MULTISCALE YOUNG MEASURES IN HOMOGENIZATION OF CONTINUOUS STATIONARY PROCESSES IN COMPACT SPACES AND APPLICATIONS

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Abstract. We introduce a framework for the study of nonlinear homogenization problems in the setting of stationary continuous processes in compact spaces. The latter are functions $f \circ T : \mathbb{R}^n \times \mathcal{Q} \to \mathcal{Q}$ with $f \circ T(x, \omega) = f(T(x)\omega)$ where $\mathcal{Q}$ is a compact (Hausdorff topological) space, $f \in C(\mathcal{Q})$ and $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, is an n-dimensional continuous dynamical system endowed with an invariant Radon probability measure μ. It can be easily shown that for almost all $\omega \in \mathcal{Q}$ the realization $f(T(x)\omega)$ belongs to an algebra with mean value, that is, an algebra of functions in $\mathrm{BUC}(\mathbb{R}^n)$ containing all translates of its elements and such that each of its elements possesses a mean value. This notion was introduced by Zhikov and Krivenko (1983). We then establish the existence of multiscale Young measures in the setting of algebras with mean value, where the compactifications of $\mathbb{R}^n$ provided by such algebras plays an important role. These parametrized measures are useful in connection with the existence of correctors in homogenization problems. We apply this framework to the homogenization of a porous medium type equation in $\mathbb{R}^n$ with a stationary continuous process as a stiff oscillatory external source. This application seems to be new even in the classical context of periodic homogenization.

1. Introduction

Continuous dynamical systems in compact spaces constitute a classical matter going back to pioneering works of Birkhoff, von Neumann, Khintchine, Kolmogorov, Markov, Hopf, Krylov and Bogolyubov, among others, during the 1930's. They provide a natural setting for stochastic homogenization problems which extends the setting of periodic and almost periodic functions and also combines topological and measure theoretic features that usually allow a better understanding of the involved questions. Following a series of important papers on stochastic homogenization of linear differential operators by Zhikov et al. [48, 49, 50] (see also [28]), Zhikov and Krivenko [51] introduced the notion of algebras with mean value which captures the essential properties of typical realizations of continuous stationary processes defined by continuous dynamical systems in compact spaces endowed with an invariant probability measure. More specifically, let $\mathcal{Q}$ be a compact (Hausdorff topological) space and $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, be an n-dimensional continuous dynamical system, that is, $T(0)\omega = \omega$, $T(x + y)\omega = T(x)T(y)\omega$, for all $\omega \in \mathcal{Q}$, and the mapping $T : \mathbb{R}^n \times \mathcal{Q} \to \mathcal{Q}$ given by $T(x, \omega) = T(x)\omega$ is continuous. A classical result of Krylov and Bogolyubov [32] establishes the existence of an invariant (regular) probability measure μ on $\mathcal{Q}$ for $T(x)$; that is $\mu(T(x)E) = \mu(E)$ for Borelian E. So we may assume that $\mathcal{Q}$ is endowed with such an invariant probability measure. A stationary continuous process is a mapping $(x, \omega) \mapsto f(T(x)\omega)$ where $f \in C(\mathcal{Q})$ and $\{T(x)\}_{x\in\mathbb{R}^n}$ is an n-dimensional continuous dynamical system on a compact space $\mathcal{Q}$ endowed with some invariant measure. The dynamical system (endowed with an invariant measure) is said to be ergodic if whenever $f \in L^2(\mathcal{Q})$ satisfies $f(T(x)\omega) = f(\omega)$ for μ-a.e. $\omega \in \mathcal{Q}$, for all $x \in \mathbb{R}^n$, then f is equivalent to a constant.

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Given any $f \in C(\mathcal{Q})$, by means of the well known Birkhoff ergodic theorem, one easily shows that for almost all $\omega \in \mathcal{Q}$ the realization $f(T(x)\omega)$ belongs to a linear subspace $\mathcal{A} \subseteq \mathrm{BUC}(\mathbb{R}^n)$, where $\mathrm{BUC}(\mathbb{R}^n)$ is the space of bounded uniformly continuous functions in $\mathbb{R}^n$, with the following properties: (i) $\mathcal{A}$ is an algebra, i.e., if $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$; (ii) if $f \in \mathcal{A}$, then its translates $f(\cdot + t)$, $t \in \mathbb{R}^n$, also belong to $\mathcal{A}$; (iii) every $f \in \mathcal{A}$ possesses a mean value. A linear subspace of $\mathrm{BUC}(\mathbb{R}^n)$ satisfying these three properties is called an algebra with mean value (algebra w.m.v., for short). Given an algebra w.m.v. $\mathcal{A}$ we may define the associated generalized Besicovitch space $\mathcal{B}^2$ as the completion of $\mathcal{A}$ with respect to the semi-norm provided by the square root of the mean value of $|f|^2$ for $f \in \mathcal{A}$. The algebra w.m.v. $\mathcal{A}$ is said to be ergodic if whenever $f \in \mathcal{B}^2$ satisfies $f(\cdot + x) = f(\cdot)$ in $\mathcal{B}^2$ for all $x \in \mathbb{R}^n$, then f is equivalent in $\mathcal{B}^2$ to a constant. It can be shown that for almost all $\omega \in \mathcal{Q}$ the realization $f(T(x)\omega)$ just mentioned belongs to an ergodic algebra, even if the dynamical system is not ergodic.

We then follow the approach in [3], defining vector valued algebras with mean value and establishing the existence of multiscale Young measures in the setting of vector valued algebras with mean value. For that, as in the case of almost periodic functions, we make essential use of the fact that associated with any algebra w.m.v. $\mathcal{A}$ there is a compact space $\mathcal{K}$ such that any $f \in \mathcal{A}$ may be viewed as an element of $C(\mathcal{K})$, which follows from a classical theorem of Stone, as is shown below (cf. Theorem 4.1). Such compact space associated with the algebra w.m.v. provides the additional parameter of the multiscale (two-scale) Young measures. The latter are useful tools for the search of corrector functions in nonlinear homogenization problems.

We show how this framework can be applied in the homogenization of nonlinear partial differential equations by considering the homogenization problem for a porous medium type equation with a stationary continuous process as a stiff oscillatory external source. In this general context we need to restrict the initial data to prepared ones, that is, those which satisfy an associated stationary equation in the oscillatory variable.

Multiscale Young measures have been introduced in periodic problems by W. E [20] as a broader tool extending the previous concept of multiscale convergence introduced by Nguetseng [38] and further developed by Allaire [1]. It refines to multiple scale analysis the classical concept of Young measures introduced in [46], so fundamentally useful, especially after its striking applications in connection with problems concerning compactness of solution operators for nonlinear partial differential equations by Tartar [45], Murat [36], DiPerna [17, 18, 19], etc.. This paper links multiscale Young measures to the recently growing interest in the more general setting of homogenization of random stationary ergodic processes (see, e.g., [41], [30], [28], [15], [44], [35], [11]).

The extension of the multiscale Young measures from the periodic setting to the almost periodic one was carried out in [3] where applications to nonlinear transport equations, scalar conservation laws with oscillatory external sources, Hamilton-Jacobi equations and fully nonlinear elliptic equations are provided. In this connection, we recall that the two-scale convergence has been extended to the context of almost periodic homogenization and, more generally, to generalized Besicovitch spaces in [13] (see also, e.g., [39, 40]). We also recall that the method of two-scale convergence was extended to the context of stochastic homogenization, under separability assumption, in [8]. The applications in the cited references [13, 39, 40, 8] are basically to linear or monotone operators.

This paper is organized as follows. In Section 2, we recall some concepts in order to state the well known Birkhoff Ergodic Theorem, which will be used in later sections, also recall the definition of continuous dynamical systems, the classical theorem of Krylov and Bogolyubov and give some elementary examples. In Section 3 we recall the definition of algebras with mean value introduced in [51]. The purpose of Section 4 is to establish the connection between algebras with mean value and continuous dynamical systems in compact spaces. We also analyse the characterization of $\mathrm{AP}(\mathbb{R}^n)$ by means of the properties of the associated compact spaces. In Section 5 we introduce the vector-valued algebras w.m.v. which are needed in the construction of the multiscale Young measures in the context of algebras w.m.v.. In Section 6 we establish the theorem on the existence of multiscale Young measures from homogenization in algebras w.m.v.. In Section 7 we apply the general framework established in the earlier sections to the homogenization problem of a porous medium type equation in $\mathbb{R}^n$ with a stationary continuous process as a stiff oscillatory external source, and oscillatory initial data satisfying a stationary equation in the oscillatory variable. We also include the Appendix A where we state without proof some basic results that are needed in Section 7.

2. Stationary Processes

We begin this section by recalling the definition of n-dimensional dynamical system in a probability measure space, as a preparation for the statement of the Birkhoff Ergodic Theorem.

Definition 2.1 (n-dimensional dynamical system)**.** Let $(\mathcal{Q}, \mathcal{M}(\mathcal{Q}), \mu)$ be any probability measure space. An *n-dimensional dynamical system on* $\mathcal{Q}$ is a family of mappings $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, which satisfies the following conditions:

(i) (Group Property) $T(0) = I$, where I is the identity mapping on $\mathcal{Q}$, and

$$T(x + y) = T(x)T(y), \qquad \forall x, y \in \mathbb{R}^n;$$

(ii) (Invariance) The mappings $T(x) : \mathcal{Q} \to \mathcal{Q}$ are measurable and μ-measure preserving, i.e.,

$$\mu(T(x)(E)) = \mu(E) \qquad \text{for every } x \in \mathbb{R}^n \text{ and every } E \in \mathcal{M}(\mathcal{Q});$$

(iii) (Measurability) Given any $F \in \mathcal{M}(\mathcal{Q})$ the set $\{(x, \omega) \in \mathbb{R}^n \times \mathcal{Q} \,:\, T(x)\omega \in F\} \subseteq \mathbb{R}^n \times \mathcal{Q}$ is measurable with respect to the product σ-algebra $\mathcal{L}_n \otimes \mathcal{M}(\mathcal{Q})$, where $\mathcal{L}_n$ is the σ-algebra of Lebesgue measurable sets.

As usual, for $p \geq 1$ we denote by $L^p(\mathcal{Q})$ be the space of the (equivalence classes of) measurable functions $f : \mathcal{Q} \to \mathbb{R}$ such that $|f|^p$ is μ-integrable on $\mathcal{Q}$, and by $L^\infty(\mathcal{Q})$ the space of the μ-essentially bounded measurable functions. For $f \in L^p(\mathcal{Q})$ and $f \in L^\infty(\mathcal{Q})$ respectively we denote

$$\|f\|_p := \left(\int_{\mathcal{Q}} |f|^p \, d\mu \right)^{1/p}, \qquad \|f\|_\infty := \operatorname*{ess\,sup}_{\omega \in \mathcal{Q}} |f(\omega)|.$$

An n-dimensional dynamical system $T(x) : \mathcal{Q} \to \mathcal{Q}$ induces an n-parameter group of transformations $T(x) : L^2(\mathcal{Q}) \to L^2(\mathcal{Q})$ defined by

$$(T(x)f)(\omega) := f(T(x)\omega), \qquad f \in L^2(\mathcal{Q}).$$

It follows that the operator $T(x) : L^2(\mathcal{Q}) \to L^2(\mathcal{Q})$ is unitary for each $x \in \mathbb{R}^n$. Moreover, it is a consequence of the Lebesgue Dominated Convergence theorem (see [28], p. 223) that the group $T(x)$ is strongly continuous, i.e.,

$$\lim_{x \to 0} \|T(x)f - f\|_2 = 0, \qquad \forall f \in L^2(\mathcal{Q}). \tag{2.1}$$

Definition 2.2 (Ergodic dynamical system)**.** Let $(\mathcal{Q}, \mathcal{M}(\mathcal{Q}), \mu)$ be any probability measure space and let $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, be an n-dimensional dynamical system on $\mathcal{Q}$. A $\mathcal{M}(\mathcal{Q})$-measurable function $f : \mathcal{Q} \to \mathbb{R}$ is called *invariant* if $f(T(x)\omega) = f(\omega)$ μ-almost everywhere in $\mathcal{Q}$, for all $x \in \mathbb{R}^n$. A dynamical system is said to be *ergodic* if every invariant function is μ-equivalent to a constant in $\mathcal{Q}$.

If f is a measurable function in $\mathcal{Q}$, for a fixed $\omega \in \mathcal{Q}$ the function $x \mapsto f(T(x)\omega)$, $x \in \mathbb{R}^n$, is called a *realization of* f and the map $(x, \omega) \mapsto f(T(x)\omega)$ is called a *stationary process*. The process is said to be *stationary ergodic* if the dynamical system is ergodic.

We will make use of the well known Birkhoff Ergodic Theorem. In order to state it we need to introduce the notion of mean value for functions defined in $\mathbb{R}^n$.

Definition 2.3. Let $g \in L^1_{\rm loc}(\mathbb{R}^n)$. A number $M(g)$ is called the *mean value of* g if

$$\lim_{\varepsilon\to 0}\int_A g(\varepsilon^{-1}x)\,dx = |A|M(g) \tag{2.2}$$

for any Lebesgue measurable bounded set $A \subseteq \mathbb{R}^n$, where $|A|$ stands for the Lebesgue measure of A. This is equivalent to say that $g(\varepsilon^{-1}x)$ converges, in the duality with L^∞ and compactly supported functions, to the constant $M(g)$. Also, if $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$ for $t > 0$ and $|A| \neq 0$, (2.2) may be written as

$$\lim_{t\to\infty}\frac{1}{t^n|A|}\int_{A_t} g(x)\,dx = M(g). \tag{2.3}$$

We now recall the Birkhoff Ergodic Theorem (see [16]).

Theorem 2.1 (Birkhoff Ergodic Theorem)**.** *Let $f \in L^p(\mathcal{Q})$, $p \geq 1$. Then for almost all $\omega \in \mathcal{Q}$ the realization $g(x) = f(T(x)\omega)$ possesses a mean value in the sense of (2.2). Moreover, the mean value $M(f(T(\cdot)\omega))$ is invariant and*

$$\int_{\mathcal{Q}} f(\omega)\,d\mu = \int_{\mathcal{Q}} M(f(T(\cdot)\omega))\,d\mu.$$

In particular, if the system $T(x)$ is ergodic, then

$$M(f(T(\cdot)\omega)) = \int_{\mathcal{Q}} f\,d\mu \qquad \text{for } \mu\text{-almost all } \omega \in \mathcal{Q}.$$

Throughout the remaining of this paper we will be dealing with continuous n-dimensional dynamical systems $T(x)$ on compact topological spaces whose definition we recall now.

Definition 2.4. Let $\mathcal{Q}$ be a compact topological space. A *continuous n-dimensional dynamical system on* $\mathcal{Q}$ is a family of mappings $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, which satisfies the following conditions:

(i) $T(0) = I$, where I is the identity mapping on $\mathcal{Q}$, and

$$T(x+y) = T(x)T(y), \qquad \forall x, y \in \mathbb{R}^n;$$

(ii) the mapping $(x, \omega) \mapsto T(x)\omega$ is continuous from $\mathbb{R}^n \times \mathcal{Q}$ to $\mathcal{Q}$.

Henceforth by *compact space* we will always mean a compact Hausdorff topological space. Moreover, in compact spaces $\mathcal{Q}$ we shall always consider *Radon* measures. By a Radon measure μ we mean that μ is defined on the σ-algebra $\mathcal{B}(\mathcal{Q})$ of Borel sets, it is σ-additive and regular, in the sense that

$$\mu(B) = \inf\{\mu(A) : A \supset B,\ A \text{ open}\}, \qquad \text{for all } B \in \mathcal{B}(\mathcal{Q}),$$

and

$$\mu(B) = \sup\{\mu(K) : K \subseteq B,\ K \text{ compact}\}, \qquad \text{for all } B \in \mathcal{B}(\mathcal{Q}).$$

We recall that for a Radon probability measure μ on a compact space $\mathcal{Q}$, the space $C(\mathcal{Q})$ is dense in the spaces $L^p(\mathcal{Q}, \mu)$ of Borel functions whose p-th power of the absolute value is μ-integrable, $1 \leq p < \infty$ (see [42], p.69).

A well known theorem of Krylov and Bogolyubov [32] (see also [37]) asserts that for any continuous dynamical system $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, there exist invariant Borel probability measures when $\mathcal{Q}$ is a compact metric space. The result holds more generally when $\mathcal{Q}$ is any compact Hausdorff topological space and the proof of the more general statement is essentially the same as that of Bogolyubov and Krylov with minor adaptations.

Theorem 2.2 (Krylov-Bogolyubov)**.** *Let $\mathcal{Q}$ be a compact Hausdorff topological space and let $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, be an n-dimensional continuous dynamical system on $\mathcal{Q}$. Then, there exists a probability Radon measure μ on $\mathcal{Q}$ invariant under $T(x)$, $x \in \mathbb{R}^n$.*

Let be given any continuous dynamical system $T(x)$, $x \in \mathbb{R}^n$, on a compact space $\mathcal{Q}$, and any probability Radon measure μ invariant under $T(x)$, $x \in \mathbb{R}^n$. Then, if we choose as $\mathcal{M}(\mathcal{Q})$ the Borel σ-algebra, $T(x)$ can be viewed as an n-dynamical system according to Definition 2.1. To prove this fact, the only nontrivial property to be checked is (iii). The class of Borel sets $E \subseteq \mathcal{Q}$ such that $\{(x,\omega) \in \mathbb{R}^n \times \mathcal{Q} : T(x)\omega \in E\}$ belongs to the product σ-algebra $\mathcal{L}_n \otimes \mathcal{B}(\mathcal{Q})$ contains the class of open sets and it is a σ-algebra; therefore, it coincides with $\mathcal{B}(\mathcal{Q})$.

Definition 2.5 (Continuous stationary process)**.** Given a compact space $\mathcal{Q}$, an n-dimensional continuous dynamical system $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, and an invariant Radon probability measure μ in $\mathcal{Q}$, by a *continuous stationary process* we mean any map $(x,\omega) \mapsto f(T(x)\omega)$ with $f \in C(\mathcal{Q})$.

We next give two basic examples of this setting.

2.1. **Periodic functions.** In this case $\mathcal{Q}$ is the torus $(S^1)^n$ and $T(x) : \mathcal{Q} \to \mathcal{Q}$ is defined as $T(x)\omega := \omega + x \pmod 1$, where we adopt the usual equivalence between S^1 and $[0,1]$ with the identification $0 \equiv 1$. We easily verify that $T(x)$ is a continuous dynamical system. The Lebesgue measure is invariant and it is also easy to see that $T(x)$ is ergodic. Observe that $C(\mathcal{Q})$ is isometrically isomorphic to the space of continuous periodic functions with period 1 in each coordinate variable.

2.2. **Almost periodic functions.** This case was extensively studied in [3]. The basic fact here is that the space of almost periodic functions is a closed subalgebra of the space of bounded uniformly continuous functions in $\mathbb{R}^n$ which induces a compactification of $\mathbb{R}^n$, called Bohr compactification, $\mathbb{G}^n$, which turns out to be a topological group with respect to the extension to $\mathbb{G}^n$ of the addition operation in $\mathbb{R}^n$. Hence, in $\mathbb{G}^n$ the Haar measure is defined and is invariant with respect to the translations $T(x) : \mathbb{G}^n \to \mathbb{G}^n$, $T(x)\omega = \omega + x$. In [3] it is shown that such maps $T(x)$ form an ergodic continuous n-dimensional dynamical system.

We leave the more general example of the algebras with mean value to be thoroughly considered in the next three sections, since the deep understanding of its relationship with continuous dynamical systems acting on compact spaces is a central point of this work.

3. Algebras with Mean Value

The concept of algebra with mean value was introduced in [51] (see also [28]) as a generalization of the concept of almost periodic functions $\mathrm{AP}(\mathbb{R}^n)$ and the corresponding Besicovitch spaces $\mathrm{BAP}^p(\mathbb{R}^n)$, $1 \le p \le \infty$ (cf. [3]), motivated by the reduction of problems of stochastic homogenization to problems of individual homogenization, in the terminology adopted in [28].

Notation: As usual, we denote by $\mathrm{BUC}(\mathbb{R}^n)$ the space of the bounded uniformly continuous real-valued functions in $\mathbb{R}^n$.

Definition 3.1. Let $\mathcal{A}$ be a linear subspace of $\mathrm{BUC}(\mathbb{R}^n)$. We say that $\mathcal{A}$ is an *algebra with mean value* (or *algebra w.m.v.*, in short), if the following conditions are satisfied:

- (A) If f and g belong to $\mathcal{A}$, then the product fg belongs to $\mathcal{A}$.
- (B) $\mathcal{A}$ is invariant with respect to translations τ_y in $\mathbb{R}^n$.
- (C) Any $f \in \mathcal{A}$ possesses a mean value.
- (D) $\mathcal{A}$ is closed in $\mathrm{BUC}(\mathbb{R}^n)$ and contains the unity, i.e., the function $e(x) := 1$ for $x \in \mathbb{R}^n$.

Remark 3.1. The definition of algebra w.m.v. as given in [28] contains only conditions (A), (B) and (C). However, since the closure of a linear subspace $\mathcal{A}$ in $\mathrm{BUC}(\mathbb{R}^n)$ satisfying (A), (B) and (C) also satisfies (A), (B) and (C) and adding the unit to such an $\mathcal{A}$ one obtains a linear subspace of $\mathrm{BUC}(\mathbb{R}^n)$ still satisfying (A), (B) and (C), the inclusion of condition (D) does not imply any restriction in the theory, and we do that here just for convenience.

For the development of the homogenization theory in algebras $\mathcal{A}$ with mean value, as is done in [51, 28] (see also [13]), in similarity with the case of almost periodic functions, one introduces, for $1 \le p < \infty$, the space $\mathcal{B}^p$ as the abstract completion of $\mathcal{A}$ with respect to the Besicovitch seminorm

$$|f|_p^p := \limsup_{L\to\infty} \frac{1}{(2L)^n} \int_{[-L,L]^n} |f|^p \, dx.$$

Both the action of translations and the mean value extend by continuity to $\mathcal{B}^p$, and we will keep using the notation $\tau_y f$ and $M(f)$ even when $f \in \mathcal{B}^p$ and $y \in \mathbb{R}^n$. Furthermore, for $p > 1$ the product in $\mathcal{A}$ extends to a bilinear operator from $\mathcal{B}^p \times \mathcal{B}^q$ into $\mathcal{B}^1$, with q equal to the dual exponent of p, satisfying

$$|fg|_1 \le |f|_p |g|_q.$$

In particular, the operator $M(fg)$ provides a nonnegative definite bilinear form on $\mathcal{B}^2$.

Remark 3.2. A classical argument going back to Besicovitch [7] (see also [28], p.239) shows that the elements of $\mathcal{B}^p$ can be represented by functions in $L^p_{\rm loc}(\mathbb{R}^n)$, $1 \le p < \infty$.

Since there is an obvious inclusion between this family of spaces, we may define the space $\mathcal{B}^\infty$ as follows:

$$\mathcal{B}^\infty = \{f \in \bigcap_{1\le p<\infty} \mathcal{B}^p \,:\, \sup_{1\le p<\infty} |f|_p < \infty\},$$

We endow $\mathcal{B}^\infty$ with the (semi)norm

$$|f|_\infty := \sup_{1\le p<\infty} |f|_p.$$

Obviously the corresponding quotient spaces for all these spaces (with respect to the null space of the seminorms) are Banach spaces, and we get an Hilbert space in the case $p = 2$. We denote by $\overset{\mathcal{B}^p}{=}$, the equivalence relation given by the equality in the sense of the $\mathcal{B}^p$ semi-norm.

A group of unitary operators $T(y) : \mathcal{B}^2 \to \mathcal{B}^2$ is then defined by setting $[T(y)f] = \tau_y \circ f$. Since the elements of $\mathcal{A}$ are uniformly continuous in $\mathbb{R}^n$, the group $\{T(y)\}$ is strongly continuous, i.e. $T(y)f \to f$ in $\mathcal{B}^2$ as $y \to 0$ for all $f \in \mathcal{B}^2$. The notion of invariant function is introduced then by simply saying that a function in $\mathcal{B}^2$ is *invariant* if $T(y)f \overset{\mathcal{B}^2}{=} f$, for all $y \in \mathbb{R}^n$. More clearly, $f \in \mathcal{B}^2$ is invariant if

$$M\big(|T(y)f - f|^2\big) = 0, \qquad \forall y \in \mathbb{R}^n. \tag{3.1}$$

The concept of ergodic algebra is then introduced as follows.

Definition 3.2. An algebra $\mathcal{A}$ w.m.v. is called *ergodic* if any invariant function f belonging to the corresponding space $\mathcal{B}^2$ is equivalent (in $\mathcal{B}^2$) to a constant.

An alternative definition of ergodic algebra is also given in [28], which is shown therein to be equivalent to Definition 3.2, by using the von Neumann's Ergodic Theorem. We state that as the following lemma, whose detailed proof may be found in [28], p.247.

Lemma 3.1. *Let $\mathcal{A} \subseteq \mathrm{BUC}(\mathbb{R}^n)$ be an algebra with mean value. Then $\mathcal{A}$ is ergodic if and only if*

$$\lim_{t\to\infty} M_y \left(\Big| \frac{1}{|B(0;t)|} \int_{B(0;t)} f(x+y)\, dx - M(f) \Big|^2 \right) = 0 \qquad \forall f \in \mathcal{A}. \tag{3.2}$$

The importance of algebras w.m.v. in connection with continuous dynamical systems is well expressed by the following result, which is a particular case of a more general assertion stated (without proof) in [28].

Theorem 3.1. *Let $\mathcal{Q}$ be a compact space, $T(x) : \mathcal{Q} \to \mathcal{Q}$, $x \in \mathbb{R}^n$, a continuous dynamical system, μ a Radon probability invariant measure in $\mathcal{Q}$, and $V \subseteq C(\mathcal{Q})$ a separable subspace. Then, for μ-almost all $\omega \in \mathcal{Q}$, there is an ergodic algebra containing the set of realizations $\big\{f(T(\cdot)\omega)\big\}_{f\in V}$.*

Proof. 1. We first show that for μ-a.a. $\omega \in \mathcal{Q}$ the closed algebra $\mathcal{A}$ generated by the family $\{f(T(\cdot + y)\omega);\ y \in \mathbb{R}^N,\ f \in V\}$ is an algebra w.m.v.. To this aim, let $\{f_j\}_{j\ge1}$ be a countable dense subset of V and let us denote by $\mathcal{B}$ the countable class of all functions $h \in C(\mathcal{Q})$ which are finite linear combinations with rational coefficients of products of the form $f_{j_1}(T(y_1)\omega)f_{j_2}(T(y_2)\omega)\cdots f_{j_k}(T(y_k)\omega)$, $j_1,\dots,j_k \ge 1$ and $y_1,\dots,y_k \in \mathbb{Q}^n$. Obviously the set

$$\mathcal{A}' := \{h(T(\cdot)\omega);\ h \in \mathcal{B}\}$$

is a countable dense subalgebra of $\mathcal{A}$. On the other hand, due to Birkhoff's Ergodic Theorem, the functions $f(T(\cdot + y)\omega)$, $y \in \mathbb{Q}^n$, possess mean values for all $f \in C(\mathcal{Q})$ as well as any finite combination $f_{j_1}(T(\cdot + y_1)\omega)f_{j_2}(T(\cdot + y_2)\omega)\cdots f_{j_k}(T(\cdot + y_k)\omega)$, $y_1,\cdots,y_k \in \mathbb{Q}^n$, for μ-a.e. $\omega \in \mathcal{Q}$. As a consequence any function in $\mathcal{A}'$ has mean value, and a density argument extends this property to the whole of $\mathcal{A}$.

2. Now we show that choosing ω adequately out of a suitable null set the algebra w.m.v. $\mathcal{A}$ is, in fact, ergodic (even if the given dynamical system is not ergodic!).To this aim, we first notice that it suffices to check the ergodicity condition (3.2) only for $g \in \mathcal{A}'$ w.m.v., by a standard density argument. The elements of $\mathcal{A}'$ are of the form $h(T(x)\omega)$ with $h \in C(\mathcal{Q})$. Therefore, given $h \in C(\mathcal{Q})$, it suffices to check that

$$\lim_{t\to\infty} M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} h(T(x+y)\omega)\,dx - M(h(T(\cdot)\omega))\Big|^2\right) = 0$$

for μ-a.a. $\omega \in \mathcal{Q}$. Now, let us define

$$\gamma_t(\omega) := \Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} h(T(x)\omega)\,dx - M(h(T(\cdot)\omega))\Big|^2,$$

and set $\Gamma_t(\omega) := M_y(\gamma_t(T(y)\omega))$. By von Neumann's Ergodic Theorem (see [16]) we have that $\lim_{t\to\infty}\Gamma_t(\omega)$ exists, for each fixed ω in a subset of $\mathcal{Q}$ of measure 1. Now, by dominated convergence and Birkhoff Ergodic Theorem, we have

$$\int_{\mathcal{Q}} \lim_{t\to\infty}\Gamma_t(\omega)\,d\mu(\omega) = \lim_{t\to\infty}\int_{\mathcal{Q}}\Gamma_t(\omega)\,d\mu(\omega) = \lim_{t\to\infty}\int_{\mathcal{Q}}\gamma_t(\omega)\,d\mu(\omega) = 0.$$

Hence, $\lim_{t\to\infty}\Gamma_t(\omega) = 0$ for μ-a.a. $\omega \in \mathcal{Q}$. Therefore, by passing to a smaller subset of $\mathcal{Q}$ with measure 1, if necessary, we have that the algebra w.m.v. $\mathcal{A}$ constructed in step 1. is in fact an ergodic algebra. □

The following result first established in [51] (see also [28]) describes the main property of ergodic algebras. We refer to [51] and [28] for the proof.

Theorem 3.2 (Zhikov & Krivenko [51])**.** *Let $\mathcal{A}$ be an ergodic algebra in* $\mathrm{BUC}(\mathbb{R}^n)$*. Then the set of functions in $\mathcal{A}$ whose distributional Fourier transform has compact support not containing* $0 \in \mathbb{R}^n$ *is dense in the space* $V = \{f \in \mathcal{B}^2 : M(f) = 0\}$.

The following lemma will be used in Section 7 where we apply our framework to the homogenization of a porous medium type equation with oscillatory external source.

Lemma 3.2. *Let $\mathcal{A}$ be an ergodic algebra in* $\mathrm{BUC}(\mathbb{R}^n)$ *and $h \in \mathcal{B}^2$ such that $M(h\Delta f) = 0$ for all $f \in \mathcal{A}$ such that $\Delta f \in \mathcal{A}$. Then h is equivalent to a constant.*

Proof. Set $V := \{f \in \mathcal{B}^2 : M(f) = 0\}$. It suffices verify that $M(hf) = 0$ for all $f \in V$. Indeed, let Y be the set of functions in $\mathcal{A}$ whose distributional Fourier transform has compact support not containing $0 \in \mathbb{R}^n$. According to Theorem 3.2 this set is dense in V. Moreover, given $f \in Y$, there exists $g \in \mathcal{A}$ such that $\Delta g = f$, as it is shown in [28], p.246. Therefore, given any $f \in Y$, we have $M(hf) = M(h\Delta g) = 0$, which then concludes the proof. □

Remark 3.3. In the case $n = 1$, a similar proof yields that if $\mathcal{A}$ is an ergodic algebra in $\mathrm{BUC}(\mathbb{R})$ and $h \in \mathcal{B}^2$ is such that $M(hf') = 0$ for all $f \in \mathcal{A}$ such that $f' \in \mathcal{A}$, then h is equivalent to a constant.

4. Compact spaces associated with algebras with mean value

We next show that any algebra w.m.v. may always be viewed as an algebra of continuous functions on a compact space $\mathcal{K}$ endowed with a continuous n-dimensional dynamical system $T(x):\mathcal{K}\to\mathcal{K}$ and an invariant Radon probability measure μ. We will make use of the following lemma which is a generalization of a lemma of [3], whose simple proof remains essentially the same and for which, therefore, we refer to [3].

Lemma 4.1. *Let X_1, X_2 be compact spaces, R_1 a dense subset of X_1 and $W:R_1\to X_2$. Suppose that for all $g\in C(X_2)$ the function $g\circ W$ is the restriction to R_1 of some (unique) $g_1\in C(X_1)$. Then W can be uniquely extended to a continuous mapping $\underline{W}:X_1\to X_2$.*
Further, suppose in addition that R_2 is a dense set of X_2, W is a bijection from R_1 onto R_2 and for all $f\in C(X_1)$, $f\circ W^{-1}$ is the restriction to R_2 of some (unique) $f_2\in C(X_2)$. Then W can be uniquely extended to a homeomorphism $\underline{W}:X_1\to X_2$.

We are now ready to prove the following result.

Theorem 4.1. *Let $\mathcal{A}\subseteq \mathrm{BUC}(\mathbb{R}^n)$ be an algebra with mean value. Then:*

- (i) *There exist a compact space $\mathcal{K}$ and an isometric isomorphism i identifying $\mathcal{A}$ with the algebra $C(\mathcal{K})$ of continuous functions on $\mathcal{K}$.*
- (ii) *$\mathcal{K}$ is a compactification of $\mathbb{T}^j\times\mathbb{R}^k$, for some integers j,k with $j+k\le n$, where $\mathbb{T}^j$ is the j-dimensional torus $\underbrace{S^1\times\cdots\times S^1}_{j\ times}$.*
- (iii) *The translations $T(y):\mathbb{R}^n\to\mathbb{R}^n$, $T(y)x=x+y$, extend to a group of homeomorphisms $T(y):\mathcal{K}\to\mathcal{K}$, $y\in\mathbb{R}^n$.*
- (iv) *The mean value operator $M(f)$ is representable by $\int_{\mathcal{K}} i(f)\,d\mathfrak{m}$ for some Radon probability measure $\mathfrak{m}$ on $\mathcal{K}$ which is invariant by the group of transformations $T(y)$, $y\in\mathbb{R}^n$.*
- (v) *The family $T(y)$, $y\in\mathbb{R}^n$, is a continuous n-dimensional dynamical system on $\mathcal{K}$.*

Proof. 1. If S is any set, denote by $B(S)$ be the Banach algebra of the bounded real-valued functions on S endowed with the sup-norm. When S is a normal topological space, a well known theorem of Stone (see [16], Theorem IV.6.18, p.274) says that if $\mathcal{U}$ is a closed subalgebra of $B(S)$ which contains the unit, then there exist a compact Hausdorff space S_1 and an isometric isomorphism between the algebras $\mathcal{U}$ and $C(S_1)$. In particular, there exist a compact topological space $\mathcal{K}$ and an isometric isomorphism between the algebra $\mathcal{A}$ and the algebra $C(\mathcal{K})$. If, in addition, the functions of the algebra $\mathcal{U}$ distinguish between the points of S, i.e., if $x\ne y$ then $f(x)\ne f(y)$ for some $f\in\mathcal{U}$, then a corollary of Stone's theorem quoted above (see [16], Corollary IV.6.19, p.276) asserts that there is a one-to-one embedding of S as a dense subset of the compact topological space $\mathcal{K}$ such that each $f\in\mathcal{U}$ has a unique continuous extension $\underline{f}$ to $\mathcal{K}$ and such correspondence $f\to\underline{f}$ is exactly the isometric isomorphism between $\mathcal{U}$ and $C(\mathcal{K})$.

2. Since in general the functions in $\mathcal{A}$ do not distinguish between the points of $\mathbb{R}^n$, we may introduce in $\mathbb{R}^n$ the equivalence relation $x\sim y$ iff $f(x)=f(y)$ for all $f\in\mathcal{A}$. We claim that $\mathbb{R}^n/\sim\,\equiv\mathcal{C}_{j,k}:=\mathbb{T}^j\times\mathbb{R}^k$, for some integers j,k with $j+k\le n$. Indeed, if $x\ne y$ and $x\sim y$, then, by the invariance of $\mathcal{A}$ under translations, $0\sim x-y$ and so

$$\mathcal{T}:=\{t\in\mathbb{R}\,:\,0\sim t(x-y)\}$$

is an additive and nontrivial subgroup of $\mathbb{R}$. So, either $\mathcal{T}=\mathbb{R}$ or $\mathcal{T}$ is a discrete subgroup $\tau\mathbb{Z}$ for some $\tau>0$. In the first case the functions of $\mathcal{A}$ are constant along all lines parallel to $x-y$. In the second case, all functions of $\mathcal{A}$ are periodic with period τ, that is, $f(z+\tau(x-y))=f(z)$ for all $z\in\mathbb{R}^n$ and all $f\in\mathcal{A}$. Continuing this procedure, we obtain a maximal independent set of vectors $v_1,\dots,v_j$, with $j\le n$, such that all functions of $\mathcal{A}$ are periodic in the direction v_i with period τ_i. Also, let l be the dimension of the space spanned by all directions along which all functions of $\mathcal{A}$ are constant. Hence, $\mathbb{R}^n/\sim$ can be naturally identified with $\mathcal{C}_{j,k}$, with $k=n-l-j$. In particular, all functions of $\mathcal{A}$ can be identified with bounded uniformly continuous function on $\mathcal{C}_{j,k}$.

3. Viewing $\mathcal{A}$ as a subalgebra of $B(\mathcal{C}_{j,k})$, the functions of $\mathcal{A}$ distinguish between the points of $\mathcal{C}_{j,k}$. Hence, there is a one-to-one embedding of $\mathcal{C}_{j,k}$ as a dense subset of the compact space $\mathcal{K}$ such that any function of $\mathcal{A}$ may be viewed as the restriction to $\mathcal{C}_{j,k}$ of a unique function belonging to $C(\mathcal{K})$.

4. There is a natural addition operation in $\mathcal{C}_{j,k}$, with respect to which it is an additive group, since it is the cartesian product of j copies of S^1, with $+(\mod 1)$, and k copies of $\mathbb{R}$, with the usual addition operation $+$. Also, for each $y \in \mathbb{R}^n$, the action of the translation $T(y) : x \mapsto x + y$ on $\mathbb{R}^n$ can also be read on the quotient space $\mathcal{C}_{j,k}$. Applying Lemma 4.1 with $X_1 = X_2 = \mathcal{K}$, $R_1 = R_2 = \mathcal{C}_{j,k} \subseteq \mathcal{K}$ we conclude that, for each $y \in \mathbb{R}^n$, $T(y)$ can be extended to a homeomorphism $\underline{T(y)} : \mathcal{K} \to \mathcal{K}$, and we keep using the notation $x + y$ for $\overline{T(y)}(x)$, $x \in \mathcal{K}$. For each $y \in \mathbb{R}^n$, $T(y)$ induces an isometry in $\mathcal{A}$ which we also denote by $T(y)$, defined by $[T(y)f](x) = f(x + y)$. Hence, this isometry extends to an isometry $T(y) : C(\mathcal{K}) \to C(\mathcal{K})$ and it turns out that $[T(y)f](z) = f(z + y)$.

5. If $y_k \to y$ in $\mathbb{R}^n$, then the isometries $T(y_k) : \mathcal{A} \to \mathcal{A}$ pointwise converge to the isometry $T(y) : \mathcal{A} \to \mathcal{A}$, as a consequence of the uniform continuity of the functions in $\mathcal{A}$. But, since $\mathcal{A}$ is isometrically isomorphic to $C(\mathcal{K})$, the corresponding sequence of isometries $T(y_n) : C(\mathcal{K}) \to C(\mathcal{K})$ converges also to the corresponding isometry $T(y) : C(\mathcal{K}) \to C(\mathcal{K})$. Hence, for any function $f \in C(\mathcal{K})$ we have $f(z + y_k) \to f(z + y)$ uniformly in $z \in \mathcal{K}$. In particular, this implies that, given a net $(z_d)_{d\in D}$ in $\mathcal{K}$ converging to $z \in \mathcal{K}$, and a sequence $(y_k) \subseteq \mathbb{R}^n$ converging to y, we have $f(T(y_k)z_d) \to f(T(y)z)$ for all $f \in C(\mathcal{K})$, because $f(z_d + y) \to f(z + y)$. Since $C(\mathcal{K})$ separates the points of $\mathcal{K}$, this is the same to say that $T(y_k)z_d \to T(y)z$ in $\mathcal{K}$, so that the mapping $(y, z) \mapsto T(y)z$ is continuous. Hence, $T(y)$, $y \in \mathbb{R}^n$, is a continuous dynamical system on $\mathcal{K}$.

6. The fact that the functions of $\mathcal{A}$ possess a mean value provides us with a linear functional $f \mapsto M(f)$ defined on $C(\mathcal{K})$. This linear functional is clearly bounded and nonnegative. Therefore, by the Riesz-Markov theorem, $M(f)$ is representable by integration with respect to a Radon measure $\mathfrak{m}$ in $\mathcal{K}$. Further, since the mean value is invariant by translations, $\mathfrak{m}$ is an invariant measure with respect to the dynamical system $T(y)$, $y \in \mathbb{R}^n$. □

As an immediate consequence of Theorem 4.1 we have the following result whose simple proof is left to the reader.

Theorem 4.2. *Let $\mathcal{A}$ be an algebra w.m.v. in $\mathbb{R}^n$ and let $\mathcal{K}$ be the compact space given by Theorem 4.1, such that $\mathcal{A}$ is isometrically isomorphic to $C(\mathcal{K})$, and $\mathfrak{m}$ be the corresponding invariant measure. Then, for $1 \le p \le \infty$, the generalized Besicovitch spaces $\mathcal{B}^p \big/ \overset{\mathcal{B}^p}{=}$ are isometrically isomorphic to $L^p(\mathcal{K}, \mathfrak{m})$. The family $T(y)$, $y \in \mathbb{R}^n$, is ergodic if and only if $\mathcal{A}$ is ergodic.*

We have seen above that for a given algebra with mean value $\mathcal{A}$ there is associated a compactification $\mathcal{K}$ of the corresponding Lie group $\mathcal{C}_{j,k} = \mathbb{T}^j \times \mathbb{R}^k$. Since $\mathbb{T}^j$ is itself compact, a natural question is whether the compact $\mathcal{K}$ could be represented as a Cartesian product $\mathcal{K} = \mathbb{T}^j \times \mathcal{K}'$ where $\mathcal{K}'$ is a compactification of $\mathbb{R}^k$. This is in fact the case as stated by the following result.

Theorem 4.3. *If $\mathcal{K}$ is a compactification of $\mathcal{C}_{j,k}$ associated with a closed subalgebra $\mathcal{A}$ of $\mathrm{BUC}(\mathcal{C}_{j,k})$ containing the unity, distinguishing between the points of $\mathcal{C}_{j,k}$, and invariant by the translations $f(\cdot+t)$, $t \in \mathcal{C}_{j,k}$, then $\mathcal{K}$ is homeomorphic to $\mathbb{T}^j \times \mathcal{K}'$ where $\mathcal{K}'$ is a compactification of $\mathbb{R}^k$. In particular, $\mathcal{A} = C(\mathbb{T}^j) \otimes \mathcal{A}'$ where $\mathcal{A}'$ is a subalgebra of $\mathrm{BUC}(\mathbb{R}^k)$ isometrically isomorphic to $C(\mathcal{K}')$.*

Proof. 1. First of all, the embedding $\mathbb{T}^j \to \mathcal{C}_{j,k} = \mathbb{T}^j \times \mathbb{R}^k$ composed with the continuous embedding $\mathcal{C}_{j,k} \to \mathcal{K}$, provides a continuous embedding of $\mathbb{T}^j$ into $\mathcal{K}$. Hence, $\mathbb{T}^j$ with the relative topology inherited from $\mathcal{K}$ is also compact. Therefore, the relative topology of $\mathbb{T}^j$, inherited from $\mathcal{K}$, being also Hausdorff, must coincide with the standard topology of $\mathbb{T}^j$.

2. We now consider the projection $\pi : \mathcal{C}_{j,k} = \mathbb{T}^j \times \mathbb{R}^k \to \mathbb{T}^j$, $\pi(\xi, \eta) = \xi$, $\xi \in \mathbb{T}^j$, $\eta \in \mathbb{R}^k$. From what we have just seen, given any function $g \in C(\mathbb{T}^j)$, this function is also continuous in the relative topology induced by the embeddings $\mathbb{T}^j \to \mathcal{C}_{j,k} \to \mathcal{K}$. By the invariance of $\mathcal{A}$ with respect to translations, and so, in particular, with respect to the "vertical" translations given the vectors $(0, \eta)$, $\eta \in \mathbb{R}^k$, we deduce that given

any pair of points $\eta_1, \eta_2 \in \mathbb{R}^k$, the sets $\mathbb{T}^j \times \{\eta_1\}$ and $\mathbb{T}^j \times \{\eta_2\}$, both with the relative topology of $\mathcal{K}$, are homeomorphic.

3. Let ξ_1, ξ_2 be any two points in $\mathbb{T}^j$ and let τ be the translation by $(\xi_2 - \xi_1, 0)$ which takes the points of $\pi^{-1}(\xi_1) \subseteq \mathcal{C}_{j,k}$ in a one-to-one way onto the points of $\pi^{-1}(\xi_2) \subseteq \mathcal{C}_{j,k}$. Due to the invariance of $\mathcal{A}$ by translations, given then any $g \in \mathcal{A}$, $[g|\pi^{-1}(\xi_2)] \circ \tau = g_1|\pi^{-1}(\xi_1)$ for some $g_1 \in \mathcal{A}$. Reciprocally, given any $f \in \mathcal{A}$, $[f|\pi^{-1}(\xi_1)] \circ \tau^{-1} = f_2|\pi^{-1}(\xi_2)$, for some $f_2 \in \mathcal{A}$. Therefore, Lemma 4.1 implies that $\tau : \pi^{-1}(\xi_1) \to \pi^{-1}(\xi_2)$ extends to a homeomorphism $\underline{\tau} : \overline{\pi^{-1}(\xi_1)} \to \overline{\pi^{-1}(\xi_2)}$, where $\overline{\pi^{-1}(\xi_1)}$ and $\overline{\pi^{-1}(\xi_2)}$ denote the closures of $\pi^{-1}(\xi_1)$ and $\pi^{-1}(\xi_2)$ in the relative topology induced by the topology of $\mathcal{K}$. Hence, all the spaces $\overline{\pi^{-1}(\xi)}$, $\xi \in \mathbb{T}^j$, are homeomorphic.

4. We also deduce from the above discussion that the relative topology of $\mathcal{C}_{j,k}$ induced by the embedding into $\mathcal{K}$ is also a product topology. In particular, the projection $\pi : \mathcal{C}_{j,k} \to \mathbb{T}^j$ is also continuous when $\mathcal{C}_{j,k}$ and $\mathbb{T}^j$ are both given the relative topology inherited from $\mathcal{K}$. Hence, π extends to a continuous surjective mapping $\underline{\pi} : \mathcal{K} \to \mathbb{T}^j$, and we have $\overline{\pi^{-1}(\xi)} = \underline{\pi}^{-1}(\xi)$, for all $\xi \in \mathbb{T}^j$.

5. Summing up the above arguments, we arrive at the conclusion that $\mathcal{K}$ is homeomorphic to $\mathbb{T}^j \times \underline{\pi}^{-1}(\xi_0)$, for an arbitrarily taken $\xi_0 \in \mathbb{T}^j$. Since $\underline{\pi}^{-1}(\xi_0)$ is clearly a compactification of $\mathbb{R}^k$, this concludes the proof of the theorem.

□

We now analyze the relationship between algebras with mean values distinguishing between points of $\mathbb{R}^n$ and the subalgebras of the algebra of almost periodic functions in $\mathbb{R}^n$.

Theorem 4.4. *Let $\mathcal{A}$ be an algebra with mean value distinguishing between the points of $\mathbb{R}^n$ and let $\mathcal{K}$ be the associated compactification of $\mathbb{R}^n$. Then $\mathcal{A}$ is a subalgebra of the algebra of the almost periodic functions in $\mathbb{R}^n$ if and only if the addition operation $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ can be extended to a continuous group operation $+ : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ giving to $\mathcal{K}$ the structure of a compact abelian topological group. In this case, the Radon measure induced by the mean value is the unique Haar measure defined in the abelian topological group $\mathcal{K}$.*

Proof. 1. First, we easily see that if $+$ can be extended continuously to $\mathcal{K} \times \mathcal{K}$ providing $\mathcal{K}$ with a structure of a compact topological abelian group, then the translates $\{f(\cdot + t) : t \in \mathbb{R}^n\}$ form a precompact family in $\mathrm{BUC}(\mathbb{R}^n)$ for any $f \in \mathcal{A}$, which is the same as saying that $\mathcal{A}$ is a subalgebra of the algebra of the almost periodic functions.

2. Indeed, these translates are the restrictions to $\mathbb{R}^n \times \{t\}$ of the composition of continuous functions $\mathcal{K} \times \mathcal{K} \xrightarrow{+} \mathcal{K} \xrightarrow{\underline{f}} \mathbb{R}$. Since the composition is uniformly continuous with respect to the (uniform) topology of $\mathcal{K} \times \mathcal{K}$, these translates are restrictions to $\mathbb{R}^n$ of the family $\underline{f}(\cdot + t) \in C(\mathcal{K})$ which is equicontinuous, in the sense that given any $\varepsilon > 0$, there is a neighborhood V of the diagonal in $\mathcal{K} \times \mathcal{K}$ such that if $(z_1, z_2) \in V$, then $|f(z_1 + t) - f(z_2 + t)| < \varepsilon$, for any $t \in \mathbb{R}^n$. In particular, the family $\{f(\cdot + t)\}$, $t \in \mathbb{R}^n$, is totally bounded in $C(\mathcal{K})$, that is, given $\varepsilon > 0$, there is a finite set $\{t_1, \dots, t_N\}$ such that for all $t \in \mathbb{R}^n$, $\|f(\cdot + t) - f(\cdot + t_j)\|_\infty < \varepsilon$, for some $j \in \{1, \dots, N\}$. Therefore, we conclude that $\{f(\cdot + t)\}$ is precompact in $\mathrm{BUC}(\mathbb{R}^n)$.

3. To prove the converse, let $\mathbb{G}^n$ be the Bohr compactification of $\mathbb{R}^n$, that is, the compactification of $\mathbb{R}^n$ induced by the whole algebra of the almost periodic functions (see [16]; also [3]). In order to take advantage of the properties of exponential functions we consider algebras of complex valued functions; the passage to the real valued case is immediate. Let then $\mathcal{A}$ be a subalgebra of almost periodic functions.

4. It is well known that the family $\mathcal{F} := \{e^{i\lambda \cdot x} : \lambda \in \mathbb{R}^n\}$ form a fundamental set in the space of almost periodic functions $\mathrm{AP}(\mathbb{R}^n)$, in the sense that any $f \in \mathrm{AP}(\mathbb{R}^N)$ may be approximated in the sup-norm by finite linear combinations of elements of $\mathcal{F}$.

5. Suppose first that $\mathcal{A}$ is a subalgebra of $\mathrm{AP}(\mathbb{R}^n)$ generated by any subset $\mathcal{F}' \subseteq \mathcal{F}$, and let $\mathcal{K}$ be the associated compactification of $\mathbb{R}^n$. We are going to apply Lemma 4.1 with $X_1 = \mathcal{K} \times \mathcal{K}$, $X_2 = \mathcal{K}$, $R_1 = \mathbb{R}^n \times \mathbb{R}^n$, $R_2 = \mathbb{R}^n$ and $W : R_1 \to R_2$ the addition operation $+$ in $\mathbb{R}^N$. For any $e^{i\lambda \cdot x} \in \mathcal{F}'$, we have $e^{i\lambda \cdot (x+y)} = e^{i\lambda \cdot x} e^{i\lambda \cdot y}$. Clearly, $e^{i\lambda \cdot x} e^{i\lambda \cdot y}$ is the restriction to $\mathbb{R}^n \times \mathbb{R}^n$ of a continuous function in $\mathcal{K} \times \mathcal{K}$.

Since $\mathcal{F}'$ is a fundamental set for $\mathcal{A}$, Lemma 4.1 implies that $+$ may be extended continuously to $\mathcal{K}\times\mathcal{K}$. It is also immediate to verify that this extension preserves the properties of an abelian group.

6. Now, we consider the case where $\mathcal{A}$ is a general subalgebra of $\mathrm{AP}(\mathbb{R}^n)$, not necessarily generated by some subset of $\mathcal{F}$. We first interpret $\mathcal{K}$ as a quotient space of $\mathbb{G}^n$ as follows. In $\mathbb{G}^n$ we consider the equivalence relation $z_1\sim z_2$ if $\underline{f}(z_1)=\underline{f}(z_2)$ for all $f\in\mathcal{A}$, where $\underline{f}$ denotes the unique continuous extension of f to $\mathbb{G}^n$. The quotient space $\tilde{\mathcal{K}}=\mathbb{G}^n/\sim$, endowed with the quotient topology, is a compact space. The functions $\underline{f}$, $f\in\mathcal{A}$, pass to the quotient and $\underline{f}\in C(\tilde{\mathcal{K}})$ for all $f\in\mathcal{A}$. Moreover, the family $\{\underline{f}\,:\,f\in\mathcal{A}\}$ distinguishes between the points of $\tilde{\mathcal{K}}$. Hence, there is an isometric isomorphism between $\mathcal{A}$ and $C(\tilde{\mathcal{K}})$ and so $\tilde{\mathcal{K}}$ is homeomorphic to $\mathcal{K}$ and we may identify these spaces.

7. Now, observe that if $z_1,z_2,\sigma_1,\sigma_2\in\mathbb{G}^n$ with $z_1\sim z_2$ and $\sigma_1\sim\sigma_2$, then $z_1+\sigma_1\sim z_2+\sigma_2$. Indeed, given any $f\in\mathcal{A}$, by the invariance of $\mathcal{A}$ by translations we have

$$f(z_1+\sigma_1)=f(z_2+\sigma_1)=f(\sigma_1+z_2)=f(\sigma_2+z_2).$$

We have seen above that for any $f\in\mathcal{A}$, $f(x+y)$ is the restriction to $\mathbb{R}^n\times\mathbb{R}^n$ of a continuous function on $\mathbb{G}^n\times\mathbb{G}^n$. Now, as we have just proved, this function may pass to the quotient $\mathbb{G}^n/\sim\times\mathbb{G}^n/\sim\equiv\mathcal{K}\times\mathcal{K}$. Hence, $f(x+y)$ is the restriction to $\mathbb{R}^n\times\mathbb{R}^n$ of a continuous function on $\mathcal{K}\times\mathcal{K}$ for all $f\in\mathcal{A}$. Therefore, an application of Lemma 4.1 with $X_1=\mathcal{K}\times\mathcal{K}$, $X_2=\mathcal{K}$, $R_1=\mathbb{R}^n\times\mathbb{R}^n$, $\mathbb{R}_2=\mathbb{R}^n$ and $W=+$, gives that $+$ may be continuously extended to $\mathcal{K}\times\mathcal{K}$ and it is again a trivial matter to prove that the abelian group properties are preserved by this extension.

8. The fact that the measure $\mathfrak{m}$ of $\mathcal{K}$ induced by the mean value on $\mathcal{A}$ is the Haar measure is a straightforward consequence of the uniqueness of the Haar measure.

□

5. Vector-valued algebras with mean value

In this section we extend the notion of algebra with mean value to vector-valued functions. We begin with the following definition.

Definition 5.1. Let $\mathcal{A}\subseteq\mathrm{BUC}(\mathbb{R}^n)$ be an algebra with mean value and let E be a Banach space. We denote by $\mathcal{A}(\mathbb{R}^n;E)$ the space of functions $f\in\mathrm{BUC}(\mathbb{R}^n;E)$ satisfying the following conditions:

(i) For all $L\in E^*$, $L_f:=\langle L,f\rangle$ belongs to $\mathcal{A}$;
(ii) The family $\mathcal{F}:=\{L_f\,:\,L\in E^*,\ \|L\|\le 1\}$ is relatively compact in $\mathcal{A}$.

For bounded Borel sets $Q\subseteq\mathbb{R}^n$ and $f\in\mathrm{BUC}(\mathbb{R}^n;E)$, it is easily checked by an approximation with Riemann sums that $L\mapsto\int_Q\langle L,f\rangle\,dx$ defines a linear functional on E^*, continuous for the weak topology $\sigma(E^*,E)$; as a consequence, there exists a unique element of E, that we shall denote by $\int_Q f\,dx$, satisfying

$$\langle L,\int_Q f\,dx\rangle=\int_Q\langle L,f\rangle\,dx\qquad\forall L\in E^*.$$

For similar reasons, if $f\in\mathcal{A}(\mathbb{R}^n;E)$ the integrals $⨍_{Q_t}f\,dx$ weakly converge in E, as $t\to+\infty$, to a vector, that we shall denote by $⨍_{\mathbb{R}^n}f\,dx$, characterized by

$$\langle L,⨍_{\mathbb{R}^n}f\,dx\rangle=⨍_{\mathbb{R}^n}\langle L,f\rangle\,dx\qquad\forall L\in E^*.$$

Theorem 5.1. *Let $\mathcal{A}\subseteq\mathrm{BUC}(\mathbb{R}^n)$ be an algebra with mean value. Let E be a Banach space and let $\mathcal{K}$ be the compact space associated with $\mathcal{A}$. There is an isometric isomorphism between $\mathcal{A}(\mathbb{R}^n;E)$ and $C(\mathcal{K};E)$. Denoting by $g\mapsto\underline{g}$ the canonical map from $\mathcal{A}$ to $C(\mathcal{K})$, the isomorphism associates to $f\in\mathcal{A}(\mathbb{R}^n;E)$ the map $\tilde{f}\in C(\mathcal{K};E)$ satisfying*

$$\underline{\langle L,f\rangle}=\langle L,\tilde{f}\rangle\in C(\mathcal{K})\qquad\forall L\in E^*.\tag{5.1}$$

Moreover $\|f\|_E \in \mathcal{A}$ *for each* $f \in \mathcal{A}(\mathbb{R}^n; E)$.

Proof. 1. For any $z \in \mathcal{K}$ we consider the map $L \mapsto \underline{L_f}(z)$. This is a linear map on E^*; we claim that the compactness of $\mathcal{F}$ implies that this map is continuous with respect to the topology $\sigma(E^*, E)$.

2. Indeed, by the well known Krein-Šmulian Theorem (see, e.g.,[16], p. 429) it suffices to check the continuity of this linear functional when restricted to bounded closed balls. Now, if $L^i \to L$ in the w^*-topology, then the maps L^i_f converge to L_f pointwise and compactness yields that they converge also in $\mathcal{A}$. As a consequence $\underline{L^i_f}$ converge uniformly in $\mathcal{K}$ to $\underline{L_f}$.

3. Hence, for any $z \in \mathcal{K}$ we can find an element of E, that we denote by $\tilde{f}(z)$, such that $\underline{L_f}(z) = \langle L, \tilde{f}(z)\rangle$ for any $L \in E^*$. This proves (5.1) and it remains to show that $\tilde{f}$ is a continuous map. This is again an argument based on the compactness of the family $\underline{\mathcal{F}} := \{\underline{L_f} : L \in E^*, \|L\| \le 1\}$: if $z_i \to z$ then, by the compactness of $\underline{\mathcal{F}}$, $\underline{L_f}(z_i) \to \underline{L_f}(z)$ uniformly with respect to L in the unit ball of E^*. As a consequence $\tilde{f}(z_i) \to \tilde{f}(z)$ in E.

4. Now we prove that $f \mapsto \tilde{f}$ is an isometry between $\mathcal{A}(\mathbb{R}^n; E)$ and $C(\mathcal{K}; E)$. This map is clearly an isomorphism. Moreover, for each $x \in \mathbb{R}^n$ we obtain from (5.1) that $\|\tilde{f}(x)\|_E = \|f(x)\|_E$. Since $\|\tilde{f}\|_E \in C(\mathcal{K})$ we have that $\|f\|_E \in \mathcal{A}$ and so $\|\tilde{f}\|_E = \underline{\|f\|_E}$. Consequently, $f \mapsto \tilde{f}$ is an isometry. □

Definition 5.2. Given a compact space $\mathcal{K}$, a probability Radon measure $\mathfrak{m}$ on $\mathcal{K}$ and a Banach space E, for $1 \le p < \infty$, we define the space $\mathbf{L}^p(\mathcal{K}; E)$ as the completion of $C(\mathcal{K}; E)$ with respect to the norm $\|\cdot\|_p$, defined as usual:

$$\|f\|_p := \left(\int_{\mathcal{K}} \|f\|_E^p \, d\mathfrak{m}\right)^{1/p}.$$

We also define $\mathbf{L}^\infty(\mathcal{K}; E)$ as the space of the functions $f : \mathcal{K} \to E$ such that $f \in \mathbf{L}^p(\mathcal{K}; E)$ for all $p \in [1, \infty)$ and $\sup_{1\le p<\infty} \|f\|_p < +\infty$. We then set

$$\|f\|_\infty := \sup_{1\le p<\infty} \|f\|_p.$$

As usual, we identify functions in $\mathbf{L}^p$ that coincide $\mathfrak{m}$-a.e. in $\mathcal{K}$.

¿From Theorem 5.1 the following analogue of Theorem 4.2 easily follows.

Theorem 5.2. *Let* $\mathcal{A}(\mathbb{R}^n)$ *be an algebra w.m.v. in* $\mathbb{R}^n$, E *be a Banach space, and* $\mathcal{K}$ *the compact space given by Theorem 4.1, such that* $\mathcal{A}(\mathbb{R}^n)$ *is isometrically isomorphic to* $C(\mathcal{K})$. *Then, for* $1 \le p \le \infty$, *the vector-valued generalized Besicovitch spaces* $\mathcal{B}^p(\mathbb{R}^n; E)/\overset{\mathcal{B}^p}{=}$ *are isometrically isomorphic to* $\mathbf{L}^p(\mathcal{K}; E)$.

6. Multiscale Young Measures from Homogenization in Algebras w.m.v.

Next we establish the theorem concerning the existence of two-scale Young measures in algebras w.m.v. . The proof follows exactly as the one of the corresponding result for almost periodic functions in [3], and so we simply refer to [3] for the proof.

We consider an algebra w.m.v. $\mathcal{A}(\mathbb{R}^n) \subseteq \mathrm{BUC}(\mathbb{R}^n)$, the associated compact space $\mathcal{K}$ that $\mathcal{A}(\mathbb{R}^n) \sim C(\mathcal{K})$ and the corresponding Radon invariant measure $\mathfrak{m}$ on $\mathcal{K}$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\{u_\varepsilon(x)\}_{\varepsilon>0}$ be a family of functions in $L^\infty(\Omega; K)$, for some compact metric space K.

Theorem 6.1. *Given any infinitesimal sequence* $\{\varepsilon_i\}_{i\in\mathbb{N}}$ *there exist a subnet* $\{u_{\varepsilon_{i(d)}}\}_{d\in D}$, *indexed by a certain directed set* D, *and a family of probability measures on* K, $\{\nu_{z,x}\}_{z\in\mathcal{K},x\in\Omega}$, *weakly measurable with respect to the product of the Borel* σ*-algebras in* $\mathcal{K}$ *and* $\mathbb{R}^n$, *such that*

$$\lim_D \int_\Omega \Phi\left(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x)\right) dx = \int_\Omega \int_{\mathcal{K}} \langle \nu_{z,x}, \underline{\Phi}(z, x, \cdot)\rangle \, d\mathfrak{m}(z)\, dx \qquad \forall \Phi \in \mathcal{A}\left(\mathbb{R}^n; C_0(\Omega \times K)\right). \tag{6.1}$$

Here $\underline{\Phi} \in C(\mathcal{K}; C_0(\Omega \times K))$ *denotes the unique extension of* Φ*. Moreover, equality* (6.1) *still holds for functions* Φ *in the following function spaces:*

(1) $\mathcal{B}^1(\mathbb{R}^n; C_0(\Omega \times K))$*;*
(2) $\mathcal{B}^p(\mathbb{R}^n; C(\bar{\Omega} \times K))$ *with* $p > 1$*;*
(3) $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; C(K)))$*.*

Remark 6.1. A similar result holds, with minor adaptations in the proof, for families $\{u_\varepsilon\}_{\varepsilon>0} \subseteq L^1(\Omega; \mathbb{R}^m)$ that satisfy the condition

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} |\{|u_\varepsilon| > R\}| = 0.$$

This happens, for instance, when a uniform bound in $L^p(\Omega; \mathbb{R}^m)$ is available. In this special case, the representation formula (6.1) is valid for all $\Phi(z, x, \lambda) \in \mathcal{A}(\mathbb{R}^n; C_0(\Omega, C(\mathbb{R}^m)))$ such that

$$\lim_{|\lambda|\to\infty} \frac{|\Phi(z, x, \lambda)|}{|\lambda|^p} = 0 \qquad \text{uniformly as } (z, x) \in \mathbb{R}^n \times \Omega.$$

This extension is analogous to the well known one in the classical theory of Young measures (see, e.g., [5], [4], [43] etc.).

As in the classical theory of Young measures we have the following consequence of Theorem 6.1.

Theorem 6.2. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set, let* $\{u_\varepsilon\} \subseteq L^\infty(\Omega; \mathbb{R}^m)$ *be uniformly bounded and let* $\nu_{z,x}$ *be a two-scale Young measure generated by a subnet* $\{u_{\varepsilon(d)}\}_{d\in D}$*, according to Theorem 6.1. Assume that* U *belongs either to* $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; \mathbb{R}^m))$ *or to* $\mathcal{B}^p(\mathbb{R}^n; C(\bar{\Omega}; \mathbb{R}^m))$ *for some* $p > 1$*. Then*

$$\nu_{z,x} = \delta_{\underline{U}(z,x)} \quad \textit{if and only if} \quad \lim_D \|u_{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} = 0. \tag{6.2}$$

7. Porous medium type equations with oscillating external sources: The Cauchy problem

Let $\mathcal{Q}$ be a compact space and let $T(x) : \mathcal{Q} \longrightarrow \mathcal{Q}$ an ergodic n-dimensional continuous dynamical system on $\mathcal{Q}$ with an invariant probability measure μ on $\mathcal{Q}$. We consider the following stochastic homogenization problem

$$\begin{cases} \partial_t u - \Delta f(u) = -\frac{1}{\varepsilon^2}\Delta_z V(T(\frac{x}{\varepsilon})\omega), & (x, t, \omega) \in \mathbb{R}^{n+1}_+ \times \mathcal{Q} \\ u(x, 0) = u_0(T(\frac{x}{\varepsilon})\omega, x), & (x, \omega) \in \mathbb{R}^n \times \mathcal{Q} \end{cases}$$

where $V, \Delta V \in C(\mathcal{Q})$ and $u_0 \in L^\infty\big(\mathbb{R}^n; C(\mathcal{Q})\big)$. Here we denote $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. As usual, $\Delta = \sum_{i=1}^n \partial^2_{x_i}$ is the Laplace operator and we denote $\Delta_z = \sum_{i=1}^n \partial^2_{z_i}$, where z represents the oscillatory variable x/ε. Here, by $\Delta V \in C(\mathcal{Q})$ we mean that the function $\tilde{V}(x, \omega) := V(T(x)\omega)$ satisfies $(\Delta_x \tilde{V})(x, \omega) = h(T(x)\omega)$, for some $h \in C(\mathcal{Q})$.

Since, by Theorem 3.1, almost all realizations of functions in $C(\mathcal{Q})$ belong to an ergodic algebra, for simplicity of notation, here and henceforth, we consider the equivalent individual homogenization problem with oscillatory functions belonging to an ergodic algebra, which in this case reduces to the problem

$$\begin{cases} \partial_t u - \Delta f(u) = -\frac{1}{\varepsilon^2}\Delta_z V(\frac{x}{\varepsilon}), & (x, t) \in \mathbb{R}^{n+1}_+ \\ u(x, 0) = u_0(\frac{x}{\varepsilon}, x), & x \in \mathbb{R}^n. \end{cases} \tag{7.1}$$

So, let $\mathcal{A}(\mathbb{R}^n)$ be an ergodic algebra, $\mathcal{K}$ be the compact space given by Theorem 4.1 such that $\mathcal{A}(\mathbb{R}^n) \sim C(\mathcal{K})$, and $\mathfrak{m}$ be the associated invariant probability measure on $\mathcal{K}$. We make the following assumptions:

(A1) The function f in (7.1) is in $C^2(\mathbb{R})$ and satisfies $f'(u) > 0$ for $u \in \mathbb{R}$.
(A2) $V \in \mathcal{A}(\mathbb{R}^n)$ and

$$u_0(z, x) = g(\varphi_0(x) + V(z)), \tag{7.2}$$

with $g := f^{-1}$, for some $\varphi_0 \in L^\infty(\mathbb{R}^n)$. In particular $u_0 \in L^\infty(\mathbb{R}^n; \mathcal{A}(\mathbb{R}^n))$.
(A3) $\Delta V \in \mathcal{A}(\mathbb{R}^n)$.

Let g be as above and let $\bar f$ be implicitly defined by the equation

$$p = \fint_{\mathbb{R}^n} g(\bar f(p) + V(z))\, dz. \tag{7.3}$$

In the sequel we shall identify V with the function $\underline{V} \in C(\mathcal{K})$, and define

$$\psi_\alpha(z) := g(V(z) + \alpha), \qquad z \in \mathcal{K}.$$

Notice that ψ_α is a steady state solution of (7.1).

Theorem 7.1. *Suppose assumptions (A1), (A2) and (A3) hold and let u_ε denote the unique weak solution of* (7.1)*. Let $\bar u$ be the unique weak solution of*

$$\begin{cases} \partial_t \bar u - \Delta \bar f(\bar u) = 0, & (x,t) \in \mathbb{R}^{n+1}_+ \\ \bar u(x,0) = \fint_{\mathbb{R}^n} u_0(z,x)\, dz, & x \in \mathbb{R}^n \end{cases} \tag{7.4}$$

and set

$$U(z,x,t) := g(\bar f(\bar u(x,t)) + V(z)). \tag{7.5}$$

Then, as $\varepsilon \to 0$, we have $u_\varepsilon \to \bar u$ in the weak star topology of $L^\infty(\mathbb{R}^{n+1}_+)$ and

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - U(\frac{x}{\varepsilon}, x, t)\|_{L^1_{loc}(\mathbb{R}^{n+1}_+)} = 0. \tag{7.6}$$

Proof. 1. First, we observe that the weak solutions u_ε, $\varepsilon > 0$, of (7.1) are bounded uniformly with respect to ε in $L^\infty(\mathbb{R}^{n+1}_+)$. For this, we note that if α_1, α_2 are such that $\alpha_1 \le \varphi_0(x) \le \alpha_2$ for $x \in \mathbb{R}$, we have

$$g(V(\frac{x}{\varepsilon}) + \alpha_1) \le u_0(\frac{x}{\varepsilon}, x) \le g(V(\frac{x}{\varepsilon}) + \alpha_2) \qquad \text{for all } x \in \mathbb{R}^n.$$

By the monotonicity of the solution operator of (7.1) (see Theorem A.3), we get

$$g(V(\frac{x}{\varepsilon}) + \alpha_1) \le u_\varepsilon(x,t) \le g(V(\frac{x}{\varepsilon}) + \alpha_2) \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+.$$

Thus, in the sequel, we denote by K a closed interval containing the image of all the functions u_ε, $\varepsilon > 0$.

Let $\nu_{z,x,t} \in \mathcal{M}(K)$, with $(z,x,t) \in \mathcal{K} \times \mathbb{R}^{n+1}_+$, be the two-scale space time Young measures associated with a subnet of $\{u_\varepsilon\}_{\varepsilon>0}$ with test functions oscillating only on the space variable. Following [21] and [3], the theorem will be proved by adapting DiPerna's method in [19], that is, by showing that $\nu_{z,x,t}$ is a Dirac measure for almost all $(z,x,t) \in \mathcal{K} \times \mathbb{R}^{n+1}_+$. Since we are going to show that $\nu_{z,x,t}$ do not depend on the chosen subnet (so that, a posteriori, a full limit as $\varepsilon \to 0$ occurs), in order to simplify our notation we will use the notation $\lim_{\varepsilon\to 0}$, not denoting the subnet.

Observe that, for every $\alpha \in \mathbb{R}$, the weak solutions u_ε and $\psi_\alpha(\frac{x}{\varepsilon})$ satisfy (see Theorem A.3)

$$\int_{\mathbb{R}^{n+1}_+} |u_\varepsilon(x,t) - \psi_\alpha(\frac{x}{\varepsilon})|\phi_t + |f(u_\varepsilon(x,t)) - f(\psi_\alpha(\frac{x}{\varepsilon}))|\Delta\phi\, dx\, dt + \int_{\mathbb{R}^n} |u_0(\frac{x}{\varepsilon}, x) - \psi_\alpha(\frac{x}{\varepsilon})|\phi(x,0)\, dx \ge 0, \tag{7.7}$$

for all $0 \le \phi \in C_c^\infty(\mathbb{R}^{n+1})$. In (7.7), we take $\phi(x,t) = \varepsilon^2 \varphi(\frac{x}{\varepsilon})\psi(x,t)$ with $0 \le \psi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, φ, $\Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$ and $\varphi \ge 0$. Observe that

$$\Delta\phi = \Delta\varphi(\frac{x}{\varepsilon})\psi(x,t) + 2\varepsilon\nabla\varphi(\frac{x}{\varepsilon})\cdot\nabla\psi(x,t) + \varepsilon^2\varphi(\frac{x}{\varepsilon})\Delta\psi(x,t).$$

Letting $\varepsilon \to 0$ and using Theorem 6.1, we get

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle \nu_{z,x,t}, |f(\cdot) - f(\psi_\alpha(z)|\rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt \ge 0.$$

Now apply the inequality above to $\|\varphi\|_\infty \pm \varphi$ to obtain

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle \nu_{z,x,t}, |f(\cdot) - V(z) - \alpha|\rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt = 0 \tag{7.8}$$

for all φ such that $\varphi, \Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$ and all $0 \le \psi \in C_c^\infty(\mathbb{R}_+^{n+1})$.

2. As in [21], we define a new family of parametrized measures $\mu_{z,x,t}$ supported on a compact set $K' \supset \{f(\lambda) - V(z) : (\lambda, z) \in K \times \mathcal{K}\}$ by

$$\langle \mu_{z,x,t}, \theta \rangle := \langle \nu_{z,x,t}, \theta(f(\cdot) - V(z)) \rangle, \qquad \theta \in C(\mathbb{R}). \tag{7.9}$$

In this way, the equation (7.8) can also be rephrased as

$$\int_{\mathbb{R}_+^{n+1}} \int_{\mathcal{K}} \psi(x,t) \langle \mu_{z,x,t}, \theta \rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt = 0, \tag{7.10}$$

where $\theta(\lambda) = |\lambda - \alpha|$.

On the other hand, inserting in the integral equation defining weak solution of (7.1) with a test function as above, we easily get letting $\varepsilon \to 0$ that (7.10) holds when θ is any affine function. Therefore, we deduce that (7.10) holds for finite linear combinations of affine functions and functions of the form $|\cdot - \alpha|$, $\alpha \in \mathbb{R}$. Since these combinations generate the piecewise affine functions, we finally conclude that (7.10) holds for all $\theta \in C(\mathbb{R})$.

Set $F(z) := \int_{\mathbb{R}_+^{n+1}} \psi(x,t) \langle \mu_{z,x,t}, \theta \rangle\, dx\, dt$ and observe that $\int_{\mathcal{K}} F(z) \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z) = 0$, for all φ such that φ, $\Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$. Then, we can apply Lemma 3.2 to obtain that F is equivalent to a constant for all $\theta \in C(\mathbb{R})$. Using this fact and defining

$$\mu_{x,t} := \int_{\mathcal{K}} \mu_{z,x,t}\, d\mathfrak{m}(z) \in \mathcal{M}(K'),$$

we have, in particular,

$$\int_{\mathbb{R}_+^{n+1}} \psi(x,t) \langle \mu_{z,x,t}, \theta \rangle\, dx\, dt = \int_{\mathcal{K}} \int_{\mathbb{R}_+^{n+1}} \psi(x,t) \langle \mu_{z,x,t}, \theta \rangle\, dx\, dt\, d\mathfrak{m}(z) = \int_{\mathbb{R}_+^{n+1}} \psi(x,t) \langle \mu_{x,t}, \theta \rangle\, dx\, dt,$$

for a.e. $z \in \mathcal{K}$, for all $\theta \in C(\mathbb{R})$.

Hence,

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, \int_{\mathcal{K}} W(z,\cdot)\, d\mathfrak{m}(z) \rangle \psi(x,t)\, dx\, dt = \sum_i \mathfrak{m}(\mathcal{K}_i) \int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, \theta_i \rangle \psi(x,t)\, dx\, dt \\ = \; &\sum_i \mathfrak{m}(\mathcal{K}_i) \int_{\mathbb{R}_+^{n+1}} \langle \mu_{z,x,t}, \theta_i \rangle \psi(x,t)\, dx\, dt = \sum_i \int_{\mathcal{K}} \int_{\mathbb{R}_+^{n+1}} \langle \mu_{z,x,t}, \theta_i \rangle \chi_{\mathcal{K}_i}(z) \psi(x,t)\, dx\, dt\, d\mathfrak{m}(z) \\ = \; &\int_{\mathbb{R}_+^{n+1}} \int_{\mathcal{K}} \langle \mu_{z,x,t}, W(z,\cdot) \rangle \psi(x,t)\, d\mathfrak{m}(z)\, dx\, dt \end{aligned} \tag{7.11}$$

for any function $W(\lambda, z) = \sum_i \theta_i(\lambda) \chi_{\mathcal{K}_i}(z)$, where $\theta_i \in C(K')$, $\mathcal{K}_i$ is any Borelian subset of $\mathcal{K}$, and $\chi_{\mathcal{K}_i}$ is the characteristic function of $\mathcal{K}_i$. By approximation (7.11) holds for any $W \in C(\mathcal{K} \times K')$.

3. From (7.7), taking the limit as $\varepsilon \to 0$, passing to a subnet if necessary, we get

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} \int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)| \rangle \varphi_t + \langle \nu_{z,x,t}, |f(\cdot) - f(\psi_\alpha(z))| \rangle \Delta\varphi(z)\, d\mathfrak{m}(z)\, dx\, dt \\ &\qquad + \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)| \varphi(x,0)\, d\mathfrak{m}(z)\, dx \ge 0 \end{aligned} \tag{7.12}$$

for all $\alpha \in \mathbb{R}$ and for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$.

We define $I(\rho, \alpha)$ and $G(\rho, \alpha)$ by

$$I(\rho, \alpha) := \int_{\mathcal{K}} |g(\rho + V(z)) - g(\alpha + V(z))|\, d\mathfrak{m}(z), \tag{7.13}$$

$$G(\rho, \alpha) := |\rho - \alpha|. \tag{7.14}$$

Now, setting $\theta(\rho) = |g(\rho + V(z)) - g(\alpha + V(z))|$, we have

$$\int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)|\rangle \varphi_t \, d\mathfrak{m}(z)\, dx\, dt = \int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} \langle \nu_{z,x,t}, \theta(f(\cdot) - V(z))\rangle \varphi_t \, d\mathfrak{m}(z)\, dx\, dt$$
$$= \int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} \langle \mu_{z,x,t}, |g(\cdot + V(z)) - g(\alpha + V(z))|\rangle \varphi_t \, d\mathfrak{m}(z)\, dx\, dt.$$

Using (7.11), we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} &\langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)|\rangle \varphi_t \, d\mathfrak{m}(z)\, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} \langle \mu_{z,x,t}, |g(\cdot + V(z)) - g(\alpha + V(z))|\rangle \varphi_t \, d\mathfrak{m}(z)\, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \Big\langle \mu_{x,t}, \int_{\mathcal{K}} |g(\cdot + V(z)) - g(\alpha + V(z))\, d\mathfrak{m}(z) \Big\rangle \varphi_t \, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, I(\cdot, \alpha)\rangle \varphi_t \, dx\, dt. \end{aligned} \tag{7.15}$$

Analogously,

$$\int_{\mathbb{R}^{n+1}_+} \int_{\mathcal{K}} \langle \nu_{z,x,t}, |f(\cdot) - f(\psi_\alpha(z))|\rangle \Delta\varphi(x,t)\, d\mathfrak{m}(z)\, dx\, dt = \int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, G(\cdot, \alpha)\rangle \Delta\varphi(x,t)\, dx\, dt. \tag{7.16}$$

Using (7.15) and (7.16) in (7.12), we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} &\langle \mu_{x,t}, I(\cdot, \alpha)\rangle \varphi_t + \langle \mu_{x,t}, G(\cdot, \alpha)\rangle \Delta\varphi \, dx\, dt \\ &+ \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)| \varphi(x,0)\, d\mathfrak{m}(z)\, dx \geq 0, \end{aligned} \tag{7.17}$$

for all $0 \leq \varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and all $\alpha \in \mathbb{R}$.

Now, choosing $\varphi(x,t) = \delta_h(t)\phi(x)$, with $0 \leq \phi \in C_c^\infty(\mathbb{R}^n)$ and $\delta_h(t) = \max\left\{\frac{h-|t|}{h}, 0\right\}$ for $h > 0$ in (7.17), we obtain

$$\overline{\lim_{h\to 0}} \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} \langle \mu_{x,t}, I(\cdot, \alpha)\rangle \phi \, dx\, dt \leq \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)| \phi \, d\mathfrak{m}(z)\, dx. \tag{7.18}$$

Using the flexibility provided by ϕ in (7.18), we deduce that the same inequality holds if $\alpha \in L^\infty(\mathbb{R}^n)$ and $\phi = \chi_{B_R}$, $R > 0$.

We have that $\varphi_0(x) = f(u_0(z,x) - V(z))$ is independent of z. Taking $\alpha(x) = \varphi_0(x)$ and recalling that $u_0(z,x) = g(\alpha(x) + V(z))$, we have $\alpha(x) = \bar{f}(\bar{u}(x,0))$. Using this and $\psi_\alpha(z) = g(\alpha + V(z))$ in (7.18), we obtain that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_{B_R} \langle \mu_{x,t}, I(\cdot, \bar{f}(\bar{u}(x,0)))\rangle \, dx\, dt = 0, \qquad \forall R > 0. \tag{7.19}$$

4. By using the Remark A.1 with $u_1 = u_\varepsilon$ and $u_2(x) = \psi_\alpha(\frac{x}{\varepsilon})$, for all $0 \le \varphi \in C_c^\infty(\mathbb{R}_+^{n+1})$ we get

$$
\begin{aligned}
(7.20)\qquad & -\int_{\mathbb{R}_+^{n+1}} B_{\vartheta_\delta}^{\psi_\alpha(\frac{x}{\varepsilon})}(u_\varepsilon(x,t))\varphi_t\,dx\,dt \\
&+\int_{\mathbb{R}_+^{n+1}} H_\delta(f(u_\varepsilon(x,t)) - f(\psi_\alpha(\frac{x}{\varepsilon})))\nabla[f(u_\varepsilon(x,t)) - f(\psi_\alpha(\frac{x}{\varepsilon}))]\cdot\nabla\varphi\,dx\,dt \\
&= -\int_{\mathbb{R}_+^{n+1}} |\nabla[f(u_\varepsilon(x,t)) - f(\psi_\alpha(\frac{x}{\varepsilon}))]|^2 H_\delta'(f(u_\varepsilon(x,t)) - f(\psi_\alpha(\frac{x}{\varepsilon})))\varphi\,dx\,dt.
\end{aligned}
$$

Now, we let $\alpha = \xi(y,s) := \bar f(\bar u(y,s))$, take $0 \le \phi \in C_c^\infty((\mathbb{R}_+^{n+1})^2)$, integrate in y, s, and send $\delta \to 0$, to get

$$
\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} -|u_\varepsilon(x,t) - \psi_{\xi(y,s)}(\frac{x}{\varepsilon})|\phi_t + \nabla_x|f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|\cdot\nabla_x\phi\,dx\,dt\,dy\,ds \\
&\quad= -\lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2} |\nabla_x[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 H_\delta'(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds.
\end{aligned}
$$

Then we use Theorem 6.1 on multiscale Young measures to obtain, as $\varepsilon \to 0$,

(7.21)

$$
\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} -\langle\mu_{x,t}, I(\cdot,\xi(y,s))\rangle\phi_t - \langle\mu_{x,t}, G(\cdot,\xi(y,s))\rangle\Delta_x\phi\,dx\,dt\,dy\,ds \\
&\quad= -\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2} |\nabla_x[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 H_\delta'(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds.
\end{aligned}
$$

5. Observe that $\nabla_y[f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] = \nabla_y[V(\frac{x}{\varepsilon}) + \xi(y,s)] = \nabla_y\xi(y,s)$. Hence

$$
0 = \int_{\mathbb{R}_+^{n+1}} \nabla_y[f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]\cdot\nabla_x[H_\delta(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))\phi]\,dx\,dt,
$$

which implies that

$$
\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \Big\{\nabla_y[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \\
&\qquad\qquad H_\delta'(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt \\
&= -\int_{\mathbb{R}_+^{n+1}} \nabla_y[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]\cdot\nabla_x\phi H_\delta(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\,dx\,dt.
\end{aligned}
$$

Integrating in y, s and letting $\delta \to 0$, we have

$$
\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} |f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|\mathrm{div}_y\nabla_x\phi\,dx\,dt\,dy\,ds \\
&= \lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2} \Big\{\nabla_y[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \\
&\qquad\qquad H_\delta'(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds.
\end{aligned}
$$

By Theorem 6.1, as $\varepsilon \to 0$, we get

$$\begin{aligned}(7.22)\quad &\int_{(\mathbb{R}^{n+1}_+)^2} \langle \mu_{x,t}, G(\cdot,\xi(y,s))\rangle \operatorname{div}_y \nabla_x \phi \, dx\, dt\, dy\, ds \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{(\mathbb{R}^{n+1}_+)^2} \nabla_y[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]H_\delta'(f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon})))\phi\, dx\, dt\, dy\, ds.\end{aligned}$$

Similarly, we have also that $f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})) = f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon}) - \xi(y,s)$ and thus

$$\nabla_x[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] = \nabla_x[f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon})],$$

is independent of y. Hence, by integrating first in (y,s) and then (x,t), proceeding as above in obtaining (7.22), yields the equality

$$\begin{aligned}(7.23)\quad &\int_{(\mathbb{R}^{n+1}_+)^2} \langle \mu_{x,t}, G(\cdot,\xi(y,s))\rangle \operatorname{div}_x \nabla_y \phi \, dx\, dt\, dy\, ds \\ &= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{(\mathbb{R}^{n+1}_+)^2} \nabla_x[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_y[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]H_\delta'(f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon})))\phi\, dx\, dt\, dy\, ds\end{aligned}$$

where u_ε and ξ are functions of x,t and y,s, respectively.

6. Let $\bar u$ be the weak solution of (7.4). From (A.5) in Theorem A.1, we have

$$\begin{aligned}(7.24)\quad &\int_{\mathbb{R}^{n+1}_+} |l-\bar u(y,s)|\phi_s + \operatorname{sgn}(\bar f(l)-\bar f(\bar u(y,s)))\nabla_y \bar f(\bar u)\cdot\nabla_y\phi\, dy\, ds \\ &= \lim_{\delta\to 0}\int_{\mathbb{R}^{n+1}_+} |\nabla_y \bar f(\bar u)|^2 H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\, dy\, ds, \qquad \text{for all } l\in\mathbb{R}.\end{aligned}$$

Now, let $k := \bar f(l)$ and notice that $l = \int_{\mathcal{K}} g(\bar f(l)+V(z))\, d\mathfrak{m}(z)$ and that $\bar u(y,s) = \int_{\mathcal{K}} g(\xi(y,s)+V(z))\, d\mathfrak{m}(z)$. Thus,

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} |l-\bar u(y,s)|\phi_s\, dy\, ds = \int_{\mathbb{R}^{n+1}_+} \left| \int_{\mathcal{K}} (g(k+V(z)) - g(\xi(y,s)+V(z))\, d\mathfrak{m}(z)\right| \phi_s\, dy\, ds \\ &= \int_{\mathbb{R}^{n+1}_+} \left(\int_{\mathcal{K}} |g(k+V(z)) - g(\xi(y,s)+V(z))|\, d\mathfrak{m}(z)\right)\phi_s\, dy\, ds = \int_{\mathbb{R}^{n+1}_+} I(k,\xi(y,s))\phi_s\, dy\, ds.\end{aligned}$$

Also,

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} \operatorname{sgn}(\bar f(l)-\bar f(\bar u(y,s)))\nabla_y \bar f(\bar u)\cdot\nabla_y\phi\, dy\, ds \\ &= -\int_{\mathbb{R}^{n+1}_+} \nabla_y|\bar f(l)-\bar f(\bar u(y,s))|\cdot\nabla_y\phi\, dy\, ds = \int_{\mathbb{R}^{n+1}_+} |k-\xi(y,s)|\Delta_y\phi\, dy\, ds \\ &= \int_{\mathbb{R}^{n+1}_+} G(k,\xi(y,s))\Delta_y\phi\, dy\, ds.\end{aligned}$$

Besides, since $\nabla_y\xi(y,s) = \nabla_y[f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} |\nabla_y \bar f(\bar u)|^2 H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\, dy\, ds = \int_{\mathbb{R}^{n+1}_+} |\nabla_y\xi(y,s)|^2 H_\delta'(k-\xi(y,s))\phi\, dy\, ds \\ &= \int_{\mathbb{R}^{n+1}_+} |\nabla_y f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2 H_\delta'(k-\xi(y,s))\phi\, dy\, ds.\end{aligned}$$

Using the two previous equalities in (7.24) we obtain

$$\int_{\mathbb{R}_+^{n+1}} I(k,\xi(y,s))\phi_s + G(k,\xi(y,s))\Delta_y\phi\,dy\,ds = \lim_{\delta\to 0}\int_{\mathbb{R}_+^{n+1}} |\nabla_y f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2 H'_\delta(k-\xi(y,s))\phi\,dy\,ds.$$

for all $k\in\mathbb{R}$ and all $0\le\phi\in C_c^\infty((\mathbb{R}_+^{n+1})^2)$.

We take $k = f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon})$ in the above equality and integrate in x,t to get

$$\int_{(\mathbb{R}_+^{n+1})^2} I(f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon}),\xi(y,s))\phi_s + G(f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon}),\xi(y,s))\Delta_y\phi\,dx\,dt\,dy\,ds$$
$$= \lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2}\left\{|\nabla_y[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 H'_\delta(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\right\}dx\,dt\,dy\,ds. \tag{7.25}$$

Applying Theorem 6.1, letting $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_{(\mathbb{R}_+^{n+1})^2} I(f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon}),\xi(y,s))\phi_s\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2}\int_{\mathcal{K}}\langle\nu_{z,x,t}, I(f(\cdot)-V(z),\xi(y,s))\rangle\phi_s\,d\mathfrak{m}(z)\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2}\int_{\mathcal{K}}\langle\mu_{z,x,t}, I(\cdot,\xi(y,s))\rangle\phi_s\,d\mathfrak{m}(z)\,dx\,dt\,dy\,ds = \int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t}, I(\cdot,\xi(y,s))\rangle\phi_s\,dx\,dt\,dy\,ds$$

Similarly

$$\lim_{\varepsilon\to 0}\int_{(\mathbb{R}_+^{n+1})^2} G(f(u_\varepsilon(x,t)) - V(\frac{x}{\varepsilon}),\xi(y,s))\Delta_y\phi\,dx\,dt\,dy\,ds = \int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t}, G(\cdot,\xi(y,s))\rangle\Delta_y\phi\,dx\,dt\,dy\,ds.$$

Using the last two equalities in (7.25), we get

$$\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t}, I(\cdot,\xi(y,s))\rangle\phi_s + \langle\mu_{x,t}, G(\cdot,\xi(y,s))\rangle\Delta_y\phi\,dx\,dt\,dy\,ds$$
$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2}|\nabla_y[f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 H'_\delta(f(u_\varepsilon(x,t)) - f(\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds. \tag{7.26}$$

7. We now prove that

$$\int_{\mathbb{R}_+^{n+1}}\langle\mu_{x,t}, I(\cdot,\xi(x,t))\rangle\varphi_t + \langle\mu_{x,t}, G(\cdot,\xi(x,t))\rangle\Delta\varphi\,dx\,dt \ge 0, \tag{7.27}$$

for all $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$.

By subtracting (7.21) from (7.22), we deduce that

$$\int_{(\mathbb{R}_+^{n+1})^2} -\langle\mu_{x,t}, I(\cdot,\xi)\rangle\phi_t - \langle\mu_{x,t}, G(\cdot,\xi)\rangle\big(\Delta_x\phi + \operatorname{div}_y\nabla_x\phi\big)\,dx\,dt\,dy\,ds$$
$$= -\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_x[f(u_\varepsilon) - f(\psi_\xi(\frac{x}{\varepsilon}))]|^2$$
$$+ \nabla_y[f(u_\varepsilon) - f(\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(u_\varepsilon) - f(\psi_\xi(\frac{x}{\varepsilon}))]\Big\}H'_\delta(f(u_\varepsilon) - f(\psi_\xi(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds. \tag{7.28}$$

where $u_\varepsilon = u_\varepsilon(x,t)$, $\xi = \xi(y,s)$.

The sum of (7.26) and (7.23)) gives

$$\begin{aligned}(7.29)\qquad &\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi)\rangle\phi_s+\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_y\phi+\operatorname{div}_x\nabla_y\phi\big)\,dx\,dt\,dy\,ds\\ &=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_y[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]|^2\\ &+\nabla_y[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\Big\}H_\delta'(f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))\phi\,dx\,dt\,dy\,ds.\end{aligned}$$

Finally, taking the difference between (7.28) and (7.29) we obtain

$$\begin{aligned}(7.30)\qquad &\int_{(\mathbb{R}_+^{n+1})^2}-\langle\mu_{x,t},I(\cdot,\xi)\rangle(\phi_t+\phi_s)-\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_x+\operatorname{div}_y\nabla_x+\operatorname{div}_x\nabla_y+\Delta_y\big)\phi\,dx\,dt\,dy\,ds\\ &=-\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{\big|\nabla_x[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\\ &+\nabla_y[f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon}))]\big|^2H_\delta'(f(u_\varepsilon)-f(\psi_\xi(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds\le0.\end{aligned}$$

Now, we take $\phi(x,t,y,s):=\varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_n(\frac{x-y}{2})\theta_n(\frac{t-s}{2})$, where $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$, and ρ_n, θ_n are classical approximations of the identity in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, as in the doubling of variables method, and observe that

$$\big(\Delta_x+\operatorname{div}_y\nabla_x+\operatorname{div}_x\nabla_y+\Delta_y\big)\phi=\rho_n(\frac{x-y}{2})\theta_n(\frac{t-s}{2})\Delta_x\varphi(\frac{x+y}{2},\frac{t+s}{2}).$$

Substituting such test function in the inequality in (7.30) and letting $n\to\infty$, we obtain (7.27), proving the assertion.

8. To conclude the proof, we set $\varphi(x,t)=\delta_h(t)\Lambda(x)$ in (7.27), with $0\le\delta_h\in C_c^\infty(\mathbb{R}_+)$ as in step 3. above and Λ given by (A.13). We define

$$\gamma(t):=\int_{\mathbb{R}^n}\langle\mu_{x,t},I(\cdot,\xi(x,t))\rangle\Lambda(x)\,dx$$

and observe that $G(\cdot,\cdot)\le CI(\cdot,\cdot)$. Then, using the properties of the weight function Λ, proceeding in a standard way and letting $h\to0$, we arrive at

$$\gamma(t)\le C\int_0^t\gamma(s)ds\qquad\text{for a.e. }t\ge0.$$

Hence, Gronwall's lemma implies $\gamma(t)=0$ for a.e. $t\ge0$ which, by the definition of γ, means that $\langle\mu_{x,t},I(\cdot,\xi(x,t))\rangle=0$ for a.e. $(x,t)\in\mathbb{R}_+^{n+1}$, and so $\langle\mu_{x,t},G(\cdot,\xi(x,t))\rangle=0$ for a.e. $(x,t)\in\mathbb{R}_+^{n+1}$. Therefore, $\mu_{x,t}$ is the Dirac mass concentrated at $\xi(x,t)$ for a.e. $(x,t)\in\mathbb{R}_+^{n+1}$. Recalling the definition of $\mu_{x,t}$ we have also that $\mu_{z,x,t}$ is the Dirac mass concentrated at $\xi(x,t)$ for a.e. (z,x,t), and thus, $\nu_{z,x,t}$ is the Dirac mass concentrated at $g(\bar f(\bar u(x,t))+V(z))$ for a.e. (z,x,t). Hence, we can apply Theorem 6.1 to conclude (7.6).

Finally, the fact that the whole sequence u_ε converges in the weak star topology of $L^\infty(\mathbb{R}_+^{n+1})$ to $\bar{u}$ follows from (7.6) observing that, for any $\varphi \in C_c(\mathbb{R}_+^{n+1})$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+^{n+1}} U(\frac{x}{\varepsilon},x,t)\varphi(x,t)dxdt &= \int_{\mathbb{R}_+^{n+1}}\int_{\mathcal{K}} U(z,x,t)\varphi(x,t)d\mathfrak{m}(z)\,dx\,dt\\ &= \int_{\mathbb{R}_+^{n+1}}\left(\int_{\mathcal{K}} g(\bar{f}(\bar{u}(x,t))+V(z)\,)d\mathfrak{m}(z)\right)\varphi(x,t)\,dx\,dt\\ &= \int_{\mathbb{R}_+^{n+1}} \bar{u}(x,t)\varphi(x,t)\,dx\,dt,\end{aligned}$$

by the definitions of $\bar{f}$ and U.

□

Appendix A. Some basic results about the nondegenerate porous medium equation

In this section we state some results about the porous medium equation which are used in Section 7. Most of them follow from more general results in [12] and in these cases for the proof we just refer to [12].

More specifically, we consider the Cauchy problem for the following quasilinear parabolic equation

$$u_t - \Delta f(u) = h(x), \qquad (x,t)\in \mathbb{R}_+^{n+1} := \mathbb{R}^n\times(0,\infty), \tag{A.1}$$

with initial data given by

$$u(x,0) = u_0(x), \qquad x\in\mathbb{R}^n, \tag{A.2}$$

where we assume that $f\in C^2(\mathbb{R})$ with $f'(u)>0$ for $u\in\mathbb{R}$, h, $u_0\in L^\infty(\mathbb{R}^n)$.

Observe that here we assume $f\in C^2(\mathbb{R})$ and we only consider the simpler nondegenerate case, i.e., $f'>0$ since this is the context we are interested in for our application in Section 7.

For smooth u_0 and h its well known the existence and uniqueness of a solution $u\in C^{2,1}(\mathbb{R}^n\times[0,\infty))$ (see, e.g., [33]). For u_0, $h\in L^\infty(\mathbb{R}^n)$, we use Aubin's compactness lemma to prove the convergence in L^2, to a limit function u, of the solutions u^n obtained by approximating u_0 and h by smooth functions u_0^n, h^n. We then deduce that the so obtained limit function u satisfies $u_t, \nabla u, \nabla^2 u\in L^p_{\rm loc}(\mathbb{R}^n\times(0,\infty))$, for all $p\in(1,\infty)$, combining Nash-De Giorgi theorem and linear theory (see, e.g., [33]). It is also easy to verify that u satisfies the equation (A.1) almost everywhere in $\mathbb{R}\times(0,\infty)$, and it is in fact a weak solution in the sense of Definition A.1 below. Hence, in this case, uniqueness follows from the doubling of variables method as in [12] (see Theorem A.3 below).

Definition A.1. A function $u\in L^\infty(\mathbb{R}_+^{n+1})$ is said to be a weak solution of the problem (A.1),(A.2) if the following hold:

(1) $f(u(x,t))\in L^2_{\rm loc}((0,\infty);H^1_{\rm loc}(\mathbb{R}^n))$;
(2) Given $\varphi\in C_c^\infty(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}_+^{n+1}} u\varphi_t - \nabla f(u)\cdot\nabla\varphi + h\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_0\varphi(x,0)\,dx = 0. \tag{A.3}$$

Let $H_\delta:\mathbb{R}\to\mathbb{R}$ be the approximation of the sgn function given by

$$H_\delta(s) := \begin{cases} 1, & \text{for } s>\delta,\\ \dfrac{s}{\delta}, & \text{for } |s|\le\delta, ,\\ -1, & \text{for } s<-\delta\end{cases}$$

and let $(H_\delta)_+$ and $(H_\delta)_-$ denote its nonnegative part and nonpositive part, respectively; $(H_\delta)_+(s) := \max\{H_\delta(s),0\}$, $(H_\delta)_-(s) := \max\{-H_\delta(s),0\}$.

Given a nondecreasing Lipschitz continuous function $\vartheta:\mathbb{R}\to\mathbb{R}$ and $k\in\mathbb{R}$, we define

$$B^k_\vartheta(\lambda):=\int_k^\lambda \vartheta(f(r))dr.$$

Let us denote

$$\vartheta_\delta(\lambda):=H_\delta(\lambda-f(k)) \qquad \text{and} \qquad (\vartheta_\delta)_+(\lambda):=(H_\delta)_+(\lambda-f(k)).$$

The following two results are essentially adaptations of more general ones established in [12] and state important properties of weak solutions of (A.1),(A.2).

Theorem A.1. *Let u be a weak solution of the problem of Cauchy* (A.1),(A.2)*, with $h,u_0\in L^\infty(\mathbb{R}^n)$. Then,*

$$\int_{\mathbb{R}^{n+1}_+} -B^k_{\vartheta_\delta}(u)\varphi_t+H_\delta(f(u)-f(k))\nabla f(u)\cdot\nabla\varphi-H_\delta(f(u)-f(k))h\varphi\,dx\,dt = -\int_{\mathbb{R}^{n+1}_+}|\nabla f(u)|^2H'_\delta(f(u)-f(k))\varphi\,dx\,dt \tag{A.4}$$

for all $k\in\mathbb{R}$ and all $0\le\varphi\in C^\infty_c(\mathbb{R}^{n+1}_+)$. Moreover, letting $\delta\to0$ in (A.4) *and using the strict increasing monotonicity of f, we obtain*

$$\int_{\mathbb{R}^{n+1}_+} -|u-k|\varphi_t+\nabla|f(u)-f(k)|\cdot\nabla\varphi-\operatorname{sgn}(u-k)h\varphi\,dx\,dt = -\lim_{\delta\to0}\int_{\mathbb{R}^{n+1}_+}|\nabla f(u)|^2H'_\delta(f(u)-f(k))\varphi\,dx\,dt, \tag{A.5}$$

for all $k\in\mathbb{R}$ and all $0\le\varphi\in C^\infty_c(\mathbb{R}^{n+1}_+)$. We have similar identities with $B^k_{\vartheta_\delta}$, H_δ replaced by $B^k_{(\vartheta_\delta)_+}$, $(H_\delta)_+$, respectively, in (A.4) *and $|u-k|$, $|f(u)-f(k)|$ replaced by $(u-k)_+$, $(f(u)-f(k))_+$, respectively, in* (A.5).

For the next result we assume that there is $V\in W^{2,\infty}(\mathbb{R}^n)$ such that, in (A.1), $h=\Delta V$. In particular, (A.1) admits stationary solutions, namely,

$$\psi_\alpha(x):=f^{-1}(V(x)+\alpha), \qquad \alpha\in\mathbb{R}.$$

The following theorem follows from (A.4), by using doubling of variables, the fact that u_2 is stationary, and the trick of completing the square in [12], theorem 13, p. 339. Because of its central role in the proof of Theorem 7.1 we will give its detailed proof.

Theorem A.2. *Let u_1,u_2 be weak solutions of the Cauchy problem for* (A.1) *with initial data $u_{01},u_{02}\in L^\infty(\mathbb{R}^n)$. Assume $h=\Delta V$ for some $V\in W^{2,\infty}(\mathbb{R}^n)$ and that $u_2=u_{02}$ is a stationary solution. Then,*

$$\begin{aligned}
&-\int_{(\mathbb{R}^{n+1}_+)^2}\Big(B^{u_2(y)}_{\vartheta_\delta}(u_1(x,t))(\phi_t+\phi_s)+H_\delta(f(u_1(x,t))-f(u_2(y)))(h(x)-h(y))\phi\Big)\,dx\,dt\,dy\,ds\\
&\quad+\int_{(\mathbb{R}^{n+1}_+)^2}H_\delta(f(u_1(x,t))-f(u_2(y)))(\nabla_x+\nabla_y)[f(u_1(x,t))-f(u_2(y))]\cdot(\nabla_x+\nabla_y)\phi\,dx\,dt\,dy\,ds\\
&=-\int_{(\mathbb{R}^{n+1}_+)^2}|(\nabla_x+\nabla_y)[f(u_1(x,t))-f(u_2(y))]|^2H'_\delta(f(u_1(x,t))-f(u_2(y)))\phi\,dx\,dt\,dy\,ds,
\end{aligned}\tag{A.6}$$

for all $0\le\phi\in C^\infty_c((\mathbb{R}^{n+1}_+)^2)$.

Proof. Let $u_1 = u(x,t)$ and $u_2 = u_2(y)$. By (A.4) applied to u_1, we have

$$\int_{\mathbb{R}^{n+1}_+} \Big\{ - B^k_{\vartheta_\delta}(u_1)\phi_t + H_\delta(f(u_1)-f(k))\nabla_x f(u_1)\cdot\nabla_x\phi - H_\delta(f(u_1)-f(k))h(x)\phi \Big\}\, dx\, dt$$
$$= -\int_{\mathbb{R}^{n+1}_+} |\nabla_x f(u_1)|^2 H'_\delta(f(u_1)-f(k))\phi\, dx\, dt$$

for all $k \in \mathbb{R}$. Setting $k = u_2(y)$ and integrating in y, s, we obtain

$$\int_{(\mathbb{R}^{n+1}_+)^2} \Big\{ - B^{u_2}_{\vartheta_\delta}(u_1)\phi_t + H_\delta(f(u_1)-f(u_2))\nabla_x f(u_1)\cdot\nabla_x\phi - H_\delta(f(u_1)-f(u_2))h(x)\phi \Big\}\, dx\, dt\, dy\, ds$$
$$= -\int_{(\mathbb{R}^{n+1}_+)^2} |\nabla_x f(u_1)|^2 H'_\delta(f(u_1)-f(u_2))\phi\, dx\, dt\, dy\, ds \tag{A.7}$$

Now, applying (A.4) to u_2, taking $k = u_1(x,t)$ and integrating x, t, we obtain

$$\int_{(\mathbb{R}^{n+1}_+)^2} \Big\{ B^{u_1}_{\vartheta_\delta}(u_2)\phi_s + H_\delta(f(u_1)-f(u_2))\nabla_y f(u_2)\cdot\nabla_y\phi - H_\delta(f(u_1)-f(u_2))h(y)\phi \Big\}\, dx\, dt\, dy\, ds$$
$$= \int_{(\mathbb{R}^{n+1}_+)^2} |\nabla_y f(u_2)|^2 H'_\delta(f(u_1)-f(u_2))\phi\, dx\, dt\, dy\, ds$$

Since $B^{u_1}_{\vartheta_\delta}(u_2)$ and $B^{u_2}_{\vartheta_\delta}(u_1)$ are independent of s, we can write the trivial equality where both members are null

$$\int_{(\mathbb{R}^{n+1}_+)^2} B^{u_1}_{\vartheta_\delta}(u_2)\phi_s\, dx\, dt\, dy\, ds = \int_{(\mathbb{R}^{n+1}_+)^2} B^{u_2}_{\vartheta_\delta}(u_1)\phi_s\, dx\, dt\, dy\, ds$$

Combining the two previous equalities yields

$$\int_{(\mathbb{R}^{n+1}_+)^2} \Big\{ B^{u_2}_{\vartheta_\delta}(u_1)\phi_s + H_\delta(f(u_1)-f(u_2))\nabla_y f(u_2)\cdot\nabla_y\phi - H_\delta(f(u_1)-f(u_2))h(y)\phi \Big\}\, dx\, dt\, dy\, ds$$
$$= \int_{(\mathbb{R}^{n+1}_+)^2} |\nabla_y f(u_2)|^2 H'_\delta(f(u_1)-f(u_2))\phi\, dx\, dt\, dy\, ds \tag{A.8}$$

Now, note that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{n+1}_+} \nabla_y f(u_2)\cdot\nabla_x[H_\delta(f(u_1)-f(u_2))\phi]\, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \Big\{ \nabla_y f(u_2)\cdot\nabla_x f(u_1) H'_\delta(f(u_1)-f(u_2))\phi + H_\delta(f(u_1)-f(u_2))\nabla_y f(u_2)\cdot\nabla_x\phi \Big\}\, dx\, dt \end{aligned}$$

and so we have

$$\int_{(\mathbb{R}^{n+1}_+)^2} H_\delta(f(u_1)-f(u_2))\nabla_y f(u_2)\cdot\nabla_x\phi\, dx\, dt\, dy\, ds = -\int_{(\mathbb{R}^{n+1}_+)^2} \nabla_y f(u_2)\cdot\nabla_x f(u_1) H'_\delta(f(u_1)-f(u_2))\phi\, dx\, dt\, dy\, ds \tag{A.9}$$

Analogously,

$$\int_{(\mathbb{R}^{n+1}_+)^2} H_\delta(f(u_1)-f(u_2))\nabla_x f(u_1)\cdot\nabla_y\phi\, dx\, dt\, dy\, ds = \int_{(\mathbb{R}^{n+1}_+)^2} \nabla_y f(u_2)\cdot\nabla_x f(u_1) H'_\delta(f(u_1)-f(u_2))\phi\, dx\, dt\, dy\, ds \tag{A.10}$$

Adding (A.7) and (A.10) yields

$$-\int_{(\mathbb{R}_+^{n+1})^2}\Big\{B_{\vartheta_\delta}^{u_2}(u_1)\phi_t - H_\delta(f(u_1)-f(u_2))h(x)\phi + H_\delta(f(u_1)-f(u_2))\nabla_x f(u_1)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$

$$= -\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_x f(u_1)|^2 + \nabla_x f(u_1)\cdot\nabla_y f(u_2)\Big\}H_\delta'(f(u_1)-f(u_2))\phi\,dx\,dt\,dy\,ds \tag{A.11}$$

Further, multiplying (A.8) by -1 and adding to (A.9) gives

$$-\int_{(\mathbb{R}_+^{n+1})^2}\Big\{B_{\vartheta_\delta}^{u_2}(u_1)\phi_s - H_\delta(f(u_1)-f(u_2))h(y)\phi + H_\delta(f(u_1)-f(u_2))\nabla_y f(u_2)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$

$$= -\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_y f(u_2)|^2 - \nabla_x f(u_1)\cdot\nabla_y f(u_2)\Big\}H_\delta'(f(u_1)-f(u_2))\phi\,dx\,dt\,dy\,ds. \tag{A.12}$$

Finally, adding (A.11) and (A.12) we obtain (A.6) concluding the proof. □

Remark A.1. From the equality in Theorem A.2, using test functions we $\phi(x,t,y,s) := \varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_n(\frac{x-y}{2})\theta_n(\frac{t-s}{2})$, where $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$, and $\rho_n,\ \theta_n$ are classical approximations of the identity in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, as in the doubling of variables method, we get

$$-\int_{\mathbb{R}_+^{n+1}} B_{\vartheta_\delta}^{u_2(x)}(u_1(x,t))\varphi_t\,dx\,dt$$
$$+\int_{\mathbb{R}_+^{n+1}} H_\delta(f(u_1(x,t))-f(u_2(x)))\nabla[f(u_1(x,t))-f(u_2(x))]\cdot\nabla\varphi\,dx\,dt$$
$$= -\int_{\mathbb{R}_+^{n+1}} |\nabla[f(u_1(x,t))-f(u_2(x))]|^2 H_\delta'(f(u_1(x,t))-f(u_2(x)))\varphi\,dx\,dt,$$

for all $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$.

Let the weight function $\Lambda:\mathbb{R}^n\to\mathbb{R}$ be defined by

$$\Lambda(x) := e^{-\sqrt{1+|x|^2}}. \tag{A.13}$$

An important feature of the weight function Λ is that

$$|D_i\Lambda(x)|\le\Lambda(x),\quad \text{for } i=1,\dots,n,\quad\text{and}\quad |\Delta\Lambda(x)|\le(n+1)\Lambda(x),\quad \text{for } x\in\mathbb{R}^n. \tag{A.14}$$

The next result establishes the existence and L^1-stability of weak solutions of (A.1),(A.2). The proof, which we will omit, is obtained by combining ideas in Volpert & Hudjaev [47], more specifically the use of the weight function Λ, and the extension of the doubling of variables method of Kruzhkov [31] to degenerate quasilinear parabolic equations obtained by Carrillo [12].

Theorem A.3. *Assume $f\in C^2(\mathbb{R})$, with $f'(u)>0$ for all $u\in\mathbb{R}$, and $h,u_0\in L^\infty(\mathbb{R}^n)$. Then we have the following:*

(i) *There exists a weak solution $u\in L^\infty(\mathbb{R}_+^{n+1})$ of the problem* (A.1),(A.2).

(ii) *If $u_1,u_2\in L^\infty(\mathbb{R}_+^{n+1})$ are weak solutions of* (A.1) *with initial data $u_{01},u_{02}\in L^\infty(\mathbb{R}^n)$, respectively, then*

$$\int_{\mathbb{R}^n}(u_1(x,t)-u_2(x,t))_+\phi_t + (f(u_1(x,t))-f(u_2(x,t)))_+\Delta\phi\,dx\,dt + \int_{\mathbb{R}^n}(u_{01}(x)-u_{02}(x))_+\phi(x,0)\,dx\ge 0, \tag{A.15}$$

for all $0\le\phi\in C_c^\infty(\mathbb{R}^{n+1})$, from which we obtain

$$\int_{\mathbb{R}^n}|u_1(x,t)-u_2(x,t|\phi_t + |f(u_1(x,t))-f(u_2(x,t))|\Delta\phi\,dx\,dt + \int_{\mathbb{R}^n}|u_{01}(x)-u_{02}(x)|\phi(x,0)\,dx\ge 0, \tag{A.16}$$

for all $0 \le \phi \in C_c^\infty(\mathbb{R}^{n+1})$.

(iii) *Therefore, there is a constant* $c > 0$*, depending only on* n *and* f*, such that for a.e.* $t \ge 0$ *we have*

$$\int_{\mathbb{R}^n} (u_1(x,t) - u_2(x,t))_+ \,\Lambda(x)\, dx \le e^{c\,t} \int_{\mathbb{R}^n} (u_{01}(x) - u_{02}(x))_+ \,\Lambda(x)\, dx. \tag{A.17}$$

In particular, we also have

$$\int_{\mathbb{R}^n} |u_1(x,t) - u_2(x,t))| \,\Lambda(x)\, dx \le e^{c\,t} \int_{\mathbb{R}^n} |u_{01}(x) - u_{02}(x)| \,\Lambda(x)\, dx. \tag{A.18}$$

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