A QUANTITATIVE FORM OF FABER-KRAHN INEQUALITY (EXTENDED VERSION)

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Abstract. The Faber-Krahn inequality states that balls are the unique minimizers of the first eigenvalue of the p-Laplacian among all sets with a fixed volume. In this paper we prove a sharp quantitative form of this inequality. This extends to the case $p > 1$ a recent result proved by Brasco, De Philippis and Velichkov [10] for the Laplacian.

Contents

1. Introduction

For an open set $\Omega \subset \mathbb{R}^n$ the *first Dirichlet eigenvalue of the p-Laplacian* is defined for $p > 1$ as

$$\lambda_p(\Omega) := \min\left\{ \int_\Omega |\nabla u|^p \, dx : \|u\|_{L^p(\Omega)} = 1 \,\text{and}\, u \in W_0^{1,p}(\Omega) \right\}$$

and coincides with the smallest number λ such that the Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution. We recall the *Faber-Krahn* inequality stating that balls are the unique minimizers of $\lambda_p(\Omega)$ among all open sets with a given volume. Denoting by B the unit ball and using the rescaling law $\lambda_p(r\Omega) = r^{-p}\lambda_p(\Omega)$, we can restate it as

$$|\Omega|^{\frac{p}{n}} \lambda_p(\Omega) \geq |B|^{\frac{p}{n}} \lambda_p(B). \tag{1.1}$$

A natural question which arises from the characterization of balls as the only sets for which the equality holds in (1.1) is the stability of this inequality. In other words one would like to prove an estimate of the type

$$|\Omega|^{\frac{p}{n}} \lambda_p(\Omega) - |B|^{\frac{p}{n}} \lambda_p(B) \geq d(\Omega),$$

Date: September 7, 2016.

2000 *Mathematics Subject Classification.* 47A75, 49R05, 35R35.

Key words and phrases. p-Laplacian, Faber-Krahn inequality, stability, free boundary problems.

where $d(\Omega)$ is a scaling invariant *asymmetry functional* measuring in a precise sense the distance from Ω to a ball.

In the case of the Dirichlet eigenvalue for the *Laplacian* this issue was first studied by Hansen and Nadirashvili in [20] for simply connected domains and by Melas in [24] for convex sets, using suitable asymmetry functionals. For a general open set it is reasonable, see [15, 14, 11], to use the so-called *Fraenkel asymmetry* which is defined as

$$\mathcal{A}(\Omega) = \min\left\{\frac{|\Omega\Delta B_r|}{|B_r|} : |B_r| = |\Omega|\right\}. \tag{1.2}$$

The quantitative estimate

$$|\Omega|^{\frac{p}{n}}\lambda_p(\Omega) - |B|^{\frac{p}{n}}\lambda_p(B) \geq \gamma\mathcal{A}(\Omega)^{\alpha}, \tag{1.3}$$

with $\alpha = 3$ and $\gamma = \gamma(p) > 0$, was first obtained by Bhattacharya in [6] in the two dimensional case. Later on his result was extended to any dimension by Maggi, Pratelli and the first author, who proved in [16] that (1.3) holds with $\alpha = p + 2$ and $\gamma(p, n) > 0$. Very recently Brasco and De Philippis in [9] gave a shorter proof of (1.3) for the Laplacian, with $\alpha = 3$ and $n \geq 2$. Although they only consider this model case, it is clear that their proof works also for the p-Laplacian.

All the exponents obtained in these papers are not sharp. Indeed, it was independently conjectured by Bhattacharya and Weitsman [7] and by Nadirashvili [25] that at least for the first eigenvalue of the Laplacian the optimal exponent in the above inequality should be 2, i.e., that there exists a positive constant $\gamma(n)$ such that for any open set of finite measure

$$|\Omega|^{\frac{2}{n}}\lambda_2(\Omega) - |B|^{\frac{2}{n}}\lambda_2(B) \geq \gamma\mathcal{A}(\Omega)^2. \tag{1.4}$$

This conjecture has been recently proved by Brasco, De Philippis and Velichkov in [10] with a deep proof inspired by the regularity theory of Alt and Caffarelli for free boundary problems. Note that the exponent 2 in (1.4) is optimal. In fact, see [9, Sect. 2.6], if one considers the family of ellipsoids

$$\Omega_\varepsilon = \{x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} : |x'|^2 + |x_n|^2(1+\varepsilon) < 1\},$$

it is not to hard to show that for $\varepsilon \to 0$

$$|\Omega_\varepsilon|^{\frac{2}{n}}\lambda_2(\Omega_\varepsilon) - |B|^{\frac{2}{n}}\lambda_2(B) \simeq \varepsilon^2 \quad \text{and} \quad \mathcal{A}(\Omega) \simeq \varepsilon.$$

In this paper we extend the result of [10] proving that (1.4) still holds with the optimal exponent 2 if one replaces $\lambda_2(\Omega)$ with the first eigenvalue of the p-Laplacian. More generally, we consider *the optimal constants for the Sobolev–Poincaré inequality*

$$\lambda_{p,q}(\Omega) := \min\left\{\int_\Omega |\nabla u|^p\,dx : \|u\|_{L^q(\Omega)} = 1 \text{ and } u \in W_0^{1,p}(\Omega)\right\}, \tag{1.5}$$

where q satisfies the following bounds

$$\begin{cases} 1 \leq q < p^* := np/(n-p) & \text{if } 1 < p < n, \\ 1 \leq q < \infty & \text{if } p \geq n. \end{cases} \tag{1.6}$$

Note that if $q = p > 1$ then $\lambda_{p,p}(\Omega)$ is the first eigenvalue of the p-Laplacian, while $1/\lambda_{2,1}(\Omega)$ coincides with the classical *torsional rigidity* of Ω, which is defined as

$$T(\Omega) = \sup\left\{\left(\int_\Omega u\,dx\right)^2 : \int_\Omega |\nabla u|^2\,dx = 1 \text{ and } u \in W_0^{1,2}(\Omega)\right\}.$$

These quantities satisfy the scaling law

$$\lambda_{p,q}(r\Omega) = r^{\frac{nq-np-qp}{q}}\lambda_{p,q}(\Omega) \qquad \text{for } r > 0.$$

Moreover a generalized version of the Faber-Krahn inequality states that

$$D_{p,q}(\Omega) := \lambda_{p,q}(\Omega)|\Omega|^{\frac{np+qp-nq}{qn}} - \lambda_{p,q}(B)|B|^{\frac{np+qp-nq}{qn}} \geq 0, \tag{1.7}$$

with the equality holding if and only if Ω is a ball.

The main result of this paper is the following *quantitative Faber-Krahn* inequality.

Theorem 1.1. *Let $n \geq 2$, $p > 1$, and let $q \geq 1$ be an exponent as in* (1.6). *There exists a constant γ, depending only on p, n, q such that for every open set Ω with finite measure*

$$D_{p,q}(\Omega) \geq \gamma \mathcal{A}(\Omega)^2. \tag{1.8}$$

The proof of this result follows the strategy originally devised in [10]. However, the nonlinear, degenerate nature of the p-Laplacian equation gives rise to substantial difficulties that require at various stages new technical tools and ideas. Let us briefly describe the main steps of the proof.

As in [10], we show that a slightly weaker quantitative estimate

$$E(\Omega) - E(B) \geq \gamma \mathcal{A}(\Omega)^2 \tag{1.9}$$

holds for any open set with $|\Omega| = |B|$, where $E(\Omega)$ is the functional

$$E(\Omega) := \min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega u \, dx \right\}. \tag{1.10}$$

Then, the general estimate (1.8) is quickly deduced by combining (1.9) with the following extension of the *Kohler-Jobin* inequality due to Brasco [8]

$$\frac{\lambda_{p,q}(\Omega)}{\lambda_{p,q}(B)} \geq \left(\frac{E(B)}{E(\Omega)} \right)^\alpha,$$

where α is a suitable exponent depending only on p, q, n and p and q are as above.

The proof of the stability inequality (1.9) is achieved in two steps. Following a strategy which is used in the proof of several quantitative isoperimetric type inequalities, we first deal with open sets $C^{2,\alpha}$ close to the unit ball. In this case we show that (1.9) holds in a stronger form

$$E(\Omega) - E(B) \geq c(n,p) \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)}, \tag{1.11}$$

where $\varphi \in C^{2,\alpha}(\partial B)$ is a function whose normal graph on the unit sphere coincides with $\partial\Omega$, i.e., $\partial\Omega = \{x + x\varphi(x) : x \in \partial B\}$. However, for the p-Laplacian even this preliminary step is by no means obvious. The reason is that the function u_Ω minimizing the right hand side of (1.10) is a weak solution of the degenerate elliptic equation

$$-\mathrm{div}(|\nabla u_\Omega|^{p-2} \nabla u_\Omega) = 1, \tag{1.12}$$

for which $C^{2,\alpha}$ global estimates are false. Thus, it is natural to regularize the functional $E(\Omega)$, setting for $\kappa \in (0,1)$

$$E_\kappa(\Omega) := \min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega (\kappa^2 + |\nabla u|^2)^{\frac{p}{2}} \, dx - \int_\Omega u \, dx \right\}.$$

Then, the key point in the proof is to show, using a delicate regularity argument, see Lemmas 2.5 and 2.7, that when Ω is sufficiently close in $C^{2,\alpha}$ to the unit ball B then

$$E_\kappa(\Omega) - E_\kappa(B) \geq c(n,p) \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)},$$

with a constant independent of κ. Hence, the validity of (1.11) for $E(\Omega)$ follows by letting $\kappa \to 0$.

Next step is to prove (1.9) for any open set. To this aim we first observe, see Lemma 4.4, that we may always assume that Ω is contained in a sufficiently large ball B_R. Then, we prove that if $\sigma > 0$ is sufficiently small then for any $\Omega \subset B_R$ with $|\Omega| = |B|$

$$E(\Omega) - E(B) \geq \sigma^4 A(\Omega), \tag{1.13}$$

where $A(\Omega)$ is a new asymmetry functional introduced in [10] by setting

$$A(\Omega) := \int_{\Omega \Delta B(x_\Omega)} |1 - |x - x_\Omega|| \, dx,$$

where $B(x_\Omega)$ is the ball of radius 1 centered at the barycenter x_Ω of Ω. Note that $A(\Omega)$ is estimated from below by the square of the Fraenkel asymmetry $\mathcal{A}(\Omega)^2$, see Lemma 3.2. Thus (1.13) immediately yields (1.9).

In order to prove (1.13) we use an argument by Cicalese and Leonardi [11], see also [2]. In short, it goes as follows. We argue by contradiction assuming that there exists a sequence of open sets $\Omega_j \subset B_R$ for which

$$|\Omega_j| = |B|, \quad \varepsilon_j := A(\Omega_j) \to 0, \quad E(\Omega_j) - E(B) < \sigma^4 A(\Omega_j). \tag{1.14}$$

Then for each j we consider a (quasi) open set U_j minimizing the following problem

$$\inf\{E(U) + \sqrt{\varepsilon_j^2 + \sigma^2 (A(U) - \varepsilon_j)^2} + f_\eta(|U|) : U \subset B_R\}, \tag{1.15}$$

where, as in [10], $f_\eta(|\Omega|)$ is a suitable penalization of the original volume constraint $|U| = |B|$. The goal is to show that the minimizers U_j converge in $C^{2,\alpha}$ to the unit ball and that

$$E(U_j) - E(B) \leq C\sigma \|\varphi_j\|^2_{H^{\frac{1}{2}}(\partial B)},$$

for a positive constant $C = C(p, n)$, where the function φ_j parametrizes the boundary of U_j. Then (1.13) is established by observing that this inequality contradicts (1.11) provided we choose σ small enough. This contradiction argument motivates the choice of the new asymmetry. In fact $A(\Omega)$ is sufficiently more regular than the Fraenkel asymmetry to guarantee the regularity of the minimizers of (1.15).

To get the $C^{2,\alpha}$ convergence of the U_j we need to establish a priori regularity estimates. To this aim we observe that each U_j solves a free boundary problem. Then, following [10], we adapt the regularity theory developed by Alt and Caffarelli [3] for a free boundary problem for the Laplacian. As a matter of fact, this theory was extended to the p-Laplacian by Danielli and Petrosyan [12]. However, we cannot use directly their result since in our case the equation (1.12) satisfied by the minimizers u_j of $E(U_j)$ is not homogeneous. Due to the nonlinearity of the p-Laplacian, the presence of a non-zero right hand side requires a series of not trivial modifications. In particular, two steps in the regularity proof must be handled with care. The first one is the global Lipschitz continuity of the u_j, see Lemmas 3.11 and 3.12 and Theorem 3.13. The second one is the regularity Theorem 3.23. The necessary changes to the proof of this latter result are given in Section 5.

2. The nearly spherical case

In the following we shall denote by $B_r(x)$ the ball with center at x and radius r. We shall also use the following simplified notation

$$B_r := B_r(0), \quad B(x) := B_1(x), \quad B := B(0)\,.$$

We say that a bounded open set $\Omega \subset \mathbb{R}^n$ is a *nearly spherical set* of class $C^{2,\alpha}$ *parametrized by* φ if there exists a function $\varphi \in C^{2,\alpha}(\partial B)$ with $\|\varphi\|_{L^\infty(\partial B)} \leq 1/2$ such that

$$\partial\Omega := \{x \in \mathbb{R}^n : x = (1+\varphi(y))y,\, y \in \partial B\}.$$

Given a function $\varphi \in H^{1/2}(\partial B)$ we define the $H^{1/2}(\partial B)$ norm of φ by setting

$$\|\varphi\|^2_{H^{1/2}(\partial B)} := \|\varphi\|^2_{L^2(\partial B)} + [\varphi]^2_{H^{1/2}(\partial B)} := \|\varphi\|^2_{L^2(\partial B)} + \int_B |\nabla\widetilde{\varphi}(x)|^2\, dx, \tag{2.1}$$

where $\widetilde{\varphi} \in H^1(B)$ is the *harmonic extension* of φ, that is the harmonic function such that $\widetilde{\varphi} = \varphi$ on ∂B. Note that this norm is equivalent to the one defined by replacing $[\cdot]_{H^{1/2}(\partial B)}$ with the so-called *Gagliardo seminorm*, see for instance [19, (1,3,3,3)].

In the following, for any integer $k \geq 0$ we denote by $y_{k,i}$, $i = 1, \dots, N(k,n)$, the harmonic polynomials of degree k on ∂B, normalized so that $\|y_{k,i}\|_{L^2(\partial B)} = 1$. Recall that for any k and i, $y_{k,i}$ is a non trivial solution of the equation

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2)y_{k,i} \qquad \text{on } \partial B, \tag{2.2}$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the *Laplace–Beltrami* operator on the unit sphere. Recall that the polynomials $y_{k,i}$ form an orthonormal base for $L^2(\partial B)$. Therefore for every $\varphi \in L^2(\partial B)$ we have

$$\|\varphi\|^2_{L^2(\partial B)} = \sum_{k=0}^{\infty} \sum_{i=1}^{N(k,n)} a^2_{k,i},$$

where $a_{k,i}$ is the Fourier coefficient of φ with respect to $y_{k,i}$. Moreover, it is well known that the seminorm $[\varphi]_{H^{1/2}(\partial B)}$ defined in (2.1) is equivalent to the square root of the following quantity

$$\sum_{k=1}^{\infty} \sum_{i=1}^{N(k,n)} k a^2_{k,i}.$$

The next lemma on a slight deformation of the unit ball is proved in [10, Lemma A.1]. Recall that if $E \subset \mathbb{R}^n$ and $\delta > 0$ the δ-*neighborhood of* E is defined by setting

$$\mathcal{N}_\delta(E) := \{x \in \mathbb{R}^n : \operatorname{dist}(x; \partial E) < \delta\}.$$

Lemma 2.1. *Given $\alpha \in (0,1)$ there exist $\delta = \delta(\alpha, n) > 0$ and a modulus of continuity ω such that for every nearly spherical set Ω parametrized by φ with $\|\varphi\|_{C^{2,\alpha}(\partial B)} \leq \delta$ and $|\Omega| = |B|$, we can find an autonomous vector field $X \in C^{2,\alpha}(\mathbb{R}^n)$ for which the following properties hold.*

(i) X is parallel to x and $\operatorname{div} X = 0$ in $\mathcal{N}_\delta(B)$;

(ii) if $\Phi_t = \Phi(t, x)$ is the flow associated to X, namely

$$\partial_t \Phi_t = X(\Phi_t) \qquad \Phi_0(x) = x,$$

then $|\Phi_t(B)| = |B|$ for all $t \in [0,1]$, $\Phi_1(\partial B) = \partial\Omega$ and

$$\int_{\partial B} X \cdot \nu_B \, d\mathcal{H}^{n-1} = 0, \tag{2.3}$$

where ν_B is the exterior normal to ∂B;

(iii) moreover, the following estimates hold:

$$\|\Phi_t - Id\|_{C^{2,\alpha}} \leq \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \quad \textit{for every } t \in [0,1]. \tag{2.4}$$

$$\Big(X \cdot \frac{x}{|x|}\Big) \circ \Phi_t - X \cdot \nu_B = \psi_t \, X \cdot \nu_B \quad \textit{on } \partial B, \tag{2.5}$$

with $\|\psi_t\|_{C^{2,\alpha}(\partial B)} \le \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)})$.

$$\|\varphi - X\cdot\nu_B\|_{L^2(\partial B)} \le \omega(\|\varphi\|_{L^\infty(\partial B)})\|\varphi\|_{L^2(\partial B)} \tag{2.6}$$

$$\|\varphi - X\cdot\nu_B\|_{H^{\frac{1}{2}}(\partial B)} \le \omega(\|\varphi\|_{L^\infty(\partial B)})\|\varphi\|_{H^{\frac{1}{2}}(\partial B)}. \tag{2.7}$$

Finally, if $\varphi\in C^\infty(\partial B)$, *then* $X\in C^\infty(\mathbb{R}^n)$.

Given $\kappa\in[0,1]$ and a bounded open set $\Omega\subset\mathbb{R}^n$, we consider the following quantity

$$E_\kappa(\Omega) := \min_{u\in W_0^{1,p}(\Omega)}\left\{\frac{1}{p}\int_\Omega(\kappa^2+|\nabla u|^2)^{\frac{p}{2}}\,dx - \int_\Omega u\,dx\right\}. \tag{2.8}$$

We shall simply write $E(\Omega)$ when $\kappa=0$. Now, given a nearly spherical set Ω parametrized by φ, with $|\Omega|=|B|$, we consider the flow Φ_t given by Lemma 2.1. Setting $\Omega_t:=\Phi_t(B)$, for all $t\in[0,1]$ we denote by $u_{\kappa,t}$ the unique minimizer of $E_\kappa(\Omega_t)$. Then $u_{\kappa,t}$ is the solution of the equation

$$\begin{cases} -\mathrm{div}\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla u_{\kappa,t}\big) = 1 & \text{in } \Omega_t\\ u_{\kappa,t}=0 & \text{on } \partial\Omega_t.\end{cases} \tag{2.9}$$

Moreover,

$$e_\kappa(t) := E_\kappa(\Omega_t) = \frac{1}{p}\int_{\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}\,dx - \int_{\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|^2\,dx. \tag{2.10}$$

Again, when $\kappa=0$ we shall drop the subscript κ by writing $e(t)$ and u_t in place of $e_\kappa(t)$ and $u_{\kappa,t}$, respectively. A direct computation shows that

$$u_0 := u_{0,0} = \frac{p-1}{p}n^{-\frac{1}{p-1}}\big(1-|x|^{\frac{p}{p-1}}\big), \quad \nabla u_0 = -n^{-\frac{1}{p-1}}|x|^{\frac{1}{p-1}}\theta, \tag{2.11}$$

where

$$\theta := \frac{x}{|x|}. \tag{2.12}$$

The next theorem contains all the regularity properties of the solutions $u_{\kappa,t}$ which we are going to use in the following.

Theorem 2.2. *Let* $\kappa\in[0,1]$, $p>1$, $\alpha\in(0,1)$. *There exist* $\beta\in(0,\alpha)$ *and a modulus of continuity* ω *depending only on* p,α,n, *but not on* κ, *such that if* Ω *is a* $C^{2,\alpha}$ *nearly spherical set parametrized by* φ *and* $\|\varphi\|_{C^{2,\alpha}(\partial B)}<\delta$, *where* δ *is the one in Lemma 2.1, then for all* $t\in[0,1]$ *and* $\kappa\in[0,1]$ *we have*

$$\|u_0 - u_{\kappa,t}\circ\Phi_t\|_{C^{1,\beta}(\overline{B})} \le \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}+\kappa). \tag{2.13}$$

Moreover, given $r\in(0,1)$, *there exist* $\delta'>0$, $0<\gamma<\alpha$ *and a modulus of continuity* ω', *depending only on* p,α,n,r, *such that if* $\|\varphi\|_{C^{2,\alpha}(\partial B)}+\kappa<\delta'$, *then for all* $t\in[0,1]$

$$\|u_0 - u_{\kappa,t}\circ\Phi_t\|_{C^{2,\gamma}(\overline{B}\setminus B_r)} \le \omega'(\|\varphi\|_{C^{2,\alpha}(\partial B)}+\kappa). \tag{2.14}$$

Proof. By applying the result in [22, Th. 1] to the solutions of (2.9) we get that there exist $\beta'\in(0,\alpha)$ and $C>0$, depending only on p,α,n and δ, such that if $\kappa\in[0,1]$ and $t\in[0,1]$ then

$$\|u_{\kappa,t}\|_{C^{1,\beta'}(\overline{\Omega}_t)} \le C. \tag{2.15}$$

Fix now $\beta\in(0,\beta')$. Then, (2.13) follows by a contradiction argument. To this aim, given three sequences φ_j, $\kappa_j\in[0,1]$ and $t_j\in[0,1]$ such that $\|\varphi_j\|_{C^{2,\alpha}(\partial B)}+\kappa_j\to0$, denote by u_j the solutions of (2.9) with $\kappa = \kappa_j$, $t = t_j$ in $\Phi_j(t_j, B)$, where the flow Φ_j is associated to φ_j as in Lemma 2.1. If for such a sequence we have that

$$\limsup_{j\to\infty} \|u_0 - u_j \circ \Phi_j(t_j, \cdot)\|_{C^{1,\beta}(\overline{B})} > 0 \tag{2.16}$$

from estimate (2.15) it follows that we may extract a not relabelled subsequence such that $v_j := u_j \circ \Phi_j(t_j, \cdot)$ converges to some function u in $C^{1,\beta}(\overline{B})$. On the other hand each v_j satisfies the equation

$$\begin{cases} -\mathrm{div}\Big(\big(\kappa_j^2 + \big|\big((\nabla\Phi_j(t_j,\cdot))^{-1}\big)^T \nabla v_j\big|^2\big)^{\frac{p-2}{2}} M_j \nabla v_j\Big) = \det \nabla\Phi_j(t_j, \cdot) & \text{in } B \\ v_j = 0 & \text{on } \partial B, \end{cases}$$

where $M_j = \big(\det \nabla\Phi_j(t_j,\cdot)\big)(\nabla\Phi_j(t_j,\cdot))^{-1}\big((\nabla\Phi_j(t_j,\cdot))^{-1}\big)^T$. Thus, from the $C^{1,\beta}$ convergence of v_j to u and (2.4) it follows that

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = 1 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

hence $u = u_0$, thus contradicting (2.16).

Finally, since $|\nabla u_0| > 0$ in $B \setminus \{0\}$ from (2.13) it follows that, given $r \in (0, 1)$, there exist δ' and c_0, depending on δ' and r, such that if $\|\varphi\|_{C^{2,\alpha}(\partial B)} + \kappa < \delta'$ then, for all $t \in [0, 1]$

$$c_0 \le |\nabla u_{\kappa,t}| \le \frac{1}{c_0} \qquad \text{in } \Omega_t \setminus B_r.$$

Then, by differentiating the equation satisfied by $u_{\kappa,t}$ and using the classical Schauder estimates, see [17, Ch. 6], we have that there exist $\gamma' \in (0, 1)$ and $C > 0$ such that, if $\|\varphi\|_{C^{2,\alpha}(\partial B)} + \kappa < \delta'$,

$$\|u_{\kappa,t}\|_{C^{2,\gamma'}(\overline{\Omega}_t\setminus B_r)} \le C.$$

From this estimate (2.14) follows by the same contradiction argument used to prove (2.13). □

Before going on we need to establish a few inequalities that will be used in the sequel.

Lemma 2.3. *Let $p > 1$. There exists $c(p) \ge 0$ such that if $\kappa \ge 0$ and $\xi, \eta \in \mathbb{R}^n$ then*

$$\big((\kappa^2 + |\xi|^2)^{\frac{p-2}{2}}\xi - (\kappa^2 + |\eta|^2)^{\frac{p-2}{2}}\eta\big) \cdot (\xi - \eta) \ge c(\kappa^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2. \tag{2.17}$$

Moreover there exists another constant $C(p) \ge 0$ such that if $\Omega \subset \mathbb{R}^n$ is an open set and for u, $v \in W^{1,p}(\Omega)$ and $0 \le s \le 1$, we set

$$u^s(x) = su(x) + (1 - s)v(x),$$

then the following two inequalities hold:

$$\int_\Omega |\nabla u - \nabla v|^p \le C \int_0^1 \frac{1}{s}\, ds \int_\Omega (|\nabla u^s|^{p-2}\nabla u^s - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u^s - v) \qquad \textit{if } p \ge 2, \tag{2.18}$$

while, if $1 < p < 2$,

$$\int_B |\nabla u - \nabla v|^p \le C\Big(\int_0^1 \frac{1}{s}\, ds \int_\Omega (|\nabla u^s|^{p-2}\nabla u^s - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u^s - v)\Big)^{\frac{p}{2}} \Big(\int_\Omega (|\nabla u| + |\nabla v|)^p\Big)^{1-\frac{p}{2}}. \tag{2.19}$$

Proof. If $1<p<2$ the proof of (2.17) is contained in the proof of Lemma 2.2 in [1], while if $p\geq 2$ it is similar and simpler.

Given $u,v\in W^{1,p}(\Omega)$, if $p\geq 2$ we may easily estimate

$$\int_\Omega|\nabla u-\nabla v|^p=p\int_0^1\frac{1}{s}\,ds\int_\Omega|\nabla(u^s-v)|^p,$$

hence (2.18) immediately follows by applying (2.17) with $\kappa=0$.

When $1<p<2$, we also use (2.17) to obtain

$$\begin{aligned}\int_\Omega|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}&=2\int_0^1 s\,ds\int_\Omega|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}\\&\leq C(p)\int_0^1\frac{1}{s}\,ds\int_\Omega|\nabla(u^s-v)|^2(|\nabla u^s|+|\nabla v|)^{p-2}\\&\leq C(p)\int_0^1\frac{1}{s}\,ds\int_\Omega(|\nabla u^s|^{p-2}\nabla u^s-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u^s-v).\end{aligned}$$

Then (2.19) follows by Hölder inequality

$$\begin{aligned}\int_B|\nabla u-\nabla v|^p&\leq\Big(\int_\Omega|\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}\Big)^{\frac{p}{2}}\Big(\int_\Omega(|\nabla u|+|\nabla v|)^p\Big)^{1-\frac{p}{2}}\\&\leq C(p)\Big(\int_0^1\frac{1}{s}\,ds\int_\Omega(|\nabla u^s|^{p-2}\nabla u^s-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u^s-v)\Big)^{\frac{p}{2}}\Big(\int_\Omega(|\nabla u|+|\nabla v|)^p\Big)^{1-\frac{p}{2}}.\end{aligned}$$

□

The next step is to calculate $e_\kappa'(t)$ and $e_\kappa''(t)$ for $\kappa>0$. The corresponding formulas are stated in the next lemma, where we have denoted by $\mathscr{H}_{\Omega_t}:=\mathrm{div}\nu_{\Omega_t}$ the *mean curvature of* Ω_t.

Lemma 2.4. *Let Ω be smooth nearly spherical set satisfying the assumptions of Lemma 2.1 and let Φ_t be a flow as in that lemma. Then, if $0<\kappa\leq 1$ and $t\in[0,1]$ we have*

$$e_\kappa'(t)=\frac{1}{p}\int_{\Omega_t}\mathrm{div}\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}X\big)\,dx-\int_{\Omega_t}\mathrm{div}\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|^2X\big)\,dx,\tag{2.20}$$

where X is as in Lemma 2.1. Moreover, denoting by X_τ the tangential component of X, we have

$$\begin{aligned}e_\kappa''(t)=&\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla\dot u_{\kappa,t}\cdot\nu_{\Omega_t})\dot u_{\kappa,t}\,d\mathcal{H}^{n-1}\\&+(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla\dot u_{\kappa,t}\cdot\nabla u_{\kappa,t})(\nabla u_{\kappa,t}\cdot\nu_{\Omega_t})\dot u_{\kappa,t}\,d\mathcal{H}^{n-1}\\&-\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla^2u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}\\&-(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2(\nabla^2u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}\\&+\int_{\partial\Omega_t}|\nabla u_{\kappa,t}|\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|\mathscr{H}_{\Omega_t}-1\big)(X\cdot\nu_{\Omega_t})^2\,d\mathcal{H}^{n-1},\end{aligned}\tag{2.21}$$

where $\dot u_{\kappa,t}$ is the unique solution in $H^1(\Omega_t)$ of the equation

$$\begin{cases}\mathrm{div}((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla\dot u_{\kappa,t}+(p-2)(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot u_{\kappa,t})\nabla u_{\kappa,t})=0 & \text{in } \Omega_t\\ \dot u_{\kappa,t}=-\nabla u_{\kappa,t}\cdot X & \text{on } \partial\Omega_t.\end{cases}\tag{2.22}$$

Proof. We start by observing that the map $t \mapsto v_{\kappa,t}(x) := u_{\kappa,t}(\Phi_t(x)) \in C^\infty(\overline{B})$ is a smooth map. Indeed this is a consequence of a classical differentiability result based on the implicit function theorem for maps with values in Banach spaces. The detailed proof of this fact can be found in [21], see in particular Proposition 5.3.7, where the authors consider the case of Laplacian. However, the same argument used therein applies to our case due to the uniform ellipticity of the equation satisfied by $v_{\kappa,t}$, which in turn holds since $\kappa > 0$. Observe also that, setting $\dot{u}_{\kappa,t}(x) := \partial_t u_{\kappa,t}(x)$, we immediately get that $\dot{u}_{\kappa,t}$ solves equation (2.22).

Let us now recall that Hadamard formula, see for instance [21, Sect. 5.2], states that if f is sufficiently smooth then

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_t} \partial_t f(t,x)\,dx + \int_{\partial\Omega_t} f(t,x)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}, \tag{2.23}$$

where ν_{Ω_t} is the exterior normal to Ω_t. Hence from (2.10) we get

$$\begin{aligned} e'_\kappa(t) = &-\int_{\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t}\,dx + \frac{1}{p}\int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &- (p-2)\int_{\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t})|\nabla u_{\kappa,t}|^2\,dx \\ &- \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|^2(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}. \end{aligned}$$

Multiplying (2.22) by $u_{k,t}$ and integrating, we get that the sum of the two volumes integrals in the above equality is zero. Therefore (2.20) follows, by applying the divergence theorem to the two boundary integrals.

Using again (2.23) to differentiate $e'_\kappa(t)$ and the divergence theorem we obtain

$$\begin{aligned} e''_\kappa(t) = &-\int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t})(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &- (p-2)\int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t})|\nabla u_{\kappa,t}|^2(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &+ \frac{1}{p}\int_{\partial\Omega_t} \operatorname{div}((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}X)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &- \int_{\partial\Omega_t} \operatorname{div}((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|^2X)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}. \end{aligned} \tag{2.24}$$

Observe that on $\partial\Omega_t$

$$\nabla u_{\kappa,t} = -|\nabla u_{\kappa,t}|\nu_{\Omega_t}, \tag{2.25}$$

and hence

$$\nabla u_{\kappa,t}\cdot\nabla\dot{u}_{\kappa,t} = -|\nabla u_{\kappa,t}|(\nabla\dot{u}_{\kappa,t}\cdot\nu_{\Omega_t}).$$

Therefore from (2.24), recalling that $\mathrm{div}X = 0$ in a neighborhood of ∂B and that $\dot u_{\kappa,t} = -\nabla u_{\kappa,t}\cdot X$ on $\partial\Omega_t$, one obtains

$$\begin{aligned} e''_\kappa(t) &= \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla \dot u_{\kappa,t}\cdot\nu_{\Omega_t})\dot u_{\kappa,t}\,d\mathcal{H}^{n-1} \\ &+(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}(\nabla\dot u_{\kappa,t}\cdot\nabla u_{\kappa,t})(\nabla u_{\kappa,t}\cdot\nu_{\Omega_t})\dot u_{\kappa,t}\,d\mathcal{H}^{n-1} \\ &-\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &-(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1}. \end{aligned} \tag{2.26}$$

Decomposing X into the sum $X_\tau + (X\cdot\nu_{\Omega_t})\nu_{\Omega_t}$, the last two integrals in (2.26) become

$$\begin{aligned} &-\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &-(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2(\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot\nu_{\Omega_t})\,d\mathcal{H}^{n-1} \\ &+\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|(\nabla^2 u_{\kappa,t}[\nu_{\Omega_t}]\cdot\nu_{\Omega_t})(X\cdot\nu_{\Omega_t})^2\,d\mathcal{H}^{n-1} \\ &+(p-2)\int_{\partial\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^3(\nabla^2 u_{\kappa,t}[\nu_{\Omega_t}]\cdot\nu_{\Omega_t})(X\cdot\nu_{\Omega_t})^2\,d\mathcal{H}^{n-1}. \end{aligned} \tag{2.27}$$

Furthermore using (2.9) and (2.25) on $\partial\Omega_t$ and recalling that $\mathrm{div}\,\nu_{\Omega_t} = \mathscr{H}_{\Omega_t}$, one has

$$\begin{aligned} -1 &= \mathrm{div}\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla u_{\kappa,t}\big) = -\mathrm{div}\big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|\nu_{\Omega_t}\big) \\ &= -(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|\mathscr{H}_{\Omega_t} + (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla^2 u_{\kappa,t}[\nu_{\Omega_t}]\cdot\nu_{\Omega_t} \\ &\qquad + (p-2)(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^2\nabla^2 u_{\kappa,t}[\nu_{\Omega_t}]\cdot\nu_{\Omega_t}. \end{aligned}$$

Then (2.21) follows from (2.26) by inserting this equality in the last two addends of (2.27). □

Next step is to let $\kappa\to 0$. To this aim, denoting by μ the measure $|x|^{\frac{p-2}{p-1}}\,dx$, we introduce the following weighted Sobolev space

$$W^{1,2}(B;\mu) := \Big\{u\in H^1_{loc}(B\setminus\{0\}) : \int_B |\nabla u|^2\,d\mu < \infty\Big\}.$$

and the corresponding subspace $W^{1,2}_0(B;\mu)$ of functions with 0 trace on ∂B. Note that when $x\to 0$ the weight $|x|^{\frac{p-2}{p-1}}$ tends to 0 if $p>2$, while it goes to ∞ if $1<p<2$. Next lemma gives the limit of the second derivatives of $e_\kappa(t)$ when both κ and t tend to 0.

Lemma 2.5. *Let Ω satisfy the assumptions of Lemma 2.4. Then $e(t)$ is differentiable for all $t\in[0,1]$ and*

$$e'(t) = \frac{1-p}{p}\int_{\partial\Omega_t}|\nabla u_{0,t}|^p X\cdot\nu_{\Omega_t}\,d\mathcal{H}^{n-1}. \tag{2.28}$$

Moreover,

$$n^{\frac{p-2}{p-1}}\lim_{\substack{t\to 0\\ \kappa\to 0}} e''_\kappa(t) = \int_B |x|^{\frac{p-2}{p-1}}|\nabla\hat w|^2\,dx + (p-2)\int_{\Omega_t}|x|^{\frac{p-2}{p-1}}|\theta\cdot\nabla\hat w|^2\,dx - \int_{\partial B}\hat w^2\,d\mathcal{H}^{n-1}, \tag{2.29}$$

where θ is as in (2.12) *and $\hat{w}$ is the unique weak solution in $W^{1,2}(B;\mu)$ of the equation*

$$\begin{cases} \operatorname{div}(|x|^{\frac{p-2}{p-1}}\nabla\hat{w} + (p-2)|x|^{\frac{p-2}{p-1}}(\theta\cdot\nabla\hat{w})\theta) = 0 & \text{in } B \\ \hat{w} = n^{-\frac{1}{p-1}}X\cdot\nu_B & \text{on } \partial B. \end{cases} \tag{2.30}$$

Proof. **Step 1.** Fix $0<r<1$. By (2.14) the functions $u_{\kappa,t}\circ\Phi_t$ are equibounded in $C^{2,\gamma}(\overline{B}\setminus B_r)$ and $|\nabla u_{\kappa,t}|\geq c(r)$ in $B\setminus B_r$ for κ and t small. Therefore, recalling (2.22), classical elliptic estimates imply that the functions $\dot{u}_{\kappa,t}\circ\Phi_t$ are equibounded in $C^{1,\gamma}(\overline{B}\setminus B_r)$. Therefore, there exists a sequence $(\kappa_i,t_i)\to(0,0)$ with $\kappa_i>0$, such that $\dot{u}_{\kappa_i,t_i}$ converge uniformly away from the origin to a function $\hat{w}\in C^1(B\setminus\{0\})$. Moreover, multiplying (2.22) by $\dot{u}_{\kappa,t}$ and integrating, we get that

$$\int_{\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}|\nabla\dot{u}_{\kappa,t}|^2\,dx+(p-2)\int_{\Omega_t}(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}}|\nabla\dot{u}_{\kappa,t}\cdot\nabla u_{\kappa,t}|^2\,dx\leq C,$$

for a suitable constant independent of κ and t. Therefore, passing to the limit as $(\kappa_i,t_i)\to(0,0)$ and recalling (2.13) and (2.11), we have from the previous inequality that

$$\int_B|\nabla\hat{w}|^2\,d\mu<\infty,$$

hence $\hat{w}\in W^{1,2}(B;\mu)$. Moreover, $\hat{w}=n^{-\frac{1}{p-1}}X\cdot\nu_B$ on ∂B.

Let us now fix a function $v\in W^{1,2}(B\setminus\{0\})$ with compact support in $B\setminus\{0\}$. Passing to the limit in the equation (2.22) we then have

$$\int_B\nabla\hat{w}\cdot\nabla v+(p-2)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla v)d\mu=0. \tag{2.31}$$

We want to show that the same equation is satisfied indeed for every $v\in W_0^{1,2}(B;\mu)$ and that $\hat{w}$ is the unique solution to (2.31) in $W^{1,2}(B;\mu)$ such that $\hat{w}=n^{-\frac{1}{p-1}}X\cdot\nu_B$ on ∂B. To this aim, let us fix $0<r<1$ and a cut-off function $\eta_r\in C^\infty(B)$ such that $0\leq\eta_r\leq1$, $\eta_r\equiv1$ in $B\setminus B_r$, $\eta_r\equiv0$ in $B_{r/2}$ and $|\nabla\eta_r|\leq c/r$. Inserting $v\eta_r$ in the above equation we get

$$\int_B\eta_r\nabla\hat{w}\cdot\nabla v+(p-2)\eta_r(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla v)d\mu=-\int_B v\nabla\hat{w}\cdot\nabla\eta_r+(p-2)v(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_r)d\mu\,. \tag{2.32}$$

In order to pass to the limit in the above equation we have to distinguish two cases.

Step 2. Let us assume that $p\geq\frac{n}{n-1}$ and that v is bounded. In this case, we may estimate the right hand side of (2.32) by

$$\begin{aligned}(p-1)\int_{B_r}|v||x|^{\frac{p-2}{p-1}}|\nabla\hat{w}||\nabla\eta_r|\,dx&\leq\frac{c\|v\|_\infty}{r}\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}|\nabla\hat{w}|^2\,dx\Big)^{\frac12}\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}\,dx\Big)^{\frac12}\\&\leq c\|v\|_\infty r^{\frac{p-2}{2(p-1)}+\frac{n}{2}-1}\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}|\nabla\hat{w}|^2\,dx\Big)^{\frac12}.\end{aligned}$$

Note that since $\frac{p-2}{2(p-1)}+\frac{n}{2}-1\geq0$ the last term in the above chain of inequalities converges to 0 as $r\to0$. Passing to the limit as $r\to0$ in (2.32) we conclude that (2.31) holds also when $v\in W_0^{1,2}(B;\mu)\cap L^\infty(B)$. Hence, by density, (2.31) holds for all $v\in W_0^{1,2}(B;\mu)$.

Let us assume now that $1<p<\frac{n}{n-1}$. This case is more delicate. Observe first that

$$\int_B\nabla\hat{w}\cdot\nabla\eta_r+(p-2)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_r)d\mu=0\,. \tag{2.33}$$

In fact, by the divergence theorem and the convergence of $u_{\kappa_i,t_i}\circ\Phi_{t_i}$ to u_0 in $C^1(\overline{B})$ and of $\dot{u}_{\kappa_i,t_i}$ to $\hat{w}$ in $C^1(\overline{B}\setminus B_{\frac{1}{2}})$, we have

$$\int_B \nabla\hat{w}\cdot\nabla\eta_r+(p-2)(\theta\cdot\nabla\hat{w})(\theta\cdot\nabla\eta_r)d\mu=\int_{\partial B}\big(|x|^{\frac{p-2}{p-1}}\partial_{\nu_B}\hat{w}+(p-2)|x|^{\frac{p-2}{p-1}}\partial_{\nu_B}\hat{w}\big)\,d\mathcal{H}^{n-1}$$
$$=n^{\frac{p-2}{p-1}}\int_{\partial B}\big(|\nabla u_0|^{p-2}\partial_{\nu_B}\hat{w}+(p-2)|\nabla u_0|^{p-4}(\nabla u_0\cdot\nabla\hat{w})\partial_{\nu_B}u_0\big)\,d\mathcal{H}^{n-1}$$
$$=\lim_{i\to\infty}n^{\frac{p-2}{p-1}}\int_{\partial B}(\kappa_i^2+|\nabla u_{\kappa_i,t_i}|^2)^{\frac{p-2}{2}}\partial_{\nu_B}\dot{u}_{\kappa_i,t_i}+(p-2)(\kappa_i^2+|\nabla u_{\kappa_i,t_i}|^2)^{\frac{p-4}{2}}(\nabla u_{\kappa_i,t_i}\cdot\nabla\dot{u}_{\kappa_i,t_i})\partial_{\nu_B}u_{\kappa_i,t_i}$$

and the last integrals are all zero by (2.22).

Observe also that if $v\in W^{1,2}(B;\mu)$ and $0<r\le 1$ we get

$$\int_{B_r}|\nabla v|\,dx\le\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{B_r}|x|^{\frac{2-p}{p-1}}\,dx\Big)^{\frac{1}{2}}\le cr^{\frac{2-p}{2(p-1)}+\frac{n}{2}}\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}.$$

Therefore $v\in W^{1,1}(B)$ and setting $v_r=\fint_{B_r}v$, we obtain

$$\fint_{B_r}|v-v_r|\,dx\le cr^{1-n}\int_{B_r}|\nabla v|\,dx\le cr^{\frac{2-p}{2(p-1)}-\frac{n}{2}+1}\Big(\int_{B_r}|x|^{\frac{p-2}{p-1}}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}\le cr^{\alpha},\tag{2.34}$$

where we have set $\alpha:=\frac{2-p}{2(p-1)}-\frac{n}{2}+1>0$ and c is a constant independent of r. From this estimate we get in particular that for all positive integers i

$$|v_{2^{-i}}-v_{2^{1-i}}|\le\fint_{B_{2^{-i}}}|v-v_{2^{-i}}|\,dx+\fint_{B_{2^{-i}}}|v-v_{2^{1-i}}|\,dx\le\frac{c}{2^{\alpha i}}\,.$$

Hence, we may conclude that the sequence $v_{2^{-i}}$ converge to some $v_0\in\mathbb{R}$. In turn, recalling (2.34), this implies that $v_r\to v_0$ as $r\to 0$ and that

$$\int_B|v(rz)-v_0|\,dz\to 0$$

as $r\to 0$. Therefore, there exists a sequence $s_i\to 0^+$ such that for

$$v(s_iz)\to v_0\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }z\in\partial B.$$

At this point we argue as follows. Let us fix $s\in(0,1)$. Without loss of generality we identify here v with its *precise representative* v^*, see [13, Sect. 4.8]. Since, for $\mathcal{H}^{n-1}$-a.e. $z\in\partial B$ the function $t\mapsto v^*(tz)$ is locally absolutely continuous in $(0,1)$, see [13, Sect. 4.9.2], we have

$$v(sz)-v(s_iz)=\int_{s_i}^s\nabla v(tz)\cdot z\,dt.$$

Therefore,

$$|v(sz)-v(s_iz)|^2\le\Big(\int_{s_i}^s t^{\frac{p-2}{p-1}+n-1}|\nabla v(tz)|^2\,dt\Big)\Big(\int_{s_i}^s t^{\frac{2-p}{p-1}-n+1}\,dt\Big).$$

Integrating this inequality on ∂B and letting $i\to\infty$ we get that

$$\int_{\partial B}|v(sz)-v_0|^2\,d\mathcal{H}^{n-1}\le cs^{\frac{2-p}{p-1}-n+2}\int_0^s t^{\frac{p-2}{p-1}+n-1}\,dt\int_{\partial B}|\nabla v(tz)|^2\,d\mathcal{H}^{n-1}$$
$$=cs^{\frac{2-p}{p-1}-n+2}\int_0^s dt\int_{\partial B_t}|x|^{\frac{p-2}{p-1}}|\nabla v(x)|^2\,d\mathcal{H}^{n-1}$$

Then, multiplying by $s^{\frac{p-2}{p-1}+n-1}$ and integrating between 0 and r , we finally get

$$\int_{B_r} |x|^{\frac{p-2}{p-1}} |v(x) - v_0|^2 \, dx \leq cr^2 \int_{B_r} |x|^{\frac{p-2}{p-1}} |\nabla v(x)|^2 \, dx. \tag{2.35}$$

By (2.33) we may assume, without loss of generality, that $v_0 = 0$. Then, using (2.35) we may estimate the right hand side of (2.32) by

$$(p-1)\int_{B_r} |v||x|^{\frac{p-2}{p-1}}|\nabla\hat{w}||\nabla\eta_r| \, dx \leq \frac{c}{r}\Big(\int_{B_r} |x|^{\frac{p-2}{p-1}}|\nabla\hat{w}|^2 \, dx\Big)^{\frac{1}{2}}\Big(\int_{B_r} |x|^{\frac{p-2}{p-1}}|v|^2 \, dx\Big)^{\frac{1}{2}}$$
$$\leq c\Big(\int_{B_r} |x|^{\frac{p-2}{p-1}}|\nabla\hat{w}|^2 \, dx\Big)^{\frac{1}{2}}\Big(\int_{B_r} |x|^{\frac{p-2}{p-1}}|\nabla v|^2 \, dx\Big)^{\frac{1}{2}}.$$

From this estimate, passing to the limit in (2.32) as $r \to 0$, we may conclude that (2.31) holds also in this case for any $v \in W_0^{1,2}(B;\mu)$. This proves that $\hat{w}$ is a solution of the Dirichlet problem (2.30).

Since $\hat{w}$ is clearly unique, by a standard compactness argument we have that $\dot{u}_{\kappa,t} \circ \Phi_t$ converge to $\hat{w}$ locally in $C^1(B \setminus \{0\})$ as $(\kappa, t) \to (0, 0)$.

Step 3. To conclude the proof, let us show that (2.28) and (2.29) holds. To this aim, observe that the same argument used in the proof of Theorem 2.2 shows that $u_{\kappa,t} \to u_{0,t}$ in $C^1(\overline{\Omega}_t)$ as $\kappa \to 0$, uniformly with respect to t. This implies in particular that $e_\kappa(t) \to e_0(t)$ as $\kappa \to 0$, uniformly with respect to t. Moreover, since by (2.20)

$$e'_\kappa(t) = \frac{1}{p}\int_{\partial\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p}{2}} X \cdot \nu_{\Omega_t} \, d\mathcal{H}^{n-1} - \int_{\partial\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}|^2 X \cdot \nu_{\Omega_t} \, d\mathcal{H}^{n-1},$$

we have that $e'_\kappa(t)$ converges uniformly for $t \in [0, 1]$ to

$$\frac{1-p}{p}\int_{\partial\Omega_t} |\nabla u_{0,t}|^p X \cdot \nu_{\Omega_t} \, d\mathcal{H}^{n-1}$$

and thus (2.28) holds. Finally, from the convergence of $\dot{u}_{\kappa,t} \circ \Phi_t$ to $\hat{w}$ as $(\kappa, t) \to (0, 0)$, passing to the limit in (2.21), recalling (2.11) and using (2.31), we get that

$$\begin{aligned}
\lim_{\substack{t\to 0\\ \kappa\to 0}} e''_\kappa(t) &= \int_{\partial B} |\nabla u_0|^{p-2}(\nabla\hat{w}\cdot\nu_B)\hat{w} \, d\mathcal{H}^{n-1} \\
&\quad + (p-2)\int_{\partial B} |\nabla u_0|^{p-4}(\nabla\hat{w}\cdot\nabla u_0)(\nabla u_0\cdot\nu_B)\hat{w} \, d\mathcal{H}^{n-1} \\
&\quad - (p-1)\int_{\partial B} |\nabla u_0|^{p-2}(\nabla^2 u_0[\nabla u_0]\cdot X_\tau)(X\cdot\nu_B) \, d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial B} |\nabla u_0|(|\nabla u_0|^{p-1}\mathscr{H}_B - 1)(X\cdot\nu_B)^2 \, d\mathcal{H}^{n-1} \\
&= \int_B |\nabla u_0|^{p-2}|\nabla\hat{w}|^2 dx + (p-2)\int_B |\nabla u_0|^{p-4}(\nabla u_0\cdot\nabla\hat{w})^2 dx - n^{\frac{-p}{p-1}}\int_{\partial B} (X\cdot\nu_B)^2 d\mathcal{H}^{n-1}.
\end{aligned}$$

This concludes the proof of (2.29). □

We next show a technical lemma.

Lemma 2.6. *Let Ω satisfy the assumptions of Lemma 2.1 and let $\kappa \in (0, 1]$, $0 < r < 1$. There exist $\delta, C_0 > 0$, depending on p, n, α and r such that if $\|\varphi\|_{C^{2,\alpha}} + \kappa \leq \delta$, then for all $t \in [0, 1]$*

$$\int_{B\setminus B_r} |\nabla\dot{u}_{\kappa,t}\circ\Phi_t|^2 + |\dot{u}_{\kappa,t}\circ\Phi_t|^2 \, dx \leq C_0\|X\cdot\nu_B\|^2_{H^{\frac{1}{2}}(\partial B)}, \tag{2.36}$$

where Φ_t is the flow associated to φ as in Lemma 2.1.

Proof. Let δ be so small that $\|\varphi\|_{C^{2,\alpha}} \leq \delta$ implies $B_r \subset\subset \Omega_t$. Let us set $w := \eta v$, where v is the harmonic extension of $-\nabla u_{\kappa,t} \cdot X$ to Ω_t and η is a smooth function such that $\eta = 1$ in $\Omega_t \setminus B_r$ and $\eta = 0$ in $B_{r/2}$. By the minimality property of $\dot{u}_{\kappa,t}$ and Theorem 2.2 we have, if $\kappa \leq \delta$ and δ is sufficiently small,

$$\begin{aligned}
&\int_{\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla \dot{u}_{\kappa,t}|^2\, dx + (p-2)\int_{\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} |\nabla \dot{u}_{\kappa,t} \cdot \nabla u_{\kappa,t}|^2\, dx \\
&\quad \leq \int_{\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla w|^2\, dx + (p-2)\int_{\Omega_t} (\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} |\nabla w \cdot \nabla u_{\kappa,t}|^2\, dx \\
&\quad \leq C \int_{\Omega_t \setminus B_{r/2}} |\nabla w|^2\, dx \leq C\|X \cdot \nabla u_{k,t}\|^2_{H^{\frac12}(\partial\Omega_t)} \leq C\|(X \cdot \nabla u_{k,t}) \circ \Phi_t)\|^2_{H^{\frac12}(\partial B)}. \qquad (2.37)
\end{aligned}$$

From this inequality and Theorem 2.2, (2.36) follows immediately if we show that the last term in the previous chain of inequalities, is controlled by the norm of $X \cdot \nu_B$ in $H^{1/2}(\partial B)$. To this aim we recall that by Lemma 2.1(i) $X = (X \cdot \theta)\theta$ in a neighborhood of ∂B. Since $\nabla u_0 = -|\nabla u_0|\theta$ on ∂B, with the help of (2.4), (2.5) and Theorem 2.2 we obtain

$$\begin{aligned}
&\|(\nabla u_{\kappa,t} \cdot X) \circ \Phi_t - \nabla u_0 \cdot X\|_{H^{\frac12}(\partial B)} \\
&\leq \|((\nabla u_{\kappa,t} \cdot \theta) \circ \Phi_t - (\nabla u_0 \cdot \theta))(X \cdot \theta) \circ \Phi_t\|_{H^{\frac12}(\partial B)} + \||\nabla u_0|((X \cdot \theta) \circ \Phi_t - X \cdot \nu_B)\|_{H^{\frac12}(\partial B)} \\
&\leq C\|(\nabla u_{\kappa,t} \cdot \theta) \circ \Phi_t - (\nabla u_0 \cdot \theta)\|_{C^1(B \setminus B_{\frac12})}\|(X \cdot \nu_B) \circ \Phi_t\|_{H^{\frac12}(\partial B)} + C\|(X \cdot \theta) \circ \Phi_t - X \cdot \nu_B\|_{H^{\frac12}(\partial B)} \\
&\leq \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)} + \kappa)\|X \cdot \nu_B\|_{H^{\frac12}(\partial B)}. \qquad (2.38)
\end{aligned}$$

The result then follows from this estimate. □

Lemma 2.7. *Let Ω satisfy the assumptions of Lemma 2.4. There exist $\delta > 0$ and a modulus of continuity ω such that, if $\|\varphi\|_{C^{2,\alpha}} \leq \delta$, then*

$$|e''_\kappa(t) - e''_\kappa(0)| \leq \omega(\|\varphi\|_{C^{2,\alpha}} + \kappa)\|X \cdot \nu_B\|^2_{H^{\frac12}(\partial B)},$$

for all κ, $t \in (0, 1]$.

Proof. **Step 1.** Observe that, from (2.21), using the divergence theorem and equation (2.22), by a change of variable, we get

$$e''_\kappa(t) = I_1(t) + I_2(t) + I_3(t),$$

where

$$\begin{aligned}
I_1(t) &:= \int_B \{(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla \dot{u}_{\kappa,t}|^2\} \circ \Phi_t \det \nabla\Phi_t\, dx \\
&\quad + (p-2)\int_B \{(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} (\nabla \dot{u}_{\kappa,t} \cdot \nabla u_{\kappa,t})^2\} \circ \Phi_t \det \nabla\Phi_t\, dx, \\
I_2(t) &:= -\int_{\partial B} \{(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} (\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}] \cdot X_\tau)(X \cdot \nu_{\Omega_t})\} \circ \Phi_t\, J_{n-1}\Phi_t\, d\mathcal{H}^{n-1} \\
&\quad - (p-2)\int_{\partial B} \{(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} |\nabla u_{\kappa,t}|^2 (\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}] \cdot X_\tau)(X \cdot \nu_{\Omega_t})\} \circ \Phi_t\, J_{n-1}\Phi_t\, d\mathcal{H}^{n-1}, \\
I_3(t) &:= \int_{\partial B} \{|\nabla u_{\kappa,t}|((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}| \mathscr{H}_{\Omega_t} - 1)(X \cdot \nu_{\Omega_t})^2\} \circ \Phi_t\, J_{n-1}\Phi_t\, d\mathcal{H}^{n-1}.
\end{aligned}$$

where $J_{n-1}\Phi_t$ is the tangential Jacobian of Φ_t. By Lemma 2.1 one has, if $\|\varphi\|_{C^{2,\alpha}}$ is small,

$$\|\mathscr{H}_{\partial\Omega_t}\circ\Phi_t-\mathscr{H}_{\partial B}\|_{L^\infty(\partial B)}+\|J_{n-1}\Phi_t-1\|_{L^\infty(\partial B)}+\|\det\nabla\Phi_t-1\|_{L^\infty(\partial B)}\le\omega(\|\varphi\|_{C^{2,\alpha}}).\quad(2.39)$$

By Lemma 2.1(i), X is parallel to θ in a neighborhood of ∂B. Hence by applying (2.5),

$$\begin{aligned}|(X\cdot\nu_{\Omega_t})\circ\Phi_t-X\cdot\nu_B| &= |((X\cdot\theta)\circ\Phi_t)(\nu_{\Omega_t}\cdot\theta)\circ\Phi_t-X\cdot\nu_B|\\ &\le |(X\cdot\theta)\circ\Phi_t||(\nu_{\Omega_t}\cdot\theta)\circ\Phi_t-1|+|(X\cdot\theta)\circ\Phi_t-X\cdot\nu_B|\\ &\le \omega(\|\varphi\|_{C^{2,\alpha}})|X\cdot\nu_B|.\end{aligned}\quad(2.40)$$

A similar computation leads to

$$|X_\tau\circ\Phi_t|\le\omega(\|\varphi\|_{C^{2,\alpha}})|X\cdot\nu_B|.\quad(2.41)$$

Combining Theorem 2.2, (2.39), (2.40) and (2.41) and observing that $I_2(0)=0$, one deduces

$$|I_2(t)|+|I_3(t)-I_3(0)|\le\omega(\|\varphi\|_{C^{2,\alpha}}+\kappa)\|X\cdot\nu_B\|^2_{L^2(\partial B)}.$$

Next we estimate $|I_1(t)-I_1(0)|$. To this aim, we set $w_{\kappa,t}=\dot u_{\kappa,t}\circ\Phi_t$, $v_{\kappa,t}=u_{\kappa,t}\circ\Phi_t$ and $N_t=(\nabla\Phi_t)^{-1}((\nabla\Phi_t)^{-1})^T$. From (2.22) we see that $w_{\kappa,t}$ is the solution to the following elliptic equation

$$\begin{cases}\operatorname{div}((\kappa^2+|((\nabla\Phi_t)^{-1})^T\nabla v_{\kappa,t}|^2)^{\frac{p-2}{2}}\det\nabla\Phi_t\,N_t\nabla w_{\kappa,t}\\ \quad+(p-2)(\kappa^2+|((\nabla\Phi_t)^{-1})^T\nabla v_{\kappa,t}|^2)^{\frac{p-4}{2}}\det\nabla\Phi_t\,(N_t\nabla v_{\kappa,t}\nabla w_{\kappa,t})N_t\nabla v_{\kappa,t})=0 & \text{in } B\\ w_{\kappa,t}=-(\nabla u_{\kappa,t}\cdot X)\circ\Phi_t & \text{on } \partial B.\end{cases}\quad(2.42)$$

Therefore we may rewrite

$$\begin{aligned}I_1(t)=&\int_B(\kappa^2+|((\nabla\Phi_t)^{-1})^T\nabla v_{\kappa,t}|^2)^{\frac{p-2}{2}}N_t\nabla w_{\kappa,t}\nabla w_{\kappa,t}\det\nabla\Phi_t\,dx\\ &+(p-2)\int_B(\kappa^2+|((\nabla\Phi_t)^{-1})^T\nabla v_{\kappa,t}|^2)^{\frac{p-4}{2}}(N_t\nabla v_{\kappa,t}\nabla w_{\kappa,t})(N_t\nabla v_{\kappa,t}\nabla w_{\kappa,t})\det\nabla\Phi_t\,dx.\end{aligned}\quad(2.43)$$

Step 2. From the estimates proved in the previous step, it is clear that we only need to show that

$$|I_1(t)-I_1(0)|\le C\omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}+\kappa)\|X\cdot\nu_B\|^2_{H^{\frac12}(\partial B)}.\quad(2.44)$$

In order to make the argument of the proof of (2.44) clearer we now highlight also the dependence on φ of Φ, X, $v_{\kappa,t}$, $w_{k,t}$ and N_t. Moreover, given any two functions $u,v\in W^{1,2}(B)$ we set

$$\begin{aligned}L_{\kappa,t,\varphi}(u,v):=&\int_B(\kappa^2+|((\nabla\Phi_{t,\varphi})^{-1})^T\nabla v_{\kappa,t,\varphi}|^2)^{\frac{p-2}{2}}N_{t,\varphi}\nabla u\nabla v\det\nabla\Phi_{t,\varphi}\,dx\\ &+(p-2)\int_B(\kappa^2+|((\nabla\Phi_{t,\varphi})^{-1})^T\nabla v_{\kappa,t,\varphi}|^2)^{\frac{p-4}{2}}(N_{t,\varphi}\nabla v_{\kappa,t,\varphi}\nabla u)(N_{t,\varphi}\nabla v_{\kappa,t,\varphi}\nabla v)\det\nabla\Phi_{t,\varphi}\,dx.\end{aligned}$$

The bilinear form $L_{\kappa,0,\varphi}$ is defined accordingly.

The proof of (2.44) is by contradiction. If (2.44) does not hold, from (2.43) it follows that there exist $\kappa_j\to0$, $t_j\to t_0\in[0,1]$ and a sequence φ_j converging to 0 in $C^{2,\alpha}(\partial B)$ such that

$$\lim_{j\to\infty}\frac{L_{\kappa_j,t_j,\varphi_j}(w_{\kappa_j,t_j,\varphi_j},w_{\kappa_j,t_j,\varphi_j})}{\|X_j\cdot\nu_B\|^2_{H^{1/2}(\partial B)}}\ne\lim_{j\to\infty}\frac{L_{\kappa_j,0,\varphi_j}(w_{\kappa_j,0,\varphi_j},w_{\kappa_j,0,\varphi_j})}{\|X_j\cdot\nu_B\|^2_{H^{1/2}(\partial B)}},\quad(2.45)$$

where X_j is the vector field associated to φ_j as in Lemma 2.1. Note that we may assume that both limits are finite, since it can be easily checked, see the proof of (2.37), that there exists a constant C indipendent of j such that

$$0 \le L_{\kappa_j,t_j,\varphi_j}(w_{\kappa_j,t_j,\varphi_j}, w_{\kappa_j,t_j,\varphi_j}),\ L_{\kappa_j,0,\varphi_j}(w_{\kappa_j,0,\varphi_j}, w_{\kappa_j,0,\varphi_j}) \le C\|X_j \cdot \nu_B\|^2_{H^{1/2}(\partial B)}.$$

We now set

$$\widetilde{w}_j := \frac{w_{\kappa_j,t_j,\varphi_j}}{\|X_j \cdot \nu_B\|_{H^{1/2}(\partial B)}}, \qquad \widetilde{w}_{0,j} := \frac{w_{\kappa_j,0,\varphi_j}}{\|X_j \cdot \nu_B\|_{H^{1/2}(\partial B)}}.$$

Observe that

$$\widetilde{w}_j = -\frac{(\nabla u_{\kappa_j,t_j,\varphi_j} \cdot X_j) \circ \Phi_{t_j,\varphi_j}}{\|X_j \cdot \nu_B\|_{H^{1/2}(\partial B)}}, \quad \widetilde{w}_{0,j} = -\frac{\nabla u_{\kappa_j,0,\varphi_j} \cdot X_j}{\|X_j \cdot \nu_B\|_{H^{1/2}(\partial B)}} \qquad \text{on } \partial B.$$

From (2.38) it follows that, up to a not relabelled subsequence, $\widetilde{w}_j$ and $\widetilde{w}_{0,j}$ both converge weakly in $H^{1/2}(\partial B)$. Moreover, as a consequence of (2.38) and (2.13), we have also that $\widetilde{w}_j - \widetilde{w}_{0,j}$ converge strongly to zero in $H^{1/2}(\partial B)$. Observe now that by Lemma 2.6 for any $r \in (0,1)$ and any sufficiently large j, we have

$$\int_{B\setminus B_r} |\nabla \widetilde{w}_j|^2 + |\widetilde{w}_j|^2\,dx \le C(r).$$

Therefore, again up to a subsequence, we may assume that there exists $w \in W^{1,2}_{loc}(B \setminus \{0\})$ such that, for every $0 < r < 1$, $\widetilde{w}_j$ converge weakly in $W^{1,2}(B \setminus B_r)$ to w.

We claim that w is the unique solution in $W^{1,2}(B, \mu)$

$$\begin{cases} \operatorname{div}(|x|^{\frac{p-2}{p-1}}\nabla w + (p-2)|x|^{\frac{p-2}{p-1}}(\theta\cdot\nabla w)\theta) = 0 & \text{in } B \\ w = Y \cdot \nu_B & \text{on } \partial B. \end{cases} \tag{2.46}$$

where $Y \cdot \nu_B$ is the weak limit in $H^{1/2}(\partial B)$ of the restriction of $\widetilde{w}_j$ on ∂B. In order to prove this we fix a function $\eta \in C^1_c(B \setminus \{0\})$. Then we multiply by η the equation satisfied by $\widetilde{w}_j$ and pass to the limit as $j \to \infty$. Thus, recalling that v_{k_j,t_j,φ_j} converge in C^1 away from the origin to the function u_0 defined in (2.11), we obtain

$$\int_B |x|^{\frac{p-2}{p-1}}\nabla w \cdot \nabla \eta + (p-2)|x|^{\frac{p-2}{p-1}}(\theta\cdot\nabla w)(\theta\cdot\nabla\eta) = 0.$$

Then, from Step 2 of the proof of Lemma 2.5 we conclude that w is the unique solution in $W^{1,2}(B, \mu)$ of (2.46).

By applying the above argument to $\widetilde{w}_{0,j}$ and using the strong convergence in $H^{1/2}(\partial B)$ to zero of $\widetilde{w}_j - \widetilde{w}_{0,j}$, we have that also $\widetilde{w}_{0,j}$ converge weakly in $W^{1,2}(B \setminus B_r)$ for every $0 < r < 1$ to the solution w of (2.46).

We claim that $\widetilde{w}_j - \widetilde{w}_{0,j}$ converge strongly in $W^{1,2}(B \setminus B_r)$ for every $0 < r < 1$.

To prove this we consider the harmonic extension z_j of $\widetilde{w}_j - \widetilde{w}_{0,j}$ from ∂B to the whole ball B. Since, $\widetilde{w}_j - \widetilde{w}_{0,j}$ converge strongly to zero in $H^{1/2}(\partial B)$, we have that $z_j \to 0$ strongly in $W^{1,2}(B)$ as $j \to \infty$. Then, given $r \in (0,1)$ we take a $C^1(\overline{B})$ function ζ such that $\zeta \equiv 1$ on $B \setminus B_r$, $0 \le \zeta \le 1$, $\zeta \equiv 0$ in a neighborhood of the origin. Recalling the equations (2.42) satisfied by $w_{\kappa_j,t_j,\varphi_j}$ and by $w_{\kappa_j,0,\varphi_j}$, respectively, and the definitions of $\widetilde{w}_j$, $\widetilde{w}_{0,j}$ and z_j, we have, using the divergence theorem twice,

$$\begin{aligned} L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j - \widetilde{w}_{0,j}, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) &= L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) - L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_{0,j}, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) \\ &= L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j, z_j\zeta) - (L_{\kappa_j,t_j,\varphi_j} - L_{\kappa_j,0,\varphi_j})(\widetilde{w}_{0,j}, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) - L_{\kappa_j,0,\varphi_j}(\widetilde{w}_{0,j}, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) \\ &= L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j, z_j\zeta) - (L_{\kappa_j,t_j,\varphi_j} - L_{\kappa_j,0,\varphi_j})(\widetilde{w}_{0,j}, (\widetilde{w}_j - \widetilde{w}_{0,j})\zeta) - L_{\kappa_j,0,\varphi_j}(\widetilde{w}_{0,j}, z_j\zeta). \end{aligned}$$

Then, using the strong convergence to 0 of z_j and the $C^{1,\alpha}$ convergence away from the origin of $v_{\kappa_j,t_j,\varphi_j}$ and $v_{\kappa_j,0,\varphi_j}$ to the function u_0 defined in (2.11), we immediately get that the last three terms in the previous chain of equalities converge to zero. Therefore we may conclude that also $L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j-\widetilde{w}_{0,j},(\widetilde{w}_j-\widetilde{w}_{0,j})\zeta)$ converges to zero. Thus, since the functions $v_{\kappa_j,t_j,\varphi_j}$ are uniformly bounded in $C^{1,\alpha}$, we easily get that, for j large enough,

$$\int_B |\nabla\widetilde{w}_j-\nabla\widetilde{w}_{0,j}|^2\zeta\,dx \le C\int_B |\nabla\widetilde{w}_j-\nabla\widetilde{w}_{0,j}||\widetilde{w}_j-\widetilde{w}_{0,j}||\nabla\zeta|\,dx.$$

From this inequality, since $\widetilde{w}_j-\widetilde{w}_{0,j}$ converge to zero strongly in L^2 away from the origin, we conclude that also $\nabla\widetilde{w}_j-\nabla\widetilde{w}_{0,j}$ converge strongly to zero in $L^2(B\setminus B_r)$.

To conclude the proof we now show that

$$\lim_{j\to\infty}(L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j,\widetilde{w}_j)-L_{\kappa_j,0,\varphi_j}(\widetilde{w}_{0,j},\widetilde{w}_{0,j}))=0, \tag{2.47}$$

thus getting a contradiction to (2.45). To this aim we fix a function ζ as above with $r=1/2$ and we observe that

$$L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j,\widetilde{w}_j)-L_{\kappa_j,0,\varphi_j}(\widetilde{w}_{0,j},\widetilde{w}_{0,j})=L_{\kappa_j,t_j,\varphi_j}(\widetilde{w}_j,\widetilde{w}_j\zeta)-L_{\kappa_j,0,\varphi_j}(\widetilde{w}_{0,j},\widetilde{w}_{0,j}\zeta).$$

Then (2.47) easily follows from the strong convergence to zero in $W^{1,2}$ of $\widetilde{w}_j-\widetilde{w}_{0,j}$ away from the origin and the $C^{1,\alpha}$ convergence of $v_{\kappa_j,t_j,\varphi_j}$ and $v_{\kappa_j,0,\varphi_j}$ to u_0. □

We are now in position to prove the main result of this section.

Theorem 2.8. *For $\alpha\in(0,1)$, there exist δ,γ_0 depending only on p,α and n, such that if Ω is a nearly spherical set of class $C^{2,\alpha}$ parametrized by φ, with $\|\varphi\|_{C^{2,\alpha}(\partial B)}\le\delta$, such that the barycenter of Ω is at the origin and $|\Omega|=|B|$, then*

$$E(\Omega)-E(B)\ge\gamma_0\|\varphi\|^2_{H^{\frac12}(\partial B)}. \tag{2.48}$$

Proof. Assume that Ω is smooth. If $\kappa\in(0,1)$. By Taylor expansion we have

$$E_\kappa(\Omega)-E_\kappa(B)=e_\kappa(1)-e_\kappa(0)=e'_\kappa(0)+\frac12 e''_\kappa(0)+\frac12\int_0^1(1-t)(e''_\kappa(t)-e''_\kappa(0))\,dt.$$

Observe now that, by (2.3) and (2.28) we have that $e'(0)=0$. Therefore, passing to the limit as $\kappa\to0$ and using Lemma 2.5 and Lemma 2.7 we get

$$E(\Omega)\ge E(B)+\frac12 Q[X\cdot\nu_B]-\omega(\|\varphi\|_{C^{2,\alpha}(\partial B)})\|X\cdot\nu_B\|^2_{H^{1/2}(\partial B)}, \tag{2.49}$$

where ω is the modulus of continuity provided by Lemma 2.7 and Q is the quadratic form defined for all functions $\psi\in H^{1/2}(\partial B)$ by setting

$$Q[\psi]=n^{\frac{-p}{p-1}}\int_B|x|^{\frac{p-2}{p-1}}|\nabla\hat u|^2\,dx+(p-2)n^{\frac{-p}{p-1}}\int_B|x|^{\frac{p-2}{p-1}}|\theta\cdot\nabla\hat u|^2\,dx-n^{\frac{-p}{p-1}}\int_{\partial B}|\hat u|^2\,d\mathcal{H}^{n-1},$$

where $\hat u$ is the unique solution in $W^{1,2}(B;\mu)$ of the equation

$$\begin{cases}\operatorname{div}(|x|^{\frac{p-2}{p-1}}\nabla\hat u+(p-2)|x|^{\frac{p-2}{p-1}}(\theta\cdot\nabla\hat u)\theta)=0 & \text{in } B\\ \hat u=\psi & \text{on } \partial B.\end{cases}$$

Given the normalized harmonic polynomials $y_{k,i}$ satisfying (2.2) we denote by $u_{k,i}$ the unique solution in $W^{1,2}(B;\mu)$ of the following Dirichlet problem

$$\begin{cases}\operatorname{div}(|x|^{\frac{p-2}{p-1}}\nabla u_{k,i}+(p-2)|x|^{\frac{p-2}{p-1}}(\theta\cdot\nabla u_{k,i})\theta)=0 & \text{in } B\\ u_{k,i}=y_{k,i} & \text{on } \partial B.\end{cases}$$

Using equation (2.2), a simple computation shows that

$$u_{k,\,i} = |x|^{\alpha_k} y_{k,\,i}(\theta),$$

where $\alpha_k > 0$ satisfies the following equation

$$(p-1)\alpha_k^2 + (pn-n-p)\alpha_k - k(k+n-2) = 0. \tag{2.50}$$

Therefore, recalling the definition of $Q[\cdot]$ and using the fact that

$$\int_{\partial B} |y_{k,i}|^2 \, d\mathcal{H}^{n-1} = 1, \qquad \int_{\partial B} |\nabla_\tau y_{k,i}|^2 \, d\mathcal{H}^{n-1} = k(k+n-2),$$

another straightforward calculation shows that for every $k \geq 1$ and $i = 1, \dots, N(k,n)$

$$n^{\frac{p}{p-1}} Q[y_{k,i}] = \frac{(p-1)\alpha_k^2 + k(k+n-2)}{2\alpha_k + n - 2 + \frac{p-2}{p-1}} - 1 = \frac{(p-1)^2\alpha_k^2 + k(k+n-2)(p-1)}{2\alpha_k(p-1) + np - n - p} - 1.$$

At this point, using equation (2.50) we may rewrite the previous equality as

$$n^{\frac{p}{p-1}} Q[y_{k,i}] = \frac{-(p-1)(np-n-p)\alpha_k + 2k(k+n-2)(p-1)}{2\alpha_k(p-1) + np - n - p} - 1.$$

Since, by (2.50) we have that

$$\alpha_k = \frac{-(np-n-p) + \sqrt{(np-n-p)^2 + 4(p-1)k(k+n-2)}}{2(p-1)},$$

inserting this value in the previous expression of $Q[y_{k,i}]$, gives, after some tedious but elementary calculations,

$$Q[y_{k,i}] = \frac{1}{2n^{\frac{p}{p-1}}} \Big(\sqrt{(np-n-p)^2 + 4(p-1)k(k+n-2)} - (np-n-p) - 2 \Big) \tag{2.51}$$

Observe that the right hand side of (2.51) is zero when $k = 1$. Moreover it is easily checked that for every $p > 1$ and $n \geq 2$ there exists a constant $c(n,p)$ such that

$$Q[y_{k,i}] \geq c(n,p)k \qquad \text{for all } k \geq 2. \tag{2.52}$$

For any integer $k \geq 0$ and any $i = 1, \dots, N(k,n)$ let $a_{k,i}$ be the Fourier coefficient of $X \cdot \nu_B$ with respect to the orthonormal system $\{y_{k,i}\}$. Observe that (2.3) yields that $a_0 = 0$. Moreover, from the assumption that the barycenter of Ω is at the origin, we have that

$$\int_{\partial B} x[(1+\varphi)^{n+1} - 1] \, d\mathcal{H}^{n-1} = 0.$$

Therefore, given $0 < \varepsilon < 1/4$ if δ is small enough, we have that

$$\Big| \int_{\partial B} x\varphi \, d\mathcal{H}^{n-1} \Big| \leq \varepsilon \|\varphi\|_{L^2(\partial B)}.$$

From this inequality, recalling (2.6), we have that if δ is small enough

$$\Big| \int_{\partial B} x \, X \cdot \nu_B \, d\mathcal{H}^{n-1} \Big| \leq 2\varepsilon \|X \cdot \nu_B\|_{L^2(\partial B)}$$

This inequality implies that if ε, hence δ, is sufficiently small, then

$$\sum_{i=1}^{n} a_{1,i}^2 \leq 2 \sum_{k=2}^{\infty} \sum_{i=1}^{N(k,n)} a_{k,i}^2$$

Therefore, recalling (2.52), we may conclude that there exists a constant c_0 such that if δ is small enough

$$Q[X \cdot \nu_B] \geq c_0 \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)}.$$

From this inequality, (2.49) and (2.7), (2.48) follows by taking δ sufficiently small.

The general case is then easily obtained by approximating Ω in $C^{2,\alpha}$ with a sequence of smooth open sets with the same volume of Ω and with barycenter at the origin. □

3. Stability for bounded sets with small asymmetry

Definition 3.1. *For a bounded, measurable set $E \subset \mathbb{R}^n$, define*

$$A(E) = \int_{E \Delta B(x_E)} |1 - |x - x_E|| \, dx,$$

where x_E is the barycenter of E.

Observe that $A(E) = 0$ if and only if E is a ball with radius 1. Moreover one may write

$$A(E) = \int_B 1 - |x| \, dx + \int_E (|x - x_E| - 1) \, dx.$$

Using this equality it is not too difficult to prove the following lemma, see [10, Lemma 4.2].

Lemma 3.2. *Let $R > 2$. There exist positive constants $c_1(n)$, $c_2(R)$, $c_3(n)$ and $\delta(n)$ such that*
- *(i) for every bounded set E, $A(E) \geq c_1 |E \Delta B_1(x_E)|^2$;*
- *(ii) for every E, $F \subset B_R$, $|A(E) - A(F)| \leq c_2 |E \Delta F|$;*
- *(iii) for every nearly spherical set Ω with $\|\varphi\|_{L^\infty(\partial B)} \leq \delta$, $A(\Omega) \leq c_3 \|\varphi\|^2_{L^2(\partial B)}$.*

The main result of this section is the following theorem whose proof is an immediate consequence of Proposition 3.4 below.

Theorem 3.3. *For every $R \geq 2$, there exists two constants $\kappa(p, n, R) > 0$ and $\varepsilon(p, n, R) > 0$ such that*

$$E(\Omega) - E(B) \geq \kappa A(\Omega), \tag{3.1}$$

for all open sets Ω contained in B_R with $|\Omega| = |B|$ and $A(\Omega) \leq \varepsilon$.

Proof. The proof is obtained by contradiction. Indeed, if the result is not true, given $0 < \sigma \leq \sigma_0$, where σ_0 is as in Proposition 3.4 below, there exists a sequence of open sets $\Omega_j \subset B_R$ such that

$$|\Omega_j| = |B|, \, \varepsilon_j := A(\Omega_j) \to 0, \, E(\Omega_j) - E(B) \leq \sigma^4 \varepsilon_j. \tag{3.2}$$

Then, using Proposition 3.4 and recalling Lemma 3.2, we can find a sequence of nearly spherical open sets U_j parametrized by φ_j such that $\varphi_j \to 0$ in $C^{2,\alpha}(\partial B)$ and

$$E(U_j) - E(B) \leq 2C_0 \sigma A(U_j) \leq 2c_3 C_0 \sigma \|\varphi_j\|^2_{L^2(\partial B)}.$$

This contradicts (2.48) if σ is chosen so that $2c_3 C_0 \sigma < \gamma_0$. □

Proposition 3.4. *Let $R \geq 2$. There exist $\sigma_0, C_0 > 0$, depending only on p, n, R, such that for any $\sigma \leq \sigma_0$ and any sequence of open sets Ω_j as in (3.2), one may find a sequence of smooth open sets $U_j \subset B_R$ such that $|U_j| = |B|$, $x_{U_j} = 0$, ∂U_j converging to ∂B in C^k for every $k \in \mathbb{N}$ and*

$$\limsup_{j \to \infty} \frac{E(U_j) - E(B)}{A(U_j)} \leq C_0 \sigma.$$

The rest of this section will be devoted to the proof of Proposition 3.4.

We will construct the sets U_j by minimizing a suitable sequence of variational problems. In principle we should impose a volume constraint in order to fulfill the condition $|U_j| = |B|$. Instead, we shall replace the volume constraint by a penalization on the volume. To this aim, we define a penalization function as in [10] setting for $0 < \eta < 1$

$$f_\eta(s) = \begin{cases} \eta(s - |B|) & s < |B| \\ \frac{1}{\eta}(s - |B|) & s \geq |B|. \end{cases}$$

Note that for any $0 \leq s_2 \leq s_1$ one has

$$\eta(s_1 - s_2) \leq f_\eta(s_1) - f_\eta(s_2) \leq \frac{1}{\eta}(s_1 - s_2) \tag{3.3}$$

The proof of next simple lemma goes exactly as the proof of Lemma 4.5 of [10] with the obvious changes due to the different rescaling in our case of the functional $E(\Omega)$ defined in (2.8).

Lemma 3.5. *For every $R \geq 2$, one can find a suitable $\eta = \eta(p, n, R) \in (0, 1)$ such that, up to a translation, B is a minimizer of*

$$F_\eta(\Omega) = E(\Omega) + f_\eta(|\Omega|)$$

among all the sets contained in B_R. Moreover, there exists a constant $C(p, n, R) > 0$ such that for any other ball B_r with $0 \leq r \leq R$

$$F_\eta(B_r) - F_\eta(B) \geq \frac{|r - 1|}{C}. \tag{3.4}$$

In the proof of Proposition 3.4 we are going to show that, up to a translation and a small dilation, the sets U_j are given by the minimizers of the following penalized problems

$$g_{\eta,j} := \inf \{G_{\eta,j}(\Omega) \mid \Omega \subset B_R, \, \Omega \text{ open}\}. \tag{3.5}$$

where

$$G_{\eta,j}(\Omega) = F_\eta(\Omega) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\Omega) - \varepsilon_j)^2} \tag{3.6}$$

with η and F_η as in Lemma 3.5.

In order to prove the existence of a minimizer for $G_{\eta,j}$, we extend the range of Ω in (3.5) to the collection of all *p-quasi open* sets contained in B_R. To this aim, similarly to what one does in the quadratic case, we say that U is p-quasi open if there exists a function $u \in W^{1,p}(\mathbb{R}^n)$ such that

$$U = \{x \mid u^*(x) > 0\},$$

where u^* is the precise representative of u. Note that u^*, hence U, is defined up to a set of zero *p-capacity*. Moreover, u^* is *p-quasi continuous*, i.e., for every $\varepsilon > 0$ there exists an open set A_ε, with $\mathrm{Cap}_p(A_\varepsilon) < \varepsilon$, such that u^* is continuous on $\mathbb{R}^n \setminus A_\varepsilon$, where $\mathrm{Cap}_p(\cdot)$ denotes the p-capacity of a set, see [13, Sect. 4.8]. If U is p-quasi open, we define the following closed subspace of $W^{1,p}(\mathbb{R}^n)$

$$W_0^{1,p}(U) = \left\{v \in W^{1,p}(\mathbb{R}^n) \mid \mathrm{Cap}_p(\{v^* \neq 0\} \setminus U) = 0\right\}.$$

Observe that since $W_0^{1,p}(U)$ is a closed subspace of $W^{1,p}(\mathbb{R}^n)$ it is also weakly closed. With this definition in hands, we may extend the energy functional $E(\Omega)$ to any p-quasi open set U as in (2.8). Then, it is easy to show that there exists a unique minimizer u_U of $E(U)$ which will be called *the energy function of U*.

Remark 3.6. Observe that if $g_{\eta,j}$ is defined as in (3.5), then

$$g_{\eta,j} = \inf \{G_{\eta,\, j}(U) \mid U \subset B_R,\, U \text{ quasi open}\}\,. \tag{3.7}$$

To see this take a quasi open set $U \subset B_R$. By definition there exists a function $u \in W_0^{1,p}(B_R)$ such that $U = \{u^* > 0\}$. Since u^* is p-quasi continuous, for every h there exists an open set $A_h \subset B_R$, with $\mathrm{Cap}_p(A_h) < 1/h$, such that u^* is continuous in $B_R \setminus A_h$. Thus, if we set

$$\Omega_h := (\{u^* > 0\} \setminus A_h) \cup A_h = \{u^* > 0\} \cup A_h,$$

Ω_h is an open subset of B_R and $\mathrm{Cap}_p(\Omega_h \setminus U) < 1/h$, hence $|\Omega_h \setminus U| \to 0$ as $h \to \infty$. Moreover, since $\mathrm{Cap}_p(\{u_U^* \neq 0\} \setminus \Omega_h) = 0$, we have that

$$E(\Omega_h) \leq \frac{1}{p}\int_{\Omega_h} |\nabla u_U|^p\, dx - \int_{\Omega_h} u_U\, dx = E(U).$$

Then (3.7) follows immediately from this inequality and Lemma 3.2 (ii).

Remark 3.7. Observe that if $U \subset B_R$ is open, setting $U_\varepsilon : \{u_U > \varepsilon\}$, we have $u_{U_\varepsilon} = (u_U - \varepsilon)^+$. Moreover U_ε is smooth for a.e. $\varepsilon > 0$. Therefore we may conclude that there exists a sequence of smooth open sets $U_h \subset U$, such that $|U \setminus U_h| \to 0$, $E(U_h) \to E(U)$ and $u_{U_h} \to u_U$ in $W^{1,p}(B_R)$.

Following the argument of [10, Lemma 4.6] one can prove the existence of a solution to (3.7).

Lemma 3.8. *There exists a constant $\sigma_0(p, n, R) > 0$ such that if $\sigma < \sigma_0$, then the infimum in* (3.7) *is attained by a p-quasi open set U_j. Moreover the perimeter of U_j is uniformly bounded with respect to j.*

Proof. Let $\mathcal{O}_k$ be a sequence minimizing the problem (3.5) such that

$$G_{\eta,\, j}(\mathcal{O}_k) < g_{\eta,\, j} + \frac{1}{k},$$

for any $k \in \mathbb{N}$. By Remark 3.6 we may assume that the sets $\mathcal{O}_k$ are open. Let u_k be the corresponding energy functions. Then by the minimum principle

$$\mathcal{O}_k = \{x \mid u_k(x) > 0\}\,.$$

By letting $t_k = \frac{1}{\sqrt{k}}$, we define

$$V_k = \{x \mid u_k(x) > t_k\}\,.$$

Note that the function $v_k = (u_k - t_k)^+$ is the energy function for V_k, and that

$$G_{\eta,\, j}(\mathcal{O}_k) < g_{\eta,\, j} + \frac{1}{k} \leq G_{\eta,\, j}(V_k) + \frac{1}{k}.$$

Hence

$$\begin{aligned} &\frac{1}{p}\int_{\{u_k>0\}} |\nabla u_k|^p\, dx - \int_{\{u_k>0\}} u_k\, dx + f_\eta(|\{u_k > 0\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\{u_k > 0\}) - \varepsilon_j)^2} \\ &\leq \frac{1}{p}\int_{\{u_k>t_k\}} |\nabla u_k|^p\, dx - \int_{\{u_k>t_k\}} u_k\, dx + f_\eta(|\{u_k > t_k\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\{u_k > t_k\}) - \varepsilon_j)^2} + \frac{1}{k}. \end{aligned}$$

Observe that from Lemma 3.2 (ii) it follows that if $E, F \subset B_R$ then

$$\left|\sqrt{\varepsilon_j^2 + \sigma^2(A(E) - \varepsilon_j)^2} - \sqrt{\varepsilon_j^2 + \sigma^2(A(F) - \varepsilon_j)^2}\right| \leq C_1\sigma|E\Delta F|, \tag{3.8}$$

for some positive constant $C_1(R)$. Therefore, from the above inequality and (3.3), we get

$$\frac{1}{p}\int_{\{0<u_k<t_k\}} |\nabla u_k|^p\,dx + \eta(|\{0<u_k<t_k\}|) \le t_k|\{u_k>0\}| + C_1\sigma|\{0<u_k<t_k\}| + \frac{1}{k}.$$

Choosing σ_0 such that $C_1\sigma_0 \le \frac{\eta}{p}$, then we have

$$\frac{1}{p}\int_{\{0<u_k<t_k\}} |\nabla u_k|^p\,dx + \frac{\eta(p-1)}{p}(|\{0<u_k<t_k\}|) \le t_k|B_R| + \frac{1}{k}.$$

From co-area formula, Young's inequality and the fact that $\eta<1$, one obtains

$$\begin{aligned}\eta\int_0^{t_k} P(\{u_k>s\})\,ds &= \eta\int_{\{0<u_k<t_k\}} |\nabla u_k|\,dx\\ &\le \frac{\eta}{p}\int_{\{0<u_k<t_k\}} |\nabla u_k|^p\,dx + \frac{\eta(p-1)}{p}|\{0<u_k<t_k\}| \le t_k|B_R| + \frac{1}{k},\end{aligned}$$

where $P(\cdot)$ denotes the perimeter of a set. Then there exists $s_k \in (0, \frac{1}{\sqrt{k}})$ such that the set

$$W_k := \{x:\, u_k(x) > s_k\}$$

satisfies

$$P(W_k) \le \frac{2}{t_k}\int_0^{t_k} P(\{u_k>s\})\,ds \le \frac{2}{\eta}|B_R| + \frac{2}{\eta\sqrt{k}}. \tag{3.9}$$

We claim that also W_k is a minimizing sequence. Indeed by (3.3), Lemma 3.2 (ii) and the fact that $(u_k-s_k)^+$ is the energy function of W_k, one gets

$$\begin{aligned}G_{\eta,j}(W_k) &= \frac{1}{p}\int_{\{u_k>s_k\}} |\nabla u_k|^p\,dx - \int_{\{u_k>s_k\}} (u_k-s_k)^+\,dx\\ &\quad + f_\eta(|\{u_k>s_k\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\{u_k>s_k\})-\varepsilon_j)^2}\\ &\le G_{\eta,j}(\mathcal{O}_k) + s_k|\{u_k>0\}| - \eta|\{0<u_k<s_k\}| + C_1\sigma|\{0<u_k<s_k\}|\\ &\le G_{\eta,j}(\mathcal{O}_k) + |B_R|\frac{1}{\sqrt{k}},\end{aligned} \tag{3.10}$$

where we used the assumption that $C_1\sigma \le \frac{\eta}{p}$.

Now by the compactness of equibounded sets with bounded perimeters, see [5, Th. 3.39], (3.9) yields that there exists a set of finite perimeter W_∞ such that $W_k \to W_\infty$ in measure. Set $w_k = (u_k-s_k)^+$. Up to a subsequence, we may assume that there exists a function $w \in W_0^{1,p}(B_R)$ such that w_k weakly converge to $w \in W_0^{1,p}(B_R)$ and strongly in $L^p(B_R)$. Set $W := \{x:\, w(x)>0\}$. Since $W_k = \{w_k>0\}$, it is easily checked that $W \subset W_\infty$ up to a set of zero measure.

We claim that the two sets actually coincide up to a set of measure zero. In fact, by (3.10), the semi-continuity of the norm and the continuity of the set function A, letting $k\to\infty$, one has

$$\begin{aligned}&E(W) + f_\eta(|W_\infty|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(W_\infty)-\varepsilon_j)^2}\\ &\qquad\le \frac{1}{p}\int_W |\nabla w|^p\,dx - \int_W w\,dx + f_\eta(|W_\infty|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(W_\infty)-\varepsilon_j)^2}\\ &\qquad\le g_{\eta,j} \le E(W) + f_\eta(|W|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(W)-\varepsilon_j)^2}.\end{aligned}$$

From this inequality, using (3.8), (3.3) and recalling that $|W\setminus W_\infty| = 0$, we get

$$\eta|W_\infty\setminus W| \le C_1\sigma|W_\infty\setminus W|.$$

Thus, the assumption $C_1\sigma < \eta$ implies that $|W\Delta W_\infty| = 0$. Therefore W is the desired set U_j. $\square$

Next result is the counterpart in our setting of Lemma 4.7 in [10]. It can be proved exactly with the same argument used therein. Therefore, we omit its proof.

Lemma 3.9. *The sequence of minimizers U_j given by Lemma 3.8 satisfies the following properties:*

(i) $|A(U_j) - \varepsilon_j| \le 3\sigma\varepsilon_j$ *and* $||U_j| - |B|| \le C\sigma^4\varepsilon_j$, *for some constant* $C = C(n, R)$;

(ii) up to a translation $U_j \to B$ *in* L^1;

(iii) for all j

$$0 \le F_\eta(U_j) - F_\eta(B) \le \sigma^4\varepsilon_j.$$

In order to prove Proposition 3.4 we need to show that the quasi open sets are U_j are smooth. To explain the strategy of the proof we start by observing that

$$U_j = \{x \mid u_j^*(x) > 0\}$$

where $u_j = u_{U_j}$ is the energy function of U_j. Since U_j minimizes (3.7), we have that for each $v \in W_0^{1,p}(B_R)$

$$\begin{aligned}\frac{1}{p}\int_{B_R}|\nabla u_j|^p\,dx - \int_{B_R} u_j\,dx + f_\eta(|\{u_j > 0\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\{u_j > 0\}) - \varepsilon_j)^2}\\ \le \frac{1}{p}\int_{B_R}|\nabla v|^p\,dx - \int_{B_R} v\,dx + f_\eta(|\{v > 0\}|) + \sqrt{\varepsilon_j^2 + \sigma^2(A(\{v > 0\}) - \varepsilon_j)^2}.\end{aligned} \tag{3.11}$$

The above minimality condition is thus telling us that u_j solves a free boundary type problem. Therefore we are going to adapt to our situation the regularity theory first established, in the case of the Laplacian, by Alt and Caffarelli in [3]. For the p-Laplacian this theory was extended in [12] and to this paper we shall also refer in the following, though the case we are studying here requires a certain number of modifications.

Remark 3.10. Using the minimality property (3.11), (ii) of Lemma 3.2 and (3.3), a standard argument shows that each function u_j is a *quasi-minimum* of the functional

$$\int_{B_R}(|\nabla u|^p + 1)\,dx$$

with zero Dirichlet condition on ∂B_R. More precisely, there exists a constant Q, depending only on p, n, R and η such that for every $v \in W_0^{1,p}(B_R)$ and every j

$$\int_{\mathrm{supp}(u_j - v)}|\nabla u_j|^p\,dx \le Q\int_{\mathrm{supp}(u_j - v)}(|\nabla v|^p + 1)\,dx.$$

Then, the De Giorgi's regularity theory yields that there exists $\alpha \in (0,1)$, depending only on p, n, R, η, such that the sequence u_j is bounded in $C^{0,\alpha}(\overline{B}_R)$, see [18, Th. 7.8].

As a consequence, we have in particular that each U_j is an open set and that

$$-\mathrm{div}(|\nabla u_j|^{p-2}\nabla u_j) = 1 \qquad \text{in } U_j.$$

Observe also that $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \le 1$ in weak sense in the whole $\mathbb{R}^n$. Indeed this is trivially true if U_j is a smooth open set, while in the general case it can be deduced immediately from Remark 3.7.

The first step in the regularity proof of the U_j is to show that when $x \in U_j$ is close to ∂U_j then $u_j(x)$ behaves like the distance from the free boundary ∂U_j. Indeed this estimate readily implies the Lipschitz continuity of u_j, see Theorem 3.13 below. To this aim, we first prove a local estimate and then a global one.

Lemma 3.11. *Let u_j be a function satisfying the minimality inequality* (3.11). *There exists a constant $C(p,n,R)$ such that if $B_r(y) \subset B_R$ and $u_j(y)=0$, then*

$$\sup_{B_{\frac{r}{4}}(y)} u_j \le Cr.$$

Proof. **Step 1.** We argue by contradiction. If the assertion were not true there would exist a sequence u_{j_k} and a sequence of balls $B_{r_k}(y_k) \subset B_R$, with $u_{j_k}(y_k)=0$, such that

$$\sup_{B_{\frac{r_k}{4}}(y_k)} u_{j_k} \ge kr_k.$$

Then, setting

$$u_k(x) := \frac{u_{j_k}(y_k + r_k x)}{r_k} \qquad \text{for } x \in \Omega_k := B_{\frac{R}{r_k}}(-y_k/r_k),$$

from (3.11) it follows that u_k is a minimizer in $W_0^{1,p}(\Omega_k)$ of the functional

$$\frac{1}{p}\int_{\Omega_k} |\nabla v|^p\,dx - r_k\int_{\Omega_k} v\,dx + r_k^{-n} f_\eta(r_k^n|\{v>0\}|) + r_k^{-n}\sqrt{\varepsilon_{j_k}^2 + \sigma^2(A(\Phi_k(\{v>0\})) - \varepsilon_{j_k})^2},$$

where

$$\Phi_k(x) = y_k + r_k x \qquad \text{for all } x \in \Omega_k.$$

Moreover, $u_k(0)=0$ and

$$\sup_{B_{\frac{1}{4}}} u_k \ge k.$$

Let us now set $d_k(x) :=$dist$(x; \{u_k=0\})$ and

$$V_k := \Big\{x \in B : \, d_k(x) \le \frac{1-|x|}{3}\Big\}.$$

Note that since $u_k(0)=0$ we have that $B_{\frac{1}{4}} \subset V_k$. Moreover,

$$m_k := \sup_{x\in V_k}(1-|x|)u_k(x) \ge \frac{3}{4}\sup_{B_{\frac{1}{4}}} u_k \ge \frac{3k}{4}.$$

Since by Remark 3.10 u_k is continuous and $(1-|x|)u_k(x)=0$ on ∂B we have that m_k is attained at some point $x_k \in V_k$. By definition, we have

$$u_k(x_k) = \frac{m_k}{1-|x_k|} \ge m_k \ge \frac{3k}{4}, \qquad \text{and} \qquad \delta_k := d_k(x_k) \le \frac{1-|x_k|}{3}. \tag{3.12}$$

Let now $z_k \in \{u_k=0\} \cap B$ be such that

$$|z_k - x_k| = \delta_k.$$

It is easily checked that $B_{2\delta_k}(z_k) \subset B$. Moreover, $B_{\frac{\delta_k}{2}}(z_k) \subset V_k$, since for any $x \in B_{\frac{\delta_k}{2}}(z_k)$ we have

$$1-|x| \ge 1-|x_k| - |x_k - x| \ge 1-|x_k| - \frac{3}{2}\delta_k \ge \frac{1-|x_k|}{2}.$$

From this inequality, recalling the definition of x_k, we immediately have

$$\sup_{B_{\frac{\delta_k}{2}}(z_k)} u_k \le 2u_k(x_k)\,. \tag{3.13}$$

Next, since $B_{\delta_k}(x_k) \subset \{u_k > 0\}$, we have that $-\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k) = r_k$ in $B_{\delta_k}(x_k)$. Therefore, from the Harnack inequality and since $r_k \le R$, we may conclude that there exist $c_0 = c_0(p,n)$ and $C = C(p,n,R)$ such that, for k large,

$$\sup_{B_{\frac{\delta_k}{4}}(z_k)} u_k \ge \inf_{B_{\frac{4\delta_k}{5}}(x_k)} u_k \ge c_0 \sup_{B_{\frac{4\delta_k}{5}}(x_k)} u_k(x_k) - C \ge \frac{c_0}{2} u_k(x_k), \tag{3.14}$$

where in the last inequality we used the first estimate in (3.12).

Step 2. We now set

$$v_k(x) := \frac{u_k(z_k + \delta_k x/2)}{u_k(x_k)} \qquad \text{for } x \in W_k := \frac{2}{\delta_k}(\Omega_k - z_k).$$

Then, from (3.13) and (3.14) we have that

$$\sup_{B} v_k \le 2, \qquad \sup_{B_{\frac{1}{2}}} v_k \ge \frac{c_0}{2}, \qquad v_k(0) = 0. \tag{3.15}$$

Moreover v_k is a minimizer in $W_0^{1,p}(W_k)$ of the functional

$$\frac{1}{p}\int_{W_k} |\nabla v|^p \, dx - \frac{\delta_k^p r_k}{2^p (u_k(x_k))^{p-1}} \int_{W_k} v \, dx + \frac{2^{n-p}}{\delta_k^{n-p}(u_k(x_k))^p r_k^n} f_\eta(2^{-n}\delta_k^n r_k^n |\{v > 0\}|)$$
$$+ \frac{2^{n-p}}{\delta_k^{n-p}(u_k(x_k))^p r_k^n} \sqrt{\varepsilon_{j_k}^2 + \sigma^2 (A(\Psi_k(\{v > 0\})) - \varepsilon_{j_k})^2},$$

where

$$\Psi_k(x) = y_k + r_k z_k + \frac{r_k \delta_k x}{2} \qquad \text{for all } x \in W_k.$$

Let us now introduce the solution $w_k \in W^{1,p}(B_{\frac{3}{4}})$ of the following Dirichlet problem

$$\begin{cases} -\mathrm{div}(|\nabla w_k|^{p-2}\nabla w_k) = \dfrac{\delta_k^p r_k}{2^p (u_k(x_k))^{p-1}} & \text{in } B_{\frac{3}{4}}, \\ w_k = v_k & \text{on } \partial B_{\frac{3}{4}}. \end{cases}$$

By the maximum principle $w_k > 0$ in $B_{\frac{3}{4}}$. Therefore $\{v_k > 0\}\Delta\{w_k > 0\} = \{v_k = 0\} \cap B_{\frac{3}{4}}$. Thus, by the minimality of v_k, using Lemma 3.2 and (3.3), and recalling that $\delta_k < 1$ and $u_k(x_k) \ge 3k/4$, we have with some simple computations

$$\frac{1}{p}\int_{B_{\frac{3}{4}}} |\nabla v_k|^p \, dx - \frac{\delta_k^p r_k}{2^p (u_k(x_k))^{p-1}} \int_{B_{\frac{3}{4}}} v_k \, dx$$
$$\le \frac{1}{p}\int_{B_{\frac{3}{4}}} |\nabla w_k|^p \, dx - \frac{\delta_k^p r_k}{2^p (u_k(x_k))^{p-1}} \int_{B_{\frac{3}{4}}} w_k \, dx + \frac{C}{k^p},$$

for some positive constant $C = C(p, n, \eta, R)$. From this estimate, setting $v_k^s = sv_k + (1-s)w_k$, for $0 < s < 1$, and testing the equation satisfied by w_k with $v_k - w_k$, we get

$$\begin{aligned}\frac{C}{k^p} &\geq \frac{1}{p}\int_{B_{\frac{3}{4}}} |\nabla v_k|^p\,dx - \frac{1}{p}\int_{B_{\frac{3}{4}}} |\nabla w_k|^p\,dx + \frac{\delta_k^p r_k}{2^p(u_k(x_k))^{p-1}}\int_{B_{\frac{3}{4}}}(w_k - v_k)\,dx \\ &= \int_0^1 \frac{1}{s}\,ds\int_{B_{\frac{3}{4}}}(|\nabla v_k^s|^{p-2}\nabla v_k^s\cdot\nabla(v_k^s - w_k)\,dx + \frac{\delta_k^p r_k}{2^p(u_k(x_k))^{p-1}}\int_{B_{\frac{3}{4}}}(w_k - v_k)\,dx \\ &= \int_0^1 \frac{1}{s}\,ds\int_{B_{\frac{3}{4}}}|\nabla v_k^s|^{p-2}\nabla v_k^s - |\nabla w_k|^{p-2}\nabla w_k)\cdot\nabla(v_k^s - w_k)\,dx. \qquad (3.16)\end{aligned}$$

From this inequality, if $p \geq 2$, recalling (2.18), we conclude that

$$\int_{B_{\frac{3}{4}}} |\nabla v_k - \nabla w_k|^p\,dx \leq \frac{C}{k^p},$$

for a suitable constant independent of k. Therefore, we may conclude that

$$v_k - w_k \to 0 \qquad \text{strongly in } W^{1,p}(B_{\frac{3}{4}}) \text{ as } k \to \infty. \qquad (3.17)$$

If $1 < p < 2$ observe that since the sequence v_k is bounded in $L^\infty(B)$, from the minimality property of v_k it follows that the sequence v_k is also bounded in $W^{1,p}(B_{\frac{3}{4}})$. Hence, also the sequence w_k is bounded in the same space. Then, recalling (2.19) and using (3.16) we immediately conclude that (3.17) holds also in this case.

Finally, by De Giorgi's regularity theorem it follows that the v_k are locally equibounded in $C^{0,\alpha}(B_{\frac{3}{4}})$. Therefore, up to a not relabelled subsequence, we may assume that they converge locally uniformly in $B_{\frac{3}{4}}$ and weakly in $W^{1,p}(B_{\frac{3}{4}})$ to a continuous function $v_0 \in W^{1,p}(B_{\frac{3}{4}})$. Then, from (3.17), we have that also the w_k converge weakly in $W^{1,p}(B_{\frac{3}{4}})$ to v_0. Moreover, by elliptic regularity the sequence w_k is also locally bounded in $C^{1,\beta}(B_{\frac{3}{4}})$ for some $\beta > 0$. Therefore, we may conclude that, up to another not relabelled subsequence, the functions w_k converge locally strongly in $W^{1,p}(B_{\frac{3}{4}})$. This implies that v_0 is a nonnegative function such that

$$\operatorname{div}(|\nabla v_0|^{p-2}\nabla v_0) = 0 \qquad \text{in } B_{\frac{3}{4}}.$$

But from (3.15) we have

$$\sup_{B_{\frac{1}{2}}} v_0 \geq \frac{c_0}{2}, \qquad \text{and} \qquad v_0(0) = 0,$$

which is a contradiction to the maximum principle. This contradiction proves the lemma. ∎

Next step is to improve the previous local estimate so to get a global estimate up to the boundary of B_R. This is the content of the next lemma.

Lemma 3.12. *Let u_j be a function satisfying the minimality inequality* (3.11)*. There exists a constant $C(p, n, R)$ such that if $y \in B_R$, $u_j(y) = 0$ and $0 < r < R$ then*

$$\sup_{B_r(y)} u_j \leq Cr.$$

Proof. **Step 1.** We first prove that there exists a constant $M(p, n, R)$ such that if $r < R/20$, and $z \in B_R$, then

$$⨍_{\partial B_r(z)} u_j\,d\mathcal{H}^{n-1} \geq Mr \qquad \text{implies that } u_j > 0 \text{ in } B_r(z). \qquad (3.18)$$

Fix j. First, observe that if the constant M is large enough then necessarily $B_{10r}(z) \subset B_R$. To see this, recall that since $-\mathrm{div}(|\nabla u_j|^{p-2}\nabla u_j) \leq 1$ from the maximum principle it follows that

$$0 \leq u_j(x) \leq \frac{p-1}{p} \frac{R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}}{n^{\frac{1}{p-1}}} \qquad \text{for all } x \in B_R.$$

Therefore there exists a constant $C_0(p, n, R)$ such that if $B_{10r}(y) \cap \partial B_R \neq 0$, then

$$\fint_{\partial B_r(z)} u_j \, d\mathcal{H}^{n-1} \leq C_0 r.$$

Thus, if we choose the constant M in (3.18) strictly bigger than C_0, inequality (3.18) yields that $B_{10r}(y) \subset B_R$. On the other hand, if there exists a point $x \in B_r(z)$ with $u_j(x) = 0$, by applying Lemma 3.11 to the ball $B_{8r}(x) \subset B_{10r}(z) \subset B_R$ we get that

$$\sup_{B_r(z)} u_j \leq \sup_{B_{2r}(x)} u_j \leq 2Cr,$$

where C is the constant provided in Lemma 3.11. From this inequality we get again a contradiction if we choose $M > 2C$.

Step 2 Fix now $y \in B_R$ such that $u_j(y) = 0$. From the Harnack inequality we know that there exists a constant $\widetilde{C}(p, n, R)$ such that if $r < R/40$

$$\sup_{B_r(y)} u_j \leq \widetilde{C}\left[\fint_{B_{2r}(y)} u \, dx + r^{\frac{p}{p-1}}\right]. \tag{3.19}$$

On the other hand, from (3.18) we have

$$\int_{B_{2r}(y)} u_j \, dx = \int_0^{2r} d\varrho \int_{\partial B_\varrho(y)} u_j \, d\mathcal{H}^{n-1} \leq \frac{n2^{n+1}|B|Mr^{n+1}}{n+1}.$$

Combining this estimate with (3.19) concludes the proof when $r < R/40$. The general case follows from the equiboundedness of the u_j. □

A consequence of Lemma 3.12 is the Lipschitz continuity of the functions u_j. Indeed this can proved arguing exactly as in the proof of [4, Th. 2.3]. Note however that while the Lipschitz estimate proved in [4] is local, in our case it goes up to the boundary of B_R. This global estimate is a consequence of the global estimate proved in Lemma 3.12 and of the fact that u_j satisfies a zero Dirichlet boundary condition on the boundary of a smooth set.

Theorem 3.13. *Let u_j be a function satisfying the minimality inequality* (3.11). *There exists a constant $\widetilde{C} = \widetilde{C}(p, n, R)$ such that for all j*

$$\|\nabla u_j\|_{L^\infty(B_R)} \leq \widetilde{C}.$$

Next step is to prove uniform density estimates for the free boundaries. To this aim we need the following non degeneracy lemma.

Lemma 3.14. *Let u_j be a function satisfying the minimality inequality* (3.11). *There exists $\sigma_0 = \sigma_0(n, p, R)$ such that for any $\gamma > 0$ and $\kappa \in (0, 1)$, there exist $\tau, r_0 > 0$, depending only on γ, κ, p, n, R, such that if $0 < \sigma < \sigma_0$, $y \in B_R$, $0 < r < r_0$ and*

$$\left(\fint_{B_r(y)} u_j^\gamma \, dx\right)^{\frac{1}{\gamma}} \leq \tau r,$$

then $u_j = 0$ in $B_{\kappa r}(y)$.

Proof. For notational simplicity let us just write u in place of u_j since all the constants involved in this proof will be indipendent of j.

Since $-\mathrm{div}(|\nabla u_j|^{p-2}\nabla u_j) \le 1$, from standard elliptic estimate, see for instance [18, Th. 7.3], we have that for every $\kappa \in (0,1)$ there exists a constant $\widetilde{C} = \widetilde{C}(p, n, \kappa, \gamma)$ such that

$$\varepsilon = \frac{1}{\sqrt{\kappa} r} \sup_{B_{\sqrt{\kappa} r}(y)} u \le \widetilde{C}\left[\frac{1}{r}\Big(\fint_{B_r(y)} u^\gamma\Big)^{\frac{1}{\gamma}} + r^{\frac{1}{p-1}}\right] \le \widetilde{C}(\tau + r^{\frac{1}{p-1}}). \tag{3.20}$$

Let w be the solution of

$$\begin{cases} \mathrm{div}(|\nabla w|^{p-2}\nabla w) = -1 & \text{in } B_{\sqrt{\kappa} r}(y) \setminus \overline{B}_{\kappa r}(y) \\ w = \sqrt{\kappa}\varepsilon r & \text{on } \partial B_{\sqrt{\kappa} r}(y) \\ w = 0 & \text{on } \overline{B}_{\kappa r}(y), \end{cases}$$

Define

$$v = \begin{cases} u & \text{on } \mathbb{R}^n \setminus B_{\sqrt{\kappa} r}(y) \\ \min\{u, w\} & \text{on } B_{\sqrt{\kappa} r}(y). \end{cases}$$

Hence $v \in W_0^{1,p}(B_R)$. Note that $w \ge u$ on $\partial B_{\sqrt{\kappa} r}(y)$. By definition of v, we conclude that

$$\{v > 0\} \subset \{u > 0\} \qquad \text{and} \qquad \{v > 0\} \setminus B_{\sqrt{\kappa} r}(y) = \{u > 0\} \setminus B_{\sqrt{\kappa} r}(y).$$

Therefore from the minimality inequality (3.11) and from (3.8),

$$\begin{aligned} &\frac{1}{p}\int_{B_{\sqrt{\kappa} r}(y)} |\nabla u|^p\,dx - \int_{B_{\sqrt{\kappa} r}(y)} u\,dx + f_\eta(|\{u>0\}|) \\ &\le \frac{1}{p}\int_{B_{\sqrt{\kappa} r}(y)} |\nabla v|^p\,dx - \int_{B_{\sqrt{\kappa} r}(y)} v\,dx + f_\eta(|\{v>0\}|) + C_1\sigma|B_{\sqrt{\kappa} r}(y) \cap (\{u>0\}\setminus\{v>0\})|. \end{aligned}$$

Now from (3.3) we infer that

$$\begin{aligned} \frac{1}{p}\int_{B_{\sqrt{\kappa} r}(y)} |\nabla u|^p\,dx - \int_{B_{\sqrt{\kappa} r}(y)} u\,dx + \frac{\eta(p-1)}{p}|(\{u>0\}\setminus\{v>0\}) \cap B_{\sqrt{\kappa} r}(y))| \\ \le \frac{1}{p}\int_{B_{\sqrt{\kappa} r}(y)} |\nabla v|^p\,dx - \int_{B_{\sqrt{\kappa} r}(y)} v\,dx, \end{aligned}$$

where, as in the proof of Lemma 3.8, we have chosen $\sigma_0 > 0$ so that $C_1\sigma_0 \le \frac{\eta}{p}$. Hence, noting that

$$\{u>0\} \cap B_{\kappa r}(y) = (\{u>0\}\setminus\{v>0\}) \cap B_{\kappa r}(y),$$

we get

$$\begin{aligned} &\frac{1}{p}\int_{B_{\kappa r}(y)} |\nabla u|^p\,dx - \int_{B_{\kappa r}(y)} u\,dx + \frac{\eta(p-1)}{p}|\{u>0\} \cap B_{\kappa r}(y)| \\ &\quad \le \frac{1}{p}\int_{B_{\kappa r}(y)} |\nabla u|^p\,dx - \int_{B_{\kappa r}(y)} u\,dx + \frac{\eta(p-1)}{p}|(\{u>0\}\setminus\{v>0\}) \cap B_{\sqrt{\kappa} r}(y)| \\ &\quad \le \frac{1}{p}\int_{B_{\sqrt{\kappa} r}(y)\setminus B_{\kappa r}(y)} (|\nabla v|^p - |\nabla u|^p)\,dx - \int_{B_{\sqrt{\kappa} r}(y)\setminus B_{\kappa r}(y)} (v-u)\,dx \\ &\quad \le \int_{B_{\sqrt{\kappa} r}(y)\setminus B_{\kappa r}(y)\cap\{u>w\}} |\nabla w|^{p-2}\nabla w\cdot(\nabla w - \nabla u)\,dx - \int_{B_{\sqrt{\kappa} r}(y)\setminus B_{\kappa r}(y)\cap\{u>w\}} (w-u)\,dx \\ &\quad = \int_{\partial B_{\kappa r}(y)} |\nabla w|^{p-2} u\partial_\nu w\,d\mathcal{H}^{n-1}. \end{aligned}$$

Observe that the last equality is obtained by testing the equation satisfied by w with $(u-w)^+$, and recalling that the boundary value of w on $\partial B_{\kappa r}(y)$ is zero, and $u \le w$ on $\partial B_{\sqrt{\kappa} r}(y)$. Note also that in the last integral ν denotes the exterior normal to $\partial B_{\kappa r}(y)$.

An explicit calculation gives that

$$w(x) = \varphi(|x-y|), \quad \text{where} \quad \varphi(\varrho) := \int_{\kappa r}^{\varrho} \Big|\frac{c_0}{t^{n-1}} - \frac{t}{n}\Big|^{\frac{2-p}{p-1}} \Big(\frac{c_0}{t^{n-1}} - \frac{t}{n}\Big)\, dt \quad \text{for } \varrho \in [\kappa r, \sqrt{\kappa} r],$$

where the constant c_0 is chosen so that $\varphi(\sqrt{\kappa} r) = \sqrt{\kappa}\varepsilon r$. Now, denote by ϱ a point in $[\kappa r, \sqrt{\kappa} r]$ such that

$$\sqrt{\kappa}\varepsilon r = \varphi(\sqrt{\kappa} r) = \varphi'(\varrho)(\sqrt{\kappa} r - \kappa r)$$

and observe that, using this equality, we may estimate for $x \in \partial B_{\kappa r}(y)$

$$\begin{aligned} |\nabla w(x)| &= \Big|\frac{c_0}{\kappa^{n-1} r^{n-1}} - \frac{\kappa r}{n}\Big|^{\frac{1}{p-1}} \le C(\kappa, p, n)\Big[\Big|\frac{c_0}{\varrho^{n-1}} - \frac{\varrho}{n}\Big|^{\frac{1}{p-1}} + r^{\frac{1}{p-1}}\Big] \\ &= C\big(\varphi'(\varrho) + r^{\frac{1}{p-1}}\big) \le C(\kappa, p, n)\big(\varepsilon + r^{\frac{1}{p-1}}\big). \end{aligned}$$

Hence,

$$\frac{1}{p}\int_{B(y,\kappa r)} |\nabla u|^p\, dx + \frac{\eta(p-1)}{p}|\{u>0\} \cap B_{\kappa r}(y)| \le C\big(\varepsilon^{p-1} + r\big)\int_{\partial B_{\kappa r}(y)} u\, d\mathcal{H}^{n-1} + \int_{B_{\kappa r}(y)} u\, dx.$$

Moreover the trace inequality and Young's inequality give

$$\begin{aligned} \int_{\partial B_{\kappa r}(y)} u\, d\mathcal{H}^{n-1} &\le C(\kappa,\, p,\, n)\Big(\frac{1}{r}\int_{B_{\kappa r}(y)} u\, dx + \int_{B_{\kappa r}(y)} |\nabla u|\, dx\Big) \\ &\le C\bigg(\Big(\sqrt{\kappa}\varepsilon + \frac{(p-1)}{p}\Big)|\{u>0\} \cap B_{\kappa r}(y)| + \frac{1}{p}\int_{B_{\kappa r}(y)} |\nabla u|^p\, dx\bigg), \end{aligned}$$

where we used that

$$\int_{B_{\kappa r}(y)} u\, dx \le \sqrt{\kappa}\varepsilon r|\{u>0\} \cap B_{\kappa r}(y)|.$$

Therefore, recalling that $\eta < 1$, we may conclude that

$$\begin{aligned} &\eta \min\Big\{\frac{1}{p}, \frac{p-1}{p}\Big\}\Big(\int_{B_{\kappa r}(y)} |\nabla u|^p\, dx + |\{u>0\} \cap B_{\kappa r}(y)|\Big) \\ &\le C_0(\varepsilon^{p-1} + r)(\varepsilon + 1)\bigg(\int_{B_{\kappa r}(y)} |\nabla u|^p\, dx + |\{u>0\} \cap B_{\kappa r}(y)|\bigg), \end{aligned}$$

for a suitable constant $C_0(p, n, \kappa)$. Then, recalling (3.20), if τ and r are chosen so that

$$C_0(\varepsilon^{p-1} + r)(\varepsilon + 1) \le \frac{\eta}{2}\min\Big\{\frac{1}{p}, \frac{p-1}{p}\Big\},$$

we conclude that $u = 0$ in $B_{\kappa r}(y)$. □

The following lemma is a simple, but essential tool for the subsequent analysis of the regularity of the free boundaries ∂U_j.

Lemma 3.15. *Let u_j be a sequence satisfying* (3.11)*, j_k a sequence of positive integers and $B_{r_k}(y_k)$ a sequence of balls such that $y_k \in \partial U_{j_k}$ and $r_k \to r_0$ as $k \to \infty$. Set*

$$u_k(x) := \frac{u_{j_k}(y_k + r_k x)}{r_k} \qquad \textit{for } x \in \Omega_k := B_{\frac{R}{r_k}}(-y_k/r_k). \tag{3.21}$$

Then, up to a subsequence, the functions u_k converge uniformly in B and weakly in $W^{1,\infty}(B)$ to a function u_0 such that $-\mathrm{div}(|\nabla u_0|^{p-1}\nabla u_0)=r_0$ in $B\cap\{u_0>0\}$ and $0\in\partial\{u_0>0\}$. Moreover,*

$$|\nabla u_k|^{p-2}\nabla u_k\to|\nabla u_0|^{p-2}\nabla u_0 \qquad \textit{locally uniformly in } B\cap\{u_0>0\}.$$

Proof. First we observe that

$$-\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k)=r_k \qquad \text{in } B\cap\{u_k>0\}. \tag{3.22}$$

Since by Theorem 3.13 the sequence u_j is equi-Lipschitz in B_R, the rescaled sequence u_k is equi-Lipschitz in B. Therefore, up to a not relabelled subsequence, it converges uniformly in B and weakly* in $W^{1,\infty}(B)$ to some function u_0. If V is an open set compactly contained in $\{u_0>0\}$ then for k large we have also that $V\subset\subset\{u_k>0\}$. From the equation (3.22) satisfied by u_k and standard elliptic estimates we have that the sequence u_k is equibounded in $C^{1,\alpha}(\overline{V})$. Therefore, $|\nabla u_k|^{p-2}\nabla u_k\to|\nabla u_0|^{p-2}\nabla u_0$ uniformly in V. The fact that $-\mathrm{div}(|\nabla u_0|^{p-1}\nabla u_0)=r_0$ in $B\cap\{u_0>0\}$ then follows from the strong convergence of the gradients and from (3.22). □

The proof of next result goes exactly as the proof of Theorem 4.4 in [12] with the obvious modification that in our case when we rescale from a ball $B_{r_k}(y_k)$ to B and replace u_{j_k} by the function u_k as in (3.21), then $-\mathrm{div}(|\nabla u_k|^{p-2}\nabla u_k)\le r_k$ in B and thus we have to compare u_k with the function v_k which agrees with u_k on ∂B and solves the equation $-\mathrm{div}(|\nabla v_k|^{p-2}\nabla v_k)=r_k$ in B (see the estimate performed in Step 2 of the proof of of Lemma 3.11). Note that, thanks to the global Lipschitz estimate provided by Theorem 3.13 and to the global estimate stated in Lemma 3.14, the density estimates (3.23) hold uniformly on $\partial U_j\cap\overline{B}_R$.

Theorem 3.16. *Let u_j be a function satisfying the minimality inequality* (3.11)*. There exist a constant $C=C(n,p,R)>1$ such that the following properties hold:*

$$\frac{1}{C}\,\mathrm{dist}\,(x,\,\partial U_j)\le u_j(x)\le C\,\mathrm{dist}\,(x,\,\partial U_j) \qquad \textit{for every } x\in B_R;$$

for every $x\in\partial U_j$ and $0<r\le R$

$$\frac{1}{C}\le\frac{|U_j\cap B_r(x)|}{|B_r|}\le 1-\frac{1}{C}. \tag{3.23}$$

If $E\subset\mathbb{R}^n$ is a measurable set we denote by $\partial^M E$ its *measure theoretic boundary*, i.e., the complement in $\mathbb{R}^n$ of the set of points where the density of E is either 0 or 1. Moreover, if E has finite perimeter, $\partial^* E$ denotes the *reduced boundary* of E, i.e., the set of points of ∂E where the *generalized exterior normal* ν_E exists (for the precise definition see [5, Ch. 3] or [23, Ch. 15]).

Remark 3.17. From Lemma 3.8 we know that each U_j is a set of finite perimeter. Moreover U_j is also open by Remark 3.10. Observe that (3.23) implies that the topological boundary of U_j coincides with its measure theoretic boundary $\partial^M U_j$. Therefore, by a well known property of sets of finite perimeter, see [5, Th. 3.61], we may conclude that

$$\mathcal{H}^{n-1}(\partial U_j\setminus\partial^* U_j)=0 \qquad \text{for all } j. \tag{3.24}$$

Next result is the counterpart in our case of Lemma 5.4 and Theorems 5.2, 5.5 and 5.6 of [12]. Though the proofs of these results are a bit long and require various intermediate results, the arguments used therein are based only on Lemma 3.14, Theorem 3.13 and the convergence of the rescaled functions u_k defined in (3.22) to a p-harmonic function u_0 when $r_k\to 0$, see Lemma 3.15. Those proofs can be reproduced in our setting without any substantial change and therefore we do not give the details here.

Theorem 3.18. *Let u_j be a function satisfying the minimality inequality* (3.11). *There exists a nonnegative Borel function q_{u_j} such that*

$$-\mathrm{div}(|\nabla u_j(x)|^{p-2}\nabla u_j(x)) = \chi_{U_j}(x) - q_{u_j}(x)^{p-1}\mathcal{H}^{n-1} \llcorner \partial U_j \qquad \text{in } B_R$$

and for all $x \in \partial U_j$

$$\frac{1}{C} \le q_{u_j}(x) \le C, \qquad \limsup_{z\to x, u_j(z)>0} |\nabla u_j(z)| = q_{u_j}(x)\,, \tag{3.25}$$

where $C = C(p, n, R) > 1$. Moreover, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U_j$, we have that

$$\frac{u_j(x+rz)}{r} \to q_{u_j}(x)(-z\cdot\nu_{U_j}(x))_+ \qquad \text{in } W^{1,q}_{loc}(\mathbb{R}^n) \text{ for all } q \ge 1, \tag{3.26}$$

where ν_{U_j} is the generalized exterior normal to the boundary of U_j.

Assume that ∂U_j is of class C^1 in a neighborhood of x and that $u_j \in C^1(\overline{U}_j)$. Then (3.26) states that $q_{u_j}(x) = |\nabla u_j(x)| = -\nabla u_j(x)\cdot\nu_{U_j}(x)$; moreover, one can show that the Euler–Lagrange equation satisfied at ∂U_j is

$$(p-1)\Big|\frac{\partial u_j(x)}{\partial \nu_{U_j}}\Big|^p - \frac{\sigma^2(A(U_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(A(U_j)-\varepsilon_j)^2}}\left[|x-x_{U_j}| - \fint_{U_j}\frac{(y-x_{U_j})\cdot x_1}{|y-x_{U_j}|}\,dy\right] = c(j),$$

for some constant $c(j)$. Thus, we proceed as in [10, Sect. 4] proving that the above formula holds indeed when replacing the absolute value of the normal derivative of u_j at the boundary by q_{u_j}. This is precisely the content of the next lemma.

Lemma 3.19. *Let $R \ge 2$ and let u_j satisfy the minimality inequality* (3.11). *Set for every $x \in \partial U_j$*

$$\Lambda_j(x) := (p-1)(q_{u_j}(x))^p - \frac{\sigma^2(A(U_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(A(U_j)-\varepsilon_j)^2}}\left[|x-x_{U_j}| - \fint_{U_j}\frac{(y-x_{U_j})\cdot x}{|y-x_{U_j}|}\,dy\right]. \tag{3.27}$$

There exists $j_0 = j_0(R)$ such that Λ_j is constant for all $j > j_0$.

Before proving this important lemma we need another technical result.

Lemma 3.20. *Let $R \ge 2$ and let u_j satisfy the minimality inequality* (3.11). *Up to a not relabelled subsequence and after a suitable translation, the sets U_j converge in L^1 to the unit ball B. Furthermore, for every $0 < 2\delta < R-1$ there exists $j_\delta > 0$ such that for $j \ge j_\delta$*

$$B_{1-\delta} \subset U_j \subset B_{1+\delta} \tag{3.28}$$

and the energy function u_j satisfies (3.11) *for every $v \in W^{1,p}_0(\mathcal{N}_\delta(U_j))$.*

This result is a simple consequence of Lemma 3.9 and (3.23). Its proof is exactly as the proof of Lemmas 4.13 and 4.14 of [10] and therefore we omit it. Let us instead give some more details of the proof of the previous lemma, which is anyway very similar to the one of Lemma 4.15 in [10].

Proof of Lemma 3.19. We choose $\delta < \frac{R-1}{4}$ and fix $j \ge j_\delta$, where j_δ is as in Lemma 3.20. As before we drop the subscript and simply write u and U instead of u_j and U_j, respectively. Moreover, up to a suitable translation, we may always assume that the barycenter x_U is at the origin.

Let us recall, see for instance [5, Th. 3.59], that since U is a set of finite perimeter, for every $x \in \partial^* U$ we have that

$$\frac{U-x}{\varrho} \to \{\nu_U(x)\cdot y \le 0\} \qquad \text{locally in } L^1 \text{ as } \varrho \to 0^+, \tag{3.29}$$

where ν_U is the generalized exterior normal to ∂U. Thus, recalling (3.24), in order to prove the assertion it is enough to show that for every $x_1, x_2 \in \partial^* U$ for which (3.26) holds we have

$$\Lambda_j(x_1) = \Lambda_j(x_2).$$

To this aim we argue as in the proof of [10, Lemma 4.15]. Therefore, since most of the calculations are exactly as in [10] we shall only indicate the changes that are needed.

We argue by contradiction assuming that $\Lambda_j(x_1) < \Lambda_j(x_2)$ and showing that this inequality violates the minimality of u. To this aim we fix a nonnegative radially symmetric smooth function $\phi(|x|)$, compactly supported in B, and define for τ, ϱ small

$$\Phi_{\tau,\varrho}(x) = \begin{cases} x - \tau\varrho\phi\Big(\dfrac{|x-x_1|}{\varrho}\Big)\nu_U(x_1) & \text{for } x \in B_\varrho(x_1) \\ x + \tau\varrho\phi\Big(\dfrac{|x-x_2|}{\varrho}\Big)\nu_U(x_2) & \text{for } x \in B_\varrho(x_2) \\ x & \text{otherwise.} \end{cases}$$

The reader should be pay attention to the fact that here, differently than in [10], by ν_U we denote the exterior normal to ∂U. If τ, ϱ are small enough, the mapping $\Phi_{\tau,\varrho}$ is a diffeomorphism. Moreover by Lemma 3.20, if τ is small the function

$$u_{\tau,\varrho} := u \circ \Phi_{\tau,\varrho}^{-1}$$

is an admissible function for (3.11). Notice that

$$U_{\tau,\varrho} := \{x \mid u_{\tau,\varrho} > 0\} = \Phi_{\tau,\varrho}(U).$$

We are now going to prove that for ϱ and τ small

$$G_{\eta,j}(U_{\tau,\varrho}) - G_{\eta,j}(U) < 0, \tag{3.30}$$

where $G_{\eta,j}$ is defined as in (3.6). This inequality will contradict the minimality of U, thus proving the lemma.

In order to prove (3.30) we estimate all the terms appearing in the functional $G_{\eta,j}$. First, by repeating exactly the calculations in the proof of [10, Lemma 4.15] we obtain, see [10, (4.37)],

$$\lim_{\varrho\to 0} \frac{||U_{\tau,\varrho}| - |U||}{\varrho^n} = o(\tau).$$

Therefore by by (3.3) we have that

$$\limsup_{\varrho\to 0} \frac{1}{\varrho^n}|f_\eta(|U_{\tau,\varrho}|) - f_\eta(|U|)| \le \lim_{\varrho\to 0} \frac{1}{\eta}\frac{||U_{\tau,\varrho}| - |U||}{\varrho^n} = o(\tau). \tag{3.31}$$

Concerning the gradient term we have, by change of variable,

$$\frac{1}{\varrho^n}\left[\int_{U_{\tau,\varrho}}|\nabla u_{\tau,\varrho}|^p\,dx-\int_U|\nabla u|^p\,dx\right]=\frac{1}{\varrho^n}\left[\int_U\left(|((\nabla\Phi_{\tau,\varrho})^{-1})^T\circ\nabla u|^p\det\nabla\Phi_{\tau,\rho}-|\nabla u|^p\right)dx\right]$$

$$\begin{aligned}&=-\tau\int_{\frac{U-x_1}{\varrho}\cap B}\Bigg[|\nabla u(x_1+\varrho y)|^p\phi'(|y|)\frac{y\cdot\nu_U(x_1)}{|y|}\\&\qquad-p|\nabla u(x_1+\varrho y)|^{p-2}\phi'(|y|)\frac{(\nabla u(x_1+\varrho y)\cdot y)(\nabla u(x_1+\varrho y)\cdot\nu_U(x_1))}{|y|}\Bigg]dy\\&\quad+\tau\int_{\frac{U-x_2}{\varrho}\cap B}\Bigg[|\nabla u(x_2+\varrho y)|^p\phi'(|y|)\frac{y\cdot\nu_U(x_2)}{|y|}\\&\qquad-p|\nabla u(x_2+\varrho y)|^{p-2}\phi'(|y|)\frac{(\nabla u(x_2+\varrho y)\cdot y)(\nabla u(x_2+\varrho y)\cdot\nu_U(x_2))}{|y|}\Bigg]dy+o(\tau).\end{aligned}$$

Thus, letting $\rho\to0$, recalling (3.26) and (3.29), and using the divergence theorem, we get

$$\begin{aligned}\lim_{\varrho\to0}\frac{1}{\varrho^n}\left[\int_{U_{\tau,\varrho}}|\nabla U_{\tau,\varrho}|^p\,dx-\int_U|\nabla u|^p\,dx\right]&=\tau(p-1)\Bigg(\int_{\{y\cdot\nu_U(x_1)\le0\}\cap B}|q_u(x_1)|^p\phi'(|y|)\frac{\nu_U(x_1)\cdot y}{|y|}\\&\qquad-\int_{\{y\cdot\nu_U(x_2)\le0\}\cap B}|q_u(x_2)|^p\phi'(|y|)\frac{\nu_U(x_2)\cdot y}{|y|}\Bigg)+o(\tau)\\&=\tau(p-1)\left(|q_u(x_1)|^p-|q_u(x_2)|^p\right)\int_{\{y_n=0\}\cap B}\phi(|y|)+o(\tau).\end{aligned}\tag{3.32}$$

A similar and actually simpler argument leads to the estimate of the L^1 term

$$\begin{aligned}&\lim_{\varrho\to0}\frac{1}{\varrho^n}\left[\int_{U_{\tau,\varrho}}u_{\tau,\varrho}\,dx-\int_U u\,dx\right]=\lim_{\varrho\to0}\frac{1}{\varrho^n}\int_U u(\det\nabla\Phi_{\tau,\rho}-1)\,dx\\&=-\tau\Bigg(\int_{\{y\cdot\nu_U(x_1)\le0\}\cap B}u(x_1)\phi'(|y|)\frac{y\cdot\nu_U(x_1)}{|y|}-\int_{\{y\cdot\nu_U(x_2)\le0\}\cap B}u(x_2)\phi'(|y|)\frac{y\cdot\nu_U(x_2)}{|y|}\Bigg)+o(\tau)\\&=o(\tau),\end{aligned}\tag{3.33}$$

where in the last equality we used the fact that $u(x_1)=u(x_2)=0$.

Concerning the asymptotic behaviour of the barycenter and of the asymmetry, the calculations are longer, but they are exactly as the ones made in [10], see (4.42) therein. One has

$$\lim_{\varrho\to0}\frac{A(U_{\tau,\varrho})-A(U)}{\varrho^n}=-\tau\left(|x_1|-|x_2|-(x_1-x_2)\cdot\fint_U\frac{y}{|y|}\,dy\right)\int_{\{y_n=0\}\cap B}\phi(|y|)+o(\tau).$$

From this inequality and the estimates (3.31), (3.32) and (3.33), recalling the definition (3.27) of Λ_j we conclude that

$$\lim_{\varrho\to0}\frac{G_{\eta,j}(U_{\tau,\rho})-G_{\eta,j}(U)}{\varrho^n}\le\tau(\Lambda_j(x_1)-\Lambda_j(x_2))\int_{\{y_n=0\}\cap B}\phi(|y|)\,dy+o(\tau).$$

This inequality immediately yields (3.30) since $\Lambda_j(x_1)<\Lambda_j(x_2)$. And this concludes the proof. ☐

Remark 3.21. Observe that by Lemma 3.20 we have that for j large

$$|x-x_{U_j}|\ge\frac{1}{2}\qquad\text{for }x\in\partial U_j.$$

Moreover

$$\frac{\sigma^2(A(U_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(A(U)-\varepsilon_j)^2}}\left[|x-x_{U_j}|-\fint_{U_j}\frac{(y-x_{U_j})\cdot x}{|y-x_{U_j}|}\,dy\right]\leq C(n,\,R)\sigma$$

when $j \geq j_\delta$. Then, if σ is small, by Lemma 3.18 Λ_j is uniformly bounded from above and away from zero . Therefore there exists a constant $\sigma_0(p,\,n,\,R)$ such that when $\sigma < \sigma_0$, the functions

$$q_{u_j}(x)=\left[\frac{1}{p-1}\left(\Lambda_j+\frac{\sigma^2(A(U_j)-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma^2(A(U_j)-\varepsilon_j)^2}}\left[|x-x_{U_j}|-\fint_{U_j}\frac{(y-x_{U_j})\cdot x}{|y-x_{U_j}|}\,dy\right]\right)\right]^{\frac{1}{p}}$$

are all smooth in some neighborhood $\mathcal{N}_\delta(U_j)$, for some $\delta > 0$ independent of j. Moreover, given $k \geq 1$, all the C^k norms of the q_{u_j} are bounded independently of j.

Following [12], we may now state the regularity property of the sets ∂U_j. To this aim, we recall the following definition originally given in [3, Definition 7.1], see also [12, Definition 6.1].

Definition 3.22. *Let $0 \leq \sigma_-,\, \sigma_+ \leq 1$ and $\tau > 0$ and let $u \in W_0^{1,p}(B_R)$ be a continuous weak solution of the equation*

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u)=\chi_U-q_u^{p-1}\mathcal{H}^{n-1}\llcorner\partial U, \tag{3.34}$$

where $U = \{u > 0\}$ is a set of finite perimeter, and $q_u : \partial U \to (0,\infty)$ a continuous function. We say that u belongs to the class $F(\sigma_+,\,\sigma_-;\,\tau)$ in $B_r(x_0)$ with respect to the direction $\nu \in \mathbb{S}^{n-1}$ if $B_r(x_0) \subset B_R$ and the following conditions hold:

(i) $x_0 \in \partial\{u > 0\}$;
(ii) for $(x - x_0)\cdot\nu \geq \sigma_+ r$, we have $u(x) = 0$;
(iii) for $(x - x_0)\cdot\nu \leq -\sigma_- r$, we have $u(x) \geq -q_u(x_0)((x - x_0)\cdot\nu + \sigma_- r)$;
(iv) we have $|\nabla u(x)| \leq q_u(x_0)(1 + \tau)$, for all $x \in B_r(x_0)$ and $\mathrm{osc}_{B_r(x_0)} q_u \leq \tau q_u(x_0)$.

Next result deals with the regularity of the sets U_j.

Theorem 3.23. *Let $u \in W_0^{1,p}(B_2)$ be a Lipschitz continuous weak solution of* (3.34)*, such that*

$$U := \{x \in B_R :\ u > 0\} \subset B_{3/2}$$

and let $q_u : \partial U \to (0,\infty)$ be Lipschitz continuous. There exist positive constants $\alpha,\,\beta,\,\omega_0,\,\tau_0$ and C, depending only on $p, n, \min q_u, \|q_u\|_{W^{1,\infty}}$ and $\|u\|_{W^{1,\infty}}$, such that if $u \in F(\omega,\,1;\,+\infty)$ in $B_\varrho(x_0) \subset B_R$ with respect to some direction $\nu \in \mathbb{S}^{n-1}$, $\omega \leq \omega_0$ and $\varrho \leq \tau_0\omega^\beta$, then there exists a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, with $\|f\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \leq C\omega$ and

$$\partial U_j \cap B_{\frac{\varrho}{4}}(x_0) = (x_0 + \mathrm{graph}_\nu f) \cap B_{\frac{\varrho}{4}}(x_0),$$

where

$$\mathrm{graph}_\nu f = \{x \in \mathbb{R}^n :\ x\cdot\nu = f(x - (x\cdot\nu)\nu)\}.$$

Moreover, if $q_u \in C^{k,\gamma}(\mathcal{N}_\delta(\partial U))$ for some integer $k \geq 1$, $\delta > 0$ and $\gamma \in (0,1)$, there exists a constant C_k, depending on all the previous quantities and also on k, δ, γ and $\|q_u\|_{C^{k,\gamma}}$, such that $f \in C^{k,\gamma}(\mathbb{R}^{n-1})$ and $\|f\|_{C^{k,\gamma}(\mathbb{R}^{n-1})} \leq C_k$.

For the proof of this result see the Appendix.

Proof of Proposition 3.4. Thanks to Lemma 3.20 and to Theorem 3.23, the proof goes exactly as the proof of Proposition 4.4 in [10]. □

4. The proof of the main theorem

In this section we prove Theorem 1.1. To this aim, following [10], we first prove a quantitative inequality for $E(\Omega)$, namely that there exists a constant γ, depending only on p and n, such that for any open set Ω with $|\Omega| = |B|$

$$E(\Omega) - E(B) \geq \gamma \mathcal{A}(\Omega)^2,$$

where $\mathcal{A}(\Omega)$ is defined as in (1.2). Then we obtain (1.8) by combining this inequality with the following extension of the *Kohler-Jobin inequality* proved in [8, Th. 1.1].

Theorem 4.1. *Let $n \geq 2$, $p > 1$ and $q \geq 1$ an exponent satisfying* (1.6)*. Then for any open set of finite measure Ω one has*

$$\frac{\lambda_{p,q}(\Omega)}{\lambda_{p,q}(B)} \geq \left(\frac{E(B)}{E(\Omega)}\right)^{\alpha},$$

where $\alpha = (np + pq - nq)(p-1)/[q(np + p - n)]$.

The proofs in this section follow very closely those given in [10] with a few technical changes. We start with a lemma which provides an estimate of the energy function in Ω and of its gradient outside a ball in terms of the measure of Ω outside a smaller ball.

Lemma 4.2. *Let Ω be an open set with $|\Omega| = |B|$ and let $u_\Omega \in W_0^{1,p}(\Omega)$ be its energy function. There exists a constant $C = C(p, n)$ such that for every $R \geq 1$ we have*

$$\|u_\Omega\|_{L^\infty(\Omega \setminus B_{R+1})} \leq C|\Omega \setminus B_R|^{\frac{1}{n}}, \qquad \int_{\Omega \setminus B_{R+1}} |\nabla u_\Omega|^p \, dx \leq C|\Omega \setminus B_R|^{1+\frac{1}{n}}. \tag{4.1}$$

Proof. The conclusion is trivial if $|\Omega \setminus B_R| = 0$, and hence we may assume that $|\Omega \setminus B_R| > 0$. We set $R_k = R + 1 - 2^{-k}$ for $k \in \mathbb{N}$ and let

$$\varphi_k(x) = \phi_k(|x|)$$

where $\phi_k(t) = 0$ if $t \in [0, R_{k-1}]$, $\phi_k(t) = 1$ if $t \geq R_k$ and in $[R_{k-1}, R_k]$ ϕ_k is the affine function connecting the values 0 and 1. We also set

$$s_k = M|\Omega \setminus B_R|^{\frac{1}{n}}(1 - 2^{-k}),$$

where M will be a constant depending only on n and p which will be chosen later.

Plugging $\varphi_k^p(u_\Omega - s_k)^+$ into the equation satisfied by u_Ω, we get

$$\int_\Omega |\nabla u_\Omega|^{p-2} \nabla u_\Omega \nabla(\varphi_k^p(u_\Omega - s_k)^+) \, dx = \int_\Omega \varphi_k^p(u_\Omega - s_k)^+ \, dx.$$

From this equality, a simple use of the Young's inequality yields

$$\int_{\{u_\Omega > s_k\}} |\nabla u_\Omega|^p \varphi_k^p \, dx \leq C(p) \int_{\{u_\Omega > s_k\}} \big((u_\Omega - s_k)^p |\nabla \varphi_k|^p + \varphi_k^p(u_\Omega - s_k)\big) \, dx. \tag{4.2}$$

From a well known elliptic estimate, see [26, Th. 1], we have

$$\|u_\Omega\|_{L^\infty(\Omega)} \leq \|u_B\|_{L^\infty(B)} \leq C(p, n).$$

Since $|\nabla\varphi_k|\leq 2^k$ and $0\leq\varphi_k\leq 1$, by a well known variant of Poincaré inequality, see [18, Cor. 3.1], we get from (4.2) that

$$\begin{aligned}\int_\Omega(\varphi_k(u_\Omega-s_k)^+)^p\,dx&\leq C\left|\{\varphi_k(u_\Omega-s_k)^+>0\}\right|^{\frac{p}{n}}\int_\Omega|\nabla(\varphi_k(u_\Omega-s_k)^+)|^p\,dx\\&\leq C\left|\{\varphi_k(u_\Omega-s_k)^+>0\}\right|^{\frac{p}{n}}\int_{\{u_\Omega>s_k\}}\big((u_\Omega-s_k)^p|\nabla\varphi_k|^p+\varphi_k^p|\nabla u_\Omega|^p\big)\,dx\\&\leq C\left|\{\varphi_k(u_\Omega-s_k)^+>0\}\right|^{\frac{p}{n}}\int_{\{u_\Omega>s_k\}}\big((u_\Omega-s_k)^p|\nabla\varphi_k|^p\,dx+\varphi_k^p(u_\Omega-s_k)\big)\,dx\\&\leq C2^{pk}\left|\{\varphi_k(u_\Omega-s_k)^+>0\}\right|^{1+\frac{p}{n}},\end{aligned}\tag{4.3}$$

for some constant $C=C(p,n)$. Observe that

$$\{(u_\Omega-s_{k+1})^+>0\}\cap(\Omega\setminus B_{R_k})\subset\{(u_\Omega-s_k)^+>s_{k+1}-s_k\}\cap(\Omega\setminus B_{R_k}),$$

and that

$$s_{k+1}-s_k=2^{-k-1}M|\Omega\setminus B_R|^{\frac{1}{n}}.$$

Therefore we conclude that

$$\begin{aligned}|\{(u_\Omega-s_{k+1})^+>0\}\cap(\Omega\setminus B_{R_k})|&\leq|\{(u_\Omega-s_k)^+>s_{k+1}-s_k\}\cap(\Omega\setminus B_{R_k})|\\&\leq\frac{2^{p(k+1)}}{M^p|\Omega\setminus B_R|^{\frac{p}{n}}}\int_\Omega(\varphi_k(u_\Omega-s_k)^+)^p\,dx\\&\leq C_0\frac{2^{pk}}{M^p|\Omega\setminus B_R|^{\frac{p}{n}}}|\{(u_\Omega-s_k)^+>0\}\cap(\Omega\setminus B_{R_{k-1}})|^{1+\frac{p}{n}},\end{aligned}$$

for a suitable constant $C_0=C_0(p,n)$. If for any $k\in\mathbb{N}$ we set

$$a_k=\frac{|\{(u_\Omega-s_k)^+>0\}\cap(\Omega\setminus B_{R_{k-1}})|}{|\Omega\setminus B_R|}\leq 1,$$

then we have

$$a_{k+1}\leq C_0\frac{2^{pk}}{M^p}a_k^{1+\frac{p}{n}}.$$

From this inequality, choosing M so that $C_0/M^p=2^{-n-p}$and arguing by induction we easily get that $a_k\leq 2^{-n(k-1)}$ for all $k\geq 1$ and thus

$$\lim_{k\to\infty}a_k=0.$$

Therefore, setting $s_\infty=M|\Omega\setminus B_R)|^{\frac{1}{n}}$ we conclude that

$$|\{(u_\Omega-s_\infty)^+>0\}\cap(\Omega\setminus B_{R+1})|=0,$$

thus proving the first inequality in (4.1).

To get the second inequality in (4.1) observe that the same argument as above shows that the following stronger inequality also holds

$$\|u_\Omega\|_{L^\infty(\Omega\setminus B_{R+1/2})}\leq C|\Omega\setminus B_R|^{\frac{1}{n}},\tag{4.4}$$

for some constant $C(p,n)$. Fix now a nonnegative C^1 function φ such that $\varphi=0$ in $B_{R+1/2}$, $\varphi=1$ in $\mathbb{R}^n\setminus B_{R+1}$ and $|\nabla\varphi|\leq C(n)$. Since $u_\Omega\varphi^p$ is an admissible test function for the equation satisfied by u_Ω, we have

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla(\varphi_k^pu)\,dx=\int_\Omega\varphi_k^pu\,dx.$$

Then, by using Young's inequality we get

$$\int_{\Omega\setminus B_{R+1}} |\nabla u_\Omega|^p\,dx \le \int_\Omega \varphi^p |\nabla u_\Omega|^p\,dx \le C(p)\int_\Omega |\nabla\varphi|^p u_\Omega^p\,dx + C(p)\int_\Omega \varphi u_\Omega\,dx$$
$$\le C(p)\int_{\Omega\setminus B_{R+1/2}} |u_\Omega|^p + |u_\Omega|\,dx.$$

The result then follows by estimating the last integral with (4.4). □

Let Ω an open set with finite measure and $r>0$. Observe that

$$E(r\Omega) = r^{n+\frac{p}{p-1}} E(\Omega). \tag{4.5}$$

Therefore, setting

$$D(\Omega) := E(\Omega)|\Omega|^{-1-\frac{p}{n(p-1)}} - E(B)|B|^{-1-\frac{p}{n(p-1)}},$$

compare with (1.7), we get from (4.5) that $D(\Omega)$ is invariant by rescaling.

Next lemma gives a not optimal quantitative inequality for $D(\Omega)$ which will be later used to get the optimal one.

Lemma 4.3. *There exists a constant γ, depending only on p,n such that for any open set $\Omega\subset\mathbb{R}^n$ with finite measure*

$$D(\Omega) \ge \gamma\mathcal{A}(\Omega)^{2+p}. \tag{4.6}$$

Proof. Since both quantities in (4.6) are scaling invariant, we may assume without loss of generality that $|\Omega| = |B|$. Observe that

$$E(\Omega) = -\frac{p-1}{p}\lambda_{p,1}(\Omega)^{-\frac{1}{p-1}}, \tag{4.7}$$

hence

$$D(\Omega) = \frac{p-1}{p}|B|^{-1-\frac{p}{n(p-1)}}\big(\lambda_{p,1}(B)^{-\frac{1}{p-1}} - \lambda_{p,1}(\Omega)^{-\frac{1}{p-1}}\big),$$

where $\lambda_{p,1}(\Omega)$ is defined as in (1.5). Therefore, if $\lambda_{p,1}(\Omega)\ge 2\lambda_{p,1}(B)$ we conclude that

$$D(\Omega) \ge \frac{p-1}{p}|B|^{-1-\frac{p}{n(p-1)}}\lambda_{p,1}(B)^{-\frac{1}{p-1}}\big(1-2^{-\frac{1}{p-1}}\big) \ge c(p,n)\mathcal{A}(\Omega)^{2+p},$$

where the last inequality follows from the fact that $\mathcal{A}(\Omega)\le 2$. On the other hand by [16, Th. 1.1] we have

$$\lambda_{p,1}(\Omega) - \lambda_{p,1}(B) \ge c(p,n)\mathcal{A}(\Omega)^{2+p}, \tag{4.8}$$

for some positive constant depending only on p,n. Therefore, if $\lambda_{p,1}(\Omega)\le 2\lambda_{p,1}(B)$, we have

$$D(\Omega) \ge \frac{p-1}{p}|B|^{-1-\frac{p}{n(p-1)}}\lambda_{p,1}(\Omega)^{-\frac{1}{p-1}}\left(\left(\frac{\lambda_{p,1}(\Omega)}{\lambda_{p,1}(B)}\right)^{\frac{1}{p-1}} - 1\right)$$
$$\ge \frac{p-1}{p}|B|^{-1-\frac{p}{n(p-1)}}(2\lambda_{p,1}(B))^{-\frac{1}{p-1}}\left(\left(\frac{\lambda_{p,1}(\Omega)}{\lambda_{p,1}(B)}\right)^{\frac{1}{p-1}} - 1\right) \ge c(p,n)\Big(\frac{\lambda_{p,1}(\Omega)}{\lambda_{p,1}(B)} - 1\Big), \tag{4.9}$$

where in the last inequality we used the fact that for all $\alpha>0$ there exists a constant $c(\alpha)$ such that

$$t^\alpha - 1 \ge c(\alpha)(t-1) \qquad \text{for all } t\in[1,2].$$

The conclusion then follows by combining (4.8) and (4.9). □

Using the two lemmas above we may then prove the following result which tells us that in estimating $D(\Omega)$ we may always reduce to the case where Ω is a bounded set.

Lemma 4.4. *There exists three positive constants C, δ, R, depending only p and n such that for every open set Ω with $|\Omega| = |B|$ and $D(\Omega) \le \delta$, one can find another open set $\widetilde{\Omega}$ with $|\widetilde{\Omega}| = |B|$ and $\widetilde{\Omega} \subset B_R$ with the property that*

$$\mathcal{A}(\Omega) \le \mathcal{A}(\widetilde{\Omega}) + CD(\Omega), \qquad D(\widetilde{\Omega}) \le CD(\Omega).$$

Proof. Let us assume that the ball achieving the asymmetry, i.e., the ball minimizing the right hand side of (1.2), is B. From (4.6) we have that if $D(\Omega) \le \delta$ with a sufficiently small δ,

$$\frac{|\Omega \setminus B|}{|B|} \le \mathcal{A}(\Omega) \le \Big(\frac{D(\Omega)}{\gamma}\Big)^{\frac{1}{p+2}} \le \frac{1}{2}. \tag{4.10}$$

Let us now estimate the energy $E(\Omega \cap B_{k+2})$ for every $k \in \mathbb{N}$. Let φ_k be the cut-off function defined by $\varphi_k(x) = \min\{1, (k+2-|x|)_+\}$, $x \in \mathbb{R}^n$, which is supported in B_{k+2} and equals to 1 in B_{k+1}. Setting $u_k := \varphi_k u_\Omega$, we have

$$\begin{aligned} E(\Omega \cap B_{k+2}) - E(\Omega) &\le \frac{1}{p}\int_\Omega |\nabla u_k|^p\,dx - \frac{1}{p}\int_\Omega |\nabla u_\Omega|^p\,dx - \int_\Omega u_k\,dx + \int_\Omega u_\Omega\,dx \\ &\le \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_k - u_\Omega)\,dx + \int_\Omega (1-\varphi_k)u_\Omega\,dx \\ &\le C(p)\int_{\Omega\setminus B_{k+1}} \big(|\nabla u_\Omega|^p + |u_\Omega|^p + |u_\Omega|\big)\,dx \le C(p,n)|\Omega \setminus B_k|^{1+\frac{1}{n}}, \end{aligned}$$

where in the last inequality we used (4.1). Set $b_k := \frac{|\Omega\setminus B_k|}{|B|}$ and recall that by (4.10) $b_k \le 1/2$. With this notation, the inequality above can be written as

$$E(\Omega \cap B_{k+2}) - E(\Omega) \le C(p,n) b_k^{1+\frac{1}{n}}. \tag{4.11}$$

By the definition of b_k and $D(\Omega)$, recalling the minimality property of B, we further have

$$\begin{aligned} E(B)(1-b_{k+2})^{1+\frac{p}{n(p-1)}} &= E(B)|B|^{-1-\frac{p}{n(p-1)}}|\Omega \cap B_{k+2}|^{1+\frac{p}{n(p-1)}} \\ &\le E(\Omega \cap B_{k+2}) \le E(\Omega) + Cb_k^{1+\frac{1}{n}} \le E(B) + CD(\Omega) + Cb_k^{1+\frac{1}{n}}. \end{aligned}$$

Since $b_k \le 1/2$ and $E(B) < 0$ we then have

$$b_{k+2} \le C(p,n)\left(1 - (1-b_{k+2})^{1+\frac{p}{n}}\right) \le \widetilde{C}(D(\Omega) + b_k^{1+\frac{1}{n}}), \tag{4.12}$$

for some positive constant $\widetilde{C}(p,n)$. Now choose $\delta > 0$ such that $2\widetilde{C}\delta \le 1/2$ and define

$$K := \max\{k \in \mathbb{N} \colon b_k \ge 2\widetilde{C}D(\Omega)\},$$

if the set on the right hand side is not empty, and $K := 0$, otherwise. The existence of such K follows from the fact that $\lim_{k\to\infty} b_k = 0$. We claim that when δ is small, there is a constant $K'(p, n)$ such that $K \le K'$. Indeed for $k+2 \le K$, we have by (4.12) and the definition of K

$$b_{k+2} \le 2(b_{k+2} - \widetilde{C}D(\Omega)) \le 2\widetilde{C}b_k^{1+\frac{1}{n}} \le b_k^{1+\frac{1}{2n}}$$

if δ is sufficiently small. Then by iteration one has that there exists $k(n) \ge 1$ such that if $k(n) \le k \le K$

$$b_k \le b_1^{1+\frac{k}{2n}}.$$

Therefore, either $K \le k(n)$ or, recalling (4.10),

$$2\widetilde{C}D(\Omega) \le b_K \le b_1^{1+\frac{K}{2n}} \le C(n,\,p)D(\Omega)^{(1+\frac{K}{2n})\frac{1}{p+2}}.$$

This implies the claim.

By the definition of K and b_k, we immediately obtain that

$$|\Omega \cap B_{K+3}| \geq |B|(1 - b_{K+1}) \geq |B| - 2\widetilde{C}|B|D(\Omega), \tag{4.13}$$

while by (4.11)

$$E(\Omega \cap B_{K+3}) \leq E(\Omega) + C b_{K+1}^{1+\frac{1}{n}} \leq E(\Omega) + C(p,\, n)D(\Omega)^{1+\frac{1}{n}}. \tag{4.14}$$

Set

$$\widetilde{\Omega} = \frac{\Omega \cap B_{K+3}}{r},$$

where

$$r = \left(\frac{|\Omega \cap B_{K+3}|}{|B|}\right)^{\frac{1}{n}} \leq 1. \tag{4.15}$$

Clearly $|\widetilde{\Omega}| = |B|$. Moreover, since $2\widetilde{C}\delta < 1/2$, from (4.13) and (4.14),

$$\operatorname{diam}(\widetilde{\Omega}) \leq d = d(p,\, n), \text{ and } D(\widetilde{\Omega}) \leq C(p,\, n)D(\Omega).$$

Let us finally compute $\mathcal{A}(\Omega) - \mathcal{A}(\widetilde{\Omega})$. To this aim let $B(x_0)$ be an optimal ball for $\widetilde{\Omega}$. Recalling the definition (4.15) of r and that $b_{K+3} < 2\widetilde{C}D(\Omega)$, we conclude that

$$\begin{aligned}
|B|\mathcal{A}(\Omega) &\leq |\Omega \Delta B(x_0/r)| \\
&\leq |\Omega \Delta (\Omega \cap B_{K+3})| + |(\Omega \cap B_{K+3})\Delta B_r(x_0/r)| + |B_r(x_0/r)\Delta B(x_0/r)| \\
&\leq 2\widetilde{C}|B|D(\Omega) + r^n|\widetilde{\Omega}\Delta B(x_0)| + \omega_n(1 - r^n) \leq |B|\mathcal{A}(\widetilde{\Omega}) + C(p,\, n)D(\Omega),
\end{aligned}$$

where in the last inequality we used the fact that

$$1 - r \leq C(p, n)D(\Omega).$$

This completes the proof of the lemma. □

We are now in position to give the proof of our main result.

Proof of Theorem 1.1. We start by proving that there exists a constant κ, depending only on p and n, such that for any open set Ω with finite measure

$$D(\Omega) \geq \kappa \mathcal{A}(\Omega)^2. \tag{4.16}$$

Since both sides of the inequality above are scaling invariant, we may assume that $|\Omega| = |B|$. Let δ be the constant in Lemma 4.4. If $D(\Omega) \geq \delta$, then since $\mathcal{A}(\Omega) < 2$ we have

$$D(\Omega) \geq \frac{\delta}{4}\mathcal{A}(\Omega)^2.$$

Thus we may assume that $D(\Omega) < \delta$. By Lemma 4.4, up to replacing Ω with $\widetilde{\Omega}$, we may further assume that $\Omega \subset B_R$.

Let ε be as in Theorem 3.3. If $A(\Omega) \leq \varepsilon$, (4.16) follows by combining (3.1) with Lemma 3.2 (i). Finally, if $A(\Omega) \geq \varepsilon$, denoting by $B(x_0)$ an optimal ball for Ω, from Lemma 3.2 (ii) we have

$$\mathcal{A}(\Omega) = \frac{|\Omega \Delta B(x_0)|}{|B|} \geq \frac{A(\Omega)}{c_2|B|} \geq \frac{\varepsilon}{c_2|B|}.$$

Therefore, recalling (4.6) we conclude that

$$D(\Omega) \geq \gamma \mathcal{A}(\Omega)^{2+p} \geq \frac{\gamma \varepsilon^p}{c_2^p|B|^p}\mathcal{A}(\Omega)^2.$$

Let now $q \geq 1$ be an exponent satisfying (1.6). From Theorem 4.1 we know that there exists $\alpha(p,q,n) > 0$ such that

$$\frac{\lambda_{p,q}(\Omega)}{\lambda_{p,q}(B)} \geq \left(\frac{E(B)}{E(\Omega)}\right)^{\alpha}.$$

From this inequality and the quantitative estimate (4.16), inequality (1.8) follows at once by the same argument used in the proof of Lemma 4.3. □

5. Appendix

This section is devoted to the proof of Theorem 3.23. The proof will follow closely the proof of Theorem 9.1 in [12], where the authors deal with solutions of the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mathcal{H}^{n-1} \llcorner \partial\{u > 0\}.$$

Although our equation (3.34) is very similar to the one above, some arguments used in [12] must be adjusted to our situation. These changes are discussed in details in this section. Throughout all this section we shall always assume that u is a solution in $W_0^{1,2}(B_2)$ of (3.34) with $\{u > 0\} \subset B_{3/2}$.

Remark 5.1. Observe that if $x_0 \in \partial\{u > 0\}$ and $r > 0$, by setting

$$u_r(x) := \frac{u(x_0 + rx)}{r}, \tag{5.1}$$

the rescaled function u_r satisfies the equation

$$-\operatorname{div}(|\nabla u_r(x)|^{p-2}\nabla u_r(x)) = r\chi_{\{u_r>0\}}(x) - q_u(x_0 + rx)^{p-1}\mathcal{H}^{n-1} \llcorner \partial\{u_r > 0\}, \tag{5.2}$$

where q_u is a Lipschitz function satisfying the assumptions of Theorem 3.23. Morover, if $u \in F(\sigma_+, \sigma_-; \tau)$ in $B_r(x_0)$ with respect to some direction ν, then $u_r \in F(\sigma_+, \sigma_-; \tau)$ in B with respect to the same direction.

The first lemma is proved with exactly the same proof of Claim 6.7 in [12].

Lemma 5.2. *If $B_r(w)$ is a ball in $\{u = 0\}$ touching $\partial\{u > 0\}$ at z, then*

$$\limsup_{x\to z,\, u(x)>0} \frac{u(x)}{\operatorname{dist}(x, B_r(w))} = q_u(z). \tag{5.3}$$

Roughly speaking, the idea of the proof of Theorem 3.23 is to show that if in a small ball $B_r(x_0)$ the free boundary $\partial\{u > 0\}$ is sufficiently flat, then it becomes even flatter in smaller balls. This amounts to prove a decay estimate for the quantity $F(\sigma_+, \sigma_-; \tau)$. This goal is achieved through a certain number of intermediate steps. The first one is contained in the next lemma.

Lemma 5.3. *There exists a constant $\gamma_0(p, n, \min q_u)$ such that for every $\varepsilon > 0$, there exists $\sigma_\varepsilon > 0$ with the property that if $0 < \sigma < \sigma_\varepsilon$, $0 < r < \gamma_0\sigma$ and $u \in F(\sigma, 1; \sigma)$ in $B_r(x_0)$ with respect to $\nu \in \mathbb{S}^{n-1}$, then $u \in F(2\sigma, \varepsilon; \sigma)$ in $B_{\frac{r}{2}}(x_0)$ with respect to the same ν.*

Proof. The proof is similar to the one of Lemma 6.5 in [12]. Yet a few changes are needed.

Up to replacing u by the function $\frac{u_r}{q_u(x_0)}$, where u_r is defined as in (5.1), by (5.2) we may assume that u satisfies in B the equation

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = \frac{r}{q_u(x_0)^{p-1}}\chi_{\{u>0\}}(x) - \left(\frac{q_u(x_0 + rx)}{q_u(x_0)}\right)^{p-1}\mathcal{H}^{n-1} \llcorner \partial\{u > 0\}.$$

Moroever, up to a rotation, we may assume that $\nu = e_n$. Thus we have that

$$\begin{cases} \text{if } x \in B \text{ and } x_n \geq \sigma, \text{ then } u(x) = 0 \\ |\nabla u(x)| \leq 1 + \sigma \ \text{ in } B \end{cases} \tag{5.4}$$

As in [12] we set, for $y \in \mathbb{R}^{n-1}$,

$$\eta(y) := \exp\biggl(- \frac{9|y|^2}{1 - 9|y|^2} \biggr)$$

for $|y| < 1/3$ and $\eta(y) = 0$ for $|y| \geq 1/3$. Then, we denote by s the largest nonnegative number such that

$$B \cap \{u > 0\} \subset D := \{x \in B : x_n < \sigma - s\eta(x')\},$$

where $x' = (x_1, \dots, x_{n-1})$. Observe that since $0 \in \partial\{u > 0\}$ we have $s \leq \sigma$. Moreover, there exists z such that

$$z \in B_{\frac{1}{2}} \cap \partial D \cap \partial\{u > 0\}, \qquad z_n = \sigma - s\eta(z'). \tag{5.5}$$

Then, given $\xi \in B_{\frac{3}{4}}$, with $\xi_n \leq -1/2$, let us denote by v the unique solution of the following problem

$$\begin{cases} -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = \dfrac{r}{q_u(x_0)^{p-1}} & \text{in } D \setminus \overline{B}_\varrho(\xi), \\ v(x) = 0 & \text{if } x \in \partial D \cap B, \\ v(x) = (1+\sigma)(\sigma - x_n) & \text{if } x \in \partial D \setminus B, \\ v(x) = -(1 - \kappa\sigma)x_n & \text{if } x \in \partial B_\varrho(\xi), \end{cases} \tag{5.6}$$

where $\varrho < 1/4$ and κ are two positive constants, to be chosen later independently of σ and r. In Lemma 5.4 below we shall prove that there exist two positive constants $C(\varrho)$ and $c(\varrho)$, depending only on ϱ, such that if $\sigma < \sigma(\kappa, \varrho)$, then

$$|\nabla v(z)| \leq 1 + C(\varrho)\sigma - c(\varrho)\kappa\sigma.$$

Observe that if $u \leq v$ on $\partial B_\varrho(\xi)$, then by construction and by the second condition in (5.4) we would have that $u \leq v$ on $\partial(D \setminus B_\varrho(\xi))$, thus concluding, by the comparison principle, that $u \leq v$ in $D \setminus B_\varrho(\xi)$. Therefore, choosing $\kappa(\varrho)$ sufficiently large and $0 < \sigma < \sigma(\varrho, \kappa)$ we would have

$$|\nabla v(z)| < 1 - \sigma,$$

where z is the point defined in (5.5). But this is impossible since, from (5.3) and the assumption $u \in F(\sigma, 1; \sigma)$, we have

$$|\nabla v(z)| \geq \frac{q_u(x_0 + rz)}{q_u(x_0)} \geq 1 - \sigma.$$

Therefore there exists a point $x_\xi \in \partial B_\varrho(\xi)$ such that

$$u(x_\xi) \geq v(x_\xi). \tag{5.7}$$

Hence for σ sufficiently small, recalling the second condition in (5.4), we have

$$\begin{aligned} u(\xi) &\geq u(x_\xi) - \varrho(1+\sigma) \geq v(x_\xi) - \varrho(1+\sigma) = -(1-\kappa\sigma)(x_\xi)_n - \varrho(1+\sigma) \\ &\geq -(x_\xi)_n - \kappa\sigma - 2\varrho \geq -\xi_n - 4\varrho. \end{aligned}$$

Integrating this inequality along vertical lines and using again the inequality $|\nabla u| \leq 1 + \sigma$ we have, still for $\sigma < \sigma(\varrho)$

$$u(\xi + te_n) \geq u(\xi) - t(1+\sigma) > -\xi_n - 4\varrho - t(1+\sigma) \geq -(\xi_n + t) - 5\varrho.$$

Choosing $\varrho = \min\{\varepsilon/5, 1/5\}$, multiplying both sides of the previous inequality by $rq_u(x_0)$ and rescaling back to $B_r(x_0)$, we obtain immediately the assertion. □

Lemma 5.4. *Let v be the function defined in* (5.6). *There exists a positive constant $\gamma_0(p, n, \min q_u)$ such that for every $\varrho < 1/4$ and every $\kappa > 0$ there exist two positive constants $C(\varrho)$ and $c(\varrho)$ and a positive constant $\sigma(\kappa, \varrho)$ such that if $0 < r < \gamma_0\sigma$ and $0 < \sigma < \sigma(\kappa, \varrho)$, then*

$$|\nabla v(z)| \le 1 + C(\varrho)\sigma - \kappa c(\varrho)\sigma,$$

where z is as in (5.5).

Proof. The proof goes as the one of Claim 6.8 in [12] with some extra work needed to estimate a few quantities in a more precise way.

As in [12], for every $\xi \in \mathbb{R}^n$ we set

$$a_{ij}(\xi) := \delta_{ij} + (p-2)\frac{\xi_i\xi_j}{|\xi|^2} \qquad \text{for all } i, j = 1, \dots, n \text{ and } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{5.8}$$

Observe that

$$\frac{1}{\lambda}|\zeta|^2 \le a_{ij}(\xi)\zeta_i\zeta_j \le \lambda|\zeta|^2 \qquad \text{for every } \ \xi, \zeta \in \mathbb{R}^n, \tag{5.9}$$

with $\lambda(p) > 0$. We are going to construct a comparison function w of the form $w = v_1 - \kappa\sigma v_2$. Let η and D be as in the proof of Lemma 5.3 and define for $x \in D$

$$v_1(x) := \frac{\gamma_1}{\mu_1}\big(1 - \exp(-\mu_1 d_1(x))\big),$$

where

$$d_1(x) := -x_n + \sigma - s\eta(x')$$

and $\mu_1 < 1$ and γ_1 are positive constants depending on σ to be chosen later. Observe that for σ small

$$v_1(x) \ge \gamma_1 d_1(x)(1 - 2\mu_1) \qquad \text{for } x \in \partial(D \setminus B_\varrho(\xi)).$$

Therefore, if we impose that

$$\gamma_1(1 - 2\mu_1) \ge 1 + 2\sigma, \tag{5.10}$$

we have

$$v_1(x) \ge (1 + 2\sigma)d_1(x) \ge v(x) \qquad \text{for } x \in \partial(D \setminus B_\varrho(\xi)). \tag{5.11}$$

Observe that there exists an absolute constant $C_1 > 0$ such that

$$1 \le |\nabla d_1| \le 1 + C_1\sigma, \qquad |D^2 d_1| \le C_1\sigma. \tag{5.12}$$

Therefore, recalling (5.10), we have

$$1 \le \gamma_1(1 - 2\mu_1) \le \gamma_1\exp(-\mu_1 d_1) \le |\nabla v_1| \le \gamma_1(1 + C_1\sigma). \tag{5.13}$$

Next, we estimate $\operatorname{div}(|\nabla v_1|^{p-2}\nabla v_1)$. We have

$$\begin{aligned}\operatorname{div}(|\nabla v_1|^{p-2}\nabla v_1) &= |\nabla v_1|^{p-2}a_{ij}(\nabla v_1)D_{ij}v_1 \\ &= \gamma_1|\nabla v_1|^{p-2}\exp(-\mu_1 d_1)a_{ij}(\nabla v_1)(D_{ij}d_1 - \mu_1 D_i d_1 D_j d_1).\end{aligned}$$

From (5.12) it follows that there exist two positive constants $C_2(p), c_2(p)$ such that

$$a_{ij}(\nabla v_1)(D_{ij}d_1 - \mu_1 D_i d_1 D_j d_1) \le C_2\sigma - c_2\mu_1.$$

Therefore, if we choose

$$\mu_1 = 2\frac{C_2}{c_2}\sigma,$$

from (5.13) we get that for σ sufficiently small

$$\operatorname{div}(|\nabla v_1|^{p-2}\nabla v_1) \le -\frac{1}{2}\gamma_1 \min\left\{1, \gamma_1^{p-2}(1+C_1\sigma)^{p-2}\right\}C_2\sigma.$$

From the above choice of μ_1 it is clear that there exists a constant C_3, depending only on p such that, if we set

$$\gamma_1 := 1 + C_3\sigma,$$

then γ_1 satisfies the constraint (5.10) and we have

$$\operatorname{div}(|\nabla v_1|^{p-2}\nabla v_1) \le -C_4\sigma,$$

for a positive constant C_4 depending only on p. Therefore, we may conclude that if $0 < r < C_4\sigma \min q_u^{p-1}$, then

$$\operatorname{div}(|\nabla v_1|^{p-2}\nabla v_1) \le -\frac{r}{q_u(x_0)^{p-1}} \qquad \text{in } D \setminus B_\varrho(\xi). \tag{5.14}$$

From this estimate, using the comparison principle and recalling (5.11) we have that

$$v_1(x) \ge v(x) \qquad \text{for all } x \in D \setminus B_\varrho(\xi).$$

Moreover, from the above choice of γ_1 and (5.13) we have that there exists a positive constant $C_5(p)$ depending only on p such that

$$1 \le |\nabla v_1| \le 1 + C_5\sigma.$$

Let us now define a function v_2, by setting

$$v_2(x) := \frac{\gamma_2}{\mu_2}(\exp(\mu_2 d_2(x)) - 1) \qquad \text{for } x \in \widetilde{D} \setminus B_\varrho(\xi),$$

where μ_2 a positive constant to be chosen later,

$$\gamma_2 := \frac{1}{4}\frac{\mu_2}{\mathrm{e}^{\mu_2} - 1}$$

$\widetilde{D} \subset D$ is a domain containing $D \setminus \mathcal{N}_{\frac{1}{10}}(\partial B \cap \{x_n = 0\})$ and d_2 is a function in $C^2(D \setminus B_\varrho(\xi))$ with values in $(0,1)$, satisfying the following conditions

$$\begin{cases} d_2 = 0 & \text{on } \partial\widetilde{D} \\ d_2 = 1 & \text{on } \partial B_\varrho(\xi) \\ \frac{1}{\widetilde{C}} \le |\nabla d_2| \le \widetilde{C} & \text{in } \widetilde{D} \setminus B_\varrho(\xi), \end{cases}$$

for a positive constant $\widetilde{C}(n,\varrho)$. Arguing as above, it is clear that we may choose μ_2 large enough, depending only on p, n and ϱ, so that for all $\xi \in \mathbb{R}^n$

$$a_{ij}(\xi)D_{ij}v_2 = \gamma_2 \exp(\mu_2 d_2) a_{ij}(\xi)(D_{ij}d_2 + \mu_2 D_i d_2 D_j d_2) \ge \overline{\mu} > 0, \tag{5.15}$$

for some positive constant $\overline{\mu}$. Moreover, since $\nabla v_2 = \gamma_2 \exp(\mu_2 d_2)\nabla d_2$, we have

$$\frac{\gamma_2}{\widetilde{C}} \le |\nabla v_2| \le \gamma_2 \mathrm{e}^{\mu_2}\widetilde{C}.$$

From this estimate, setting

$$w := v_1 - \kappa\sigma v_2$$

and recalling (5.15) we may easily conclude, arguing as in the proof of (5.14), that for $\sigma < \sigma(\kappa, \varrho)$, with $\sigma(\kappa,\varrho)$ sufficiently small,

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) \le -C_6\sigma,$$

for some positive constant C_6 depending only p, n, ϱ. Hence, if $0 < r < \gamma_0\sigma$, for some constant γ_0 depending only on p, n, ϱ and $\min q_u^{p-1}$, we may conclude that

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) \leq -\frac{r}{q_u(x_0)^{p-1}}. \tag{5.16}$$

Finally observe that from the definition of d_2 and (5.11) we have that

$$w = v_1 \geq v \qquad \text{on } \partial\widetilde{D},$$

while, observing that $v_2 \equiv 1/4$ on $\partial B_\varrho(\xi)$ and using (5.11) again, we have that for all $x \in \partial B_\varrho(\xi)$

$$w(x) \geq d_1(x) - \frac{\kappa\sigma}{4} \geq -(1-\kappa\sigma)x_n = v(x),$$

since $\xi_n \leq -1/2$ and $\varrho < 1/4$. Thus, recalling (5.16), we may conclude that $w \geq v$ in $\widetilde{D} \setminus B_\varrho(\xi)$. Therefore, if z is as in (5.5) and $\sigma < \sigma(\kappa, \varrho)$,

$$|\nabla v(z)| \leq |\nabla w(z)| = |\nabla v_1(z)| - \kappa\sigma|\nabla v_2(z)| \leq 1 + C_5\sigma - \kappa\sigma\frac{\gamma_2}{\widetilde{C}},$$

where the equality in the above formula follows by observing that $\nabla d_1(z)$ and $\nabla d_2(z)$ have the same direction. This estimate concludes the proof of the lemma. □

Next lemma provides an estimate from below on ∇u_j near the free boundary.

Lemma 5.5. *For every $\varepsilon, \delta > 0$ there exists $\sigma_{\varepsilon,\delta}$ such that if $0 < \sigma < \sigma_{\varepsilon,\delta}$ and $0 < r < \gamma_0\sigma$, with γ_0 as in Lemma 5.3, and $u \in F(\sigma, 1; \sigma)$ in $B_r(x_0)$ with respect to some direction ν, then $|\nabla u| \geq q_u(x_0)(1-\delta)$ in $B_{\frac{19r}{20}}(x_0) \cap \{(x-x_0)\cdot\nu \leq -\varepsilon r\}$.*

Proof. We argue as in the proof of Lemma 6.6 in [12] with the some technical modifications.

First, up to a translation and a rotation we may assume that $x_0 = 0$ and that $\nu = e_n$. Then, we argue by contradiction, assuming that there exist a sequence of Lipschitz functions u_k, satisfying the assumptions of Theorem 3.23 with constants uniformly bounded above and away from zero, and a sequence of radii $0 < r_k < \frac{\gamma_0}{k}$ such that $u_k \in F(\frac{1}{k}, 1; \frac{1}{k})$ with respect to e_n in B_{r_k} and

$$|\nabla u_k(\widetilde{x}_k)| < q_{u_k}(0)(1-\delta) \qquad \text{for some } \widetilde{x}_k \in B_{\frac{19r_k}{20}} \cap \{x_n \leq -\varepsilon r_k\}.$$

By rescaling u_k as in the proof of Lemma 5.3, we set for all $x \in B$

$$v_k(x) = \frac{u_k(r_k x)}{r_k q_{u_k}(0)}$$

and we have that

$$-\operatorname{div}(|\nabla v_k(x)|^{p-2}\nabla v_k(x)) = \frac{r_k}{q_{u_k}(0)^{p-1}}\chi_{\{v_k>0\}} - \left(\frac{q_{u_k}(r_k x)}{q_{u_k}(0)}\right)^{p-1}\mathcal{H}^{n-1} \llcorner \partial\{v_k > 0\} \quad \text{in } B$$

and that

$$|\nabla v_k(x_k)| \leq 1-\delta \qquad \text{for some } x_k \in B_{\frac{19}{20}} \cap \{x_n \leq -\varepsilon\}.$$

Letting $k \to \infty$, from Lemma 5.3 we obtain that

$$v_k \to v_0 \qquad \text{uniformly in } \overline{B},$$

where v_0 is the p-harmonic function such that $v_0(x) = 0$ if $x_n \geq 0$, $v_0(x) = -x_n$ if $x_n < 0$. Note that by elliptic regularity $v_k \to v_0$ locally in $C^{1,\alpha}$ in $B \cap \{x_n < 0\}$. Therefore, if $\overline{x}$ is a limit point of the sequence x_k we have that $|\nabla v_0(\overline{x})| \leq 1-\delta$ thus getting a contradiction since $|\nabla v_0(\overline{x})| = 1$. □

Next step in the proof of the regularity of the free boundary is to improve the decay estimates contained in Lemma 5.3 and Lemma 5.5. This is the content of the following two results.

Proposition 5.6. *There exist $\sigma_0, C_0 > 0$, depending only on p, n and $\min q_u$ such that if $u \in F(\sigma, 1; \sigma)$ in B_r, then $u \in F(2\sigma, C_0\sigma; \sigma)$ in $B_{\frac{r}{2}}$, provided that $0 < \sigma < \sigma_0$ and $0 < r < \gamma_0\sigma$, where γ_0 is as in Lemma 5.3.*

Proof. Following the proof of Theorem 6.3 in [12], we start with the same rescaling as in the proof of Lemma 5.3 and with the same definition of the function v as in (5.6). Then, we choose $\varrho = 1/10$, and κ accordingly, so that (5.7) holds. Then, we set for $x \in D$

$$w(x) := (1+\sigma)(\sigma - x_n) - u(x).$$

Then, the second condition in (5.4) implies that $w(x) \geq 0$ in $B_{2\varrho}(\xi)$, while from (5.7) and the definition of v we get

$$w(x_\xi) \leq (1+\sigma)(\sigma - (x_\xi)_n) - v(x_\xi) \leq C_1(\kappa)\sigma,$$

for some positive constant C_1 depending only on the choice of κ. If σ is sufficiently small we have from Lemma 5.5 that $|\nabla u| > \frac{1}{2}$ in $B_{2\varrho}(\xi)$, hence u satisfies the linear equation in nondivergence form

$$a_{ij}(\nabla u(x))D_{ij}u(x) = -\frac{r}{q_u(x_0)^{p-1}|\nabla u(x)|^{p-2}},$$

where the coefficients a_{ij} are defined as in (5.8). From the Harnack inequality, see Corollary 9.21 and Theorem 9.22 of [17], and the bounds on $|\nabla u|$ we may conclude that

$$w(\xi) \leq C(w(x_\xi) + r),$$

where C depends on p, n and $\min q_u$. Therefore, for a possibly different constant C, we have

$$u(\xi) \geq -\xi_n - C\sigma \quad \text{on } \{\xi \in \partial B_{\frac{3}{4}} : \xi_n \leq -1/2\}.$$

The assertion then follows integrating on vertical lines as at the end of the proof of Lemma 5.3. □

Next result improves the statement of Lemma 5.5. Its proof follows from the above Proposition 5.6 with exactly the same proof of Theorem 6.4 in [12], by using the same rescaling used in the proof of Lemma 5.5.

Proposition 5.7. *For every $\delta \in (0,1)$ there exist $\sigma_\delta, C_\delta > 0$ such that if $0 < \sigma < \sigma_\delta$ and $0 < r < \gamma_0\sigma$, with γ_0 as in Lemma 5.3, and $u \in F(\sigma, 1; \sigma)$ in $B_r(x_0)$ with respect to some direction ν, then $|\nabla u| \geq q_u(x_0)(1-\delta)$ in $B_{\frac{r}{2}}(x_0) \cap \{(x - x_0)\cdot\nu \leq -C_\delta r\}$.*

Next result is a crucial step in the iteration process needed for the proof of Theorem 3.34. It is the counterpart in our case of Theorem 7.1 of [12] and its proof, which follows closely the one given therein, requires nevertheless some technical adjustments.

Proposition 5.8. *There exist $C > 0$ and $0 < \alpha < 1/2$ depending only on $p, n, \|\nabla u\|_{L^\infty}$ and $\|q_u\|_{W^{1,\infty}}$ such that if $B_r(x_0) \cap \partial\{u > 0\} \neq \emptyset$*

$$\sup_{B_r(x_0)} |\nabla u| \leq q_u(x_0) + Cr^\alpha.$$

Proof. Since q_u is Lipschitz, we may assume without loss of generality that $x_0 \in \partial\{u > 0\}$. Moreover, we may also assume that $x_0 = 0$ and that $r \leq 1$.

Given $\varepsilon \geq 0$ we consider in B_{2r} the function

$$U_\varepsilon := (|\nabla u|^2 - Q_{2r}^2 - \varepsilon)^+,$$

where $Q_\varrho := \sup_{B_\varrho(x_0)} q_u$. Since u satisfies the equality in (3.25), we may conclude that U_ε vanishes in a neighborhood of $\partial\{u > 0\}$. Moreover, $\{U_\varepsilon > 0\} \subset \big\{|\nabla u| > \sqrt{Q_{2r}^2 + \varepsilon/2}\big\}$. Therefore u is a solution in $B_{2r} \cap \{U_\varepsilon > 0\}$ of the equation

$$a_{ij}(\nabla u) D_{ij} u = -\frac{1}{|\nabla u|^{p-2}},$$

where the coefficients a_{ij} are defined in (5.8) and satisfy the ellipticity condition (5.9). Observe that if γ is sufficiently large, depending only on p, n and $\|\nabla u\|_\infty$, the function $v := |\nabla u|^2$ satisfies the inequality

$$Mv := D_i(\mathrm{e}^{\gamma v} a_{ij}(\nabla u) D_j v) \geq -C_0 \qquad \text{in } \big\{|\nabla u| > \sqrt{Q_{2r}^2 + \varepsilon/2}\big\},$$

for some constant $C_0(p, n, \|\nabla u\|_\infty)$. Indeed this fact follows from a much more general estimate, see [17, Sect. 13.3], but in our case it can be easily checked with a direct computation. Thus, we have also

$$MU_\varepsilon \geq -C_0 \qquad \text{in } \big\{|\nabla u| > \sqrt{Q_{2r}^2 + \varepsilon/2}\big\}.$$

Therefore, if we extend M to a uniformly elliptic operator in divergence form with measurable coefficients defined in the whole ball B_{2r}

$$\widetilde{M} w = D_i(\widetilde{a}_{ij}(x) D_j w)$$

so that

$$\widetilde{a}_{ij}(x) = \mathrm{e}^{\gamma v} a_{ij}(\nabla u(x)) \qquad \text{in } \big\{|\nabla u| > \sqrt{Q_{2r}^2 + \varepsilon/2}\big\}$$

and the ellipticity constant of $\widetilde{M}$ is controlled from below and from above by the ellipticity constant of M, we may conclude that

$$\widetilde{M} U_\varepsilon \geq -C_0 \qquad \text{in } B_{2r}.$$

Therefore, the function

$$w_\varepsilon := \sup_{B_{2r}} U_\varepsilon - U_\varepsilon$$

is a nonnegative supersolution in B_{2r} of the operator $\widetilde{M}$. Observe also that

$$w_\varepsilon \equiv \sup_{B_{2r}} U_\varepsilon \qquad \text{on } B_{2r} \cap \{u = 0\} \text{ and that } |B_{2r} \cap \{u = 0\}| \geq c r^n, \tag{5.17}$$

for some constant $c(n, p)$, where the last inequality follows from the density estimate (3.23). By the weak Harnack inequality for supersolutions, see Th. 8.18 in [17] we have that there exists a constant C depending only on p, n and C_0 such that

$$r^{-n} \|w_\varepsilon\|_{L^1(B_{2r})} \leq C\big(\inf_{B_r} w_\varepsilon + r^2\big).$$

Therefore, using (5.17), we have that

$$\sup_{B_{2r}} U_\varepsilon \leq C_1\big(\sup_{B_{2r}} U_\varepsilon - \sup_{B_r} U_\varepsilon + r^2\big)$$

for a suitable constant $C_1 > 1$ depending again only on p, n and C_0. Letting $\varepsilon \to 0^+$ we have that

$$\sup_{B_r} U_0 \leq \frac{C_1 - 1}{C_1} \sup_{B_{2r}} U_0 + r^2,$$

from which, in turn, we have

$$\begin{aligned}\sup_{B_r}(|\nabla u|^2 - q_u(0)^2) &\leq \frac{C_1-1}{C_1}\sup_{B_{2r}}(|\nabla u|^2 - q_u(0)^2) + \frac{Q_{2r}^2 - q_u(0)^2}{C_1} + r^2 \\ &\leq \frac{C_1-1}{C_1}\sup_{B_{2r}}(|\nabla u|^2 - q_u(0)^2) + C_2 r,\end{aligned}$$

where C_2 is a constant depending on $\|q_u\|_{W^{1,\infty}}$. Then, by a standard iteration argument we get that there exist $0<\beta<1$ and a positive constant C depending only on C_1 and C_2 such that for all $0<r\leq 1$

$$\sup_{B_r}(|\nabla u|^2 - q_u(0)^2) \leq Cr^\beta.$$

From this inequality the conclusion follows at once. □

From now on the proof continues as in [12] by considering the so-called nonhomogeneous blow-up at the free boundary. For the readers's convenience let us just recall what are the main steps. To this aim, we denote a point in $\mathbb{R}^n$ by (y,h), with $y\in\mathbb{R}^{n-1}$ and $h\in\mathbb{R}$ and by B' the unit ball in $\mathbb{R}^{n-1}$. Next lemma states the continuity property of the limit of the function measuring the signed distance of the free boundary from the hyperplane $\{h=0\}$.

Lemma 5.9. *Let u_k, $k\in\mathbb{N}$, be Lipschitz functions satisfying the assumptions of Theorem 3.23 with constants uniformly bounded above and away from zero. Assume that $u_k\in F(\sigma_k,\sigma_k;\tau_k)$ in direction e_n in the ball B_{r_k}, where $\sigma_k\to 0$ as $k\to\infty$ and $0<r_k\leq\tau_k=o(\sigma_k^2)$. For $y\in B'$, set*

$$\begin{aligned}f_k^+(y) &:= \sup\{h:\ (r_k y, \sigma_k r_k h)\in\partial\{u_k>0\}\},\\ f_k^-(y) &:= \inf\{h:\ (r_k y, \sigma_k r_k h)\in\partial\{u_k>0\}\}.\end{aligned}$$

Then, there exists a not relabelled subsequence such that for all $y\in B'$

$$f(y) := \limsup_{\substack{z\to y\\ k\to\infty}} f_k^+(z) = \liminf_{\substack{z\to y\\ k\to\infty}} f_k^-(z).$$

Moreover, $f_k^+\to f$ and $f_k^-\to f$ uniformly, $f(0)=0$ and f is continuous.

After performing the same rescaling as in (5.1), the proof of this lemma goes exactly as the proof of Lemma 7.3 in [3]. Indeed the the only ingredient used in the proof is Theorem 5.6. for which the stronger assumptions $r_k=o(\sigma_k^2)$ and $\tau_k=o(\sigma_k^2)$ are not really needed. However these assumptions are important in the proof of the following result.

Lemma 5.10. *Under the assumptions of Lemma 5.9 the function f is subharmonic in B'.*

Proof. The proof is essentially the same as the proof of Lemma 5.4 of [4] with some small differences. Nevertheless, we indicate the details for the reader's convenience. Again, we replace the function u_k in B_{r_k} by the function

$$v_k(x) := \frac{u_k(r_k x)}{r_k q_{u_k}(0)} \qquad \text{for } x\in B.$$

We argue by contradiction assuming that the function f is not subharmonic. If this is the case, there exist a ball $B'_\varrho(y_0)\subset\subset B'$ and a harmonic function g in the neighborhood of this ball such that

$$g>f \quad \text{on } \partial B'_\varrho(y_0) \quad \text{and} \quad f(y_0)>g(y_0).$$

Following the notation of [4], we set $Z:=B'_\varrho(y_0)\times\mathbb{R}$ and, given a function $\varphi: B'_\varrho(y_0)\to\mathbb{R}$,

$$Z^\pm(\varphi) := \{(y,h)\in Z:\ h\gtrless\varphi(y)\} \qquad Z_0(\varphi) := \{(y,h)\in Z:\ h=\varphi(y)\}.$$

Given an open set $X \subset \mathbb{R}^n$ and $\delta > 0$, we set $d_\delta(X)(x) := \min\{1, (1/\delta)\mathrm{dist}(x, \mathbb{R}^n \setminus X)\}$, so that $d_\delta(X)$ converges to χ_X as $\delta \to 0^+$. Testing the equation satisfied by v_k with the function $d_\delta(Z^+(\sigma_k g))$ we have

$$\begin{aligned}\int_B |\nabla v_k|^{p-2}\nabla v_k \cdot \nabla d_\delta(Z^+(\sigma_k g))\, dx = \frac{r_k}{q_{u_k}(0)^{p-1}} \int_B d_\delta(Z^+(\sigma_k g))\, dx \\ - \int_{B\cap\partial\{v_k>0\}} \left(\frac{q_{u_k}(r_k x)}{q_{u_k}(0)}\right)^{p-1} d_\delta(Z^+(\sigma_k g))\, d\mathcal{H}^{n-1}.\end{aligned}$$

Observe that up to replacing g by $g+\varepsilon$ for some arbitrarily small ε we may assume without loss of generality that $\mathcal{H}^{n-1}(Z_0(\sigma_k g) \cap \partial\{v_k > 0\}) = 0$. Therefore, letting $\delta \to 0^+$ in the equality above we easily get, denoting by ν the exterior normal to $Z^+(\sigma_k g)$,

$$\begin{aligned}-\int_{Z_0(\sigma_k g)\cap\{v_k>0\}} |\nabla v_k|^{p-2}\nabla v_k \cdot \nu\, d\mathcal{H}^{n-1} = \frac{r_k}{q_{u_k}(0)^{p-1}} |Z^+(\sigma_k g)| \\ - \int_{Z^+(\sigma_k g)\cap\partial\{v_k>0\}} \left(\frac{q_{u_k}(r_k x)}{q_{u_k}(0)}\right)^{p-1} d\mathcal{H}^{n-1}.\end{aligned}$$

From this equality, recalling that $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ in B_{r_k} and thus that $|\nabla v_k| \le (1+\tau_k)$ in B, we have

$$\begin{aligned}&(1 - Cr_k)\mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) \\ &\qquad \le (1+\tau_k)^{p-1}\mathcal{H}^{n-1}(Z_0(\sigma_k g) \cap \{v_k > 0\}) + \frac{r_k}{q_{u_k}(0)^{p-1}} |Z^+(\sigma_k g)|,\end{aligned}$$

where $C > 0$ depends only on p and $\|q_u\|_{W^{1,\infty}}$. From this inequality we readily infer

$$\mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) \le (1 + C(r_k + \tau_k))\mathcal{H}^{n-1}(Z_0(\sigma_k g) \cap \{v_k > 0\}) + Cr_k, \tag{5.18}$$

where C still denotes a positive constant independent of k. Defining

$$E_k := Z^-(\sigma_k g) \cup \{v_k > 0\},$$

the set E_k has finite perimeter in the cylinder Z and

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) \le \mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{v_k > 0\}) + \mathcal{H}^{n-1}(Z_0(\sigma_k g) \cap \{v_k = 0\}). \tag{5.19}$$

We now recall the following excess estimate, proved in [3, p. 136].

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) \ge \mathcal{H}^{n-1}(Z_0(\sigma_k g)) + c_0\sigma_k^2,$$

where c_0 is an absolute positive constant. Note that the proof of this estimate uses only the fact that g is harmonic, that $f(y_0) > g(y_0)$ and that $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ and therefore holds verbatim also in our case. Comparing this inequality with (5.19) and (5.18) above, we then conclude that

$$c_0\sigma_k^2 \le C(r_k + \tau_k)\mathcal{H}^{n-1}(Z_0(\sigma_k g) \cap \{v_k > 0\}) + Cr_k,$$

thus getting a contradiction, since both r_k and τ_k are $o(\sigma_k^2)$. This contradiction proves the lemma. □

The proof of next and final lemma is obtained arguing exactly as in the proof of Lemma 8.3 in [12] with very minor changes, after performing the same rescaling as in the proof of Lemma 5.10 above and makes use of Theorem 5.7 and of the assumption that r_k is converging to zero faster than τ_k.

Lemma 5.11. *There exists a positive constant C such that, for any $y \in B'_{\frac{r}{2}}$*

$$\int_0^{\frac{1}{2}} \frac{dr}{r^2} \left(\fint_{\partial B'_r(y)} (f(x) - f(y))\, dy \right) \leq C.$$

At this point we have all the ingredients to start the decay argument needed to prove Theorem 3.23. Indeed the proof goes exactly as the proof of Theorem 8.2 in [3].

Acknowledgment

The work of N. F. has been carried on at the University of Jyväskylä and supported by the FiDiPro project 2100002028. The work of Y. Z. was partially supported by the Academy of Finland via the Centre of Excellence in Analysis and Dynamics Research, grant no. 271983. The authors want also to thank the Mathematics and Statistics Department of Jyväskylä and the Department of Mathematics of the University of Napoli for providing them hospitality and a friendly atmosphere.

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