A PRIMER ON CARNOT GROUPS: HOMOGENOUS GROUPS, CC SPACES, AND REGULARITY OF THEIR ISOMETRIES

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Abstract. Carnot groups are distinguished spaces that are rich of structure: they are those Lie groups equipped with a path distance that is invariant by left-translations of the group and admit automorphisms that are dilations with respect to the distance. We present the basic theory of Carnot groups together with several remarks. We consider them as special cases of graded groups and as homogeneous metric spaces. We discuss the regularity of isometries in the general case of Carnot-Carathéodory spaces and of nilpotent metric Lie groups.

Contents

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This essay is a survey written for a summer school on metric spaces.

Carnot groups are special cases of Carnot-Carathéodory spaces associated with a system of bracket-generating vector fields. In particular, they are geodesic metric spaces. Carnot groups are also examples of homogeneous groups and stratified groups. They are simply connected nilpotent groups and their Lie algebras admit special gradings: stratifications. The stratification can be used to define a left-invariant metric on each group in such a way that with respect to this metric the group is self-similar. Namely, there is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under Euclidean dilations.

Carnot groups, and more generally homogeneous groups and Carnot-Carathéodory spaces, appear in several mathematical contexts. Such groups appear in harmonic analysis, in the study of hypoelliptic differential operators, as boundaries of strictly pseudo-convex complex domains, see the books [Ste93, CDPT07] as initial references. Carnot groups, with sub-Finsler distances, appear in geometric group theory as asymptotic cones of nilpotent finitely generated groups, see [Gro96, Pan89]. SubRiemannian Carnot groups are limits of Riemannian manifolds and are metric tangents of subRiemannian manifolds. SubRiemannian geometries arise in many areas of pure and applied mathematics (such as algebra, geometry, analysis, mechanics, control theory, mathematical physics), as well as in applications (e.g., robotics), for references see the book [Mon02]. The literature on geometry and analysis on Carnot groups is plentiful. In addition to the previous references, we also cite some among the more authoritative ones [RS76, FS82, NSW85, KR85, VSCC92, Hei95, Mag02, Vit08, BLU07, Jea14, Rif14, ABB15].

The setting of Carnot groups has both similarities and differences compared to the Euclidean case. In addition to the geodesic distance and the presence of dilations and translations, on each Carnot group one naturally considers Haar measures, which are unique up to scalar multiplication and are translation invariant. Moreover, in this setting they have the important property of being Ahlfors-regular measures. In fact, the measure of each r-ball is the measure of the unit ball multiplied by r^Q, where Q is an integer depending only on the group. With such a structure of metric measure space, it has become highly interesting to study geometric measure theory, and other aspects of analysis or geometry, in Carnot groups. On the one hand, with so much structure many results in the Euclidean setting generalize to Carnot groups. The most celebrated example is Pansu's version of Rademacher's theorem for Lipschitz maps (see [Pan89]). Other results that have been generalized are the Isoperimetric Inequality [Pan82b], the Poincaré Inequality [Jer86], the Nash Embedding Theorem [LD13], the Myers-Steenrod Regularity Theorem [CL14], and, at least partially, the De-Giorgi Structure Theorem [FSS03]. On the other hand, Carnot groups exhibit fractal behavior, the reason being that on such metric spaces the only curves of finite length are the horizontal ones. In fact, except for the Abelian groups, even if the distance is defined by a smooth subbundle, it is not smooth and the Hausdorff dimension differs from the topological dimension. Moreover, these spaces contain no subset of positive measure that is biLipschitz equivalent to a subset of a Euclidean space and do not admit biLipschitz embedding into reflexive Banach spaces, nor into L^1, see [Sem96b, AK00a, CK06, CK10a, CK10b]. For this reason, Carnot groups, together with the classical fractals and boundaries of hyperbolic groups, are the main examples in an emerging field called 'non-smooth analysis', in addition see [Sem96a, HK98, Che99, LP01, Hei01, Laa02, GS92, BGP92, CC97, BM91, MM95, KL97, Gro99, BP00, KB02, BK02]. A non-exhaustive, but long-enough list of other contribution in geometric measure theory on Carnot groups is [Pan82a, KR85, KR95, AK00b, HK00, Amb01, Hei01, Amb02, Pau04, KS04, CP06, ASV06, BSV07, MSV08, DGN08, MV08, Mag08, RR08, Rit09, AKL09, MZGZ09, LZ13, BeL13, LR14, LN15, LR15], and more can be found in [SC16, LD15a].

In this essay, we shall distinguish between Carnot groups, homogeneous groups, and stratified groups. In fact, the latter are Lie groups with a particular kind of grading, i.e., a stratification. Hence, they are only an algebraic object. Instead, homogeneous groups are Lie groups that are arbitrarily graded but are equipped with distances that are one-homogeneous with respect to the dilations induced by the grading (see (0.1)). Finally, for the purpose of this paper, the term Carnot group will be reserved for those stratified groups that are equipped with homogeneous distances making them Carnot-Carathéodory spaces. It is obvious that up to biLipschitz equivalence every Carnot group has one unique geometric structure. This is the reason why this term sometimes replaces the term stratified group.

From the metric viewpoint, Carnot groups are peculiar examples of isometrically homogeneous spaces. In this context we shall differently use the term 'homogeneous': it means the presence of a transitive group action. In fact, on Carnot groups left-translations act transitively and by isometries. In some sense, these groups are quite prototypical examples of geodesic homogeneous spaces. An interesting result of Berestovskii gives that every isometrically homogeneous space whose distance is geodesic is indeed a Carnot-Carathéodory space, see Theorem 4.5. Hence, Carnot-Carathéodory spaces and Carnot groups are natural examples in metric geometry. This is the reason why they appear in different contexts.

A purpose of this essay is to present the notion of a Carnot group from different viewpoints. As we said, Carnot groups have very rich metric and geometric structures. However, they are easily characterizable as geodesic spaces with self-similarity. Here is an equivalent definition, which is intermediate between the standard one (Section 2.3) and one of the simplest axiomatic ones (Theorem 5.2). A *Carnot group* is a Lie group G endowed with a left-invariant geodesic distance d admitting for each $\lambda > 0$ a bijection $\delta_\lambda : G \to G$ such that

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q), \qquad \forall p, q \in G. \tag{0.1}$$

In this paper we will pursue an Aristotelian approach: we will begin by discussing the examples, starting from the very basic ones. We first consider Abelian Carnot groups, which are the finite-dimensional normed spaces, and then the basic non-Abelian group: the Heisenberg group. After presenting these examples, in Section 1 we will formally discuss the definitions from the algebraic viewpoint, with some basic properties. In particular, we show the uniqueness of stratifications in Section 1.2. In Section 2 we introduce the homogeneous distances and the general definition of Carnot-Carathéodory spaces. In Section 2.5 we show the continuity of homogeneous distances with respect to the manifold topology. Section 3 is devoted to present Carnot groups as limits. In particular, we consider tangents of Carnot-Carathéodory spaces. In Section 4 we discuss isometrically homogeneous spaces and explain why Carnot-Carathéodory spaces are the only geodesic examples. In Section 5 we provide a metric characterization of Carnot groups as the only spaces that are locally compact geodesic homogeneous and admit a dilation. Finally, in Section 6 we overview several results that provide the regularity of distance-preserving homeomorphisms in Carnot groups and more generally in homogeneous groups and Carnot-Carathéodory spaces.

0. Prototypical examples

We start by reviewing basic examples of Carnot groups. First we shall point out that Carnot groups of step one are in fact finite-dimensional normed vector spaces. Afterwards, we shall recall the simplest non-commutative example: the Heisenberg group, equipped with various distances.

Example 0.1. Let V be a finite dimensional vector space, so V is isomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}$. In such a space we have

- a group operation (sum) $p, q \mapsto p + q$,
- dilations $p \mapsto \lambda p$ for each factor $\lambda > 0$.

We are interested in the distances d on V that are

(i) translation invariant:

$$d(p + q, p + q') = d(q, q'), \qquad \forall p, q, q' \in V$$

(ii) one-homogeneous with respect to the dilations:

$$d(\lambda p, \lambda q) = \lambda d(p, q), \qquad \forall p, q \in V, \forall \lambda > 0.$$

These are the distances coming from norms on V. Indeed, setting $\|v\| = d(0, v)$ for $v \in V$, the axioms of d being a distance together with properties (i) and (ii), give that $\|\cdot\|$ is a norm.

A geometric remark: for such distances, straight segments are geodesic. By definition, a geodesic is an isometric embedding of an interval. Moreover, if the norm is strictly convex, then straight segments are the only geodesic.

An algebraic remark: each (finite dimensional) vector space can be seen as a Lie algebra (in fact, a commutative Lie algebra) with Lie product $[p, q] = 0$, for all $p, q \in V$. Via the obvious identification points/vectors, this Lie algebra is identified with its Lie group $(V, +)$.

One of the theorem that we will generalize in Section 6 is the following.

Theorem 0.2. *Every isometry of a normed space fixing the origin is a linear map.*

An analytic remark: the definition of (directional) derivatives

$$\lim_{h\to 0}\frac{f(x+hy)-f(x)}{h}=\frac{\partial f}{\partial y}(x)$$

of a function f between vector spaces makes use of the group operation, the dilations, and the topology. We shall consider more general spaces on which these operations are defined.

Example 0.3. Consider $\mathbb{R}^3$ with the standard topology and differentiable structure. Consider the operation

$$(\bar{x},\bar{y},\bar{z})\cdot(x,y,z):=\left(\bar{x}+x,\bar{y}+y,\bar{z}+z+\tfrac{1}{2}(\bar{x}y-\bar{y}x)\right).$$

This operation gives a non-Abelian group structure on $\mathbb{R}^3$. This group is called *Heisenberg group.*

The maps $\delta_\lambda(x,y,z)=(\lambda x,\lambda y,\lambda^2 z)$ give a one-parameter family of group homomorphisms:

$$\delta_\lambda(pq)=\delta_\lambda(p)\delta_\lambda(q)\qquad\text{and}\qquad\delta_\lambda\circ\delta_\mu=\delta_{\lambda\mu}.$$

We shall consider distances that are

(i) left (translation) invariant:

$$d(p\cdot q,p\cdot q')=d(q,q'),\qquad\forall p,q,q'\in G$$

(ii) one-homogeneous with respect to the dilations:

$$d(\delta_\lambda(p),\delta_\lambda(q))=\lambda d(p,q),\qquad\forall p,q\in G,\forall\lambda>0.$$

These distances, called ‘homogeneous’, are completely characterized by the distance from a point, in fact, just by the sphere at a point.

Example 0.3.1. (Box distance) Set

$$d_{box}(0,p):=\|p\|_{box}:=\max\left\{|x_p|,|y_p|,\sqrt{|z_p|}\right\}.$$

This function is δ_λ-homogeneous and satisfies the triangle inequality. To check that it satisfies the triangle inequality we need to show that $\|p\cdot q\|\leq\|p\|+\|q\|$. First,

$$|x_{p\cdot q}|=|x_p+x_q|\leq|x_p|+|x_q|\leq\|p\|+\|q\|,$$

and analogously for the y component. Second,

$$\begin{aligned}\sqrt{|z_{p\cdot q}|}&=\sqrt{\left|z_p+z_q+\frac{1}{2}(x_py_q-x_qy_p)\right|}\\&\leq\sqrt{|z_p|+|z_q|+|x_p||y_q|+|x_q||y_p|}\\&\leq\sqrt{\|p\|^2+\|q\|^2+2\|p\|\|q\|}\qquad=\|p\|+\|q\|.\end{aligned}$$

The group structure induces left-invariant vector fields, a basis of which is

$$X=\partial_1-\frac{y}{2}\partial_3,\qquad Y=\partial_2+\frac{x}{2}\partial_3,\qquad Z=\partial_3.$$

Example 0.3.2. (Carnot-Carathéodory distances) Fix a norm $\|\cdot\|$ on $\mathbb{R}^2$, e.g., $\|(a,b)\| = \sqrt{a^2+b^2}$. Consider

$$d(p,q) := \inf \left\{ \int_0^1 \|(a(t), b(t))\| \, \mathrm{d}t \right\},$$

where the infimum is over the piece-wise smooth curves $\gamma \in C^\infty_{\mathrm{pw}}([0,1];\mathbb{R}^3)$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\dot\gamma(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$. Such a d is a homogenous distance and for all $p, q \in \mathbb{R}^3$ there exists a curve γ realizing the infimum. These distances are examples of CC distances also known as subFinsler distances.

Example 0.3.3. (Korányi distance) Another important example is given by the Cygan-Korányi distance:

$$d_K(0,p) := \|p\|_K := \left((x_p^2 + y_p^2)^2 + 16\, z_p^2\right)^{1/4}.$$

The feature of such a distance is that it admits a conformal inversion, see [CDPT07, p.27].

The space $\mathbb{R}^3$ with the above group structure is an example of Lie group, i.e., the group multiplication and the group inversion are smooth maps. The space $\mathfrak{g}$ of left-invariant vector fields (also known as the Lie algebra) has a peculiar structure: it admits a stratification. Namely, setting

$$V_1 := \mathrm{span}\{X, Y\} \qquad \text{and} \qquad V_2 := \mathrm{span}\{Z\},$$

we have

$$\mathfrak{g} = V_1 \oplus V_2, \quad [V_1, V_1] = V_2, \quad [V_1, V_2] = \{0\}.$$

As an exercise, verify that $Z = [X, Y]$.

1. Stratifications

In this section we discuss Lie groups, Lie algebras, and their stratifications. We recall the definitions and we point out a few remarks. In particular, we show that a group can be stratified in a unique way, up to isomorphism.

1.1. **Definitions.** Given a group G we denote by gh or $g \cdot h$ the product of two elements $g, h \in G$ and by g^{-1} the inverse of g. A *Lie group* is a differentiable manifold endowed with a group structure such that the map $G \times G \to G$, $(g,h) \mapsto g^{-1} \cdot h$ is C^∞. We shall denote by e the identity of the group and by $L_g(h) := g \cdot h$ the left translation. Any vector X in the tangent space at the identity extends uniquely to a left-invariant vector field $\tilde X$, as $\tilde X_g = (dL_g)_e X$, for $g \in G$.

The *Lie algebra associated with a Lie group* G is the vector space T_eG equipped with the bilinear operation defined by $[X, Y] := [\tilde X, \tilde Y]_e$, where the last bracket denotes the Lie bracket of vector fields, i.e., $[\tilde X, \tilde Y] := \tilde X \tilde Y - \tilde Y \tilde X$.

The general notion of Lie algebra is the following: A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space together with a bilinear operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the *Lie bracket*, such that, for all $x, y, z \in \mathfrak{g}$, one has

(1) anti-commutativity: $[x, y] = -[y, x]$,
(2) Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

All Lie algebras considered here are over $\mathbb{R}$ and finite-dimensional. In what follows, given two subspaces V, W of a Lie algebra, we set $[V,W] := \mathrm{span}\{[X,Y];\ X \in V,\ Y \in W\}$.

Definition 1.1 (Stratifiable Lie algebras)**.** A *stratification* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

for some integer $s \geq 1$, where $V_s \neq \{0\}$ and $[V_1, V_j] = V_{j+1}$ for all integers $j \in \{1,\ldots,s\}$ and where we set $V_{s+1} = \{0\}$. We say that a Lie algebra is *stratifiable* if there exists a stratification of it. We say that a Lie algebra is *stratified* when it is stratifiable and endowed with a fixed stratification called the *associated stratification*.

A stratification is a particular example of grading. Hence, for completeness, we proceed by recalling the latter. More considerations on this subject can be found in [LR15].

Definition 1.2 (Positively graduable Lie algebras)**.** A *positive grading* of a Lie algebra $\mathfrak{g}$ is a family $(V_t)_{t\in(0,+\infty)}$ of linear subspaces of $\mathfrak{g}$, where all but finitely many of the V_t's are $\{0\}$, such that $\mathfrak{g}$ is their direct sum

$$\mathfrak{g} = \bigoplus_{t\in(0,+\infty)} V_t$$

and where

$$[V_t, V_u] \subset V_{t+u}, \qquad \text{for all } t,u > 0.$$

We say that a Lie algebra is *positively graduable* if there exists a positive grading of it. We say that a Lie algebra is *graded* (or *positively graded*, to be more precise) when it is positively graduable and endowed with a fixed positive grading called the *associated positive grading*.

Given a positive grading $\mathfrak{g} = \oplus_{t>0} V_t$, the subspace V_t is called the *layer of degree t* (or *degree-t layer*) of the positive grading and non-zero elements in V_t are said to *have degree t*. The *degree* of the grading is the maximum of all positive real numbers $t > 0$ such that $V_t \neq \{0\}$.

Given two graded Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with associated gradings $\mathfrak{g} = \oplus_{t>0} V_t$ and $\mathfrak{h} = \oplus_{t>0} W_t$, a *morphism of the graded Lie algebras* is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi(V_t) \subseteq W_t$ for all $t > 0$. Hence, two graded Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic as graded Lie algebras* if there exists a bijection $\phi : \mathfrak{g} \to \mathfrak{h}$ such that both ϕ and ϕ^{-1} are morphisms of the graded Lie algebras.

Recall that for a Lie algebra $\mathfrak{g}$ the terms of the lower central series are defined inductively by $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}^{(s+1)} = \{0\}$ for some integer $s \geq 1$ and more precisely we say that $\mathfrak{g}$ *nilpotent of step s* if $\mathfrak{g}^{(s+1)} = \{0\}$ but $\mathfrak{g}^{(s)} \neq \{0\}$.

Remark 1.3. A positively graduable Lie algebra is nilpotent (simple exercise similar to Lemma 1.16). On the other hand, not every nilpotent Lie algebra is positively graduable, see Exercise 1.8. A stratification of a Lie algebra $\mathfrak{g}$ is equivalent to a positive grading whose degree-one layer generates $\mathfrak{g}$ as a Lie algebra. Therefore, given a stratification, the degree-one layer uniquely determines the stratification and satisfies

$$\mathfrak{g} = V_1 \oplus [\mathfrak{g}, \mathfrak{g}]. \tag{1.4}$$

However, an arbitrary vector space V_1 that is in direct sum with $[\mathfrak{g},\mathfrak{g}]$ (i.e., satisfying (1.4)) may not generate a stratification, see Example 1.9. Any two stratifications of a Lie algebra are isomorphic, see Section 1.2. A stratifiable Lie algebra with s non-trivial layers is nilpotent of step s. Every 2-step nilpotent Lie algebra is stratifiable. However, not all graduable Lie algebras are stratifiable, see Example 1.7. Stratifiable and graduable Lie algebras admit several different grading: given a grading $\oplus V_t$, one can define the so-called s-power as the new grading $\oplus W_t$ by setting $W_t = V_{t/s}$, where $s > 0$. Moreover, for a stratifiable algebra it is not true that any grading is a power of a stratification, see Example 1.6.

Example 1.5. Free-nilpotent Lie algebras are stratifiable. Namely, having fixed $r, s \in \mathbb{N}$, one considers formal elements $e_1, \dots, e_r$ and all possible formal iterated Lie bracket up to length s, modulo the anti-commutativity and Jacobi relations, e.g., $[e_1, e_2]$ equals $-[e_2, e_1]$ and both have length 2. The span of such vectors is the *free-nilpotent Lie algebra of rank r and step s*. The strata of a stratification of such an algebra are formed according to the length of formal bracket considered. Any nilpotent Lie algebra is a quotient of a free-nilpotent Lie algebra, however, the stratification may not pass to this quotient.

Example 1.6. The Heisenberg Lie algebra $\mathfrak{h}$ is the 3-dimensional lie algebra spanned by three vectors X, Y, Z and with only non-trivial relation $Z = [X, Y]$, cf Example 0.3. This stratifiable algebra also admits positive gradings that are not power of stratifications. In fact, for $\alpha \in (1, +\infty)$, the *non-standard grading of exponent α* is

$$\mathfrak{h} = W_1 \oplus W_\alpha \oplus W_{\alpha+1}$$

where

$$W_1 := \operatorname{span}\{X\}, \quad W_\alpha := \operatorname{span}\{Y\}, \quad W_{\alpha+1} := \operatorname{span}\{Z\}.$$

Up to isomorphisms of graded Lie algebras and up to powers, these non-standard gradings give all the possible positive gradings of $\mathfrak{h}$ that are not a stratification.

Example 1.7. Consider the 7-dimensional Lie algebra $\mathfrak{g}$ generated by $X_1, \dots, X_7$ with only non-trivial brackets

$$\begin{array}{rclrclrcl}
[X_1, X_2] &=& X_3, & [X_1, X_3] &=& 2X_4, & [X_1, X_4] &=& 3X_5 \\
[X_2, X_3] &=& X_5, & [X_1, X_5] &=& 4X_6, & [X_2, X_4] &=& 2X_6 \\
[X_1, X_6] &=& 5X_7, & [X_2, X_5] &=& 3X_7, & [X_3, X_4] &=& X_7.
\end{array}$$

This Lie algebra $\mathfrak{g}$ admits a grading but it is not stratifiable.

Example 1.8. There exist nilpotent Lie algebras that admit no positive grading. For example, consider the Lie algebra of dimension 7 with basis $X_1, \dots, X_7$ with only non-trivial relations given by

$$[X_1, X_j] = X_{j+1} \text{ if } 2 \le j \le 6, \;\; [X_2, X_3] = X_6, \;\; [X_2, X_4] = -[X_2, X_5] = [X_3, X_4] = X_7.$$

Example 1.9. We give here an example of a stratifiable Lie algebra $\mathfrak{g}$ for which one can find a subspace V in direct sum with $[\mathfrak{g},\mathfrak{g}]$ but that does not generate a stratification. We consider $\mathfrak{g}$ the stratifiable Lie algebra of step 3 generated by e_1, e_2 and e_3 and with the relation $[e_2, e_3] = 0$. Then $\dim \mathfrak{g} = 10$ and a stratification of $\mathfrak{g}$ is generated by $V_1 := \operatorname{span}\{e_1, e_2, e_3\}$. Taking $V := \operatorname{span}\{e_1, e_2 + [e_1, e_2], e_3\}$, one has (1.4), but since $[V, V]$ and $[V, [V, V]]$ both contain $[e_2, [e_1, e_3]]$, V does not generate a stratification of $\mathfrak{g}$.

Definition 1.10 (Positively graduable, graded, stratifiable, stratified groups)**.** We say that a Lie group G is a *positively graduable* (respectively *graded*, *stratifiable*, *stratified*) *group* if G is a connected and simply connected Lie group whose Lie algebra is positively graduable (respectively graded, stratifiable, stratified).

For the sake of completeness, in Theorem 1.14 below we present an equivalent definition of positively graduable groups in terms of existence of a contractive group automorphism. The result is due to Siebert.

Definition 1.11 (Dilations on graded Lie algebras)**.** Let $\mathfrak{g}$ be a graded Lie algebra with associated positive grading $\mathfrak{g} = \oplus_{t>0} V_t$. For $\lambda > 0$, we define the *dilation on* $\mathfrak{g}$ *(relative to the associated positive grading) of factor* λ as the unique linear map $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta_\lambda(X) = \lambda^t X \qquad \forall X \in V_t.$$

Dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ are Lie algebra isomorphisms, i.e., $\delta_\lambda([X,Y]) = [\delta_\lambda X, \delta_\lambda Y]$ for all $X, Y \in \mathfrak{g}$. The family of all dilations $(\delta_\lambda)_{\lambda>0}$ is a one-parameter group of Lie algebra isomorphisms, i.e., $\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}$ for all $\lambda, \eta > 0$.

Exercise 1.12. Let $\mathfrak{g}$ and $\mathfrak{h}$ be graded Lie algebras with associated dilations $\delta_\lambda^{\mathfrak{g}}$ and $\delta_\lambda^{\mathfrak{h}}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then ϕ is a morphism of graded Lie algebras if and only if $\phi \circ \delta_\lambda^{\mathfrak{g}} = \delta_\lambda^{\mathfrak{h}} \circ \phi$, for all $\lambda > 0$. [Solution: Let $\mathfrak{g} = \oplus_{t>0} V_t$ be the grading of $\mathfrak{g}$. If $x \in V_t$ then $\phi(\delta_\lambda x) = \phi(\lambda^t x) = \lambda^t \phi(x)$, which gives the equivalence.]

Given a Lie group homomorphism $\phi : G \to H$, we denote by $\phi_* : \mathfrak{g} \to \mathfrak{h}$ the associated Lie algebra homomorphism. If G is simply connected, given a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group homomorphism $\phi : G \to H$ such that $\phi_* = \psi$ (see [War83, Theorem 3.27]). This allows us to define dilations on G as stated in the following definition.

Definition 1.13 (Dilations on graded groups)**.** Let G be a graded group with Lie algebra $\mathfrak{g}$. Let $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ be the dilation on $\mathfrak{g}$ (relative to the associated positive grading of $\mathfrak{g}$) of factor $\lambda > 0$. The *dilation on* G *(relative to the associated positive grading) of factor* λ is the unique Lie group automorphism, also denoted by $\delta_\lambda : G \to G$, such that $(\delta_\lambda)_* = \delta_\lambda$.

For technical simplicity, one keeps the same notation for both dilations on the Lie algebra $\mathfrak{g}$ and the group G. There will be no ambiguity here. Indeed, graded groups being nilpotent and simply connected, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from $\mathfrak{g}$ to G (see [CG90, Theorem 1.2.1] or [FS82, Proposition 1.2]) and one has $\delta_\lambda \circ \exp = \exp \circ \delta_\lambda$ (see [War83, Theorem 3.27]), hence dilations on $\mathfrak{g}$ and dilations on G coincide in exponential coordinates.

For the sake of completeness, we give now an equivalent characterization of graded groups due to Siebert. If G is a topological group and $\tau : G \to G$ is a group isomorphism, we say that τ is *contractive* if, for all $g \in G$, one has $\lim_{k\to\infty} \tau^k(g) = e$. We say that G is *contractible* if G admits a contractive isomorphism.

For graded groups, dilations of factor $\lambda < 1$ are contractive isomorphisms, hence positively graduable groups are contractible. Conversely, Siebert proved (see Theorem 1.14 below) that if G is a connected locally compact group and $\tau : G \to G$ is a contractive isomorphism then

G is a connected and simply connected Lie group and τ induces a positive grading on the Lie algebra $\mathfrak{g}$ of G (note however that τ itself may not be a dilation relative to the induced grading).

Theorem 1.14. [Sie86, Corollary 2.4] *A topological group G is a positively graduable Lie group if and only if G is a connected locally compact contractible group.*

Sketch of the proof. Regarding the non-trivial direction, by the general theory of locally compact groups one has that G is a Lie group. Hence, the contractive group isomorphism induces a contractive Lie algebra isomorphism ϕ. Passing to the complexified Lie algebra, one considers the Jordan form of ϕ with generalized eigenspaces V_α, $\alpha \in \mathbb{C}$. The t-layer V_t of the grading is then defined as the real part of the span of those V_α with $-\log|\alpha| = t$. $\square$

Remark 1.15. A distinguished class of groups in Riemannian geometry are the so-called Heintze's groups. They are those groups that admit a structure of negative curvature. It is possible to show that such groups are precisely the direct product of a graded group N times $\mathbb{R}$ where $\mathbb{R}$ acts on N via the grading.

1.2. **Uniqueness of stratifications.** We start by observing the following simple fact.

Lemma 1.16. *If $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ is a stratified Lie algebra, then*
$$\mathfrak{g}^{(k)} = V_k \oplus \cdots \oplus V_s.$$
In particular, $\mathfrak{g}$ is nilpotent of step s.

Now we show that the stratification of a stratifiable Lie algebra is unique up to isomorphism. Hence, also the structure of a stratified group is essentially unique.

Proposition 1.17. *Let $\mathfrak{g}$ be a stratifiable Lie algebra with two stratifications,*
$$V_1 \oplus \cdots \oplus V_s = \mathfrak{g} = W_1 \oplus \cdots \oplus W_t.$$
Then $s = t$ and there is a Lie algebra automorphism $A : \mathfrak{g} \to \mathfrak{g}$ such that $A(V_i) = W_i$ for all i.

Proof. We have $\mathfrak{g}^{(k)} = V_k \oplus \cdots \oplus V_s = W_k \oplus \cdots \oplus W_t$. Then $s = t$. Moreover, the quotient mappings $\pi_k : \mathfrak{g}^{(k)} \to \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ induce linear isomorphisms $\pi_k|_{V_k} : V_k \to \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ and $\pi_k|_{W_k} : W_k \to \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$. For $v \in V_k$ define $A(v) := (\pi_k|_{W_k})^{-1} \circ \pi_k|_{V_k}(v)$. Explicitly, for $v \in V_k$ and $w \in W_k$ we have
$$A(v) = w \quad \iff \quad v - w \in \mathfrak{g}^{(k+1)}.$$
Extend A to a linear map $A : \mathfrak{g} \to \mathfrak{g}$. This is clearly a linear isomorphism and $A(V_i) = W_i$ for all i. We need to show that A is a Lie algebra morphism, i.e., $[Aa, Ab] = A([a,b])$ for all $a, b \in \mathfrak{g}$. Let $a = \sum_{i=1}^s a_i$ and $b = \sum_{i=1}^s b_i$ with $a_i, b_i \in V_i$. Then
$$A([a,b]) = \sum_{i=1}^s \sum_{j=1}^s A([a_i, b_j])$$
$$[Aa, Ab] = \sum_{i=1}^s \sum_{j=1}^s [Aa_i, Ab_j],$$

therefore we can just prove $A([a_i, b_j]) = [Aa_i, Ab_j]$ for $a_i \in V_i$ and $b_j \in W_j$. Notice that $[a_i, b_j]$ belongs to V_{i+j} and $[Aa_i, Ab_j]$ belongs to W_{i+j}. Therefore we have $A([a_i, b_j]) = [Aa_i, Ab_j]$ if and only if $[a_i, b_j] - [Aa_i, Ab_j] \in \mathfrak{g}^{(i+j+1)}$. On the one hand, we have $a_i - Aa_i \in \mathfrak{g}^{(i+1)}$ and $b_j \in W_j$, so $[a_i - Aa_i, b_j] \in \mathfrak{g}^{(i+j+1)}$. On the other hand, we have $Aa_i \in W_i$ and $Ab_j - b_j \in \mathfrak{g}^{(j+1)}$, so $[Aa_i, Ab_j - b_j] \in \mathfrak{g}^{(i+j+1)}$. Hence, we have

$$[a_i, b_j] - [Aa_i, Ab_j] = [a_i - Aa_i, b_j] - [Aa_i, Ab_j - b_j] \in \mathfrak{g}^{(i+j+1)}. \qquad \square$$

2. Metric groups

In this section we review homogeneous distances on groups. The term *homogeneous* should not be confused with its use in the theory of Lie groups. Indeed, clearly a Lie group is a homogeneous space since it acts on itself by left translations and in fact we will only consider distances that are left-invariant. We shall use the term homogeneous as it is done in harmonic analysis to mean that there exists a transformation that dilates the distance. In the literature there is another definition of *homogeneous group*, but even if this definition is algebraic, it coincides with our definition. Namely, around 1970 Stein introduced a definition of homogeneous group as a graded group with degrees greater or equal than 1. After [HS90], we know that these are exactly those groups groups that admit a homogeneous distance, which is defined as follows.

Definition 2.1 (Homogeneous distance)**.** Let G be a Lie group. Let $\{\delta_\lambda\}_{\lambda\in(0,\infty)}$ be a one-parameter family of continuous automorphisms of G. A distance d on G is called *homogeneous* if

(i) it is *left-invariant*, i.e.,

$$d(p \cdot q, p \cdot q') = d(q, q'), \qquad \forall p, q, q' \in G;$$

(ii) it is *one-homogeneous* with respect to δ_λ, i.e.,

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q), \forall p, q \in G, \forall \lambda > 0.$$

Some remarks are due:

(1) Conditions (i) and (ii) imply that the distance is *admissible*, i.e., it induces the manifold topology, and in fact d is a continuous function[1];

(2) Because of the existence of one such a distance, the automorphisms δ_λ, with $\lambda \in (0, 1)$, are contractive.

(3) The existence of a contractive automorphism implies the existence of a positive grading of the Lie algebra of the group, see Theorem 1.14.

(4) The existence of a homogeneous distance implies that the smallest degree of the grading should be ≥ 1. These kind of groups are called *homogeneous* by Stein and collaborators.

(5) Any homogeneous group admits a homogeneous distance, see [HS90] and [LR15, Section 2.5]

[1] This claim is not at all trivial. However, in Section 2.5 we provide the proof in the case the group is positively graded and the maps $\{\delta_\lambda\}$ are the dilations relative to the grading.

2.1. **Abelian groups.** The basic example of a homogeneous group is provided by Abelian groups, which admit the stratification where the degree-one stratum V_1 is the whole Lie algebra. Homogenous distances for this stratification are the distances induced by norms, cf. Section 0.1.

2.2. **Heisenberg group.** Heisenberg group equipped with CC-distance, or box distance, or Korányi distance is the next important example of homogeneous group, cf. Section 0.3.

2.3. **Carnot groups.** Since on stratified groups the degree-one stratum V_1 of the stratification of the Lie algebra generates the whole Lie algebra, on these groups there are homogeneous distances that have a length structure defined as follows.

Let G be a stratified group. So G is a simply connected Lie group and its Lie algebra $\mathfrak{g}$ has a stratification: $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$. Therefore, we have $\mathfrak{g} = V_1 \oplus [V_1, V_1] \oplus \cdots \oplus V_1^{(s)}$.

Fix a norm $\|\cdot\|$ on V_1. Identify the space $\mathfrak{g}$ as T_eG so $V_1 \subseteq T_eG$. By left translation, extend V_1 and $\|\cdot\|$ to a left-invariant subbundle Δ and a left-invariant norm $\|\cdot\|$:

$$\Delta_p := (\mathrm{d}L_p)_e V_1 \qquad \text{and} \qquad \|(\mathrm{d}L_p)_e(v)\| := \|v\|, \qquad \forall p \in G, \forall v \in V_1.$$

Using piece-wise C^∞ curves tangent to Δ, we define the *CC-distance* associated with Δ and $\|\cdot\|$ as

$$d(p,q) := \inf \left\{ \int_0^1 \|\dot\gamma(t)\| \, \mathrm{d}t \;\middle|\; \gamma \in C^\infty_{\mathrm{pw}}([0,1];G), \begin{array}{l} \gamma(0) = p, \\ \gamma(1) = q, \end{array} \dot\gamma \in \Delta \right\}. \tag{2.2}$$

Since Δ and $\|\cdot\|$ are left-invariant, d is left-invariant. Since, $\delta_\lambda v = \lambda v$ for $v \in V_1$, and hence $\|\delta_\lambda v\| = \lambda \|v\|$ for $v \in \Delta$, the distance d is one-homogeneous with respect to δ_λ.

The fact that V_1 generates $\mathfrak{g}$, implies that for all $p, q \in G$ there exists a curve $\gamma \in C^\infty([0,1];G)$ with $\dot\gamma \in \Delta$ joining p to q. Consequently, d is finite valued.

We call the data $(G, \delta_\lambda, \Delta, \|\cdot\|, d)$ a *Carnot group*, or, more explicitly, *subFinsler Carnot group*. Usually, the term Carnot group is reserved for *subRiemannian Carnot group*, i.e., when the norm comes from a scalar product.

2.4. **Carnot-Carathéodory spaces.** Carnot groups are particular examples of a more general class of spaces. These spaces have been named after Carnot and Carathéodory by Gromov. However, in the work of Carnot and Carathéodory there is very little about these kind of geometries (one can find some sprout of a notion of contact structure in [Car09] and in the adiabatic processes' formulation of Carnot). Therefore, we should say that the pioneer work has been done mostly by Gromov and Pansu, see [Pan82a, Gro81, Pan82b, Pan83b, Pan83a, Str86, Pan89, Ham90], see also [Gro96, Gro99].

Let M be a smooth manifold. Let Δ be a subbundle of the tangent bundle of M. Let $\|\cdot\|$ be a 'smoothly varying' norm on Δ. Analogously, using (2.2) with G replaced by M, one defines the *CC-distance* associated with Δ and $\|\cdot\|$. Then $(M, \Delta, \|\cdot\|, d)$ is called *Carnot-Carathéodory space* (or also *CC-space* or *subFinsler manifold*).

2.5. **Continuity of homogeneous distances.** We want to motivate now the fact that a homogeneous distance on a group induces the correct topology. Such a fact is also true when one considers quasi distances and dilations that are not necessarily $\mathbb{R}$-diagonalizable automorphisms. For the sake of simplicity, we present here the simpler case of distances that are homogeneous with respect to the dilations relative to the associated positive grading. The argument is taken from [LR15]. The complete proof will appear in a forthcoming paper.

Proposition 2.3. *Every homogeneous distance induces the manifold topology.*

Proof in the case of standard dilations. Let G be a group with identity element e and Lie algebra graded by $\mathfrak{g} = \oplus_{s>0} V_s$. Let d be a distance homogeneous with respect to the dilations relative to the associated positive grading. It is enough to show that the topology induced by d and the manifold topology give the same neighborhoods of the identity.

We show that if p converges to e then $d(e,p) \to 0$. Since G is nilpotent, we can consider coordinates using a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ adapted to the grading. Namely, first for all $i = 1, \dots, n$ there is $d_i > 0$ such that $X_i \in V_{d_i}$, and consequently, $\delta_\lambda(X) = \lambda^{d_i} X$. There is a diffeomorphism $p \mapsto (P_1(p), \dots, P_n(p))$ from G to $\mathbb{R}^n$ such that for all $p \in G$

$$p = \exp(P_1(p)X_1) \cdot \ldots \cdot \exp(P_n(p)X_n).$$

Notice that $P_i(e) = 0$. Then, using the triangle inequality, the left invariance, and the homogeneity of the distance, we get

$$d(e,p) \leq \sum_i d(e, \exp(P_i(p)X_i)) = \sum_i |P_i(p)|^{1/d_i} d(e, \exp(X_i)) \to 0, \qquad \text{as } p \to e.$$

We show that if $d(e, p_n) \to 0$ then p_n converges to e. By contradiction and up to passing to a subsequence, there exists $\epsilon > 0$ such that $\|p_n\| > \epsilon$, where $\|\cdot\|$ denotes any auxiliary Euclidean norm. Since the map $\lambda \mapsto \|\delta_\lambda(q)\|$ is continuous, for all $q \in G$, then for all n there exists $\lambda_n \in (0,1)$ such that $\|\delta_{\lambda_n}(p_n)\| = \epsilon$. Since the Euclidean ϵ-sphere is compact, up to subsequence, $\delta_{\lambda_n}(p_n) \to q$, with $\|q\| = \epsilon$ so $q \neq e$ and so $d(e,q) > 0$. However,

$$0 < d(e,q) \leq d(e, \delta_{\lambda_n}(p_n)) + d(\delta_{\lambda_n}(p_n), q) = \lambda_n d(e, p_n) + d(\delta_{\lambda_n}(p_n), q) \to 0,$$

where at the end we used the fact, proved in the first part of this proof, that if q_n converges to q then $d(q_n, q) \to 0$. □

Remark 2.4. To deduce that a homogeneous distance is continuous it is necessary to require that the automorphisms used in the definition are continuous with respect to the manifold topology. Otherwise, consider the following example. Via a group isomorphism from $\mathbb{R}$ to $\mathbb{R}^2$, pull back the standard dilations and the Euclidean metric from $\mathbb{R}^2$ to $\mathbb{R}$. In this way we get a one-parameter family of dilations and a homogeneous distance on $\mathbb{R}^2$. Clearly, this distance gives to $\mathbb{R}^2$ the topology of the standard $\mathbb{R}$.

3. Limits of Riemannian manifolds

3.0.1. *A topology on the space of metric spaces.* Let X and Y be metric spaces, $L > 1$ and $C > 0$. A map $\phi : X \to Y$ is an (L,C)-*quasi-isometric embedding* if for all $x, x' \in X$

$$\frac{1}{L} d(x,x') - C \leq d(\phi(x), \phi(x')) \leq L d(x,x') + C.$$

If $A, B \subset Y$ are subsets of a metric space Y and $\epsilon > 0$, we say that A is an *ϵ-net* for B if

$$B \subset \mathrm{Nbhd}_\epsilon^Y(A) := \{y \in Y : d(x, A) < \epsilon\}.$$

Definition 3.1 (Hausdorff approximating sequence)**.** Let $(X_j, x_j), (Y_j, y_j)$ be two sequences of pointed metric spaces. A sequence of maps $\phi_j : X_j \to Y_j$ with $\phi(x_j) = y_j$ is said to be *Hausdorff approximating* if for all $R > 0$ and all $\delta > 0$ there exists ϵ_j such that

(1) $\epsilon_j \to 0$ as $j \to \infty$;
(2) $\phi_j|_{B(x_i,R)}$ is a $(1, \epsilon_j)$-quasi isometric embedding;
(3) $\phi_j(B(x_j, R))$ is an ϵ_j-net for $B(y_j, R - \delta)$.

Definition 3.2. We say that a sequence of pointed metric spaces (X_j, x_j) converges to a pointed metric space (Y, y) if there exists an Hausdorff approximating sequence $\phi_j : (X_j, x_j) \to (Y, y)$.

This notion of convergence was introduced by Gromov [Gro81] and it is also called *Gromov-Hausdorff convergence*. It defines a topology on the collection of (locally compact, pointed) metric spaces, which extends the notion of uniform convergence on compact sets of distances on the same topological space.

Definition 3.3. If $X = (X, d)$ is a metric space and $\lambda > 0$, we set $\lambda X = (X, \lambda d)$.

Let X, Y be metric spaces, $x \in X$ and $y \in Y$.

We say that (Y, y) is the *asymptotic cone* of X if for all $\lambda_j \to 0$, $(\lambda_j X, x) \to (Y, y)$.

We say that (Y, y) is the *tangent space* of X at x if for all $\lambda_j \to \infty$, $(\lambda_j X, x) \to (Y, y)$.

Remark 3.4. The notion of asymptotic cone is independent from x. In general, asymptotic cones and tangent spaces may not exist. In the space of boundedly compact metric spaces, limits are unique up to isometries.

3.1. **Tangents to CC-spaces: Mitchell Theorem.** The following result states that Carnot groups are infinitesimal models of CC-spaces: the tangent metric space to an equiregular subFinsler manifold is a subFinsler Carnot group. We recall that a subbundle $\Delta \subseteq TM$ is *equiregular* if for all $k \in \mathbb{N}$

$$\bigcup_{q \in M} \mathrm{span}\{[Y_1, [Y_2, [\ldots [Y_{l-1}, Y_l]]]]_q \ : \ l \leq k, Y_j \in \Gamma(\Delta), j = 1, \ldots, l\}$$

defines a subbundle of TM. For example, on a Lie group G any G-invariant subbundle is equiregular.

For the next theorem see [Mit85, Bel96, MM95, MM00, Jea14].

Theorem 3.5 (Mitchell)**.** *Let M be an equiregular subFinsler manifold and $p \in M$. Then the tangent space of M at p exists and is a subFinsler Carnot group.*

3.2. **Asymptotic cones of nilmanifolds.** We point out another theorem showing how Carnot groups naturally appear in Geometric Group Theory. The result is due to Pansu, and proofs can be found in [Pan83a, BrL13, BrL].

Theorem 3.6 (Pansu)**.** *Let G be a nilpotent Lie group equipped with a left-invariant subFinsler distance. Then the asymptotic cone of G exists and is a subFinsler Carnot group.*

4. Isometrically homogeneous geodesic manifolds

4.1. **Homogeneous metric spaces.** Let $X = (X, d)$ be a metric space. Let $\mathrm{Iso}(X)$ be the group of self-isometries of X, i.e., distance-preserving homeomorphisms of X.

Definition 4.1. A metric space X is *isometrically homogeneous* (or is a *homogeneous metric space*) if for all $x, x' \in X$ there exists $f \in \mathrm{Iso}(X)$ with $f(x) = x'$.

The main examples are the following. Let G be Lie group and $H < G$ compact subgroup. So the collection of the right cosets $G/H := \{gH : g \in G\}$ is an analytic manifold on which G acts (on the left) by analytic maps. Since H is compact, there always exists a G-invariant distance d on G/H.

Let us present some regularity results for isometries of such spaces. Later, in Section 6.1 we shall give more details.

Theorem 4.2 (LD, Ottazzi)**.** *The isometries of G/H as above are analytic maps.*

The above result is due to the author and Ottazzi, see [LO16]. The argument is based on the structure theory of locally compact groups, see next two sections. Later, in Theorem 6.1, we will discuss a stronger statement.

We present now an immediate consequence saying that in these examples the isometries are also Riemannian isometries for some Riemannian structure. In general, the Riemannian isometry group may be larger, e.g., in the case of $\mathbb{R}^2$ with ℓ_1-norm.

Corollary 4.3. *For each G-invariant distance d on G/H there is a Riemannian G-invariant distance d_R on G/H such that*

$$\mathrm{Iso}(G/H, d) < \mathrm{Iso}(G/H, d_R).$$

Sketch of the proof of the corollary. Fix $p \in G/H$ and ρ_p a scalar product on $T_p(G/H)$. Let K be the stabilizer of p in $\mathrm{Iso}(G/H, d)$, which is compact by Ascoli-Arzelà. Let μ be a Haar measure on K. Set

$$\tilde{\rho}_p = \int_K F^* \rho_p \, \mathrm{d}\mu(F)$$

and for all $q \in G/H$ set $\tilde{\rho}_q = F^* \tilde{\rho}_p$, for some $F \in \mathrm{Iso}(G/H, d)$ with $F(q) = p$. Then $\tilde{\rho}$ is a Riemannian metric tensor that is $\mathrm{Iso}(G/H, d)$-invariant. □

4.2. **Berestovskii's characterization.** Carnot groups are examples of isometrically homogeneous spaces. In addition, their CC-distances have the property of having been constructed by a length structure. Hence, these distances are intrinsic in the sense of the following definition. For more information on length structures and intrinsic distances, see [BBI01].

Definition 4.4 (Intrinsic distance)**.** A distance d on a set X is *intrinsic* if for all $x, x' \in X$

$$d(x, x') = \inf\{L_d(\gamma)\},$$

where the infimum is over all curves $\gamma \in C^0([0, 1]; X)$ from x to x' and

$$L_d(\gamma) := \sup \sum d(\gamma(t_i), \gamma(t_{i-1})),$$

where the supremum is over all partitions $t_0 < \ldots < t_k$ of $[0, 1]$.

Examples of intrinsic distances are given by length structures, subRiemannian structures, Carnot-Carathéodory spaces, and geodesic spaces. A metric space whose distance is intrinsic is called *geodesic* if the infimum in Definition 4.4 is always attained.

The significance of the next result is that CC-spaces are natural objects in the theory of homogeneous metric spaces.

Theorem 4.5 (Berestovskii, [Ber88])**.** *If a homogeneous Lie space G/H is equipped with a G-invariant intrinsic distance d, then d is subFinsler, i.e., there exist a G-invariant subbundle Δ and a G-invariant norm $\|\cdot\|$ such that d is the CC distance associated with Δ and $\|\cdot\|$.*

How to prove Berestovskii's result. In three steps:

Step 1. Show that locally d is $\geq$ to some Riemannian distance d_R.
Step 2. Deduce that curves that have finite length with respect to d are Euclidean rectifiable and define the horizontal bundle Δ using velocities of such curves. Definite $\|\cdot\|$ similarly.
Step 3. Conclude that d is the CC distance for Δ and $\|\cdot\|$. □

The core of the argument is in Step 1. Hence we describe its proof in an exemplary case.

Simplified proof of Step 1. We only consider the following simplification: $G = G/H = \mathbb{R}^2$, i.e., d is a translation invariant distance on the plane. Let d_E be the Euclidean distance. We want to show that d_E/d is locally bounded. If not, there exists $p_n \to 0$ such that $\dfrac{d_E(p_n,0)}{d(p_n,0)} > n$. Since the topologies are the same, there exists $r > 0$ such that the closure of $B_d(0,r)$ is contained in $B_E(0,1)$. Since $p_n \to 0$, there is $h_n \in \mathbb{N}$ such that eventually $h_n p_n \in B_E(0,1) \setminus B_d(0,r)$. Hence,

$$\begin{aligned} 0 < r < d(h_n p_n, 0) = h_n d(p_n, 0) &\leq \frac{h_n}{n} d_E(p_n, 0) \\ &= \frac{1}{n} d_E(h_n p_n, 0) \leq \frac{1}{n} \to 0, \end{aligned}$$

which gives a contradiction. □

5. A metric characterization of Carnot groups

5.1. **Characterizations of Lie groups.** Providing characterization of Lie groups among topological groups was one of the Hilbert problems: the 5^{th} one. Nowadays, it is considered solved in various forms by the work of John von Neumann, Lev Pontryagin, Andrew Gleason, Deane Montgomery, Leo Zippin, and Hidehiko Yamabe. See [MZ74] and references therein. For the purpose of characterizing Carnot groups among homogeneous metric spaces, we shall make use of the following.

Theorem 5.1 (Gleason - Montgomery - Zippin. 1950's)**.** *Let X be a metric space that is connected, locally connected, locally compact, of finite topological dimension, and isometrically homogeneous. Then its isometry group* $\mathrm{Iso}(X)$ *is a Lie group.*

As a consequence, the isometrically homogeneous metric spaces considered in Theorem 5.1 are all of the form discussed in Section 4 after Definition 4.1.

How to prove Theorem 5.1. In many steps:

Step 1. $\mathrm{Iso}(X)$ is locally compact, by Ascoli-Arzelá.
Step 2. Main Approximation Theorem: There exists an open and closed subgroup G of $\mathrm{Iso}(X)$ that can be approximated by Lie groups:

$$G = \varprojlim G_i, \qquad \text{(inverse limit of continuous epimorphisms with compact kernel)}$$

by Peter-Weyl and Gleason.
Step 3. $G \curvearrowright X$ transitively, by Baire; so $X = G/H = \varprojlim G_i/H_i$.
Step 4. Since the topological dimension of X is finite and X is locally connected, for i large $G_i/H_i \to G/H$ is a homeomorphism.
Step 5. $G = G_i$ for i large, so G is a Lie group so G is NSS, i.e., G has no small subgroups.
Step 6. G is NSS so $\mathrm{Iso}(X)$ is NSS
Step 7. Locally compact groups with NSS are Lie groups, by Gleason. □

5.2. **A metric characterization of Carnot groups.** In what follows, we say that a metric space (X, d) is *self-similar* if there exists $\lambda > 1$ such that the metric space (X, d) is isometric to the metric space $(X, \lambda d)$. In other words, there exists a homeomorphism $f : X \to X$ such that

$$d(f(p), f(q)) = \lambda d(p, q), \text{ for all } p, q \in X.$$

When this happens for all $\lambda > 0$ (and all maps $f = f_\lambda$ fix a point) X is said to be a *cone*. Homogeneous groups are examples of cones.

The following result is a corollary of the work of Gleason-Montgomery-Zippin, Berestovskii, and Mitchell, see [LD15b]. It gives a metric characterization of Carnot groups.

Theorem 5.2. *SubFinsler Carnot groups are the only metric spaces that are*

(1) *locally compact,*
(2) *geodesic,*
(3) *isometrically homogeneous, and*
(4) *self-similar.*

Sketch of the proof. Each such metric space X is connected and locally connected. Using the conditions of local compactness, self-similarity, and homogeneity, one can show that X is a doubling metric space. In particular, X is finite dimensional. By the result of Gleason-Montgomery-Zippin (Theorem 5.1) the space X has the structure of a homogeneous Lie space G/H and, by Berestovskii's result (Theorem 4.5), as a metric space X is an equiregular subFinsler manifold. By Mitchell's result (Theorem 3.5), the tangents of X are subFinsler Carnot groups. Since X is self-similar, X is isometric to its tangents. □

6. Isometries of metric groups

6.1. **Regularity of isometries for homogeneous spaces.** Let X be a metric space that is connected, locally connected, locally compact, with finite topological dimension, and isometrically homogeneous. By Montgomery-Zippin, $G := \mathrm{Iso}(X)$ has the structure of (analytic) Lie group, which is unique. Fixing $x_0 \in X$, the subgroup $H := \mathrm{Stab}_G(x_0)$ is compact.

Therefore, G/H has an induced structure of analytic manifolds. We have that X is homeomorphic to G/H and G acts on G/H analytically. One may wonder if X could have had a different differentiable structure. The next result says that this cannot happen.

Theorem 6.1 (LD, Ottazzi, [LO16]). *Let $M = G/H$ be a homogeneous manifold equipped with a G-invariant distance d, inducing the manifold topology. Then the isometry group* $\mathrm{Iso}(M)$ *is a Lie group, the action*

$$\begin{array}{rcl} \mathrm{Iso}(M) \times M & \to & M \\ (F, p) & \mapsto & F(p) \end{array} \tag{6.2}$$

is analytic, and, for all $p \in M$, the space $\mathrm{Iso}_p(M)$ *is a compact Lie group.*

The above result is just a consequence of Theorem 5.1 and the uniqueness of analytic structures for homogeneous Lie spaces, see [LO16, Proposition 4.5].

6.2. **Isometries of Carnot groups.** We want to explain in further details what are the isometries of Carnot groups. We summarize our knowledge by the following results.

- [MZ74] implies that the global isometries are smooth, since Carnot groups are homogeneous spaces.
- [CC06] implies that isometries between open subsets of Carnot groups are smooth, since 1-quasi-conformal maps are.
- [CL14] implies that isometries between equiregular subRiemannian manifolds are smooth.
- [LO16] implies that isometries between open subsets of subFinsler Carnot groups are affine, i.e., composition of translations and group homomorphisms.
- [KL16] implies that isometries of nilpotent connected Lie groups are affine.

In the rest of this exposition we give more explanation on the last three points.

6.3. **Local isometries of Carnot groups.**

Theorem 6.3 (LD, Ottazzi, [LO16]). *Let G_1, G_2 be subFinsler Carnot groups and for $i = 1, 2$ consider $\Omega_i \subset G_i$ open sets. If $F : \Omega_1 \to \Omega_2$ is an isometry, then there exists a left translation τ on G_2 and a group isomorphism ϕ between G_1 and G_2, such that F is the restriction to Ω_1 of $\tau \circ \phi$, which is a global isometry.*

Sketch of the proof. In four steps:

Step 1. We may assume that the distance is subRiemannian, regularizing the subFinsler norm.

Step 2. F is smooth; this is a PDE argument using the regularity of the subLaplacian, see the next section. Alternatively, one can use [CC06].

Step 3. F is completely determined by its horizontal differential $(\mathrm{d}F)_{H_e G}$, see the following Corollary 6.6.

Step 4. The Pansu differential $(PF)_e$ exists and is an isometry with the same horizontal differential as F at e, see [Pan89]. □

We remark that in Theorem 6.3 the assumption that Ω_i are open is necessary, unlike in the Euclidean case. However, these open sets are not required to be connected.

6.4. **Isometries of subRiemannian manifolds.** The regularity of subRiemannian isometries should be thought of as two steps, where as an intermediate result one obtains the preservation of a good measure: the Popp measure. A good introduction to the notion of Popp measure can be found in [BR13]. Both of the next results are due to the author in collaboration with Capogna, see [CL14].

Theorem 6.4 (LD, Capogna)**.** *Let $F : M \to N$ be an isometry between two subRiemannian manifolds. If there exist two C^∞ volume forms* vol_M *and* vol_N *such that $F_* \mathrm{vol}_M = \mathrm{vol}_N$, then F is a C^∞ diffeomorphism.*

Theorem 6.5 (LD, Capogna)**.** *Let $F : M \to N$ be an isometry between equiregular subRiemannian manifolds. If* vol_M *and* vol_N *are the Popp measures on M and N, respectively, then $F_* \mathrm{vol}_M = \mathrm{vol}_N$.*

How to prove Theorem 6.5. In two steps:

Step 1. Carnot group case: Popp is a Haar measure and hence a fixed multiple of the Hausdorff measure. The latter is a metric invariant.

Step 2. There is a representation formula ([ABB12, pages 358-359], [GJ14, Section 3.2]]) for the Popp volume in terms of the Hausdorff measure and the tangent measures in the tangent metric space, which is a Carnot group:

$$\mathrm{d}\,\mathrm{vol}_M = 2^{-Q}\mathcal{N}_p(\mathrm{vol}_M)(B_{\mathcal{N}_p(M)}(e,1))\mathrm{d}\mathcal{S}^Q_M.$$

One then makes use of Step 1. □

How to prove Theorem 6.4. In many steps:

Step 1. F is an isomorphism of metric *measure* spaces, so F preserves minimal upper gradients, Dirichlet energy, harmonic maps, and subLaplacian. Recall the horizontal gradient: If $X_1, \dots, X_m$ is an orthonormal frame for the horizontal bundle, define

$$\nabla_{\mathrm{H}} u := (X_1 u)X_1 + \ldots + (X_m u)X_m.$$

From the horizontal gradient one defines the subLaplacian: $\Delta_{\mathrm{H}} u = g$ means

$$\int_M gv \,\mathrm{d}\,\mathrm{vol}_M = \int_M \langle \nabla_{\mathrm{H}} u, \nabla_{\mathrm{H}} v\rangle \,\mathrm{d}\,\mathrm{vol}_M, \qquad \forall v \in \mathrm{Lip}_c(M),$$

We remark that $\Delta_{\mathrm{H}}(\cdot)$ depends on the choice of the measure vol_M.

Step 2. Hajlasz and Koskela's result, [HK00, page 51 and Section 11.2]: $\|\nabla_{\mathrm{H}} u\|$ coincides almost everywhere with the minimal upper gradient of u. Consequently, since $F_* \mathrm{vol}_M = \mathrm{vol}_N$,

$$\Delta_{\mathrm{H}} u = g \implies \Delta_{\mathrm{H}}(u \circ F) = g \circ F,$$

where the first subLaplacian is with respect to vol_N and the second one with respect to vol_M.

Step 3. Rothschild and Stein's version of Hörmander's Hypoelliptic Theorem, [RS76, Theorem 18]: Let $X_0, X_1, ..., X_r$ bracket generating vector fields in $\mathbb{R}^n$. Let u be a distributional solution to the equation $(X_0 + \sum_{i=1}^r X_i^2)u = g$ and let $k \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$. Then, considering the horizontal local Sobolev spaces $W^{k,p}_{\mathrm{H,loc}}$,

$$g \in W^{k,p}_{\mathrm{H}}(\mathbb{R}^n, \mathcal{L}^n) \implies u \in W^{k+2,p}_{\mathrm{H,loc}}(\mathbb{R}^n, \mathcal{L}^n).$$

Corollary: If vol_M is a C^∞ volume form,

$$\Delta_{\mathrm{H}} u \in W^{k,p}_{\mathrm{H,loc}}(M, \mathrm{vol}_M) \implies u \in W^{k+1,p}_{\mathrm{H,loc}}(M, \mathrm{vol}_M).$$

Step 4. Bootstrap argument: F isometry implies that $F \in W^{1,p}_{\mathrm{H}}$. Taking x_j some C^∞ coordinate system, we get that $\Delta_{\mathrm{H}} x_i =: g_i \in C^\infty \subset W^{k,p}_{\mathrm{H,loc}}$, for all k and p. Then, by Step 3

$$\Delta_{\mathrm{H}}(x_i \circ F) = g_i \circ F \in W^{1,p}_{\mathrm{H}} \implies x_i \circ F \in W^{2,p}_{\mathrm{H}}.$$

Then we iterate Rothschild and Stein's regularity of Step 3:

$$g_i \circ F \in W^{2,p}_{\mathrm{H,loc}} \implies x_i \circ F \in W^{3,p}_{\mathrm{H,loc}}$$

and by induction we complete. □

6.5. **SubRiemannian isometries are determined by the horizontal differential.**

Corollary 6.6. *Let M and N be two connected equiregular subRiemannian manifolds. Let $p \in M$ and let Δ be the horizontal bundle of M. Let $f, g : M \to N$ be two isometries. If $f(p) = g(p)$ and $df|_{\Delta_p} = dg|_{\Delta_p}$, then $f = g$.*

The proof can be read in [LO16, Proposition 2.8]. Once we know that the isometries fixing a point are a (smooth) compact Lie group, the argument is an easy exercise in differential geometry.

6.6. **Isometries of nilpotent groups.** The fact that Carnot isometries are affine (Theorem 6.3) is a general feature of the fact that we are dealing with a nilpotent group. In fact, isometries are affine whenever they are globally defined on a nilpotent connected group. Here we obviously require that the distances are left-invariant and induce the manifold topology. For example, this is the case for arbitrary homogeneous groups.

Theorem 6.7 (LD, Kivioja). *Let N_1 and N_2 be two nilpotent connected metric Lie groups. Any isometry $F\colon N_1 \to N_2$ is affine.*

This result is proved in [KL16] with algebraic techniques by studying the nilradical, i.e., the biggest nilpotent ideal, of the group of self-isometries of a nilpotent connected metric Lie group. The proof leads back to a Riemannian result of Wolf, see [Wol63].

6.7. **Two isometric non-isomorphic groups.** In general isometries of a subFinsler Lie group G may not be affine, not even in the Riemannian setting. As counterexample, we take the universal covering group $\tilde{G}$ of the group $G = E(2)$ of Euclidean motions of the plane. This group is also called *roto-translation* group. One can see that there exists a Riemannian distance on $\tilde{G}$ that makes it isometric to the Euclidean space $\mathbb{R}^3$. In particular, they have the same isometry group. However, a straightforward calculation of the automorphisms shows that not all isometries fixing the identity are group isomorphisms of $\tilde{G}$. As a side note, we remark that the group $\tilde{G}$ admits a left-invariant subRiemannian structure and a map into the subRiemannian Heisenberg group that is locally biLipschitz. However, these two spaces are not quasi-conformal, see [FKL14].

Examples of isometric Lie groups that are not isomorphic can be found also in the strict subRiemannian context. There are three-dimensional examples, see [AB12]. Also the analogue of the roto-translation construction can be developed.

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