Continuum limits of discrete thin films with superlinear growth densities

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1 Introduction

Object of this paper is the description of the overall behaviour of variational pair-interaction lattice systems defined on 'thin' domains of $\mathbb{Z}^N$; *i.e.* on domains consisting on a finite number M of mutually interacting copies of a portion of a $N-1$-dimensional discrete lattice (see Figure 1). On one hand we draw a parallel with

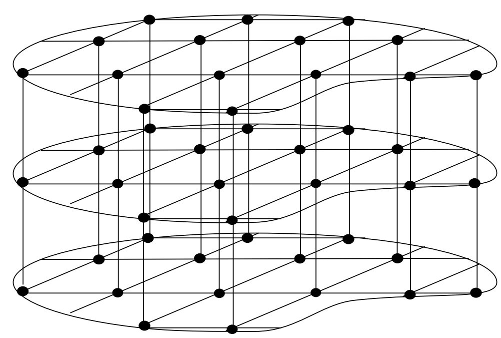

Figure 1: A discrete thin film.

the analog theories for 'continuous' thin films showing that general compactness and homogenization results can be proven by adapting the techniques commonly used for problems on Sobolev spaces; on the other hand we show that new phenomena arise due to the different nature of the microscopic interactions, and in particular that for long-range interactions a surface energy on the free surfaces of the film due to boundary layer effects renders the effective behaviour depend in a non-trivial way on the number M of layers.

In a more precise notation, we consider energies depending on functions parameterized on a portion of the lattice $\mathbb{Z}^N$ consisting of a 'cylindrical' set

$$\mathcal{Z}(\omega, M) = (\omega \cap \mathbb{Z}^{N-1}) \times \{1, \dots M\}$$

(see Fig. 1) of the form

$$\sum_{\alpha,\beta \in \mathcal{Z}(\omega,M)} \varphi_{\alpha,\beta}(u_\alpha - u_\beta), \qquad u : \mathcal{Z}(\omega, M) \to \mathbb{R}^d$$

when the size of ω is large. Upon introducing a small positive parameter ε and scaling ω to a fixed size, obtaining the discrete thin domain

$$\mathcal{Z}_\varepsilon(\omega, M) = (\omega \cap \varepsilon\mathbb{Z}^{N-1}) \times \{\varepsilon, 2\varepsilon, \dots M\varepsilon\}, \tag{1.1}$$

this problem can be reformulated as the description of the Γ-limit of energies of the form

$$F_\varepsilon(u) = \sum_{\alpha,\beta \in \mathcal{Z}_\varepsilon(\omega,M)} \varepsilon^{N-1} f^\varepsilon_{\alpha,\beta}\Big(\frac{u_\alpha - u_\beta}{\varepsilon}\Big), \qquad u : \mathcal{Z}_\varepsilon(\omega, M) \to \mathbb{R}^d \tag{1.2}$$

(at this point a more general dependence of the energy densities on ε is introduced for the sake of generality).

The energies above may be viewed as the discrete analog of thin-film energies of the form

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon} \int_{\omega\times(0,M\varepsilon)} W_\varepsilon(x, Du(x))\, dx \qquad u \in W^{1,p}(\omega \times (0, M\varepsilon); \mathbb{R}^d). \tag{1.3}$$

For such functionals a number of results in the framework of variational convergence have been obtained since the pioneering work of Le Dret and Raoult [22]; in particular if W_ε uniformly satisfy some p-growth conditions, a general compactness result by Braides, Fonseca and Francfort [9] shows that, upon subsequences, we can always suitably define a Γ-limit energy in the lower-dimensional set ω of the form

$$\mathcal{F}(u) = \int_\omega W(\hat{x}, Du(\hat{x}))\, d\hat{x}, \qquad u \in W^{1,p}(\omega; \mathbb{R}^d) \tag{1.4}$$

(here, $\hat{x} = (x_1, \dots, x_{n-1})$). Particular cases are when $W_\varepsilon = W_0(Du)$ is independent of ε and x, in which the limit W is simply given [22] by

$$W(A) = M\, Q\overline{W}_0(A), \qquad \overline{W}_0(A) = \inf_z W_0(A\,|\,z), \tag{1.5}$$

where Q stands for the operation of quasiconvexification (note the trivial dependence on M), and when $W_\varepsilon(x, A) = W(x/\varepsilon, A)$, in which suitable homogenization formulas hold [9].

On the other hand, a general compactness theory for discrete systems with energy densities of polynomial growth defined on 'thick' domains by Alicandro and Cicalese [1] is also available. In particular, that theory can be applied in the case above when $M = 1$ and the energies F_ε are simply interpreted as defined on $\omega \cap \varepsilon\mathbb{Z}^{N-1}$. In this case again an energy of the same form as $\mathcal{F}$ above can be proven to be the Γ-limit in a suitable sense of such F_ε (discrete functions must be identified with piecewise-constant interpolations). Appropriate homogenization formulas apply as well if the discrete interactions possess some periodicity. It must be noted that homogenized energy densities in many ways behave differently than the corresponding usual integral homogenization energies [8], as shown by the computation of bounds for simple discrete mixtures of two types of quadratic interactions by Braides and Francfort [10]. Even for simple non-convex next-to-nearest interactions the formulas defining the limit energy densities highlight microscopic oscillations (see Braides, Gelli and Sigalotti [14] and Pagano and Paroni [23] for one-dimensional examples, and Friesecke and Theil [21] for a simple optically non-linear example in dimension two).

In the general case $M > 1$ we note that the energies F_ε can also be seen as defined on M copies of $\omega \cap \varepsilon\mathbb{Z}^{N-1}$ interacting through pair-interactions corresponding to $\beta - \alpha$ with a non-zero component in the N-th variable. We first show that functions on which the limit energy is finite, that are thus defined on M copies of ω, are actually equal on each of these copies, so that the limit energy can be defined on the only set ω. To prove a general compactness and representation theorem for the limit we adapt both the localization techniques on cylindrical domains used by Braides, Fonseca and Francfort [9] to prove a compactness result for energies on continuous thin films, and those used by Alicandro and Cicalese [1] for thick discrete systems, where the main difficulty is taking care of long-range interactions. More precise homogenization formulas are given in the case when the energy densities are periodic; *i.e.* $f^\varepsilon_{\alpha,\beta} = f_{\alpha/\varepsilon,\beta/\varepsilon}$ and there exists an integer k such that $f_{i,j} = f_{i',j'}$ if $i - i' = j - j' \in k\mathbb{Z}^{N-1}$. As in [9] these formulas are defined through minimum problems on rectangles with boundary conditions on the lateral boundaries only. In the discrete case it must be noted that these formulas are necessary also in the 'trivial' case when $f_{i,j} = f_{j-i}$; *i.e.* the energy densities depend only on the distance of α and β in the unscaled reference lattice $\mathbb{Z}^N$, as already observed for 'thick' domains, except when only nearest-neighbour interactions are taken into account (see [5]). In the case of thin domains an additional scale effect must be taken into account, since long-range interactions (next-to nearest interactions and further) produce different effects close to the upper and lower free boundaries than in the interior (see Figure 2). These effects can be viewed as generating a surface energy through a boundary layer (see [6, 15]) that for thin films is of the same order as the bulk energy. Note that this effect is present also for simple quadratic interactions, as observed by Charlotte and Truskinovsky for one-dimensional systems [18].

The paper is organized as follows: after introducing the necessary notation in

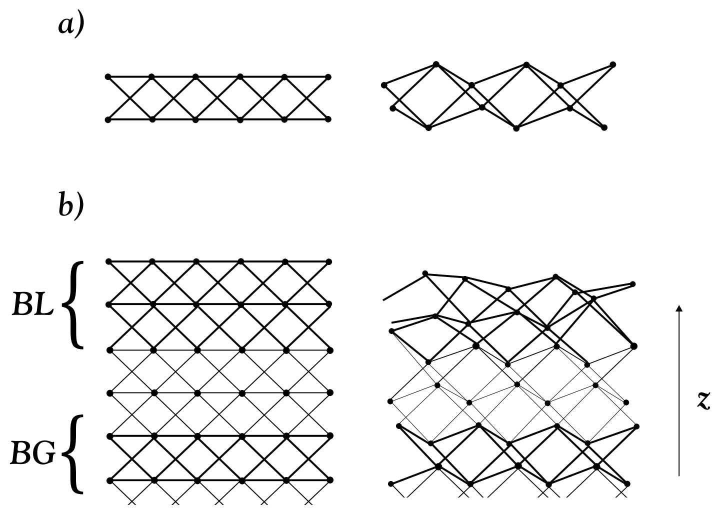

Figure 2: Different effects in a simple model for thin films with nearest and next to nearest interactions: a) ground state geometry for a two layers thin film; b) bulk geometry (BG) and boundary layer effect (BL) for a multi-layer thin film subject to a vertical deformation gradient z.

order to make the energies F_ε more tractable, the compactness and representation theorem is proved in Section 2 together with the convergence of minimum problems. Section 3 is devoted to homogenization; in particular the simple case when only 'horizontal' and 'vertical' interactions are present is treated, when a formula can be given showing explicitly the non-trivial dependence of the homogenization formula on the number M of layers. The asymptotic behaviour of homogenization formulas as the number of layers increases is treated in Section 4, showing in some cases a separation of scale effect so that we may first let M tend to infinity, thus obtaining a bulk energy by the results of Alicandro and Cicalese [1] and then applying the dimension-reduction formula by Le Dret And Raoult (1.5). Various examples are presented in Section 5, where in particular the Cauchy-Born rule is discussed, stating that microscopic arrays corresponding to affine boundary conditions are regularly spaced on a lattice with the same overall affine deformation. It is shown that the domain of affine boundary condition where the Cauchy-Born rule fails decreases with the number M of layers.

Our techniques can be used also to cover types of non central interaction energies even in the case of linear elasticity (see [2]). The extension of the result to sub-linear growth densities will require to consider limit energies on discontinuous functions (see [7],[15], [11], [12], [13] [17], [18], [25]) Finally, we mention that our work has connections with a number of papers where variational methods for thin structures are dealt with but in a different perspectives, most notably that of Friesecke and James [20], some works by Blanc *et al.* [3] and more recently by Schmidt [24].

2 Compactness and integral representation

In this section we introduce the family of discrete energies we will consider in the rest of the paper and then prove a compactness theorem, asserting that any sequence of energies in this class has a subsequence whose Γ-limit F is an integral functional.

2.1 Notation and Preliminaries

Given $N,\ d \in \mathbb{N}$, we denote by $\{e_1, e_2, \dots, e_N\}$ the standard basis in $\mathbb{R}^N$, by $|\cdot|$ the usual euclidean norm and by $\mathcal{M}^{d\times N}$ the space of $d \times N$ matrices. For $x,\ y \in \mathbb{R}^N$, $[x, y]$ denotes the segment between x and y. If Ω is an open subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω while $\mathcal{A}_0(\Omega)$ denotes the family of all open subsets of Ω whose closure is a compact subset of Ω. If $B \subset \mathbb{R}^N$ is a Borel set, we will denote by $|B|$ its Lebesgue measure. We use standard notation for L^p and Sobolev spaces.

We also introduce a useful notation for difference quotient along any direction. Fix $\xi \in \mathbb{R}^N$; for $\varepsilon > 0$ and for every $u : \mathbb{R}^N \to \mathbb{R}^d$ we define

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon|\xi|}.$$

Moreover we set

$$\mathcal{A}_\varepsilon(\Omega) := \{u : \mathbb{R}^N \to \mathbb{R}^d\ :\ u \text{ constant on } \alpha + [0, \varepsilon)^N \text{ for any } \alpha \in \varepsilon\mathbb{Z}^N \cap \Omega\}.$$

2.2 Compactness Theorem

In what follows, for the sake of simplicity, ω will denote a bounded convex open set of $\mathbb{R}^{N-1}$ with Lipschitz boundary, the general case of a non-convex ω being dealt with similarly (see [1]). Given $M \in \mathbb{N}$ and $\varepsilon > 0$, we set

$$\Omega_\varepsilon := \omega \times [0, \varepsilon(M - 1)].$$

In the following we will identify the set of functions $\mathcal{A}_\varepsilon(\Omega_\varepsilon)$ with $[\mathcal{A}_\varepsilon(\omega)]^M$ through the bijection $u \mapsto (u^0, u^1, \dots, u^{M-1})$ where, for any $i \in \{0, 1, \dots, M - 1\}$, $u^i \in \mathcal{A}_\varepsilon(\omega)$ is defined by

$$u^i(\beta) = u(\beta, \varepsilon i) \quad \forall \beta \in \varepsilon\mathbb{Z}^{N-1} \cap \omega.$$

We consider the family of functionals $F_\varepsilon : [L^p(\omega;\mathbb{R}^d)]^M \to [0,+\infty]$ defined as

$$F_\varepsilon(u) = \begin{cases} \sum_{\xi\in\mathbb{Z}^N} \sum_{\alpha,\alpha+\varepsilon\xi\in\Omega_\varepsilon\cap\varepsilon\mathbb{Z}^N} \varepsilon^{N-1} f_\varepsilon^\xi\left(\alpha, D_\varepsilon^\xi u(\alpha)\right) & \text{if } u\in\mathcal{A}_\varepsilon(\Omega_\varepsilon) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.1}$$

and $f_\varepsilon^\xi : (\varepsilon\mathbb{Z}^N\cap\Omega_\varepsilon)\times\mathbb{R}^d \to [0,+\infty)$ is a given function. Let $p>1$, on f_ε^ξ we make the following assumptions:

$$f_\varepsilon^{e_i}(\alpha,z) \geq c_1(|z|^p-1) \quad \forall(\alpha,z)\in(\varepsilon\mathbb{Z}^N\cap\Omega)\times\mathbb{R}^d,\ i\in\{1,\dots,N\} \tag{2.2}$$

$$f_\varepsilon^\xi(\alpha,z) \leq C_\varepsilon^\xi(|z|^p+1) \quad \forall(\alpha,z)\in(\varepsilon\mathbb{Z}^N\cap\Omega)\times\mathbb{R}^d,\ \xi\in\mathbb{Z}^N, \tag{2.3}$$

where $c_1>0$ and $\{C_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfies:

$$\limsup_{\varepsilon\to0^+} \sum_{\xi\in\mathbb{Z}^N} C_\varepsilon^\xi < +\infty; \tag{H1}$$

$$\forall\delta>0 \quad \exists\, M_\delta>0 : \quad \limsup_{\varepsilon\to0^+} \sum_{|\xi|>M_\delta} C_\varepsilon^\xi < \delta. \tag{H2}$$

We define also a "localized" version of our energies: given an open set $A\in\mathcal{A}(\omega)$, set

$$A_\varepsilon := A\times[0,\varepsilon(M-1)].$$

We isolate the contributions due to interactions within A_ε as follows. For $u\in\mathcal{A}_\varepsilon(\Omega_\varepsilon)$, $A\in\mathcal{A}(\omega)$ and $\xi\in\mathbb{Z}^N$, set

$$\mathcal{F}_\varepsilon^\xi(u,A) := \sum_{\alpha\in R_\varepsilon^\xi(A_\varepsilon)} \varepsilon^{N-1} f_\varepsilon^\xi\left(\alpha, D_\varepsilon^\xi u(\alpha)\right), \tag{2.4}$$

where

$$R_\varepsilon^\xi(A_\varepsilon) := \{\alpha\in\varepsilon\mathbb{Z}^N :\ [\alpha,\alpha+\varepsilon\xi]\subset A_\varepsilon\}.$$

The function $\mathcal{F}_\varepsilon^\xi$ represents the energy due to the interactions within A_ε along the direction ξ. Then the local version of the functional in (2.1) is given by

$$F_\varepsilon(u,A) = \begin{cases} \sum_{\xi\in\mathbb{Z}^N} \mathcal{F}_\varepsilon^\xi(u,A) & \text{if } u\in\mathcal{A}_\varepsilon(\Omega_\varepsilon) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

In the following, referring to [4] and [19] for all the definitions and properties of Γ-convergence, we will denote by $F'(u,A)$ and $F''(u,A)$ the $\Gamma\text{-}\liminf_{\varepsilon\to0+} F_\varepsilon(u,A)$ and the $\Gamma\text{-}\limsup_{\varepsilon\to0+} F_\varepsilon(u,A)$ with respect to the $[L^p(\omega;\mathbb{R}^d)]^M$-topology. Moreover if $u\in[L^p(\omega;\mathbb{R}^d)]^M$ is such that $u^0=u^1=\ldots=u^{M-1}$ we will simply write $u\in L^p(\omega;\mathbb{R}^d)$. The main result of this section is stated in the following theorem.

Theorem 2.1 **(compactness)** *Let* $\{f^{\xi}_{\varepsilon}\}_{\varepsilon,\xi}$ *satisfy (2.2), (2.3) and let* (H1)-(H2) *hold. Then:*

(i) if $u \in [L^p(\omega;\mathbb{R}^d)]^M$ *is such that* $F'(u,A) < +\infty$*, then*

$$u^0 = u^1 = \ldots = u^{M-1} \in W^{1,p}(A;\mathbb{R}^d)$$

and

$$F'(u,A) \geq c \ \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N-1})} - |A| \right) \tag{2.6}$$

for some positive constant c *independent on* u *and* A*.*

(ii) for every $u \in W^{1,p}(\omega;\mathbb{R}^d)$ *there holds*

$$F''(u,A) \leq C \ \left(\|\nabla u\|^p_{L^p(A;\mathbb{R}^{d\times N-1})} + |A| \right) \tag{2.7}$$

for some positive constant C *independent on* u *and* A*.*

(iii) for every sequence (ε_j) *of positive real numbers converging to* 0*, there exists a subsequence* (ε_{j_k}) *and a Carathéodory function quasiconvex in the second variable* $f : \omega \times \mathbb{R}^{d\times N-1}$ *satisfying*

$$c(|S|^p - 1) \leq f(x,S) \leq C(|S|^p + 1),$$

with $0 < c < C$*, such that* $(F_{\varepsilon_{j_k}}(\cdot))$ Γ*-converges with respect to the* $[L^p(\omega;\mathbb{R}^d)]^M$*-topology to the functional* $F : [L^p(\omega;\mathbb{R}^d)]^M \to [0,+\infty]$ *defined as*

$$F(u) = \begin{cases} \displaystyle\int_{\omega} f(x,\nabla u)\,dx & \textit{if } u \in W^{1,p}(\omega;\mathbb{R}^d) \\ +\infty & \textit{otherwise.} \end{cases} \tag{2.8}$$

Proof. (i) Let $\varepsilon_n \to 0^+$ and let u_n converge to u in $[L^p(\omega;\mathbb{R}^d)]^M$ and be such that $\liminf_n F_{\varepsilon_n}(u_n,A) < +\infty$. By the growth condition (2.2) we get

$$\begin{aligned} F_{\varepsilon_n}(u_n,A) &\geq c_1 \sum_{i=1}^{N} \sum_{\alpha\in R^{e_i}_{\varepsilon_n}(A_{\varepsilon_n})} \varepsilon_n^{N-1} |D^{e_i}_{\varepsilon_n} u_n(\alpha)|^p - c_1 N|A| \\ &= c_1 \sum_{j=0}^{M-1} \sum_{i=1}^{N-1} \sum_{\beta\in \tilde{R}^{e_i}_{\varepsilon_n}(A)} \varepsilon_n^{N-1} |D^{e_i}_{\varepsilon_n} u^j_n(\beta)|^p \\ &\quad + c_1 \sum_{j=0}^{M-2} \sum_{\beta\in\varepsilon\mathbb{Z}^{N-1}\cap A} \varepsilon_n^{N-1} \left| \frac{u^{j+1}_n(\beta) - u^j_n(\beta)}{\varepsilon_n} \right|^p - c_1 N|A|. \end{aligned} \tag{2.9}$$

where

$$\tilde{R}^{\xi}_{\varepsilon_n}(A) := \{\alpha \in \varepsilon\mathbb{Z}^{N-1} : \ [\alpha, \alpha+\varepsilon\xi] \subset A\}.$$

For any $\eta > 0$, set

$$A_\eta := \{x \in A : dist(x, A^c) > \eta\}.$$

Note that for any $j \in \{0, 1, \ldots, M-2\}$ and for n large enough it holds

$$\frac{1}{\varepsilon_n^p} \int_{A_\eta} |u_n^{j+1}(x) - u_n^j(x)|^p \, dx \leq \sum_{\beta \in \varepsilon \mathbb{Z}^{N-1} \cap A} \varepsilon_n^{N-1} \left| \frac{u_n^{j+1}(\beta) - u_n^j(\beta)}{\varepsilon_n} \right|^p$$

Thus, by (2.9) and the arbitrariness of η, passing to the limit in the previous inequality, we get that $u^0 = u^1 = \ldots = u^{M-1}$. For any j, by (2.9) we have,

$$\liminf_n \sum_{i=1}^{N-1} \sum_{\beta \in \tilde{R}_{\varepsilon_n}^{e_i}(A)} \varepsilon_n^{N-1} |D_{\varepsilon_n}^{e_i} u_n^j(\beta)|^p < +\infty.$$

Then, by a slicing argument (see the proof of Proposition 3.4 in [1]), one can prove that $u \in W^{1,p}(A; \mathbb{R}^d)$ and that

$$\liminf_n \sum_{i=1}^{N-1} \sum_{\beta \in \tilde{R}_{\varepsilon_n}^{e_i}(A)} \varepsilon_n^{N-1} |D_{\varepsilon_n}^{e_i} u_n^j(\beta)|^p \geq \|\nabla u\|_{L^p(A;\mathbb{R}^d)}^p.$$

Hence, by (2.9), (2.6) holds.

(ii) By a density argument it suffices to prove (2.7) for $u \in C_c^\infty(\mathbb{R}^{N-1}; \mathbb{R}^d)$. For such a u, consider the family $(u_\varepsilon) \subset \mathcal{A}_\varepsilon$ defined as

$$u_\varepsilon(\alpha) := u(\alpha_\pi), \quad \alpha \in \varepsilon \mathbb{Z}^N.$$

Then $u_\varepsilon \to u$ in $[L^p(\omega; \mathbb{R}^d)]^M$ as $\varepsilon \to 0^+$ and, following the same steps as in the proof of Proposition 3.5 in [1], one can prove that

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, A) \leq C \left(\int_A |\nabla u(x)|^p \, dx + |A| \right),$$

from which we deduce (2.7).

In order to prove *(iii)* we need some preliminary result we will state in the next propositions. □

The next technical lemma is analogous to Lemma 3.6 in [1] and asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions. We omit the proof since it is the same as that of Lemma 3.6 in [1] up to small changes.

Lemma 2.2 *Let $A \in \mathcal{A}(\omega)$ and set $A^\varepsilon := \{x \in A : \ dist(x, \partial A) > 2\varepsilon\sqrt{N-1}\}$. Then for any $\xi \in \mathbb{Z}^N$ and $u \in \mathcal{A}_\varepsilon$ there holds*

$$\sum_{\alpha \in \tilde{R}_\varepsilon^\xi(A^\varepsilon)} |D_\varepsilon^\xi u(\alpha)|^p \leq C \sum_{i=1}^N \sum_{\alpha \in \tilde{R}_\varepsilon^{e_i}(A)} |D_\varepsilon^{e_i} u(\alpha)|^p.$$

In the next three propositions we establish the subadditivity, the inner regularity and the locality of the set function $F''(u,\cdot)$ which are the analogue of Propositions 3.7, 3.8 and 3.9 in [1]. In the proofs we will exploit the same cut-off argument used in [1], the main difference being in the choice of the cut-functions depending only on the planar variables.

Proposition 2.3 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (2.2),(2.3) and let* (H1)-(H2) *hold. Let $A, B \in \mathcal{A}(\omega)$ and let $A', B' \in \mathcal{A}(\omega)$ be such that $A' \subset\subset A$ and $B' \subset\subset B$. Then for any $u \in W^{1,p}(\omega;\mathbb{R}^d)$,*

$$F''\left(u, A' \cup B'\right) \leq F''\left(u, A\right) + F''\left(u, B\right). \tag{2.10}$$

Proof. Without loss of generality, we may suppose $F''(u,A)$ and $F''(u,B)$ finite. Let $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon$ both converge to u in $[L^p(\omega;\mathbb{R}^d)]^M$ and be such that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u,A), \quad \limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, B) = F''(u,B).$$

By (2.2) and Lemma 2.2, we infer that

$$\sup_{\xi\in\mathbb{Z}^N} \sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi((A^\varepsilon)_\varepsilon)} \varepsilon^{N-1} |D_\varepsilon^\xi u_\varepsilon(\alpha)|^p < +\infty,$$

$$\sup_{\xi\in\mathbb{Z}^N} \sup_{\varepsilon>0} \sum_{\alpha\in R_\varepsilon^\xi((B^\varepsilon)_\varepsilon)} \varepsilon^{N-1} |D_\varepsilon^\xi v_\varepsilon(\alpha)|^p < +\infty,$$

where A^ε and B^ε are defined as in Lemma 2.2. Moreover, since (u_ε) and (v_ε) converge to u in the $[L^p(\omega;\mathbb{R}^d)]^M$-topology, we have

$$\begin{aligned}\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap(\omega')_\varepsilon} \varepsilon^{N-1} & \left(|u_\varepsilon(\alpha)|^p + |v_\varepsilon(\alpha)|^p\right) \\ & \leq \quad \|u_\varepsilon\|^p_{[L^p(\omega;\mathbb{R}^d)]^M} + \|v_\varepsilon\|^p_{[L^p(\omega;\mathbb{R}^d)]^M} \leq C < +\infty,\end{aligned} \tag{2.11}$$

$$\sum_{\alpha\in\varepsilon\mathbb{Z}^N\cap(\omega')_\varepsilon} \varepsilon^{N-1} \left(|u_\varepsilon(\alpha) - v_\varepsilon(\alpha)|^p\right) \leq \|u_\varepsilon - v_\varepsilon\|^p_{[L^p(\omega;\mathbb{R}^d)]^M} \to 0^+$$

for any $\omega' \subset\subset \omega$. Set

$$d := \operatorname{dist}\left(A', A^c\right)$$

and, given $L \in \mathbb{N}$, for any $i \in \{1,\ldots,L\}$ define

$$A_i := \{x \in A : \ \operatorname{dist}(x, A') < i\frac{d}{L}\}.$$

Let φ_i be a cut-off function between A_i and A_{i+1}, with $\|\nabla\varphi_i\|_\infty \leq 2\frac{L}{d}$. Then for any $i \in \{1,\dots,L\}$ consider the family of functions $w^i_\varepsilon \in \mathcal{A}_\varepsilon$ still converging to u in $[L^p(\omega;\mathbb{R}^d)]^M$ defined as

$$w^i_\varepsilon(\alpha) := \varphi_i(\alpha_\pi)u_\varepsilon(\alpha) + (1-\varphi_i(\alpha_\pi))\, v_\varepsilon(\alpha).$$

Then, proceeding as in the proof of Proposition 3.7 in [1], we can choose $i(\varepsilon) \in \{1,\dots,L-3\}$ such that $w^{i(\varepsilon)}_\varepsilon$ still converges to u in $[L^p(\omega;\mathbb{R}^d)]^M$ and,

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(w^{i(\varepsilon)}_\varepsilon, A'\cup B') \leq F''(u,A) + F''(u,B) + \frac{C}{L-3},$$

from which we easily deduce (2.10). □

Proposition 2.4 *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (2.2),(2.3) and let* (H1)-(H2) *hold. Then for any $u \in W^{1,p}(\omega;\mathbb{R}^d)$ and for any $A \in \mathcal{A}(\omega)$, there holds*

$$\sup_{A'\subset\subset A} F''(u,A') = F''(u,A).$$

Proof. Since $F''(u,\cdot)$ is an increasing set function, it suffices to prove that

$$\sup_{A'\subset\subset A} F''(u,A') \geq F''(u,A).$$

To do this, we apply the same argument of the proof of Proposition 2.3. Given $\delta > 0$, there exists $A'' \subset\subset A$ such that

$$|A\setminus \overline{A''}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A''})} \leq \delta.$$

Let $\tilde\omega \supset\supset \omega$ and let $\tilde u \in W^{1,p}(\tilde\omega;\mathbb{R}^d)$ an extension of u. By Theorem 2.1 (*ii*), we may find $v_\varepsilon \in \mathcal{A}_\varepsilon(\tilde\omega)$ such that v_ε converges to $\tilde u$ in $[L^p(\tilde\omega;\mathbb{R}^d)]^M$ and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, A\setminus\overline{A''}) \leq C\left(|A\setminus\overline{A''}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A''})}\right) \leq C\delta. \tag{2.12}$$

We remark that this extension on $\tilde\omega$ is just a technical tool to exploit an analogue of inequality (2.11) and obtain a control of the interactions near the boundary of ω. Let $A' \in \mathcal{A}(\omega)$ be such that $A'' \subset\subset A' \subset\subset A$ and let $u_\varepsilon \in \mathcal{A}_\varepsilon$ converge to u in $[L^p(\omega;\mathbb{R}^d)]^M$, with

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A') = F''(u,A'). \tag{2.13}$$

Set

$$d := \mathrm{dist}(A'', A'^c).$$

Given $L \in \mathbb{N}$, for any $i \in \{1, \ldots, L\}$ let A_i , φ_i and w^i_ε be defined as in the previous proposition. By reasoning as before, taking into account (2.12), we can choose $i(\varepsilon) \in \{1, \ldots, L-3\}$ such that $w^{i(\varepsilon)}_\varepsilon$ converges to u in $[L^p(\omega; \mathbb{R}^d)]^M$ and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(w^{i(\varepsilon)}_\varepsilon, A) \le \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A') + C(\frac{1}{L-3} + \delta + \delta L^p). \tag{2.14}$$

Then, by using (2.13) we deduce that

$$F''(u, A) \le \sup_{A' \subset\subset A} F''(u, A') + C(\frac{1}{L-3} + \delta + \delta N^p).$$

Eventually, letting δ go to zero and L go to infinity, we obtain the thesis. □

Proposition 2.5 *Let* $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ *satisfy (2.2),(2.3) and let* (H1)-(H2) *hold. Then for any* $A \in \mathcal{A}(\omega)$ *and for any* $u,\ v \in W^{1,p}(\omega; \mathbb{R}^d)$*, such that* $u = v$ *a.e. there holds*

$$F''(u, A) = F''(v, A).$$

Proof. Thanks to Proposition 2.4, we may assume that $A \in \mathcal{A}_0(\omega)$. We first prove

$$F''(u, A) \ge F''(v, A). \tag{2.15}$$

Once more we apply the argument used in the previous proposition. Given $\delta > 0$, there exists $A_\delta \subset\subset A$ such that

$$|A \setminus \overline{A_\delta}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A_\delta})} \le \delta.$$

Let $v_\varepsilon \in \mathcal{A}_\varepsilon(\omega)$ and $u_\varepsilon \in \mathcal{A}_\varepsilon(\omega)$ be such that

$$v_\varepsilon \to v \text{ in}[L^p(\omega; \mathbb{R}^d)]^M$$
$$u_\varepsilon \to u \text{ in}[L^p(\omega; \mathbb{R}^d)]^M$$

and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) = F''(u, A), \tag{2.16}$$

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon, A \setminus \overline{A_\delta}) = F''(v, A \setminus \overline{A_\delta}) \le C\left(|A \setminus \overline{A_\delta}| + \|\nabla u\|^p_{L^p(A\setminus\overline{A_\delta})}\right) \le C\delta$$

Set

$$d := \mathrm{dist}(A_\delta, A^c)$$

and, given $L \in \mathbb{N}$, for any $i \in \{1, \ldots, L\}$ define

$$A_i := \{x \in A : \ \mathrm{dist}(x, A_\delta) < i\frac{d}{L}\}.$$

Let then φ_i and $w^i_\varepsilon \in \mathcal{A}_\varepsilon$ be as in the previous proposition. Hence, as in the proof of Propositions 2.3 and 2.4, we can choose $i(\varepsilon) \in \{1, \dots, L-3\}$ such that $w^{i(\varepsilon)}_\varepsilon$ converges to v in $[L^p(\omega; \mathbb{R}^d)]^M$ and

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(w^{i(\varepsilon)}_\varepsilon, A) \le \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) + C\left(\frac{1}{L-3} + \delta + \delta L^p\right).$$

By (2.16), we get

$$F''(v, A) \le F''(u, A) + C\left(\frac{1}{L-3} + \delta + \delta L^p\right).$$

Eventually, letting first $\delta \to 0^+$ and then $L \to +\infty$, we obtain (2.15). Reversing the roles of u and v we obtain the thesis. □

Proof of Theorem 2.1 (iii). By the compactness property of the Γ-convergence and by Proposition 2.4, there exists a subsequence (ε_{j_k}) such that, for any $(u, A) \in W^{1,p}(\omega; \mathbb{R}^d) \times \mathcal{A}(\omega)$, there holds

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) =: F(u, A)$$

(see [8] Theorem 10.3). Moreover, by (i),

$$\Gamma(L^p)\text{-}\lim_k F_{\varepsilon_{j_k}}(u) = +\infty$$

for $u \in L^p(\omega; \mathbb{R}^d) \setminus W^{1,p}(\omega; \mathbb{R}^d)$. So far, it suffices to check that, for every $(u, A) \in W^{1,p}(\omega; \mathbb{R}^d) \times \mathcal{A}(\omega)$, $F(u, A)$ satisfies the following hypotheses:

(1) (locality) F is local, i.e. $F(u, A) = F(v, A)$ if $u = v$ *a.e.* on $A \in \mathcal{A}(\Omega)$;

(2) (measure property) for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;

(3) (growth condition) there exists $c > 0$ and $a \in L^1(\Omega)$ such that

$$F(u, A) \le c \int_A (a(x) + |Du|^p)\, dx$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$;

(4) (translation invariance in u) $F(u + z, A) = F(u, A)$ for all $z \in \mathbb{R}^d$, $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$;

(5) (lower semicontinuity) for all $A \in \mathcal{A}(\Omega)$ $F(\cdot, A)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}(\omega; \mathbb{R}^d)$.

In fact (see [16]), if all these hypotheses are fulfilled, there exists a Carathéodory function $f : \omega \times \mathbb{M}^{d\times(N-1)} \to [0, +\infty)$ satisfying the growth condition

$$c(|S|^p - 1) \le f(x, S) \le C(|S|^p + 1),$$

with $0 < c < C$ for all $x \in \omega$ and $S \in M^{d\times(N-1)}$ and such that

$$F(u, A) = \int_A f(x, Du(x))\, dx$$

for all $u \in W^{1,p}(\omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\omega)$.
It can be easily seen that the superadditivity property of $F_\varepsilon(u, \cdot)$ is conserved in the limit. Thus, as an easy consequence of (i), (ii) and Propositions 2.3, 2.4, 2.5 and thanks to De Giorgi - Letta Criterion (see [8]), hypotheses (1), (2), (3) hold true. Moreover, as $F_\varepsilon(u, A)$ depends on u only through its difference quotients, hypothesis (4) is satisfied and finally, by the lower semicontinuity property of Γ-limit, also hypothesis (5) is fulfilled. □

2.3 Convergence of minimum problems

In order to treat minimum problems with boundary data, we also derive a compactness theorem in the case that our functionals are subject to Dirichlet boundary conditions.

Given $\varphi \in Lip\ (\mathbb{R}^{N-1})$ and $l \in \mathbb{N}$, set, for any $\varepsilon > 0$ and $A \in \mathcal{A}(\omega)$

$$\mathcal{A}^l_{\varepsilon,\varphi}(A) := \{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N)\ : u(\alpha) = \varphi(\alpha_\pi) \text{ if } \big(\alpha_\pi + [-l\varepsilon, l\varepsilon]^{N-1}\big) \cap A^c \neq \emptyset\big)\}. \tag{2.17}$$

Then define $F^{\varphi,l}_\varepsilon : [L^p(\omega; \mathbb{R}^d)]^M \times \mathcal{A}(\omega) \to [\prime, +\infty]$ as

$$F^{\varphi,l}_\varepsilon(u, A) = \begin{cases} F_\varepsilon(u, A) & \text{if } u \in \mathcal{A}^l_{\varepsilon,\varphi}(A) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.18}$$

By simplicity of notation we set $\mathcal{A}_{\varepsilon,\varphi}(A) := \mathcal{A}^1_{\varepsilon,\varphi}(A)$ and $F^\varphi_\varepsilon := F^{\varphi,1}_\varepsilon$.

Theorem 2.6 *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy (2.2), (2.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to* 0*, let (ε_{j_k}) and f be as in Theorem 2.1 (iii). For any $\varphi \in Lip\ (\mathbb{R}^{N-1})$, let $F^\varphi : [L^p(\omega; \mathbb{R}^d)]^M \times \mathcal{A}(\omega) \to [0, +\infty]$ be defined as*

$$F^\varphi(u, A) = \begin{cases} \displaystyle\int_A f(x, \nabla u)\, dx & \textit{if } u - \varphi \in W^{1,p}_0(A; \mathbb{R}^d) \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, for any $A \in \mathcal{A}(\omega)$ with Lipschitz boundary and $l \in \mathbb{N}$, $(F^{\varphi,l}_{\varepsilon_{j_k}}(\cdot, A))$ Γ-converges with respect to the $[L^p(\omega; \mathbb{R}^d)]^M$-topology to the functional $F^\varphi(\cdot, A)$.

Proof. For the sake of simplicity we prove the Theorem with $l = 1$, the proof being the same in the other cases. Let us first prove the Γ-liminf inequality. Let (u_k) be a sequence of functions belonging to $\mathcal{A}_{\varepsilon_{j_k},\varphi}(A)$ converging to u in the $[L^p(\omega;\mathbb{R}^d)]^M$ topology such that

$$\liminf_k F^{\varphi}_{\varepsilon_{j_k}}(u_k, A) = \lim_k F^{\varphi}_{\varepsilon_{j_k}}(u_k, A) < +\infty.$$

Then, from (2.2), we get in particular that

$$\sup_k \sum_{i=1}^{N} \sum_{\alpha\in R^{e_i}_{\varepsilon_{j_k}}(A_{\varepsilon_{j_k}})} \varepsilon^{N-1}_{j_k} |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty. \tag{2.19}$$

Thanks to the boundary conditions on u_k it is easy to deduce that

$$\sup_k \sum_{i=1}^{N} \sum_{\alpha,\alpha+\varepsilon_{j_k}e_i\in\omega_{\varepsilon_{j_k}}\cap\varepsilon_{j_k}\mathbb{Z}^N} \varepsilon^{N-1}_{j_k} |D^{e_i}_{\varepsilon_{j_k}} u_n(\alpha)|^p < +\infty.$$

Then, by reasoning as in the proof of Theorem 2.1 (i), we can prove that $u \in W^{1,p}(\omega;\mathbb{R}^d)$ and, since (u_k) converge to φ in $[L^p(\omega \setminus A;\mathbb{R}^d)]^M$, we get that $u - \varphi \in W^{1,p}_0(A;\mathbb{R}^d)$. By Theorem 2.1 one has

$$\liminf_k F^{\varphi}_{\varepsilon_{j_k}}(u_k, A) = \liminf_k F_{\varepsilon_{j_k}}(u_k, A) \geq F^{\varphi}(u, A).$$

To prove the Γ-limsup inequality, let us first consider $u \in W^{1,p}(\omega;\mathbb{R}^d)$ such that supp $(u - \varphi) \subset\subset A$. Let $u_k \in \mathcal{A}_{\varepsilon_{j_k}}(\omega)$, be such that (u_k) converges to u in $[L^p(\omega;\mathbb{R}^d)]^M$ and

$$\limsup_k F_{\varepsilon_{j_k}}(u_k, A) = F^{\varphi}(u, A).$$

Then, by reasoning as in the proof of Proposition 2.4, given $\delta > 0$, we can find suitable cut-off functions ϕ_k with supp $(u - \varphi) \subset\subset$ supp $\phi_k \subset\subset A$ such that, set

$$v_k(\alpha) := \phi_k(\alpha_\pi)u_k(\alpha) + (1 - \phi_k(\alpha_\pi))\varphi(\alpha_\pi),$$

then (v_k) still converges to u in $[L^p(\omega;\mathbb{R}^d)]^M$, $v_k \in \mathcal{A}_{\varepsilon_{j_k},\varphi}(\omega)$ for k large enough and

$$\limsup_k F_{\varepsilon_{j_k}}(v_k, A) \leq \limsup F_{\varepsilon_{j_k}}(u_k, A) + \delta.$$

Thus, thanks to the definition of Γ-limsup we have

$$\Gamma\text{-limsup} F^{\varphi}_{\varepsilon_{j_k}}(u, A) \leq F^{\varphi}(u, A) + \delta.$$

By the arbitrariness of δ, we obtain the required inequality. In the general case the thesis follows by a density argument, thanks to the lower semicontinuity of Γ-limsup and to the continuity of F with respect to the strong convergence in $W^{1,p}(\omega;\mathbb{R}^d)$. □

As a consequence of the previous theorem we derive the following result about the convergence of minimum problems with boundary data.

Corollary 2.7 *Under the hypotheses of Theorem 2.6 we get that, for any* $\varphi \in Lip\ (\mathbb{R}^{N-1})$*,* $l \in \mathbb{N}$ *and* $A \in \mathcal{A}(\omega)$ *with Lipschitz boundary*

$$\lim_k \inf\{F_{\varepsilon_{j_k}}(u,A): \ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}(A)\} = \min\{F(u,A): \ u-\varphi \in W^{1,p}_0(A;\mathbb{R}^d)\}.$$

Moreover, if (u_k) *is a converging sequence such that*

$$\lim_k F_{\varepsilon_{j_k}}(u_k,A) = \lim_k \inf\{F_{\varepsilon_{j_k}}(u,A): \ u \in \mathcal{A}^l_{\varepsilon_{j_k},\varphi}(A)\},$$

then its limit is a minimizer for $\min\{F(u,A): \ u-\varphi \in W^{1,p}_0(A;\mathbb{R}^d)\}$.

Proof. Let (u_k) be a sequence such that $F^{\varphi}_{\varepsilon_{j_k}}(u_k,A) < +\infty$. Then, by (2.2) and by the boundary conditions on u_k, it is easy to show that, for any $j \in \{0,1,\dots,M-1\}$

$$\sup_k \sum_{i=1}^{N-1} \sum_{\beta \in \varepsilon_{j_k}\mathbb{Z}^{N-1}\cap K} \varepsilon^{N-1} |D^{e_i}_{\varepsilon_{j_k}} u^j_k(\beta)|^p < +\infty,$$

for any compact set K of $\mathbb{R}^{N-1}$. By virtue of this property, up to passing to a continuous extension of u^j_k vanishing outside a bounded open set containing ω, we get

$$\lim_{|h|\to 0} \sup_k \|\tau_h u^j_k - u^j_k\|_{L^p(\mathbb{R}^{N-1};\mathbb{R}^d)} = 0,$$

where we have set

$$(\tau_h u)(x) := u(x+h), \ x \in \mathbb{R}^{N-1}, \ h \in \mathbb{R}^{N-1}.$$

Then, by Frechét-Kolmogorov Theorem and by (i) of Theorem 2.1, there exists a subsequence (u_{k_n}) converging in $[L^p(\omega;\mathbb{R}^d)]^M$ to a function $u \in L^p(\omega;\mathbb{R}^d)$. Arguing as in the previous proof it is easy to show that $u-\varphi \in W^{1,p}_0(\omega)$. The thesis follows thanks to Theorem 2.6. □

We can also derive the analogue of Theorem 2.6 and Corollary 2.7 about the convergence of minimum problems with periodic conditions. We omit the proof since it is similar to that of the previous proposition up to small changes.

Let $\mathcal{Q}(\omega)$ be the family of all open $N-1$-cubes contained in ω. For any $\varepsilon > 0$, $r > 0$, $Q = (x_0, x_0+r)^{N-1} \in \mathcal{Q}(\omega)$ and $\varphi \in Lip\ (\mathbb{R}^{N-1})$, set

$$r_\varepsilon = \varepsilon\left(\left[\frac{r}{\varepsilon}\right] - 2\right),$$

$$\mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) = \left\{u \in \mathcal{A}_\varepsilon(\mathbb{R}^N): \ u^j - \hat{\varphi} \ \ r_\varepsilon - \text{periodic}, \ j \in \{0,\dots,M-1\}\right\}, \quad (2.20)$$

where $\hat{\varphi} \in \mathcal{A}_\varepsilon(\mathbb{R}^{N-1})$, $\hat{\varphi}(\alpha) = \varphi(\alpha)$ for any $\alpha \in \varepsilon\mathbb{Z}^{N-1}$. Then define $F^{\varphi,\#}_\varepsilon : [L^p(\omega;\mathbb{R}^d)]^M \times \mathcal{Q}(\omega) \to [0,+\infty]$ as

$$F^{\varphi,\#}_\varepsilon(u,Q) = \begin{cases} F_\varepsilon(u,Q) & \text{if } u \in \mathcal{A}^{\#}_{\varepsilon,\varphi}(Q) \\ +\infty & \text{otherwise.} \end{cases} \quad (2.21)$$

Theorem 2.8 *Let $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ satisfy (2.2), (2.3) and let* (H1)-(H2) *hold. Given (ε_j) a sequence of positive real numbers converging to 0, let (ε_{j_k}) and f be as in Theorem 2.1. Then, for any $\varphi \in Lip\ (\mathbb{R}^{N-1})$, let $F^{\#} : [L^p(\omega;\mathbb{R}^d)]^M \times \mathcal{Q}(\omega) \to [0,+\infty]$ be defined as*

$$F^{\varphi,\#}(u,Q) = \begin{cases} \displaystyle\int_Q f(x,\nabla u)\,dx & \text{if } u \in W^{1,p}_{\#}(Q;\mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for any $Q \in \mathcal{Q}(\omega)$, $(F^{\varphi,\#}_{\varepsilon_{j_k}}(\cdot,Q))$ Γ-converges with respect to the $[L^p(\omega;\mathbb{R}^d)]^M$-topology to the functional $F^{\varphi,\#}(u,Q)$.

As a consequence of the previous theorem, by reasoning as in the proof of Corollary 2.7 one can prove the following result.

Corollary 2.9 *Under the hypotheses of Theorem 2.8 we get that, for any $\varphi \in Lip\ (\mathbb{R}^{N-1})$ and $Q \in \mathcal{Q}(\omega)$*

$$\lim_k \inf\{F_{\varepsilon_{j_k}}(u,Q):\ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)\} = \min\{F(u,Q):\ u-\varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}.$$

Moreover, if (u_k) is a converging sequence such that

$$\lim_k F_{\varepsilon_{j_k}}(u_k,Q) = \lim_k \inf\{F_{\varepsilon_{j_k}}(u,Q):\ u \in \mathcal{A}^{\#}_{\varepsilon_{j_k},\varphi}(Q)\},$$

then its limit is a minimizer for $\min\{F(u,Q):\ u-\varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^d)\}$.

3 Homogenization

In this section we will give a homogenization result by presenting a Γ-convergence theorem for the energies F_ε in the case that the functions f_ε^ξ are obtained by rescaling by ε functions f^ξ periodic in the space variable.

Let $\mathbf{k} = (k_1,\dots,k_{N-1}) \in \mathbf{N}^{N-1}$ be given and set

$$\mathcal{R}_{\mathbf{k}} := (0,k_1) \times \cdots \times (0,k_{N-1}).$$

For any $\xi \in \mathbf{Z}^N$, let $f^\xi : \mathbf{Z}^N \times \mathbf{R}^d \to [0,+\infty)$ be such that $f^\xi((\cdot,\alpha_N),z)$ is $\mathcal{R}_{\mathbf{k}}$-periodic for any $\alpha_N \in \mathbf{Z}$ and $z \in \mathbf{R}^d$. Then we consider f_ε^ξ of the following form

$$f_\varepsilon^\xi(\alpha,z) := f^\xi\left(\frac{\alpha}{\varepsilon},z\right). \tag{3.1}$$

In this case, the growth conditions (2.2) and (2.3) and hypotheses (H1) and (H2) can be rewritten as follows:

$$f^{e_i}(\alpha,z) \geq c_1(|z|^p - 1), \quad \forall i \in \{1,\dots,N\}, \tag{3.2}$$

$$f^{\xi}(\alpha, z) \le C^{\xi}(|z|^p + 1), \tag{3.3}$$

where

$$\sum_{\xi \in \mathbf{Z}^N} C^{\xi} < +\infty. \tag{$H3$}$$

In the sequel we will use the following notation: for any $x = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}$ define

$$[x]_{\mathbf{k}} := \left(\left[\frac{x_1}{k_1} \right] k_1, \dots, \left[\frac{x_{N-1}}{k_{N-1}} \right] k_{N-1} \right).$$

Moreover, for any $A \in \mathcal{A}(\omega)$, $\varepsilon > 0$, $l \in \mathbf{N}$ and $S \in \mathcal{M}^{d \times N-1}$ we denote by $\mathcal{A}^l_{\varepsilon,S}(A)$ the set defined in formula (2.17) with $\varphi(x) = Sx$. By simplicity of notation, we set $\mathcal{A}^1_{\varepsilon,S}(A) := \mathcal{A}_{\varepsilon,S}(A)$. Finally for every $r > 0$ we set $Q_r := (0, r)^{N-1}$. The following homogenization theorem is the main result of this section. Since, up to minor changes, it can be proved as Theorem 4.1 in [1], we only sketch the proof for reader's convenience.

Theorem 3.1 *Let $\{f^{\xi}_{\varepsilon}\}_{\varepsilon,\xi}$ satisfy (3.1)-(3.3) and let* (H3) *hold. Then, (F_ε) Γ-converges with respect to the $[L^p(\omega; \mathbf{R}^d)]^M$-topology to the functional $F : [L^p(\omega; \mathbf{R}^d)]^M \to [0, +\infty]$ defined as*

$$F(u) = \begin{cases} \displaystyle\int_{\Omega} f_{hom}(\nabla u)\, dx & \text{if } u \in W^{1,p}(\omega; \mathbf{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

where $f_{hom} : \mathcal{M}^{d \times N-1} \to [0, +\infty)$ is given by the following homogenization formula

$$f_{hom}(S) := \lim_{h \to +\infty} \frac{1}{h^{N-1}} \inf \Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta,\ \beta + \xi \in (Q_h)_1 \cap \mathbf{Z}^N} f^{\xi}(\beta, D^{\xi}_1 v(\beta)), \quad v \in \mathcal{A}_{1,S}(Q_h) \Big\} \tag{3.5}$$

Proof. Let (ε_n) be a sequence of positive numbers converging to 0. Then, by Theorem 2.1, we can extract a subsequence (not relabelled) such that (F_{ε_n}) Γ-converges to a functional F defined as in (2.8) and such that, for any $u \in W^{1,p}(\omega; \mathbf{R}^d)$, $A \in \mathcal{A}(\omega)$

$$\Gamma\text{-}\lim_n F_{\varepsilon_n}(u, A) = \int_A f(x, \nabla u)\, dx.$$

The theorem is proved if we show that

(1) f does not depend on the space variable x,

(2) $f \equiv f_{hom}$.

By the periodicity assumptions one shows that

$$F(Sx, B(y,\rho)) = F(Sx, B(z,\rho))$$

for all $S \in \mathcal{M}^{d\times N-1}$, $y, z \in \omega$ and $\rho > 0$ such that $B(y,\rho) \cup B(z,\rho) \subset \omega$ and then (1) follows. Let $x_0 \in \omega$ and $r > 0$ be such that $Q_r(x_0) = (x_0, x_0 + r)^{N-1} \subseteq \omega$.

By the quasiconvexity of f and by Corollary 2.7, one has

$$\begin{aligned} f(S) &= \frac{1}{r^{N-1}} \min \left\{ \int_{Q_r(x_0)} f(\nabla u)\, dx : \quad u - Sx \in W_0^{1,p}(Q_r(x_0); \mathbf{R}^d) \right\} \\ &= \frac{1}{r^{N-1}} \min \left\{ F(u, Q_r(x_0)) : \quad u - Sx \in W_0^{1,p}(Q_r(x_0); \mathbf{R}^d) \right\} \\ &= \frac{1}{r^{N-1}} \liminf_n \left\{ F_{\varepsilon_n}(u, Q_r(x_0)) : \quad u \in \mathcal{A}_{\varepsilon_n, S}(Q_r(x_0)) \right\}. \end{aligned}$$

Eventually, through the change of variable

$$\beta = \frac{\alpha}{\varepsilon}, \quad v(\beta) = \frac{1}{\varepsilon} u(\varepsilon\beta) \tag{3.6}$$

and setting

$$T_n := \left[\frac{r}{\varepsilon_n}\right] + 1,$$

one can show that

$$f(S) = \lim_n \frac{1}{T_n^{N-1}} \inf \left\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta,\ \beta+\xi \in (Q_{T_n})_1 \cap \mathbf{Z}^N} f^\xi(\beta, D_1^\xi v(\beta)), \quad v \in \mathcal{A}_{1,S}\left(Q_{T_n}\right) \right\}.$$

Then (2) follows by showing that there exists the limit in (3.5). The proof of this fact is analogous to that of Proposition 4.2 in [1]. □

Remark 3.2 In formula (3.5) we can replace $\mathcal{A}_{1,S}(Q_h)$ by $\mathcal{A}^l_{1,S}(Q_h)$ for any fixed $l \in \mathbf{N}$, the proof being exactly the same.

Remark 3.3 The function f_{hom} in Theorem 3.1 also satisfies

$$\begin{aligned} f_{hom}(S) = \lim_{h\to+\infty} \frac{1}{h^{N-1}} \inf \Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta,\ \beta+\xi \in (Q_h)_1 \cap \mathbf{Z}^N} f^\xi \left(\beta, S\frac{\xi}{|\xi|} + D_1^\xi v(\beta)\right), \\ v \in \mathcal{A}_1^{\#}\left(Q_h\right) \Big\}, \end{aligned} \tag{3.7}$$

where $\mathcal{A}_1^{\#}(Q)$ is given by (2.20) with $\varepsilon = 1$ and $\varphi = 0$. The proof of this characterization is analogous to that of Theorem 3.1, taking into account Corollary 2.9 and the following alternative formula for a quasiconvex function f

$$f(S) = \frac{1}{r^{N-1}} \min \left\{ \int_{Q_r} f(S + \nabla\psi)\, dx : \psi \in W_{\#}^{1,p}(Q_r; \mathbf{R}^d) \right\}.$$

As a consequence of Theorem 2.6, Corollary 2.7 and Theorem 3.1 we immediately derive the following result about Γ-convergence and convergence of minimum problems for homogeneous functionals subject to Dirichlet boundary conditions.

Theorem 3.4 *For any $\varphi \in Lip\ (\mathbf{R}^{N-1})$ and $l \in \mathbf{N}$ let $F_\varepsilon^{\varphi,l}$ be defined by (2.18) and let $F^\varphi : [L^p(\omega; \mathbf{R}^d)]^M \times \mathcal{A}(\omega) \to [0, +\infty]$ be defined as*

$$F^\varphi(u, A) = \begin{cases} \displaystyle\int_A f_{hom}(\nabla u)\, dx & \text{if } u - \varphi \in W_0^{1,p}(A; \mathbf{R}^d) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

Under the hypotheses of Theorem 3.1, $F_\varepsilon^\varphi(\cdot, A)$ Γ-converges with respect to the $[L^p(\omega; \mathbf{R}^d)]^M$-topology to $F^\varphi(\cdot, A)$ for any $A \in \mathcal{A}$.

Corollary 3.5 *Under the hypotheses of Theorem 3.4, for any $\varphi \in Lip\ (\mathbf{R}^{N-1})$, $l \in \mathbf{N}$ and $A \in \mathcal{A}(\omega)$*

$$\lim_{\varepsilon \to 0} \inf\{F_\varepsilon(u, A) :\ u \in \mathcal{A}^l_{\varepsilon,\varphi}\} = \min\{F(u, A) :\ u - \varphi \in W_0^{1,p}(A; \mathbf{R}^d)\}.$$

Moreover, for any (ε_j) converging to zero as j tends to infinity, if (u_j) is a converging sequence such that

$$\lim_j F_{\varepsilon_j}(u_j, A) = \lim_j \inf\{F_{\varepsilon_j}(u, A) :\ u \in \mathcal{A}^l_{\varepsilon_j,\varphi}\},$$

then its limit is a minimizer for $\min\{F(u, A) :\ u - \varphi \in W_0^{1,p}(A; \mathbf{R}^d)\}$.

An analogous result about the convergence of minimum problems with periodic conditions follows by Theorem 2.8 and Corollary 2.9.

3.1 The convex case: a cell problem formula

In this section we will see that in the convex case the function f_{hom} can be rewritten by a single periodic minimization problem on $\mathcal{R}_\mathbf{k} \times \{0, 1, \dots, M-1\}$. Set

$$\hat{k} := \prod_{i=1}^{N-1} k_i,$$

$$I_\mathbf{k} := \prod_{i=1}^{N-1} \{0, \dots, k_i - 1\}$$

and

$$\mathcal{A}_{1,\#}(\mathcal{R}_\mathbf{k}) := \{u \in \mathcal{A}_1(\mathbf{R}^{N-1})\ :\ u^j \text{ is } \mathcal{R}_\mathbf{k} - \text{periodic},\ j \in \{0, \dots, M-1\}\}.$$

We remark that the following result is analogous to Theorem 5.1 in [1].

Theorem 3.6 *Let $(f^\xi_\varepsilon)_{\varepsilon,\xi}$ satisfies all the assumptions of Theorem 3.1 and in addition let $f^\xi_\varepsilon(\alpha,\cdot)$ be convex for all $\alpha \in \varepsilon\mathbf{Z}^N$, $\varepsilon > 0$ and $\xi \in \mathbf{Z}^N$. Then the conclusion of Theorem 3.1 holds with f_{hom} satisfying*

$$f_{hom}(S) = \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}\times I^\xi_M} f^\xi\left(\beta, S\frac{\xi_\pi}{|\xi|} + D^\xi_1 v(\beta)\right), \quad v \in \mathcal{A}^\#_1(\mathcal{R}_{\mathbf{k}})\Big\},$$

for all $S \in \mathcal{M}^{d\times N-1}$, where $I^\xi_M := \{-\xi_N \vee 0, \dots, M-1+(-\xi_N \wedge 0)\}$.

Proof. Set

$$\overline{f}(S) := \frac{1}{\hat{k}} \inf\Big\{ \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}\times I^\xi_M} f^\xi\left(\beta, S\frac{\xi_\pi}{|\xi|} + D^\xi_1 v(\beta)\right), \quad v \in \mathcal{A}^\#_1(\mathcal{R}_{\mathbf{k}})\Big\}.$$

By the characterization of f_{hom} given by (3.7) and noting that $\mathcal{A}^\#_1(\mathcal{R}_{\mathbf{k}}) \subset \mathcal{A}^\#_1(Q_{n\hat{k}})$ for all $n \in \mathbf{N}$, one can show that

$$f_{hom}(S) \leq \overline{f}(S).$$

To prove the opposite inequality for simplicity we will suppose that there exists $R > 0$ such that $f^\xi_\varepsilon = 0$ if $|\xi| > R$ and that we can replace $\mathcal{A}_{1,S}(Q_h)$ by $\mathcal{A}^{[R]}_{1,S}(Q_h)$ in the definition of f_{hom}. For the general case we refer to the proof of Theorem 5.1 in [1]. For $n \in \mathbf{N}$, let $u \in \mathcal{A}^{[R]}_{1,S}(Q_{n\hat{k}})$ and let $v \in \mathcal{A}^\#_1(Q_{n\hat{k}})$ be such that

$$v(\alpha) = u(\alpha) - S\alpha_\pi, \quad \forall \alpha \in Q_{n\hat{k}} \times \{0, 1, \dots M-1\}.$$

Moreover we set

$$I^n_{\mathbf{k}} := \prod_{i=1}^{N-1} \{0, \dots, n\prod_{j\neq i} k_j - 1\}.$$

Then, we get

$$\begin{aligned}
&\frac{1}{(n\hat{k})^{N-1}} \sum_{|\xi|\leq R} \sum_{\beta,\ \beta+\xi\in(Q_{n\hat{k}})_1} f^\xi\left(\beta, S\frac{\xi}{|\xi|} + D^\xi_1 v(\beta)\right) \\
&\quad = \frac{1}{(n\hat{k})^{N-1}} \sum_{|\xi|\leq R} \sum_{\beta\in\{0,\dots,n\hat{k}\}^{N-1}\times I^\xi_M} f^\xi\left(\beta, S\frac{\xi}{|\xi|} + D^\xi_1 v(\beta)\right) - O(\frac{1}{n}) \\
&\quad = \frac{1}{\hat{k}} \sum_{|\xi|\leq R} \sum_{\beta\in I_k\times I^\xi_M} \frac{1}{\hat{k}^{N-2}n^{N-1}} \sum_{\gamma\in I^n_{\mathbf{k}}} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + D^\xi_1 v(\beta + \sum_{i=1}^{N-1}\gamma_i k_i e_i)\right) - O(\frac{1}{n}) \\
&\quad \geq \frac{1}{\hat{k}} \sum_{|\xi|\leq R} \sum_{\beta\in I_k\times I^\xi_M} f^\xi\left(\beta, M\frac{\xi}{|\xi|} + \frac{1}{\hat{k}^{N-2}n^{N-1}} \sum_{\gamma\in I^n_{\mathbf{k}}} D^\xi_1 v(\beta + \sum_{i=1}^{N-1}\gamma_i k_i e_i)\right) - O(\frac{1}{n}),
\end{aligned}$$

where in the last inequality we have used the convexity hypothesis on f^{ξ}. Eventually, set

$$v_n(\beta) := \frac{1}{\hat{k}^{N-2} n^{N-1}} \sum_{\gamma \in I^n_{\mathbf{k}}} v(\beta + \sum_{i=1}^{N-1} \gamma_i k_i e_i).$$

It is easy to show that $v_n \in \mathcal{A}_1^{\#}(\mathcal{R}_k)$ and so, by the previous inequality, we get

$$\begin{aligned}
&\frac{1}{(n\hat{k})^{N-1}} \sum_{|\xi|\leq R} \sum_{\beta,\ \beta+\xi \in (Q_{n\hat{k}})_1} f^{\xi}\left(\beta, S\frac{\xi}{|\xi|} + D_1^{\xi} v(\beta)\right) \\
&\geq \frac{1}{\hat{k}} \sum_{|\xi|\leq R} \sum_{\beta \in I_k \times I_M^{\xi}} f^{\xi}\left(\beta, S\frac{\xi}{|\xi|} + D_1^{\xi} v_n(\beta)\right) - O(\frac{1}{n}) \\
&\geq \overline{f}(S) - O(\frac{1}{n}).
\end{aligned}$$

Passing to the inf with respect to $u \in \mathcal{A}^{[R]}_{1,S}(Q_{n\hat{k}})$ and then, letting n tend to $+\infty$, we obtain that

$$f_{hom}(S) \geq \overline{f}(S).$$

□

Remark 3.7 Note that if $\mathcal{R}_{\mathbf{k}} = (0,1)^{N-1}$, that is

$$f^{\xi}(\alpha, z) = f^{\xi}(\alpha_N, z),$$

in Theorem 3.6 we obtain

$$\begin{aligned}
&f_{hom}(S) = \sum_{j=0}^{M-1} \sum_{\{\xi_N=0\}} f^{\xi}\left(j, \frac{S\xi_\pi}{|\xi|}\right) \\
&+ \inf\{\sum_{\{\xi_N\neq 0\}} \sum_{j=-\xi_N\vee 0}^{M-1+(-\xi_N\wedge 0)} f^{\xi}\left(j, \frac{S\xi_\pi}{|\xi|} + \frac{v(j+\xi_N)-v(j)}{|\xi|}\right) :\ v : \{0,\dots,M-1\} \to \mathbf{R}^d\}.
\end{aligned}$$

3.2 Interactions along independent directions

In the section we will see how the formula defining f_{hom} can be simplified in some particular case.

On f^{ξ} we will consider the following additional hypotheses:

(I1) $f^{\xi}(\alpha, z) \not\equiv 0$ if and only if $\xi_N = 0$ or $\xi_\pi = 0$;

(I2) $f^{\xi}(\alpha, z) = f^{\xi}(\alpha_\pi, z)$ if $\xi_N = 0$;

(I3) $f^{\xi}(\alpha, z) = f^{\xi}((\alpha_N, z)$ if $\xi_\pi = 0$.

Assumption (I1) asserts that only planar or vertical interactions are allowed, while assumptions (I2) and (I3) state that the potentials accounting for planar interactions do not depend on the vertical space variable and the potentials accounting for vertical interactions do not depend on the planar space variables.

Theorem 3.8 *Let f^ξ satisfy the hypotheses of Theorem 3.1 and (I1)–(I3). Then the conclusion of Theorem 3.1 holds with f_{hom} given by*

$$f_{hom}(S) = M f^{\pi}_{hom}(S) + \inf\{\sum_{k=1}^{M-1}\sum_{j=0}^{M-1-k} f^{ke_N}(j, \frac{u(j+k)-u(j)}{k} : \ u : \{0,\dots,M-1\} \to \mathbf{R}^d\}, \tag{3.9}$$

where f^{π}_{hom} is given by (4.4) by replacing N with $N-1$.

Proof. It suffices to observe that, thanks to (I1)-(I3), in the definition of $f_{hom}(S)$ one can reduce to consider test functions of the type

$$u(\alpha) = v(\alpha_\pi) + w(\alpha_N),$$

with $v \in \mathcal{A}_1(\mathbf{R}^{N-1})$ such that $v(\alpha_\pi) = S\alpha_\pi$ if $\alpha_\pi \in \partial Q_h + [-1,1]^{N-1}$ and with $w : \{0,\dots,M-1\} \to \mathbf{R}^d$. Moreover, for such u one has

$$\sum_{\xi\in\mathbf{Z}^N}\ \sum_{\alpha,\alpha+\xi\in(Q_h)_1} f^\xi(\alpha, D^\xi_1 u(\alpha)) = M \sum_{\xi\in\mathbf{Z}^{N-1}}\ \sum_{\alpha_\pi,\alpha_\pi+\xi\in Q_h} f^\xi(\alpha_\pi, D^\xi_1 v(\alpha_\pi)) + \sum_{k=1}^{M-1}\sum_{j=0}^{M-1-k} f^{ke_N}(j, \frac{w(j+k)-w(j)}{k}.$$

Then one can easily get the conclusion taking into account (4.1). □

Remark 3.9 If $f^{ke_n}(j,z) = f^{ke_n}(z)$ and $f^{ke_n} \equiv 0$ for all $k \neq 1$, that is only homogeneous nearest neighbours interactions along the vertical direction are taken into account, then

$$f_{hom}(S) = M f^{\pi}_{hom}(S) + \inf_{z\in\mathbf{R}^d} f^{e_N}(z).$$

4 Layer dependence: asymptotic formulas

In this section we are interested in analyzing the asymptotic behavior of the function f_{hom} given by formula (3.5) when the number of layers M tends to infinity. To this aim, we will assume that the energy densities f^ξ_ε satisfy the hypotheses of Theorem 3.1 and, given $k_N \in \mathbf{N}$, the additional condition

$$f^\xi((\alpha_\pi,\cdot),z) \quad \text{is } (0,k_N)\text{-periodic } \forall z\in\mathbf{R}^d, \tag{4.1}$$

that is f^ξ_ε fulfill all the hypotheses of Theorem 4.1 in [1] whose statement is written in the following subsection to fix some notation.

4.1 Homogenization result for thick domains

Let $\mathbf{k} = (k_1, \dots, k_N) \in \mathbf{Z}^N$ be given and set

$$\mathcal{P}_{\mathbf{k}} := (0, k_1) \times \cdots \times (0, k_N).$$

For any $\xi \in \mathbf{Z}^N$, let $f^\xi : \mathbf{Z}^N \times \mathbf{R}^d \to [0, +\infty)$ be such that $f^\xi(\cdot, z)$ is $\mathcal{P}_{\mathbf{k}}$-periodic for any $z \in \mathbf{R}^d$ and satisfies hypotheses (3.2), (3.3) and (H3) of the previous section. Given a convex bounded open set $\Omega \subset \mathbf{R}^N$, consider the family of functionals $F_\varepsilon : L^p(\Omega; \mathbf{R}^d) \to [0, +\infty]$ defined as

$$F^b_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi \in \mathbf{Z}^N} \sum_{\alpha, \alpha + \varepsilon\xi \in \Omega} \varepsilon^N f^\xi\left(\tfrac{\alpha}{\varepsilon}, D^\xi_\varepsilon u(\alpha)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

The following Theorem has been proved in [1].

Theorem 4.1 *(F^b_ε) Γ-converges with respect to the $L^p(\Omega; \mathbf{R}^d)$-topology to the functional $F^b : L^p(\Omega; \mathbf{R}^d) \to [0, +\infty]$ defined as*

$$F^b(u) = \begin{cases} \displaystyle\int_\Omega f^b_{hom}(\nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbf{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.3}$$

where $f^b_{hom} : \mathcal{M}^{d\times N} \to [0, +\infty)$ is given by the following homogenization formula

$$f^b_{hom}(S) := \lim_{h \to +\infty} \frac{1}{h^N} \min\Big\{ \sum_{\xi \in \mathbf{Z}^N} \sum_{\beta, \beta + \xi \in Q_h} f^\xi(\beta, D^\xi_1 v(\beta)), \quad v \in \mathcal{A}^S(Q_h) \Big\}, \tag{4.4}$$

with $h \in \mathbf{N}$, $Q_h = (0, h)^N$ and

$$\mathcal{A}^S(Q_h) := \{u \in \mathcal{A}_1(\mathbf{R}^N) : \ u(\alpha) = S\alpha \text{ if } \alpha \in \partial Q_h + [-1, 1]^N\}.$$

In the following, given $A \in \mathcal{A}(\mathbf{R}^N)$, we will use the notation $F^b_\varepsilon(u, A)$ to denote the energy in (2.1) with Ω replaced by A.

4.2 Discrete and continuous models for dimension reduction

From now on we will denote F_ε by F^M_ε and f_{hom} by f^M_{hom} thus highlighting the dependence on M in all the formulas we obtained in the previous sections for the energy densities.

Given $\delta > 0$, suppose that in the definition (2.1) we replace M by $M_\varepsilon = \frac{\delta}{\varepsilon}$. Then $\frac{1}{M_\varepsilon} F^{M_\varepsilon}_\varepsilon(u) = F^b_\varepsilon(u)$ where F^b_ε is given by (4.2) with $\Omega = \omega \times (0, \delta)$. By Theorem 4.1 we have that, for all $u \in W^{1,p}(\omega \times (0, \delta); \mathbf{R}^d)$,

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \frac{1}{M_\varepsilon} F^{M_\varepsilon}_\varepsilon(u) = \frac{1}{\delta} \int_{\omega \times (0,\delta)} f^b_{hom}(\nabla u(x))\, dx =: F_\delta(u).$$

In the pioneering paper by Le Dret and Raoult [22] it has been proved that F_δ Γ-converges, as δ tends to zero, to the functional

$$\int_\omega Q\overline{f}_{hom}(\nabla u(x))\,dx \quad u\in\ W^{1,p}(\omega;\mathbf{R}^d),$$

where

$$\overline{f}_{hom}(S) := \inf\{f^b_{hom}(S|z),\ z\in\mathbf{R}^d\} \tag{4.5}$$

and $Q\overline{f}_{hom}$ denotes the quasi-convex envelope of $\overline{f}_{hom}$. This considerations lead us to ask ourselves if

$$\lim_{M\to+\infty}\frac{1}{M}f^M_{hom}(S) = Q\overline{f}_{hom}(S). \tag{4.6}$$

Here we provide a partial answer to this problem by showing that (4.6) holds under some additional assumptions on the energy densities f^ξ_ε, while in the general case we are only able to prove the following inequality.

Proposition 4.2 *Under the hypotheses of Theorem 4.1, for all $S\in\mathbf{R}^{d\times(N-1)}$ there holds*

$$\lim_{M\to+\infty}\frac{1}{M}f^M_{hom}(S)\le Q\overline{f}_{hom}(S),$$

where $\overline{f}_{hom}$ is given by (4.5).

Proof. One can write $Q\overline{f}_{hom}$ as (see [9])

$$Q\overline{f}_{hom}(S) = \inf_{t>0}\inf\Big\{t\int_{Q'\times(0,\frac{1}{t})} f^b_{hom}(\nabla u(x))\,dx,\ u\in W^{1,p}(Q'\times(0,\frac{1}{t});\mathbf{R}^d), \tag{4.7}$$

$$u(x_\pi,x_N) = Sx_\pi\ if x_\pi\in\partial Q'\Big\},$$

where $Q'=(0,1)^{N-1}$. Thus, by Theorem 4.1 and by the convergence of minimum problems proved in [1], we get

$$Q\overline{f}_{hom}(S) = \inf_{t>0}\lim_{\varepsilon\to0}\inf\Big\{tF^b_\varepsilon(u,Q'\times(0,\frac{1}{t})),\ u\in\mathcal{A}_{\varepsilon,S}(Q')\Big\}.$$

On the other hand, by Corollary 2.7,

$$f^M_{hom}(S) = \lim_{\varepsilon\to0}\inf\Big\{F^M_\varepsilon(u,Q'),\ u\in\mathcal{A}_{\varepsilon,S}(Q')\Big\}.$$

With fixed $t>0$ and $M\in\mathbf{N}$, given $u_\varepsilon\in\mathcal{A}_{\varepsilon,S}(Q')$ we set $v^i_\varepsilon(\alpha)=u_\varepsilon(\alpha_\pi,\alpha_N+iM\varepsilon)$ for $i\in\{0,1,\dots,[\frac{1}{tM\varepsilon}]-1\}$. Since $v^i_\varepsilon\in\mathcal{A}_{\varepsilon,S}(Q')$, it holds

$$\begin{aligned} tF^b_\varepsilon(u_\varepsilon,Q'\times(0,\frac{1}{t})) &\ge t\varepsilon\sum_{i=0}^{[\frac{1}{tM\varepsilon}]-1}F^M_\varepsilon(v^i_\varepsilon,Q')\\ &\ge \varepsilon t([\frac{1}{tM\varepsilon}]-1)\inf\Big\{F^M_\varepsilon(u,Q'),\ u\in\mathcal{A}_{\varepsilon,S}(Q')\Big\}\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0}\inf\{tF^b_\varepsilon(u_\varepsilon,Q'\times(0,\frac{1}{t}))\ u\in\mathcal{A}_{\varepsilon,S}(Q')\}\geq\frac{1}{M}f_{hom}(S)$$

and the conclusion holds by letting M go to infinity and passing to the infimum on t. □

In the following two propositions we provide two particular cases in which (4.6) holds.

Proposition 4.3 *Under the hypotheses of Theorem 3.8, if $f^{ke_n}(j,z)=f^{ke_n}(z)$, then*

$$\lim_{M\to+\infty}\frac{1}{M}f^M_{hom}(S)=f^\pi_{hom}(S)+\inf_{z\in\mathbf{R}^d}\tilde{f}(z),$$

where $\tilde{f}$ is given by formula (4.4) when $N=1$ that is

$$\tilde{f}(z):=\lim_{M\to\infty}\frac{1}{M}\inf\Big\{\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}f^{ke_N}\big(\frac{u(i+k)-u(i)}{k}\big),\ u(0)=0\ \ u(M-1)=(M-1)z\Big\}$$

Remark 4.4 Note that, under the hypotheses of Proposition 4.3,

$$f^b_{hom}(S|z)=f^\pi_{hom}(S)+\tilde{f}(z).$$

Since f^π_{hom} is quasiconvex, we get that

$$Q\overline{f}_{hom}(S)=f^\pi_{hom}(S)+\inf_{z\in\mathbf{R}^d}\tilde{f}(z),$$

where $\overline{f}_{hom}$ is given by (4.5).

Proof of Proposition 4.3. Set

$$F^M(u):=\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}f^{ke_N}\big(\frac{u(i+k)-u(i)}{k}\big)$$

and $c_M:=\inf\{F^M(u):\ u:\{0,\dots,M-1\}\to\mathbf{R}^d\}$. By (3.9)

$$\frac{f^M_{hom}(S)}{M}=f^\pi_{hom}(S)+\frac{c_M}{M}.$$

By Proposition 4.2 and Remark 4.4 it suffices to prove that

$$\lim_{M\to+\infty}\frac{c_M}{M}\geq\inf_z\tilde{f}(z).$$

Set

$$\mathcal{F}^M(u)=\frac{1}{M-1}\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}f^{ke_N}\Big(\frac{u(\frac{1}{M-1}(i+k))-u(\frac{i}{M-1})}{\frac{k}{M-1}}\Big).$$

By Theorem 4.1 with $N = 1$ we have

$$\mathcal{F}^M(u) \overset{\Gamma(L^p)}{\longrightarrow} \int_0^1 \tilde{f}(u'(t))\, dt. \tag{4.8}$$

By a change of variables it holds that

$$\frac{c_M}{M-1} = \inf\{\mathcal{F}^M(u) : \ u(0) = 0 \ u : \frac{1}{M-1}\mathbf{Z} \cap [0,1] \to \mathbf{R}^d\},$$

then there exist $C > 0$ such that

$$\frac{c_M}{M-1} \leq F^M(0) \leq C. \tag{4.9}$$

For $\delta > 0$, let u_M be such that $u_M(0) = 0$ and

$$\frac{c_M}{M-1} + \delta \quad \geq \quad \mathcal{F}^M(u_M) \tag{4.10}$$

Note that, thanks to growth conditions f^ξ and by convexity, it holds

$$\begin{aligned} \mathcal{F}^M(u_M) \quad &\geq \quad \frac{1}{M-1} \sum_{i=0}^{M-1-\frac{1}{M-1}} \Big| \frac{u_M^{i+\frac{1}{M-1}} - u_M^i}{\frac{1}{M-1}} \Big|^p - \frac{C}{M-1} \\ &\geq \quad C|u_M(1) - u_M(0)|^p - \frac{C}{M-1}. \end{aligned}$$

Set $z_M = |u_M(1) - u_M(0)|$, by (4.9) and the previous chain of inequalities, we have that, up to subsequences, $z_M \to \overline{z}$ and $u_M \to \overline{u}$ with $\overline{u} \in W^{1,p}((0,1); \mathbf{R}^d)$ and $u(0) = 0$, $u(1) = \overline{z}$. Then, by (4.8) and the convexity of $\tilde{f}$ we get

$$\liminf_M \mathcal{F}_M(u_M) \geq \int_0^1 \tilde{f}(u'(t))\, dt \geq \tilde{f}(\overline{z}) \geq \inf_z \tilde{f}(z).$$

By the previous inequality and (4.10) we have that

$$\lim_M \frac{c_M}{M} \geq \liminf_M \mathcal{F}_M(u_M) - \delta \geq \inf_z \tilde{f}(z) - \delta.$$

This estimate proves the claim by the arbitrariness of δ. □

Remark 4.5 By [14] we have that, if $f^{ke_n} \equiv 0$ for all $k \neq 1, 2$, then the homogenization formula defining $\tilde{f}$ reduces to a finite formula

$$\tilde{f}(z) := f^{2e_N}(z) + \frac{1}{2} \inf\{f^{e_N}(z_1) + f^{e_N}(z_2) : \ z_1 + z_2 = 2z\}.$$

Proposition 4.6 (Convex case) *Under the hypotheses of Theorem 4.1 and Theorem 3.6,*

$$\lim_{M\to+\infty} \frac{1}{M} f^M_{hom}(S) = \overline{f}_{hom}(S),$$

where $\overline{f}_{hom}$ is given by (4.5).

Remark 4.7 *Note that, under the hypotheses of Proposition 4.6, since f^b_{hom} is convex, one can easily show that also $\overline{f}_{hom}$ is convex and then*

$$Q\overline{f}_{hom}(S) = \overline{f}_{hom}(S).$$

Proof of Proposition 4.6. Given $\delta > 0$, let $v_M \in \mathcal{A}_{1,\#}(\mathcal{R}_{\mathbf{k}})$ be such that

$$\frac{1}{\hat{k}} \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in I_{\mathbf{k}}\times I^\xi_M} f^\xi\left(\beta, S\frac{\xi_\pi}{|\xi|} + D^\xi_1 v_M(\beta)\right) \leq f^M_{hom}(S) + \delta$$

Let $u_M(\alpha) = \frac{1}{M-1} v_M((M-1)\alpha)$. Then $u_M \in \mathcal{A}_{\frac{1}{M-1},\#}(\mathcal{R}_{\frac{\mathbf{k}}{\mathbf{M-1}}})$ and

$$f^M_{hom}(S) \geq \frac{1}{\hat{k}} \sum_{\xi\in\mathbf{Z}^N} \sum_{\beta\in\frac{1}{M-1}(I_{\mathbf{k}}\times I^\xi_M)} f^\xi\left(\beta, S\frac{\xi_\pi}{|\xi|} + D^\xi_1 v_M(\beta)\right) - \delta$$

Let us denote by $u^\#_M$ the periodic extension of u_M on Q', then

$$\frac{1}{M-1} f^M_{hom}(S) \quad \geq \quad F^b_{\frac{1}{M-1}}(u_M(\alpha) + (S|0)\alpha, Q'\times(0,1)) - \delta. \qquad (4.11)$$

By the coerciveness assumption, up to additive constants and subsequences, $u_M \to u$ with $u \in W^{1,p}_\#(Q';\mathbf{R}^d)$. By Γ-convergence and convexity it holds

$$\begin{aligned}
\liminf_M F^b_{\frac{1}{M-1}}(u_M(\alpha) + (S|0)\alpha, Q'\times(0,1)) \quad &\geq \quad \int_{Q'\times(0,1)} f^b_{hom}((\nabla_\pi u|\partial_N u) + (S|0))\,dx \\
&\geq \quad \int_0^1 dx_N \int_{Q'} \overline{f}_{hom}(\nabla_\pi u + S)\ dx_\pi \\
&\geq \quad \overline{f}_{hom}(S).
\end{aligned}$$

By (4.11) and the previous inequality, thanks to the arbitrariness of δ we get that

$$\lim_M \frac{1}{M} f^M_{hom}(S) \geq \overline{f}_{hom}(S).$$

$\square$

Remark 4.8 The proof of Theorem 4.6 is based on the fact that, in the convex case, f^M_{hom} has a non-asymptotic formula. Then the conclusion of Theorem 4.6 holds whenever f^M_{hom} enjoys this property.

Proposition 4.9 (Symmetric case) *Let f_ε^ξ satisfy the hypotheses of Theorem 4.1 and*

$$\begin{array}{rcl} f^\xi(\alpha,z) &=& f^\xi(\alpha_\pi,z), \quad \forall \xi\in\mathbf{Z}^N \\ f^{(\xi_\pi,\xi_N)} &\equiv& f^{(\xi_\pi,-\xi_N)}. \end{array} \tag{4.12}$$

Then

$$\lim_{M\to+\infty}\frac{1}{M}f^M_{hom}(S)=\overline{f}_{hom}(S),$$

where $\overline{f}_{hom}$ is given by (4.5).

Remark 4.10 Note that if (4.12) holds, then, for all $S\in\mathbf{R}^{d\times(N-1)}$, $f^b_{hom}(S|y)=f^b_{hom}(S|-y)$ and, since f^b_{hom} is quasi-convex, then $Q\overline{f}_{hom}(S)=f^b_{hom}(S|0)$.

Proof. We suppose that there exists $R\in\mathbf{N}$ such that

$$f^\xi\equiv 0,\quad \text{if } |\xi_N|>R,$$

the general case being dealt with similarly up to minor changes. Let $S\in\mathbf{R}^{d\times(N-1)}$. By Corollary 3.5, there exists $u_\varepsilon\in\mathcal{A}_{\varepsilon,S}(Q')$ such that

$$\lim_\varepsilon F^M_\varepsilon(u_\varepsilon,Q')=f^M_{hom}(S).$$

Moreover, by the growth assumptions on f^ξ we get

$$F^M_\varepsilon(u_\varepsilon,Q')\le C(1+\|S\|^p)M.$$

Set

$$G_\varepsilon(u,[i,j]):=\sum_{\xi\in\mathbf{Z}^N}C^\xi\sum_{\alpha,\alpha+\varepsilon\xi\in Q'\times[i\varepsilon,j\varepsilon]}\varepsilon^{N-1}|D^\xi_\varepsilon u(\alpha)|^p.$$

By the growth hypotheses on f^ξ and by Lemma 2.2, we have

$$\sup_\varepsilon G_\varepsilon(u_\varepsilon,[0,M-1])\le C\|S\|^pM.$$

With fixed $L\in\mathbf{N}$, there exist $i^1_\varepsilon,i^2_\varepsilon\in\{0,\dots,L-1\}$ such that

$$G_\varepsilon(u_\varepsilon,[i^1_\varepsilon R,(i^1_\varepsilon+1)R])\le\frac{1}{L}\sum_{i=0}^{L-1}G_\varepsilon(u_\varepsilon,[iR,(i+1)R])\le\frac{C}{L}\|S\|^pM,$$

$$\begin{array}{rl} & G_\varepsilon(u_\varepsilon,[\,M-1-(i^2_\varepsilon+1)R,M-1-i^2_\varepsilon R]) \\ \le & \displaystyle\frac{1}{L}\sum_{i=0}^{L-1}G_\varepsilon(u_\varepsilon,[M-1-(i+1)R,M-1-iR])\le\frac{C}{L}\|S\|^pM. \end{array}$$

Set

$$M_\varepsilon = M - 1 - (i^2_\varepsilon + i^1_\varepsilon)R$$

and

$$v_\varepsilon(\alpha_\pi, \alpha_N) = u_\varepsilon(\alpha_\pi, \alpha_N + i^1_\varepsilon R\varepsilon) \quad 0 \le \alpha_N \le M_\varepsilon \varepsilon.$$

Since

$$M - 1 - 2(L-1)R \le M_\varepsilon \le M - 1,$$

up to subsequences, we may suppose that $M_\varepsilon \to \tilde{M}$ with $M - 1 - 2(L-1)R \le \tilde{M} \le M - 1$,

$$F^{M_\varepsilon}_\varepsilon(v_\varepsilon, Q') \le F^M_\varepsilon(u_\varepsilon, Q'),$$

$$G_\varepsilon(v_\varepsilon, [0, R] \vee G_\varepsilon(v_\varepsilon, [M_\varepsilon - R, M_\varepsilon] \le \frac{C}{L} \|S\|^p M.$$

Set

$$\tilde{v}_\varepsilon(\alpha_\pi, \alpha_N) = v_\varepsilon(\alpha_\pi, \varepsilon M_\varepsilon - \alpha_N),$$

we note that by (4.12), we get

$$F^{M_\varepsilon}_\varepsilon(\tilde{v}_\varepsilon, Q') = F^{M_\varepsilon}_\varepsilon(v_\varepsilon, Q').$$

Let us define $w_\varepsilon \in \mathcal{A}_\varepsilon(Q' \times \mathbf{R})$ as

$$w_e(\alpha) = \begin{cases} v_\varepsilon(\alpha_\pi, \alpha_N - jM_\varepsilon \varepsilon) & \text{if } jM_\varepsilon\varepsilon \le \alpha_N \le (j+1)M_\varepsilon\varepsilon,\ j \text{ even} \\ \tilde{v}_\varepsilon(\alpha_\pi, \alpha_N - jM_\varepsilon \varepsilon) & \text{if } jM_\varepsilon\varepsilon \le \alpha_N \le (j+1)M_\varepsilon\varepsilon,\ j \text{ odd,} \end{cases}$$

then, by (4.12)

$$F^b_\varepsilon(w_\varepsilon, Q' \times (0,1)) \le \varepsilon \left(\left[\frac{1}{M_\varepsilon \varepsilon} \right] + 1 \right) F^{M_\varepsilon}_\varepsilon(v_\varepsilon, Q') + \varepsilon \sum_{j=1}^{[\frac{1}{M_\varepsilon\varepsilon}]} B^j_\varepsilon(w_\varepsilon),$$

where

$$B^j_\varepsilon(w_\varepsilon) = \sum_{|\xi_N| > 1} \sum_{\alpha \in \mathcal{B}^{j,\xi}_\varepsilon} \varepsilon^{N-1} f^\xi(\frac{\alpha_\pi}{\varepsilon}, D^\xi_\varepsilon w_\varepsilon(\alpha)),$$

with

$$\mathcal{B}^{j,\xi}_\varepsilon = \{\alpha \in \varepsilon\mathbf{Z}^N :\ \alpha_\pi, \alpha_\pi + \varepsilon\xi \in Q', jM_\varepsilon - \varepsilon(\xi_N \vee 0) < \alpha_N < jM_\varepsilon - \varepsilon(\xi_N \wedge 0)\}.$$

Note that for $\alpha \in \mathcal{B}^{j,\xi}_\varepsilon$, we have that

$$D^\xi_\varepsilon w_\varepsilon(\alpha) = \frac{\|\hat{\xi}\|}{\|\xi\|} D^{\hat{\xi}}_\varepsilon v_\varepsilon(\alpha_\pi, \alpha_N - (j-1)M_\varepsilon\varepsilon),$$

where $\hat{\xi} = (\xi_\pi, 2(jM_\varepsilon - \frac{\alpha_N}{\varepsilon}) - \xi_N)$. Since $\|\hat{\xi}\| \le \|\xi\|$ we get

$$|D^\xi_\varepsilon w_\varepsilon(\alpha)|^p \le |D^{\hat{\xi}}_\varepsilon v_\varepsilon(\alpha_\pi, \alpha_N - (j-1)M_\varepsilon\varepsilon)|^p$$

and thus, by the growth hypotheses on f^ξ,

$$B^j_\varepsilon(w_\varepsilon) \leq C(1 + G_\varepsilon(v_\varepsilon, [0,R] \vee G_\varepsilon(v_\varepsilon, [M_\varepsilon - R, M_\varepsilon]) \leq C(1 + \frac{1}{L}\|S\|^p M).$$

Hence

$$F^b_\varepsilon(w_\varepsilon, Q' \times (0,1)) \leq \varepsilon\left(\left[\frac{1}{M_\varepsilon \varepsilon}\right] + 1\right) F^{M_\varepsilon}_\varepsilon(v_\varepsilon, Q') + \varepsilon\left[\frac{1}{M_\varepsilon \varepsilon}\right] C(1 + \frac{1}{L}\|S\|^p M).$$

By the construction of w_ε one can easily show that, up to subsequences (not relabelled), there exists $w \in W^{1,p}(Q';\mathbf{R}^d)$ with $w(x) = Sx$ on $\partial Q'$ such that $w_\varepsilon \to w$ in $L^p(Q' \times (0,1);\mathbf{R}^d)$. Then, by Theorem 4.1, from the previous estimate we get that

$$\begin{aligned} Q\overline{f}_{hom}(S) &\leq \int_{Q'\times(0,1)} f^b_{hom}(\nabla w|0)\,dx \leq \liminf_\varepsilon F^b_\varepsilon(w_\varepsilon, Q' \times (0,1)) \\ &\leq \frac{M}{\tilde{M}}(\frac{1}{M} f^M_{hom}(S) + \frac{C}{L}\|S\|^p). \end{aligned}$$

Letting first M and then L tend to infinity, we get

$$Q\overline{f}_{hom}(S) \leq \lim_M \frac{f^M_{hom}(S)}{M}.$$

Then the thesis follows by Proposition 4.2. □

5 Examples

In this section we will consider two special cases of nearest and next to nearest neighbours interactions in a $2D-1D$ dimension reduction problem in which we show that $\frac{f^M_{hom}}{M}$ has a non-asymptotic formula for $M = 2$ and for $M \to +\infty$. As highlighted in Figure 3, were the geometry of these discrete systems is shown, both the examples can be seen as particular cases of the most simple geometry in $2D$ which takes into account nearest and next to nearest interactions and which, at the same time, is invariant under discrete rotations compatible with the lattice structure (see [21]).

Example 5.1 In the following example we will consider energies defined for $u \in \mathcal{A}_\varepsilon((0,1) \times [0, \varepsilon(M-1)])$ as

$$\begin{aligned} F^M_\varepsilon(u) = & \sum_{\alpha,\alpha+\varepsilon e_1 \in (0,1)_\varepsilon} \varepsilon f_1(D^{e_1}_\varepsilon u(\alpha)) + \sum_{\alpha,\alpha+\varepsilon(e_1+e_2) \in (0,1)_\varepsilon} \varepsilon f_2(D^{e_1+e_2}_\varepsilon u(\alpha)) \\ & + \sum_{\alpha,\alpha+\varepsilon(e_1-e_2) \in (0,1)_\varepsilon} \varepsilon f_3(D^{e_1-e_2}_\varepsilon u(\alpha)), \end{aligned} \tag{5.1}$$

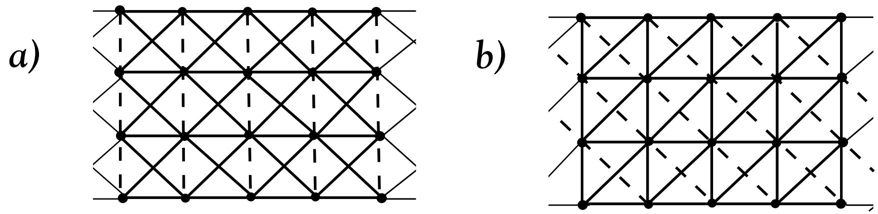

Figure 3: Reference configurations of the discrete system in Example 5.1 and 5.5 (the dashed lines represent negligible interactions).

with $f_i : \mathbf{R}^d \to [0, +\infty)$ satisfying

$$c(|z|^p - 1) \le f_i(z) \le C(|z|^p + 1), \quad i = 1, 2, \qquad f_3(z) \le C(|z|^p + 1).$$

Note that in this example interaction along direction e_2 are not taken into account (see Figure 3 a)). Thus, hypotheses of Section 3 are not satisfied. Anyway, thanks to the additional coerciveness condition on f_2, it is easy to see that again finite difference quotients along e_2 can be controlled and so the conclusion of Theorem (3.1) still holds true.

Proposition 5.2 *Let F^M_ε be defined by (5.1) and let f^M_{hom} be the density energy of its Γ-limit. Then it holds*

$$2\Big((M-2)(\widetilde{f}\,)^{**}(z) + (\widehat{f}\,)^{**}(z)\Big) \le f^M_{hom}(z) \le 2\Big((M-2)\widetilde{f} + \widehat{f}\,\Big)^{**}(z) \quad (5.2)$$

for any $z \in \mathbf{R}^d$, where $\widetilde{f} : \mathbf{R}^d \to [0, +\infty)$ and $\widehat{f} : \mathbf{R}^d \to [0, +\infty)$ are defined by the following formulas

$$\widetilde{f}(z) := \frac{1}{4}\inf\{f_1(z_1) + f_1(z_2) : z_1 + z_2 = 2z\} + \frac{1}{2}\inf\left\{f_2\Big(\frac{z_1}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_1 + z_2 = 2z\right\},$$
$$\widehat{f}(z) := \frac{1}{2}\inf\{f_1(z_1) + f_1(z_2) : z_1 + z_2 = 2z\} + \frac{1}{2}\inf\left\{f_2\Big(\frac{z_1}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_1 + z_2 = 2z\right\}. \quad (5.3)$$

Proof. We first prove the lower bound estimate. For $\beta \in (0, 1) \cap \varepsilon\mathbf{Z}$ set

$$D^1_\varepsilon u^j(\beta) = D^{e_1}_\varepsilon u(\beta, \varepsilon j), \quad D^2_\varepsilon u^j(\beta) = D^{e_1+e_2}_\varepsilon u(\beta, \varepsilon j), \quad D^3_\varepsilon u^j(\beta) = D^{e_1-e_2}_\varepsilon u(\beta, \varepsilon j).$$

$$F^M_\varepsilon(u) \quad \ge \quad \sum_{j=1}^{M-2} \sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}} \frac{\varepsilon}{4}\big(f_1(D^1_\varepsilon u^j(\beta)) + f_1(D^1_\varepsilon u^j(\beta + \varepsilon))\big)$$

$$
\begin{aligned}
&+\frac{\varepsilon}{2}\left(f_2(D^2_\varepsilon u^j(\beta))+f_3(D^3_\varepsilon u^{j+1}(\beta+\varepsilon))\right)\\
+&\sum_{j=1}^{M-2}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}\frac{\varepsilon}{4}\left(f_1(D^1_\varepsilon u^j(\beta))+f_1(D^1_\varepsilon u^j(\beta+\varepsilon))\right)\\
&+\frac{\varepsilon}{2}\left(f_2(D^2_\varepsilon u^{j-1}(\beta+\varepsilon))+f_3(D^3_\varepsilon u^j(\beta)\right)\\
+&\sum_{j=1}^{M-2}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}\frac{\varepsilon}{2}\left(f_1(D^1_\varepsilon u^0(\beta))+f_1(D^1_\varepsilon u^j(\beta+\varepsilon))\right)\\
&+\frac{\varepsilon}{2}\left(f_2(D^2_\varepsilon u^0(\beta))+f_3(D^3_\varepsilon u^1(\beta+\varepsilon))\right)\\
+&\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}\frac{\varepsilon}{2}\left(f_1(D^1_\varepsilon u^{M-1}(\beta))+f_1(D^1_\varepsilon u^{M-1}(\beta+\varepsilon))\right)\\
&+\frac{\varepsilon}{2}\left(f_2(D^2_\varepsilon u^{M-2}(\beta+\varepsilon))+f_3(D^3_\varepsilon u^{M-1}(\beta)\right).
\end{aligned}
$$

Then by (5.3) we get

$$
\begin{aligned}
F^M_\varepsilon(u) &\geq 2\sum_{j=1}^{M-2}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}\varepsilon\widetilde{f}(D^1_{2\varepsilon}u^j(\beta))+\sum_{j=0,M-1}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}\varepsilon\widehat{f}(D^1_{2\varepsilon}u^j(\beta))\\
&= \sum_{j=1}^{M-2}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}2\varepsilon\widetilde{f}(D^1_{2\varepsilon}u^j(\beta))+\frac{1}{2}\sum_{j=0,M-1}\sum_{\frac{\beta}{\varepsilon}\ \text{even/odd}}2\varepsilon\widehat{f}(D^1_{2\varepsilon}u^j(\beta)).
\end{aligned}
$$

This estimate provides a lower bound of our energies F^M_ε by one-dimensional discrete energies involving only nearest neighbours interactions on a lattice of spacing 2ε. Then by the Γ-convergence result in [14] we get that, if $u_\varepsilon \to zt$ in $(L^p(0,1))^M$, then

$$
\liminf_{\varepsilon} F^M_\varepsilon(u_\varepsilon) \geq 2((M-2)\widetilde{f}(z)+\widehat{f}(z)).
$$

We now prove the upper bound estimate. For the sake of simplicity we consider the case f_i are lower semicontinuous functions. Fix $z \in \mathbf{R}$ such that

$$
2\left((M-2)\widetilde{f}(z)+\widehat{f}(z)\right) = 2\left((M-2)\widetilde{f}+\widehat{f}\right)^{**}(z)
$$

and set

$$
\begin{aligned}
z_1, z_2 \in \mathbf{R} \ &: \ \inf\{f_1(z_1)+f_1(z_2) : z_1+z_2=2z\} = f_1(z_1)+f_1(z_2)\\
z_1', z_2' \in \mathbf{R} \ &: \ \inf\left\{f_2\Big(\frac{z_1}{\sqrt{2}}\Big)+f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_1+z_2=2z\right\} = f_2\Big(\frac{z_1'}{\sqrt{2}}\Big)+f_3\Big(\frac{z_2'}{\sqrt{2}}\Big).
\end{aligned}
$$

Let $u_\varepsilon \in \mathcal{A}_\varepsilon((0,1)_\varepsilon) \to \mathbf{R}^d$ be such that

$$
u^j_\varepsilon(\beta) = \begin{cases} ((\beta-1)z+z_1+\frac{j}{2}(z_1'-z_2'))\varepsilon & \text{if } \frac{\beta}{\varepsilon} \text{ is odd}\\ (\beta z+\frac{j}{2}(z_1'-z_2'))\varepsilon & \text{if } \frac{\beta}{\varepsilon} \text{ is even,}\end{cases} \tag{5.4}
$$

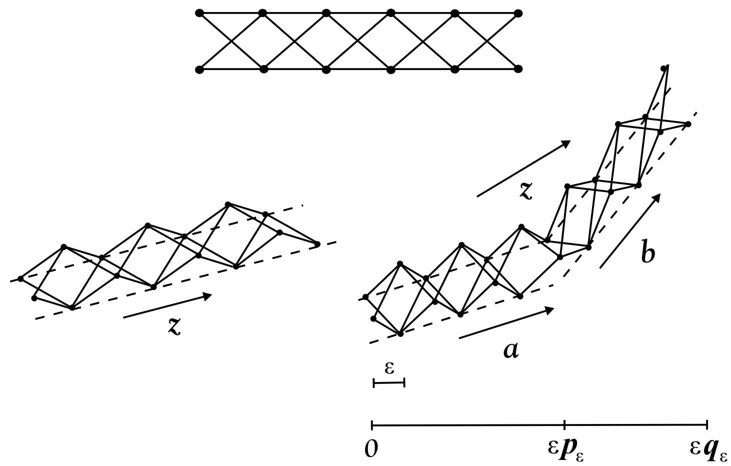

Figure 4: Reference configuration and optimizing sequences of the discrete system in Proposition 5.2.

(see Figure 4). Then $u_\varepsilon \to zt$ in $(L^p(0,1))^M$ and

$$\begin{aligned} F_\varepsilon^M(u_\varepsilon) &= 2(M-2)\left(\frac{1}{4}(f_1(z_1)+f_1(z_2))+\frac{1}{2}f_2\Big(\frac{z_1'}{\sqrt{2}}\Big)+f_3\Big(\frac{z_2'}{\sqrt{2}}\Big)\right)\\ &+ 2\left(\frac{1}{2}(f_1(z_1)+f_1(z_2))+\frac{1}{2}f_2\Big(\frac{z_1'}{\sqrt{2}}\Big)+f_3\Big(\frac{z_2'}{\sqrt{2}}\Big)\right)+O(1)\\ &= 2(M-2)\widetilde{f}(z)+2\widehat{f}(z)+O(1). \end{aligned}$$

Hence we get that

$$f_{hom}^M(z) \le \limsup_{\varepsilon} F_\varepsilon^M(u_\varepsilon) = 2\left((M-2)\widetilde{f}(z)+\widehat{f}(z)\right).$$

If otherwise z is such that

$$2\left((M-2)\widetilde{f}(z)+\widehat{f}(z)\right) > 2\left((M-2)\widetilde{f}+\widehat{f}\,\right)^{**}(z)$$

there exist $a,\ b \in \mathbf{R}$ and $\lambda \in (0,1)$ such that

$$z = \lambda a + (1-\lambda)b$$

and that

$$\begin{aligned} 2\left((M-2)\widetilde{f}(a)+\widehat{f}(a)\right) &= 2\left((M-2)\widetilde{f}+\widehat{f}\,\right)^{**}(a)\\ 2\left((M-2)\widetilde{f}(b)+\widehat{f}(b)\right) &= 2\left((M-2)\widetilde{f}+\widehat{f}\,\right)^{**}(b). \end{aligned}$$

Let $\lambda_\varepsilon \in \mathbf{Q}\cap(0,1)$ be such that $\lambda_\varepsilon \to \lambda$ and let $p_\varepsilon,\ q_\varepsilon \in \mathbf{N}$ be such that

$$\lambda_\varepsilon = \frac{p_\varepsilon}{q_\varepsilon}, \quad \lim_\varepsilon \varepsilon q_\varepsilon = 0.$$

Let $z_1^a, z_2^a, z_1'^a, z_2'^a, z_1^b, z_2^b, z_1'^b, z_2'^b \in \mathbf{R}$ be such that

$$\begin{array}{ll} \inf & \{f_1(z_1)+f_1(z_2) : z_1+z_2=2a\} = f_1(z_1^a)+f_1(z_2^a) \\ \inf & \left\{ f_2\Big(\frac{z_1}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_1+z_2=2a \right\} = f_2\Big(\frac{z_1'^a}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2'^a}{\sqrt{2}}\Big) \\ \inf & \{f_1(z_1)+f_1(z_2) : z_1+z_2=2b\} = f_1(z_1^b)+f_1(z_2^b) \\ \inf & \left\{ f_2\Big(\frac{z_1}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_1+z_2=2b \right\} = f_2\Big(\frac{z_1'^b}{\sqrt{2}}\Big) + f_3\Big(\frac{z_2'^b}{\sqrt{2}}\Big). \end{array}$$

Let u_ε^a and u_ε^b be defined as in (5.4) with z_1, z_1', z_2' replaced respectively by $z_1^a, z_1'^a, z_2'^a$ and by $z_1^b, z_1'^b, z_2'^b$. Let $u_\varepsilon \in \mathcal{A}_\varepsilon(0,1)$ be such that

$$\begin{array}{lll} u_\varepsilon(\alpha) & = & \left[\dfrac{\alpha_1}{\varepsilon q_\varepsilon}\right] z\varepsilon q_\varepsilon + u_\varepsilon(\alpha_1 - \left[\dfrac{\alpha_1}{\varepsilon q_\varepsilon}\right]\varepsilon q_\varepsilon, \alpha_2) \\ u_\varepsilon(\alpha) & = & \begin{cases} u_\varepsilon^a(\alpha) & \text{if } \alpha \in [0,p_\varepsilon]_\varepsilon \\ u_\varepsilon^b(\alpha) & \text{if } \alpha \in [p_\varepsilon,q_\varepsilon]_\varepsilon \end{cases}. \end{array} \tag{5.5}$$

Then $u_\varepsilon \to zt$ in $(L^p(0,1))^M$ and

$$\begin{array}{lll} f^M_{hom}(z) & \le & \limsup_\varepsilon F^M_\varepsilon(u_\varepsilon) \\ & = & 2\lambda\Big((M-2)\widetilde{f}(a)+\widehat{f}(a)\Big) + 2(1-\lambda)\Big((M-2)\widetilde{f}(b)+\widehat{f}(b)\Big) \\ & = & 2\Big((M-2)\widetilde{f}(z)+\widehat{f}(z)\Big)^{**}. \end{array}$$

□

Remark 5.3 Note that, by (5.2) we deduce that

$$f^2_{hom}(z) = 2(\widehat{f}\ )^{**}(z)$$

and that

$$\lim_{M\to+\infty} \frac{f^M_{hom}(z)}{M} = 2(\widetilde{f}\ )^{**}(z).$$

with $\widehat{f}$ and $\widetilde{f}$ as in (5.3).

Remark 5.4 (Cauchy-Born rule) Let $N=2$ and let F_ε and f_{hom} be as in Theorem 3.1 with $\omega=(0,1)$. We say that $z \in \mathbf{R}^d$ is a *strong Cauchy-Born (sCB) state* or a *weak Cauchy-Born (wCB) state* if respectively, for any

$$u_\varepsilon(t) \to zt \text{ such that } F_\varepsilon(u_\varepsilon) \to f_{hom}(z),$$

it holds

(i) $\#\{\alpha \in (0,1)\cap \varepsilon\mathbf{Z}:\ u_\varepsilon(\alpha+\varepsilon)-u_\varepsilon(\alpha)\neq \varepsilon z\} = o(\frac{1}{\varepsilon})$,

(ii) $\#\{\alpha \in (0,1)\cap \varepsilon\mathbf{Z}:\ u_\varepsilon(\alpha+2\varepsilon)-u_\varepsilon(\alpha)\neq 2\varepsilon z\} = o(\frac{1}{\varepsilon})$.

In the previous two cases we simply say that the Cauchy-Born (CB) rule holds at z or, shortly, that z is a CB state.

From a mechanical point of view, in this two different cases, to a macroscopic strain z it corresponds a micro-structure of

(i) uniformly displaced material points,

(ii) periodically displaced material points on the microscopic scale 2ε.

Let now u_ε be as in (5.4) or in (5.5). Since $u_\varepsilon \to zt$ and

$$\begin{aligned}\lim_\varepsilon F^2_\varepsilon(u_\varepsilon) &= f^2_{hom}(z),\\ \lim_\varepsilon \frac{F^M_\varepsilon(u_\varepsilon)}{M} &= \frac{2}{M}\Big((M-2)\widetilde{f}(z)+\widehat{f}(z)\Big)^{**} = \frac{f^M(z)}{M}+O(\frac{1}{M}),\end{aligned}$$

the validity or failure of the CB rule related to F^M_ε given by (5.1), when $M=2$ or $M\to+\infty$, can be reduced to study whether the cases (i), (ii) hold for such a u_ε. Then the set

$$C(M) := \{z\in\mathbf{R}^d:\ z \text{ is not a CB state}\},$$

is given, by all $z\in\mathbf{R}^d$ such that

$$(\widehat{f}\ )^{**}(z) < \widehat{f}(z)$$

if $M=2$ or such that

$$(\widetilde{f}\ )^{**}(z) < \widetilde{f}(z)$$

if $M\to+\infty$.

We now provide an example giving us some insight in the dependence of the Cauchy-Born rule to the number of layers M. Let

$$f_1(z) = (z+1)^2\wedge(z-1)^2 \qquad f_2(z)=f_3(z)=z^2.$$

Then an easy computation gives

$$\begin{aligned} C(M) \subseteq C(2) &= \Big(-\frac{5}{6},-\frac{1}{6}\Big)\cup\Big(\frac{1}{6},\frac{5}{6}\Big)\\ \lim_{M\to\infty} C(M) &= \Big(-\frac{3}{4},-\frac{1}{4}\Big)\cup\Big(\frac{1}{4},\frac{3}{4}\Big)\subset C(2).\end{aligned}$$

This means that a scale transition phenomenon occurs: there exists a macroscopic state with uniform strain z which is not a CB state for $M=2$, corresponding to an oscillation on a mesoscopic scale δ_ε such that $\delta_\varepsilon\to 0$, $\frac{\delta_\varepsilon}{\varepsilon}\to+\infty$, and which, asymptotically in M, becomes a CB state, thus corresponding to an oscillation on the microscopic scale ε.

Example 5.5 In the following example we will consider energies defined for $u \in \mathcal{A}_\varepsilon((0,1)\times[0,\varepsilon(M-1)])$ as

$$\begin{aligned} F^M_\varepsilon(u) = & \sum_{\alpha,\alpha+\varepsilon e_1\in(0,1)_\varepsilon} \varepsilon f_1(D^{e_1}_\varepsilon u(\alpha)) + \sum_{\alpha,\alpha+\varepsilon e_1\in(0,1)_\varepsilon} \varepsilon f_2(D^{e_2}_\varepsilon u(\alpha)) \\ & + \sum_{\alpha,\alpha+\varepsilon(e_1+e_2)\in(0,1)_\varepsilon} \varepsilon f_3(D^{e_1+e_2}_\varepsilon u(\alpha)), \end{aligned} \tag{5.6}$$

with $f_i : \mathbf{R}^d \to [0,+\infty)$ satisfying

$$c(|z|^p - 1) \le f_i(z) \le C(|z|^p + 1), \quad i = 1,2, \qquad f_3(z) \le C(|z|^p + 1).$$

This is a particular homogeneous case of the model considered in Section 3 with $N = 2$, $\omega = (0,1)$, $f^\xi \equiv 0$ if $\xi \ne e_1, e_2, e_1 + e_2$ (see Figure 3 b)).

Proposition 5.6 *Let F^M_ε be defined by (5.6) and let f^M_{hom} be the density energy of its Γ-limit. Then it holds*

$$2\Big((M-2)(\widetilde{f}\,)^{**}(z) + (\widehat{f}\,)^{**}(z)\Big) \le f^M_{hom}(z) \le 2\Big((M-2)\widetilde{f} + \widehat{f}\,\Big)^{**}(z) \tag{5.7}$$

for any $z \in \mathbf{R}^d$, where $\widetilde{f} : \mathbf{R}^d \to [0,+\infty)$ and $\widehat{f} : \mathbf{R}^d \to [0,+\infty)$ are defined by the following formulas

$$\begin{aligned} \widetilde{f}(z) &:= \frac{f_1(z)}{2} + \frac{1}{2}\inf\Big\{ f_2(z_1) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_2 - z_1 = z \Big\}, \\ \widehat{f}(z) &:= f_1(z) + \frac{1}{2}\inf\Big\{ f_2(z_1) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_2 - z_1 = z \Big\}. \end{aligned} \tag{5.8}$$

Proof. The proof is analogous to that of Proposition 5.2. In particular, fixed $z \in \mathbf{R}^d$ such that

$$2\Big((M-2)\widetilde{f} + \widehat{f}\,\Big)(z) = 2\Big((M-2)\widetilde{f} + \widehat{f}\,\Big)^{**}(z),$$

and given z_1, z_2 such that

$$\inf\Big\{ f_2(z_1) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big) : z_2 - z_1 = z \Big\} = f_2(z_1) + f_3\Big(\frac{z_2}{\sqrt{2}}\Big),$$

the upper bound inequality can be obtained by arguing as before with (5.4) replaced by

$$u^j_\varepsilon(\beta) = z\beta + jz_1\varepsilon.$$

□

Remark 5.7 Note that, by (5.7) we deduce that

$$f^2_{hom}(z) = 2(\widehat{f}\,)^{**}(z)$$

and that

$$\lim_{M\to+\infty} \frac{f^M_{hom}(z)}{M} = 2(\widetilde{f}\,)^{**}(z).$$

with $\widehat{f}$ and $\widetilde{f}$ as in (5.8).

Remark 5.8 By Remark 4.8, if f^M_{hom} is as in Example 5.1 and 5.5, one can prove that

$$\lim_{M\to+\infty} \frac{1}{M} f^M_{hom}(z) = \overline{f}_{hom}(z).$$

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