilon| \leq C \left( \frac{1}{\mathcal{L}(2B)} \int_{2B} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \right)^{\frac{1}{Q}},$$

where  $2B$  denotes the ball with the same center of  $B$  and twice the radius.



The proof of this theorem is developed across several lemmata in this section.

For  $\epsilon, \delta > 0$  and  $i = 1, 2, 3$  set  $v_i = X_i^\epsilon u^\epsilon$  and observe that by differentiating (6.11) along  $X_i^\epsilon$ ,  $i = 1, 2, 3$  one has

$$(6.20) \quad \sum_{i,j=1}^3 X_i^\epsilon \left( A_{i,\xi_j}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon v_1 \right) + \sum_{i=1}^3 X_i^\epsilon \left( A_{i,\xi_2}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \right) + X_3 \left( A_2^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \right) \\ + \sum_{i=1}^3 X_i^\epsilon \left( A_{i,x_1}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) - \frac{x_2}{2} A_{i,x_3}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \right) = 0;$$

$$(6.21) \quad \sum_{i,j=1}^3 X_i^\epsilon \left( A_{i,\xi_j}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon v_2 \right) - \sum_{i=1}^3 X_i^\epsilon \left( A_{i,\xi_1}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \right) - X_3 \left( A_1^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \right) \\ + \sum_{i=1}^3 X_i^\epsilon \left( A_{i,x_2}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) + \frac{x_1}{2} A_{i,x_3}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \right) = 0;$$

and

$$(6.22) \quad \sum_{i,j=1}^3 X_i^\epsilon \left( A_{i,\xi_j}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon v_3 \right) + \epsilon \sum_{i=1}^3 X_i^\epsilon \left( A_{i,x_3}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \right) = 0.$$

*Remark 6.23.* Note that the terms containing  $X_3$  in the equations above are not bounded as  $\epsilon \rightarrow 0$  in the  $g_\epsilon$  metric. In the following it will be crucial to obtain estimates that are stable as  $\epsilon \rightarrow 0$ .

The following results were originally proved for the case with no dependence of  $x$ , in [MM07, Theorem 7], [MZGZ09, Lemma 5.1] and then again in [Zho09] with a more direct argument bypassing the difference quotients method. The proofs in our setting are very similar and we omit most of the details.

**Lemma 6.24.** *For every  $\beta \geq 0$  and  $\eta \in C_0^\infty(B)$  one has*

$$\int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon v_3|^2 |v_3|^\beta \eta^2 \, d\mathcal{L} \leq \left( \frac{2\Lambda}{\lambda(\beta+1)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \eta|^2 |v_3|^{\beta+2} \, d\mathcal{L} \\ + 2\epsilon^2 \Lambda \left( 1 + \frac{1}{\lambda(\beta+1)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta \eta^2 \, d\mathcal{L}.$$

*Proof.* Multiply both sides of (6.22) by  $\phi = \eta^2 |X_3^\epsilon u^\epsilon|^\beta X_3^\epsilon u^\epsilon$  and integrate over  $B$ . The result follows in a standard way from Young's inequality and from the structure conditions (6.13).  $\square$

Note that dividing both sides of the inequality above by  $\epsilon^{\beta+2}$  and letting  $\beta \rightarrow 0$  one recovers the Manfredi-Mingione original lemma (see for instance [Zho09, Lemma 3.3]).

**Lemma 6.25.** *For every  $\beta \geq 0$  and  $\eta \in C_0^\infty(B)$  one has*

$$\begin{aligned} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^2 \, d\mathcal{L} &\leq C(\beta+1)^4 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} |X_3 u^\epsilon|^2 \eta^2 \, d\mathcal{L} \\ &+ C \int_B (\eta^2 + |\nabla_\epsilon \eta|^2) (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} + C \int_B \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta+1}{2}} \, d\mathcal{L}, \end{aligned}$$

for some constant  $C = C(\lambda, \Lambda) > 0$ .

*Proof.* The proof follows the arguments in [Zho09] and [MZGZ09], multiplying both sides of (6.20), (6.21) and (6.22) by  $\phi = \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{\beta}{2}} v_i$  for  $i = 1, 2, 3$ , integrating over  $B$  and then using Young inequality and the structure conditions (6.13).  $\square$

The next step provides a crucial reverse Hölder-type inequality.

**Lemma 6.26.** *For every  $\beta \geq 2$  and  $\eta \in C_0^\infty(B)$  one has*

$$\begin{aligned} &\int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \\ &\leq C (\beta+1)^2 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2 \left( \epsilon^2 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 (\eta^{\beta+2} + \eta^\beta) \, d\mathcal{L} \right. \\ &\quad \left. + \epsilon^\beta \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta \, d\mathcal{L} \right). \end{aligned}$$

Note that dividing by  $\epsilon^\beta$  and letting  $\epsilon \rightarrow 0$  one recovers Zhong's estimate.

*Proof.* Differentiating (6.11) along  $X_1^\epsilon$ , recalling that  $[X_1^\epsilon, X_2^\epsilon] = X_3$ , and multiplying by a test function  $\phi \in C_0^\infty(B)$  yields

$$(6.27) \quad \int_B X_1^\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon \phi \, d\mathcal{L} = \int_B X_3 A_2^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) \phi \, d\mathcal{L}.$$

Next set  $\phi = \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta X_1^\epsilon u^\epsilon$  in the previous identity to obtain in the left-hand side

$$\begin{aligned} \int_B X_1^\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon \phi \, d\mathcal{L} &= \int_B A_{i,\xi_j}^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_1^\epsilon X_j^\epsilon u^\epsilon X_1^\epsilon X_i^\epsilon u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \, d\mathcal{L} \\ &\quad - \int_B X_1^\epsilon A_2^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \, d\mathcal{L} \\ &\quad + \beta \int_B X_1^\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon X_3^\epsilon u^\epsilon |X_3^\epsilon u^\epsilon|^{\beta-2} X_3^\epsilon u^\epsilon X_1^\epsilon u^\epsilon \eta^{\beta+2} \, d\mathcal{L} \\ &\quad + (\beta+2) \int_B X_1^\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon \eta |X_3^\epsilon u^\epsilon|^\beta X_1^\epsilon u^\epsilon \eta^{\beta+1} \, d\mathcal{L}. \end{aligned}$$

Substituting in (6.27) and using the structure conditions (6.13) one obtains

$$\begin{aligned}
(6.28) \quad & \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta \sum_{j=1}^3 |X_1^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \\
& \leq \int_B X_1^\epsilon A_2^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \, d\mathcal{L} \\
& \quad + \beta \int_B |X_1^\epsilon A_i^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon X_3^\epsilon u^\epsilon| X_3^\epsilon u^\epsilon |X_3^\epsilon u^\epsilon|^{\beta-1} X_1^\epsilon u^\epsilon \eta^{\beta+2} \, d\mathcal{L} \\
& \quad + (\beta + 2) \int_B |X_1^\epsilon A_i^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon \eta| X_3^\epsilon u^\epsilon |X_3^\epsilon u^\epsilon|^\beta X_1^\epsilon u^\epsilon \eta^{\beta+1} \, d\mathcal{L} \\
& \quad \quad \quad + \int_B |X_3 A_2^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \eta^{\beta+2}| X_3^\epsilon u^\epsilon |X_3^\epsilon u^\epsilon|^\beta X_1^\epsilon u^\epsilon \, d\mathcal{L} \\
& \leq \sum_{h,k=1}^3 \left| \int_B X_h^\epsilon A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \, d\mathcal{L} \right| \\
& \quad + \beta \int_B |\nabla_\epsilon A_i^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon)| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} |\nabla_\epsilon u^\epsilon| \eta^{\beta+2} \, d\mathcal{L} \\
& \quad + (\beta + 2) \int_B |\nabla_\epsilon A_i^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon)| |\nabla_\epsilon \eta| |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+1} \, d\mathcal{L} \\
& \quad + \int_B \sum_{j=1}^2 |X_3 A_j^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon)| \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \, d\mathcal{L} = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

In a similar fashion, differentiating (6.11) along  $X_2^\epsilon$  and  $X_3^\epsilon$ , and using the test function  $\phi = \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta X_h^\epsilon u^\epsilon$  with  $h = 2, 3$ , one arrives at a similar estimate for  $X_h^\epsilon X_j^\epsilon u^\epsilon$  in the left-hand side. The combination of such estimate and (6.28) yields

$$\int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \leq I_1 + I_2 + I_3 + I_4.$$

Next, for any  $\tau > 0$ , we estimate each single component  $|I_k|$  in the following way

$$\begin{aligned}
(6.29) \quad |I_h| & \leq \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \\
& \quad + \epsilon^2 \frac{C(\beta+1)^2 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2}{\tau} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 (\eta^{\beta+2} + \eta^\beta) \, d\mathcal{L} \\
& \quad \quad \quad + \tau^{-1} \epsilon^2 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta \, d\mathcal{L},
\end{aligned}$$

from which the conclusion will follow immediately. We begin by looking at  $I_1$ .

- **Estimate of  $I_1$ .** Proceeding as in [Zho09] we integrate by parts to obtain

(6.30)

$$\begin{aligned} \int_B X_h^\epsilon A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \, d\mathcal{L} &= - \int_B A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_h^\epsilon \left( X_3 u^\epsilon \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \right) \, d\mathcal{L} \\ &= -\epsilon^{-1}(\beta+1) \int_B A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta X_3^\epsilon X_h^\epsilon u^\epsilon \, d\mathcal{L} \\ &\quad - (\beta+2) \int_B A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \eta^{\beta+1} X_h^\epsilon \eta |X_3^\epsilon u^\epsilon|^\beta X_3 u^\epsilon \, d\mathcal{L} = \mathcal{I} + \mathcal{II} \end{aligned}$$

- **Estimate of  $\mathcal{I}$ .** Using Young inequality one has

$$\begin{aligned} &\left| \epsilon^{-1}(\beta+1) \int_B A_k^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta X_3^\epsilon X_h^\epsilon u^\epsilon \, d\mathcal{L} \right| \\ &\leq \epsilon^{-1}(\beta+1) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon X_3^\epsilon u^\epsilon| \eta^{\beta+2} \, d\mathcal{L} \\ &\leq \tau \|\nabla_\epsilon \eta\|_{L^\infty(B)}^{-2} \epsilon^{-2}(\beta+1) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon X_3^\epsilon u^\epsilon|^2 \eta^{\beta+4} \, d\mathcal{L} \\ &\quad + \frac{(\beta+1) \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2}{\tau} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^\beta \eta^\beta \, d\mathcal{L} = \mathcal{A} + \mathcal{B} \end{aligned}$$

Next, we invoke Lemma 6.24 to estimate the first integral  $\mathcal{A}$  as

$$\begin{aligned} &\int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon X_3^\epsilon u^\epsilon|^2 \eta^{\beta+4} \, d\mathcal{L} \\ &\leq \left( \frac{4\Lambda}{\lambda(\beta+6)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |\nabla_\epsilon \eta|^2 |X_3^\epsilon u^\epsilon|^{\beta+2} \, d\mathcal{L} \\ &\quad + 2\epsilon^2 \Lambda \left( 1 + \frac{2}{\lambda(\beta+3)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^\beta \eta^\beta \, d\mathcal{L} \\ &\leq \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2 \left( \frac{4\Lambda}{\lambda(\beta+6)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta |X_3^\epsilon u^\epsilon|^2 \, d\mathcal{L} \\ &\quad + 2\epsilon^2 \Lambda \left( 1 + \frac{2}{\lambda(\beta+3)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} |X_3^\epsilon u^\epsilon|^2 \eta^\beta \, d\mathcal{L} \end{aligned}$$

(using the fact that  $|X_3^\epsilon u^\epsilon| \leq \epsilon \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|$  one concludes)

$$\begin{aligned} &\leq \epsilon^2 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2 \left( \frac{4\Lambda}{\lambda(\beta+6)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \, d\mathcal{L} \\ &\quad + 2\epsilon^4 \Lambda \left( 1 + \frac{2}{\lambda(\beta+3)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^\beta \, d\mathcal{L} \end{aligned}$$

To estimate  $\mathcal{B}$  we simply observe that

$$|\mathcal{B}| \leq \frac{\epsilon^2(\beta+1)\|\nabla_\epsilon\eta\|_{L^\infty(B)}^2}{\tau} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L}$$

In conclusion we have proved

$$\begin{aligned} |\mathcal{I}| \leq & \tau(\beta+1) \left[ \left( \frac{4\Lambda}{\lambda(\beta+6)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \right. \\ & \left. + 2\|\nabla_\epsilon\eta\|_{L^\infty(B)}^{-2} \epsilon^2 \Lambda \left( 1 + \frac{2}{\lambda(\beta+3)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^2 d\mathcal{L} \right] \\ & + \frac{\epsilon^2(\beta+1)\|\nabla_\epsilon\eta\|_{L^\infty(B)}^2}{\tau} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \end{aligned}$$

– **Estimate of  $\mathcal{II}$ .** Observe that, in view of Young's inequality, one has

$$\begin{aligned} |\mathcal{II}| \leq & \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |\nabla_\epsilon\eta| |X_3^\epsilon u^\epsilon|^\beta |X_3 u^\epsilon|^2 d\mathcal{L} \\ & + \frac{(\beta+2)^2}{\tau} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \eta^\beta |\nabla_\epsilon\eta|^2 |X_3^\epsilon u^\epsilon|^\beta d\mathcal{L} \\ \leq & \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |\nabla_\epsilon\eta| |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \\ & + \epsilon^2 \frac{(\beta+2)^2}{\tau} \|\nabla_\epsilon\eta\|_{L^\infty(B)}^2 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \eta^\beta |X_3^\epsilon u^\epsilon|^{\beta-2} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \end{aligned}$$

This concludes the estimate of  $I_1$ , as in (6.29).

- **Estimate of  $I_2$ .** To estimate  $I_2$  we will note that in view of the structure conditions (6.13) there exists a constant  $C$  depending on  $B$  (essentially  $\max_B |x_i|$ ) such that

$$\begin{aligned} & \left| \int_B |\nabla_\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon)| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} |\nabla_\epsilon u^\epsilon| \eta^{\beta+2} d\mathcal{L} \right| \\ & \leq \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon| |\nabla_\epsilon u^\epsilon| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\ & \quad + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |\nabla_\epsilon u^\epsilon| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\ & \leq \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\ & \quad + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L}. \end{aligned}$$

Note that the second integral occurs only because of the dependence of  $A_i$  on the space variable  $x$ . The first integral is estimated exactly as in [Zho09], by means of

Young's inequality and Lemma 6.24. In fact one has

$$\begin{aligned}
& \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon| |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\
& \leq \epsilon^{-2} \|\nabla_\epsilon \eta\|_{L^\infty}^{-2} \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon|^2 |X_3^\epsilon u^\epsilon|^\beta \eta^{\beta+4} d\mathcal{L} \\
& \quad + C\epsilon^2 \beta^2 \|\nabla_\epsilon \eta\|_{L^\infty}^2 \tau^{-1} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta d\mathcal{L} \\
& \leq C\epsilon^{-2} \|\nabla_\epsilon \eta\|_{L^\infty}^{-2} \tau (\beta+2)^4 \left( \frac{2\Lambda}{\lambda(\beta+1)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \eta^{\beta+2} |\nabla_\epsilon \eta|^2 |X_3^\epsilon u^\epsilon|^{\beta+2} d\mathcal{L} \\
& \quad + 2C\beta^2 \|\nabla_\epsilon \eta\|_{L^\infty}^{-2} \tau \Lambda \left( 1 + \frac{1}{\lambda(\beta+1)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^\beta \eta^{\beta+4} d\mathcal{L} \\
& \quad + C\epsilon^2 \beta^2 \|\nabla_\epsilon \eta\|_{L^\infty}^2 \tau^{-1} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta d\mathcal{L}.
\end{aligned}$$

Estimate (6.29) then follows once one assumes (without loss of generalization) that  $\|\nabla_\epsilon \eta\|_{L^\infty} \geq 1$  and using the fact that  $|X_3^\epsilon u^\epsilon| \leq \epsilon \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|$ .

For the second integral we first use Young inequality and obtain

$$\begin{aligned}
& \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\
& \leq \tau \epsilon^{-2} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon|^2 |X_3^\epsilon u^\epsilon|^\beta \eta^{\beta+4} d\mathcal{L} \\
& \quad + \epsilon^2 \tau^{-1} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+2}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta d\mathcal{L}.
\end{aligned}$$

Invoking Lemma 6.24 and Young inequality one then has

$$\begin{aligned}
(6.31) \quad & \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^{\beta-1} \eta^{\beta+2} d\mathcal{L} \\
& \leq \tau \epsilon^{-2} \left[ \left( \frac{2\Lambda}{\lambda(\beta+1)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \eta|^2 |v_3|^{\beta+2} d\mathcal{L} \right. \\
& \quad \left. + 2\epsilon^2 \Lambda \left( 1 + \frac{1}{\lambda(\beta+1)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta \eta^2 d\mathcal{L} \right] \\
& \quad + \tau^{-1} \epsilon^2 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+2}{2}} |X_3^\epsilon u^\epsilon|^{\beta-2} \eta^\beta d\mathcal{L}
\end{aligned}$$

$$\begin{aligned}
&\leq \tau \epsilon^{-2} \left[ \left( \frac{2\Lambda}{\lambda(\beta+1)} + 2\Lambda \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \eta|^2 |v_3|^{\beta+2} d\mathcal{L} \right. \\
&\quad \left. + 2\epsilon^2 \Lambda \left( 1 + \frac{1}{\lambda(\beta+1)^2} \right) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta \eta^2 d\mathcal{L} \right] \\
&\quad + \tau^{-1} \frac{\beta-2}{\beta} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^\beta \eta^\beta d\mathcal{L} \\
&\quad + \tau^{-1} \frac{2}{\beta} \epsilon^\beta \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta d\mathcal{L}.
\end{aligned}$$

From the latter, estimate (6.29) follows once one recalls that  $|X_3^\epsilon u^\epsilon| \leq \epsilon \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|$ .

- **Estimate of  $I_3$ .** Using the structure conditions (6.13) one has

$$\begin{aligned}
&(\beta+2) \int_B |\nabla_\epsilon A_i^{\epsilon,\delta}(x, \nabla_\epsilon u^\epsilon)| |\nabla_\epsilon \eta| |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+1} d\mathcal{L} \\
&\leq (\beta+2) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+1} |\nabla_\epsilon \eta| d\mathcal{L} \\
&\quad + C(\beta+2) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+1} |\nabla_\epsilon \eta| d\mathcal{L}
\end{aligned}$$

The second integrand in the right hand side is estimated as in (6.31). To estimate the first integral we use Young inequality to obtain

$$\begin{aligned}
&\int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+1} |\nabla_\epsilon \eta| d\mathcal{L} \\
&\leq \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 |X_3^\epsilon u^\epsilon|^\beta \eta^{\beta+2} d\mathcal{L} \\
&\quad + C\tau^{-1} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |X_3^\epsilon u^\epsilon|^\beta \eta^\beta |\nabla_\epsilon \eta|^2 d\mathcal{L}
\end{aligned}$$

and consequently invoke  $|X_3^\epsilon u^\epsilon| \leq \epsilon \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|$  to conclude that (6.29) holds.

- **Estimate of  $I_4$ .** The structure conditions (6.13) yield

$$\begin{aligned}
&\int_B \sum_{j=1}^2 |X_3 A_j^\epsilon(x, \nabla_\epsilon u^\epsilon)| \eta^{\beta+2} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| d\mathcal{L} \\
&\leq (\beta+2) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3^\epsilon u^\epsilon| |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+2} d\mathcal{L} \\
&\quad + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |X_3^\epsilon u^\epsilon|^\beta |\nabla_\epsilon u^\epsilon| \eta^{\beta+2} d\mathcal{L},
\end{aligned}$$

which are estimated as for (6.30) and using  $|X_3^\epsilon u^\epsilon| \leq \epsilon \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|$ .  $\square$

The argument in the previous proof can be adapted to the case  $\beta = 0$  to obtain

**Corollary 6.32.** *There exists a constant  $C > 0$  depending only on  $\lambda, \Lambda, Q$  such that for every  $\eta \in C_0^\infty(B)$  with  $0 \leq \eta \leq 1$  one has*

$$\begin{aligned} & \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^2 \, d\mathcal{L} \\ & \leq C (1 + \|\nabla_\epsilon \eta\|_{L^\infty(B)}^2 + \|X_3 \eta\|_{L^\infty(B)}) \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} \, d\mathcal{L}. \end{aligned}$$

Note that the previous result immediately implies part (2) of Proposition 6.18.

The following corollary is a straightforward consequence of Lemma 6.26 and the Young inequality applied to the right hand side of inequality of the lemma.

**Corollary 6.33.** *For every  $\beta \geq 2$  and  $\eta \in C_0^\infty(B)$  with  $0 \leq \eta \leq 1$ , one has*

$$\begin{aligned} & \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u^\epsilon|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \\ & \leq \epsilon^\beta C^\beta (\beta + 1)^4 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^\beta \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^\beta \, d\mathcal{L} \right. \\ & \quad \left. + \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta \, d\mathcal{L} \right). \end{aligned}$$

**Theorem 6.34** (Caccioppoli Inequality, [Zho09]). *For every  $\beta \geq 2$  and  $\eta \in C_0^\infty(B)$  with  $0 \leq \eta \leq 1$ , one has*

$$\begin{aligned} & \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^2 \, d\mathcal{L} \\ & \leq C(\beta + 1)^8 (\|\nabla_\epsilon \eta\|_{L^\infty(B)} + \|\eta X_3 \eta\|_{L^\infty(B)}) \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} \\ & \quad + C \int_B \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta+1}{2}} \, d\mathcal{L}. \end{aligned}$$



*Proof.* We use Hölder inequality and Corollary 6.33 to obtain

$$\begin{aligned}
& \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} |X_3 u^\epsilon|^2 \eta^2 \, d\mathcal{L} \\
& \leq \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3 u|^{\beta+2} \eta^{\beta+2} \, d\mathcal{L} \right)^{\frac{2}{\beta+2}} \left( \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} \right)^{\frac{\beta}{\beta+2}} \\
& \leq \left( \epsilon^{-\beta} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3^\epsilon u|^\beta \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^{\beta+2} \, d\mathcal{L} \right)^{\frac{2}{\beta+2}} \left( \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} \right)^{\frac{\beta}{\beta+2}} \\
& \leq \left[ C^\beta (\beta+1)^4 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^\beta \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^\beta \, d\mathcal{L} \right. \right. \\
& \quad \left. \left. + \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta \, d\mathcal{L} \right) \right]^{\frac{2}{\beta+2}} \left( \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} \right)^{\frac{\beta}{\beta+2}}.
\end{aligned}$$

Recalling Lemma 6.25 the previous estimate then yields

$$\begin{aligned}
& \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^2 \, d\mathcal{L} \leq C(\beta+1)^4 \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} |X_3 u^\epsilon|^2 \eta^2 \, d\mathcal{L} \\
& \quad + C \int_B (\eta^2 + |\nabla_\epsilon \eta|^2) (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} + C \int_B \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta+1}{2}} \, d\mathcal{L} \\
& \leq C(\beta+1)^4 \left[ C^\beta (\beta+1)^4 \|\nabla_\epsilon \eta\|_{L^\infty(B)}^\beta \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2+\beta}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 \eta^\beta \, d\mathcal{L} \right. \right. \\
& \quad \left. \left. + \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \eta^\beta \, d\mathcal{L} \right) \right]^{\frac{2}{\beta+2}} \left( \int_{\text{Supp}(\eta)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{2}} \, d\mathcal{L} \right)^{\frac{\beta}{\beta+2}} + C \int_B \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta+1}{2}} \, d\mathcal{L}.
\end{aligned}$$

The conclusion follows immediately from the latter and from Young inequality.  $\square$

**Lemma 6.35.** *Let  $u^\epsilon \in W_{\epsilon,loc}^{1,Q}(B) \cap C^\infty(B)$  be the unique solution of (6.17). For every  $\beta \geq 2$  set  $w = (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+\beta}{4}}$ . If  $\eta \in C_0^\infty(B)$  with  $0 \leq \eta \leq 1$ , and  $\kappa = Q/(Q-2)$ , then one has*

$$\left( \int_B w^{2\kappa} \eta^2 \, d\mathcal{L} \right)^{\frac{1}{\kappa}} \leq C(\beta+1)^8 (\|\nabla_\epsilon \eta\|_{L^\infty(B)} + \|\eta X_3 \eta\|_{L^\infty(B)}) \int_{\text{Supp}(\eta)} w^2 \, d\mathcal{L},$$

where  $C > 0$  is a constant depending only on  $Q$ .

*Proof.* Recall that the Sobolev constant depends only on the constants in the Poincaré inequality and in the doubling inequality [HK00], both of which are stable in this Riemannian approximation scheme (see [CCR13]). The result follows immediately applying Sobolev inequality and invoking Theorem 6.34.  $\square$

The proof of Theorem 6.19 now follows in a standard fashion, as described in [Zho09], from the Moser iteration scheme (see for instance [GT01, Theorem 8.18]) and from [HKM06, Lemma 3.38]. Note that the constant involved in such iteration are stable as  $\epsilon \rightarrow 0$  (see [CCR13]).

*Uniform  $C_\epsilon^{1,\alpha}$  regularity.* Throughout this section we will implicitly use the uniform (in  $\epsilon$ ) local Lipschitz regularity of solutions of (6.17) and set for every  $B(x_0, 2r_0) \subset B$ ,  $k \in \mathbb{R}$ ,  $l = 1, 2, 3$ , and  $0 < r < r_0/4 < 1$ ,

$$\begin{aligned} \mu^\epsilon(r) &= \text{osc}_{B(x_0, r)} |\nabla_\epsilon u^\epsilon|; A_{l, k, r}^- = \{x \in B(x_0, r) \text{ such that } X_l^\epsilon u^\epsilon < k\} \\ &\text{and } A_{l, k, r}^+ = \{x \in B(x_0, r) \text{ such that } X_l^\epsilon u^\epsilon > k\}. \end{aligned}$$

The proof of Proposition 6.18 and in particular of the  $C^{1,\alpha}$  estimate in part (1) follows immediately from the following theorem, which is the main result of the section:

**Theorem 6.36.** *Let  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1, Q}(B) \cap C^\infty(B)$  be the unique solution of (6.17). There exists a constant  $s > 0$  depending only on  $Q, \lambda, \Lambda, r_0$  such that*

$$\mu(r) \leq (1 - 2^{-s})\mu(4r) + 2^s(\delta + \mu(r_0)^2)^{\frac{Q}{2}} \left(\frac{r}{r_0}\right)^{\frac{1}{Q}},$$

for all  $0 < r < r_0/8$ .

Our first step in the proof of this theorem consists in establishing a Caccioppoli inequality, in Proposition 6.57 for second order derivatives on super level sets  $A_{l, k, r}^+$ . This result will imply that the gradient  $\nabla_\epsilon u^\epsilon$  is in a De Giorgi-type class and then Theorem 6.36 will follow from well known results in the literature.

We begin with some preliminary lemmata. We indicate by  $|A|$  the Lebesgue measure  $\mathcal{L}(A)$  of a set  $A$ .

**Lemma 6.37.** *Let  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1, Q}(B) \cap C^\infty(B)$  be the unique solution of (6.17). For any  $q \geq 4$  there exists a positive constant  $C$  depending only on  $q, \lambda, \Lambda$  such that for all  $k \in \mathbb{R}$ ,  $l = 1, 2, 3$  and  $0 < r' < r < r_0/2$ ,  $\eta \in C_0^\infty(B(x_0, r))$  such that  $\eta = 1$  on  $B(x_0, r')$  one has*

$$\begin{aligned} (6.38) \quad & \int_{A_{l, k, r'}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_l|^2 \eta^2 \, d\mathcal{L} \\ & \leq \int_{A_{l, k, r}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_l|^2 |\nabla_\epsilon \eta|^2 \, d\mathcal{L} + C(\delta + \mu(r_0)^2)^{\frac{Q}{2}} |A_{l, k, r}^+|^{1-\frac{2}{q}} + I_3 \end{aligned}$$

where we have set  $\omega_l = (X_l^\epsilon u^\epsilon - k)^+$  and

$$(6.39) \quad I_3 = \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3 u^\epsilon| |\omega_1| \eta^2 \, d\mathcal{L}.$$

*Proof.* We study the case  $l = 1$ , since  $l = 2, 3$  is similar. Select a cut-off function  $\eta \in C_0^\infty(B(x_0, r))$  such that  $\eta = 1$  on  $B(x_0, r')$  and  $|\nabla_\epsilon \eta| \leq M(r - r')^{-1}$ , for some  $M > 0$  independent of  $\epsilon$ . Substitute  $\phi = \eta^2 \omega_1$  in the weak form of (6.20) to obtain

$$\begin{aligned} \int_B A_{i, \xi_j}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon X_1^\epsilon u^\epsilon X_i^\epsilon \omega_1 \eta^2 d\mathcal{L} &= -2 \int_B A_{i, \xi_j}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon X_1^\epsilon u^\epsilon X_i^\epsilon \eta \omega_1 d\mathcal{L} \\ &\quad - \int_B A_{i, \xi_2}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_3 u^\epsilon X_i^\epsilon (\omega_1 \eta^2) d\mathcal{L} \\ &\quad + \int_B X_3 A_i^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \eta^2 \omega_1 d\mathcal{L} \\ &\quad - \int_B \left( A_{i, x_1}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) - \frac{x_2}{2} A_{i, x_3}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) \right) X_i^\epsilon (\eta^2 \omega_1) d\mathcal{L}. \end{aligned}$$

Using Young inequality and the structure conditions (6.13) one easily obtains the estimate

$$\begin{aligned} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_1|^2 \eta^2 d\mathcal{L} &\leq C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \eta|^2 \omega_1^2 d\mathcal{L} \\ &\quad + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3 u^\epsilon|^2 \eta^2 d\mathcal{L} + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3 u^\epsilon| |\omega_1| \eta^2 d\mathcal{L} \\ &\quad + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} \left( \omega_1 \eta^2 + 2\omega_1 \eta |\nabla_\epsilon \eta| + \eta^2 |\nabla_\epsilon \omega_1| \right) d\mathcal{L} \leq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

The terms  $I_1$  and  $I_3$  are already in the form needed for (6.58). To estimate  $I_4$  we observe that for every  $\tau > 0$  one can estimate

$$\begin{aligned} I_4 &\leq \tau \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_1|^2 \eta^2 d\mathcal{L} + \\ &\quad C\tau^{-1} \int_{A_{1, k, r}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} d\mathcal{L} + C \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \eta|^2 \omega_1^2 d\mathcal{L}, \end{aligned}$$

thus leading to the correct left hand side for (6.58). To estimate  $I_2$  we argue as in [Zho09] and invoke Theorem 6.34 and Corollary 6.33 to show

$$\begin{aligned}
I_2 &\leq \left( \int_{A_{1,k,r}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} d\mathcal{L} \right)^{1-\frac{2}{q}} \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3 u^\epsilon|^q \eta^2 d\mathcal{L} \right)^{\frac{2}{q}} \\
&\leq (\delta + \mu(r_0)^2)^{\frac{Q-2}{2} \frac{q-2}{q}} |A_{1,k,r}^+|^{1-\frac{2}{q}} \left( \int_{B(x_0, r_0/2)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |X_3 u^\epsilon|^{q-2} |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \right)^{\frac{2}{q}} \\
&\leq (\delta + \mu(r_0)^2)^{\frac{Q-2}{2} \frac{q-2}{q}} |A_{1,k,r}^+|^{1-\frac{2}{q}} \left[ C^{q-2} (q-1)^4 r_0^{2-q} \left( \int_{B(x_0, \frac{2}{3}r_0)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-4+q}{2}} \sum_{i,j=1}^3 |X_i^\epsilon X_j^\epsilon u^\epsilon|^2 d\mathcal{L} \right. \right. \\
&\quad \left. \left. + \int_{B(x_0, \frac{2}{3}r_0)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+q-2}{2}} d\mathcal{L} \right) \right]^{\frac{2}{q}} \\
&\leq C^q (q-1)^{12} r_0^{-q} (\delta + \mu(r_0)^2)^{\frac{Q-2}{2} \frac{q-2}{q}} |A_{1,k,r}^+|^{1-\frac{2}{q}} \left[ \int_{B(x_0, r_0)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+q-2}{2}} d\mathcal{L} \right. \\
&\quad \left. + \int_{B(x_0, r_0)} \eta^2 (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+q-3}{2}} d\mathcal{L} + \int_{B(x_0, r_0)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+q-2}{2}} d\mathcal{L} \right]^{\frac{2}{q}} \\
&\leq C^q (q-1)^{12} r_0^{-q} (\delta + \mu(r_0)^2)^{\frac{Q}{2}} |A_{1,k,r}^+|^{1-\frac{2}{q}}.
\end{aligned}$$

□

In order to obtain from the previous lemma a Cacciopoli inequality we only need to obtain an estimate of  $I_3$ . The proof of the previous lemma yields the following

**Corollary 6.40.** *In the hypothesis and notation of the previous lemma, one has that for any  $q \geq 4$  there exists a positive constant  $C$  depending only on  $q, \lambda, \Lambda$  such that for all  $k \in \mathbb{R}$ ,  $l = 1, 2, 3$  and  $\eta \in C_0^\infty(B(x_0, r))$ ,*

$$(6.41) \quad I_3 \leq C (\delta + \mu(r_0)^2)^{\frac{Q-2}{4}} |A_{l,k,r}^+|^{\frac{1}{2}} G_0^{\frac{1}{2}}$$

where

$$G_0 = \int (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \omega_l^2 |\nabla_\epsilon v_3|^2 \eta^2 d\mathcal{L}.$$

*Proof.* From Hölder inequality one has,

$$\begin{aligned}
(6.42) \quad &\int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon X_3 u^\epsilon| |\omega_1| \eta^2 d\mathcal{L} \\
&\leq C (\delta + \mu(r_0)^2)^{\frac{Q-2}{4}} |A_{l,k,r}^+|^{\frac{1}{2}} \left( \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \omega_1^2 |\nabla_\epsilon X_3 u^\epsilon|^2 \eta^2 d\mathcal{L} \right)^{\frac{1}{2}}
\end{aligned}$$

□

**Lemma 6.43.** *In the hypothesis and notations of Lemma 6.40, for every  $m \in \mathbb{N}$ ,  $m \geq 1$  one has that there exists a constant  $C$  depending on  $m, Q, \lambda, \Lambda$ , such that*

$$(6.44) \quad G_0 \leq C \sum_{h=0}^m K^{2-\frac{1}{2m+h}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2m+h+2}}$$

where

$$K = \left( \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \omega_l^2 (\eta^2 + |\nabla_\epsilon \eta|^2) d\mathcal{L} + \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_l|^2 \eta^2 d\mathcal{L} \right)^{\frac{1}{2}}.$$

*Proof.* In the following we will denote by  $C$  a series of positive constants depending only on  $m, Q, \lambda, \Lambda$ . We study the case  $l = 1$ , since  $l = 2$  is similar and  $l = 3$  is slightly easier.

The bound (6.44) follows from a bootstrap argument, whose main step is the subject of the following estimates.

For  $\beta \geq 0$  and for any cut-off function  $\eta \in C_0^\infty(B(x_0, r))$ , let

$$G_\beta = \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \omega_l^2 |\nabla_\epsilon v_3|^2 |v_3|^\beta \eta^2 d\mathcal{L},$$

$$F_\beta = \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta |\omega_l|^2 \eta^2 d\mathcal{L},$$

where we recall that  $\omega_l = (X_l^\epsilon u^\epsilon - k)^+$ , for  $l = 1, 2, 3$ .

We claim that there exists a constant  $C > 0$ , depending only on  $Q, \lambda, \Lambda$  such that

$$(6.45) \quad G_\beta \leq \begin{cases} CK \left( G_{2\beta+2}^{\frac{1}{2}} + F_{2\beta+2}^{\frac{1}{2}} + (\delta + \mu(r_0)^2)^{\frac{1}{2}} F_{2\beta}^{\frac{1}{2}} \right), & \text{if } \beta > 0 \\ CK \left( G_2^{\frac{1}{2}} + F_2^{\frac{1}{2}} + (\delta + \mu(r_0)^2)^{1+\frac{Q\sigma}{4}} K^{1-\sigma} \right), & \text{if } \beta = 0 \text{ and for any } \sigma \in [0, 2), \end{cases}$$

and

$$(6.46) \quad F_\beta \leq \begin{cases} CK (\delta + \mu(r_0)^2)^{\frac{1}{2}} F_{2\beta}^{\frac{1}{2}} & \text{if } \beta > 0, \\ C (\delta + \mu(r_0)^2) K^2 & \text{if } \beta = 0, \end{cases}$$

In particular, for every  $\beta > 0$  and  $m \geq 2$ , it will follow that one has

$$(6.47) \quad F_\beta \leq (CK)^{2(1-\frac{1}{2m})} (\delta + \mu(r_0)^2)^{1-\frac{1}{2m}} F_{2m\beta}^{\frac{1}{2m}}.$$

Estimate (6.46) follows directly from Hölder inequality and from the gradient bounds in Theorem 6.19,

$$(6.48) \quad \begin{aligned} F_\beta &= \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta |\omega_l|^2 \eta^2 d\mathcal{L} \\ &\leq \left( \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_l|^2 \eta^2 d\mathcal{L} \right)^{\frac{1}{2}} \left( \int_{B(x_0, r)} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q+2}{2}} |v_3|^{2\beta} |\omega_l|^2 \eta^2 d\mathcal{L} \right)^{\frac{1}{2}} \\ &\leq CK (\delta + \mu(r_0)^2)^{\frac{1}{2}} F_{2\beta}^{\frac{1}{2}} \end{aligned}$$

To prove (6.45) substitute  $\phi = \eta^2 \omega_l^2 |v_3|^\beta v_3$  in the weak form of (6.22) to obtain

$$(6.49) \quad (\beta+1) \int_B A_{i\xi_j}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_j^\epsilon v_3 X_i^\epsilon v_3 \omega_l^2 |v_3|^\beta \eta^2 \, d\mathcal{L} \leq \int_B |A_{i\xi_j}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon)| |X_j^\epsilon v_3| |X_i^\epsilon [\eta^2 \omega_l^2]| |v_3|^{\beta+1} \, d\mathcal{L} \\ + \epsilon \int_B |A_{i, x_3}^{\epsilon, \delta}(x, \nabla_\epsilon u^\epsilon) X_i^\epsilon \left[ \eta^2 \omega_l^2 |v_3|^\beta v_3 \right]| \, d\mathcal{L} = A + B.$$

The first term on the left hand side is estimated via Young's inequality

$$A \leq CK \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |v_3|^{2\beta+2} |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L} \right)^{\frac{1}{2}}$$

For the second term we note that

$$(6.50) \quad B \leq C\epsilon \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} \left| \nabla_\epsilon \left[ \eta^2 \omega_l^2 |v_3|^\beta v_3 \right] \right| \, d\mathcal{L} \\ \leq C\epsilon \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |v_3|^{\beta+1} |\omega_1|^2 \eta |\nabla_\epsilon \eta| \, d\mathcal{L} \\ + C\epsilon \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |v_3|^\beta |\nabla_\epsilon v_3| |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ + C\epsilon \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-1}{2}} |v_3|^{\beta+1} |\nabla_\epsilon \omega_1| |\omega_1| \eta^2 \, d\mathcal{L} = T_1 + T_2 + T_3.$$

For any  $\bar{\epsilon} > 0$ , Young inequality and (6.46) yield the estimate

$$(6.51) \quad T_2 \leq \bar{\epsilon} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |v_3|^\beta |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ + C_{\bar{\epsilon}} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^\beta |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ \leq \bar{\epsilon} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |v_3|^\beta |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ + C_{\bar{\epsilon}} K (\delta + \mu(r_0)^2)^{\frac{1}{2}} F_{2\beta}^{\frac{1}{2}}.$$

The other two terms are estimated through Hölder inequality as

$$T_1 + T_3 \leq K \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^{2\beta+2} |\omega_1|^2 \eta^2 \, d\mathcal{L} \right)^{\frac{1}{2}}.$$

In view of the structure conditions (6.13), of (6.49), and of the estimates above for  $A$  and  $B$  one has

$$\begin{aligned} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} \omega_1^2 |\nabla_\epsilon v_3|^2 |v_3|^\beta \eta^2 \, d\mathcal{L} &\leq K \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |v_3|^{2\beta+2} |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L} \right)^{\frac{1}{2}} \\ &\quad + K \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |v_3|^{2\beta+2} |\omega_1|^2 \eta^2 \, d\mathcal{L} \right)^{\frac{1}{2}} \\ &\quad + C_{\bar{\epsilon}} K (\delta + \mu(r_0)^2)^{\frac{1}{2}} F_{2\beta}^{\frac{1}{2}} + \bar{\epsilon} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |v_3|^\beta |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L}. \end{aligned}$$

Bringing the last term on the right hand side over to the left hand side one obtains (6.45) in the case  $\beta > 0$ . For the case  $\beta = 0$ , the estimate on  $T_2$  above can be improved. We let  $\sigma \in [0, 2)$  and observe that

$$\begin{aligned} (6.52) \quad T_2 &\leq \bar{\epsilon} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon v_3|^2 |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ &\quad + C_{\bar{\epsilon}} \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ &\leq \bar{\epsilon} G_0 + C_{\bar{\epsilon}} (\delta + \mu(r_0)^2) \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_1|^2 \eta^2 \, d\mathcal{L} \\ &\leq \bar{\epsilon} G_0 + C_{\bar{\epsilon}} K^{2(1-\frac{\sigma}{2})} (\delta + \mu(r_0)^2) \left( \int_B (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_1|^2 \eta^2 \, d\mathcal{L} \right)^{\frac{\sigma}{2}} \\ &\leq \bar{\epsilon} G_0 + C_{\bar{\epsilon}} K^{2(1-\frac{\sigma}{2})} (\delta + \mu(r_0)^2)^{1+\frac{Q\sigma}{4}}. \end{aligned}$$

The latter concludes the proof of the estimates (6.45) and (6.46). At this point we can proceed with the description of the bootstrap argument needed to prove the bound on  $G_0$ .

In view of Lemma 6.24, Corollary 6.32, Corollary 6.33 and Theorem 6.34 one has the following

$$(6.53) \quad G_\beta \leq C(\delta + \mu(r_0)^2)^{\frac{Q+\beta+2}{2}} |B(x_0, r_0)| \quad \text{and} \quad F_\beta \leq C(\delta + \mu(r_0)^2)^{\frac{Q+\beta+2}{2}} |B(x_0, r_0)|.$$

Combining (6.53) with (6.45) and (6.46) yields for all  $\beta > 0$  and  $m \geq 1$ ,

$$\begin{aligned} (6.54) \quad G_\beta &\leq CK G_{2\beta+2}^{\frac{1}{2}} + (CK)^{2-\frac{1}{2m}} (\delta + \mu^2(r_0))^{\frac{1}{2}(1-\frac{1}{2m})} F_{2^m(2\beta+2)}^{\frac{1}{2m+1}} \\ &\quad + (CK)^{2-\frac{1}{2m}} (\delta + \mu^2(r_0))^{\frac{1}{2}+\frac{1}{2}(1-\frac{1}{2m})} F_{2^m(2\beta)}^{\frac{1}{2m+1}} \\ &\leq CK G_{2\beta+2}^{\frac{1}{2}} + (CK)^{2-\frac{1}{2m}} (\delta + \mu^2(r_0))^{\frac{\beta+2}{2}+\frac{1}{2m+1}\frac{Q}{2}} |B(x_0, r_0)|^{\frac{1}{2m+1}} \end{aligned}$$

Iterating the latter  $m$  times and setting  $\beta_m = 2^m - 2$  one obtains

$$G_{\beta_2} \leq C \left[ K^{2(1-\frac{1}{2^m})} G_{\beta_{m+2}}^{\frac{1}{2^m}} + \sum_{h=1}^m K^{2(1-\frac{1}{2^{m+h}})} (\delta + \mu^2(r_0))^{2(1+\frac{Q}{2^{m+h+2}})} \right].$$

From the latter, (6.47), and keeping in mind the starting point (6.45) corresponding to  $\beta = 0$ , one concludes that for any  $\sigma \in [0, 2)$ ,

(6.55)

$$\begin{aligned} G_0 &\leq C \left[ K^{2(1-\frac{1}{2^{m+1}})} G_{\beta_{m+2}}^{\frac{1}{2^{m+1}}} + \sum_{h=1}^m K^{1-\frac{1}{2^{m+h}}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2^{m+h+2}}} + F_2^{\frac{1}{2}} + K^2(\delta + \mu^2(r_0)) \right]. \\ &\leq C \left[ K^{2(1-\frac{1}{2^{m+1}})} G_{\beta_{m+2}}^{\frac{1}{2^{m+1}}} + \sum_{h=1}^m K^{2-\frac{1}{2^{m+h}}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2^{m+h+2}}} \right. \\ &\quad \left. + K^{1-\frac{1}{2^m}} (\delta + \mu^2(r_0))^{\frac{1}{2}-\frac{1}{2^{m+1}}} F_{2^{m+1}}^{\frac{1}{2^{m+1}}} + K^{2-\sigma} (\delta + \mu^2(r_0))^{1+\frac{Q\sigma}{4}} \right]. \end{aligned}$$

Applying (6.53) to the latter and letting  $\sigma = \frac{1}{2^m}$ , yields the estimate

$$(6.56) \quad G_0 \leq C \left[ K^{2-\frac{1}{2^m}} (\delta + \mu^2(r_0))^{1+\frac{Q+2}{2^{m+2}}} + \sum_{h=1}^m K^{2-\frac{1}{2^{m+h}}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2^{m+h+1}}} \right. \\ \left. + K^{2-\frac{1}{2^m}} (\delta + \mu^2(r_0))^{\frac{1}{2}-\frac{1}{2^{m+1}}+\frac{Q+2^{m+1}+2}{2^{m+2}}} + K^{2-\frac{1}{2^m}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2^{m+2}}} \right],$$

concluding the proof.  $\square$

**Proposition 6.57** (Caccioppoli inequality on super-level sets). *Let  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1, Q}(B) \cap C^\infty(B)$  be the unique solution of (6.17). For any  $q \geq 4$  there exists a positive constant  $C$  depending only on  $q, \lambda, \Lambda$  such that for all  $k \in \mathbb{R}$ ,  $l = 1, 2, 3$  and  $0 < r' < r < r_0/2$  one has*

$$(6.58) \quad \int_{A_{l, k, r'}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_l|^2 \eta^2 \, d\mathcal{L} \leq C \int_{A_{l, k, r}^+} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_l|^2 |\nabla_\epsilon \eta|^2 \, d\mathcal{L} \\ + C(\delta + \mu(r_0)^2)^{\frac{Q}{2}} |A_{l, k, r}^+|^{1-\frac{2}{q}},$$

where we have set  $\omega_l = (X_l^\epsilon u^\epsilon - k)^+$ .

*Proof.* As above, we study the case  $l = 1$ , since  $l = 2, 3$  is similar. Denote by  $\mathcal{A}$  the right hand side of (6.58), then in view of (6.38) one only needs to show  $I_3 \leq \mathcal{A}$ . From Lemma 6.37 one has

$$K \leq (\mathcal{A} + I_3)^{\frac{1}{2}}.$$

In view of (6.44) and Corollary 6.40 one obtains

$$(6.59) \quad I_3 \leq C(\delta + \mu(r_0)^2)^{\frac{Q-2}{4}} |A_{l, k, r}^+|^{\frac{1}{2}} \left( \sum_{h=0}^m K^{2-\frac{1}{2^{m+h}}} (\delta + \mu^2(r_0))^{1+\frac{Q}{2^{m+h+2}}} \right)^{\frac{1}{2}} \\ \leq C(\delta + \mu(r_0)^2)^{\frac{Q}{4}} |A_{l, k, r}^+|^{\frac{1}{2}} \sum_{h=0}^m \left( \mathcal{A}^{\frac{1}{2}} + I_3^{\frac{1}{2}} \right)^{1-\frac{1}{2^{m+h+1}}} (\delta + \mu^2(r_0))^{\frac{Q}{2^{m+h+3}}},$$



Next we observe that in view of Young inequality, for every  $h = 1, \dots, m$

$$\begin{aligned}
(6.60) \quad & C(\delta + \mu(r_0)^2)^{\frac{Q}{4}} |A_{l,k,r}^+|^{\frac{1}{2}} \left( \mathcal{A}^{\frac{1}{2} - \frac{1}{2m+h+2}} + I_3^{\frac{1}{2} - \frac{1}{2m+h+2}} \right) (\delta + \mu^2(r_0))^{\frac{Q}{2m+h+3}} \\
& \leq \frac{1}{2} I_3 + \frac{1}{2} \mathcal{A} + C \left( (\delta + \mu(r_0)^2)^{\frac{Q}{4}} |A_{l,k,r}^+|^{\frac{1}{2}} (\delta + \mu^2(r_0))^{\frac{Q}{2m+h+3}} \right)^{\frac{2m+h+2}{1+2m+h+1}} \\
& \leq \frac{1}{2} I_3 + \frac{1}{2} \mathcal{A} + C(\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q}{2}} |A_{l,k,r}^+|^{\frac{1}{2} \left( \frac{2m+h+2}{1+2m+h+1} \right)}
\end{aligned}$$

To complete the proof of (6.58) we choose  $m$  sufficiently large so that

$$1 - \frac{2}{q} \leq \frac{1}{2} \left( \frac{2m+h+2}{1+2m+h+1} \right).$$

□

A similar argument yields the corresponding result for sub-level sets:

**Corollary 6.61.** *Let  $u^\epsilon \in W_{\epsilon, \text{loc}}^{1,Q}(B) \cap C^\infty(B)$  be the unique solution of (6.17). For any  $q \geq 4$  there exists a positive constant  $C$  depending only on  $q, \lambda, \Lambda$  such that for all  $k \in \mathbb{R}$ ,  $l = 1, 2, 3$  and  $0 < r' < r < r_0/2$  one has*

$$\begin{aligned}
(6.62) \quad & \int_{A_{l,k,r'}^-} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\nabla_\epsilon \omega_l|^2 \, d\mathcal{L} \leq C(r-r')^{-2} \int_{A_{l,k,r}^-} (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}} |\omega_l|^2 \, d\mathcal{L} \\
& \quad + C(\delta + \mu(r_0)^2)^{\frac{Q}{2}} |A_{l,k,r}^-|^{1-\frac{2}{q}},
\end{aligned}$$

where we have set  $\omega_l = (X_l^\epsilon u^\epsilon - k)^-$ .

From this point on, the rest of the argument does not rely on the function  $u_\epsilon$  being a solution of the equation anymore but only on the Caccioppoli inequality above. The proof of Theorem 6.36 is very similar to the Euclidean case as developed in [LU68], and [DiB83]. It ultimately relies on the properties of De Giorgi classes in the general setting of metric spaces, as developed in [KS01] and [KMMP12]. We recall that a function  $f \in W_H^{1,2}(B(x_0, r_0)) \cap L^\infty(B(x_0, r_0))$  is in the De Giorgi class  $DG^+(\chi, q, \gamma)$  if there exists constants  $\chi, q, \gamma > 0$  such that for every  $0 < r' < r < r_0/4 < 1/2$  and  $k \in \mathbb{R}$  one has

$$(6.63) \quad \int_{B(x_0, r')} |\nabla_\epsilon w|^2 \, d\mathcal{L} \leq \gamma(r-r')^{-2} \int_{B(x_0, r)} w^2 \, d\mathcal{L} + \chi |\{x \in B(x_0, r) \text{ such that } w > 0\}|^{1-\frac{2}{q}},$$

where  $\omega = (f - k)^+$ . A function  $f \in W_H^{1,2}(B(x_0, r_0)) \cap L^\infty(B(x_0, r_0))$  is in the De Giorgi class  $DG^-(\chi, q, \gamma)$  if (6.63) holds for  $\omega = (f - k)^-$ . We set  $DG(\chi, q, \gamma) = DG^+(\chi, q, \gamma) \cap DG^-(\chi, q, \gamma)$ . It is well known, see for instance [KMMP12] and references therein, that functions in  $DG$  satisfy a scale invariant Harnack inequality and the following oscillation bounds: *If  $f \in DG(\chi, q, \gamma)$  then there exists  $s = s(q, \gamma, Q, r_0) > 0$  such that*

$$\text{osc}_{B(x_0, r/2)} f \leq (1 - 2^{-s}) \text{osc}_{B(x_0, r)} f + \chi r^{1-\frac{Q}{q}}.$$

From the latter, the Hölder continuity follows immediately assuming  $q$  is large enough. We need to show that (6.58) and (6.62) imply  $X_l^\epsilon u^\epsilon \in DG(\chi, q, \gamma)$ . To do this we need to prove a result analogue to [DiB83, Proposition 4.1]:

**Lemma 6.64.** *In the notation established above, there exists  $\tau > 0$  depending on  $Q, \lambda, \Lambda, r_0$  such that if for at least one  $k = 1, 2, 3$ ,*

$$\left| \left\{ x \in B(x, r) \text{ such that } X_k^\epsilon u^\epsilon < \frac{1}{8} \text{osc}_{B(x, 2r)} |\nabla_\epsilon u^\epsilon| \right\} \right| \leq \tau r^Q,$$

then

$$\sup_{B(x, \frac{r}{2})} X_k^\epsilon u^\epsilon \geq \frac{\text{osc}_{B(x, 2r)} |\nabla_\epsilon u^\epsilon|}{100}.$$

Analogously, if for at least one  $k = 1, 2, 3$ ,

$$\left| \left\{ x \in B(x, r) \text{ such that } X_k^\epsilon u^\epsilon > -\frac{1}{8} \text{osc}_{B(x, 2r)} |\nabla_\epsilon u^\epsilon| \right\} \right| \leq \tau r^Q,$$

then

$$\sup_{B(x, \frac{r}{2})} X_k^\epsilon u^\epsilon \leq -\frac{\text{osc}_{B(x, 2r)} |\nabla_\epsilon u^\epsilon|}{100}.$$

This result is proved exactly as in [DiB83, Proposition 4.1] (see also [Zho09, Lemma 4.4]) and it yields essentially the equivalence

$$(\delta + \mu(2r)^2)^{\frac{Q-2}{2}} \approx (\delta + |\nabla_\epsilon u^\epsilon|^2)^{\frac{Q-2}{2}},$$

for all  $x \in B(x_0, r)$ , when  $|\nabla_\epsilon u^\epsilon|$  is small with respect to  $\text{osc}_{B(x, 2r)} |\nabla_\epsilon u^\epsilon|$ . This equivalence, together with (6.58) and (6.62) implies  $X_l^\epsilon u^\epsilon \in DG(\chi, q, \gamma)$ , thus concluding the proof of the Hölder regularity of the gradient in Theorem 6.36.  $\square$

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