A PIECEWISE KORN INEQUALITY IN SBD AND APPLICATIONS TO EMBEDDING AND DENSITY RESULTS

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Abstract. We present a piecewise Korn inequality for generalized special functions of bounded deformation ($GSBD^2$) in a planar setting generalizing the classical result in elasticity theory to the setting of functions with jump discontinuities. We show that for every configuration there is a partition of the domain such that on each component of the cracked body the distance of the function from an infinitesimal rigid motion can be controlled solely in terms of the linear elastic strain. In particular, the result implies that $GSBD^2$ functions have bounded variation after subtraction of a piecewise infinitesimal rigid motion. As an application we prove a density result in $GSBD^2$. Moreover, for all $d \geq 2$ we show $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d$ and the embedding $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d)$ into the space of special functions of bounded variation (SBV). Finally, we present a Korn-Poincaré inequality for functions with small jump sets in arbitrary space dimension.

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1. Introduction

A natural framework for the investigation of damage and fracture models in a geometrically linear setting is given by the space of *special functions of bounded deformation* investigated in [3, 5]. The space $SBD^p(\Omega)$ consists of all functions $u \in L^1(\Omega;\mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\rm sym}$-valued Radon measure which can be written as the sum of an $L^p(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ function $e(u) := \frac{1}{2}((\nabla u)^T + \nabla u)$ and a part concentrated on a rectifiable set J_u with finite $\mathcal{H}^{d-1}$ measure. Starting with the seminal paper [25] various variational problems for fracture mechanics in the realm of linearized elasticity have recently been investigated in the literature (see e.g. [6, 10, 11, 24, 41]), where the common ground of all these models is that the main energy term is essentially of the form

$$\int_\Omega |e(u)|^2\,dx + \mathcal{H}^{d-1}(J_u) \tag{1.1}$$

for $u \in SBD^2(\Omega)$. These so-called Griffith functionals comprise elastic bulk contributions for the unfractured regions of the body represented by the linear elastic strain $e(u)$ and surface terms that assign energy contributions on the crack paths comparable to the size of the 'jump set' J_u.

For technical reasons models are often formulated in $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$, e.g. in [5, 11, 41], since for this setting a compactness result in SBD was proved in [5, Theorem 1.1] and hereby the existence of solutions to minimization problems related to (1.1) is guaranteed. To overcome this restriction, the space of *generalized special functions of bounded deformation*, denoted by $GSBD(\Omega)$, was introduced in [18], admitting a compactness result under weaker assumptions and leading to the investigation of fracture models [28, 35] in a more general

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setting. (For an exact definition of $GSBD$, which is based on properties of one-dimensional slices, we refer to Section 3.3.)

On the one hand, models of the form (1.1) with linearized elastic energies are in general easier to treat than their nonlinear counterparts since in the regime of finite elasticity the energy density of the elastic contributions is genuinely geometrically nonlinear due to frame indifference rendering the problem highly non-convex. On the other hand, a major difficulty of these models in contrast to nonlinear problems formulated in the space SBV of *special functions of bounded variation* (see [4]) is given by the fact that one controls only the linear elastic strain.

Indeed, as discussed in [15], it appears that various properties being well established in SBV are only poorly understood in SBD due to the lack of control on the skew symmetric part of the distributional derivative $(Du)^T - Du$. Especially, to the best of our knowledge, for the *coarea formula* in BV (see [4, Theorem 3.40]), being useful in various applications as [7, 8, 19, 21, 26], no BD analog has been obtained in the literature.

Therefore, it is a natural and highly desirable issue to gain a deeper understanding of the relation between SBD and SBV or even to show that in certain circumstances SBD functions have bounded variation. In fact, one may expect that hereby various problems in SBD could be solvable by a reduction to corresponding results in SBV.

Clearly, a generic function of bounded deformation does not necessarily lie in SBV as one can already construct a function with $J_u = \emptyset$ such that $e(u) \in L^1(\Omega)$, but $\nabla u \notin L^1(\Omega)$ (cf. [14, 40]). Moreover, following the examples in [3, 18] one can define functions in $GSBD^2(\Omega)$ not having bounded variation (see Example 2.6 and Lemma 2.8 below), i.e. also the higher integrability for the elastic strain and the finiteness of the energy (1.1) is in general not sufficient. However, the examples involve unbounded configurations and it has therefore been conjectured that $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$ is a subspace of $SBV(\Omega;\mathbb{R}^d)$.

A profound understanding of the connection between SBV and SBD functions is directly related to the validity of a Korn-type inequality in the setting of functions exhibiting jump discontinuities. In elasticity theory Korn's inequality is the key estimate providing a relation between the symmetric and the full part of the gradient (see e.g. [39]). It states that the distance of ∇u from a skew symmetric matrix can be controlled solely in terms of the linear elastic strain $e(u)$. In the recently appeared contributions [16, 29] this well-know estimate has been generalized in a planar setting to configurations in SBD with small jump set controlling ∇u away from a small exceptional set.

However, for general functions of bounded deformation Korn's inequality in its basic form is doomed to fail. In fact, the domain may be disconnected into various parts by the jump set with completely different behavior on each component. In the special case that a brittle material does not store elastic energy, i.e. $e(u) = 0$, Chambolle, Giacomini, and Ponsiglione [13] showed that the body behaves piecewise rigidly, i.e. the only possibility that u is not an affine mapping is that the body is divided into at most countably many parts each of which subject to a different infinitesimal rigid motion.

Consequently, this observation already shows that an analogous statement of Korn's inequality in $(G)SBD$ has to be formulated in a considerably more complex way involving a partition of the domain. The problem is related to the result in [31], where a quantitative piecewise rigidity result in a geometrically nonlinear setting is established stating that in the planar case the distance of the deformation gradient from a piecewise rigid motion can be controlled.

The main goal of the present work is the derivation of an analogous result in the geometrically linear setting which we call a *piecewise Korn inequality.* We show in the planar case that for each $u \in GSBD^2(\Omega)$ there is an associated partition whose boundary length is controlled by

$\mathcal{H}^1(J_u)$ and a corresponding piecewise infinitesimal rigid motion a, being constant on each connected component of the cracked body, such that the distance of u and ∇u from a can be estimated in terms of $\|e(u)\|_{L^2(\Omega)}$. This result for configurations storing elastic energy and exhibiting cracks may be seen as a suitable combination of Korn's inequality for elastic materials and the qualitative result in [13].

The estimate proves to be useful to gain a deeper understanding of the relation between SBV and SBD functions. Although, as discussed above, $GSBD^2$ functions do not have bounded variation in general, the piecewise Korn inequality yields that after subtraction of a piecewise infinitesimal rigid motion the function lies in the space SBV^p for all $p < 2$. Hereby we particularly derive that each function in $GSBD^2$ has bounded variation away from an at most countable union of sets of finite perimeter with arbitrarily small Lebesgue measure.

It turns out that for the subspace of bounded functions this observation can be substantially improved. In this case it is possible to control the piecewise infinitesimal rigid motion in terms of $\|u\|_\infty$ and $\mathcal{H}^{d-1}(J_u)$ and we indeed obtain the embedding

$$SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d), \tag{1.2}$$

which is contrast to the piecewise Korn inequality is proved in arbitrary space dimension using a slicing technique. Moreover, similar arguments yield that without the L^∞-bound one may derive the inclusion $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d$ for $d \geq 2$. (See Section 3.3 below for the definition of *generalized functions of bounded variation*.) In particular, from (1.2) we deduce that in the space of bounded SBD^2 functions, being a natural and widely adopted space in the investigation of linear fracture models, the *coarea formula* is applicable.

Apart from the derivation of embeddings a major motivation for the derivation of the piecewise Korn inequality are applications to Griffith models. We show that the embedding result and the coarea formula allow to derive a Korn-Poincaré inequality in SBD stating that for a $GSBD^2$ function with small jump set the distance from a single infinitesimal rigid motion can be controlled outside a small exceptional set. This estimate, established in arbitrary space dimension, enhances a result recently obtained by Chambolle, Conti, and Francfort [12] in the sense that also the length of the boundary of the exceptional set can be bounded in terms of $\mathcal{H}^{d-1}(J_u)$ and therefore compactness results in $GSBD$ (see [18]) are applicable.

We also present an approximation result for $GSBD^2$ functions in a planar setting improving [35] in the sense that no L^2-bound on the function is needed. Hereby we can complete the Γ-convergence result for the linearization of Griffith energies [28] by providing recovery sequences for every $GSBD^2$ function. Moreover, the main statement of this paper will be a fundamental ingredient to prove a general compactness theorem and to analyze quasistatic crack growth for energies of the form (1.1) (see [32]).

The major difficulties in the derivation of the result come from the fact we treat a full free discontinuity problem in the language of Ambrosio and De Giorgi [20] deriving an estimate without any a priori assumptions on the the crack geometry. Already simplified situations, in which the jump set decomposes the body into a finite number of sets with Lipschitz boundary, are subtle since there are no uniform bounds on the constants in Poincaré's and Korn's inequality. In fact, in the basic case of simply connected sets, in [30] a lot of effort was needed to derive a decomposition result into domains for which uniform bounds on the constants can be derived. Even more challenging difficulties occur for highly irregular jump sets forming, e.g., infinite crack patterns on various mesoscopic scales.

At first sight the derivation of the piecewise Korn inequality appears to be easier than the related problem [31] due to the geometrical linearity. However, whereas in [31] the statement is established only for a suitable, arbitrarily small modification of the configuration, in the present context the estimates hold for the original function, where this stronger result is indispensable to prove the announced embedding and approximation results. In particular, the modification scheme for the deformation and jump set, which was iteratively applied in [31] on mesoscopic scales becoming gradually larger, is not adequate in the present context and we need to use comparably different proof techniques.

The general strategy will be to first derive an auxiliary piecewise Korn inequality, which similarly as in [16, 29] holds up to a small exceptional set. Then the main result follows by an iterative application of the estimate on various mesocopic scales. For the derivation of the auxiliary statement we first identify the regions where the jump set is too large. Hereby we can (1) construct a partition of the domain into simply connected sets and (2) use the Korn inequality [16, 29] for functions with small jump sets to find a modification of the configuration, which is smooth on each component of the domain. To control the shape of the components we apply the main result of the paper [30] which allows to pass to a refined partition consisting of *John domains* with uniformly controlled John constant. The auxiliary statement then follows by using a Korn inequality for John domains (see e.g. [1]).

Let us remark that the piecewise Korn inequality is only proved in a planar setting since also the major proof ingredients, the Korn inequality for functions with small jump sets [16, 29] and the decomposition result [30], have only been shown in dimension two. Although the derivation of a higher dimensional analog seems currently out of reach, the majority of the applications in this contribution, namely the embedding (1.2) and the Korn-Poincaré inequality, hold in arbitrary dimensions due to slicing arguments.

The paper is organized as follows. In Section 2 we present the main results including the embedding and approximation for $GSBD^2$ functions. In Example 2.6 we also recall a standard construction for an SBD function not having bounded variation, which is based on the idea to cut out small balls from the bulk part. Hereby we see that in view of the piecewise Korn inequality examples of this type essentially represent the only possibility to define SBD^2 functions not lying in SBV. Moreover, the construction shows that the embedding (1.2) is sharp in the sense that the result is false if (1) SBV is replaced by SBV^p, $p > 1$, and if (2) L^∞ is replaced by L^q, $q < \infty$.

Section 3 is devoted to some preliminaries. We first recall the definition of John domains and formulate the decomposition result established in [30]. Then we state some properties of affine mappings being especially important for the derivation of (1.2), where we need fine estimates how affine mappings and its derivatives can by controlled in terms of the volume and diameter of the underlying set. Afterwards, we recall the definition of $(G)SBV$ and $(G)SBD$ functions and discuss basic properties.

In Section 4 we prove a version of the piecewise Korn inequality outside a small exceptional set. Together with the construction of a partition into simply connected sets we also provide a covering of Whitney-type such that the jump set J_u in each element of the covering is small. This then allows us to apply the inequality [29] in order to define a modification of the deformation being smooth in each part of the domain. We discuss properties of the covering and the exceptional set being crucial for the iterative application of the auxiliary result in Section 5.1.

Section 5 then contains the proof of the piecewise Korn inequality. In Section 5.1 we first prove the result for configurations in SBD with a regular jump set consisting of a finite number of segments. Then in Section 5.2 we derive the general case by an approximation argument similar to the one in [29]. Here we also prove our main approximation result.

Finally, Section 6 contains the proof of the embedding results and the Korn-Poincaré inequality. As an ingredient for the derivation of (1.2) we also provide a *piecewise Poincaré* *inequality*, which will allow to control the distance of configurations from piecewise infinitesimal rigid motions in terms of the L^∞-norm. For the latter as well as for the Korn-Poincaré inequality the coarea formula in BV will turn out to be a key ingredient.

2. Main results

2.1. **Piecewise Korn inequality.** Before we formulate the main results of this article, let us introduce some notions. We say a partition $(P_j)_{j=1}^\infty$ of $\Omega \subset \mathbb{R}^d$ is a *Caccioppoli partition* if each P_j is a set of finite perimeter such that $\sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$ (see Section 3.3 for details). Given corresponding affine mappings $a_j = a_{A_j,b_j} : \mathbb{R}^d \to \mathbb{R}^d$ with $a_j(x) = A_j\, x + b_j$ for $A_j \in \mathbb{R}^{d\times d}_{\rm skew}$, $b_j \in \mathbb{R}^d$, we say a_j is an *infinitesimal rigid motion* and the function $\sum_{j=1}^\infty a_j \chi_{P_j}$ is a *piecewise infinitesimal rigid motion.*

For the definition and properties of special functions of bounded variation (SBV) and deformation (SBD) we refer to Section 3.3. In particular, for $u \in SBD^2(\Omega)$ we denote by $e(u)$ the part of the strain $Eu = \frac{1}{2}(Du^T + Du)$ which is absolutely continuous with respect to $\mathcal{L}^d$ and let J_u be the jump set. Our main goal is to show the following result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary and $p \in [1,2)$. Then there is a constant $c = c(p) > 0$ and a constant $C > 0$ only depending on* $\mathrm{diam}(\Omega)$ *and p such that for each $u \in GSBD^2(\Omega)$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω with*

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)) \tag{2.1}$$

and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that

$$v := u - \sum\nolimits_{j=1}^\infty a_j \chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2) \tag{2.2}$$

and

$$\begin{aligned} &(i) \quad \|v\|_{L^\infty(\Omega)} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}, \\ &(ii) \quad \|\nabla v\|_{L^p(\Omega)} \le C\|e(u)\|_{L^2(\Omega)}. \end{aligned} \tag{2.3}$$

In its most general form the result holds in the generalized space $GSBD^2(\Omega)$, which loosely speaking arises from $SBD^2(\Omega)$ by requiring that one-dimensional slices have bounded variation. (We again refer to Section 3.3 for the exact definition.) The reader not interested in the generalized space may readily replace $GSBD^2(\Omega)$ by $SBD^2(\Omega)$ here and in the following (except for Lemma 2.8). The result will first be proved for configurations with a regular jump set consisting of a finite number of segments. Only at the very end we pass to the general case by using a density result (see Theorem 3.11, Lemma 5.4) and Ambrosio's compactness theorem in SBV. Consequently, the article is in large part accessible for readers without familiarity with the properties of the above function spaces.

Remark 2.2. Let us comment the results of Theorem 2.1.

(i) An important special case is given by functions not storing linearized elastic energy, i.e. $e(u) = 0$ a.e. Then (2.3) gives that u is a piecewise infinitesimal rigid motion and thus Theorem 2.1 can be interpreted as a *piecewise rigidity result.* It shows that the only way that global rigidity may fail is that the body is divided into at most countably many parts each of which subject to a different infinitesimal rigid motion. This result has been discussed for SBD functions in [13] and Theorem 2.1 may be seen as a quantititive extension in the $GSBD$ setting.

(ii) It is a well known fact that in general ∇u and u are not summable for $u \in GSBD^2(\Omega)$. The result shows that one may gain control over the function and its derivative after subtraction of a piecewise infinitesimal rigid motion. We hope that such an estimate will be useful to reduce various problems from the $(G)SBD$ to the SBV setting.

(iii) In contrast to [31], where a related result in a geometrically nonlinear setting is established, in Theorem 2.1 no modification of the configuration u is necessary. This will be crucial for the embedding results announced in Section 2.3.

(iv) In general one has $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j \setminus J_u) > 0$ even if J_u already induces a partition $(P'_j)_j$ of Ω with $J_u = \bigcup_j \partial^* P'_j \cap \Omega$ up to a set of negligible $\mathcal{H}^1$-measure. Indeed, it is well known that in irregular domains, e.g. in domains with external cusps, Korn's inequality may fail (cf. [34]). Also in the case that each P'_j is a Lipschitz set, $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j \setminus J_u) > 0$ might be necessary since possibly the constants $C_{\rm korn}(P'_j)$ involved in the $W^{1,p}$ Korn inequality tend to infinity as $j \to \infty$.

(v) Note that the constant in (2.3) depends on Ω only through its diameter and $p \in [1, 2)$. Consequently, the above result may be also interesting in the case of $u \in H^1(\Omega)$ and varying domains Ω.

(vi) At some points of the proof we have to pass to a slightly smaller exponent by Hölder's inequality and therefore we derive the result only for $p < 2$. In particular, we do not know if a similar result holds in the critical case $p = 2$. Note, however, that for the applications we have in mind, it is sufficient that (2.3) holds for $p = 1$.

The result is first shown up to a small exceptional set (see Theorem 4.1) and then Theorem 2.1 for functions with regular jump set follows from an iterative application of the estimate in Theorem 4.1 (see Section 5.1). Finally, the general version is proved by approximation arguments given in Section 5.2. For a more detailed outline of the proof we refer to the beginning of Section 4 and Section 5.

In Remark 2.2(ii) we have discussed that in general $GSBD$ functions are not summable. A similar problem occurs in $GSBV$ (see Section 3.3 for the exact definition). However, we get that $GSBV$ functions are integrable after subtraction of a *piecewise constant function*.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $\rho > 0$ and $m \in \mathbb{N}$. Then there is a constant $c = c(m) > 0$ such that for each $u \in (GSBV(\Omega;\mathbb{R}))^m$ with $\|\nabla u\|_{L^1(\Omega;\mathbb{R}^{d\times m})} + \mathcal{H}^{d-1}(J_u) < +\infty$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω and corresponding translations $(b_j)_{j=1}^\infty \subset \mathbb{R}^m$ such that $v := u - \sum_{j=1}^\infty b_j \chi_{P_j} \in SBV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and*

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=1}^\infty \mathcal{H}^{d-1}((\partial^* P_j \cap \Omega) \setminus J_u) \le c\rho, \\ &(ii) \quad \|v\|_{L^\infty(\Omega;\mathbb{R}^m)} \le c\rho^{-1} \|\nabla u\|_{L^1(\Omega;\mathbb{R}^{d\times m})}. \end{aligned} \tag{2.4}$$

Note that for the special choice $\rho = \theta(\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega))$, $\theta > 0$, (2.4)(ii) is similar to (2.3)(i). In particular, in the proofs we will use Theorem 2.3 to derive (2.3)(i) once (2.3)(ii) is established. The estimate is essentially based on the *coarea formula* in BV (see [4, Theorem 3.40]) and the argumentation in the proof goes back to [8, Proposition 6.2]. We remark that in contrast to Theorem 2.1 the result is derived in arbitrary space dimension. The proof will be given in Section 6.1.

Remark 2.4. Before we proceed with some applications of Theorem 2.1 let us comment on the estimates (2.1) and (2.4)(i). In (2.4)(i) for the choice $\rho = \theta(\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega))$ with θ small we obtain a fine estimate in the sense that the additional surface energy associated to the partition may be taken arbitrarily small with respect to the original jump set, where the constant in (2.4)(ii) blows up for $\theta \to 0$.

In contrast to the embedding and approximation results addressed in this article, for certain applications of the piecewise Korn inequality it is crucial to have also a refined version of (2.1), e.g. for a jump transfer theorem in SBD or a general compactness and existence result for Griffith energies in the realm of linearized elasticity (see [32]). The derivation of such an estimate needs additional techniques and we refer to [32] for a deeper analysis.

2.2. **Approximation of GSBD functions.** For each $u \in GSBD^2(\Omega)$ we can consider the sequence

$$v_n = u - \sum\nolimits_{j \geq n} a_j \chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2) \tag{2.5}$$

with $(P_j)_j$ and $(a_j)_j$ as in Theorem 2.1. Hereby we can approximate $GSBD^2$ functions by bounded functions of bounded variation which coincide with the original function up to a set of arbitrarily small measure. In particular, combining this observation with the density result proved in [35] (see Theorem 3.11) we obtain the following improved approximation result for $GSBD^2$ functions in a planar setting (see Section 5.2 for the proof).

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_k)_k \subset SBV^2(\Omega;\mathbb{R}^2)$ such that each J_{u_k} is the union of a finite number of closed connected pieces of C^1-curves, each u_k belongs to $W^{1,\infty}(\Omega \setminus J_{u_k};\mathbb{R}^2)$ and the following properties hold:*

$$\begin{aligned} (i) &\quad u_k \to u \text{ in measure on } \Omega,\\ (ii) &\quad \|e(u_k) - e(u)\|_{L^2(\Omega)} \to 0,\\ (iii) &\quad \mathcal{H}^1(J_{u_k} \triangle J_u) \to 0. \end{aligned} \tag{2.6}$$

Here the symbol $\triangle$ denotes the symmetric difference of two sets. Observe that in [11, 35] the additional assumption $u \in L^2(\Omega)$ was needed which can be circumvented in the planar setting by (1) first approximating u by $(v_n)_n \subset GSBD^2(\Omega) \cap L^2(\Omega)$ as given in (2.5) and then (2) applying the original density result [11, 35]. In fact, in various applications in fracture mechanics approximation results are essential, e.g. for the construction of recovery sequences. Consequently, to establish complete Γ-limits it is indispensable to have a density result for all admissible functions. In particular, Theorem 2.5 allows us to complete the picture in the linearization result for Griffith energies presented in [28].

Recall that we establish Theorem 2.1 first for functions with a regular jump set and the general case then follows by approximation of any $u \in GSBD^2(\Omega)$. Although Theorem 2.5 provides such a density result, in the proof we have to argue differently since we already use the general version of Theorem 2.1 to prove Theorem 2.5. The strategy in the proof of Theorem 2.1 is to provide an adaption of the arguments in [11, 35] and to approximate each $u \in GSBD^2(\Omega)$ (without $L^2(\Omega)$ assumption) by functions $(u_k)_k$ with regular jump set such that (2.6)(i) holds and $\|e(u_k)\|_{L^2(\Omega)} \leq c\|e(u)\|_{L^2(\Omega)}$, $\mathcal{H}^1(J_{u_k}) \leq c\mathcal{H}^1(J_u)$ (see Lemma 5.4), which is sufficient for (2.1) and (2.3).

2.3. **Relation between SBV and SBD: Embedding results.** The standard examples for functions of bounded deformation not having bounded variation are given by configurations where small balls are cut out from the bulk part with an appropriate choice of the functions on these specific sets (see e.g. [3, 15, 18]). We include the construction here for convenience of the reader.

Example 2.6. Let $B_k = B_{r_k}(x_k) \subset B_1(0) \subset \mathbb{R}^d$ be pairwise disjoint balls for $k \in \mathbb{N}$ with x_k converging to some limiting point x_∞. For $k \in \mathbb{N}$ let $A_k = (a^k_{ij}) \in \mathbb{R}^{d\times d}_{\rm skew}$ with $a^k_{12} = -a^k_{21} =$

$d_k \in \mathbb{R}$ and $a^k_{ij} = 0$ otherwise. Define the piecewise infinitesimal rigid motion

$$u(x) = \sum\nolimits_{k=1}^{\infty} (A_k\,(x - x_k))\chi_{B_k}(x). \tag{2.7}$$

With the choice $r_k = (\frac{1}{k(\log(k))^2})^{\frac{1}{d-1}}$ and $d_k = r_k^{-1}$ we get $u \in L^\infty(B_1(0);\mathbb{R}^d)$ and $u \in SBV(B_1(0);\mathbb{R}^d) \cap SBD(B_1(0))$ with $e(u) = 0$ a.e. and $\mathcal{H}^{d-1}(J_u) < +\infty$. However, $\nabla u \notin L^p(B_1(0);\mathbb{R}^{d\times d})$ for $p > 1$.

In view of Theorem 2.1 we see that the above construction is essentially the only way to define $(G)SBD$ functions with $e(u) \in L^2(\Omega)$ and $\mathcal{H}^1(J_u) < +\infty$ not having bounded variation. In particular, by considering $u\chi_{\bigcup_{j=1}^n P_j}$, $n \in \mathbb{N}$, with $(P_j)_j$ as in (2.1), we observe that each $u \in GSBD^2(\Omega)$ has bounded variation away from an at most countable union of sets of finite perimeter with arbitrarily small Lebesgue measure.

Observe that $\nabla u \in L^1(B_1(0);\mathbb{R}^{d\times d})$ for the function given in Example 2.6. An L^∞ bound indeed implies that functions are of bounded variation as the following embedding result shows.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Then $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d)$. More precisely, there is a constant $C > 0$ only depending on Ω such that*

$$\|\nabla u\|_{L^1(\Omega)} \le C\|e(u)\|_{L^2(\Omega)} + C\|u\|_{L^\infty(\Omega)}(\mathcal{H}^{d-1}(J_u) + 1). \tag{2.8}$$

Note that in contrast to Theorem 2.1 our main embedding result as well as the following theorems are valid in arbitrary space dimension, where we apply a slicing technique to extend the estimate from the planar to the d-dimensional setting. As a direct consequence we get that one may apply the *coarea formula* in BV for functions in $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$. This estimate can be potentially applied in many situations and will hopefully reveal useful.

Note that the property $u \in SBV(\Omega;\mathbb{R}^d)$ does not follow directly from (2.8). However, by first deriving Theorem 2.7 for functions with regular jump set and then using a density argument (see Theorem 3.11), we can show $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \subset BV(\Omega;\mathbb{R}^d)$. The claim then follows from Alberti's rank one property in BV (see [2]) implying $|D^c u|(\Omega) \le \sqrt{2}|E^c u|(\Omega)$.

The first term on the right hand side in (2.8) controls the distance of u from a piecewise infinitesimal rigid motion $(a_j)_j = (a_{A_j,b_j})_j$ (cf. (2.3)(ii)) and the last term allows to estimate the skew symmetric matrices $(A_j)_j$. Let us note that the above inequality is optimal in following sense: by Example 2.6 the L^1-norm on the left cannot be replaced by any other L^p-norm. Moreover, the bound $\mathcal{H}^{d-1}(J_u) < +\infty$ is essential by [15, Theorem 3.1], where a function in $u \in SBD(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$ is constructed with $e(u) = 0$ a.e., $\mathcal{H}^{d-1}(J_u) = \infty$ and $\nabla u \notin L^1(\Omega;\mathbb{R}^{d\times d})$. Finally, the following lemma based on the construction given in Example 2.6 shows that the L^∞-norm cannot be replaced by any other L^q-norm.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^d$ open, bounded. Then for all $1 \le q < \infty$ we find some $u \in GSBD^2(\Omega) \cap L^q(\Omega;\mathbb{R}^d)$ such that the approximate differential ∇u exists a.e. and $\nabla u \notin L^1(\Omega;\mathbb{R}^{d\times d})$.*

Consequently, $GSBD^2(\Omega) \cap L^q(\Omega;\mathbb{R}^d) \not\subset BV(\Omega;\mathbb{R}^d)$ for all $q < \infty$. In Theorem 2.1 we have seen that $v = u - a \in SBV^p(\Omega;\mathbb{R}^2) \subset GBV(\Omega;\mathbb{R}^2)$ for a piecewise infinitesimal rigid motion $a := \sum_{j=1}^\infty a_j\chi_{P_j}$. Moreover, one has that $a \in GBV(\Omega;\mathbb{R}^2)$ (see [15, Theorem 2.2]). Of course, herefrom we cannot directly deduce that $u = v + a \in GBV(\Omega;\mathbb{R}^2)$ since $GBV(\Omega;\mathbb{R}^2)$ is not a vector space. However, the property holds as the following inclusion shows.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Then $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d \subset GBV(\Omega;\mathbb{R}^d)$. More precisely, there is a constant $C > 0$ only depending on Ω*

such that for all $i = 1, \ldots, d$ and all $M > 0$

$$|Du_i^M|(\Omega) \le CM(\mathcal{H}^{d-1}(J_u) + 1) + C\|e(u)\|_{L^2(\Omega)}. \tag{2.9}$$

Here for $u \in GSBD^2(\Omega)$ the functions $u_1, \ldots, u_d$ denote the components of u and $u_i^M := \min\{\max\{u_i, -M\}, M\}$ for $M > 0$. Theorem 2.9 shows that the coarea formula in GBV (see [4, Theorem 4.34]) is applicable in $GSBD^2(\Omega)$. The results announced in this section will be proved in Section 6.2.

2.4. **A Korn-Poincaré inequality for functions with small jump set.** As an application of our embedding results and the coarea formula we present a Korn-Poincaré inequality for functions with small jump set.

Theorem 2.10. *Let $Q \subset \mathbb{R}^d$ be a cube. Then there is a constant $C > 0$ only depending on* $\mathrm{diam}(Q)$ *and $d \in \mathbb{N}$ such that for all $u \in GSBD^2(Q)$ there is a set of finite perimeter $E \subset Q$ with*

$$\mathcal{H}^{d-1}(\partial^* E \cap Q) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}, \qquad |E| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \tag{2.10}$$

and an infinitesimal rigid motion $a = a_{A,b}$ such that

$$\begin{aligned} &(i) \quad \|u - a\|_{L^{\frac{2d}{d-1}}(Q \setminus E)} \le C\|e(u)\|_{L^2(Q)}, \\ &(ii) \quad \|u - a\|_{L^\infty(Q \setminus E)} \le C(\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}. \end{aligned} \tag{2.11}$$

Moreover, if $\mathcal{H}^{d-1}(J_u) > 0$, $\bar{u} := (u - a)\chi_{Q \setminus E} \in SBV(Q;\mathbb{R}^d) \cap SBD^2(Q) \cap L^\infty(Q;\mathbb{R}^d)$ and

$$|D\bar{u}|(Q) \le C\|e(u)\|_{L^2(Q)}. \tag{2.12}$$

Note that the exceptional set E is associated to the parts of Q being detached from the bulk part of Q by J_u. A variant of the proof shows that $(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$ in (2.10), (2.11)(ii) can be replaced by $(\mathcal{H}^{d-1}(J_u))^{\frac{p}{2}}$ if in (2.11)(i) we replace $\frac{2d}{d-1}$ by $\frac{2d}{p(d-1)}$ for $1 \le p \le 2$.

This result improves an estimate recently obtained by Chambolle, Conti, and Francfort [12] in the sense that $\mathcal{H}^{d-1}(\partial^* E)$ can be controlled and therefore compactness results in $GSBD$ (see [18]) are applicable. In addition, we provide a Korn-type estimate in (2.12). Similar results in a planar setting with an even finer estimate for the perimeter of E have been investigated in [27] and [16, 29].

The statement essentially follows by combining (1) the result in [12] and (2) applying the coarea formula in BV together with (2.9). In particular, the argument uses a truncation at a specific level set (cf. (2.11)(ii)) which is reminiscent of the Poincaré inequality in SBV due to De Giorgi, Carriero, Leaci (see [21]). The proof will be given in Section 6.3.

3. Preparations

3.1. **John domains.** A key step in our analysis will be the construction of a partition and a corresponding modification of the deformation being in $W^{1,p}$ on each component of the partition. Then in the application of Poincaré's and Korn's inequality on each component it is essential to provide uniform bounds for the constants involved in the inequalities. To this end, we introduce the notion of *John domains.*

Definition 3.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $x_0 \in \Omega$. We say Ω is a *ϱ-John domain* with respect to the *John center* x_0 and with the constant $\varrho > 0$ if for all $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, l_\gamma] \to \Omega$, parametrized by arc length, such that $\gamma(0) = x$, $\gamma(l_\gamma) = x_0$ and $t \le \varrho\,\mathrm{dist}(\gamma(t), \partial\Omega)$ for all $t \in [0, l_\gamma]$.

Domains of this form were introduced by John [36] to study problems in elasticity theory and the term was first used by Martio and Sarvas [37]. Roughly speaking, a domain is a John domain if it is possible to connect two arbitrary points without getting too close to the boundary of the set. This class is much larger than the class of Lipschitz domains and contains sets which may possess fractal boundaries or internal cusps (external cusps are excluded), e.g. the interior of Koch's *snow flake* is a John domain. Although in the following we will only consider domains with Lipschitz boundary, it is convenient to introduce the much more general notion of John domains as the constants in Poincaré's and Korn's inequality only depend on the John constant. More precisely, we have the following statement (see e.g. [1, 9, 22]).

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a c-John domain. Let $p \in (1,\infty)$ and $q \in (1,d)$. Then there is a constant $C = C(c,p,q) > 0$ such that for all $u \in W^{1,p}(\Omega)$ there is some $A \in \mathbb{R}^{d\times d}_{\rm skew}$ such that*

$$\|\nabla u - A\|_{L^p(\Omega)} \le C\|e(u)\|_{L^p(\Omega)}.$$

Moreover, for all $u \in W^{1,q}(\Omega)$ there is some $b \in \mathbb{R}^d$ such that

$$\|u - b\|_{L^{q^*}(\Omega)} \le C\|\nabla u\|_{L^q(\Omega)},$$

where $q^* = \frac{dq}{d-q}$. The constant is invariant under rescaling of the domain.

We recall a result about the decomposition of sets into John domains.

Theorem 3.3. *There is a universal constant $c > 0$ such that for all simply connected, bounded domains $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and all $\varepsilon > 0$ there is a partition $\Omega = \Omega_0 \cup \ldots \cup \Omega_N$ (up to a set of negligible measure) such that $|\Omega_0| \le \varepsilon$ and the sets $\Omega_1,\ldots,\Omega_N$ are c-John domains with Lipschitz boundary such that*

$$\sum\nolimits_{j=0}^N \mathcal{H}^1(\partial\Omega_j) \le c\mathcal{H}^1(\partial\Omega). \tag{3.1}$$

Observe that in general it is necessary to introduce an (arbitrarily) small exceptional set Ω_0 as can be seen, e.g., by considering polygons with very acute interior angles. The statement is given explicitly in [30, Theorem 6.4], but can also easily be derived from the main result of that paper ([30, Theorem 1.1]) since Lipschitz sets can be approximated from within by smooth sets.

The essential step in the proof of Theorem 3.3 is to consider polygonal domains. One observes that convex polygons can be (iteratively) separated into convex polygons such that (3.1) holds and each set contains a ball whose size is comparable to the diameter of the set, whereby Definition 3.1 (for fixed $\varrho > 0$) can be confirmed. For general, nonconvex polygons the property in Definition 3.1 may be violated if a concave vertex is 'too close to the opposite part of the boundary'. It is shown that in this case the set may be (iteratively) separated into smaller polygons with less concave vertices, for which Definition 3.1 holds. We refer to [30] for more details.

3.2. **Properties of infinitesimal rigid motions.** In this section we collect some properties of infinitesimal rigid motions. As before $a = a_{A,b}$ stands for the mapping $a(x) = A\,x + b$ with $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $b \in \mathbb{R}^2$. The following lemma is shown in [32, Lemma 2.3].

Lemma 3.4. *Let $M > 0$, $\delta > 0$ $F \subset \mathbb{R}^2$ bounded, measurable and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ a continuous nondecreasing function satisfying $\lim_{s\to\infty}\psi(s) = +\infty$. Then there is a constant $C = C(M,\delta,\psi,F)$ such that for every Borel set $E \subset F$ with $|E| \ge \delta$ and every infinitesimal rigid motion $a_{A,b}$ one has*

$$\int_E \psi(|Ax+b|)\,dx \le M \quad \Rightarrow \quad |A| + |b| \le C. \tag{3.2}$$

In the following we denote by $d(E)$ the diameter of a set $E \subset \mathbb{R}^2$.

Lemma 3.5. *Let $q \in [1,\infty)$. There is a constant $c = c(q) > 0$ such that for every Borel set $E \subset \mathbb{R}^2$ and every infinitesimal rigid motion $a = a_{A,b}$ one has for all $x \in E$*

$$
\begin{aligned}
&(i) \quad |A| \le c|E|^{-\frac{1}{2}-\frac{1}{q}}\|a\|_{L^q(E)}, \quad |A| \le c|E|^{-\frac{1}{2}}\|a\|_{L^\infty(E)},\\
&(ii) \quad |A\,x + b| \le c|E|^{-\frac{1}{2}-\frac{1}{q}} d(E)\|a\|_{L^q(E)}.
\end{aligned}
\tag{3.3}
$$

Proof. Without restriction assume that $A \neq 0$ as otherwise the statement is clear. The assumption $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ implies that A is invertible and that $|Ay| = \frac{\sqrt{2}}{2}|A||y|$ for all $y \in \mathbb{R}^2$. Setting $z := -A^{-1}b$ we find for all $x \in E$

$$\tfrac{1}{\sqrt{2}}|A||x - z| = |a(x)| \le \|a\|_{L^\infty(E)}. \tag{3.4}$$

Consequently, as $|E \setminus B_\rho(z)| \ge \frac{1}{2}|E|$ with $\rho = (\frac{|E|}{2\pi})^{\frac{1}{2}}$, we find $|E|\rho^q|A|^q \le c\|a\|^q_{L^q(E)}$ for $q < \infty$, which implies the first part of (3.3)(i). In the case $q = \infty$ we immediately get $|A| \le c|E|^{-\frac{1}{2}}\|a\|_{L^\infty(E)}$ from (3.4). As there is some $x_0 \in E$ with $|A\,x_0 + b| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)}$, we conclude for all $x \in E$

$$|A\,x + b| \le |A\,x_0 + b| + \tfrac{1}{\sqrt{2}}|A||x - x_0| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)} + cd(E)\rho^{-1}|E|^{-\frac{1}{q}}\|a\|_{L^q(E)}.$$

This concludes the proof since $|E|^{\frac{1}{2}} \le d(E)$ by the isodiametric inequality. □

Remark 3.6. Analogous estimates hold on lines. If, e.g., $\Gamma \subset \mathbb{R} \times \{0\}$ with $l := \mathcal{H}^1(\Gamma) < \infty$, then for a constant $c = c(q) > 0$ we have for all infinitesimal rigid motions $a = a_{A,b}$

$$l^q|A|^q \le cl^{-1}\int_\Gamma |a(x)|^q \, d\mathcal{H}^1(x).$$

With Lemma 3.5 at hand we now see that the L^q-norm on larger sets can be controlled.

Lemma 3.7. *Let $q \in [1,\infty)$. Then there is a constant $c = c(q) > 0$ such that for all $y \in \mathbb{R}^2$, $R > 0$, Borel sets $E \subset Q^y_R := y + (-R,R)^2$ and $a = a_{A,b}$ one has*

$$\|a\|_{L^q(Q^y_R)} \le c(R^2|E|^{-1})^{\frac{1}{2}+\frac{1}{q}}\|a\|_{L^q(E)}.$$

Proof. Define $F = Q^y_R \supset E$. We repeat the last estimate of the previous proof for each $x \in F$ and obtain with $\rho = (\frac{|E|}{2\pi})^{\frac{1}{2}}$

$$|A\,x + b| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)} + cd(F)\rho^{-1}|E|^{-\frac{1}{q}}\|a\|_{L^q(E)}.$$

This implies with $|F| = 4R^2$ and $d(F) = 2\sqrt{2}R$

$$\|a\|^q_{L^q(F)} \le cR^2|E|^{-1}\|a\|^q_{L^q(E)} + cR^{2+q}|E|^{-\frac{q}{2}-1}\|a\|^q_{L^q(E)}.$$

In view of $R^q|E|^{-\frac{q}{2}} \ge 4^{-\frac{q}{2}}$ this concludes the proof. □

3.3. **(G)SBV and (G)SBD functions.** In this section we collect the definitions and fundamental properties of the function spaces needed in this article. In the following let $\Omega \subset \mathbb{R}^d$ be an open, bounded set.

SBV- and GSBV-functions. The space $BV(\Omega;\mathbb{R}^d)$ consists of the functions $u \in L^1(\Omega;\mathbb{R}^d)$ such that the distributional gradient Du is a $\mathbb{R}^{d\times d}$-valued finite Radon measure on Ω. BV-functions have an approximate differential $\nabla u(x)$ at $\mathcal{L}^d$-a.e. $x \in \Omega$ ([4, Theorem 3.83]) and their jump set J_u is $\mathcal{H}^{d-1}$-rectifiable in the sense of [4, Definition 2.57]. The space $SBV(\Omega;\mathbb{R}^d)$, often abbreviated hereafter as $SBV(\Omega)$, of *special functions of bounded variation* consists of those $u \in BV(\Omega;\mathbb{R}^d)$ such that

$$Du = \nabla u \mathcal{L}^d + [u] \otimes \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

where ν_u is a normal of J_u and $[u] = u^+ - u^-$ (the 'crack opening') with $u^\pm$ being the one-sided limits of u at J_u. If in addition $\nabla u \in L^p(\Omega)$ for $1 < p < \infty$ and $\mathcal{H}^{d-1}(J_u) < +\infty$, we write $u \in SBV^p(\Omega)$. Moreover, $(S)BV_{\rm loc}(\Omega)$ denotes the space of functions which belong to $(S)BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

We define the space $GBV(\Omega;\mathbb{R}^d)$ of *generalized functions of bounded variation* consisting of all $\mathcal{L}^d$-measurable functions $u : \Omega \to \mathbb{R}^d$ such that for every $\phi \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $BV_{\rm loc}(\Omega)$ (see [20]). Likewise, we say $u \in GSBV(\Omega;\mathbb{R}^d)$ if $\phi \circ u$ belongs to $SBV_{\rm loc}(\Omega)$ and $u \in GSBV^p(\Omega;\mathbb{R}^d)$ for $u \in GSBV(\Omega;\mathbb{R}^d)$ if $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$. As usual we write for shorthand $GSBV(\Omega) := GSBV(\Omega;\mathbb{R})$. See [4] for the basic properties of these function spaces.

We now state a version of Ambrosio's compactness theorem in $GSBV$ adapted for our purposes (see e.g. [4, 19]):

Theorem 3.8. *Let $1 < p < \infty$. Let $(u_k)_k$ be a sequence in $GSBV^p(\Omega;\mathbb{R}^d)$ with*

$$\|\nabla u_k\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k}) + \|u_k\|_{L^1(\Omega)} \le C$$

for some $C > 0$ not depending on k. Then there is a subsequence (not relabeled) and a function $u \in GSBV^p(\Omega;\mathbb{R}^d)$ such that $u_k \to u$ a.e., $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. If in addition $\|u_k\|_\infty \le C$ for all $k \in \mathbb{N}$, we find $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$.

Caccioppoli-partitions. We say a partition $\mathcal{P} = (P_j)_{j=1}^\infty$ of Ω is a *Caccioppoli partition* of Ω if $\sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$, where $\partial^* P_j$ denotes the *essential boundary* of P_j (see [4, Definition 3.60]). We say a partition is *ordered* if $|P_i| \ge |P_j|$ for $i \le j$. In the whole paper we will always tacitly assume that partitions are ordered. We now state a compactness result for ordered Caccioppoli partitions (see [4, Theorem 4.19, Remark 4.20]).

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $\mathcal{P}_i = (P_{j,i})_{j=1}^\infty$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of Ω such that $\sup_i \sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_{j,i} \cap \Omega) < +\infty$. Then there exists a Caccioppoli partition $\mathcal{P} = (P_j)_{j=1}^\infty$ and a not relabeled subsequence such that $\chi_{P_{j,i}} \to \chi_{P_j}$ in measure for all $j \in \mathbb{N}$ as $i \to \infty$.*

SBD- and GSBD-functions. We say that a function $u \in L^1(\Omega;\mathbb{R}^d)$ is in $BD(\Omega;\mathbb{R}^d)$ if the symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\rm sym}$-valued Radon measure. Likewise, we say u is a *special function of bounded deformation* if Eu has vanishing Cantor part $E^c u$. Then Eu can be decomposed as

$$Eu = e(u)\mathcal{L}^d + [u] \odot \nu_u \mathcal{H}^{d-1}|_{J_u}, \tag{3.5}$$

where $e(u)$ is the absolutely continuous part of Eu with respect to the Lebesgue measure $\mathcal{L}^d$, $[u]$, ν_u, J_u as before and $\odot$ denotes the symmetrized tensor product. If in addition $e(u) \in L^2(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$, we write $u \in SBD^2(\Omega)$. For basic properties of this function space we refer to [3, 5].

We now introduce the space of *generalized functions of bounded variation*. Observe that it is not possible to follow the approach in the definition of $GSBV$ since for $u \in SBD(\Omega)$ the composite $\phi \circ u$ typically does not lie in $SBD(\Omega)$. In [18] another approach is suggested which is based on certain properties of one-dimensional slices. For fixed $\xi \in S^{d-1}$ we set

$$\Pi^\xi := \{y \in \mathbb{R}^d : y \cdot \xi = 0\}\,, \quad \Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\} \text{ for } y \in \Pi^\xi\,,$$
$$\Omega^\xi := \{y \in \Pi^\xi : \Omega^\xi_y \neq \emptyset\}\,.$$

Definition 3.10. *An $\mathcal{L}^d$-measurable function $u : \Omega \to \mathbb{R}^d$ belongs to $GBD(\Omega)$ if there exists a positive bounded Radon measure λ_u such that, for all $\tau \in C^1(\mathbb{R}^d)$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$, and all $\xi \in S^{d-1}$, the distributional derivative $D_\xi(\tau(u \cdot \xi))$ is a bounded Radon measure on Ω whose total variation satisfies*

$$|D_\xi(\tau(u \cdot \xi))|\,(B) \leq \lambda_u(B)$$

for every Borel subset B of Ω. A function $u \in GBD(\Omega)$ belongs to the subset $GSBD(\Omega)$ if in addition for every $\xi \in S^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, the function $u^\xi_y(t) := u(y + t\xi)$ belongs to $SBV_{\mathrm{loc}}(\Omega^\xi_y)$.

As before we say $u \in GSBD^2(\Omega)$ if in addition $e(u) \in L^2(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$. For later we note that there is a compactness result in $GSBD^2(\Omega)$ similar to Theorem 3.8 (see [18, Theorem 11.3]).

Density results. We recall a density result in $GSBD^2$ (see [35] and also [11, Theorem 3, Remark 5.3].

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(\Omega) \cap L^2(\Omega;\mathbb{R}^d)$. Then there exists a sequence $(u_k)_k \subset SBV^2(\Omega;\mathbb{R}^d)$ such that each J_{u_k} is the union of a finite number of closed connected pieces of C^{d-1}-hypersurfaces, each u_k belongs to $W^{1,\infty}(\Omega \setminus J_{u_k};\mathbb{R}^d)$ and*

$$\begin{aligned} &(i) \quad \|u_k - u\|_{L^2(\Omega)} \to 0,\\ &(ii) \quad \|e(u_k) - e(u)\|_{L^2(\Omega)} \to 0,\\ &(iii) \quad \mathcal{H}^{d-1}(J_{u_k} \triangle J_u) \to 0. \end{aligned} \tag{3.6}$$

If in addition $u \in L^\infty(\Omega)$, one can ensure $\|u_k\|_\infty \leq \|u\|_\infty$ for all $k \in \mathbb{N}$.

Remark 3.12. The result together with [17] shows that the approximating sequence $(u_k)_k$ can also be chosen such that J_{u_k} is the finite union of closed $(d-1)$-simplices intersected with Ω. In this case (3.6)(iii) has to be replaced by $\mathcal{H}^{d-1}(J_{u_k}) \to \mathcal{H}^{d-1}(J_u)$.

Korn inequality in SBD. As a final preparation we recall a Korn inequality in SBD^2 in a planar setting for functions with small jump set.

Theorem 3.13. *Let $p \in [1,2]$. Then there is a universal constant $c > 0$ such that for all squares $Q_\mu = (-\mu,\mu)^2$, $\mu > 0$, and all $u \in SBD^2(Q_\mu)$ there is a set of finite perimeter $E \subset Q_\mu$ with*

$$\mathcal{H}^1(\partial E) \leq c\mathcal{H}^1(J_u), \qquad |E| \leq c(\mathcal{H}^1(J_u))^2 \tag{3.7}$$

and $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $b \in \mathbb{R}^2$ such that

$$\begin{aligned} &(i) \quad \|u - (A\,\cdot + b)\|_{L^p(Q_\mu \setminus E)} \leq c\mu^{\frac{2}{p}}\|e(u)\|_{L^2(Q_\mu)},\\ &(ii) \quad \|\nabla u - A\|_{L^p(Q_\mu \setminus E)} \leq c\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{3.8}$$

Moreover, there is a Borel set $\Gamma \subset \partial Q_\mu$ such that

$$\mathcal{H}^1(\Gamma) \leq c\mathcal{H}^1(J_u), \qquad \int_{\partial Q_\mu \setminus \Gamma} |Tu - (A\,x + b)|^2 \, d\mathcal{H}^1(x) \leq c\mu\|e(u)\|^2_{L^2(Q_\mu)}, \tag{3.9}$$

where Tu denotes the trace of u on ∂Q_μ.

The fact that the constant is independent of μ and p follows from a standard scaling argument and Hölder's inequality. More generally, the result holds on connected, bounded Lipschitz sets Ω for a constant C also depending on Ω.

Proof. The result stated in (3.7)-(3.8) has first been derived in [29] for $p \in [1,2)$ and was then improved in [16] by showing the estimate for $p = 2$. In [12] the trace estimate has been established. We need to confirm that in both estimates (3.8), (3.9) one can indeed take the same infinitesimal rigid motion.

Let E and $a_{A,b}$ be given such that (3.7)-(3.8) hold for $p = 2$. By [12] we find Borel sets $F \subset Q_\mu$, $\Gamma \subset \partial Q_\mu$ and $a' = a_{A',b'}$ such that (3.9) holds with a' in place of a and $|F| \le c(\mathcal{H}^1(J_u))^2$ as well as $\|u - a'\|_{L^2(Q_\mu \setminus F)} \le c\mu\|e(u)\|_{L^2(Q_\mu)}$. We can assume that $\mathcal{H}^1(J_u)$ is so small that $|E \cup F| \le c(\mathcal{H}^1(J_u))^2 \le \frac{1}{2}|Q_\mu|$ since otherwise passing to a larger $c > 0$ in (3.7) we could choose $E = Q_\mu$ and the statement was trivially satisfied. Consequently, we have $\|a - a'\|_{L^2(Q_\mu \setminus (E \cup F))} \le c\mu\|e(u)\|_{L^2(Q_\mu)}$ and by Lemma 3.7 $\|a - a'\|_{L^2(Q_\mu)} \le c\mu\|e(u)\|_{L^2(Q_\mu)}$ and thus by Lemma 3.5 $\|a - a'\|_{L^\infty(Q_\mu)} \le c\|e(u)\|_{L^2(Q_\mu)}$. This yields (3.9) for the mapping a. □

4. A piecewise Korn inequality up to a small exceptional set

As a key step for the proof of Theorem 2.1 we first derive a piecewise Korn inequality up to a small exceptional set. By iterative application of this result in Section 5 we then derive the main inequality. In this section we will consider configurations on a square with regular jump set consisting of a finite number of segments and defer the general case also to Section 5, where we will make use of a density result (see Theorem 3.11 and Lemma 5.4).

For $\Omega \subset \mathbb{R}^d$ open, bounded we let $\mathcal{W}(\Omega) \subset SBV^2(\Omega)$ be the functions u such that $J_u = \bigcup_{j=1}^n \Gamma^u_j$ is the finite union of closed $(d-1)$-simplices intersected with Ω and $u|_{\Omega \setminus J_u} \in W^{1,\infty}(\Omega \setminus J_u)$. (In dimension $d = 2$ each Γ^u_j is a closed segment.) In the following we say $(P_j)_{j=1}^n$ is a partition of Ω if the sets are open, pairwise disjoint and satisfy $|\Omega \setminus \bigcup_{j=1}^n P_j| = 0$. For convenience we often write $\Omega = \bigcup_{j=1}^n P_j$ although the identity only holds up to a set of negligible $\mathcal{L}^2$-measure.

Theorem 4.1. *Let $p \in [1,2)$ and $\theta > 0$. Then there is a universal constant $c > 0$ and some $C = C(p,\theta) > 0$ such that the following holds:*

(1) For each square $Q_\mu = (-\mu,\mu)^2$ for $\mu > 0$ and each $u \in \mathcal{W}(Q_\mu)$ one finds an exceptional set $E \subset Q_\mu$ with

$$|E| \le c\mu\theta^2\, \mathcal{H}^1(J_u), \qquad \mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u) \tag{4.1}$$

such that there is a partition $Q_\mu = \bigcup_{j=1}^n P_j$ and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that the function $v := u - \sum_{j=1}^n a_j \chi_{P_j}$ satisfies

$$\begin{aligned} &(i) \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \partial P_j \cap Q_\mu\Big) \le C\mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \partial P_j \cap J_u\Big),\\ &(ii) \quad \|\nabla v\|_{L^p(Q_\mu \setminus E)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{4.2}$$

(2) One can choose E, $(P_j)_{j=1}^n$ and v such that we also have

$$\|v\|_{L^\infty(Q_\mu)} \le C\|u\|_{L^\infty(Q_\mu)}. \tag{4.3}$$

Note that (4.2)(ii) is similar to (2.3)(ii) with the correct scaling. Since (2.3)(i) will eventually follow from (2.3)(ii) by Theorem 2.3 it is not necessary at this stage to derive an analog of (2.3)(i). However, we establish (4.3) which may be of independent interest. Although not needed and not derived explicitly in Section 5, let us remark that also in Theorem 2.1, if

$u \in L^\infty(\Omega)$, the piecewise infinitesimal rigid motions can be chosen such that $\|v\|_{L^\infty(\Omega)} \le C\|u\|_{L^\infty(\Omega)}$ with v as in (2.2).

Estimate (4.2)(i) differs from (2.1) in the sense that the length of the boundary of the partition can be controlled solely by the part of J_u contained in $\bigcup_{j=1}^n \partial P_j$. This property will be crucial for the iterative application of the arguments in Section 5 since hereby a blow up of the length of the boundary of the partition can be avoided. Also the fact the volume of the exceptional set in (4.1) is controlled in terms of a (small) parameter $\theta > 0$ will be convenient for the subsequent analysis. Note that therefore it will not be restrictive to concentrate on the case $\theta \le 1$ and

$$\mu\theta^2 \le \mathcal{H}^1(J_u) \le c\mu\theta^{-2}. \tag{4.4}$$

In fact, for functions with smaller jump sets Theorem 4.1 directly follows from Theorem 3.13 for a partition only consisting of one element, and if the jump set is too large, one can choose $E = Q_\mu$ and (4.2) trivially holds.

The strategy for the proof of Theorem 4.1 is the following. We first identify squares Q of various mesoscopic sizes where $J_u \cap Q$ is 'too large', i.e. comparable to the diameter of the square. This will allow us to construct a partition of Q_μ consisting of simply connected sets and a corresponding Whitney covering such that $\mathcal{H}^1(J_u \cap Q) \ll d(Q)$ for the squares of the covering (see Section 4.1). Then applying the Korn inequality for functions with small jump set (Theorem 3.13) on each of these squares, we can modify the configuration on each component of the partition to a Sobolev function $\bar{u}$ whose distance from u can be controlled outside a small exceptional set (see Section 4.2). Then we use Theorem 3.3 to find another refined partition consisting of John domains whose John constant can be uniformly controlled. Theorem 4.1 then follows from Theorem 3.2 applied on $\bar{u}$ and the fact that the difference of ∇u and $\nabla \bar{u}$ is small on the bulk part of the domain (see Section 4.3).

Before we start with the construction of the auxiliary partition we introduce some further notation. For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1,1)^2$ for $p \in s(1,1) + 2s\mathbb{Z}^2$ and write

$$\mathcal{Q}^s = \{Q = Q^s(p) : \ p \in s(1,1) + 2s\mathbb{Z}^2\}.$$

Let $\theta > 0$ be given and assume without restriction that $\theta \in 2^{-\mathbb{N}}$ and θ small with respect to 1. For all $i \in \mathbb{N}_0$ we define $s_i = \mu\theta^i$ and we consider the coverings of dyadic squares $\mathcal{Q}^i := \mathcal{Q}^{s_i}$. For each square Q of the above form we also introduce the corresponding enlarged squares $Q \subset Q' \subset Q'' \subset Q'''$ defined by

$$Q' = \tfrac{3}{2}Q, \quad Q'' = 3Q, \quad Q''' = 5Q, \tag{4.5}$$

where λQ denotes the square with the same center and orientation and λ-times the sidelength of Q. By $d(A)$ we again indicate the diameter of a set $A \subset \mathbb{R}^2$ and by $\mathrm{dist}(A,B)$ we denote the euclidian distance between $A, B \subset \mathbb{R}^2$.

4.1. **Construction of an auxiliary partition.** To ensure that we provide an estimate which is valid up to a small exceptional set (cf. (4.1)), it will be convenient to decompose Q_μ into smaller squares. To this end, we introduce the auxiliary jump set

$$J_u^* := J_u \cup J_0 \quad \text{with } J_0 = \bigcup\nolimits_{Q \in \mathcal{Q}^7, Q \subset Q_\mu} \partial Q \tag{4.6}$$

and observe that together with (4.4) we find

$$\mathcal{H}^1(J_u^*) \le \mathcal{H}^1(J_u) + c\theta^{-7}\mu \le c\theta^{-9}\mathcal{H}^1(J_u) \tag{4.7}$$

for a universal constant $c > 0$. One of the main strategies will be the application of Theorem 3.13. As the result only provides an estimate for functions with small jump set, we introduce for $i \geq 1$ the set of 'bad squares'

$$\mathcal{Q}^i_{\rm bad} = \{Q \in \mathcal{Q}^i : \mathcal{H}^1(J_u^* \cap Q') \geq \theta^3 s_i\}. \tag{4.8}$$

In particular, note that each square with $Q' \cap J_0 \neq \emptyset$ satisfies $\mathcal{H}^1(Q' \cap J_u^*) \geq 2s_i$ and thus lies in $\mathcal{Q}^i_{\rm bad}$. Let $B^i = \bigcup_{Q \in \mathcal{Q}^i_{\rm bad}} \overline{Q'''}$ and observe that B^i is the union of squares in $\mathcal{Q}^i$ up to a set of negligible measure (see Figure 1(a) below). In particular, we have $B^i \supset Q_\mu$ for $i = 1, \ldots, 7$. We state a lemma about the union of 'bad sets'.

Lemma 4.2. *There is universal constant $c > 0$ such that for all $1 \leq i_1 \leq i_2 < \infty$ and for all $\mathcal{R}^j \subset \mathcal{Q}^j_{\rm bad}$ we have for $R := \bigcup_{j=i_1}^{i_2} \bigcup_{Q \in \mathcal{R}^j} \overline{Q'''}$*

$$\mathcal{H}^1(\partial R) \leq c\theta^{-3}\mathcal{H}^1(J_u^* \cap R).$$

Moreover, there is a set $\Gamma_R \supset \partial R$ being a finite union of closed segments with $\mathcal{H}^1(\Gamma_R) \leq c\theta^{-3}\mathcal{H}^1(J_u^* \cap R)$ such that for each connected component R_k of R the set $\Gamma_R \cap R_k$ is connected.

Proof. For $i_1 \leq j \leq i_2$ let $\hat{\mathcal{R}}^j = \{Q \in \mathcal{R}^j : \overline{Q'''} \not\subset \bigcup_{i=i_1}^{j-1} \bigcup_{Q \in \mathcal{R}^i} \overline{Q'''}\}$ and $\hat{R}^j = \bigcup_{Q \in \hat{\mathcal{R}}^j} Q'$. Note that in view of (4.5) for θ small the sets $(\hat{R}^j)_{j=i_1}^{i_2}$ are pairwise disjoint and by (4.8) we have for all $i_1 \leq j \leq i_2$

$$\#\hat{\mathcal{R}}^j \leq \theta^{-3}s_j^{-1} \sum\nolimits_{Q \in \hat{\mathcal{R}}^j} \mathcal{H}^1(J_u^* \cap Q') \leq 4\theta^{-3}s_j^{-1}\mathcal{H}^1(J_u^* \cap \hat{R}^j),$$

where we used that each $x \in \mathbb{R}^2$ is contained in at most four Q', $Q \in \mathcal{Q}^j$. Since $\bigcup_{j=i_1}^{i_2} \hat{R}^j \subset R$, $(\hat{R}^j)_j$ are pairwise disjoint and $\mathcal{H}^1(\partial Q''') = 40 s_j$ for $Q \in \hat{\mathcal{R}}^j$, we then derive

$$\mathcal{H}^1(\partial R) \leq \sum\nolimits_{j=i_1}^{i_2} \sum\nolimits_{Q \in \hat{\mathcal{R}}^j} \mathcal{H}^1(\partial Q''') \leq \sum\nolimits_{j=i_1}^{i_2} c\theta^{-3}\mathcal{H}^1(J_u^* \cap \hat{R}^j) \leq c\theta^{-3}\mathcal{H}^1(J_u^* \cap R).$$

To see the second part of the statement, we define $\Gamma_R = \bigcup_{j=i_1}^{i_2} \bigcup_{Q \in \hat{\mathcal{R}}^j} \partial Q'''$ and note that $\Gamma_R \cap R_k$ is connected for each connected component R_k of R. □

Let $\mathcal{B}^i := (B^i_k)_k$ be the connected components of B^i for $i \geq 1$. By the remark below (4.8) for each $i \in \mathbb{N}$ there is a component B^i_k with $J_0 \subset B^i_k$ (see Figure 1(a)). Given $p \in [1, 2)$ let $r = \frac{1}{24}(2 - p)$ for shorthand. We say a set $B^i_k \in \mathcal{B}^i$ is an *isolated component* if

$$d(B^i_k) \leq \theta^{-ir} s_i. \tag{4.9}$$

Let $\mathcal{B}^i_{\rm iso} \subset \mathcal{B}^i$ be the subset consisting of the isolated components and see Figure 1(a),(b) for an illustration. Note that the component $B^i_k \in \mathcal{B}^i$ containing J_0 satisfies $B^i_k \notin \mathcal{B}^i_{\rm iso}$. Consequently, each set in $\mathcal{B}^i_{\rm iso}$ is contained in a square $Q \in \mathcal{Q}^7$ and thus

$$d(B^i_k) \leq 2\sqrt{2}\mu\theta^7 \quad \text{ for all } B^i_k \in \mathcal{B}^i_{\rm iso}. \tag{4.10}$$

Recall the structure of J_u^* (see the definition of $\mathcal{W}(Q_\mu)$ before Theorem 4.1) and denote the connected components of J_u^* by $(\Gamma^*_j)_{j=1}^N$, each of which being the union of finitely many closed segments. Choosing $I \in \mathbb{N}$ sufficiently large we see that for $i \geq I$ each set $(B^i_k)_k$ contains exactly one connected component of J_u^*. In particular, we can assume that $I \in \mathbb{N}$ is so large that

$$\begin{aligned} &(i) \quad \mathrm{dist}(\Gamma^*_{j_1}, \Gamma^*_{j_2}) \geq \theta^{-1}s_I \quad \text{for} \quad 1 \leq j_1 < j_2 \leq N, \\ &(ii) \quad \mathcal{B}^i_{\rm iso} = \emptyset \quad \text{for} \quad i \geq I. \end{aligned} \tag{4.11}$$

Indeed, for (ii) we observe $d(B^i_k) \geq d(J_u^* \cap B^i_k) \geq \min_{j=1,\ldots,N} d(\Gamma^*_j) > 0$ and $\theta^{-ir}s_i \to 0$ for $i \to \infty$. Finally, we introduce the *saturation* of a bounded set $A \subset \mathbb{R}^2$ defined by $\mathrm{sat}(A) =$

$\mathbb{R}^2 \setminus A'$, where A' denotes the (unique) unbounded connected component of $\mathbb{R}^2 \setminus A$. Loosely speaking, $\mathrm{sat}(A)$ arises from A by 'filling the holes of A'. In the construction of the partition we will neglect the part of J_u^* contained in

$$U_i := \mathrm{sat}\Big(\bigcup\nolimits_{B_k^i \in \mathcal{B}^i_{\rm iso}} B_k^i\Big), \quad i \ge 1 \tag{4.12}$$

as in these regions the discontinuity set of u can be removed by modifying the function u in a suitable way. This will be described in detail in Section 4.2 below.

We now show that Q_μ can be partitioned into simply connected components $(P'_j)_j$ with a corresponding covering of Whitney-type in terms of squares in $\bigcup_{i\ge 1} \mathcal{Q}^i$ such that the size of J_u is controllable in the squares not associated to the isolated components U_i.

Theorem 4.3. *Let $\mu > 0$, $\theta > 0$ and $p \in [1,2)$. Then there is a universal constant $c > 0$ and a constant $C = C(\theta,p) > 0$ such that for all $u \in \mathcal{W}(Q_\mu)$ with (4.4) there is a partition $(P'_j)_{j=1}^m$ of Q_μ consisting of open, simply connected sets satisfying*

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P'_j\big) \le C\mathcal{H}^1(J_u) \tag{4.13}$$

with the following properties: the set $\bigcup_{j=1}^m \partial P'_j$ is a finite union of closed segments. Moreover, there is a covering $\mathcal{C} \subset \bigcup_{i=8}^\infty \mathcal{Q}^i$ of Q_μ with pairwise disjoint dyadic squares, a closed set $Z \subset Q_\mu$ and an index $I = I(u,\theta,p) \in \mathbb{N}$ such that

$$Z \subset \bigcup\nolimits_{i=8}^I U_i \tag{4.14}$$

for U_i as in (4.12) and with $\mathcal{C}^i := \mathcal{C} \cap \mathcal{Q}^i$ we have

$$\begin{aligned}
&(i) \quad Q_\mu \setminus \big(J_u \cap \bigcup\nolimits_{j=1}^m \partial P'_j\big) \subset \bigcup\nolimits_{Q\in\mathcal{C}} Q' \subset Q_\mu,\\
&(ii) \quad Q'_1 \cap Q'_2 \neq \emptyset \ \ \textit{for} \ \ Q_1, Q_2 \in \mathcal{C} \ \Rightarrow \ \theta d(Q_1) \le d(Q_2) \le \theta^{-1} d(Q_1),\\
&(iii) \quad \#\{Q \in \mathcal{C} : x \in Q'\} \le 12 \quad \textit{for all} \ \ x \in Q_\mu,\\
&(iv) \quad \mathcal{H}^1\big(J_u \cap Q'\big) \le \theta^2 s_i \quad \textit{for all} \ \ Q \in \mathcal{C}^i \ \textit{with} \ Q'' \not\subset Z,\\
&(v) \quad \mathcal{H}^1\big(J_u \cap Q'\big) \le \theta^2 s_i \quad \textit{for all} \ \ Q \in \mathcal{C}^i \ \textit{with} \ Q'' \cap \hat{Q}'' \neq \emptyset \ \textit{for some} \ \hat{Q} \in \mathcal{C}^{i-1} \cup \mathcal{C}^{i+1},\\
&(vi) \quad J_u \ \cap \ \bigcup\nolimits_{i \ge I+1}^\infty \bigcup\nolimits_{Q\in\mathcal{C}^i} Q' \ = \emptyset,
\end{aligned} \tag{4.15}$$

where Q', Q'' denote the enlarged squares corresponding to $Q \in \mathcal{C}$ (cf. (4.5)).

The fact that the partition consists of simply connected sets will be essential since hereby Theorem 3.3 will be applicable (see Section 4.3). Observe that (4.15)(ii),(iii) are the typical properties of a Whitney covering, where $\mathcal{C}$ possibly does not cover the part of J_u contained in the boundary of the partition (see (4.15)(i)). The crucial condition (4.15)(iv) states that outside of Z the jump set in each square is small compared to its diameter. Later this will allow us to apply Theorem 3.13 (in Z we will have to argue differently). Before we proceed with the proof of Theorem 4.3 we give a more precise meaning to the exceptional set Z, which is associated to the isolated components.

Lemma 4.4. *Let be given the situation of Theorem 4.3 with $(P'_j)_j$ and Z as in (4.13)-(4.14). Then the covering $\mathcal{C}$ can be chosen such that (4.15) holds and that there is a partition $Z = \bigcup_{i=8}^I Z^i$ into pairwise disjoint, closed sets such that each Z^i is the union of squares in $\mathcal{C}^i =$*

$\mathcal{C} \cap \mathcal{Q}^i$ up to a set of measure zero. We have

$$\begin{aligned} &(i) \quad \big|Z\big| \le c\mu\theta^5 \mathcal{H}^1(J_u), \qquad |Z^i| \le C\theta^{-ir} s_i \mathcal{H}^1(J_u), \\ &(ii) \quad \sum\nolimits_{i=8}^I \mathcal{H}^1(\partial Z^i) \le C\mathcal{H}^1(J_u) \end{aligned} \tag{4.16}$$

for a universal $c > 0$ and $C = C(\theta) > 0$ only depending on θ. Denote by $(X^i_k)_k$ the connected components of Z^i and let $\mathcal{X}^i_k = \{Q \in \mathcal{C}^i : Q \subset X^i_k,\ \partial Q \cap \partial X^i_k \neq \emptyset\}$. We get for each $i = 8, \ldots, I$ and each X^i_k

$$\begin{aligned} &(i) \quad d(X^i_k) \le \theta^{-ir} s_i, \quad \#\mathcal{X}^i_k \le c\theta^{-2ir}, \\ &(ii) \quad Q' \cap Z \subset X^i_k \quad \text{for all} \quad Q \in \mathcal{X}^i_k, \\ &(iii) \quad \sum\nolimits_k \#\mathcal{X}^i_k \le C\mathcal{H}^1(J_u) s_i^{-1}. \end{aligned} \tag{4.17}$$

We now prove Theorem 4.3 and Lemma 4.4.

Proof of Theorem 4.3. Let u and the corresponding sets J_u^*, $(\mathcal{B}^i)_{i\ge1}$ and $(U_i)_{i\ge1}$ be given as defined in (4.6)-(4.12). We first concern ourselves with the 'bad sets' $\mathcal{B}^i$ and the union of bad sets for different $i \in \mathbb{N}$ (Step I). The sets not associated to isolated components will then be the starting point for the construction of a partition into simply connected components $(P'_j)_j$ (Step II and III). In this context, the fact that the length of not isolated components $\mathcal{B}^i \setminus \mathcal{B}^i_{\rm iso}$ compared to s_i increases in $i \in \mathbb{N}$ (see (4.9)) will be crucial to control the length of the segments, which are introduced to define the partition (cf. (4.25) below).

Finally, we define a covering consisting of pairwise disjoint dyadic squares and prove property (4.15) (Step IV). In particular, the fact that the area of the 'bad sets' becomes gradually smaller for larger $i \in \mathbb{N}$ allows us to confirm (4.15)(i),(iv). For (4.15)(v) we will exploit that the 'bad sets' are defined in terms of enlarged squares (cf. (4.5)) and (4.15)(vi) will follow from (4.11). In the following $c > 0$ stands for a universal constant and $C = C(\theta, p) > 0$ for a generic constant independent of all other parameters.

Step I (Definition of 'bad sets'): Let $I \in \mathbb{N}$ be given such that (4.11) holds, where in general the choice of I depends on u, θ and p. We define sets forming the starting point for the construction of the partition. Recall the definition of B^i below (4.8). For $1 \le i \le I$ let $\mathcal{D}^i = (D^i_k)_k$ be the connected components of $\bigcup_{l=i}^I B^l$ having nonempty intersection with B^i and satisfying

$$d(D^i_k) > \theta^{-ir} s_i. \tag{4.18}$$

Observe that by definition of $\mathcal{B}^i_{\rm iso}$, in particular (4.9), each $B^i_k \in \mathcal{B}^i \setminus \mathcal{B}^i_{\rm iso}$ is contained in a component of $\mathcal{D}^i$. As each connected component is a finite union of squares, we find $d(D^i_k) \le \mathcal{H}^1(\partial D^i_k)$ and thus by Lemma 4.2 (for $\mathcal{R}^j = \mathcal{Q}^j_{\rm bad}$, $i \le j \le I$) we obtain

$$\sum\nolimits_k d(D^i_k) \le \mathcal{H}^1\Big(\partial \bigcup\nolimits_{l=i}^I B^l\Big) \le c\theta^{-3}\mathcal{H}^1\big(J_u^* \cap \bigcup\nolimits_{l=i}^I B^l\big) \le c\theta^{-3}\mathcal{H}^1(J_u^*) \tag{4.19}$$

independently of $1 \le i \le I$.

Recall that by the definition of J_0 and (4.8) we have $B^i \supset Q_\mu$ for $i = 1, \ldots, 7$. We start with the set $E^7 := \bigcup_k D^7_k \supset Q_\mu$ and assume that E^k, $7 \le k \le i-1$, have been constructed. Let

$$\mathcal{E}^i = \{D^i_k : D^i_k \cap E^{i-1} \neq \emptyset\} \quad \text{and} \quad E^i = \bigcup\nolimits_{D^i_k \in \mathcal{E}^i} D^i_k. \tag{4.20}$$

(See also Figure 1(b),(c).) We notice that $E^i \subset E^j$ for $7 \le j < i \le I$. Indeed, each $D^i_k \in \mathcal{E}^i$ is contained in a connected component of $\bigcup_{l=i-1}^I B^l$, denoted by $D^{i-1}_{k'}$. Since D^i_k intersects E^{i-1}, $D^{i-1}_{k'}$ intersects E^{i-1} and the definition of E^{i-1} implies $D^{i-1}_{k'} \in \mathcal{E}^{i-1}$ and $D^{i-1}_{k'} \subset E^{i-1}$.

This yields $E^i \subset E^{i-1}$, as desired. Moreover, note that each E^i contains J_0 and therefore particularly $\partial Q_\mu \subset E^i$ for $i \geq 1$. For later we also introduce the associated 'removed sets'

$$\mathcal{P}^i = \{Q \in \bigcup\nolimits_{l=i}^{I} \mathcal{Q}^l_{\rm bad} : \overline{Q'''} \subset E^i,\ \overline{Q'''} \cap E^{i+1} = \emptyset\}, \quad R^i = \bigcup\nolimits_{Q \in \mathcal{P}^i} \overline{Q'''} \tag{4.21}$$

for $7 \leq i \leq I-1$ and observe that

$$R^j \cap R^i = \emptyset \quad \text{for } 7 \leq j < i \leq I-1. \tag{4.22}$$

To see this, consider $j < i$ and note that $\overline{Q'''} \cap E^i = \emptyset$ for $Q \in \mathcal{P}^j$ since $E^i \subset E^{j+1}$. Thus, we find $R^j \cap E^i = \emptyset$ and therefore (4.22) holds as $R^i \subset E^i$.

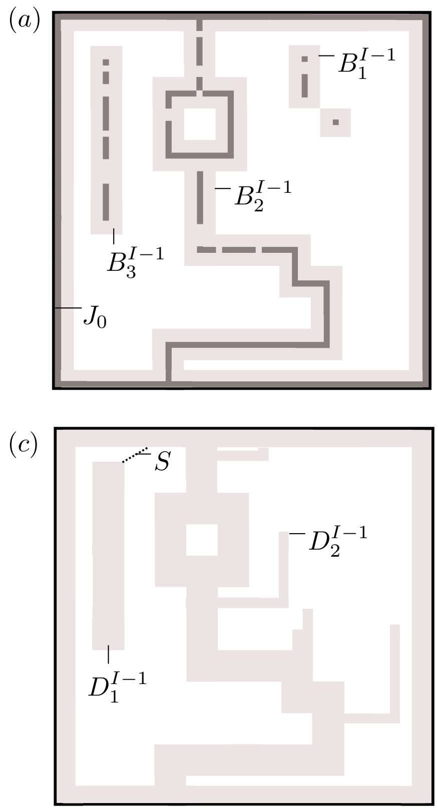

FIGURE 1. (a) A part of Q_μ is depicted with $B^{I-1} = \bigcup_{k=1}^3 B_k^{I-1}$ in light gray and the squares $\mathcal{Q}^{I-1}_{\rm bad}$ in dark gray. Note that $B_1^{I-1} \in \mathcal{B}^{I-1}_{\rm iso}$ and $J_0 \subset B_2^{I-1}$. (b) In light gray we see B^{I-1} and in dark gray B^I. The sets B_1^I, B_2^I, B_3^I are contained in $\mathcal{B}^I_{\rm iso}$ with $B_2^I \subset {\rm sat}(B_1^I)$. We suppose $B_4^I \notin \mathcal{B}^I_{\rm iso}$ and observe $B_4^I \cap B^{I-1} = \emptyset$. (c) The set $E^{I-1} = D_1^{I-1} \cup D_2^{I-1}$ is sketched. Note that $B_4^I \cap E^{I-1} = \emptyset$ although $B_4^I \notin \mathcal{B}^I_{\rm iso}$. Whereas E^{I-1} consists of two components, $E^{I-1}_{\mathcal{S}} := E^{I-1} \cup S$ is connected. (d) In light gray $E^I \cup S$ is depicted and in dark gray $R^{I-1} = R_1^{I-1} \cup R_2^{I-1}$.

Essentially, E^i arises from E^{i-1} by removing the squares $\mathcal{P}^{i-1}$. Moreover, E^i still intersects the squares $\{Q \in \mathcal{Q}^{i-1} \setminus \mathcal{P}^{i-1} : \overline{Q'''} \subset E^{i-1}\}$, but in general covers a smaller portion compared to E^{i-1} (cf. Figure 1(d)). More precisely, we get

$$\mathrm{dist}(x, E^i) \le 10\sqrt{2}s_{i-1} \quad \text{for all } \ x \in E^{i-1} \setminus R^{i-1}. \tag{4.23}$$

In fact, since $x \in E^{i-1}$, by (4.20) we find $Q \in \bigcup_{l=i-1}^{I} \mathcal{Q}^l_{\mathrm{bad}}$ such that $x \in \overline{Q'''} \subset E^{i-1}$. As $x \notin R^{i-1}$, we obtain $Q \notin \mathcal{P}^{i-1}$ and therefore $\overline{Q'''} \cap E^i \neq \emptyset$, which implies (4.23). Before we proceed with the construction of the partition, we notice $E^7 = \bigcup_k D^7_k \supset Q_\mu$ and E^7 is connected. Note, however, that this property in general fails for $E^i, i \ge 8$. The strategy in the following will be to connect the components of E^i by suitable segments lying in E^{i-1}.

Step II (Construction of the partition, induction step): We construct the partition inductively. To this end, we show that for $i \ge 7$ there is a family of closed sets $\mathcal{S}_i$ with

$$\#\mathcal{S}_i \le \theta^{-4}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*), \tag{4.24}$$

such that each $S \in \mathcal{S}_i$ is a finite union of closed segments and $E^i_{\mathcal{S}} := E^i \cup \bigcup_{S\in\mathcal{S}_i} S$ is connected. Since $\partial Q_\mu \subset E^i$, we then get that the components of $Q_\mu \setminus E^i_{\mathcal{S}}$ are simply connected sets. Moreover, we assume (recall (4.21))

$$\mathcal{H}^1\big(\bigcup_{S\in\mathcal{S}_i} S\big) \le c\theta^{-5}\mathcal{H}^1(J_u^*)\sum_{j=8}^{i}\theta^{jr} + c\theta^{-3}\sum_{j=7}^{i-1}\mathcal{H}^1(J_u^*\cap R^j). \tag{4.25}$$

We begin with $i = 7$ and set $\mathcal{S}_7 = \emptyset$. As seen at the end of Step I, $E^7_{\mathcal{S}}$ is connected.

Now assume the above assertions, in particular (4.24)-(4.25), hold for $i-1$, $i \ge 8$. We construct $\mathcal{S}_i$ and $E^i_{\mathcal{S}}$ as follows. Recall that each connected component $D^i_k \in \mathcal{E}^i$ satisfies $d(D^i_k) \ge \theta^{-ir}s_i$ by (4.18). Consequently, by (4.19) we get for θ small

$$\#\mathcal{E}^i \le \theta^{ir}s_i^{-1}\sum_k d(D^i_k) \le c\theta^{-3}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*) \le \tfrac{1}{4}\theta^{-4}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*). \tag{4.26}$$

We denote the connected components of $E^i \cup \bigcup_{S\in\mathcal{S}_{i-1}} S$ by $\mathcal{F}^i := (F^i_k)_{k=1}^{M_i}$ in the following. Observe that each F^i_k is a union of closed sets with Hausdorff-dimension one or two. By (4.24) (for $i-1$) and (4.26) we have

$$M_i = \#\mathcal{F}^i \le \tfrac{1}{4}\theta^{-4}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*) + \theta^{-4}\theta^{(i-1)r}s_{i-1}^{-1}\mathcal{H}^1(J_u^*) \le \tfrac{1}{2}\theta^{-4}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*) \tag{4.27}$$

for θ small enough recalling that $0 < r \le \frac{1}{24}$ (see before (4.9)) and $s_i = \theta s_{i-1}$. Now the strategy is to add segments in order to eventually obtain a connected set. By the definition of $\mathcal{F}^i$, $E^{i-1}_{\mathcal{S}} = E^{i-1} \cup \bigcup_{S\in\mathcal{S}_{i-1}} S$ and the fact that $E^i \subset E^{i-1}$ (see before (4.21)) we get

$$E^{i-1}_{\mathcal{S}} = \bigcup_{k=1}^{M_i} F^i_k \cup (E^{i-1}\setminus E^i). \tag{4.28}$$

Recall that $E^{i-1}_{\mathcal{S}}$ is connected by hypothesis. We denote the connected components of R^{i-1} (see (4.21)) by $\mathcal{R}^{i-1} = (R^{i-1}_k)_k$ and let $\bar{c} = 10\sqrt{2}$ for shorthand. Observe that by (4.23) and the fact that $E^i \subset \bigcup_{k=1}^{M_i} F^i_k$

$$\mathrm{dist}\,\big(x, \bigcup_{k=1}^{M_i} F^i_k\big) \le \bar{c}s_{i-1} \quad \text{for all } \ x \in E^{i-1}\setminus R^{i-1}. \tag{4.29}$$

We now claim that for all $F^i_j \in \mathcal{F}^i$ there is another component $F^i_k \in \mathcal{F}^i$ such that (a) $\mathrm{dist}(F^i_j, F^i_k) \le 4\bar{c}s_{i-1}$ or (b) there is a corresponding $R^{i-1}_l \in \mathcal{R}^{i-1}$ such that

$$\mathrm{dist}(F^i_j, R^{i-1}_l) \le 4\bar{c}s_{i-1}, \qquad \mathrm{dist}(F^i_k, R^{i-1}_l) \le 4\bar{c}s_{i-1}. \tag{4.30}$$

In fact, fix $F^i_j \in \mathcal{F}^i$ and assume (a) does not hold, i.e. $\mathrm{dist}(F^i_j, F^i_k) > 4\bar{c}s_{i-1}$ for all $k \neq j$. By (4.28) we can find $F^i_k \in \mathcal{F}^i$, $j \neq k$, and a (closed) curve γ in $\overline{E^{i-1}\setminus E^i}$ such that $F^i_j \cup F^i_k \cup \gamma$

is connected (see Figure 2(a)). We let γ' be the (unique) connected component of

$$\gamma \cap \{x \in Q_\mu : \mathrm{dist}(x, F_j^i) \geq 2\bar{c}s_{i-1}\}$$

having nonempty intersection with F_k^i. By (4.29) and the fact that (a) does not hold, we then find some $x \in R^{i-1} \cap \gamma'$ with $2\bar{c}s_{i-1} \leq \mathrm{dist}(x, F_j^i) \leq 4\bar{c}s_{i-1}$.

Let $R_l^{i-1} \in \mathcal{R}^{i-1}$ be the component containing x and define $\gamma'' = \gamma' \cap R_l^{i-1}$. If γ'' intersects F_k^i, we see that R_l^{i-1} has the desired property (4.30). Otherwise, we find $y \in \gamma''$ with $y \in \partial R_l^{i-1}$ and therefore by (4.29) and the definition of γ' we get some $F_{k'}^i \in \mathcal{F}^i$, $k' \neq j$ (and possibly $k' \neq k$), so that $\mathrm{dist}(y, F_{k'}^i) \leq \bar{c}s_{i-1}$. Herefrom we again deduce (4.30) with $F_{k'}^i$ in place of F_k^i.

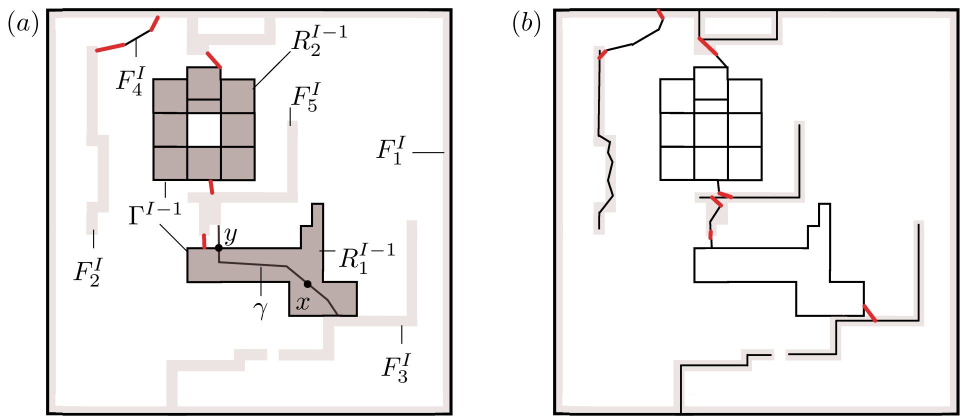

Figure 2. (a) The components of F^I and $\mathcal{S}_*$ are depicted, where the (union of) segments S_l, $l = 1, \ldots, 4$, connecting $(F^I)_{k=1}^5$ are highlighed in red and (a possible choice of) the set Γ^{I-1} is illustrated in black. While in the example we can choose $\Gamma^{I-1} \cap R_1^{I-1} = \partial R_1^{I-1}$, in R_2^{I-1} a more elaborated definition of Γ^{I-1} is necessary (cf. Lemma 4.2) since R_2^{I-1} is not simply connected. We also sketched a curve γ as introduced below (4.30) such that $F_3^I \cup F_5^I \cup \gamma$ is connected. (b) In black the segments in $\mathcal{S}_I$ as well as the components $(\Gamma_k^*)_k$ contained in E^I are illustrated. The components are combined to a connected set by the red segments.

We observe that in case (a) one can choose a (closed) segment S with $\mathcal{H}^1(S) \leq 4\bar{c}s_{i-1}$ such that $F_j^i \cup S \cup F_k^i$ is connected and in case (b) we can find two segments S^1, S^2 with $\mathcal{H}^1(S^1 \cup S^2) \leq 8\bar{c}s_{i-1}$ such that $S^1 \cup S^2 \cup R_l^{i-1} \cup F_j^i \cup F_k^i$ is connected, where the component R_l^{i-1} is chosen as above. Thus, for a universal $c > 0$ large enough we can find $M_i - 1$ sets $(\hat{R}_l^{i-1})_{l=1}^{M_i-1} \subset \mathcal{R}^{i-1} \cup \{\emptyset\}$ and sets $(S_l)_{l=1}^{M_i-1}$ with

$$\mathcal{H}^1(S_l) \leq cs_{i-1}, \tag{4.31}$$

where each S_l consists of at most two segments, so that $\bigcup_{k=1}^{M_i} F_k^i \cup \bigcup_{l=1}^{M_i-1}(\hat{R}_l^{i-1} \cup S_l)$ is connected. Indeed, the construction of sets connecting two different components $(F_k^i)_k$ has been addressed above and the fact that it suffices to consider $M_i - 1 = \#\mathcal{F}^i - 1$ sets may be seen by induction over the number of components M_i. Recalling (4.21) we apply Lemma 4.2 for the squares $\mathcal{R}^j = \mathcal{P}^{i-1} \cap \mathcal{Q}^j \subset \mathcal{Q}^j_{\rm bad}$, $i-1 \le j \le I$ and obtain a set $\Gamma^{i-1} \supset \partial R^{i-1}$, being a finite union of closed segments, such that $\Gamma^{i-1} \cap \hat{R}^{i-1}_l$ is connected for $l = 1, \ldots, M_i - 1$ and

$$\mathcal{H}^1\big(\bigcup\nolimits_{l=1}^{M_i-1} \hat{R}^{i-1}_l \cap \Gamma^{i-1}\big) \le \mathcal{H}^1\big(\Gamma^{i-1}\big) \le c\theta^{-3}\mathcal{H}^1(R^{i-1} \cap J_u^*). \tag{4.32}$$

Then we introduce the new jump components

$$\mathcal{S}_* = \big(S_l \cup (\hat{R}^{i-1}_l \cap \Gamma^{i-1})\big)_{l=1}^{M_i-1}, \qquad \mathcal{S}_i = \mathcal{S}_{i-1} \cup \mathcal{S}_*.$$

We are now in the position to confirm (4.24)-(4.25). First, each component in $\mathcal{S}_i$ is closed and a finite union of closed segments. Moreover, we observe that $E^i_{\mathcal{S}} := E^i \cup \bigcup_{S \in \mathcal{S}_i} S$ is connected by the definition of $(F^i_k)_k$ and $\mathcal{S}_*$ (cf. Figure 2(a)). By (4.24) (for $i-1$) and (4.27) we find

$$\#\mathcal{S}_i \le \#\mathcal{S}_{i-1} + \#\mathcal{F}^i - 1 \le \theta^{-4}\theta^{r(i-1)}s_{i-1}^{-1}\mathcal{H}^1(J_u^*) + \tfrac{1}{2}\theta^{-4}\theta^{ir}s_i^{-1}\ \mathcal{H}^1(J_u^*) \le \theta^{-4}\theta^{ir}s_i^{-1}\mathcal{H}^1(J_u^*)$$

for θ small enough. This gives (4.24). Moreover, we use (4.27) as well as (4.31) to find

$$\sum\nolimits_{l=1}^{M_i-1} \mathcal{H}^1(S_l) \le c(M_i - 1)s_{i-1} \le c\theta^{-5}\theta^{ir}\mathcal{H}^1(J_u^*).$$

This together with (4.32) and (4.25) for step $i-1$ yield (4.25) for step i.

Step III (Construction of the partition, final step): We are now in a position to define a partition $(P'_j)_j$ with the desired properties. After iteration step I we have the connected set $E^I_{\mathcal{S}} = E^I \cup \bigcup_{S \in \mathcal{S}_I} S$. Note that by the definition of $\mathcal{D}^I$ in (4.18) and the choice of I (see (4.11)(ii)) we have $\mathcal{D}^I = \mathcal{B}^I$. Consequently, recalling (4.20) we see that E^I consists of the connected components of B^I having nonempty intersection with E^{I-1}. Similarly as before, by $\mathcal{E}^I \subset \mathcal{D}^I = \mathcal{B}^I$ we denote the connected components D^I_k with $D^I_k \cap E^{I-1} \neq \emptyset$.

In view of the remark before (4.11) we observe that each D^I_k contains exactly one connected component $\Gamma^*_{j_k}$ of J_u^*. More precisely, by (4.8), (4.11)(ii) we have

$$\mathrm{dist}(x, \Gamma^*_{j_k}) \le cs_I \quad \text{ for all } x \in D^I_k \tag{4.33}$$

for a universal constant $c > 0$. Moreover, possibly passing to a larger I we may assume $\#\mathcal{E}^I \le s_I^{-1}\mathcal{H}^1(J_u^*)$ since $\#\mathcal{E}^I$ is bounded from above by the number of components of J_u^* independently of $I \in \mathbb{N}$.

Consequently, arguing similarly as in Step II we can connect the components $\mathcal{S}_I$ and $(J_u^* \cap D^I_k)_k$ with segments (cf. Figure 2(b)). By (4.24) we get $\#\mathcal{E}^I + \#\mathcal{S}_I \le c\theta^{-4}s_I^{-1}\mathcal{H}^1(J_u^*)$ and thus in view of (4.33) we can find a family of closed segments $\hat{\mathcal{S}}$ with $\#\hat{\mathcal{S}} \le c\theta^{-4}s_I^{-1}\mathcal{H}^1(J_u^*)$ and $\mathcal{H}^1(\hat{S}) \le cs_I$ for all $\hat{S} \in \hat{\mathcal{S}}$ such that

$$\bigcup\nolimits_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k} \cup \bigcup\nolimits_{\hat{S} \in \hat{\mathcal{S}}} \hat{S} \cup \bigcup\nolimits_{S \in \mathcal{S}_I} S$$

is connected, where as before $\Gamma^*_{j_k}$ denotes the component contained in D^I_k. Note that $\partial Q_\mu \subset \bigcup_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k}$ since $\partial Q_\mu \subset J_0 \subset E^I$. Thus, letting

$$\mathcal{S} = \mathcal{S}_I \cup \hat{\mathcal{S}} \cup \{\Gamma^*_{j_k} : D^I_k \in \mathcal{E}^I\} \tag{4.34}$$

we finally find that the connected components of $Q_\mu \setminus \bigcup_{S \in \mathcal{S}} S$, denoted by $(P'_j)_{j=1}^m$, are simply connected and form a partition of Q_μ (up to a set of negligible measure). By construction $\bigcup_{j=1}^m \partial P'_j$ is a finite union of closed segments. Using (4.25), $\sum_{S \in \hat{\mathcal{S}}} \mathcal{H}^1(S) \le c\theta^{-4}\mathcal{H}^1(J_u^*)$ and the fact that the sets $(R^i)_i$ are pairwise disjoint (cf. (4.22)) we obtain for $C = C(\theta, p)$

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P'_j\big) \le \sum\nolimits_{j=1}^m \mathcal{H}^1(\partial P'_j) \le C\mathcal{H}^1(J_u^*) \le C\mathcal{H}^1(J_u),$$

where in the last step we employed (4.7). This gives (4.13).

Step IV (Covering): We now finally define a Whitney-type covering of Q_μ associated to the partition $(P'_j)_{j=1}^m$. For each $8 \le i \le I$ we define the sets

$$\begin{aligned} T^i &= \bigcup\nolimits_{Q \in \mathcal{T}^i} \overline{Q} && \text{with} \quad \mathcal{T}^i = \{Q \in \mathcal{Q}^i : \ Q \subset Q_\mu \setminus E^i\}, \\ T^i_- &= \bigcup\nolimits_{Q \in \mathcal{T}^i_-} \overline{Q} && \text{with} \quad \mathcal{T}^i_- = \{Q \in \mathcal{T}^i : Q'' \subset T^i\}, \\ T^i_{--} &= \bigcup\nolimits_{Q \in \mathcal{T}^i_{--}} \overline{Q} && \text{with} \quad \mathcal{T}^i_{--} = \{Q \in \mathcal{T}^i_- : Q'' \subset T^i_-\}, \end{aligned} \tag{4.35}$$

where loosely speaking T^i_-, T^i_{--} arise from T^i by removing 'layers' of squares. (We refer to Figure 3 for an illustration.) As a preparation we observe that $T^j \supset T^i$ for $8 \le i < j \le I$ since $E^j \subset E^i$ and $s_j < s_i$. This particularly implies

$$T^{i-1}_- \subset T^i_-, \qquad \mathrm{dist}(\partial T^{i-1}_-, \partial T^i_-) \ge s_{i-1} \tag{4.36}$$

for $9 \le i \le I$. Moreover, we find

$$T^i \setminus T^i_{--} \cap \bigcup\nolimits_{Q \in \mathcal{Q}^i_{\mathrm{bad}}} Q = \emptyset. \tag{4.37}$$

In fact, for $Q \subset T^i \setminus T^i_{--}$ we have $\overline{Q'''} \not\subset E^i$ and $\overline{Q'''} \cap E^i \neq \emptyset$ (see Figure 3(a)). If $Q \in \mathcal{Q}^i_{\mathrm{bad}}$, however, the connected component D^i_k of $\bigcup_{l=i}^I B^l$ containing $\overline{Q'''}$ has nonempty intersection with E^i and then the definition in (4.20) implies $\overline{Q'''} \subset D^i_k \subset E^i$. This gives a contradiction.

We now introduce the covering as follows. Recall that each set $T^i_- \setminus T^{i-1}_-$ for $8 \le i \le I$ is the union of squares in $\mathcal{Q}^i$ (up to a negligible set), where we set $T^7_- = \emptyset$. We define for $8 \le i \le I$

$$\mathcal{C}^i = \{Q \in \mathcal{Q}^i : Q \subset T^i_- \setminus T^{i-1}_-\}. \tag{4.38}$$

Hereby we obtain a covering of $Q_\mu \setminus T^I_-$. More precisely we have

$$T^I_- \subset \bigcup\nolimits_{Q \in \bigcup_{k \le I} \mathcal{C}^k} Q' \subset Q_\mu, \tag{4.39}$$

where the second inclusion follows from (4.35) and the fact that $\partial Q_\mu \subset E^i$ for all $i \in \mathbb{N}$. To complete the covering, we introduce for all $k \ge I+1$

$$\mathcal{G}_k = \{Q \in \mathcal{Q}^k : \ Q'' \subset Q_\mu, \ \ Q'' \cap (E^I \cap J_u) = \emptyset\}. \tag{4.40}$$

Assuming that we have already constructed $\mathcal{C}^I, \ldots, \mathcal{C}^n$ for $n \ge I$ we define

$$\mathcal{C}^{n+1} = \big\{Q \in \mathcal{G}_{n+1} : Q \cap \bigcup\nolimits_{k=8}^{n} \bigcup\nolimits_{\hat{Q} \in \mathcal{C}^k} \hat{Q} = \emptyset\big\}. \tag{4.41}$$

Clearly, $\mathcal{C} := \bigcup_{k=8}^\infty \mathcal{C}^k$ is a covering of Q_μ (up to a set of negligible measure) consisting of pairwise disjoint, dyadic squares. Finally, for $8 \le i \le I$ we define with U_i as given in (4.12)

$$\mathcal{Y}^i = \{Q \in \mathcal{C}^i : Q'' \subset U_i, \ \mathcal{H}^1(J_u \cap Q') > \theta^2 s_i\}, \qquad Y^i = \mathrm{sat}\Big(\bigcup\nolimits_{Q \in \mathcal{Y}^i} \overline{Q''}\Big). \tag{4.42}$$

Let $Z = \bigcup_{8 \le i \le I} Y^i$ and observe that (4.14) holds since $Y^i \subset U_i$.

We now confirm (4.15). First, we already noticed (4.39). Then taking (4.40) into account, using $E^I \cap J_u \subset \bigcup_{D^I_k \in \mathcal{E}^I} \Gamma^*_{jk} \cap J_u \subset \big(\bigcup_{j=1}^m \partial P'_j \cap J_u\big)$ (see (4.34)) and arguing as in the construction of a Whitney covering for open sets, we obtain (4.15)(i).

To see (4.15)(ii), we fix $Q_1 \in \mathcal{C}^i$ and $Q_2 \in \mathcal{C}^j$ with $j \ge i$ and $Q'_1 \cap Q'_2 \neq \emptyset$. If $i \ge I+1$, we deduce $j \le i+1$ in view of (4.40)-(4.41), where one argues as in the construction of a Whitney covering. If $i \le I-1$, we suppose we had $j \ge i+2$. Then by (4.38) we get $Q_1 \subset T^i_-$ and $Q_2 \cap T^{i+1}_- = \emptyset$. Consequently, using (4.36) we obtain $\mathrm{dist}(Q_1, Q_2) \ge s_i$ and thus the contradiction $Q'_1 \cap Q'_2 = \emptyset$.

Finally, for the case $i = I$ it suffices to show that all $Q \in \mathcal{Q}^{I+1}$ with $Q \subset Q_\mu \setminus T^I_-$ and $Q'_1 \cap Q' \neq \emptyset$ fulfill $Q \in \mathcal{C}^{I+1}$. First, Q satisfies $Q'' \subset T^I$ by (4.35) and (4.38). Then $Q'' \cap E^I = \emptyset$ by (4.35), which also implies $Q'' \subset Q_\mu$ since $\partial Q_\mu \subset E^I$. Consequently, $Q \in \mathcal{G}_{I+1}$ and thus $Q \in \mathcal{C}^{I+1}$ as $Q \subset Q_\mu \setminus T^I_-$.

Moreover, property (4.15)(iii) directly follows from (4.15)(ii) recalling the fact that each $x \in Q_\mu$ is contained in at most four sets Q', $Q \in \mathcal{Q}^i$.

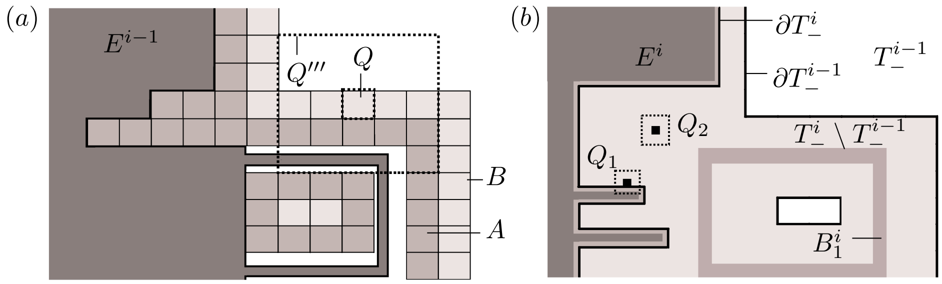

FIGURE 3. (a) The set E^{i-1} and the 'layers' $A = T^{i-1} \setminus T^{i-1}_-$, $B = T^{i-1}_- \setminus T^{i-1}_{--}$ are sketched, where in general E^{i-1} is not a union of squares in $\mathcal{Q}^{i-1}$. Note that a square Q in $A \cup B$ is not in $\mathcal{Q}^{i-1}_{\rm bad}$ since $\overline{Q'''}$ would then belong to E^{i-1}. (b) In dark gray we have depicted E^i, which is a subset of E^{i-1}. The sets T^{i-1}_- and $T^i_- \setminus T^{i-1}_-$ are illustrated in white and light gray, respectively. Note that ∂T^{i-1}_-, ∂T^i_- satisfy (4.36). Let $B^i_1 \in \mathcal{B}^i_{\rm iso}$ be the only isolated component in the example. Observe that B^i_1 is also contained in $T^i_- \setminus T^{i-1}$ and does not belong to E^i, although $B^i_1 \subset E^{i-1}$. Moreover, the black squares Q_1, Q_2 are not in $\mathcal{Q}^i_{\rm bad}$ since otherwise they would either be contained in E^i or in an isolated component in $\mathcal{B}^i_{\rm iso}$.

To see (4.15)(vi), we note that each $Q \in \mathcal{C}^i$ for $i \geq I+1$ satisfies $Q \cap (Q_\mu \setminus T^I_-) \neq \emptyset$. Recalling (4.33)-(4.34) and (4.40) we derive

$$\mathrm{dist}(Q', \bigcup\nolimits_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k}) \leq c s_I, \qquad \mathrm{dist}(Q', \bigcup\nolimits_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k} \cap J_u) > 0.$$

Consequently, (4.11)(i) for θ small enough yields (vi). It now remains to prove (4.15)(iv),(v), where by (4.15)(vi) it suffices to consider $8 \leq i \leq I$.

We confirm (4.15)(iv). To this end, fix $Q \in \mathcal{C}^i$, $8 \leq i \leq I$, with $Q'' \not\subset Z$. By (4.42) we can even assume that $Q'' \not\subset U_i$ since $Q'' \not\subset Z$ together with $Q'' \subset U_i$ already implies (4.15)(iv). Let $Q_* \in \mathcal{Q}^{i-1}$ with $Q \subset Q_*$. First, if $Q \subset T^{i-1} \setminus T^{i-1}_-$, we see $Q_* \notin \mathcal{Q}^{i-1}_{\rm bad}$ by (4.37). Then by (4.8) we get

$$\mathcal{H}^1(J_u \cap Q') \leq \mathcal{H}^1(J^*_u \cap Q'_*) \leq \theta^3 s_{i-1} = \theta^2 s_i. \tag{4.43}$$

Otherwise, we have $Q \subset T^i_- \setminus T^{i-1}$. Thus, as T^{i-1} consists of squares in $\mathcal{Q}^{i-1}$, we get $Q_* \cap T^{i-1} = \emptyset$ and then $Q_* \cap E^{i-1} \neq \emptyset$ by (4.35). If now $Q \notin \mathcal{Q}^i_{\rm bad}$ or $Q_* \notin \mathcal{Q}^{i-1}_{\rm bad}$, the assertion follows directly from (4.8) or similarly as in (4.43), respectively.

Otherwise, we get $Q \in \mathcal{Q}^i_{\rm bad}$, $Q_* \in \mathcal{Q}^{i-1}_{\rm bad}$ and thus $\overline{Q'''_*} \subset B^{i-1}$. Recalling $Q_* \cap E^{i-1} \neq \emptyset$ and (4.20) (for $i-1$) we get that $\overline{Q'''_*}$ is contained in a component of $\mathcal{E}^{i-1}$, which implies $Q \subset \overline{Q'''_*} \subset E^{i-1}$. Recall also that $Q'' \not\subset U_i$. Consequently, $\overline{Q'''}$ is contained in a component $\mathcal{B}^i \setminus \mathcal{B}^i_{\rm iso}$ and therefore by the remark below (4.18), $\overline{Q'''}$ is contained in a component of $\mathcal{D}^i$.

Since $Q \subset E^{i-1}$, again by (4.20) (for i) we derive $\overline{Q'''} \subset E^i$ (cf. also Figure 3(b)). This, however, gives a contradiction as $Q \in \mathcal{C}^i$ and thus $Q \subset T^i_-$, where $T^i_- \cap E^i = \emptyset$. This concludes the proof of (iv).

Finally, to see (4.15)(v) we suppose that for $Q \in \mathcal{C}^i$, $8 \le i \le I$, there is a neighbor $\hat{Q} \in \mathcal{C}^{i-1}$ such that $Q'' \cap \hat{Q}'' \neq \emptyset$. Then by construction we find $\hat{Q} \subset T^{i-1}_- \setminus T^{i-1}_{--}$ and a further square $Q_* \in \mathcal{Q}^{i-1}$ with $Q \subset Q_* \subset T^{i-1} \setminus T^{i-1}_-$. In view of (4.37) the desired property follows as in (4.43). Likewise, if there is a neighbor $\hat{Q} \in \mathcal{C}^{i+1}$ with $Q'' \cap \hat{Q}'' \neq \emptyset$, we see $Q \subset T^i_- \setminus T^i_{--}$ and then again by (4.37) we get the claim from (4.8). □

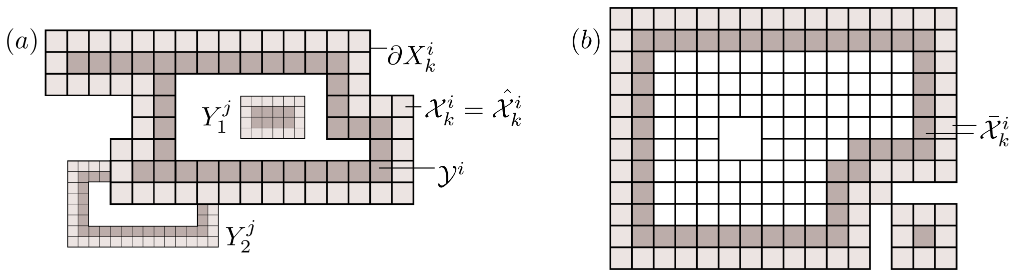

FIGURE 4. (a) The squares $\mathcal{X}^i_k$ are depicted in light gray and the dark gray squares are contained in $\mathcal{Y}^i$. It is possible that $Y^j_1 \subset X^i_k$, whereas Y^j_2, which overlaps with X^i_k, cannot exist since this contradics (4.15)(v). (b) In light and dark gray the squares in $\bar{\mathcal{X}}^i_k$ are depicted (see (4.56) below). We have also illustrated the part of $\mathcal{C}$ contained in X^i_k, which may (in contrast to $\mathcal{C}_*$) contain squares in $\mathcal{Q}^j$, $j < i$.

Proof of Lemma 4.4. Let $\mathcal{C}$ be given satisfying (4.15) and I as in (4.11). Recall (4.42). We denote the (closed) connected components of Y^i by $(Y^i_k)_k$. We first show that

$$Q \in \mathcal{C}^i, Q \subset Y^i_k \quad \text{for all} \quad Q \in \hat{\mathcal{Y}}^i_k := \{Q \in \mathcal{C} : Q \cap Y^i_k \neq \emptyset,\ \partial Q \cap \partial Y^i_k \neq \emptyset\}. \tag{4.44}$$

In fact, for each $Q \in \hat{\mathcal{Y}}^i_k$ there is $\tilde{Q} \in \mathcal{Y}^i$, $\tilde{Q}'' \subset Y^i_k$ such that $|\tilde{Q}'' \cap Q| > 0$. Since $\tilde{Q}$ fulfills the condition stated in (4.42), property (4.15)(v) yields $\tilde{Q}'' \subset \bigcup_{Q_* \in \mathcal{C}^i} \overline{Q_*}$. As $\mathcal{C}$ consists of pairwise disjoint dyadic squares, this yields $Q \in \mathcal{C}^i$ and $Q \subset \tilde{Q}'' \subset Y^i_k$, as desired. We now see that each pair Y^i_k, Y^j_l, $i < j$, either satisfies

$$\mathrm{dist}_\infty(Y^i_k, Y^j_l) \ge \tfrac{1}{2} s_i \tag{4.45}$$

or one set is contained in the other. (Here dist_∞ denotes the distance with respect to the maximum norm.) If not, we would find $Q_i \in \hat{\mathcal{Y}}^i_k$ and $Q_j \in \hat{\mathcal{Y}}^j_l$ such that $Q'_j \cap Q'_i \neq \emptyset$. Then $j = i + 1$ by (4.15)(iv) and (4.44). Moreover, there would be $Q_* \in \mathcal{Y}^j$, $Q''_* \subset Y^j_l$, such that $Q''_* \cap Q''_i \neq \emptyset$. This, however, contradicts (4.15)(v) and the fact that $\mathcal{H}^1(J_u \cap Q'_*) > \theta^2 s_j$ by (4.42). (We also refer to Figure 4(a).)

Now for each $8 \le i \le I$ let $(X^i_k)_k \subset (Y^i_k)_k$ be the components such that for all $j \neq i$ we have $X^i_k \not\subset Y^j_l$ for all components Y^j_l. Accordingly, we denote the sets defined in (4.44) by $\hat{\mathcal{X}}^i_k$. Define $Z^i = \bigcup_k X^i_k$ and note that by (4.45) the sets $(Z^i)_i$ are pairwise disjoint with $Z = \bigcup_{i=8}^I Z^i$. Recalling the definition of $\mathcal{X}^i_k$ before (4.17) we then see that $\hat{\mathcal{X}}^i_k = \mathcal{X}^i_k$. We now show (4.16) and (4.17).

As each component $(X^i_k)_k$ of Z^i is contained in a component of U_i, (4.17)(i) follows directly from (4.9). By (4.45) and the fact that $\mathrm{dist}_\infty(X^i_k, X^i_l) \ge 2s_i$ for all i and $k \neq l$, we obtain (4.17)(ii). The proof of (4.44) showed that for each square in $\mathcal{X}^i_k$ there is an adjacent square in $\mathcal{Y}^i$ (cf. Figure 4(a)). Consequently, we obtain by (4.42) for $c > 0$ sufficiently large

$$\begin{aligned}\sum\nolimits_k \#\mathcal{X}^i_k &\le 8\sum\nolimits_k \#\{Q \in \mathcal{Y}^i : Q'' \subset X^i_k\} \le c\theta^{-2}s_i^{-1}\sum\nolimits_{Q\in\mathcal{Y}^i}\mathcal{H}^1(Q'\cap Z^i \cap J_u)\\ &\le c\theta^{-2}s_i^{-1}\mathcal{H}^1(J_u\cap Z^i),\end{aligned}$$

where we again used that each $x \in Q_\mu$ is contained in at most four enlarged squares. Since $(Z^i)_i$ are pairwise disjoint, the previous estimate also yields

$$\begin{aligned}\sum\nolimits_{i=8}^I \mathcal{H}^1(\partial Z^i) &\le \sum\nolimits_{i=8}^I\sum\nolimits_k \mathcal{H}^1(\partial X^i_k) \le \sum\nolimits_{i=8}^I\big(\sum\nolimits_{Q\in\bigcup_k\mathcal{X}^i_k}\mathcal{H}^1(\partial Q)\big)\\ &\le \sum\nolimits_{i=8}^I c\theta^{-2}\mathcal{H}^1(J_u\cap Z^i) \le c\theta^{-2}\mathcal{H}^1(J_u),\end{aligned}$$

i.e. (4.16)(ii) holds. Moreover, by (4.17)(i) and the fact that $d(X^i_k) \le \mathcal{H}^1(\partial X^i_k)$ (see (4.19) for a similar argument)

$$|Z^i| = \sum\nolimits_k |X^i_k| \le \sum\nolimits_k d(X^i_k)^2 \le \theta^{-ir}s_i\sum\nolimits_k\mathcal{H}^1(\partial X^i_k) \le c\theta^{-ir}s_i\theta^{-2}\mathcal{H}^1(J_u).$$

Likewise, as each component $(X^i_k)_{i,k}$ also satisfies $d(X^i_k) \le 2\sqrt{2}\mu\theta^7$ by (4.10), we conclude the proof of (4.16) by calculating

$$|Z| = \big|\bigcup\nolimits_{i=8}^I Z^i\big| \le c\mu\theta^7\sum\nolimits_{i=8}^I\sum\nolimits_k\mathcal{H}^1(\partial X^i_k) \le c\mu\theta^5\mathcal{H}^1(J_u).$$

Finally, we now define a modification $\mathcal{C}_*$ of $\mathcal{C}$ such that each Z^i consists of squares $\mathcal{C}^i_* := \mathcal{C}_*\cap\mathcal{Q}^i$. To this end, fix a component X^i_k. By (4.38) and (4.44) we have $\bigcup_{Q\in\hat{\mathcal{X}}^i_k} Q \subset T^i_- \setminus T^{i-1}_-$, which by (4.35) has empty intersection with E^i. Since the diameter of each connected component of E^i exceeds $s_i\theta^{-ir}$ (see (4.18)), (4.17)(i) then yields $X^i_k \cap E^i = \emptyset$. Consequently, $X^i_k \subset T^i_-$ and again by (4.38) we then obtain $X^i_k \subset \bigcup_{j=8}^i\bigcup_{Q\in\mathcal{C}^j}\overline{Q}$ (see Figure 3(b) and Figure 4). This together with the fact that $\hat{\mathcal{X}}^i_k \subset \mathcal{Q}^i$ by (4.44) shows that we can replace the covering $\mathcal{C}$ in each component X^i_k by a covering $\mathcal{C}_*$ consisting exclusively of squares in $\mathcal{Q}^i$ such that all conditions in (4.15) remain true. □

4.2. **Modification of the deformation.** In the last section we have seen that for a configuration $u \in \mathcal{W}(Q_\mu)$ the jump J_u outside of $\bigcup_{j=1}^m \partial P'_j \cup Z$ can be controlled in a suitable way. We now show that we can provide a modification $\bar{u}$ of u such that $\bar{u}|_{P'_j}$ is smooth for every P'_j. To this end, we will proceed similarly as in [29] with the additional difficulty, that we have to treat the isolated components with particular care.

Theorem 4.5. *Let $\mu > 0, \theta > 0$ and $p \in [1,2)$. Then there are a universal $c > 0$ and a constant $C = C(\theta,p) > 0$ such that for all $u \in \mathcal{W}(Q_\mu)$ with* (4.4) *and for the corresponding partition $(P'_j)_{j=1}^m$ and covering $\mathcal{C}$ as given by Theorem 4.3 and Lemma 4.4 the following holds: There is a modification $\bar{u} : Q_\mu \to \mathbb{R}^2$, being smooth in $\bigcup_{Q\in\mathcal{C}} Q' \supset Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P'_j)$, and an exceptional set $F \subset Q_\mu$ with*

$$|F| \le c\mu\theta^5\,\mathcal{H}^1(J_u), \qquad \mathcal{H}^1(\partial F) \le C\mathcal{H}^1(J_u) \tag{4.46}$$

such that $\bar{u} = u$ on ∂Q_μ (in the sense of traces) and

$$\begin{aligned} &(i) \quad \|e(\bar{u})\|_{L^p(Q_\mu)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)},\\ &(ii) \quad \|\nabla\bar{u} - \nabla u\|_{L^p(Q_\mu\setminus F)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)},\\ &(iii) \quad \|\bar{u} - u\|_{L^p(Q_\mu\setminus F)} \le C\mu^{\frac{2}{p}}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{4.47}$$

Proof. Let $(P_j')_j$, $Z = \bigcup_{i=8}^I Z^i$ and a Whitney-type covering $\mathcal{C}$ of $Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P_j')$ be given such that (4.15) holds and each Z^i is a union of squares in $\mathcal{C}^i$. We will first use Theorem 3.13 to define infinitesimal rigid motions for each $Q \in \mathcal{C}$ (Step I), where on each connected component of Z we choose a single infinitesimal rigid motion. Afterwards, we estimate the difference of the affine mappings on adjacent squares (Step II).

Finally, with the help of a partition of unity associated to $\mathcal{C}$ we will construct a function being smooth in $\bigcup_{Q\in\mathcal{C}} Q'$ and confirm (4.47) (Step III). In this context, we explicitly exploit $p<2$ and Hölder's inequality as we hereby can compensate the difficulty that $d(X_k^i)s_i^{-1} \to \infty$ as $i \to \infty$ is possible, where $(X_k^i)_{k,i}$ denote the connected components of Z (cf. (4.17)(i) and (4.58) below).

In the following $C = C(\theta, p) > 0$ denotes a generic constant and $c > 0$ is universal. We can suppose that θ is small with respect to c.

Step I (Definition of infinitesimal rigid motions): Let us first consider the subset of 'good squares'

$$\mathcal{C}_{\rm g} := \big\{Q \in \mathcal{C} : Q'' \not\subset Z\big\} \subset \bigcup\nolimits_{i\ge1} \big\{Q \in \mathcal{C}^i : \mathcal{H}^1(J_u \cap Q') \le \theta^2 s_i\big\}, \tag{4.48}$$

where the inclusion follows from (4.15)(iv). We apply Theorem 3.13 on Q', $Q \in \mathcal{C}_{\rm g}$, to find infinitesimal rigid motions $a_Q = a_{A_Q,b_Q}$ and exceptional sets $E_Q \subset Q'$ such that

$$d(Q)^{-\frac{2}{p}}\|u - a_Q\|_{L^p(Q'\setminus E_Q)} + d(Q)^{1-\frac{2}{p}}\|\nabla u - A_Q\|_{L^p(Q'\setminus E_Q)} \le c\|e(u)\|_{L^2(Q')}. \tag{4.49}$$

Moreover, taking (4.48) into account we find for all $Q \in \mathcal{C}_{\rm g}$

$$|E_Q| \le cd(Q)\theta^2\mathcal{H}^1(J_u \cap Q') \le c\theta^4 d(Q)^2, \quad \mathcal{H}^1(\partial E_Q) \le c\mathcal{H}^1(J_u \cap Q'). \tag{4.50}$$

As a preparation for Step II, we now estimate the difference of the infinitesimal rigid motions for neighboring sets in $\mathcal{C}_{\rm g}$. For $Q \in \mathcal{C}_{\rm g}$ we let

$$\mathcal{N}_{\rm g}(Q) = \{\hat{Q} \in \mathcal{C}_{\rm g} \setminus \{Q\} : Q' \cap \hat{Q}' \neq \emptyset\} \tag{4.51}$$

and observe that by (4.15)(ii) we have $\theta d(\hat{Q}) \le d(Q) \le \theta^{-1}d(\hat{Q})$ for all $\hat{Q} \in \mathcal{N}_{\rm g}(Q)$. This also implies $\#\mathcal{N}_{\rm g}(Q) \le c\theta^{-2}$. Moreover, as the covering consists of dyadic squares, $Q' \cap \hat{Q}'$ contains a ball B with radius larger than $c\min\{d(Q), d(\hat{Q})\} \ge c\theta d(Q)$ for some small $c > 0$. Consequently, by (4.50) we find $|\hat{E} \cap B| \le \frac{1}{2}|B|$ for θ sufficiently small, where $\hat{E} = E_{\hat{Q}} \cup E_Q$. Thus, $|B \setminus \hat{E}| \ge c\theta^2|Q'|$ and then we find by Lemma 3.7, (4.49) and the triangle inequality

$$\|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C\|a_Q - a_{\hat{Q}}\|^p_{L^p(B\setminus\hat{E})} \le Cd(Q)^2\|e(u)\|^p_{L^2(Q'\cup\hat{Q}')} \tag{4.52}$$

for all $\hat{Q} \in \mathcal{N}_{\rm g}(Q)$. Therefore, by $\#\mathcal{N}_{\rm g}(Q) \le c\theta^{-2}$ we get

$$\sum\nolimits_{\hat{Q}\in\mathcal{N}_{\rm g}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le Cd(Q)^2\|e(u)\|^p_{L^2(N_Q)}, \tag{4.53}$$

where $N_Q := \bigcup_{\hat{Q}\in\mathcal{N}_{\rm g}(Q)\cup\{Q\}} \hat{Q}'$.

To conclude Step I, it remains to define infinitesimal rigid motions for the squares in $\mathcal{C} \setminus \mathcal{C}_{\rm g}$. Consequently, in view of (4.48) and Lemma 4.4 we have to concern ourselves with the behavior of u inside the isolated components $(X^l_k)_k$ for $8 \le l \le I$. Since $Q' \not\subset X^l_k$ for all $Q \in \mathcal{X}^l_k$, (4.17)(ii) yields $\mathcal{X}^l_k \subset \mathcal{C}_{\rm g}$. Consequently, we can derive an estimate of the form (4.53) for all $Q \in \mathcal{X}^l_k$. Note that $N^l_k := \bigcup_{Q \in \mathcal{X}^l_k} Q'$ is connected (cf. Figure 4) and since $d(X^l_k) \le \theta^{-lr} s_l$ by (4.17)(i) we have by Lemma 3.7

$$\|a_{Q_1} - a_{Q_2}\|_{L^p(N^l_k)} \le C\theta^{-(1+\frac{2}{p})lr}(\|a_{Q_1} - a_{Q_2}\|_{L^p(Q_1')} + \|a_{Q_1} - a_{Q_2}\|_{L^p(Q_2')})$$

for all $Q_1, Q_2 \in \mathcal{X}^l_k$. Thus, the triangle inequality and the fact that N^l_k is connected yield

$$\max_{Q_1,Q_2 \in \mathcal{X}^l_k} \|a_{Q_1} - a_{Q_2}\|_{L^p(N^l_k)} \le C\theta^{-(1+\frac{2}{p})lr} \sum_{\tilde{Q} \in \mathcal{X}^l_k} \sum_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|a_{\tilde{Q}} - a_{\hat{Q}}\|_{L^p(\tilde{Q}')}$$

and therefore the discrete Hölder inequality together with $\#\mathcal{X}^l_k \le c\theta^{-2lr}$ (see (4.17)(i)), $\#\mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k \le 8$ for all $\tilde{Q} \in \mathcal{X}^l_k$ and (4.52) implies

$$\begin{aligned}\max_{Q_1,Q_2 \in \mathcal{X}^l_k} \|a_{Q_1} - a_{Q_2}\|^p_{L^p(N^l_k)} &\le C\theta^{-3plr} \sum_{\tilde{Q} \in \mathcal{X}^l_k} \sum_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|a_{\tilde{Q}} - a_{\hat{Q}}\|^p_{L^p(\tilde{Q}')} \\ &\le C s_l^2 \theta^{-3plr} \|e(u)\|^p_{L^2(N^l_k)}. \end{aligned} \tag{4.54}$$

(See [33, Section 4] for similar arguments.) For each X^l_k we fix an infinitesimal rigid motion by setting $a^l_k = a_Q$ for an arbitrary $Q \in \mathcal{X}^l_k$.

We can now define affine mappings associated to each $Q \in \mathcal{C} \setminus \mathcal{C}_g$. Given $Q \in \mathcal{C}^l$ with $Q'' \subset Z^l$ we choose the component X^l_k with $Q \subset X^l_k$ and define $a_Q = a^l_k$.

Step II (Difference of affine mappings on neighboring squares): Similarly as in (4.51) we introduce the neighbors of each $Q \in \mathcal{C}$ defined by

$$\mathcal{N}(Q) = \{\hat{Q} \in \mathcal{C} \setminus \{Q\} : Q' \cap \hat{Q}' \neq \emptyset\}.$$

Let us compare the infinitesimal rigid motions on neighboring squares. First, consider $Q \in \mathcal{C}$ with $Q \cap Z = \emptyset$. Then $\mathcal{N}(Q) \subset \mathcal{C}_{\rm g}$ since each $\hat{Q} \in \mathcal{N}(Q)$ satisfies $\hat{Q}'' \not\subset Z$. Consequently, the results of Step I are applicable and we find by (4.53) for all $Q \in \mathcal{C}$ with $Q \cap Z = \emptyset$

$$\sum_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C d(Q)^2 \|e(u)\|^p_{L^2(N_Q)}. \tag{4.55}$$

We now concern ourselves with the squares contained in the isolated components. Recall the definition of N^l_k below (4.53). Let $\bar{\mathcal{X}}^l_k = \{Q \in \mathcal{Q}^l : Q \subset X^l_k, Q \cap N^l_k \neq \emptyset\}$, which consists of the squares $\mathcal{X}^l_k$ and its neighbors contained in X^l_k (cf. Figure 4(b)). Fix $Q \in \bar{\mathcal{X}}^l_k$ and recall that a_Q as in (4.49) if $Q \in \mathcal{X}^l_k$ and $a_Q = a^l_k$ otherwise. Moreover, by (4.17)(ii) we get $a_{\hat{Q}} = a^l_k$ for all $\hat{Q} \in \mathcal{N}(Q) \setminus \mathcal{N}_{\rm g}(Q)$. We apply (4.53)-(4.54) and find for all $Q \in \bar{\mathcal{X}}^l_k$ using $\#\mathcal{N}(Q) \le c\theta^{-2}$ and the definition of a^l_k

$$\sum_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C s_l^2 \|e(u)\|^p_{L^2(N_Q)} + C s_l^2 \theta^{-3plr} \|e(u)\|^p_{L^2(N^l_k)}. \tag{4.56}$$

Finally, if $Q \subset X^l_k$, $Q \notin \bar{\mathcal{X}}^l_k$ (see the white squares in Figure 4(b)), we observe that all neighbors lie in the same isolated component and thus by definition

$$\sum_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} = 0. \tag{4.57}$$

We now sum over all components and obtain collecting (4.55)-(4.57)

$$\begin{aligned} H &:= \sum_{Q \in \mathcal{C}} \sum_{\hat{Q} \in \mathcal{N}(Q)} d(Q)^{-p} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \\ &\le C \sum_{l \ge 8} \sum_k \#\bar{\mathcal{X}}^l_k \, s_l^{2-p} \theta^{-3plr} \|e(u)\|^p_{L^2(N^l_k)} + C \sum_{Q \in \mathcal{C}} d(Q)^{2-p} \|e(u)\|^p_{L^2(N_Q)}. \end{aligned}$$

Since $\#\bar{\mathcal{X}}_k^l \le c\#\mathcal{X}_k^l \le c\theta^{-2lr}$ by (4.17)(iii), we get

$$H \le C\sum\nolimits_{l\ge 8}\sum\nolimits_k (\#\bar{\mathcal{X}}_k^l)^{1-\frac{p}{2}}\ s_l^{2-p}\theta^{-4plr}\|e(u)\|^p_{L^2(N_k^l)} + C\sum\nolimits_{Q\in\mathcal{C}} d(Q)^{2-p}\|e(u)\|^p_{L^2(N_Q)}$$

Taking (4.15)(iii) into account we see that each $x \in Q_\mu$ is contained in a bounded number of different neighborhoods N_Q, $Q \in \mathcal{C}_{\rm g}$, with N_Q as defined in (4.53). Moreover, we note that for each $8 \le l \le I$ the sets $(N_k^l)_k$ are pairwise disjoint. We observe that $M_l := \sum_k \#\bar{\mathcal{X}}_k^l \le c\sum_k \#\mathcal{X}_k^l \le Cs_l^{-1}\mu$ by (4.17)(iii) and the assumption that $\mathcal{H}^1(J_u) \le c\theta^{-2}\mu$ (see (4.4)). Consequently, using the discrete Hölder inequality we derive

$$\begin{aligned} H &\le C\sum\nolimits_{l\ge 8} M_l^{1-\frac{p}{2}} s_l^{2-p}\theta^{-4plr}\big(\sum\nolimits_k \|e(u)\|^2_{L^2(N_k^l)}\big)^{\frac{p}{2}} + C\big(\sum\nolimits_{Q\in\mathcal{C}} d(Q)^2\big)^{1-\frac{p}{2}}\|e(u)\|^p_{L^2(Q_\mu)} \\ &\le C\sum\nolimits_{l\ge 8}(\mu s_l)^{1-\frac{p}{2}}\theta^{-4plr}\|e(u)\|^p_{L^2(Q_\mu)} + C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)} \\ &\le C\mu^{2-p}\big(\sum\nolimits_{l\ge 8}\theta^{plr}+1\big)\|e(u)\|^p_{L^2(Q_\mu)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}, \end{aligned} \tag{4.58}$$

where in the second step we used that $\sum_{Q\in\mathcal{C}} d(Q)^2 \le c|Q_\mu| \le c\mu^2$. In the last two steps we employed $s_l = \mu\theta^l$, used $1-\frac{p}{2} \ge 6pr \ge 5pr$ as $r = \frac{1}{24}(2-p)$ (see before (4.9)), as well as $\sum_{l\ge 8}\theta^{plr} \le C = C(\theta,p)$.

Step III (Definition of the modification): We are now in a position to define the modification. First, we choose a partition of unity $(\varphi_Q)_{Q\in\mathcal{C}} \subset C^\infty(\mathbb{R}^2)$ with $\sum_{Q\in\mathcal{C}}\varphi_Q(x) = 1$ for $x \in \bigcup_{Q\in\mathcal{C}} Q'$ and

$$\begin{aligned} &(i)\ \ \operatorname{supp}(\varphi_Q) \subset Q' \text{ for all } Q \in \mathcal{C}, \\ &(ii)\ \ \|\nabla\varphi_Q\|_\infty \le cd(Q)^{-1} \text{ for all } Q \in \mathcal{C}. \end{aligned} \tag{4.59}$$

As the proof of the existence of such a partition is very similar to the construction of a partition of unity for Whitney coverings (see [23, 42]), we omit it here. Let us just briefly mention that the idea it to take a cut-off function $\bar\varphi \in C_c^\infty((0,1)^2)$ and to define the functions φ_Q as suitably rescaled versions of $\bar\varphi$ taking the fact into account that each point is contained in only a bounded number of different squares Q', $Q \in \mathcal{C}$. We now define the modification $\bar u : Q_\mu \to \mathbb{R}^2$ by

$$\bar u = \sum\nolimits_{Q\in\mathcal{C}}\varphi_Q a_Q = \sum\nolimits_{Q\in\mathcal{C}}\varphi_Q(A_Q\ \cdot + b_Q) \tag{4.60}$$

in Q_μ. First, observe that $\bar u$ is smooth in $\bigcup_{Q\in\mathcal{C}} Q'$ and that $\bigcup_{Q\in\mathcal{C}} Q' \supset Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P_j')$ by (4.15)(i). Let

$$F := \bigcup\nolimits_{Q\in\mathcal{C}_{\rm g}} E_Q \cup Z. \tag{4.61}$$

By (4.15)(iii), (4.16), (4.50) and $d(Q) \le 2\sqrt{2}\mu\theta^8$ for all $Q \in \mathcal{C} \subset \bigcup_{j\ge 8}\mathcal{Q}^j$ we derive $|F| \le c\mu\theta^5\mathcal{H}^1(J_u)$ and $\mathcal{H}^1(\partial F) \le C\mathcal{H}^1(J_u)$. This gives (4.46). We obtain $\nabla\bar u = \sum_{\hat Q\in\mathcal{C}}\big(\varphi_{\hat Q}A_{\hat Q} + a_{\hat Q}\otimes\nabla\varphi_{\hat Q}\big)$ and using that $\nabla(\sum_{\hat Q\in\mathcal{C}}\varphi_{\hat Q}) = 0$ we find for $x \in Q$, $Q \in \mathcal{C}$

$$\nabla\bar u(x) = \sum\nolimits_{\hat Q\in\mathcal{C}}\big(\varphi_{\hat Q}(x)A_{\hat Q} + (a_{\hat Q}(x) - a_Q(x))\otimes\nabla\varphi_{\hat Q}(x)\big).$$

Then we get by (4.15)(ii),(iii), (4.58), (4.59) and the discrete Hölder inequality

$$\|e(\bar u)\|^p_{L^p(Q_\mu)} \le C\sum\nolimits_{Q\in\mathcal{C}}\sum\nolimits_{\hat Q\in\mathcal{N}(Q)} d(Q)^{-p}\|a_Q - a_{\hat Q}\|^p_{L^p(Q')} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}, \tag{4.62}$$

which implies (4.47)(i). Similarly, we find using (4.15)(ii),(iii), (4.49), (4.58), (4.59) and the fact that $\bigcup_{Q\in\mathcal{C}_{\rm g}} Q' \supset Q_\mu \setminus F$

$$\begin{aligned}\|\nabla\bar{u} - \nabla u\|^p_{L^p(Q_\mu\setminus F)} &\le C\sum\nolimits_{Q\in\mathcal{C}_{\rm g}} \|\nabla u - A_Q\|^p_{L^p(Q'\setminus F)} + CH\\ &\le C\sum\nolimits_{Q\in\mathcal{C}_{\rm g}} d(Q)^{2-p}\|e(u)\|^p_{L^2(Q')} + CH \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)},\end{aligned}$$

where we repeated the Hölder-type estimate in (4.58). Let $S^k = \{x\in Q_\mu : \mathrm{dist}(x,\partial Q_\mu)\le s_k\}$ and observe $d(Q)\le 2\sqrt{2}s_k$ for all $Q\in\mathcal{C}$ with $Q\cap S^k\neq\emptyset$ by (4.15)(i). Then we recall (4.60) and get by (4.15)(iii), (4.49)

$$\|\bar{u}-u\|^p_{L^p(S^k\setminus F)} \le C\sum\nolimits_{l\ge k}\sum\nolimits_{Q\in\mathcal{C}_{\rm g}\cap\mathcal{C}^l}\|u-a_Q\|^p_{L^p(Q'\setminus F)} \le Cs_k^2\|e(u)\|^p_{L^2(Q_\mu)}. \tag{4.63}$$

In particular for $k=0$ this implies (4.47)(iii).

Finally, to show that $\bar{u}=u$ on ∂Q_μ one may argue, e.g., as in the proof of [16, Theorem 2.1]. We briefly sketch the argument for the reader's convenience. Choose $\psi_k\in C^\infty(Q_\mu)$ such that $\psi_k=1$ in a neighborhood of ∂Q_μ and $\psi_k=0$ on $Q_\mu\setminus S^k$ with $\|\nabla\psi_k\|_\infty\le cs_k^{-1}$. Note that (4.15)(vi) and (4.61) imply $F\cap S_k=\emptyset$ for k large enough. We define $v_k=\psi_k(u-\bar{u})\in SBD(Q_\mu)$ and show $v_k\to 0$ strongly in BD which implies $v_k|_{\partial Q_\mu}\to 0$ in L^1 and implies the assertion. In fact, by (4.63) and Hölder's inequality we get for k sufficiently large

$$\|\bar{u}-u\|_{L^1(S^k)} \le C(\mu s_k)^{1-\frac{1}{p}}\|\bar{u}-u\|_{L^p(S^k)} \le C\mu^{1-\frac{1}{p}}s_k^{1+\frac{1}{p}}\|e(u)\|_{L^2(Q_\mu)}.$$

Now we derive that

$$|Ev_k|(Q_\mu) \le |Eu|(S^k) + |E\bar{u}|(S^k) + cs_k^{-1}\|\bar{u}-u\|_{L^1(S^k)}$$

vanishes for $k\to\infty$ and likewise $\|v_k\|_{L^1(Q_\mu)}\le\|u-\bar{u}\|_{L^1(S^k)}\to 0$. This implies $v_k\to 0$ in BD, as desired. ☐

Remark 4.6. (i) For later reference we recall that the exceptional set F consists of the isolated components and the exceptional sets E_Q for $Q\in\mathcal{C}_{\rm g}$ (cf. (4.61)). In particular, by (4.15)(ii),(iii) and (4.50) we find (for θ small) that $|F\cap Q'|\le\theta|Q|$ for all $Q\in\mathcal{C}$ with $Q\cap Z=\emptyset$.

(ii) Moreover, in view of (4.48), each $Q\in\mathcal{C}$ with $Q\cap Z=\emptyset$ satisfies (4.49).

We close this section with the observation that also $\|\bar{u}\|_\infty$ can be controlled, which is necessary for the proof of (4.3). The reader not interested in the derivation of (4.3) (which will indeed not be needed in the sequel for the proof of Theorem 2.1) may readily skip the following lemma.

Lemma 4.7. *Let be given the situation of Theorem 4.5. Then the modification $\bar{u}: Q_\mu\to\mathbb{R}^2$ can be chosen such that* (4.46)-(4.47) *hold and $\|\bar{u}\|_{L^\infty(Q_\mu\setminus F)}\le C\|u\|_\infty$ for $C=C(\theta,p)$.*

Proof. The goal is to show that for each $Q\in\mathcal{C}_{\rm g}$ there is an infinitesimal rigid motion $\hat{a}_Q$ such that

$$\|\hat{a}_Q\|_{L^\infty(Q')}\le C\|u\|_\infty \tag{4.64}$$

and (4.49) still holds for a possibly larger constant with $\hat{a}_Q$ in place of a_Q. Then we can repeat the previous proof with $\hat{a}_Q$ in place of a_Q and since for each $x\in Q_\mu\setminus F$ we have $Q\in\mathcal{C}_{\rm g}$ for all $Q\in\mathcal{C}$ with $x\in Q'$, we obtain $|\bar{u}(x)|\le C\|u\|_\infty$ in view of (4.60). Let us now show (4.64). Fix $Q\in\mathcal{C}_{\rm g}$. Using (4.49) for the affine mapping a_Q we find by (4.50) and Lemma 3.7

$$\|a_Q\|^p_{L^p(Q')}\le C\|a_Q\|^p_{L^p(Q'\setminus E_Q)}\le Cd(Q)^2\|e(u)\|^p_{L^2(Q')}+C|Q'|\|u\|^p_\infty.$$

Applying Lemma 3.5 we find for all $x \in Q'$

$$d(Q)|A_Q| + |A_Q\, x + b_Q| \leq C(\|e(u)\|_{L^2(Q')} + \|u\|_\infty). \tag{4.65}$$

If now $\|e(u)\|_{L^2(Q')} \leq \|u\|_\infty$, then (4.64) directly follows with $\hat{a}_Q = a_Q$. Otherwise, (4.64) is clearly satisfied with $\hat{a}_Q := 0$ and in view of (4.65) also (4.49) holds for a larger constant since then

$$\|a_Q\|_{L^p(Q')} \leq C d(Q)^{\frac{2}{p}} \|e(u)\|_{L^2(Q')}, \quad \|A_Q\|_{L^p(Q')} \leq C d(Q)^{\frac{2}{p}-1} \|e(u)\|_{L^2(Q')}.$$

□

4.3. **Partitions into John domains.** The essential goals of Section 4.1 and Section 4.2 were to provide a partition of Q_μ into simply connected sets and an associated configuration, which is smooth on each component. We will now apply Theorem 3.3 to obtain a refined decomposition into John domains such that we can apply Korn's inequality (see Theorem 3.2) with uniform constants. This allows to control the distance of the modification from an associated infinitesimal rigid motion on each component and leads to the proof of Theorem 4.1. At the end of this section we will collect all the properties needed for the subsequent analysis in Section 5. The following result is the key ingredient for the derivation of Theorem 4.1.

Theorem 4.8. *Let $\mu, \theta, \eta > 0$ and $p \in [1,2)$. There are a universal $c > 0$ and a constant $C = C(\theta, p) > 0$ such that for all $u \in \mathcal{W}(Q_\mu)$ with (4.4) and for the corresponding partition $(P'_j)_{j=1}^m$, covering $\mathcal{C}$ and modification $\bar{u}$ as derived in Theorem 4.3 and Theorem 4.5, respectively, the following holds:*
There is a partition $(P_j)_{j=1}^n$ of Q_μ and a Borel set $R \subset Q_\mu$ such that each P_j with $P_j \not\subset R$ is a c-John domain with Lipschitz boundary. We have

$$\begin{aligned} &(i) \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \partial P_j\Big) \leq C\mathcal{H}^1(J_u), \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^m \partial P'_j \setminus \bigcup\nolimits_{j=1}^n \partial P_j\Big) = 0, \\ &(ii) \quad |R| \leq \eta, \qquad \mathcal{H}^1(\partial R) \leq C\mathcal{H}^1(J_u), \end{aligned} \tag{4.66}$$

and there are infinitesimal rigid motions a_j, $j = 1, \ldots, n$, such that

$$\|\nabla \bar{v}\|_{L^p(Q_\mu \setminus R)} \leq C\mu^{\frac{2}{p}-1} \|e(u)\|_{L^2(Q_\mu)}, \qquad \|\bar{v}\|_{L^p(P_j \setminus R)} \leq C d(P_j) \|\nabla \bar{v}\|_{L^p(P_j)}, \tag{4.67}$$

where $\bar{v} := \sum_{j=1}^n \chi_{P_j}(\bar{u} - a_j)$.

Note that the occurrence of a 'rest set' R is unavoidable as we have discussed below Theorem 3.3.

Proof. We apply Theorem 4.3 and Theorem 4.5 to obtain a partition $(P'_j)_{j=1}^m$, a covering $\mathcal{C}$, and a modification $\bar{u}$ associated to u. Let us first note that Theorem 3.3 cannot be applied directly since the sets are possibly not Lipschitz. However, we can introduce a refined partition as follows. (Note that this is just a technical point and not the core of the argument.)

For fixed $i \in \mathbb{N}$ let $\mathcal{R}^i$ be the squares $\mathcal{Q}^i$ whose closure have nonempty intersection with $\bigcup_{j=1}^m \partial P'_j$. Since $\bigcup_{j=1}^m \partial P'_j$ consists of finitely many closed segments, we clearly get for $I \in \mathbb{N}$ large enough that $R_1 := \bigcup_{Q \in \mathcal{R}^I} \overline{Q''}$ satisfies $|R_1| \leq \frac{\eta}{2}$, $\mathcal{H}^1(\partial R_1) \leq c\mathcal{H}^1(\bigcup_{j=1}^m \partial P'_j)$ and there is a partition $(P''_j)_{j=1}^M$ of Q_μ such that

$$\begin{aligned} &(i) \quad \bigcup\nolimits_{j=1}^M \partial P''_j = \bigcup\nolimits_{j=1}^m \partial P'_j \cup \partial R_1, \\ &(ii) \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^M \partial P''_j\Big) \leq c\mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^m \partial P'_j\Big) \end{aligned} \tag{4.68}$$

for a universal $c > 0$. We order the partition such that $P''_j = P'_j \setminus R_1$ for $j = 1, \ldots, m$, and note that (again for $I \in \mathbb{N}$ large enough) the sets $(P''_j)_{j=1}^m$ are simply connected domains with

Lipschitz boundary, in particular the union of squares. Moreover, we have that $R_1 \cup \bigcup_{j=1}^m P_j''$ forms a partition of Q_μ.

Now we apply Theorem 3.3 for $\varepsilon = \frac{\eta}{2m}$ on each set in $(P_j'')_{j=1}^m$ and find a partition $Q_\mu = R_1 \cup R_2 \cup P_1''' \cup \ldots \cup P_N'''$ (up to a set of negligible measure) with $R_2 = \bigcup_{j=1}^m \Omega_0^j$ such that $|\Omega_0^j| \le \frac{\eta}{2m}$, $\mathcal{H}^1(\partial\Omega_0^j) \le c\mathcal{H}^1(\partial P_j'')$ for $j = 1, \ldots, m$ and $P_1''', \ldots, P_N'''$ are c-John domains with Lipschitz boundary. Then $|R_2| \le \frac{\eta}{2}$ and since $\sum_{j=1}^m \mathcal{H}^1(\partial P_j'') \le 2\mathcal{H}^1(\bigcup_{j=1}^m \partial P_j'')$, we get by (4.13) and (4.68)

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial\Omega_0^j\big) + \mathcal{H}^1\big(\bigcup\nolimits_{j=1}^N \partial P_j'''\big) \le c\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^M \partial P_j''\big) \le c\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P_j'\big) \le C\mathcal{H}^1(J_u).$$

By $(P_j)_{j=1}^n$ we denote the partition consisting of $(P_j''')_{j=1}^N$, $(\Omega_0^j)_{j=1}^m$, and $(P_j'')_{j=m+1}^M$. Then in view of (4.68) and the previous estimate, (4.66) follows. We let $R = R_1 \cup R_2$ and observe $|R| \le \eta$ as well as $\mathcal{H}^1(\partial R) \le C\mathcal{H}^1(J_u)$. Recall that $\bar{u}$ is smooth on each P_j''', $j = 1, \ldots, N$. Applying Theorem 3.2 on each P_j''' we obtain infinitesimal rigid motions $(a_j)_{j=1}^N$ such that by Hölder's inequality and the fact that $Q_\mu \setminus R = \bigcup_{j=1}^N P_j'''$

$$\|\nabla\bar{v}\|_{L^p(Q_\mu\setminus R)} \le C\|e(\bar{u})\|_{L^p(Q_\mu)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \tag{4.69}$$

where $\bar{v} := \bar{u} - \sum_{j=1}^N \chi_{P_j'''} a_j$. By Poincaré's inequality and a scaling argument we also have $\|\bar{v}\|_{L^p(P_j''')} \le Cd(P_j''')\|\nabla\bar{v}\|_{L^p(P_j''')}$ for $j = 1, \ldots, N$. With $a_j = 0$ for all other components we indeed get $\bar{v} = \sum_{j=1}^n \chi_{P_j}(\bar{u} - a_j)$. □

We are now in a position to give the proof of Theorem 4.1. The reader not interested in the derivation of (4.3) may readily skip the second part of the proof.

Proof of Theorem 4.1. (1) Let $p \in [1, 2)$. We may without restriction assume that $\mathcal{H}^1(J_u) \le c\theta^{-2}\mu$ as otherwise the claim is trivially satisfied (cf. (4.4)). Let $(P_j')_{j=1}^m$, $\mathcal{C}$ be the partition and covering constructed in Theorem 4.3 and let $\bar{u}$ be the modification given by Theorem 4.5. We distinguish two cases:

(a) Suppose first that $\mathcal{H}^1(J_u \setminus \bigcup_{Q\in\mathcal{C}} Q') \ge \mu\theta^2$. Then $\mathcal{H}^1(J_u \cap \bigcup_{j=1}^m \partial P_j') \ge \mu\theta^2$ by (4.15)(i). We apply Theorem 4.8 to obtain the partition $Q_\mu = \bigcup_{j=1}^n P_j$ such that by (4.4) and (4.66)(i)

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^n \partial P_j\big) \le C\mathcal{H}^1(J_u) \le C\mu \le C\mathcal{H}^1(J_u \cap \bigcup\nolimits_{j=1}^m \partial P_j') \le C\mathcal{H}^1(J_u \cap \bigcup\nolimits_{j=1}^n \partial P_j),$$

where in the last step we used $\bigcup_{j=1}^m \partial P_j' \subset \bigcup_{j=1}^n \partial P_j$ up to a set of negligible $\mathcal{H}^1$-measure. This gives (4.2)(i). Let F be the exceptional set derived in Theorem 4.5 (see (4.46)), let R the set in (4.66)(ii) and define

$$E = F \cup R. \tag{4.70}$$

Then (4.1) follows directly from (4.46) and (4.66)(ii) since η can be chosen arbitrarily small. To see (4.2)(ii), we apply (4.47)(ii) and (4.67) to find

$$\|\nabla v\|_{L^p(Q_\mu\setminus E)} \le \|\nabla\bar{v}\|_{L^p(Q_\mu\setminus R)} + \|\nabla u - \nabla\bar{u}\|_{L^p(Q_\mu\setminus F)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \tag{4.71}$$

where $v = u - \sum_{j=1}^n \chi_{P_j} a_j$ and $\bar{v}$ as in (4.67) for infinitesimal rigid motions $(a_j)_{j=1}^n$.

(b) We now suppose $\mathcal{H}^1(J_u \setminus \bigcup_{Q\in\mathcal{C}} Q') \le \mu\theta^2$. (Observe that this includes the case $\mathcal{H}^1(J_u) \le \mu\theta^2$). By the construction of $\bar{u}$ in Theorem 4.5 this implies $\mathcal{H}^1(J_{\bar{u}}) \le \mu\theta^2$. Consequently, we may directly apply Theorem 3.13 on $\bar{u}$ and define $E = \tilde{E} \cup F$ with $\tilde{E}$ as in (3.7) and F as in (4.46). Observe that also in this case (4.1) holds and that (4.2)(i) is trivially satisfied for the partition consisting only of Q_μ. Similarly as in case (a) property (4.2)(ii) follows from (3.8) and (4.47)(ii).

(2) We now show that we can find an exceptional set E' and a configuration $v' = u - \sum_{j=1}^n a'_j \chi_{P_j}$ such that (4.1)-(4.3) hold. We only treat the case (a) where $\mathcal{H}^1(J_u \setminus \bigcup_{Q \in \mathcal{C}} Q') \geq \mu\theta^2$ since (b) is similar. With E as in part (1) we define $\mathcal{P} = \{P_j : |P_j \setminus E| \geq \frac{1}{2}|P_j|\}$ and define $E' = E \cup \bigcup_{P \notin \mathcal{P}} P$. Observe that $\mathcal{H}^1(\partial E') \leq C\mathcal{H}^1(J_u)$ by (4.1), (4.66)(i) and $|E'| \leq 2|E|$. This yields (4.1) for E'.

Note that $\mathcal{P} \subset (P'''_j)_{j=1}^N$ for the components considered before (4.69). Since P'''_j is a c-John domain, we get $|P'''_j| \geq c' d(P'''_j)^2$ for $c' = c'(c)$. Therefore, $|P_j \setminus E| \geq \frac{1}{2}c'(d(P_j))^2$ for all $P_j \in \mathcal{P}$ and by Lemma 3.7 we get $\|a_j\|_{L^p(Q)} \leq C\|a_j\|_{L^p(P_j \setminus E)}$, where Q is a square containing P_j with $|P_j \setminus E| \geq c'|Q|$ for a possibly smaller c'. By Lemma 3.5, (4.67) and the fact that $\|\bar{u}\|_{L^\infty(Q_\mu \setminus F)} \leq C\|u\|_\infty$ (see Lemma 4.7) we then derive

$$\begin{aligned} d(P_j)|A_j| + \|a_j\|_{L^\infty(P_j)} &\leq C d(P_j)|P_j|^{-\frac{1}{2}-\frac{1}{p}}\|a_j\|_{L^p(P_j)} \leq C|P_j|^{-\frac{1}{p}}\|a_j\|_{L^p(P_j)} \\ &\leq C|P_j|^{-\frac{1}{p}}\|a_j\|_{L^p(P_j \setminus E)} \leq C|P_j|^{-\frac{1}{p}}\|\bar{u} - a_j\|_{L^p(P_j \setminus E)} + C\|\bar{u}\|_{L^\infty(Q_\mu \setminus F)} \\ &\leq C(d(P_j))^{1-\frac{2}{p}}\|\nabla \bar{v}\|_{L^p(P_j)} + C\|u\|_\infty. \end{aligned}$$

If $(d(P_j))^{1-\frac{2}{p}}\|\nabla \bar{v}\|_{L^p(P_j)} \leq \|u\|_\infty$, we indeed obtain $\|v\|_{L^\infty(P_j)} \leq C\|u\|_\infty$ and set $v' = v$ on P_j, i.e. $a'_j = a_j$. Otherwise, we set $v' = u$ on P_j, i.e $a'_j = 0$, and derive by a short calculation using the previous estimate

$$\|\nabla v'\|^p_{L^p(P_j \setminus E)} \leq C\|\nabla v\|^p_{L^p(P_j \setminus E)} + C|P_j||A_j|^p \leq C\|\nabla v\|^p_{L^p(P_j \setminus E)} + C\|\nabla \bar{v}\|^p_{L^p(P_j)}.$$

Consequently, summing over all components $P_j \in \mathcal{P}$ and using (4.67), (4.71) we see that also v' satisfies (4.2)(ii) and additionally $\|v'\|_\infty \leq C\|u\|_\infty$, which gives (4.3). □

The goal in the next section will be to apply Theorem 4.1 iteratively on various mesoscopic scales. In this context, however, we will not only need the result of Theorem 4.1, but also the structure of the Whitney-type covering and the exceptional sets analyzed in this section. In the following lemma we investigate the relation between the partition given by Theorem 4.8, the 'rest set' and the covering $\mathcal{C}$ constructed in Theorem 4.3. As a preparation we define

$$R' = R \cup \bigcup_{Q \in \mathcal{C} : |Q' \cap R| \geq \theta |Q|} Q \tag{4.72}$$

and note that $|R'| \leq 12\theta^{-1}\eta$ by (4.66) and (4.15)(iii).

Lemma 4.9. *Let be given the situation of Theorem 4.8 with $(P_j)_{j=1}^n$, R as in (4.66) and R' as in (4.72). Then there is another partition $(P^*_j)_{j=1}^N = \mathcal{P}_R \cup \mathcal{P}_{\rm good} \cup \mathcal{P}_{\rm small}$ of Q_μ and $(A_j)_{j=1}^N \subset \mathbb{R}^{2\times 2}_{\rm skew}$ such that $\mathcal{P}_{\rm good}$ consists of c-John domains, $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P^*_j\big) \leq C\mathcal{H}^1(J_u)$ and with $P^*_{j,\mathcal{C}} := \bigcup_{Q \in \mathcal{C}, Q \cap P^*_j \neq \emptyset} Q$ we get*

$$\begin{aligned} &(i) \quad \sum_{j=1}^N \|\nabla \bar{u} - A_j\|^p_{L^p(P^*_j \setminus R)} \leq C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}, \\ &(ii) \quad P^*_j \subset R' \text{ for all } P^*_j \in \mathcal{P}_R, \\ &(iii) \quad d(Q) \leq 4\sqrt{2}d(P^*_j) \text{ for all } Q \in \mathcal{C} : Q \cap P^*_j \neq \emptyset \text{ and for all } P^*_j \in \mathcal{P}_{\rm good}, \\ &(iv) \quad \text{for all } P^*_k \in \mathcal{P}_{\rm small} \text{ there is } P^*_j \in \mathcal{P}_{\rm good} \text{ with } P^*_{k,\mathcal{C}} \subset P^*_{j,\mathcal{C}}. \end{aligned} \tag{4.73}$$

Remark 4.10. Combining (4.73)(i) with (4.47) and letting E as in (4.1) we also get (see (4.71) for a similar argument)

$$\sum_{j=1}^N \|\nabla u - A_j\|^p_{L^p(P^*_j \setminus E)} \leq C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}. \tag{4.74}$$

Proof of Lemma 4.9. With $(P_j)_{j=1}^n$ from Theorem 4.8 we let

$$\mathcal{P}'_{\rm good} = \{P_j : d(Q) \le 4\sqrt{2}d(P_j) \ \text{ for all } \ Q \in \mathcal{C} : Q \cap P_j \neq \emptyset\} \tag{4.75}$$

and by $\mathcal{P}'_{\rm small}$ we denote the remaining components. Recalling (4.5) we find for each $P_j \in \mathcal{P}'_{\rm small}$ some $Q \in \mathcal{C}$ such that $P_j \subset Q'$. Then (4.15)(ii) shows that $P_j \in \mathcal{P}'_{\rm small}$ intersects at most $c\theta^{-2}$ different squares of $\mathcal{C}$. Consequently, indicating by $\mathcal{P}''_{\rm small}$ the nonempty sets in $\{Q \cap P_j : P_j \in \mathcal{P}'_{\rm small},\ Q \in \mathcal{C}\}$ we derive for $c > 0$ large enough

$$\sum\nolimits_{P \in \mathcal{P}''_{\rm small}} \mathcal{H}^1(\partial P) \le c\theta^{-2} \sum\nolimits_{P_j \in \mathcal{P}'_{\rm small}} \mathcal{H}^1(\partial P_j). \tag{4.76}$$

Note that $\mathcal{P}'_{\rm good} \cup \mathcal{P}''_{\rm small}$ forms a partition. We let $\mathcal{P}_R = \{P \in \mathcal{P}'_{\rm good} \cup \mathcal{P}''_{\rm small} : P \subset R'\}$ and let $\mathcal{P}''_{\rm good} = \mathcal{P}'_{\rm good} \setminus \mathcal{P}_R$, $\mathcal{P}'''_{\rm small} = \mathcal{P}''_{\rm small} \setminus \mathcal{P}_R$. By $\mathcal{C}_{\rm small}$ we denote the squares $Q \in \mathcal{C}$ with $Q \not\subset R'$ and $Q \cap \bigcup_{P_j \in \mathcal{P}''_{\rm good}} P_j = \emptyset$. As then $|Q \cap \bigcup_{P \in \mathcal{P}'''_{\rm small}} P| \ge |Q \setminus R'| \ge (1-\theta)|Q| = \frac{1-\theta}{16}(\mathcal{H}^1(\partial Q))^2$ for $Q \in \mathcal{C}_{\rm small}$ by (4.72), the isoperimetric inequality and (4.76) yield for $C = C(\theta) > 0$

$$\sum_{Q \in \mathcal{C}_{\rm small}} \mathcal{H}^1(\partial Q) \le C \sum_{Q \in \mathcal{C}_{\rm small}} \sum_{P \in \mathcal{P}'''_{\rm small}} \mathcal{H}^1(\partial P \cap \overline{Q}) \le C \sum_{P_j \in \mathcal{P}'_{\rm small}} \mathcal{H}^1(\partial P_j). \tag{4.77}$$

We define $\mathcal{P}_{\rm good} = \mathcal{P}''_{\rm good} \cup (Q)_{Q \in \mathcal{C}_{\rm small}}$ and let $\mathcal{P}_{\rm small} \subset \mathcal{P}'''_{\rm small}$ be the components not contained in some $Q, Q \in \mathcal{C}_{\rm small}$. We denote the sets of the partition $\mathcal{P}_R \cup \mathcal{P}_{\rm good} \cup \mathcal{P}_{\rm small}$ by $(P_j^*)_{j=1}^N$. Note that by Theorem 4.8, $\mathcal{P}_{\rm good}$ consists of c-John domains. The assertion $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P_j^*\big) \le C\mathcal{H}^1(J_u)$ follows from (4.66)(i) and (4.76)-(4.77). Moreover, (4.73)(ii) follows directly from the definition of $\mathcal{P}_R$, (4.73)(iii) is obvious for $(Q)_{Q \in \mathcal{C}_{\rm small}}$ and otherwise we recall (4.75). Likewise, (4.73)(iv) follows from the construction of $\mathcal{P}_{\rm small}$, in particular the definition of $\mathcal{C}_{\rm small}$.

Finally, by (4.47), the fact that $\bar{u}$ is smooth in $Q, Q \in \mathcal{C}_{\rm small}$, and Korn's inequality there are matrices $A_Q \in \mathbb{R}^{2\times 2}_{\rm skew}$ such that

$$\sum\nolimits_{Q \in \mathcal{C}_{\rm small}} \|\nabla \bar{u} - A_Q\|^p_{L^p(Q)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}.$$

For the components $\mathcal{P}'_{\rm good} \cup \mathcal{P}_{\rm small}$ we apply (4.67) and then get that there are matrices $(A_j)_{j=1}^N$ such that (4.73)(i) holds. ☐

The following lemma collects the properties needed for the following analysis in Section 5. Similarly as in the proof of Theorem 4.1 we will distinguish two cases depending on whether the jump set is large or not. This will be reflected in (4.79) below.

Given a covering $\mathcal{C}$ as in Theorem 4.3, a partition $(P_j)_j$ as in Theorem 4.8 (or Lemma 4.9) and an exceptional set R' as in (4.72) we define

$$\mathcal{Q}(P_j; \mathcal{C}, R') = \{Q \in \mathcal{C} : Q \cap P_j \neq \emptyset,\ Q \cap P_j \not\subset R'\} \tag{4.78}$$

and the sets $P_{j,\rm cov} = \bigcup_{Q \in \mathcal{Q}(P_j;\mathcal{C},R')} Q$, which will play a crucial role in Section 5 to control the difference of infinitesimal rigid motions. Note that the definition of $P_{j,\rm cov}$ always depends on $\mathcal{C}, R'$ although not made explicit in the notation.

Lemma 4.11. *Let $\mu, \theta > 0$, $\eta > 0$, $p \in [1,2)$ and r as in* (4.9). *Then there are a universal constant $c > 0$ and $C = C(\theta, p) > 0$ such the the following holds:*

(1) For each $u \in \mathcal{W}(Q_\mu)$ with $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu$ there is a partition $Q_\mu = \bigcup_{j=0}^n P_j$ with

$$P_0 = \emptyset \quad \textit{or} \quad P_0 = Q_\mu, \tag{4.79}$$

a covering $\mathcal{C}_u \subset \bigcup_{i\ge1}\mathcal{Q}^i$ *of* Q_μ *and sets* Z^l_u, $l \ge 8$, *as given in Lemma 4.4 such that with* $S_u := J_u \setminus \bigcup_{Q\in\mathcal{C}_u} Q'$ *we have*

$$\begin{aligned} &(i) \ \ \mathcal{H}^1\big(\bigcup\nolimits_{j=1}^n \partial P_j\big) \le C\mathcal{H}^1(S_u),\\ &(ii) \ \ Q \subset Q_\mu \setminus S_u \ \text{for all } Q \in \mathcal{C}_u, \ \ \overline{Q} \subset Q_\mu \setminus S_u \ \text{for all } Q \subset \bigcup\nolimits_{l\ge8} Z^l_u,\\ &(iii) \ \ \bigcup\nolimits_{l\ge8}\bigcup\nolimits_k \partial X^l_k \subset \bigcup\nolimits_{j=1}^n \partial P_j, \end{aligned} \tag{4.80}$$

where $(X^l_k)_k$ *denote the connected components of* Z^l_u. *Each* Z^l_u *is the union of squares in* $\mathcal{Q}^l\cap\mathcal{C}_u$ *up to a set of measure zero. Moreover, there are exceptional sets* E_u, R_u *such that for* $l \ge 8$ *one has*

$$\begin{aligned} &(i) \ \ |E_u| \le c\mu\theta^2\mathcal{H}^1(J_u) \le cd(Q_\mu)\theta^2\mathcal{H}^1(J_u\cap Q_\mu) \le C\mu^2, \quad |R_u| \le \eta\\ &(ii) \ \ |Q'\cap E_u| \le c\theta|Q| \ \text{for all } Q\in\mathcal{C}_u \text{ with } Q \not\subset R_u, \ Q\cap\bigcup\nolimits_{l\ge8} Z^l_u = \emptyset,\\ &(iii) \ \ d(X^l_k) \le \theta^{-rl}s_l \quad \text{for all } X^l_k,\\ &(iv) \ \ |Z^l_u| \le C\theta^{-lr}s_l\mathcal{H}^1(J_u) \le C\theta^{-rl}s_l\mu, \end{aligned} \tag{4.81}$$

and there are infinitesimal rigid motions $a_j = a_{A_j,b_j}$, $j = 1,\dots,n$, *such that*

$$\begin{aligned} &(i) \ \ \sum\nolimits_{j=0}^n \|\nabla u - A_j\|^p_{L^p(P_{j,\mathrm{cov}}\setminus E_u)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)},\\ &(ii) \ \ \#\{P_{j,\mathrm{cov}} : x\in P_{j,\mathrm{cov}}\} \le c \quad \text{for all} \quad x\in Q_\mu, \end{aligned} \tag{4.82}$$

where $P_{j,\mathrm{cov}}$ *as defined in (4.78) with respect to* $\mathcal{C}_u, R_u$.

(2) If $P_0 = Q_\mu$, *then* $P_{0,\mathrm{cov}} = Q_\mu$, $S_u = \emptyset$, $\bigcup_{l\ge8} Z^l_u = \emptyset$ *and there is a set* $\Gamma_u \subset \partial Q_\mu$ *with* $\mathcal{H}^1(\Gamma_u) \le c\theta\mu$ *such that*

$$\int_{\partial Q_\mu\setminus\Gamma_u} |Tu - a_0|^2\, d\mathcal{H}^1 \le c\mu\|e(u)\|^2_{L^2(Q_\mu)}, \tag{4.83}$$

where Tu *denotes the trace of* u *on* ∂Q_μ.

Proof. Similarly as in the proof of Theorem 4.1 we distinguish two cases depending on whether the jump set is large or not. Let $\mathcal{C}$ be the covering constructed in Lemma 4.4 and $\bar u$ the modification given by Theorem 4.5.

(a) First assume $\mathcal{H}^1(J_u \setminus \bigcup_{Q\in\mathcal{C}} Q') \ge \mu\theta^2$. We define $\mathcal{C}_u = \mathcal{C}$ and $(Z^l_u)_{l\ge8} = (Z^l)_{l\ge8}$ as in Lemma 4.4 such that (4.15) holds. We let E_u as in Theorem 4.1 and $R_u = R'$ as in (4.72). Denote the partition given in Lemma 4.9 by $(P^*_j)_{j=1}^N = \mathcal{P}_R \cup \mathcal{P}_{\mathrm{good}} \cup \mathcal{P}_{\mathrm{small}}$ and note that $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P^*_j\big) \le C\mathcal{H}^1(J_u)$.

For each $P^*_j \in \mathcal{P}_{\mathrm{good}}$ choose a set $\mathcal{T}(P^*_j) \subset \mathcal{P}_{\mathrm{small}}$ with $P^*_{k,\mathcal{C}} \subset P^*_{j,\mathcal{C}}$ for $P^*_k \in \mathcal{T}(P^*_j)$ (cf. (4.73)(iv)) such that $\mathcal{P}_{\mathrm{small}} = \dot\bigcup_{P^*_j\in\mathcal{P}_{\mathrm{good}}} \mathcal{T}(P^*_j)$ is a partition of $\mathcal{P}_{\mathrm{small}}$. Set $\mathcal{T}(P^*_j) = \emptyset$ for $P^*_j \in \mathcal{P}_R$. Let $(P_j)_{j=0}^n$ be the partition of Q_μ with $P_0 = \emptyset$ consisting of the components $(X^l_k)_{l,k}$ of $(Z^l_u)_{l\ge8}$ and the sets

$$P_j := \big(P^*_j \cup \bigcup\nolimits_{P^*_k\in\mathcal{T}(P^*_j)} P^*_k\big) \setminus \bigcup\nolimits_{l\ge8} Z^l_u, \quad \text{for } P^*_j \in \mathcal{P}_R\cup\mathcal{P}_{\mathrm{good}}. \tag{4.84}$$

We now confirm (4.80)-(4.82). First, the definition of the partition directly implies (4.80)(iii). By (4.16)(ii) and $\mathcal{H}^1(J_u) \le c\mu\theta^{-2}$ we see $\mathcal{H}^1(\bigcup_{j=1}^n \partial P_j) \le C\mathcal{H}^1(J_u) \le C\mu$. Then (4.80)(i) follows from the assumption $\mathcal{H}^1(S_u) \ge \mu\theta^2$, where $S_u = J_u \setminus \bigcup_{Q\in\mathcal{C}_u} Q'$. Likewise, the definition of S_u and the fact that $Q' \subset Q_\mu$ for all $Q \in \mathcal{C}_u$ (see (4.15)(i)) imply $Q' \subset Q_\mu \setminus S_u$ and particularly (4.80)(ii).

Moreover, the first part of (4.81)(i) follows from (4.1) and $\mathcal{H}^1(J_u) \le c\mu\theta^{-2}$. Applying (4.66)(ii) with $\frac{\theta}{12}\eta$ in place of η, we get $|R_u| \le \eta$ by (4.72). Likewise, (4.81)(iii),(iv) are consequences of Lemma 4.4. Recalling (4.70) we find $E_u \setminus R \subset F$ with F as defined in Theorem 4.5 and R as in (4.66)(ii). Thus, Remark 4.6(i) implies $|Q' \cap (E_u \setminus R)| \le \theta|Q|$ for all $Q \cap \bigcup_{l\ge 8} Z_u^l = \emptyset$. Then the fact that $|Q' \cap R| \le \theta|Q|$ for all $Q \in \mathcal{C}_u$ with $Q \not\subset R_u$ (see (4.72)) yields (4.81)(ii).

It remains to show (4.82). First, each component $P_j \in (X_k^l)_{l,k}$ of $Z_u := \bigcup_{l\ge 8} Z_u^l$ consists of a union of squares in $\mathcal{Q}^l \cap \mathcal{C}_u$ and thus $P_{j,\mathrm{cov}} \subset P_j \subset F \subset E_u$ (cf. Remark 4.6(i) and (4.61)).

Now consider P_j such that $P_j^* \in \mathcal{P}_R$ with P_j^* as in (4.84). As $P_j \subset P_j^* \subset R_u$, (4.78) implies $P_{j,\mathrm{cov}} = \emptyset$. For P_j with $P_j^* \in \mathcal{P}_{\mathrm{good}}$ we recall that P_j^* is a c-John domain. Let $Q \in \mathcal{Q}(P_j; \mathcal{C}_u, R_u)$ and note that by (4.73)(iv), (4.84), and the choice of $\mathcal{T}(P_j^*)$ we have $Q \cap P_j^* \neq \emptyset$. Since $d(Q) \le 4\sqrt{2}d(P_j^*)$ by (4.73)(iii), Definition 3.1 implies that $|Q' \cap P_j^*| \ge c'|Q|$ for some c' small only depending on the John constant c. As for $Q \in \mathcal{Q}(P_j; \mathcal{C}_u, R_u)$ we have $Q \cap Z_u = \emptyset$ (see (4.84)) and $Q \not\subset R_u$, this also yields

$$|(P_j^* \cap Q') \setminus E_u| \ge c|Q| \tag{4.85}$$

for θ small by (4.81)(ii). Consequently, each $Q \in \mathcal{C}_u$ intersects only a bounded number of different $(P_{j,\mathrm{cov}})_{j=1}^n$, which establishes (4.82)(ii). Recall that by Remark 4.10 we get $(A_j)_{j=1}^N \subset \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ such that (4.74) holds (with E_u in place of E).

For $Q \in \mathcal{Q}(P_j; \mathcal{C}_u, R_u)$ we have $Q \cap Z_u = \emptyset$ and thus by Remark 4.6(ii) there is $A_Q \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ such that (4.49) holds. This together with Lemma 3.7 and (4.85) yields

$$\begin{aligned}
\|\nabla u - A_j\|^p_{L^p(Q\setminus E_u)} &\le C\|\nabla u - A_Q\|^p_{L^p(Q\setminus E_Q)} + C\|A_Q - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}\\
&\le C\|\nabla u - A_Q\|^p_{L^p(Q'\setminus E_Q)} + C\|\nabla u - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}\\
&\le C d(Q)^{2-p}\|e(u)\|^p_{L^2(Q')} + C\|\nabla u - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}.
\end{aligned}$$

Summing over all P_j with $P_j^* \in \mathcal{P}_{\mathrm{good}}$ and $Q \in \mathcal{Q}(P_j; \mathcal{C}_u, R_u)$ we finally get (4.82)(i). Indeed, for the term on the right we use (4.74) and the fact that each $x \in Q_\mu$ is contained in a bounded number of different Q', $Q \in \mathcal{C}_u$. For the left term we apply the discrete Hölder inequality to compute (cf. (4.58) for a similar argument)

$$\sum\nolimits_{Q\in\mathcal{C}_u} (d(Q))^{2-p}\|e(u)\|^p_{L^2(Q')} \le \big(\sum\nolimits_{Q\in\mathcal{C}_u} |Q|\big)^{1-\frac{p}{2}}\|e(u)\|^p_{L^2(Q_\mu)}. \tag{4.86}$$

(b) We now concern ourselves with the case $\mathcal{H}^1(J_u \setminus \bigcup_{Q\in\mathcal{C}} Q') \le \mu\theta^2$. In particular, we have $\mathcal{H}^1(J_{\bar u}) \le \mu\theta^2$. We proceed as in case (b) in the proof of Theorem 4.1 and let $E_u = E' \cup F$ with E' as in (3.7) and F as given by Theorem 4.5. Moreover, we let $P_0 = Q_\mu$, a_0 as in (3.8), $R_u = \emptyset$, $\mathcal{C}_u = \{Q \in \mathcal{Q}^1 : Q \subset Q_\mu\}$ and $Z_u^l = \emptyset$ for all $l \ge 8$.

First, (4.80)(i),(iii) are trivially satisfied since the partition only consists of P_0. Moreover, (4.80)(ii) follows from the fact that $\bigcup_{l\ge 8} Z_u^l = \emptyset$ and $S_u = \emptyset$. Properties (4.81)(iii),(iv) are trivial. By $\mathcal{H}^1(J_u) \le c\mu\theta^{-2}$, (3.7) and (4.46) we get

$$|E_u| \le |E'| + |F| \le c(\mathcal{H}^1(J_{\bar u}))^2 + c\mu\theta^5\mathcal{H}^1(J_u) \le c\mu^2\theta^3.$$

This implies (4.81)(i) and also (4.81)(ii) since $|Q| = 4\mu^2\theta^2$ for all $Q \in \mathcal{C}_u$. As $P_{0,\mathrm{cov}} = P_0$, (4.82) follows from (3.8). Finally, (3.9) applied on $\bar u$ together with $\mathcal{H}^1(J_{\bar u}) \le \mu\theta^2$ and the fact that $Tu = T\bar u$ on ∂Q_μ yields (4.83). $\square$

5. Derivation of the piecewise Korn inequality

This section is devoted to the proof of our piecewise Korn inequality. We first assume that $\Omega = (-\mu_0, \mu_0)^2$ is a square with $\mu_0 > 0$. Similarly as in (4.5) we define $\mathcal{Q}^i := \mathcal{Q}^{t_i}$ for $t_i = \mu_0\theta^i$, $\theta > 0$, and denote by $Q' = \frac{3}{2}Q$ the enlarged square corresponding to $Q \in \mathcal{Q}^i$. (We will apply the results of Section 4 on squares $(-\mu, \mu)^2$ of different sizes and thus in general $s_i \neq t_i$, where s_i as defined before (4.5).) Recall the definition of the set $\mathcal{W}(\Omega)$ before Theorem 4.1. We first establish the main result for functions in $\mathcal{W}(\Omega)$ which additionally satisfy

$$\mathcal{H}^1(Q \cap J_u) \leq \theta^{-1} d(Q) \qquad \text{for all } \; Q \in \bigcup\nolimits_{i\geq 1} \mathcal{Q}^i. \tag{5.1}$$

Afterwards, we drop assumption (5.1) and show the result for all functions in $\mathcal{W}(\Omega)$. Finally, we pass to general domains with Lipschitz boundary and Theorem 2.1 will be derived using a density argument (see Section 5.2).

5.1. **Proof for configurations with regular jump set.** In this section we let $\Omega = Q_{\mu_0} = (-\mu_0, \mu_0)^2$. We first prove Theorem 2.1 for configurations in $\mathcal{W}(Q_{\mu_0})$ satisfying (5.1), which already represents the core of the argument. If (5.1) is violated on a square Q, we add ∂Q to the boundary of the partition and treat the problem on Q as a separate problem independent from the analysis on the rest of the domain (see Theorem 5.2 below for details).

The strategy is to apply first Lemma 4.11 on Q_{μ_0} whereby we derive a partition and corresponding infinitesimal rigid motions. Moreover, we obtain a Whitney covering and an associated exceptional set (Step I). We then apply the arguments iteratively on the squares of the covering in order to establish a partition and estimates which hold up to an exceptional set progressively becoming smaller (cf. (4.81)(i),(ii)). Here we will crucially need the properties (4.80)(i),(ii) which ensure that each part of J_u is only 'used' in one single iteration step to control the length of the boundary of the partition (Step II). Moreover, the validity of Korn's inequality on the enlarged components $P_{j,\mathrm{cov}}$ (see (4.82)(i)) will allow to control the difference of infinitesimal rigid motions given in different iteration steps (Step III). In this context it is again fundamental that $p < 2$ as hereby using Hölder's inequality a blow up of the energy can be avoided (cf. (5.26) below). As before we have to treat the contributions of isolated components separately, where we exploit (4.80)(iii) and (4.81)(iii),(iv). Finally, as $\nabla u \in L^\infty(Q_{\mu_0})$, after a finite number of iteration steps the exceptional set, denoted by E_I, is small enough such that the contribution of $\|\nabla u\|_{L^p(E_I)}$ is negligible (Step IV).

Theorem 5.1. *Let $p \in [1, 2)$. Then Theorem 2.1 holds for all $u \in \mathcal{W}(Q_{\mu_0})$ satisfying* (5.1).

Proof. Let $p \in [1, 2)$ and $u \in \mathcal{W}(Q_{\mu_0})$ be given satisfying (5.1). Note that this particularly implies $\mathcal{H}^1(J_u) \leq 2\sqrt{2}\theta^{-2}\mu_0$. A classical result states that u is piecewise rigid if $\|e(u)\|_{L^2(Q_{\mu_0})} = 0$ (see also Remark 2.2(i)), so we can concentrate on the case $\|e(u)\|_{L^2(Q_{\mu_0})} > 0$. Since $\nabla u \in L^p(Q_{\mu_0})$, we can select ε sufficiently small such that for all Borel sets $B \subset Q_{\mu_0}$ with $|B| \leq 4\varepsilon$ one has

$$\|\nabla u\|^p_{L^p(B)} \leq \mu_0^{2-p} \|e(u)\|^p_{L^2(Q_{\mu_0})}. \tag{5.2}$$

We introduce $p' = 1 + \frac{p}{2} \in (p, 2)$. Later we will pass from p' to p by Hölder's inequality. We let $r = \frac{2-p}{24}$ as in (4.9) and set $\lambda = \frac{(1-r)(2-p)}{3p+2}$. In the following $c > 0$ stands for a universal constant and $C = C(\theta, p) > 0$ for a generic constant independent of ε and μ_0. We can suppose that θ is small with respect to c and λ, particularly that $c\theta^\lambda \leq \frac{1}{6}$ (cf. (5.22) below). At the end of the proof we will fix θ depending on p such that eventually the constant C depends only on p.

Step I (Induction hypothesis): We first formulate the induction hypothesis for step i. Assume we have a partition $\mathcal{P}^i = (P^i_j)_{j=1}^\infty$ of Q_{μ_0} (up to a set of negligible measure), a covering $\mathcal{C}_i$ of Q_{μ_0} with $\mathcal{C}_i \subset \bigcup_{j \ge i+1} \mathcal{Q}^j$ consisting of pairwise disjoint dyadic squares, exceptional sets $E_i, R_i \subset Q_{\mu_0}$ with $|R_i| \le \frac{\varepsilon}{2}\sum_{j=0}^i 2^{-j}$, a set $S_i \subset J_u$ and sets $(Z^l_i)_{l \ge i+8}$ such that

$$\begin{aligned} &(i) \ \ \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^\infty \partial P^i_j \setminus \partial Q_{\mu_0}\Big) \le C_1 \mathcal{H}^1(S_i),\\ &(ii) \ \ Q \subset Q_{\mu_0} \setminus S_i \text{ for all } Q \in \mathcal{C}_i, \ \ \overline{Q} \subset Q_{\mu_0} \setminus S_i \text{ for all } Q \subset \bigcup\nolimits_{l \ge i+8} Z^l_i,\\ &(iii) \ \ \bigcup\nolimits_{l \ge i+8} \bigcup\nolimits_k \partial X^{l,i}_k \subset \bigcup\nolimits_{j=1}^\infty \partial P^i_j \end{aligned} \tag{5.3}$$

for some $C_1 = C_1(\theta, p)$, where $(X^{l,i}_k)_k$ denote the connected components of Z^l_i. Each set Z^l_i is the union of squares in $\mathcal{Q}^l \cap \mathcal{C}_i$ up to a set of measure zero. Moreover, we have for $l \ge i+8$

$$\begin{aligned} &(i) \ \ |E_i| \le C_1 t_i \mu_0,\\ &(ii) \ \ |Z^l_i| \le C_1 \theta^{-rl} t_l \mu_0,\\ &(iii) \ \ d(X^{l,i}_k) \le \theta^{-rl} t_l \quad \text{for all } X^{l,i}_k,\\ &(iv) \ \ |Q \cap E_i| \le c\theta |Q| \ \text{ for } Q \in \mathcal{C}_i \text{ with } Q \not\subset R_i, \ Q \cap \bigcup\nolimits_{l \ge i+8} Z^l_i = \emptyset. \end{aligned} \tag{5.4}$$

We further suppose that that there is a decomposition $Q_{\mu_0} = \bigcup_{l=0}^i D^i_l$ of Q_{μ_0} such that for some $C_2 = C_2(\theta, p)$

$$|D^i_l| \le C_2 \theta^{-rl} t_l \mu_0 \quad \text{ for all } 1 \le l \le i. \tag{5.5}$$

For each P^i_j we set $P^i_{j,\mathrm{cov}} = \bigcup_{Q \in \mathcal{Q}(P_j;\mathcal{C}_i,R_i)} Q$ with $\mathcal{Q}(P_j;\mathcal{C}_i,R_i)$ as defined in (4.78). We assume that for each P^i_j there is $A^i_j \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ such that for all $0 \le l \le i$

$$\begin{aligned} &(i) \ \ \sum\nolimits_{j=1}^\infty \|\nabla u - A^i_j\|^{p'}_{L^{p'}((P^i_{j,\mathrm{cov}} \cap D^i_l)\setminus E_i)} \le C_3 \theta^{-\lambda l} \mu_0^{2-p'} \|e(u)\|^{p'}_{L^2(Q_{\mu_0})},\\ &(ii) \ \ \#\{P^i_{j,\mathrm{cov}} : x \in P^i_{j,\mathrm{cov}}\} \le N_0 \quad \text{for all} \quad x \in Q_{\mu_0}, \end{aligned} \tag{5.6}$$

for some $N_0 \in \mathbb{N}$ and a constant $C_3 = C_3(\theta, p)$ large enough, which will eventually be specified in (5.18) and (5.22). Note that the constant in (5.6)(i) blows up for $l \to \infty$, but that D^i_l covers only a comparably small set (see (5.5)).

First, the properties hold for $i = 0$. To see this, we apply Lemma 4.11 on Q_{μ_0} with $\eta = \frac{\varepsilon}{2}$ to find a partition $Q_{\mu_0} = \bigcup_{j=0}^{n_0} P^0_j$ and define $\mathcal{C}_0 = \mathcal{C}_u$, $E_0 = E_u$, $R_0 = R_u$, $Z^l_0 = Z^l_u$ for $l \ge 8$ as well as $S_0 := J_u \setminus \bigcup_{Q \in \mathcal{C}_0} Q'$. (Lemma 4.11 is applicable as $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$ by (5.1).)

Then with $\eta = \frac{\varepsilon}{2}$ we get $|R_0| \le \frac{\varepsilon}{2}$ and (5.3)-(5.4) follow from (4.80)-(4.81). Moreover, (5.6) is a consequence of (4.82) with $D^0_0 = Q_{\mu_0}$ and p' in place of p if we choose C_3 large enough.

Step II (Induction step: Partition): We now pass from step $i-1$ to i. For each $Q \in \mathcal{C}_{i-1}$ we apply Lemma 4.11 with p' in place of p, $\eta = \varepsilon 2^{-i-1}|Q_{\mu_0}|^{-1}|Q|$ and $\mu = t_k$ with k such that $Q \in \mathcal{Q}^k$. Note that Lemma 4.11 is applicable due to (5.1). We obtain a partition $Q = \bigcup_{j=0}^{n_Q} P^Q_j$ up to a set of negligible measure satisfying (4.79), a set $R^Q \subset Q$ with

$$|R^Q| \le \varepsilon 2^{-i-1} |Q_{\mu_0}|^{-1} |Q|, \tag{5.7}$$

an exceptional set E^Q, a covering $\mathcal{C}^Q$ with $\mathcal{C}^Q \subset \bigcup_{j \ge i+1} \mathcal{Q}^j$, the set $S^Q = (J_u \cap Q) \setminus \bigcup_{\hat{Q} \in \mathcal{C}^Q} \hat{Q}'$, sets $(Z^l_Q)_{l \ge 8}$ as in Lemma 4.4 and infinitesimal rigid motions $(a^Q_j)_{j=0}^{n_Q}$ such that (4.80)-(4.82) hold (with $s_l = t_k \theta^l$ for $l \in \mathbb{N}$ and $\mu = t_k$). Recall particularly that depending on the case in

(4.79) either P_0^Q or $\bigcup_{j=1}^{n_Q} P_j^Q$ forms a partition of Q. We split up the covering into different sets as follows. Let $\mathcal{C}'_{i-1}$ be the squares $Q \in \mathcal{C}_{i-1}$ with $P_0^Q = \emptyset$, define

$$\mathcal{C}'''_{i-1} := \big\{Q \in \mathcal{C}_{i-1} \setminus \mathcal{C}'_{i-1} : Q \subset \bigcup\nolimits_{l \ge i+7} Z^l_{i-1}, \quad \mathcal{H}^1(J_u \cap \partial Q) \ge \theta d(Q)\big\} \tag{5.8}$$

and $\mathcal{C}''_{i-1} = \mathcal{C}_{i-1} \setminus (\mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1})$. For $\mathcal{C}''_{i-1} \cup \mathcal{C}'''_{i-1}$ the associated partition is trivial (see Lemma 4.11 (2)). (Note that the introduction of $\mathcal{C}'''_{i-1}$ is only a technical point. The core of the argument lies in the separation of $\mathcal{C}'_{i-1}$ and $\mathcal{C}''_{i-1}$.)

Let $R_i = R_{i-1} \cup \bigcup_{Q \in \mathcal{C}_{i-1}} R^Q$ and observe that $|R_i| \le \frac{\varepsilon}{2} \sum_{j=0}^i 2^{-j}$ by (5.7). Recall that the connected components $(X_k^{l,i-1})_{k \ge 1}$ of Z^l_{i-1} consist of squares $\mathcal{Q}^l \cap \mathcal{C}_{i-1}$, i.e. $\{Q \in \mathcal{Q}^l : Q \subset X_k^{l,i-1}\} = \{Q \in \mathcal{C}_{i-1} : Q \subset X_k^{l,i-1}\}$. Then by $\mathcal{P}^l_k$ we denote the connected components of the set

$$Y^l_k := \mathrm{int}\Big(\bigcup\nolimits_{Q \in \mathcal{C}''_{i-1}, Q \subset X_k^{l,i-1}} \overline{Q}\Big) = \mathrm{int}\Big(\bigcup\nolimits_{Q \in \mathcal{Q}^l \setminus (\mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}), Q \subset X_k^{l,i-1}} \overline{Q}\Big). \tag{5.9}$$

Note that by (4.80)(i) we have $\mathcal{H}^1(\partial Q) \le \mathcal{H}^1(\bigcup_{j=1}^{n_Q} \partial P_j^Q) \le C\mathcal{H}^1(S^Q)$ for all $Q \in \mathcal{C}'_{i-1}$. Then

$$\begin{aligned} \sum\nolimits_{P \in \mathcal{P}^l_k} \mathcal{H}^1(\partial P \setminus \partial X_k^{l,i-1}) &\le \sum\nolimits_{Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}, Q \subset X_k^{l,i-1}} \mathcal{H}^1(\partial Q) \\ &\le C \sum_{Q \in \mathcal{C}'_{i-1}, Q \subset X_k^{l,i-1}} \mathcal{H}^1(S^Q) + \sum_{Q \in \mathcal{C}'''_{i-1}, Q \subset X_k^{l,i-1}} \mathcal{H}^1(\partial Q). \end{aligned} \tag{5.10}$$

We now introduce the partition $\mathcal{P}^i$ for iteration step i and show (5.3)-(5.4). We set

$$\begin{aligned} &\mathcal{P}_a = \{P_j^Q : Q \in \mathcal{C}'_{i-1}, \ j = 1, \ldots, n_Q\} \cup \{P_0^Q = Q : Q \in \mathcal{C}'''_{i-1}\}, \\ &\mathcal{P}_b = \{P \in \mathcal{P}^l_k : l \ge i+7, k \ge 1\} \\ &P'_j := P_j^{i-1} \setminus \bigcup\nolimits_{P \in \mathcal{P}_a \cup \mathcal{P}_b} \overline{P} \quad \text{for all} \quad P_j^{i-1} \in \mathcal{P}^{i-1}, \quad \mathcal{P}_c = (P'_j)_{j=1}^\infty. \end{aligned} \tag{5.11}$$

Define $\mathcal{P}^i = \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c$ and denote the sets also by $\mathcal{P}^i = (P^i_j)_{j=1}^\infty$. We observe that $\mathcal{P}^i$ is a partition of Q_{μ_0} up to a set of negligible measure and that the sets in $\mathcal{P}_c$ do not intersect $\bigcup_{l \ge i+7} Z^l_{i-1}$.

By (5.8) with $\hat{S}^Q := J_u \cap \partial Q$ we find $\mathcal{H}^1(\partial Q) = 2\sqrt{2}d(Q) \le C\mathcal{H}^1(\hat{S}^Q)$ for $Q \in \mathcal{C}'''_{i-1}$. Consequently, using (4.80)(i) for the components $\mathcal{C}'_{i-1}$ and applying (5.3)(iii) for step $i-1$ as well as (5.10) we get

$$\mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^\infty \partial P^i_j \setminus \big(\bigcup\nolimits_{j=1}^\infty \partial P^{i-1}_j \cup \partial Q_{\mu_0}\big)\Big) \le C \sum\nolimits_{Q \in \mathcal{C}'_{i-1}} \mathcal{H}^1(S^Q) + C\sum\nolimits_{Q \in \mathcal{C}'''_{i-1}} \mathcal{H}^1(\hat{S}^Q).$$

Letting $S_i = S_{i-1} \cup S^*_i$ with $S^*_i := \bigcup_{Q \in \mathcal{C}'_{i-1}} S^Q \cup \bigcup_{Q \in \mathcal{C}'''_{i-1}} \hat{S}^Q$ we obtain $S_i \subset J_u$ and confirm (5.3)(i) as follows. Since $S^Q = \emptyset$ for all $Q \in \mathcal{C}''_{i-1} \cup \mathcal{C}'''_{i-1}$ (see Lemma 4.11 (2)), $S^Q \subset Q \subset Q_{\mu_0} \setminus S_{i-1}$ for all $Q \in \mathcal{C}'_{i-1}$ and $\hat{S}^Q \subset \overline{Q} \subset Q_{\mu_0} \setminus S_{i-1}$ for all $Q \in \mathcal{C}'''_{i-1}$ by (5.3)(ii) for step $i-1$, we get $S_{i-1} \cap S^*_i = \emptyset$. Moreover, each $x \in Q_{\mu_0}$ is contained in at most four different $\hat{S}^Q$. Then the claim follows from (5.3)(i) for step $i-1$ (if we choose $C_1 \ge C$).

Moreover, we define $\mathcal{C}_i = \bigcup_{Q \in \mathcal{C}_{i-1}} \mathcal{C}^Q \subset \bigcup_{j \ge i+1} \mathcal{Q}^j$ and for $l \ge i+8$

$$Z^l_i = \bigcup\nolimits_{k \ge i} \bigcup\nolimits_{Q \in \mathcal{C}'_{i-1} \cap \mathcal{Q}^k} Z_Q^{l-k}, \qquad E_i = \bigcup\nolimits_{Q \in \mathcal{C}_{i-1}} E^Q. \tag{5.12}$$

Each Z^l_i consists of squares in $\mathcal{Q}^l \cap \mathcal{C}_i$ up to a set of negligible measure since, as noted below (4.80), for each $Q \in \mathcal{C}_{i-1} \cap \mathcal{Q}^k$ the sets Z^j_Q consist of squares with sidelength $2s_j$, where according to the notation in the previous section we have $s_j = t_k\theta^j$ for $j \in \mathbb{N}$.

We now confirm (5.3)(ii). Consider $Q \in \mathcal{C}_i$. First, the fact that $\mathcal{C}_i$ is a refinement of $\mathcal{C}_{i-1}$ yields some $\hat{Q} \in \mathcal{C}_{i-1}$ such that $Q \subset \hat{Q}$. Then (5.3)(ii) for step $i-1$ implies $Q \subset Q_{\mu_0} \setminus S_{i-1}$. Likewise, recalling the definition of the set S_i^*, we get $Q \cap S_i^* = \emptyset$ by (4.80)(ii) and therefore $Q \cap S_i = \emptyset$.

In the special case that $\hat{Q} \in \mathcal{C}'_{i-1}$ and $Q \subset \bigcup_{l \ge 8} Z^l_{\hat{Q}} \subset \bigcup_{l \ge i+8} Z^l_i$, we get $\overline{Q} \subset \hat{Q}$ by (4.80)(ii) and thus as before by (5.3)(ii), (4.80)(ii) we also derive $\overline{Q} \subset Q_{\mu_0} \setminus S_{i-1}$ and $\overline{Q} \cap S_i^* = \emptyset$.

Property (5.3)(iii) directly follows from (4.80)(iii) and the definition of $\mathcal{P}_a$ in (5.11). By (4.81)(iv) and (5.12) we get for each square $Q \in \mathcal{C}_{i-1} \cap \mathcal{Q}^k$, $k \ge i$ and $l \ge i+8$

$$|Z^l_i \cap Q| \le C\theta^{-(l-k)r} s_{l-k} \mathcal{H}^1(J_u \cap Q) = Ct_k \theta^{(l-k)(1-r)} \mathcal{H}^1(J_u \cap Q) \le Ct_l \theta^{-rl} \mathcal{H}^1(J_u \cap Q),$$

where as before we have $s_j = t_k \theta^j$ for $j \in \mathbb{N}$. Then taking the sum over all squares we derive (5.4)(ii), where we use the fact that $\mathcal{H}^1(J_u) \le c\theta^{-2}\mu_0$. Likewise, (5.4)(iii) follows from (4.81)(iii) and a similar calculation. Moreover, (4.81)(ii) yields (5.4)(iv) by the definition in (5.12) and the fact that $R^Q \subset R_i$. Finally, to see (5.4)(i) we recall that for all $Q \in \mathcal{C}_{i-1}$

$$|E^Q| \le c\theta^2 d(Q) \mathcal{H}^1(J_u \cap Q) \le Ct_i \mathcal{H}^1(J_u \cap Q)$$

by (4.81)(i). Consequently, in view of (5.12) and $\mathcal{H}^1(J_u) \le c\theta^{-2}\mu_0$ the result follows when we take the sum over all $Q \in \mathcal{C}_{i-1}$ (if we choose $C_1 \ge C$).

We close this step by defining a decomposition of Q_{μ_0} and confirming (5.5). Let $D^i_i = E_{i-1} \cup \bigcup_{P \in \mathcal{P}_a \cup \mathcal{P}_b} P$ and $D^i_l = D^{i-1}_l \setminus D^i_i$ for $0 \le l \le i-1$. Then (5.5) for $0 \le l \le i-1$ follows directly. Moreover, recalling (5.3)(i), (5.11) and $\mathcal{H}^1(J_u) \le c\theta^{-2}\mu_0$ we compute

$$\begin{aligned} |\bigcup\nolimits_{P \in \mathcal{P}_a} P| = |\bigcup\nolimits_{Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}} Q| &\le ct_i \sum\nolimits_{Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}} \mathcal{H}^1(\partial Q \setminus \partial Q_{\mu_0}) \\ &\le cC_1 t_i \mathcal{H}^1(S_i) \le cC_1 t_i \mathcal{H}^1(J_u) \le C \cdot C_1 t_i \mu_0. \end{aligned} \tag{5.13}$$

Likewise, by (5.11) and (5.4)(ii) we see $|\bigcup_{P \in \mathcal{P}_b} P| \le \sum_{l \ge i+7} |Z^l_{i-1}| \le C \cdot C_1 \theta^{-ri} t_i \mu_0$. This together with the estimate for $|E_{i-1}|$ in (5.4)(i) yields (5.5) for D^i_i if one chooses $C_2 \ge C \cdot C_1$.

Step III (Induction step: Korn inequalities): It remains to define matrices $(A^i_j)_{j=1}^\infty \subset \mathbb{R}^{2\times 2}_{\rm skew}$ and to establish (5.6). First, we observe that by (4.82) we have for all $Q \in \mathcal{C}_{i-1}$

$$\sum\nolimits_{j=0}^{n_Q} \|\nabla u - A^Q_j\|^{p'}_{L^{p'}(P^Q_{j,\rm cov} \setminus E^Q)} \le Cd(Q)^{2-p'} \|e(u)\|^{p'}_{L^2(Q)} \tag{5.14}$$

for suitable $(a^Q_j)_j$, where $P^Q_{j,\rm cov} = \bigcup_{\hat{Q} \in \mathcal{Q}(P^Q_j; \mathcal{C}^Q, R^Q)} \hat{Q}$ is defined as in (4.78). As a preparation we analyze the behavior in the 'isolated components'. Fix some $X^{l,i-1}_k$ with $l \ge i+7$ and recalling (5.9) we consider a corresponding connected component $P \in \mathcal{P}^l_k$. By definition there is a set $\mathcal{Q}(P) \subset \{Q \in \mathcal{Q}^l : Q \subset X^{l,i-1}_k\}$ such that $P = {\rm int}(\bigcup_{Q \in \mathcal{Q}(P)} \overline{Q})$.

As $Q \in \mathcal{C}''_{i-1}$ for $Q \in \mathcal{Q}(P)$, a single infinitesimal rigid motion a^Q_0 is given such that (5.14) holds with $P^Q_{0,\rm cov} = Q$ (see Lemma 4.11 (2)) and by (4.83)

$$\int_{\partial Q \setminus \Gamma^Q} |Tu - a^Q_0|^2 \, d\mathcal{H}^1 \le ct_l \|e(u)\|^2_{L^2(Q)}$$

for a set Γ^Q with $\mathcal{H}^1(\Gamma^Q) \le c\theta t_l$. We can estimate the difference of the infinitesimal rigid motions on adjacent squares as follows. Consider $Q_1, Q_2 \in \mathcal{Q}(P)$ such that ∂Q_1 and ∂Q_2 have a common edge. Then the previous estimate implies with $\Gamma := \Gamma^{Q_1} \cup \Gamma^{Q_2} \cup (J_u \cap \partial Q_1 \cap \partial Q_2)$

$$\int_{(\partial Q_1 \cap \partial Q_2) \setminus \Gamma} |a^{Q_1}_0 - a^{Q_2}_0|^2 \, d\mathcal{H}^1 \le c \sum\nolimits_{k=1,2} \int_{\partial Q_k \setminus \Gamma} |Tu - a^{Q_k}_0|^2 \, d\mathcal{H}^1 \le ct_l \|e(u)\|^2_{L^2(Q_1 \cup Q_2)},$$

where we used that the traces Tu on ∂Q_1 and ∂Q_2 coincide on $(\partial Q_1 \cap \partial Q_2)\setminus\Gamma$. By (5.8) we get $\mathcal{H}^1(\Gamma) \le c\theta t_l$ as $Q_1, Q_2 \notin \mathcal{C}'''_{i-1}$. Therefore, by Remark 3.6 we get for θ small $t_l^2|A_0^{Q_1} - A_0^{Q_2}|^2 \le C\|e(u)\|^2_{L^2(Q_1\cup Q_2)}$. Consequently, recalling that P is connected, $\#\mathcal{Q}(P) \le c\theta^{-2rl}$ by (5.4)(iii) and using the discrete Hölder inequality we find

$$\max_{Q_1,Q_2\in\mathcal{Q}(P)} t_l^2|A_0^{Q_1} - A_0^{Q_2}|^{p'} \le ct_l^{2-p'}\theta^{-2rl(p'-1)}\sum\nolimits_{Q\in\mathcal{Q}(P)}\|e(u)\|^{p'}_{L^2(Q)}. \tag{5.15}$$

Applying (5.14), $\#\mathcal{Q}(P) \le c\theta^{-2rl}$, $p' \le 2$ and recalling (5.12) we derive for each $\hat{Q} \in \mathcal{Q}(P)$

$$\|\nabla u - A_0^{\hat{Q}}\|^{p'}_{L^{p'}(P\setminus E_i)} \le (C + c\theta^{-4rl})t_l^{2-p'}\sum\nolimits_{Q\in\mathcal{Q}(P)}\|e(u)\|^{p'}_{L^2(Q)}. \tag{5.16}$$

We are now in the position to define matrices associated to $\mathcal{P}^i$. Recalling the definition of the partition $\mathcal{P}^i$ in (5.11) we distinguish the the following cases: (a) For $P^i_j = P^Q_k$ for some $Q \in \mathcal{C}'_{i-1}$ and $k = 1,\dots,n_Q$ we let $A^i_j = A^Q_k$ and for $P^i_j = P^Q_0$ for some $Q \in \mathcal{C}'''_{i-1}$ we let $A^i_j = A^Q_0$ as given by (5.14). (b) If $P^i_j \in \mathcal{P}^l_k$, we set $A^i_j = A_0^{\hat{Q}}$ for an (arbitrary) $\hat{Q} \in \mathcal{Q}(P^i_j)$. (c) Finally, for all P^i_j with $P^i_j = P'_k$ for some $P^{i-1}_k \in \mathcal{P}^{i-1}$ we let $A^i_j = A^{i-1}_k$.

We now show (5.6)(i). As a preparation we recall that the discrete Hölder inequality together with $\#\{Q \in \mathcal{C}_i : Q \subset Z^l_i\} \le ct_l^{-2}|Z^l_i| \le cC_1t_l^{-1}\theta^{-rl}\mu_0$ (see (5.4)(ii)), $r = \frac{1}{24}(2-p)$, and $t_l = \mu_0\theta^l$ implies

$$(i)\quad \sum\nolimits_{Q\in\mathcal{C}_{i-1}}(d(Q))^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} \le c\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}, \tag{5.17}$$

$$\begin{aligned}(ii)\quad t_l^{2-p'}\theta^{-4rl}\sum\nolimits_{Q\in\mathcal{C}_i:Q\subset Z^l_i}\|e(u)\|^{p'}_{L^2(Q)} &\le cC_1\theta^{-4rl}(\theta^{-rl}t_l\mu_0)^{1-p'/2}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}\\ &\le cC_1\theta^{rl}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}.\end{aligned}$$

For (i) we refer to (4.86) and for (ii) to (4.58) for similar arguments, where we particularly use $1 - \frac{p'}{2} = \frac{1}{2}(1-\frac{p}{2}) = 6r$. We first consider the set D^i_i as defined before (5.13) and show that for $k = a, b, c$

$$\sum\nolimits_{P^i_j\in\mathcal{P}_k}\|\nabla u - A^i_j\|^{p'}_{L^{p'}((P^i_{j,\mathrm{cov}}\cap D^i_i)\setminus E_i)} \le \tfrac{1}{3}C_3\theta^{-\lambda i}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}. \tag{5.18}$$

First, for the components $\mathcal{P}_a$ the property in (5.18) directly follows from (5.14) and (5.17)(i) provided that C_3 is chosen large enough, where in this case the additional factor $\theta^{-\lambda i}$ is not needed. In fact, $P^i_j = P^Q_k$ for some $Q \in \mathcal{C}'_{i-1}\cup\mathcal{C}'''_{i-1}$ for all $P^i_j \in \mathcal{P}_a$ and $P^i_{j,\mathrm{cov}} = P^Q_{k,\mathrm{cov}}$ due to the definition of $\mathcal{C}_i$ with $P^i_{j,\mathrm{cov}}$ as defined before (5.6). Likewise, for sets $P^i_j \in \mathcal{P}_b$ we get (5.18) by (5.16) and (5.17)(ii) using $P^i_{j,\mathrm{cov}} \subset \bigcup_{Q\in\mathcal{Q}(P^i_j)} Q \subset P^i_j$ as well as the fact that the sets $\mathcal{Q}(P)$ for $P \in \mathcal{P}_b$ are pairwise disjoint. Note that again the additional factor $\theta^{-\lambda i}$ is not needed, but the choice of C_3 also depends on C_1 and $\sum_l \theta^{rl} < +\infty$. (We omit details here as a very similar computation has already been performed in (4.58).)

Finally, we concern ourselves with a component $P^i_j = P'_k \in \mathcal{P}_c$. Note $P'_k \subset P^{i-1}_k$ with $P^{i-1}_k \in \mathcal{P}^{i-1}$. Recall the definition of the sets $P^i_{j,\mathrm{cov}}, P^{i-1}_{k,\mathrm{cov}}$ before (5.6). As $\mathcal{C}_i$ is a refinement of $\mathcal{C}_{i-1}$ and $R_i \supset R_{i-1}$, we obtain

$$P^i_{j,\mathrm{cov}} \subset P^*_j := \bigcup\{Q \in \mathcal{C}_{i-1}:\ Q \cap P^i_j \neq \emptyset,\ Q \subset P^{i-1}_{k,\mathrm{cov}}\} \subset P^{i-1}_{k,\mathrm{cov}}. \tag{5.19}$$

Fix $Q \in \mathcal{C}_{i-1}\cap\mathcal{Q}^l$ with $Q \subset P^*_j$. As $P^i_j \cap Q \neq \emptyset$, (5.11) implies $Q \cap Z^l_{i-1} = \emptyset$ and that the partition associated to Q, introduced before (5.7), is trivial, i.e. $P^Q_0 = P^Q_{0,\mathrm{cov}} = Q$. We observe

$$(i)\quad |E_{i-1}\cap Q| \le c\theta t_l^2, \qquad (ii)\quad |E^Q| \le c\theta t_l^2.$$

In fact, (i) follows from (5.4)(iv) for step $i-1$ and the fact that $Q \subset P^{i-1}_{k,\mathrm{cov}}$ (cf. (4.78)). By (4.81)(i) and (5.1) we obtain (ii). Thus, for θ sufficiently small with respect to c we obtain by (5.14) and the triangle inequality

$$\begin{aligned}|Q||A^Q_0 - A^{i-1}_k|^{p'} &\le c\|\nabla u - A^Q_0\|^{p'}_{L^{p'}(Q\setminus(E^Q\cup E_{i-1}))} + c\|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(Q\setminus(E^Q\cup E_{i-1})}\\ &\le Ct_l^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} + c\|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(Q\setminus E_{i-1})}.\end{aligned} \tag{5.20}$$

Consequently, we find using again (5.14)

$$\|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(Q\setminus E_i)} \le C(d(Q))^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} + c\|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(Q\setminus E_{i-1})}. \tag{5.21}$$

Now we are in the position to confirm (5.18). Note that each $Q \in \mathcal{C}_{i-1}$ is contained only in a bounded number N_0 of different P^*_j by (5.6)(ii) and (5.19). Summing over all squares $Q \subset P^*_j$ and all $P^i_j = P'_k \in \mathcal{P}_c$ we deduce using (5.6) for step $i-1$ as well as (5.17)-(5.21)

$$\begin{aligned}&\sum\nolimits_{P^i_j = P'_k \in \mathcal{P}_c} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}((P^i_{j,\mathrm{cov}}\cap D^i_i)\setminus E_i)} \le \sum\nolimits_{P^i_j = P'_k\in\mathcal{P}_c} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}((P^*_j\cap D^i_i)\setminus E_i)}\\ &\quad\le N_0\cdot C\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} + c\sum\nolimits_{P^{i-1}_k\in\mathcal{P}^{i-1}} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(P^{i-1}_{k,\mathrm{cov}}\setminus E_{i-1})}\\ &\quad\le (CN_0 + cC_3\theta^{-(i-1)\lambda})\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} \le \tfrac{1}{3}C_3\theta^{-i\lambda}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})},\end{aligned} \tag{5.22}$$

where the last step holds for $C_3 = C_3(\theta, p)$ large enough and θ small (depending on λ) such that $CN \le \frac{1}{6}C_3$ and $c\theta^\lambda \le \frac{1}{6}$.

To conclude the proof of (5.6)(i) it remains to consider the sets $(D^i_l)^{i-1}_{l=0}$. Recalling the definition of the decomposition before (5.13), we see $D^i_l \setminus E_i \subset D^{i-1}_l \setminus E_{i-1}$ for $0 \le l \le i-1$. Consequently, (5.6)(i) for $0 \le l \le i-1$ follows directly from the corresponding estimates in step $i-1$ and (5.19).

To show (5.6)(ii), it suffices to treat the cases (a)-(c) separately. Indeed, by the definition of $\mathcal{C}_i$ we have $P_{j_1,\mathrm{cov}}$ and $P_{j_2,\mathrm{cov}}$ are disjoint if they lie in different sets $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c$. For $\mathcal{P}_a$ the desired property follows directly from (4.82)(ii). For $\mathcal{P}_b$ it is obvious as the sets $P^i_{j,\mathrm{cov}} \subset P^i_j \in \mathcal{P}_b$ are pairwise disjoint. Finally, by (5.19) for each $P^i_j \in \mathcal{P}_c$ we have a (different) $P^{i-1}_{k,\mathrm{cov}}$ with $P^i_{j,\mathrm{cov}} \subset P^{i-1}_{k,\mathrm{cov}}$ and the property follows from (5.6)(ii) for step $i-1$.

Step IV (Conclusion): We are now in a position to define the partition and the infinitesimal rigid motions such that the assertion of Theorem 2.1 holds. As $\limsup_{i\to\infty}|E_i| = 0$ by (5.4)(i) we can choose $I \in \mathbb{N}$ so large that $|E_I| \le \varepsilon$. By $\mathcal{P}^I = (P^I_j)^\infty_{j=1}$ we denote the partition given in iteration step I and get by (5.3)(i) that

$$\sum\nolimits^\infty_{j=1} \mathcal{H}^1(\partial P^I_j \setminus \partial Q_{\mu_0}) \le C\mathcal{H}^1(S_I) \le C\mathcal{H}^1(J_u), \tag{5.23}$$

where in the last step we used $S_I \subset J_u$. In view of the definition (4.78) we get

$$\sum\nolimits^\infty_{j=1} |P^I_j \setminus P^I_{j,\mathrm{cov}}| \le \sum\nolimits^\infty_{j=1}\sum\nolimits_{Q\in\mathcal{C}_I} |P^I_j \cap Q \cap R_I| \le |R_I| \le \varepsilon.$$

Thus, letting $E' = E_I \cup \bigcup^\infty_{j=1}(P^I_j \setminus P^I_{j,\mathrm{cov}})$ we get $|E'| \le 2\varepsilon$. We set $\mathcal{P}^* = \{P^I_j \in \mathcal{P}^I : |P^I_j \setminus E'| \ge \frac{1}{2}|P^I_j|\}$ and define

$$v' := u - \sum\nolimits_{P^I_j\in\mathcal{P}^*} (A^I_j\, x)\chi_{P^I_j}, \qquad E'' = E' \cup \bigcup\nolimits_{P^I_j\notin\mathcal{P}^*} P^I_j, \tag{5.24}$$

where $(A^I_j)_j \subset \mathbb{R}^{2\times 2}_{\rm skew}$ are the matrices corresponding to $(P^I_j)_j$ (cf. (5.6)). We now show (2.3)(ii) for v'. We deduce from (5.6)

$$\|\nabla v'\|^{p'}_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D^I_l)} \le C\mu_0^{2-p'}\theta^{-l\lambda}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} \tag{5.25}$$

for all $0 \le l \le I$ for a constant $C = C(\theta,p)$. Moreover, by (5.5)

$$|D^I_l| \le C\theta^{-rl}t_l\mu_0 \le C\theta^{l(1-r)}\mu_0^2$$

for all $0 \le l \le I$. Recall the definition $p' = 1+\frac{p}{2}$, $r = \frac{2-p}{24}$ and $\lambda = \frac{(1-r)(2-p)}{3p+2}$. Passing from p' to p and using (5.24), (5.25) we obtain by Hölder's inequality

$$\begin{aligned}
\|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} &= \sum\nolimits_{l=0}^I \|\nabla v'\|^p_{L^p((Q_{\mu_0}\setminus E'')\cap D^I_l)} \le \sum\nolimits_{l=0}^I |D^I_l|^{1-p/p'}\|\nabla v'\|^p_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D^I_l)}\\
&\le C\mu_0^{2-p}\sum\nolimits_{l\ge 0}\theta^{l(1-r)(1-p/p')-l\lambda p/p'}\|e(u)\|^p_{L^2(Q_{\mu_0})}\\
&\le C\mu_0^{2-p}\sum\nolimits_{l\ge 0}\theta^{l\lambda}\|e(u)\|^p_{L^2(Q_{\mu_0})} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}
\end{aligned}\tag{5.26}$$

with $C = C(\theta,p,\lambda) = C(\theta,p)$. Now we have to analyze the behavior in E''. To this end, we fix some $P^I_j \in \mathcal{P}^I$. If $P^I_j \in \mathcal{P}^*$, we can choose a measurable set F_j with $P^I_j \cap E' \subset F_j \subset P^I_j$ and $|F_j| = 2|P^I_j \cap E'|$. We find by $|F_j| \le 2|F_j\setminus E'| = 2|F_j \setminus E''|$

$$|F_j||A^I_j|^p \le c\|\nabla u - A^I_j\|^p_{L^p(F_j\setminus E'')} + c\|\nabla u\|^p_{L^p(F_j\setminus E'')}. \tag{5.27}$$

If $P^I_j \notin \mathcal{P}^*$, we set $F_j = P^I_j$. Then we see that $F := \bigcup_{j=1}^\infty F_j$ satisfies $|F| \le 2|E'| \le 4\varepsilon$ and thus by (5.2) we derive $\|\nabla u\|^p_{L^p(F)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$. Consequently, as $E'' \subset F$ we obtain by (5.26)-(5.27)

$$\begin{aligned}
\|\nabla v'\|^p_{L^p(Q_{\mu_0})} &\le \|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} + \|\nabla v'\|^p_{L^p(F)}\\
&\le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} + c\|\nabla u\|^p_{L^p(F)} + c\sum\nolimits_{P^I_j\in\mathcal{P}^*}|F_j||A^I_j|^p\\
&\le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} + c\|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}.
\end{aligned}\tag{5.28}$$

All arguments hold for a fixed θ sufficiently small and thus the constant C eventually only depends on p. To obtain (2.3)(i) we apply Theorem 2.10 on v' and $\rho = \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})$ to get a Caccioppoli partition $(P'_j)_j$ and corresponding translations $(b_j)_j$ such that with $v := v' - \sum_j b_j\chi_{P'_j}$ by (5.23) and Hölder's inequality

$$\begin{aligned}
&\sum\nolimits_j \mathcal{H}^1(\partial^* P'_j) \le \mathcal{H}^1(J_{v'}) + \mathcal{H}^1(\partial Q_{\mu_0}) + c\rho \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})),\\
&\|v\|_{L^\infty(Q_{\mu_0})} \le c\rho^{-1}\|\nabla v'\|_{L^1(Q_{\mu_0})} \le c\mu_0\rho^{-1}\|e(u)\|_{L^2(Q_{\mu_0})}.
\end{aligned}$$

(For the proof of Theorem 2.10, which is completely independent from the arguments in Section 4 and Section 5, we refer to Section 6.1 below.) Let $(P_j)_j$ be the Caccioppoli partition consisting of the sets $(P^I_i \cap P'_j)_{i,j}$ and observe that (2.1) follows. Choosing corresponding $(a_j)_j$ such that $v = u - \sum_j a_j\chi_{P_j}$, we obtain (2.3) as $\nabla v' = \nabla v$. Finally, since $u \in \mathcal{W}(Q_{\mu_0})$, we also have that $v \in SBV^2(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$. $\square$

We now drop assumption (5.1). The idea is to add the boundary of the squares violating (5.1) to the boundary of the partition. As in each such square $\mathcal{H}^1(J_u)$ is large, eventually the length of the added boundary can be controlled, cf. (5.34) below. On each of these squares we then separately proceed as in Theorem 5.1. For the following proof we introduce the notation $\mathcal{F}^k := \{F = {\rm int}(\bigcup_{Q\in\mathcal{Q}_F}\overline{Q}) : \mathcal{Q}_F \subset \mathcal{Q}^k\}$ for $k \in \mathbb{N}$.

Theorem 5.2. *Let $p \in [1,2)$. Then Theorem 2.1 holds for all $u \in \mathcal{W}(Q_{\mu_0})$.*

Proof. We reduce the problem to Theorem 5.1 by passing to suitable modifications of u. Since $u \in \mathcal{W}(Q_{\mu_0})$, we have $J_u = \bigcup_{j=1}^n \Gamma_j^u$ consists of closed segments and $\nabla u \in L^p(Q_{\mu_0})$. As in Theorem 5.1 it suffices to treat the case $\|e(u)\|_{L^2(Q_{\mu_0})} > 0$. We choose $I \in \mathbb{N}$ such that for $\mathcal{R}_0 = \{Q \in \mathcal{Q}^I : Q \cap J_u \neq \emptyset\}$

$$\begin{aligned}
&(i) \quad \|\nabla u\|^p_{L^p(A)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} \quad \text{for } A \subset \mathbb{R}^2 \text{ with } |A| \le \theta^{-1}\mathcal{H}^1(J_u)t_I,\\
&(ii) \quad \sum\nolimits_{Q\in\mathcal{R}_0} \mathcal{H}^1(\partial Q) \le c\mathcal{H}^1(J_u).
\end{aligned} \tag{5.29}$$

Let $R_0 = \bigcup_{Q\in\mathcal{R}_0} Q$. Clearly, we have $|R_0| \le c\mathcal{H}^1(J_u)t_I \le \theta^{-1}\mathcal{H}^1(J_u)t_I$ for θ small (with respect to c). Moreover, $J_u \cap Q = \emptyset$ for each $Q \in \bigcup_{j\ge I} \mathcal{Q}^j$ with $Q \cap R_0 = \emptyset$. For later we also define $J_u' = J_u \cup \partial R_0$.

Set $u_0 = u\chi_{Q_{\mu_0}\setminus R_0}$. Assume $u_j \in \mathcal{W}(Q_{\mu_0})$ and $\mathcal{R}_j \subset \mathcal{Q}^{I-j}$, $0 \le j \le i$, have already been constructed with $R_j := \bigcup_{Q\in\mathcal{R}_j} Q$. For all $j = 1,\dots,i$ we define the sets S^j_τ, $\tau = (\tau_0,\dots,\tau_{j-1}) \in J_j := \{0,1\}^j$, by

$$S^j_\tau = \big(\overline{R_j} \cap \bigcup\nolimits_{k:\tau_k=1} \overline{R_k}\big) \setminus \bigcup\nolimits_{k:\tau_k=0} \overline{R_k} \text{ for } \tau \neq 0, \qquad S^j_0 = \overline{R_j} \setminus \bigcup\nolimits_{k=0}^{j-1} \overline{R_k}.$$

For $j = 0$ we let $J_0 = \{0\}$ and $S^0_0 = \overline{R_0}$. Denoting by $\#\tau = \sum_{k=0}^{j-1}\tau_k$ for $\tau \in J_j$ we assume that for all $j = 0,\dots,i$

$$\begin{aligned}
&(i) \quad J_{u_j} \subset \Gamma_j := \bigcup\nolimits_{k=0}^{j} \big(\partial R_k \setminus \bigcup\nolimits_{l=k+1}^{j} \overline{R_l}\big) \cup \big(J_u \setminus \bigcup\nolimits_{k=0}^{j} \overline{R_k}\big) \subset Q_{\mu_0} \setminus \bigcup\nolimits_{k=0}^{j} R_k,\\
&(ii) \quad \mathcal{H}^1(J_{u_j} \cap Q) \le \theta^{-1} d(Q) \quad \text{for all } Q \in \bigcup\nolimits_{k=I-j}^{\infty} \mathcal{Q}^k,\\
&(iii) \quad \mathcal{H}^1(\partial R_j \cap F) \le \sum\nolimits_{k=0}^{j} \sum\nolimits_{\tau\in J_j:\#\tau=k} 2^{-k}\, \mathcal{H}^1\big(J_u' \cap S^j_\tau \cap F\big) \text{ for all } F \in \mathcal{F}^{I-(j+1)}.
\end{aligned} \tag{5.30}$$

From the above discussion we get that the properties hold for $i = 0$ where particularly (iii) follows from the fact that $J_u' \supset \partial R_0$. We define u_{i+1} and R_{i+1} as follows. Let

$$\mathcal{R}_{i+1} = \{Q \in \mathcal{Q}^{I-i-1} : \mathcal{H}^1(\Gamma_i \cap Q) > 16t_{I-i-1}\} \tag{5.31}$$

and $R_{i+1} = \bigcup_{Q\in\mathcal{R}_{i+1}} Q$. Set $u_{i+1} = u_i\chi_{Q_{\mu_0}\setminus R_{i+1}}$. Then (for θ small) (5.30)(i),(ii) hold by construction and (5.30)(i),(ii) for step i. We now show (iii). Fix $F \in \mathcal{F}^{I-(i+2)}$ and recall that F is open. Then (5.31) and the fact that $\mathcal{H}^1(\partial Q) = 8t_{I-i-1}$ yield by (5.30)(i)

$$\begin{aligned}
\mathcal{H}^1(\partial R_{i+1} \cap F) \le \frac{1}{2}\mathcal{H}^1(\Gamma_i \cap F \cap \overline{R_{i+1}}) &\le \frac{1}{2}\mathcal{H}^1\Big(J_u \cap F \cap \Big(\overline{R_{i+1}} \setminus \bigcup\nolimits_{k=0}^{i} \overline{R_k}\Big)\Big)\\
&\quad + \frac{1}{2}\sum\nolimits_{j=0}^{i} \mathcal{H}^1\Big((\partial R_j \cap F) \setminus \bigcup\nolimits_{k=j+1}^{i} \overline{R_k}\Big).
\end{aligned}$$

We then note that by definition of S^{i+1}_0

$$\mathcal{H}^1\big(J_u \cap F \cap \big(\overline{R_{i+1}} \setminus \bigcup\nolimits_{k=0}^{i} \overline{R_k}\big)\big) = \mathcal{H}^1(J_u \cap F \cap S^{i+1}_0). \tag{5.32}$$

Moreover, for each $\tau \in J_j$ we have $S^j_\tau \setminus \bigcup_{k=j+1}^{i} \overline{R_k} = S^{i+1}_{\tau'} = S^{i+1}_{\tau'} \cap (\overline{R_j} \setminus \bigcup_{k=j+1}^{i} \overline{R_k})$ with $\tau' = (\tau, 1, 0\dots, 0) \in J_{i+1}$. Consequently, since $\#\tau' = \#\tau + 1$ we find by (5.30)(iii) for step

$i-1$ and the fact that $F \setminus \bigcup_{l=j+1}^{i} \overline{R_l} \in \mathcal{F}^{I-(j+1)}$

$$\begin{aligned}
&\frac{1}{2}\sum\nolimits_{j=0}^{i} \mathcal{H}^1\big((\partial R_j \cap F) \setminus \bigcup\nolimits_{l=j+1}^{i} \overline{R_l}\big)\\
&\quad\le \sum\nolimits_{j=0}^{i}\sum\nolimits_{k=0}^{j}\sum\nolimits_{\tau\in J_j:\#\tau=k} 2^{-k-1}\,\mathcal{H}^1\big(J_u'\cap S^j_\tau \cap (F\setminus \bigcup\nolimits_{l=j+1}^{i}\overline{R_l})\big)\\
&\quad\le \sum\nolimits_{j=0}^{i}\sum\nolimits_{k=1}^{j+1}\sum\nolimits_{\tau'\in J_{i+1}:\#\tau'=k} 2^{-k}\,\mathcal{H}^1\big(J_u'\cap S^{i+1}_{\tau'} \cap F\cap(\overline{R_j}\setminus \bigcup\nolimits_{l=j+1}^{i}\overline{R_l})\big)\\
&\quad\le \sum\nolimits_{k=1}^{i+1}\sum\nolimits_{\tau':\#\tau'=k} 2^{-k}\,\mathcal{H}^1\big(J_u'\cap S^{i+1}_{\tau'}\cap F\big).
\end{aligned}$$

This together with (5.32) yields (5.30)(iii). For later we also note that for all $Q \in \mathcal{Q}^{I-i-1}\setminus\mathcal{R}_{i+1}$ we get for θ small by (5.31)

$$\big|Q\cap \bigcup\nolimits_{k\le i} R_k\big| \le 2t_{I-i}\mathcal{H}^1(\Gamma_i\cap\overline{Q}) \le 2t_{I-i}(16t_{I-i-1}+8t_{I-i-1}) \le 2t^2_{I-i-1} = \tfrac{1}{2}|Q|, \tag{5.33}$$

where in the first step we used that $\bigcup_{k\le i} R_k$ consists of squares with sidelength smaller than $2t_{I-i}$ whose boundaries are contained in Γ_i.

We proceed in this way until step $i = I-1$ and finally let $R_I = Q_{\mu_0}$, $\mathcal{R}_I = \{Q_{\mu_0}\}$. By (5.30)(iii) and the fact that $(S^I_{\tau'})_{\tau'\in J_I}$ is a partition of Q_{μ_0} we derive

$$\begin{aligned}
\sum\nolimits_{j=0}^{I-1}\mathcal{H}^1(\partial R_j\cap Q_{\mu_0}) &\le \sum\nolimits_{j=0}^{I-1}\sum\nolimits_{k=0}^{j}\sum\nolimits_{\tau\in J_j:\#\tau=k} 2^{-k}\,\mathcal{H}^1\big(J_u'\cap S^j_\tau\big)\\
&= \sum\nolimits_{\tau'\in J_I}\sum\nolimits_{j=0}^{I-1}\sum\nolimits_{k=0}^{j}\sum\nolimits_{\tau\in J_j:\#\tau=k} 2^{-k}\,\mathcal{H}^1\big(J_u'\cap S^j_\tau\cap S^I_{\tau'}\big).
\end{aligned}$$

Note that $S^j_\tau \cap S^I_{\tau'} \neq \emptyset$ if and only if τ coincides with the first j entries of τ' and $\tau'_j = 1$. Consequently, with $\mathcal{H}^1(J_u') \le c\mathcal{H}^1(J_u)$ we find

$$\begin{aligned}
\sum\nolimits_{j=0}^{I-1}\mathcal{H}^1(\partial R_j\cap Q_{\mu_0}) &\le \sum\nolimits_{\tau'\in J_I}\sum\nolimits_{j:\tau'_j=1} 2^{-(\sum_{l=0}^{j-1}\tau'_l)}\,\mathcal{H}^1\big(J_u'\cap S^I_{\tau'}\big) + \mathcal{H}^1\big(J_u'\cap S^I_0\big)\\
&\le c\sum\nolimits_{\tau'\in J_I}\mathcal{H}^1\big(J_u'\cap S^I_{\tau'}\big) \le c\mathcal{H}^1(J_u).
\end{aligned}\tag{5.34}$$

We now apply Theorem 5.1 separately on each square $Q\in\mathcal{R}_j$, $j=1,\ldots,I$, for the function u_{j-1}. In fact, in view of (5.30)(ii) condition (5.1) is satisfied (replacing Q_{μ_0} by Q and u by u_{j-1}). Hereby, by (5.23) we obtain a partition of the square $Q\in\mathcal{R}_j$, which we can restrict to $Q\setminus\bigcup_{k=0}^{j-1}\overline{R_k}$. Consequently, taking the union over all partitions defined on each $Q\in\bigcup_{j=1}^{I}\mathcal{R}_j$ we obtain a partition $(P_j')_{j=0}^\infty$ of Q_{μ_0} with $\bigcup_{j=0}^{I}\partial R_j \subset \bigcup_{j=0}^\infty \partial P_j'$, where we set $P_0' = R_0$. By (5.30)(i) and the fact that $\sum_{j=0}^{I-1}\mathcal{H}^1(\partial R_j\cap Q_{\mu_0}) \le c\mathcal{H}^1(J_u)$ we get

$$\begin{aligned}
&\sum\nolimits_{j=1}^{I-1}\sum\nolimits_{Q\in\mathcal{R}_j}\big(\mathcal{H}^1(\partial Q\cap Q_{\mu_0}) + \mathcal{H}^1(J_{u_{j-1}}\cap Q)\big) + \mathcal{H}^1(u_{I-1}\cap Q_{\mu_0})\\
&\quad\le \sum\nolimits_{j=1}^{I-1}\mathcal{H}^1(\partial R_j\cap Q_{\mu_0}) + \sum\nolimits_{j=1}^{I}\mathcal{H}^1(\Gamma_{j-1}\cap R_j)\\
&\quad\le c\mathcal{H}^1(J_u) + \sum_{j=1}^{I}\mathcal{H}^1\Big(\bigcup\nolimits_{k=0}^{j-1}\partial R_k\cap\big(R_j\setminus\bigcup\nolimits_{k=0}^{j-1}R_k\big)\Big) + \sum_{j=1}^{I}\mathcal{H}^1\Big(J_u\cap\big(R_j\setminus\bigcup\nolimits_{k=0}^{j-1}\overline{R_k}\big)\Big)\\
&\quad\le \sum\nolimits_{j=0}^{I-1}\mathcal{H}^1(\partial R_j\cap Q_{\mu_0}) + c\mathcal{H}^1(J_u) \le c\mathcal{H}^1(J_u).
\end{aligned}$$

Consequently, applying (5.23) for each $Q\in\bigcup_{j=1}^{I}\mathcal{R}_j$ we find that $\sum_{j=1}^\infty\mathcal{H}^1(\partial P_j'\setminus\partial Q_{\mu_0}) \le C\mathcal{H}^1(J_u)$. Moreover, Theorem 5.1 yields piecewise constant $\mathbb{R}^{2\times2}_{\rm skew}$-valued functions $\bar{A}_Q$ on each $Q \in \bigcup_{j=1}^I \mathcal{R}_j$ (being constant on each component of the partition) such that by (5.28) and the definition of u_{j-1} we have

$$\|\nabla u - \bar{A}_Q\|^p_{L^p(Q\setminus\bigcup_{k<j}\overline{R_k})} \le \|\nabla u_{j-1} - \bar{A}_Q\|^p_{L^p(Q)} \le Ct^{2-p}_{I-j}\|e(u_{j-1})\|^p_{L^2(Q)}$$
$$= Ct^{2-p}_{I-j}\|e(u)\|^p_{L^2(Q\setminus\bigcup_{k<j}\overline{R_k})}.$$

If $|Q\setminus\bigcup_{k<j}\overline{R_k}| > 0$, we find some $\hat{Q} \in \mathcal{Q}^{I-j+1}\setminus\mathcal{R}_{j-1}$ with $\hat{Q} \subset Q$. Then (5.33) (for $i = j-2$) implies $|Q\setminus\bigcup_{k<j}\overline{R_k}| \ge |\hat{Q}\setminus\bigcup_{k<j-1}\overline{R_k}| \ge 2t^2_{I-j+1} = 2\theta^2 t^2_{I-j}$. Thus, for each $Q \in \bigcup_{j=1}^I\mathcal{R}_j$

$$\|\nabla u - \bar{A}_Q\|^p_{L^p(Q\setminus\bigcup_{k<j}\overline{R_k})} \le C|Q\setminus\bigcup\nolimits_{k<j}\overline{R_k}|^{1-\frac{p}{2}}\|e(u)\|^p_{L^2(Q\setminus\bigcup_{k<j}\overline{R_k})}.$$

Therefore, for each $(P'_j)_{j\ge1}$ we find a corresponding $A'_j \in \mathbb{R}^{2\times2}_{\mathrm{skew}}$ such that for $v' := u - \sum_{j=1}^\infty (A'_j\,x)\chi_{P'_j}$ we have by summing over all squares

$$\|v'\|^p_{L^p(Q_{\mu_0}\setminus R_0)} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})},$$

where similarly as before we have used the discrete Hölder inequality. (See e.g. (4.86), (5.17). We omit the details.) It remains to analyze the behavior on R_0. Observe that Theorem 5.1 is not applicable in the squares contained in R_0 since property (5.1) might not hold. However, we may argue similarly as in Step IV of the previous proof. By (5.29)(i) and the fact that $|R_0| \le \theta^{-1}\mathcal{H}^1(J_u)t_I$ we find $\|\nabla u\|^p_{L^p(R_0)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$. Since $u = v'$ on $R_0 = P'_0$, we get $\|\nabla v'\|^p_{L^p(Q_{\mu_0})} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$ and therefore have re-derived (5.28). To conclude the proof of (2.3), it now remains to apply Theorem 2.10 on v' as in the previous proof. □

Remark 5.3. The proof of Theorem 5.1 and Theorem 5.2, in particular (5.23), show that applying Theorem 2.10 with $\rho = \mathcal{H}^1(J_u)$ instead of $\rho = \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})$ yields a partition $(P_j)_j$ and a function v such that (2.3)(ii) still holds and (2.1), (2.3)(i) are replaced by

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j\setminus Q_{\mu_0}) \le C\mathcal{H}^1(J_u), \qquad \|v\|_{L^\infty(Q_{\mu_0})} \le C(\mathcal{H}^1(J_u))^{-1}\|e(u)\|_{L^2(\Omega)}.$$

5.2. **Density arguments.** This section is devoted to the proof of the general version of Theorem 2.1 and to the proof of Theorem 2.5. The strategy is to approximate a given function $u \in GSBD^2(\Omega)$ by a sequence $(u_n)_n \subset \mathcal{W}(\Omega)$. (Recall the definition before Theorem 4.1.) We then apply Theorem 5.2 on $(u_n)_n$ and show that the desired properties can be recovered for the limiting function u. We first present a variant of Theorem 3.11 which yields an approximation result for every $GSBD^2$ function at the expense of a nonoptimal estimate for the surface energy. The reader only interested in the main result for functions in $(G)SBD^2(\Omega)\cap L^2(\Omega)$ may skip the following lemma and corollary and may replace Corollary 5.5 by Theorem 3.11 in the proof of Theorem 2.1 below.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ open, bounded. Let $u \in GSBD^2(\Omega)$ and $\Omega' \subset\subset \Omega$ with Lipschitz boundary. Then there exists a sequence $(u_k)_k \subset \mathcal{W}(\Omega')$ such that for a universal constant $c > 1$ one has*

$$\begin{aligned} &(i)\ \ u_k \to u \text{ a.e. in } \Omega' \text{ as } k\to\infty,\\ &(ii)\ \ \|e(u_k)\|_{L^2(\Omega')} \le c\|e(u)\|_{L^2(\Omega)},\ \mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(J_{u_k}) \le c\mathcal{H}^{d-1}(J_u),\ k\in\mathbb{N}. \end{aligned} \tag{5.35}$$

Proof. We follow closely the proofs in [11, Theorem 1]), [35, Theorem 3.5] and only indicate the necessary changes. By [35, Lemma 2.2] one can find a basis $\mathbf{e}_1,\ldots,\mathbf{e}_d$ of $\mathbb{R}^d$ such that

$$\mathcal{H}^{d-1}(\{x \in J_u : [u](x)\cdot e = 0\}) = 0$$

for all $e \in D := \{\mathbf{e}_i, 1 \le i \le d, \mathbf{e}_i + \mathbf{e}_j, 1 \le i < j \le d\}$. For $h > 0$ small and each $y \in [0,1)^d$ we introduce the discretized function of u defined by

$$u_h^y(\xi) = u(hy + \xi), \quad \xi \in h\mathbb{Z}^d \cap (\Omega - hy).$$

Moreover, letting $\triangle(x) = \prod_{i=1}^d \max\{(1 - |x_i|), 0\}$ we define the continuous interpolation

$$w_h^y(x) = \sum\nolimits_{\xi \in h\mathbb{Z}^d \cap \Omega} u_h^y(\xi) \triangle\Big(\frac{x - (\xi + hy)}{h}\Big).$$

Note that $w_h^y \in W^{1,\infty}(\Omega')$ and for h small enough w_h^y is indeed well defined for all $x \in \Omega'$ since $\Omega' \subset\subset \Omega$. We now show that $w_h^y \to u$ in measure on Ω' as $h \to 0$ for a.e. $y \in [0,1)^d$ which is equivalent to

$$\int_{y \in [0,1)^d} \|\varphi_1(|w_h^y - u|)\|_{L^1(\Omega')} \to 0, \tag{5.36}$$

where $\varphi_t(s) = \min\{s, \frac{1}{t}\}$ for $s \ge 0$ and $t > 0$. Using the subadditivity of φ_1, a change of variables and the fact that $\sum_{\xi \in h\mathbb{Z}^d \cap \Omega} \triangle\Big(\frac{x-\xi}{h} - y\Big) = 1$ for all $x \in \Omega'$, $y \in [0,1)^d$ we deduce

$$\begin{aligned}
&\int_{y \in [0,1)^d} dy \int_{\Omega'} \varphi_1(|w_h^y(x) - u(x)|)\, dx \\
&\quad = \int_{y \in [0,1)^d} dy \int_{\Omega'} \varphi_1\Big(\Big|\sum\nolimits_{\xi \in h\mathbb{Z}^d \cap \Omega} \triangle\Big(\frac{x-\xi}{h} - y\Big)(u(x) - u(hy + \xi))\Big|\Big)\, dx \\
&\quad \le \int_{y \in [0,1)^d} dy \int_{\Omega'} \sum\nolimits_{\xi \in h\mathbb{Z}^d \cap \Omega} \varphi_1\Big(\Big|\triangle\Big(\frac{x-\xi}{h} - y\Big)(u(x) - u(hy + \xi))\Big|\Big)\, dx \\
&\quad \le \sum\nolimits_{\xi \in h\mathbb{Z}^d \cap \Omega} \int_{\frac{x-\xi}{h} - [0,1)^d} \int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x - hz))|)\, dx\, dz \\
&\quad \le \int_{(-1,1)^d} \int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x - hz))|)\, dx\, dz.
\end{aligned}$$

In the last step we used that the sets $\frac{x-\xi}{h} - [0,1)^d$ are pairwise disjoint for $\xi \in h\mathbb{Z}^d \cap \Omega$. Since

$$\int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x - hz))|)\, dx = \int_{\Omega'} \triangle(z) \varphi_{\triangle(z)}(|u(x) - u(x - hz)|)\, dx \to 0$$

for $h \to 0$ and is uniformly bounded by $|\Omega'|$ for all $z \in (-1,1)^d$, we obtain (5.36) by dominated convergence. We now follow closely the proof in [35]. For a specific choice of $y \in [0,1)^d$ and a set of 'bad cubes' (see (3.13)-(3.15) in [35] for details)

$$\mathcal{Q}_h \subset \{Q_\xi = \xi + hy + [0,h)^d : \xi \in h\mathbb{Z}^d\} \tag{5.37}$$

with $\#\mathcal{Q}_h \le Ch^{-(d-1)}$ one defines the function $u_h = w_h^y \chi_{\Omega' \setminus \bigcup_{Q \in \mathcal{Q}_h} Q}$. Clearly, $u_h \in \mathcal{W}(\Omega')$ as the jump set is given by the boundary of the cubes $\mathcal{Q}_h$. (Strictly speaking J_{u_h} is only contained in the boundary of the cubes. However, the desired property may always be achieved by an infinitesimal perturbation of u_h, see [11, Remark 5.3]). In view of (5.36) and $|\bigcup_{Q \in \mathcal{Q}_h} Q| \le Ch$ we get, possibly passing to a not relabeled subsequence, $u_h \to u$ a.e. on Ω' which gives (5.35)(i).

Following the lines of [35], for a suitable choice of $y \in [0,1)^d$ we get $\|e(u_h)\|_{L^2(\Omega')} \le c\|e(u)\|_{L^2(\Omega)}$ and $\mathcal{H}^{d-1}(J_{u_h}) \le c\mathcal{H}^{d-1}(J_u)$ for a universal constant $c > 1$ after passage to a not relabeled subsequence. In fact, in the estimates for the elastic energy, which are based on a slicing technique, the assumption $u \in L^2(\Omega)$ is not needed. (This assumption was only needed to define an extension of u, which is not necessary in our setting.) To conclude the proof of (5.35)(ii) we remark that the inequality $\mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(J_{u_h})$ has not been stated explicitly in [35], but can always be achieved by introducing arbitrary additional 'bad squares' in (5.37). $\square$

The proof of the following corollary is now straightforward.

Corollary 5.5. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary and let $Q \subset \mathbb{R}^d$ be a cube with $\Omega \subset\subset Q$. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_k)_k \subset \mathcal{W}(Q)$ such that for a universal constant $c > 1$ one has*

$$\begin{aligned} &(i) \quad u_k \to u \text{ a.e. in } \Omega \text{ as } k \to \infty,\\ &(ii) \quad \|e(u_k)\|_{L^2(Q)} \le c\|e(u)\|_{L^2(\Omega)},\\ &(iii) \quad \mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(J_{u_k}) \le c\mathcal{H}^{d-1}(J_u) + c\mathcal{H}^{d-1}(\partial\Omega). \end{aligned} \tag{5.38}$$

Proof. Choose a sequence of Lipschitz sets $\Omega_n \subset\subset \Omega$ with $\mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(\partial\Omega_n) \le c\mathcal{H}^{d-1}(\partial\Omega)$ and $|\Omega \setminus \Omega_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$ such that $\partial\Omega_n$ consists of a finite number of closed $(d-1)$-simplices. For each $n \in \mathbb{N}$ we apply Lemma 5.4 on $\Omega_n \subset\subset \Omega$ and obtain sequences $(v^n_l)_l \subset \mathcal{W}(\Omega_n)$ converging in measure to u on Ω_n such that $\|e(v^n_l)\|_{L^2(\Omega_n)}$ and $\mathcal{H}^{d-1}(J_{v^n_l})$ are uniformly controlled by $c\|e(u)\|_{L^2(\Omega)}$ and $c\mathcal{H}^{d-1}(J_u)$, respectively. Define the extensions $\hat{v}^n_l = v^n_l \chi_{\Omega_n} \in \mathcal{W}(Q)$. Possibly replacing $\hat{v}^n_l$ by $\hat{v}^n_l + t^n_l$ for a suitable $t^n_l \in \mathbb{R}^d$, $|t^n_l| \le \frac{1}{l}$, we obtain (not relabeled) sequences still converging to u on Ω_n such that $\mathcal{H}^{d-1}(\partial\Omega_n \setminus J_{\hat{v}^n_l}) = 0$ and thus

$$\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(J_{\hat{v}^n_l}) \le c\mathcal{H}^{d-1}(J_u) + c\mathcal{H}^{d-1}(\partial\Omega).$$

Consequently, by a standard diagonal sequence argument taking into account that convergence in measure is metrizable (take $(f,g) \mapsto \int_\Omega \min\{|f-g|,1\}$) we get a sequence $(w_k)_k \subset \mathcal{W}(Q)$ satisfying (5.38). $\square$

We are now in the position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $p \in (1,2)$ and let $u \in GSBD^2(\Omega)$ be given. Without restriction we assume that Ω is connected as otherwise the following arguments are applied on each connected component of Ω. Choose a square $Q_{\mu_0} \supset\supset \Omega$ with $\mathcal{H}^1(\partial Q_{\mu_0}) \le c\mathcal{H}^1(\partial\Omega)$ for a universal constant $c > 0$.

By Corollary 5.5 we find a sequence $(u_k)_k \subset \mathcal{W}(Q_{\mu_0})$ satisfying (5.38). (The reader only interested in the case $(G)SBD^2(\Omega) \cap L^2(\Omega)$ can instead apply Theorem 3.11 and Remark 3.12.) We apply Theorem 5.2 on each u_k and obtain a sequence of (ordered) Caccioppoli partitions $(P^k_j)_j$ of Q_{μ_0} and corresponding infinitesimal rigid motions $(a^k_j)_j = (a_{A^k_j,b^k_j})_j$ such that $v_k := u_k - \sum_{j=1}^\infty a^k_j \chi_{P^k_j} \in SBV^p(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$ satisfies by (2.1) and (2.3)

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P^k_j) \le c(\mathcal{H}^1(J_{u_k}) + \mathcal{H}^1(\partial Q_{\mu_0})) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)),\\ &(ii) \quad \|\nabla v_k\|_{L^p(Q_{\mu_0})} \le C\|e(u_k)\|_{L^2(Q_{\mu_0})} \le C\|e(u)\|_{L^2(\Omega)},\\ &(iii) \quad \|v_k\|_{L^\infty(Q_{\mu_0})} \le C(\mathcal{H}^1(J_{u_k}))^{-1}\|e(u_k)\|_{L^2(Q_{\mu_0})} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}. \end{aligned} \tag{5.39}$$

Possibly passing to a (not relabeled) refinement of the partition (consisting of the sets $(P^k_j \cap \Omega)_j \cup (P^k_j \setminus \Omega)_j$) we may assume that each component P^k_j satisfies $P^k_j \subset \Omega$ or $P^k_j \cap \Omega = \emptyset$ and (5.39)(i) still holds. Therefore, Theorem 3.9 implies the existence of a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω and a (not relabeled) subsequence such that $\chi_{P^k_j} \to \chi_{P_j}$ in $L^1(\Omega)$, when $k \to \infty$, for all $j \in \mathbb{N}$ and such that, passing to the limit in (5.39)(i) via the lower semicontinuity of the perimeter

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)).$$

This gives (2.1). Applying Ambrosio's compactness theorem (Theorem 3.8) on the sequence $(v_k)_k$ we find $v \in SBV^p(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$ such that $v_k \to v$ a.e. and $\nabla v_k \rightharpoonup \nabla v$ weakly in $L^p(Q_{\mu_0})$ up to a not relabeled subsequence. In particular, we have considering the restriction of v to Ω

$$\|v\|_{L^\infty(\Omega)} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}, \qquad \|\nabla v\|_{L^p(\Omega)} \le C\|e(u)\|_{L^2(\Omega)}. \tag{5.40}$$

Since $(P_j)_j$ is a partition of Ω, it now suffices to show the existence of infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that

$$(u - v)\chi_{P_j} = a_j\chi_{P_j} \tag{5.41}$$

a.e. in P_j for all $j \in \mathbb{N}$. Indeed, (2.3) then is immediate by (5.40). Clearly, if $|P_j| = 0$ it suffices to set $a_j = 0$. If instead $|P_j| > 0$, then it exists $\delta > 0$ independently of k such that $|P_j^k| \ge \delta$. As $u_k\chi_{P_j^k} \to u\chi_{P_j}$ a.e. and $v_k\chi_{P_j^k} \to v\chi_{P_j}$ a.e., the sequence

$$a_j^k\chi_{P_j^k} = (u_k - v_k)\chi_{P_j^k}$$

converges in measure to $(u - v)\chi_{P_j}$. Therefore, it exists a positive nondecreasing continuous function ψ with $\lim_{s\to\infty}\psi(s) = +\infty$ such that $\sup_{k\ge1}\int_{P_j^k}\psi(|a_j^k|)\,\mathrm{d}x \le 1$ (see e.g. [32, Remark 2.2]). By Lemma 3.4 we infer that $(a_j^k)_k$ are bounded in $W^{1,\infty}(\Omega)$ for a constant independent of k. Consequently, we find an infinitesimal rigid motion a_j such that $a_j^k\chi_{P_j^k} \to a_j\chi_{P_j}$ in $L^1(\Omega)$. By the convergence of $a_j^k\chi_{P_j^k}$ to $(u - v)\chi_{P_j}$, this implies (5.41) and concludes the proof. □

Now with Theorem 2.1 at hand the proof of our density result Theorem 2.5 is straightforward.

Proof of Theorem 2.5. Let $u \in GSBD^2(\Omega)$. By Theorem 2.1 applied for some $p \in [1, 2)$ we obtain a Caccioppoli partition $(P_j)_j$ of Ω and corresponding infinitesimal rigid motions $(a_j)_j$ such that $v := u - \sum_j a_j\chi_{P_j} \in SBV^p(\Omega) \cap L^\infty(\Omega)$. As motivated in (2.5) we consider the sequence

$$v_k = u - \sum\nolimits_{j\ge k} a_j\chi_{P_j} \in SBV^p(\Omega) \cap L^\infty(\Omega)$$

and observe that $v_k \to u$ in measure on Ω, $e(v_k) = e(u_k)$ for all $k \in \mathbb{N}$ and $\mathcal{H}^1(J_{v_k} \triangle J_u) \to 0$ when $k \to \infty$. Using Theorem 3.11 each function v_k can be approximated in $L^2(\Omega)$ by a sequence with the properties stated in Theorem 3.11 such that (3.6) holds. Now the assertion follows from a diagonal sequence argument. □

6. Proof of further results

6.1. **Piecewise Poincaré inequality.** We start with the proof of the piecewise Poincaré inequality which is essentially based on the coarea formula for BV functions and can be derived completely independently from the results discussed in the previous sections.

Proof of Theorem 2.3. Without restriction we assume $\|\nabla u\|_{L^1(\Omega)} > 0$ as otherwise u is piecewise constant (see [4, Theorem 4.23], [13]) and there is nothing to show. We start with the case $m = 1$ and $u \in SBV(\Omega;\mathbb{R})$. Following ideas in [8, 26] we can use the coarea formula in BV (see [4, Theorem 3.40]) to write

$$\|\nabla u\|_{L^1(\Omega)} = |Du|(\Omega \setminus J_u) = \int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^*\{u > t\}\big)\,dt.$$

Consequently, with $M := \rho^{-1}\|\nabla u\|_{L^1(\Omega)}$ we find $t_i \in (iM, (i+1)M]$ for all $i \in \mathbb{Z}$ such that

$$\mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^*\{u > t_i\}\big) \le \frac{1}{M}\int_{iM}^{(i+1)M} \mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^*\{u > t\}\big)\,dt. \tag{6.1}$$

Let $E_i = \{u > t_i\} \setminus \{u > t_{i+1}\}$ and note that each set has finite perimeter in Ω since it is the difference of two sets of finite perimeter. Now (6.1) implies

$$\sum\nolimits_{i\in\mathbb{Z}} \mathcal{H}^{d-1}\big((\Omega \cap \partial^* E_i) \setminus J_u\big) \le \frac{2}{M}\|\nabla u\|_{L^1(\Omega)} = 2\rho. \tag{6.2}$$

Since $|\Omega \setminus \bigcup_{i\in\mathbb{Z}} E_i| = 0$, $(E_i)_i$ is a Caccioppoli partition of Ω. Moreover, we note that the function $v := u - \sum_i t_i\chi_{E_i}$ lies in $L^\infty(\Omega)$ and satisfies $\|v\|_{L^\infty(E_i)} \le 2M$. This implies (2.4). Consider the sequence $v_n = \sum_{|i|\le n}(u-t_i)\chi_{E_i}$ for $n \in \mathbb{N}$ with $v_n \in SBV(\Omega;\mathbb{R})$ by [4, Theorem 3.84]. Since $v_n \to v$ in BV norm and SBV is a closed subspace of BV, we conclude $v \in SBV(\Omega;\mathbb{R}) \cap L^\infty(\Omega)$. This concludes the proof in the case $m = 1$ and $u \in SBV(\Omega;\mathbb{R})$.

If $u \in GSBV(\Omega;\mathbb{R})$, we apply the analog of the coarea for $GSBV$ functions (see [4, Theorem 4.34]) and again obtain $(t_i)_{i\in\mathbb{Z}}$, $(E_i)_{i\in\mathbb{Z}}$ such that $v := u - \sum_i t_i\chi_{E_i} \in L^\infty(\Omega)$ and (2.4) holds. (Note that ∇u, J_u have to be understood in a weaker sense, cf. [4, Section 4.5].) The characterization of scalar $GSBV$ functions (see [4, Remark 4.27]) together with $\nabla u \in L^1(\Omega)$, $\mathcal{H}^{d-1}(J_u) < \infty$, yields $\min\{\max\{-n, u\}, n\} \in SBV(\Omega;\mathbb{R})$ for all $n \in \mathbb{N}$. Consequently, similarly as before the sequence $v_n = \sum_{|i|\le n}(u-t_i)\chi_{E_i}$ lies in $SBV(\Omega;\mathbb{R})$, converges to v in BV norm and thus $v \in SBV(\Omega;\mathbb{R}) \cap L^\infty(\Omega)$.

Now let $u \in (GSBV(\Omega;\mathbb{R}))^m$ for $m \ge 2$. We repeat the above argumentation for each component u^j, $j = 1,\ldots,m$, and obtain corresponding $(E^j_i)_{i\in\mathbb{Z}}$ and $(t^j_i)_{i\in\mathbb{Z}}$ such that (6.2) holds with E^j_i in place of E_i and $v^j := u^j - \sum_i t^j_i \chi_{E^j_i}$ satisfies $\|v^j\|_\infty \le 2\rho^{-1}\|\nabla u\|_{L^1(\Omega)}$. We introduce the Caccioppoli partition of Ω, denoted by $(P_j)_{j\ge1}$, consisting of the (nonempty) sets in

$$\{E^1_{i_1} \cap \ldots \cap E^m_{i_m} : i_1, \ldots, i_m \in \mathbb{Z}\}.$$

In view of (6.2) we get that (2.4)(i) holds for a constant also depending on the dimension m. Fix $P_j = E^1_{i_1} \cap \ldots \cap E^m_{i_m}$ and define the corresponding translation $b_j \in \mathbb{R}^m$ by $b_j = (t_{i_1}, \ldots, t_{i_m})$. Then we conclude that $v := u - \sum_j b_j\chi_{P_j}$ satisfies (2.4)(ii) for a constant depending on m. Arguing as before for each component we find $v \in SBV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. □

6.2. **Embedding results.** We now prove the results stated in Section 2.3.

Proof of Theorem 2.7. We first prove (2.8) for $u \in \mathcal{W}(Q)$ for a square $Q \subset \mathbb{R}^2$ in dimension two. Afterwards, we use a slicing and density argument to derive the result for domains in $\mathbb{R}^d$. By Theorem 5.2 for $p = 1$ we find a Caccioppoli partition $(P_j)_j$ of $Q \subset \mathbb{R}^2$ and corresponding infinitesimal rigid motions $(a_j)_j$ such that $v := u - \sum_j a_j\chi_{P_j} \in SBV(Q) \cap L^\infty(Q)$ and

$$\begin{aligned} &\|v\|_\infty \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q))^{-1}\|e(u)\|_{L^2(Q)}, \quad \|\nabla v\|_{L^1(Q)} \le C\|e(u)\|_{L^2(Q)}, \\ &\sum\nolimits_j \mathcal{H}^1(\partial^* P_j) \le C\mathcal{H}^1(J_u) + C\mathcal{H}^1(\partial Q), \end{aligned} \tag{6.3}$$

where C only depends on the diameter of Q. To conclude the proof of (2.8) for $d = 2$, it now suffices to show that

$$\sum\nolimits_j |A_j||P_j| \le C(\mathcal{H}^1(J_u) + 1)\|u\|_\infty + C\|e(u)\|_{L^2(Q)} \tag{6.4}$$

for $C = C(Q)$. To this end, we use Lemma 3.5 and the isoperimetric inequality to obtain

$$\sum\nolimits_j |A_j||P_j| \le c\sum\nolimits_j |P_j|^{\frac12}\|a_j\|_{L^\infty(P_j)} \le c(\|u\|_\infty + \|v\|_\infty)\sum\nolimits_j \mathcal{H}^1(\partial^* P_j). \tag{6.5}$$

Then (6.4) follows from (6.3).

We now treat the case $Q = (-\mu,\mu)^d$ and $u \in \mathcal{W}(Q) \subset SBV(Q) \cap L^\infty(Q)$. To prove the assertion, we need to control $\|\partial_j u_i\|_{L^1(\Omega)}$ for each $1 \le i,j \le d$. For notational convenience we only treat the case $i,j \in \{1,2\}$. The other terms follow analogously due to the symmetry of the problem. For $x \in Q$ we write $x = (x_1, x_2, y)$ with $y \in (-\mu,\mu)^{d-2}$ and introduce the functions $w^y : (-\mu,\mu)^2 \to \mathbb{R}^2$, $w^y(x_1,x_2) = (u_1(x),u_2(x))$ for $y \in (-\mu,\mu)^{d-2}$. Applying the result in $d=2$ we obtain for a.e. $y \in (-\mu,\mu)^{d-2}$

$$\sum\nolimits_{1\le i,j\le 2} \|\partial_j w^y_i\|_{L^1((-\mu,\mu)^2;\mathbb{R})} \le C\|e(w^y)\|_{L^2((-\mu,\mu)^2;\mathbb{R}^{2\times 2}_{\rm sym})} + C\|u\|_\infty(\mathcal{H}^1(J_{w^y})+1),$$

where $C=C(\mu)$. Once we have proved

$$\int_{(-\mu,\mu)^{d-2}} \mathcal{H}^1(J_{w^y})\, d\mathcal{H}^{d-2}(y) \le C\mathcal{H}^{d-1}(J_u), \tag{6.6}$$

we take the integral over $(-\mu,\mu)^{d-2}$, use Fubini's theorem and Hölder's inequality to conclude

$$\sum\nolimits_{1\le i,j\le 2} \|\partial_j u_i\|_{L^1(Q;\mathbb{R})} \le C\|e(u)\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})} + C\|u\|_\infty(\mathcal{H}^{d-1}(J_u)+1).$$

To see (6.6), we apply slicing techniques for BV functions (see [4, Section 3.11]): for $f \in SBV((-\mu,\mu)^n;\mathbb{R}^m)$ and $j=1,\ldots,n$ we define $f_{j,s} : (-\mu,\mu) \to \mathbb{R}^m$ by $f_j(t) = f(s+t\mathbf{e}_j)$ for $s \in \Pi^n_j := \{s \in (-\mu,\mu)^n : s\cdot \mathbf{e}_j = 0\}$. We obtain

$$\int_{\Pi^n_j} \#J_{f_{j,s}}\, d\mathcal{H}^{n-1}(s) = \int_{J_f} |\xi_f \cdot \mathbf{e}_j|\, d\mathcal{H}^{n-1},$$

where ξ_f denotes a normal of the jump set J_f. First, applying this estimate on $w^y \in SBV((-\mu,\mu)^2;\mathbb{R}^2)$ for $j=1,2$ and $y \in (-\mu,\mu)^{n-2}$ a.e. we obtain

$$\int_{\Pi^2_1} \#J_{w^y_{1,s}}\, d\mathcal{H}^1(s) + \int_{\Pi^2_2} \#J_{w^y_{2,s}}\, d\mathcal{H}^1(s) = \int_{J_{w^y}} (|\xi_{w^y}\cdot \mathbf{e}_1| + |\xi_{w^y}\cdot\mathbf{e}_2|)\, d\mathcal{H}^1 \ge \mathcal{H}^1(J_{w^y}).$$

Repeating the argument for the function $w=(u_1,u_2) \in SBV((-\mu,\mu)^d;\mathbb{R}^2)$ for $j=1,2$ we also get

$$\begin{aligned}\int_{\Pi^d_1} \#J_{w_{1,s}}\, d\mathcal{H}^{d-1}(s) + \int_{\Pi^d_2} \#J_{w_{2,s}}\, d\mathcal{H}^{d-1}(s) &= \int_{J_w} (|\xi_w\cdot\mathbf{e}_1| + |\xi_w\cdot\mathbf{e}_2|)\, d\mathcal{H}^{d-1}\\ &\le 2\mathcal{H}^{d-1}(J_w) \le 2\mathcal{H}^{d-1}(J_u).\end{aligned}$$

Taking the integral over $(-\mu,\mu)^{d-2}$ we derive (6.6) from the last two estimates.

It remains to consider general Lipschitz domains $\Omega \subset \mathbb{R}^d$ and $u \in SBD^2(\Omega)\cap L^\infty(\Omega)$. First, we choose a cube Q containing Ω and define the extension $\bar u = u\chi_\Omega \in SBD^2(Q)\cap L^\infty(Q)$. Clearly, the choice of Q depends only on Ω. Note that $\mathcal{H}^{d-1}(J_{\bar u}) \le (\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega))$. By Theorem 3.11, Remark 3.12 we find a sequence $(u_k)_k \subset \mathcal{W}(Q)$ with $u_k \to \bar u$ in $L^2(Q)$ and $\|u_k\|_\infty \le \|u\|_\infty$ such that by the above arguments

$$|Du_k|(Q) = \|\nabla u_k\|_{L^1(Q)} + \int_{J_{u_k}} |[u_k]|\, d\mathcal{H}^{d-1} \le C\|e(u)\|_{L^2(\Omega)} + C\|u\|_\infty(\mathcal{H}^{d-1}(J_u)+1) \tag{6.7}$$

for $C=C(\Omega)$, where we used $|[u_k]| \le 2\|u\|_\infty$ a.e. As $(u_k)_k$ is uniformly bounded in BV norm, we deduce $u \in BV(\Omega)$ and that (2.8) holds by lower semicontinuity. Finally, by Alberti's rank one property $|D^cu| \le \sqrt{2}|E^cu|$ and the fact that $u \in SBD(\Omega)$ we conclude $u \in SBV(\Omega)$. □

Proof of Theorem 2.9. Let $u \in GSBD^2(\Omega)$. We show that each component u_i, $i=1,\ldots,d$, satisfies (2.9) for the truncation $u^M_i = \min\{\max\{u_i,-M\},M\}$. Herefrom we particularly deduce $u_i \in GBV(\Omega;\mathbb{R})$ since in the scalar case this property is equivalent to $u^M_i \in BV_{\rm loc}(\Omega)$ for all $M>0$ (see [4, Remark 4.27]). As in the previous proof it essentially suffices to show

$$|Du^M_i|(Q) \le \|\nabla u^M_i\|_{L^1(Q)} + 2M\mathcal{H}^{d-1}(J_{u^M_i}) \le CM(\mathcal{H}^{d-1}(J_u)+1) + C\|e(u)\|_{L^2(Q)}, \tag{6.8}$$

where $\Omega = Q$ is a cube, $C=C(Q)$ and $u \in \mathcal{W}(Q) \subset SBV(Q)$. In fact, we then establish the general case using the approximation of u given by Corollary 5.5 and repeating the argument in (6.7). (Note, however, that in contrast to (6.7) we cannot apply Alberti's theorem and therefore only obtain $u_i^M \in BV(\Omega)$.) Finally, in view of the slicing argument (6.6) it is enough to treat the planar case $u \in \mathcal{W}(Q)$ for a square $Q \subset \mathbb{R}^2$.

By Theorem 5.2 for $p = 1$ we get that $v := u - \sum_j a_j \chi_{P_j} \in SBV(Q) \cap L^\infty(Q)$ satisfying $\|v\|_{L^\infty(Q)} \le M'$ with $M' := C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q))^{-1}\|e(u)\|_{L^2(Q)}$ by (2.3). Let $a = \sum_j a_j \chi_{P_j}$ and a_i be the i-th component for $i = 1, 2$. From [15, Theorem 2.2] we obtain $a \in (GSBV(Q))^2$, in particular it is shown that for a universal $c > 0$

$$|Da_i^M|(Q) \le cM \sum\nolimits_j \mathcal{H}^1(\partial^* P_j) \le cM(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q)). \tag{6.9}$$

Up to sets of negligible $\mathcal{L}^2$-measure we have

$$T := \{\nabla u_i^M \neq 0\} \subset \{|u_i| \le M\} \subset \{|a_i| \le M + M'\}$$

since $\|v\|_{L^\infty(Q)} \le M'$. Therefore, we compute using (6.9) and (2.3)

$$\begin{aligned} \|\nabla u_i\|_{L^1(T)} &\le \|\nabla v_i\|_{L^1(T)} + \|\nabla a_i\|_{L^1(T)} \le \|\nabla v_i\|_{L^1(Q)} + \|\nabla a_i^{M+M'}\|_{L^1(Q)} \\ &\le c(M + M')(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q)) + C\|e(u)\|_{L^2(Q)} \\ &\le CM(\mathcal{H}^1(J_u) + 1) + C\|e(u)\|_{L^2(Q)}. \end{aligned} \tag{6.10}$$

As $\|\nabla u_i^M\|_{L^1(Q)} = \|\nabla u_i\|_{L^1(T)}$ and $\mathcal{H}^1(J_{u_i^M} \setminus J_u) = 0$, we obtain the second inequality in (6.8). The first inequality follows from the decomposition of the distributional derivative and the fact that $\|u_i^M\|_\infty \le M$. This concludes the proof. $\square$

We also obtain the following variant of (2.9) needed in Section 6.3.

Corollary 6.1. *Let $Q \subset \mathbb{R}^d$ be a cube. Then there is a constant $C = C(Q) > 0$ such that for all $u \in \mathcal{W}(Q)$ and Borel sets $F \subset Q$ one has*

$$|Du_i^M|(Q) \le CM\big(\mathcal{H}^{d-1}(J_u) + |F|\big) + C\big(\|e(u)\|_{L^2(Q)} + \|u\|_{L^1(Q\setminus F)}\big).$$

for all $i = 1, \ldots, d$ and $M > 0$, where $u_i^M =: \min\{\max\{u_i, -M\}, M\}$.

Proof. Similarly as in the proof of Theorem 2.9 it suffices to show this estimate in the planar setting $d = 2$ for a square $Q \subset \mathbb{R}^2$ as the general case then follows by Fubini and the slicing argument (6.6) for a constant depending on the dimension. Let $\bar{c} = \bar{c}(Q) > 0$ to be specified below. If $\mathcal{H}^1(J_u) + |F| \ge \bar{c}$, the result follows directly from (6.8) for a constant depending on $\bar{c}$.

Now let $\mathcal{H}^1(J_u) + |F| \le \bar{c}$. By Theorem 5.2 for $p = 1$ in the version of Remark 5.3 we obtain an ordered Caccioppoli partition $(P_j)_j$ and $v := u - \sum_j a_j \chi_{P_j}$ such that for $C = C(Q) > 0$

$$\|\nabla v\|_{L^1(Q)} + \mathcal{H}^1(J_u)\|v\|_{L^\infty(Q)} \le C\|e(u)\|_{L^2(Q)}, \qquad \sum\nolimits_j \mathcal{H}^1(\partial^* P_j \cap Q) \le C\mathcal{H}^1(J_u).$$

The essential step is now to show

$$\|\nabla a_i^{M+M'}\|_{L^1(Q)} \le C\big((M + M')\mathcal{H}^1(J_u) + \|u\|_{L^1(Q\setminus F)} + \|e(u)\|_{L^2(Q)}\big) \tag{6.11}$$

for $a := \sum_j a_j \chi_{P_j}$ and $M' := C(\mathcal{H}^1(J_u))^{-1}\|e(u)\|_{L^2(Q)}$. Indeed, the claim then follows by repeating the argument in (6.10) noting that $\{\nabla u_i^M \neq 0\} \subset \{|a_i| \le M + M'\}$ since $\|v\|_\infty \le M'$. For notational convenience we set $\bar{M} = M + M'$.

Since $\sum_j \mathcal{H}^1(\partial^* P_j \cap Q) \le C\bar{c}$, for $\bar{c}$ small enough the relative isoperimetric inequality implies $|P_1| > \frac{1}{2}|Q|$ and $|P_j| \le C(\mathcal{H}^1(\partial^* P_j \cap Q))^2$ for $j \ge 2$. Without restriction we assume that the sets $(P_j)_{j\ge 2}$ are connected (more precisely indecomposable, see [4, Example 4.18]) as otherwise we consider the indecomposable components. By [32, Lemma 4.8] we get that the diameter of each P_j is controlled in terms of $C\mathcal{H}^1(\partial^* P_j \cap Q)$ for $C = C(Q)$, which then also yields $\mathcal{H}^1(\partial^* P_j) \le C\mathcal{H}^1(\partial^* P_j \cap Q)$ for all $j \ge 2$. Then again by [15, Theorem 2.2] (cf. also (6.9))

$$\sum_{j\ge 2} \|\nabla a_i^{\bar M}\|_{L^1(P_j)} \le c\bar M \sum_{j\ge 2} \mathcal{H}^1(\partial^* P_j) \le C\bar M \sum_{j\ge 2} \mathcal{H}^1(\partial^* P_j \cap Q) \le C\bar M \mathcal{H}^1(J_u). \tag{6.12}$$

By Theorem 3.13 we find an infinitesimal rigid motion $a' = a_{A',b'}$ and an exceptional set $F' \subset Q$ with

$$\|\nabla u - A'\|_{L^1(Q\setminus F')} + \|u - a'\|_{L^1(Q\setminus F')} \le C\|e(u)\|_{L^2(Q)}, \qquad |F'| \le C(\mathcal{H}^1(J_u))^2 \le C\bar c^2.$$

Consequently, for $\bar c$ small we have $|F \cup F'| \le C\bar c \le \frac14 |Q|$ and derive using Lemma 3.5 and the triangle inequality with $G := F \cup F'$

$$\begin{aligned}
\|\nabla a_i^{\bar M}\|_{L^1(P_1)} &\le |P_1||A_1| \le 2|P_1 \setminus G||A_1| \\
&\le 2|Q\setminus G||A'| + 2\|\nabla u - A_1\|_{L^1(P_1)} + 2\|\nabla u - A'\|_{L^1(Q\setminus F')} \\
&\le C|Q\setminus G|^{-\frac12}\|a'\|_{L^1(Q\setminus G)} + C\|e(u)\|_{L^2(Q)} \\
&\le C\|u - a'\|_{L^1(Q\setminus F')} + C\|u\|_{L^1(Q\setminus F)} + C\|e(u)\|_{L^2(Q)} \\
&\le C\|u\|_{L^1(Q\setminus F)} + C\|e(u)\|_{L^2(Q)}.
\end{aligned}$$

This together with (6.12) concludes the proof of (6.11). □

We close this section with the proof of Lemma 2.8.

Proof of Lemma 2.8. Let $q \in [1,\infty)$. Consider the function defined in (2.7) with $r_k = k^{-p}$ and $d_k = k^{-1+dp}$, where $p = \frac{1}{d-1} + \frac{1}{q(d-1)^2}$. (Note that the existence of pairwise disjoint balls with this property is guaranteed by [18, Lemma 12.2] and the approximate differential of u in the sense [4, Definition 3.70] exists a.e.) We first see that $\mathcal{H}^{d-1}(J_u) < +\infty$ as $\sum_k r_k^{d-1} < \infty$ due to the fact that $p > \frac{1}{d-1}$. To see that $u \in L^q(\Omega)$ and $\nabla u \notin L^1(\Omega)$, it suffices to show

$$\sum\nolimits_k r_k^d (r_k d_k)^q < \infty, \qquad \sum\nolimits_k r_k^d d_k = \infty.$$

The latter is immediate since $-pd - 1 + pd = -1$. To see the first property we calculate $-p(d+q) - q + dpq = \frac{1}{d-1} - pd < -1$, where in the last step we used $p > \frac{1}{d-1}$.

It remains to show that $u \in GSBD^2(\Omega)$. To this end, we consider the sequence of functions

$$u_j = \sum\nolimits_{k=1}^{j} (A_k\,(x - x_k))\chi_{B_k}(x) \in GSBD^2(\Omega) \cap L^q(\Omega)$$

converging to u and by the compactness theorem for $GSBD$ (see [18, Theorem 11.3]) we see that $u \in GSBD^2(\Omega)$ since $e(u_j) = 0$ a.e., $\sup_j \mathcal{H}^{d-1}(J_{u_j}) < \infty$ and $\sup_j \|u\|_{L^q(\Omega)} < \infty$. □

6.3. **A Korn-Poincaré inequality for functions with small jump set.** We finally give the proof of the Korn-Poincaré inequality for functions with small jump set.

Proof of Theorem 2.10. As in the previous sections we first treat the case $u \in \mathcal{W}(Q) \subset SBV(Q)$ with $\mathcal{H}^{d-1}(J_u) > 0$ and at the end of the proof we indicate the adaptions if $u \in GSBD^2(Q)$. We may assume that $\mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(\partial Q)$ since otherwise the theorem trivially holds with $E = Q$ for $C = C(Q) > 0$ sufficiently large. We apply the Korn-Poincaré inequality due to Chambolle, Conti, and Francfort (see [12, Theorem 1]) which is as the assertion of Theorem 2.10 with the difference that only the volume of the exceptional set can be controlled. We find $F \subset Q$ with $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$ for $C = C(Q) > 0$ and an infinitesimal rigid motion a such that with $q = \frac{2d}{d-1}$ and $v := u - a$

$$\|v\|_{L^q(Q\setminus F)} \le C\|e(u)\|_{L^2(Q)}. \tag{6.13}$$

We now consider a suitable truncation of v. As a preparation let $\eta \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$ be a function with

$$\eta(z) = z \ \text{ for } \ |z| \le 1, \quad |\eta(z)| \le |z| \ \text{ for } z \in \mathbb{R}^d, \quad \eta(z) = 0 \ \text{ for } \ |z| \ge 2 \tag{6.14}$$

and let $\eta_i(z) = i\eta(z/i)$ for $i \in \mathbb{N}$. Then $\eta_i \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$, $\eta_i(z) = z$ for $|z| \le i$, $\eta_i = 0$ on $\{|z| \ge 2i\}$ and $\|\nabla\eta_i\|_\infty \le c$ for c independent of i. We now define

$$M = (\mathcal{H}^{d-1}(J_u))^{-\frac{d}{q(d-1)}}\|e(u)\|_{L^2(Q)} = (\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}$$

and observe that by (6.13)

$$|\{|v| > M\}| \le |F| + \tfrac{1}{M^q}\|v\|^q_{L^q(Q\setminus F)} \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}. \tag{6.15}$$

Define $v' = \eta_{2\sqrt{d}M}(v)$ and note $v' \in SBV(Q)$ with $\mathcal{H}^{d-1}(J_{v'} \setminus J_u) = 0$. We apply Corollary 6.1 and then by $\|\nabla\eta_{2\sqrt{d}M}\|_\infty \le c$, $\|\eta_{2\sqrt{d}M}\|_\infty \le 4\sqrt{d}M$, $\nabla\eta_{2\sqrt{d}M} = 0$ on $\{|z| \ge 4\sqrt{d}M\}$ and the chain rule in BV (see [4, Theorem 3.96])

$$\begin{aligned}|Dv'|(Q) &\le c\sum\nolimits_{i=1}^d \|\nabla v_i^{4\sqrt{d}M}\|_{L^1(Q)} + 2\|\eta_{2\sqrt{d}M}\|_\infty\mathcal{H}^{d-1}(J_u)\\ &\le CM(\mathcal{H}^{d-1}(J_u) + |F|) + C\|e(v)\|_{L^2(Q)} + C\|v\|_{L^1(Q\setminus F)}\\ &\le CM\mathcal{H}^{d-1}(J_u) + C\|e(u)\|_{L^2(Q)},\end{aligned} \tag{6.16}$$

where in the last step we used (6.13) and $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \le C\mathcal{H}^{d-1}(J_u)$. (Recall $\mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(\partial Q)$.) Moreover, (6.14) yields $v' = v$ on $\{|v| < 2\sqrt{d}M\}$. Now arguing similarly as in the proof of Theorem 2.3 (see Section 6.1), using the coarea formula, we can find $t_i \in (M, 2M)$ such that $G_i := \{-t_i < v_i' < t_i\}$ is a set of finite perimeter and

$$\begin{aligned}\mathcal{H}^{d-1}(Q\cap\partial^* G_i) &\le \frac{1}{M}\int_{-\infty}^{\infty}\mathcal{H}^{d-1}\big(Q\cap\partial^*\{v_i' > t\}\big)\,dt \le \frac{1}{M}|Dv'|(Q)\\ &\le \frac{1}{M}\big(CM\mathcal{H}^{d-1}(J_u) + C\|e(u)\|_{L^2(Q)}\big) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}},\end{aligned} \tag{6.17}$$

where in the last step we used $\mathcal{H}^{d-1}(J_u) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$ for a constant $C = C(Q)$.

We define $E = Q\setminus\bigcap_{i=1}^d G_i$. As $Q\setminus G_i \subset \{|v'| > M\}$ and $\{|v'| > M\} \subset \{|v| > M\}$ by (6.14), (6.15) yields the second part of (2.10). By (6.17) we get $\mathcal{H}^{d-1}(\partial^* E\cap Q) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$, which shows the first part of (2.10). Since $v' = v$ on $\{|v'| < 2\sqrt{d}M\}$ and

$$Q\setminus E \subset \{|v_i'| < 2M,\ i = 1,\dots,d\} \subset \{|v'| < 2\sqrt{d}M\},$$

we observe $\|v\|_{L^\infty(Q\setminus E)} \le C(\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}$ by the definition of M. This gives (2.11)(ii) and together with (6.13), $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$ we obtain

$$\|v\|_{L^q(Q\setminus E)} \le \|v\|_{L^q(Q\setminus F)} + \|v\|_{L^q(F\cap\{|v|\le 2\sqrt{d}M\})} \le C\|e(u)\|_{L^2(Q)} + CM|F|^{\frac{1}{q}} \le C\|e(u)\|_{L^2(Q)},$$

which establishes (2.11)(i). Finally, $\bar{u} := (u-a)\chi_{Q\setminus E} = v\chi_{Q\setminus E} = v'\chi_{Q\setminus E}$ clearly lies in $SBV^2(Q)\cap L^\infty(Q)$ since $u \in \mathcal{W}(Q)$. We use (6.16) and [4, Theorem 3.84] to calculate

$$|D\bar{u}|(Q) \le |Dv'|(Q) + CM\mathcal{H}^{d-1}(\partial^* E\cap Q) \le CM(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}} + C\|e(u)\|_{L^2(Q)},$$

which by the definition of M gives (2.12). The variant announced below Theorem 2.10 with $p \in [1,2]$ follows by replacing $q = \frac{2d}{d-1}$ by $q' = \frac{2d}{p(d-1)}$ in the above proof.

Finally, we treat the case $u \in GSBD^2(Q)$ with $\mathcal{H}^{d-1}(J_u) > 0$. We proceed similarly as in the density argument in the proof of Theorem 2.1 (see Section 5.2) and refer therein for details. Let $(u^j)_j$ be a sequence of rescaled versions of u defined on squares $Q_j \supset\supset Q$ with

$|Q_j \setminus Q| \to 0$ for $j \to \infty$. Using Lemma 5.4 we approximate $(u^j)_j$ with a (diagonal) sequence $(u_k)_k \subset \mathcal{W}(Q)$ such that (5.35) holds with $u_k \to u$ a.e. on Q.

We then obtain infinitesimal rigid motions $(a_k)_k$ and exceptional sets $(E_k)_k$ such that (2.10)-(2.12) hold for each $k \in \mathbb{N}$. By compactness we find a set of finite perimeter E satisfying (2.10) such that $\chi_{E_k} \to \chi_E$ in measure after extracting a not relabeled subsequence. Moreover, applying Lemma 3.4 we find that also the infinitesimal rigid motions converge to some a and (2.11) follows by lower semicontinuity.

The sequence $\bar{u}_k := (u_k - a_k)\chi_{Q \setminus E_k}$ is uniformly bounded in BV norm and we therefore deduce $\bar{u} := (u - a)\chi_{Q \setminus E} \in BV(Q)$ satisfies (2.12) by lower semicontinuity. Likewise, a compactness result in SBD^2 together with the uniform bound on $\|\bar{u}_k\|_\infty$ gives $\bar{u} \in SBD^2(Q)$ (see [5]). Finally, Alberti's rank one property also yields $\bar{u} \in SBV(Q)$. $\square$

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