Homogenization by blow-up

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Abstract

In this paper we highlight how the Fonseca and Müller blow-up technique is particularly well suited for homogenization problems. As examples we give a simple proof of the nonlinear homogenization theorem for integral functionals and we prove a homogenization theorem for sets of finite perimeter.

Keywords: Γ-convergence; homogenization; blow up.

1 Introduction

The use of Γ-convergence techniques allows to state and prove homogenization theorems in very general settings, with minimal convexity, regularity and structure assumptions on the functionals under examination. The price to pay for such a general approach is the use of more complex localization and representation arguments, sketched in their main lines in a seminal paper by De Giorgi in 1975 [11], and subsequently refined by a number of authors (this approach is at the basis of the books by Dal Maso [10] and Braides and Defranceschi [6]). Such methods are designed to ensure compactness for integral functionals under only growth conditions (tipically, functionals on Sobolev spaces on an open set Ω with polynomial growth conditions on the integrands). The output of such a process is that a Γ-limit exists and can be localized on all subsets of Ω. Once this general result is obtained, the particular case of (e.g., periodic) homogenization can be studied by characterizing the limit energy density, first proving that it is homogeneous (i.e., invariant by translations) and then providing an asymptotic homogenization formula.

The localization and representation method applied to homogenization problems has nevertheless the disadvantage of not exploiting the particular form of periodic energies. As is well known, the proof of Γ-convergence results relies on showing two inequalities; one of the two – the so-called limsup inequality – is quite simple for homogenization problems, as it is suggested by the homogenization formulas themselves. In the case of recovery sequences for an affine function for Sobolev functionals, in fact, such construction is obtained just by scaling optimal test functions for the periodic minimum problems giving the homogenization formula. The same construction is then easily extended to piecewise-affine functions, and then by density to the whole target

Sobolev space. The proof of the other inequality – the liminf inequality – is the object of the Fonseca and Müller blow-up technique [14]. This method was introduced to deal with relaxation problems, but actually works for any type of problems for which, at the end of the process, we obtain a lower estimate with an energy whose density is characterized by a formula which in turn can be used as above to construct a recovery sequence (a relaxation or homogenization formula in the two cases).

This paper aims at popularizing such a technique, until now little used in this context.

1.1 The blow-up technique for homogenization problems

In this section we describe the guidelines of the blow-up technique. This applies to sequences of functionals $F_j = F_j(u)$ defined on a space of functions u on an open set Ω, which can be written as *set functions*; i.e., such that for all u we have $F_j(u) = \mu_j(\Omega)$, where μ_j is defined for all open subsets of Ω. In the much studied case of periodic integrals

$$F_j(u) = \int_\Omega f\Big(\frac{x}{\varepsilon_j}, \nabla u\Big)\, dx$$

we clearly have

$$\mu_j(A) = \int_A f\Big(\frac{x}{\varepsilon_j}, \nabla u\Big)\, dx.$$

The blow-up argument can de divided into five steps.

Step 1: definition of a limit measure. We only have to consider the non-trivial case when $\liminf_j F_j(u_j)$ is finite and $u_j \to u_0$. We suppose that the set functions μ_j weak* converge to some measure μ; i.e., that $\sup_j \mu_j(A) < +\infty$ and we have

$$\mu(A) = \lim_j \mu_j(A)$$

for all A such that $\mu(A) = \mu(\overline{A})$ (we refer to De Giorgi and Letta [12] for the notions of convergence of set functions). This definition of weak*-convergence of set functions coincides with the usual notion of weak*-convergence of measures if the set functions μ_j are already (restriction to open sets of) measures, but in this process we may include also "mildly non-local" energies (e.g. convolutions or finite-range discrete systems), which cannot be directly written as measures, but whose limit can. Note that weak*-convergence is compact on equi-bounded measures, so that the existence of a limit measure μ is often trivially ensured, up to subsequences, by the equi-boundedness of the energies.

We fix some positive measure λ, whose choice is driven by the target function u and the framework of the problem, and we consider the Radon-Nikodym decomposition of μ with respect to λ; i.e.,

$$\mu = \frac{d\mu}{d\lambda}\lambda + \mu^s,$$

where $\mu^s \perp \lambda$. In the case of integral functionals we simply choose $\lambda = \mathcal{L}^n$.

Step 2: local analysis. We fix $x_0 \in \Omega$ such that x_0 is a Lebesgue point for μ with respect to λ; i.e.,

$$\frac{d\mu}{d\lambda}(x_0) = \lim_{\rho \to 0^+} \frac{\mu(x_0 + \rho D)}{\lambda(x_0 + \rho D)},$$

where D is a suitable open set properly chosen for the problem (usually a cube or a ball).

Step 3: blow up. By a diagonalization argument we can choose $\rho_j \to 0$ such that

$$\frac{d\mu}{d\lambda}(x_0) = \lim_j \frac{F_j(u_j, x_0 + \rho_j D)}{\lambda(x_0 + \rho_j D)}.$$

We rescale the functionals F_j and define the family G_j as follows:

$$G_j(v, D) = \frac{F_j(u, x_0 + \rho_j D)}{\lambda(x_0 + \rho_j D)},$$

where v and u are linked by a scaling argument driven by the requirement that the v_j corresponding to u_j converge to a *meaningful* v_0, whose form depends only on u_0 and x_0. In the Sobolev case this v_0 can be chosen a linear function with gradient $\nabla u_0(x_0)$.

Step 4: local estimates. Depending on the type of energies, we find an appropriate way to estimate the scaled energies $G_j(v_j, D)$ as $v_j \to v_0$ with minimum problems. To this end it is often useful to be able to modify v_j in such a way that they satisfy the same boundary conditions as v_0. In this way a simple inequality is obtained by minimization:

$$G_j(v_j, D) \geq \inf\{G_j(v, D) : v = v_0 \text{ on } \partial D\}.$$

We obtain an inequality of the form

$$\frac{d\mu}{d\lambda}(x_0) \geq \varphi^\lambda(x_0),$$

where

$$\varphi^\lambda(x_0) = \liminf_j \inf\{G_j(v, D) : v = v_0 \text{ on } \partial D\}.$$

This can be recognized to be a formula of homogenization type. In the case of Sobolev functionals, in fact, we obtain, after scaling,

$$\varphi^\lambda(x_0) = \liminf_j \Big(\frac{\varepsilon_j}{\rho_j}\Big)^n \inf\Big\{\int_{\frac{\rho_j}{\varepsilon_j} D} f(y, \nabla w(y))\, dy : w(y) = \langle \nabla u(x_0), y\rangle \text{ on } \partial \frac{\rho_j}{\varepsilon_j} D\Big\},$$

with the minimum computed on $w \in W^{1,p}(\frac{\rho_j}{\varepsilon_j} D; \mathbb{R}^m)$.

At this point, an independent side computation is needed to show that indeed such a formula does not depend on the subsequence, as customary in homogenization problems.

Step 5: global estimates. The conclusion follows from integrating the local estimates above.

1.2 The Nonlinear Homogenization Theorem revisited

In Section 2 we will expand the reasoning outlined above to re-obtain the Homogenization Theorem for non-linear energies on vector-valued functions by Braides [4] and Müller [16]. Despite the restriction on the growth of the functions (that must be polynomial, and hence not covering functionals defined e.g. on functions satisfying a positive-determinant constraint, which are the natural class to consider in Non-linear Elasticity) this result, proved in 1985, still provides the model for nonlinear homogenization theorems. The proof by Braides uses the localization method by De Giorgi hinted at the beginning of the Introduction, noting that the result immediately follows after representing the limit by a homogeneous quasiconvex function, and using the property of convergence of minima of Γ-convegence. The proof of Müller is self-contained, but essentially follows the same localization idea. It must be mentioned that Γ-convergence homogenization results have also been obtained by other methods, but mainly restricted to the convex case (see Allaire [1] for an approach by two-scale convergence) or obtaining a weaker representation in terms of Young measures. A recent paper by Cioranescu et al. [9] uses a slight variant of two-scale convergence to recover the theorem by Braides and Müller, showing how a much more complex proof than the original one is needed in that framework. In addition, that proof already uses the knowledge of the asymptotic homogenization formula (which, on the contrary, is naturally derived in the original case).

The use of the blow-up method limits the technical points to the possibility of slightly varying boundary conditions and of defining the asymptotic homogenization formula as a limit. Both facts are well known and can be proved by simple and general methods. A similar use of the blow-up method had already been performed by Alvarez and Mandallena [2], who use slighly more complex hypotheses than periodicity.

1.3 Homogenization of perimeter functionals

In order to show the generality of the blow-up method we apply it to a different type of problems; namely, that of computing the Γ-limit of functionals of the form

$$F_\varepsilon(E) = \int_{\partial^* E\cap\Omega} f\Big(\frac{x}{\varepsilon}, \nu_E\Big) d\mathcal{H}^{n-1},$$

with $\partial^* E$ denoting the reduced boundary of the set E of finite perimeter in Ω. This type of energies can be rephrased as defined on characteristic functions $u = \chi_E$ and hence re-set in the framework above. In this framework

$$\mu_j(A) = \int_{S(u_j)\cap A} f\Big(\frac{x}{\varepsilon}, \nu_{E_j}\Big) d\mathcal{H}^{n-1},$$

where $S(u_j) = \partial^* E_j$ denotes the set of essential discontinuity points of u_j, and u_j converges in L^1.

A homogenization theorem for this type of perimeter functionals has been proved by Ambrosio and Braides [3], under some continuity assumptions on f that allow to include the almost-periodic case. Here we remove those assumption in the periodic case. In this case, if $u = \chi_E$, the target measure is

$$\lambda(A) = \mathcal{H}^{n-1}(A \cap S(u)),$$

which then gives a different scaling in Step 3 above (so that v_0 is actually the characteristic function of a half-space) and hence also influences the definition of the homogenized energy density.

2 The Nonlinear Homogenization Theorem

In all that follows $n, m \geq 1$ are fixed integers, $p > 1$. With $\mathbb{M}^{m\times n}$ we denote the space of $m\times n$ matrices with real entries. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $\mathcal{L}^n(E)$ or alternatively $|E|$ denote its n-dimensional Lebesgue measure. $B_r(x)$ is the open ball of centre x and radius r; if $x = 0$ we will write B_r in place of $B_r(x)$. $Q_r(x)$ is the open n-cube $Q_r(x) = x + \big(-\frac{r}{2}, \frac{r}{2}\big)^n$; if $x = 0$ we will write Q_r in place of $Q_r(0)$. The letter c denotes a generic strictly positive constant. We refer to [13] for the necessary measure-theoretical machinery, and to [18] for fine properties of Sobolev functions.

The Nonlinear Homogenization Theorem reads as follows

Theorem 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $f : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0, +\infty)$ be a Borel function such that*

$$x \mapsto f(x, \xi) \text{ is 1-periodic for all } \xi \in \mathbb{M}^{m\times n}. \tag{1}$$

Assume that there exists a constant $C > 0$ such that

$$\frac{1}{C}(|\xi|^p - 1) \leq f(x, \xi) \leq C(1 + |\xi|^p) \text{ for all } \xi \in \mathbb{M}^{m\times n}, \text{ for a.e. } x \in \Omega. \tag{2}$$

Then, for all $\xi \in \mathbb{M}^{m\times n}$ there exists the limit

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{(0,T)^n} f(y, \xi + \nabla v(y))\, dy : \ v \in W_0^{1,p}((0,T)^n; \mathbb{R}^m)\Big\} \tag{3}$$

and the functionals $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ defined by

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)\, dx \tag{4}$$

Γ-converge, with respect to the L^p convergence to the homogenized functional F_0 : *$W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ given by*

$$F_0(u) = \int_\Omega f_{\rm hom}(\nabla u)\, dx. \tag{5}$$

2.1 Auxiliary results

As mentioned in the Introduction, we will need a couple of technical results to prove the Nonlinear Homogenization Theorem, which are classical and with an easy independent proof. We recall them in the form we need.

Lemma 2.2 (De Giorgi's Lemma to match boundary conditions) *Let D be a bounded open subset of $\mathbb{R}^n$ with $|\partial D| = 0$. Let (f_j) be a sequence of Borel functions $f_j : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0, +\infty)$ such that there exists a positive constant $C > 0$ independent of j such that* (2) *holds for f_j, and consider the functionals $F_j : W^{1,p}(D; \mathbb{R}^m) \to [0, +\infty)$, defined by*

$$F_j(u) = \int_D f_j\big(x, \nabla u(x)\big)\, dx. \tag{6}$$

If u_j is a sequence with $\sup_j F_j(u_j) < +\infty$ weakly converging to u_0 in $W^{1,p}$, then there exists a sequence $\widetilde{u}_j$ still converging to u_0 and with $\widetilde{u}_j = u_0$ in a neighbourhood of ∂D such that

$$\lim_j \int_D |f_j(x, \nabla u_j) - f_j(x, \nabla \widetilde{u}_j)|\, dx = 0.$$

Proof. The proof is achieved by a cut-off argument, setting $\widetilde{u}_j = \phi_j u_j + (1-\phi_j)u_0$, where ϕ_j are suitable functions compactly supported in D and converging to the constant 1 chosen in such a way that $\int_{C_j} |\nabla u_j|^p\, dx \to 0$, where $C_j = \{\nabla \phi_j \neq 0\}$ (for details see [6] Section 11.1).

The second result is the validity of an asymptotic homogenization formula which provides the candidate energy density for the Γ-limit.

Proposition 2.3 (homogenization formula) *Let $f : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0, +\infty)$ be such that* (1) *holds. Assume that there exists a constant $C > 0$ such that*

$$0 \le f(x, \xi) \le C(1 + |\xi|^p) \quad \textit{for all } \xi \in \mathbb{M}^{m\times n}, \ \textit{for a.e. } x \in \mathbb{R}^n. \tag{7}$$

For all $T > 0$ consider an arbitrary point $x_T \in \mathbb{R}^n$. Then, for all $\xi \in \mathbb{M}^{m\times n}$ there exists the limit

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{Q_T(x_T)} f(y, \xi + \nabla v(y))\, dy : \ v \in W_0^{1,p}(Q_T(x_T); \mathbb{R}^m)\Big\} \tag{8}$$

Proof. The proof can be easily performed using a "subadditivity argument" (see [6] Proposition 14.4). This argument has been recast as a general subadditivity theorem by Licht and Michaille in [15].

2.2 Proof of the lower bound by blow-up

Let (ε_j) be a positive infinitesimal sequence. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $u_j \to u$ in $L^p(\Omega; \mathbb{R}^m)$. It is not restrictive to assume that $\liminf_j F_{\varepsilon_j}(u_j)$ is finite, so that we may also suppose that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$.

For all $j \in \mathbb{N}$ and for all $A \in \mathcal{B}(\mathbb{R}^n)$ we define the localized functional

$$F_{\varepsilon_j}(u, A) = \int_A f\Big(\frac{x}{\varepsilon_j}, \nabla u\Big)\, dx. \tag{9}$$

Moreover, the sequence of positive measures μ_j is defined as follows

$$\mu_j = f\Big(\frac{x}{\varepsilon_j}, \nabla u_j\Big)\mathcal{L}^n; \tag{10}$$

i.e., $\mu_j(A) = F_{\varepsilon_j}(u_j, A)$. We divide the blow-up argument into five steps.

Step 1. Definition of a limit measure. Note that the measures (μ_j) are equi-bounded; i.e., there exists $c > 0$ such that $\sup_j |\mu_j|(\Omega) \leq c < +\infty$. By the weak* compactness of measures, there exists a positive measure μ on Ω such that $\mu_j \rightharpoonup \mu$, upon possibly passing to subsequences. We consider the Radon-Nikodym decomposition of the limit measure μ with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$:

$$\mu = \frac{d\mu}{dx}\mathcal{L}^n + \mu^s, \tag{11}$$

where $\mu^s \perp \mathcal{L}^n$. We recall that since μ is a positive measure, then its singular part is a positive measure as well.

Step 2. Local analysis. Let $x_0 \in \Omega$ be a Lebesgue point for μ with respect to $\mathcal{L}^n$; i.e.,

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^n(Q_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\rho^n}. \tag{12}$$

The Besicovitch Derivation Theorem states that $\mathcal{L}^n$-almost every $x \in \Omega$ is a Lebesgue point for μ with respect to $\mathcal{L}^n$. Up to a set of zero Lebesgue measure, we can assume that in addition x_0 satisfies the following condition:

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\Big(\frac{1}{|Q_\varepsilon(x_0)|}\int_{Q_\varepsilon(x_0)} |u(x) - u(x_0) - \langle\nabla u(x_0), x - x_0\rangle|^p\Big)^{\frac{1}{p}} = 0; \tag{13}$$

see e.g. [18] Theorem 3.4.2. Since μ is finite, we have $\mu(\partial Q_\rho(x_0)) = 0$ for all $\rho > 0$ but a countable set. For all such ρ we then have

$$\mu(Q_\rho(x_0)) = \lim_j \mu_j(Q_\rho(x_0)). \tag{14}$$

Step 3. Blow-up. The arguments of Step 2 imply that for $\mathcal{L}^n$-almost every $x_0 \in \Omega$ there exists a positive infinitesimal sequence (ρ_j) such that

$$\frac{d\mu}{dx}(x_0) = \lim_j \frac{\mu_j(Q_{\rho_j}(x_0))}{\mathcal{L}^n(Q_{\rho_j}(x_0))}. \tag{15}$$

In fact, $\mathcal{L}^n$-almost every $x_0 \in \Omega$ is a Lebesgue point for μ; i.e., (12) holds. By a diagonalization argument on (12) and (14), we can extract a sequence (ρ_j) satisfying (15). Recalling the expression of μ_j we can note that (15) is equivalent to

$$\frac{d\mu}{dx}(x_0) = \lim_j \frac{1}{\rho_j^n}\int_{Q_{\rho_j}(x_0)} f\Big(\frac{x}{\varepsilon_j}, \nabla u_j(x)\Big)\,dx. \tag{16}$$

Now we modify u_j in order to define a sequence (v_j) weakly converging in $W^{1,p}(Q_1;\mathbb{R}^m)$ to the linear function $w_0(x) = \langle\nabla u(x_0), x\rangle$.

For all $j \in \mathbb{N}$ and $\rho > 0$, let w_j^ρ be given by

$$w_j^\rho = \frac{u_j(x_0 + \rho x) - u(x_0)}{\rho}. \tag{17}$$

We define the affine functions $u_0(x) = u(x_0) + \langle \nabla u(x_0), x\rangle = u(x_0) + w_0(x)$ as well. Note that by construction

$$w_j^\rho = \frac{u_j(x_0 + \rho x) - u_0(\rho x)}{\rho} + w_0(x). \tag{18}$$

We first prove that

$$\lim_{\rho\to 0^+} \lim_j \int_{Q_1} |w_j^\rho(x) - w_0(x)|^p\, dx = 0. \tag{19}$$

By (18) and the fact that $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$, we deduce that the above limit is equivalent to

$$\lim_{\rho\to 0^+} \lim_j \int_{Q_1} \frac{1}{\rho^p} |u_j(x_0+\rho x) - u_0(\rho x)|^p\, dx = \lim_{\rho\to 0^+} \frac{1}{\rho^p} \int_{Q_1} |u(x_0+\rho x) - u_0(\rho x)|^p\, dx.$$

By a change of variables we set $y = x_0 + \varepsilon x$ and get that the limit equals

$$\lim_{\rho\to 0} \frac{1}{\rho^p} \int_{Q_\rho(x_0)} |u(y) - u_0(y - x_0)|^p \frac{1}{\rho^n}\, dy.$$

Recalling the expression of u_0 we deduce that

$$\begin{aligned}\lim_{\rho\to 0} \lim_j \int_{Q_1} &|w_j^\rho(x) - w_0(x)|^p\, dx \\ &= \lim_{\rho\to 0} \frac{1}{\rho^p} \Big(\frac{1}{\rho^n} \int_{Q_\rho(x_0)} |u(x) - u(x_0) - \langle \nabla u(x_0), x - x_0\rangle|^p\, dx \Big).\end{aligned}$$

By (13) we conclude that (19) holds.

We can extract a further subsequence $\rho_j \to 0$ and define

$$v_j(x) := w_j^{\rho_j}(x) = \frac{u_j(x_0 + \rho_j x) - u(x_0)}{\rho_j}$$

such that $v_j \to w_0(x)$ in $L^p(Q_1;\mathbb{R}^m)$. Since $\nabla v_j(x) = \nabla u_j(x_0 + \rho_j x)$ the gradients $(|\nabla v_j|)$ are equi-bounded in $L^p(Q_1;\mathbb{R}^m)$, hence $v_j \rightharpoonup w_0(x)$ in $W^{1,p}(Q_1;\mathbb{R}^m)$ up to subsequences. By a scaling argument in (16) we get

$$\frac{d\mu}{dx}(x_0) = \lim_j \int_{Q_1} f\Big(\frac{x_0 + \rho_j y}{\varepsilon_j}, \nabla v_j(y)\Big)\, dy.$$

By Lemma 2.2 we can modify (v_j) to get a sequence $(\widetilde{v}_j)$ such that $\widetilde{v}_j \in w_0(x) + W_0^{1,p}(Q_1;\mathbb{R}^m)$ and

$$\int_{Q_1} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nabla \widetilde{v}_j(y)\Big)\, dy \le \int_{Q_1} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nabla v_j(y)\Big)\, dy + o(1) \quad \text{as } j \to +\infty.$$

There follows that

$$\begin{aligned}\frac{d\mu}{dx}(x_0) &\ge \liminf_j \int_{Q_1} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nabla \widetilde{v}_j(y)\Big)\, dy \\ &\ge \liminf_j \inf\Big\{ \int_{Q_1} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nabla u(x_0) + \nabla w(y)\Big)\, dy : w \in W_0^{1,p}(Q_1;\mathbb{R}^m)\Big\}.\end{aligned}$$

By a scaling argument we set $z = \frac{\rho_j y + x_0}{\varepsilon_j}$ in the previous inequality. Thus we get

$$\frac{d\mu}{dx}(x_0) \ge \liminf_j \inf\Big\{ \int_{Q_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j})} f\big(z, \nabla u(x_0) + \nabla w(y)\big)\, dy : w \in W_0^{1,p}(Q_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j});\mathbb{R}^m)\Big\}.$$

Step 4. Local estimates. We can apply Proposition 2.3 with T and x_T replaced by ρ_j/ε_j and x_0/ε_j respectively. Thus we get that there exists the limit

$$\lim_j \inf\Big\{\int_{Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)} f\big(y, \nabla u(x_0) + \nabla w(y)\big)\,dy : w \in W_0^{1,p}(Q_{\frac{\rho_j}{\varepsilon_j}}\big(\frac{x_0}{\varepsilon_j}\big);\mathbb{R}^m)\Big\}$$
$$= f_{\text{hom}}(\nabla u(x_0)).$$

Note that the lower estimate is independent of the sequences (ε_j) and (ρ_j). To sum up: for $\mathcal{L}^n$-almost every $x_0 \in \Omega$ we have

$$\frac{d\mu}{dx}(x_0) \geq f_{\text{hom}}(\nabla u(x_0)). \tag{20}$$

Step 5. Global estimates. The conclusion follows from integrating (20) on Ω. Taking into account (11) and (20) we get

$$\mu(\Omega) \geq \int_\Omega \frac{d\mu}{dx}\,dx \geq \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx.$$

Since $\mu_j \rightharpoonup \mu$ we have $\liminf_j \mu_j(\Omega) \geq \mu(\Omega)$, and hence

$$\liminf_j F_{\varepsilon_j}(u_j) = \liminf_j \mu_j(\Omega) \geq \mu(\Omega) \geq \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx = F_0(u),$$

as desired.

2.3 Proof of the Γ-limsup inequality

In this section we include the proof of the opposite inequality in the Γ-convergence result for the sake of completeness. Since this part is independent of the blow-up argument, we only sketch it. Details can be traced e.g. in [6]. The proof is divided into three steps.

(i) We consider a target function u of the form $u(x) = \langle\xi, x\rangle$ with $\xi \in \mathbb{M}^{m\times n}$. Let $T \in \mathbb{N}$ be fixed. We denote by m_T the rescaled infimum

$$m_T = \frac{1}{T^n}\inf\Big\{\int_{Q_T} f(x, \xi + \nabla w(x))\,dx : w \in W_0^{1,p}(Q_T;\mathbb{R}^m)\Big\} = f_{\text{hom}}(\xi) + o(1)$$

as $T \to +\infty$. Let $w_T \in W_0^{1,p}(Q_T;\mathbb{R}^m)$ be such that

$$\frac{1}{T^n}\int_{Q_T} f(x, \xi + \nabla w_T(x))\,dx < m_T + o(1) \tag{21}$$

as $T \to +\infty$. We extend the function w_T by T-periodicity to the whole $\mathbb{R}^n$, thus getting a function $w_T \in W^{1,p}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$. For all $\varepsilon > 0$ we set

$$u^T_\varepsilon(x) = \varepsilon w_T\Big(\frac{x}{\varepsilon}\Big).$$

By construction $u^T_\varepsilon \in W^{1,p}(\Omega;\mathbb{R}^m)$, and $u^T_\varepsilon \to 0$. Having set

$$v^T_{\varepsilon_j}(x) = u^T_{\varepsilon_j}(x) + \langle\xi, x\rangle, \tag{22}$$

we deduce that $v^T_{\varepsilon_j} \rightharpoonup \langle \xi, x\rangle$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, and we have

$$\begin{aligned} F_{\varepsilon_j}(v^T_{\varepsilon_j}) &= \int_\Omega f\Big(\frac{x}{\varepsilon_j}, \nabla u^T_{\varepsilon_j}(x) + \xi\Big)\, dx \\ &= \int_\Omega f\Big(\frac{x}{\varepsilon_j}, \nabla w_T\Big(\frac{x}{\varepsilon_j}\Big) + \xi\Big)\, dx = |\Omega| m_T + o(1). \end{aligned}$$

We can conclude that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(v^T_\varepsilon) \le |\Omega| m_T + o(1) = F_0(u) + o(1),$$

as desired.

We point out that by Lemma 2.2 we may assume that the (approximate) recovery sequence $(v^T_\varepsilon) \in \langle \xi, \cdot\rangle + W^{1,p}_0(\Omega;\mathbb{R}^m)$. Moreover, we can note that the arguments hold unvaried if we fix a target function of the form $u(x) = \langle \xi, x\rangle + b$, with $\xi \in \mathbb{M}^{m\times n}$ and $b \in \mathbb{R}^n$.

(ii) Now we show how to extend the construction above to piecewise-affine functions. We consider a target function

$$u(x) = \sum_{k=1}^N \chi_{\Omega_k}\big(\langle \xi_k, x\rangle + b_k\big),$$

where $N \in \mathbb{N}$, $\xi_k \in \mathbb{M}^{m\times n}$, $b_k \in \mathbb{R}^n$, $\bigcup_{k=1}^N \Omega_k = \Omega$, $\Omega_k \cap \Omega_h = \emptyset$ if $k \neq h$.

By Step **(i)** there exists a recovery sequence u^k_ε for $u_k := \langle \xi_k, x\rangle + b_k$ on Ω_k with $u^\varepsilon_k = u_k$ on $\partial\Omega_k$. It suffices then to define

$$u_\varepsilon = \sum_k \chi_{\Omega_k} u^\varepsilon_k,$$

and repeat the computations for each Ω_k.

(iii) In this step we conclude the proof of the Γ-limsup inequality using a density argument.

Let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Let (u_k) be piecewise-affine functions $u_k : \Omega \to \mathbb{R}^m$ such that $u_k \to u$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. From its definition it is easily deduced that $f_{\rm hom}$ is a continuous function satisfying a polynomial growth condition of order p. In particular the functional F_0 in (5) is continuous with respect to the strong convergence in $W^{1,p}$. Since Γ-$\limsup_j F_{\varepsilon_j}$ is lower semicontinuous with respect to the strong convergence of $W^{1,p}(\Omega;\mathbb{R}^m)$, the results of Step **(ii)** imply that

$$\Gamma\text{-}\limsup_j F_{\varepsilon_j}(u) \le \liminf_{k\to\infty} \Big(\Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}(u_k)\Big) \le \liminf_{k\to+\infty} F_0(u_k) = F_0(u).$$

3 Homogenization of perimeter functionals

In what follows we will consider sets of finite perimeter E in a bounded open set Ω of $\mathbb{R}^n$ (see e.g. [5]). With $\partial^* E$ we indicate the *reduced boundary* of E, by ν_E the *inner normal* to E, defined at all points of $\partial^* E$. The $(n-1)$-dimensional (surface) *Hausdorff measure* will be denoted by $\mathcal{H}^{n-1}$. A set of finite perimeter can also be identified with its characteristic function $u = \chi_E$, which is a function of bounded variation. In this case $\partial^* E = S(u)$, the set of *approximate discontinuity points* of u, up to a set of $\mathcal{H}^{n-1}$-measure zero. The convergence of sets E_j will be understood as the L^1-convergence of the corresponding characteristic functions.

With $Q^\nu_T(x)$ we denote a cube with centre x, side length T, and one face orthogonal to ν. If $x = 0$ we use the notation Q^ν_T, and if also $T = 1$ the notation Q^ν. If needed, we will tacitly assume that for fixed ν the cubes $Q^\nu_T(x)$ and $Q^\nu_S(x)$ differ by a dilation and a translation only. With $H^\nu(x) = \{y \in \mathbb{R}^n : \langle y - x, \nu\rangle > 0\}$ we denote the half space with boundary the hyperplane $\Pi^\nu(x)$ through x, with inner normal ν. If $x = 0$ we drop x from the notation.

We will consider a Borel function $f : \mathbb{R}^n \times S^{n-1} \to [\alpha, \beta]$, with $0 < \alpha < \beta$, 1-periodic in the first variable, and the energies

$$F_\varepsilon(E) = \int_{\Omega\cap\partial^* E} f\Big(\frac{x}{\varepsilon}, \nu_E\Big) d\mathcal{H}^{n-1},$$

which represent a perimeter functional in a inhomogeneous environment. The Homogenization Theorem for such energies reads as follows.

Theorem 3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, and let f be as above. Then for all $\nu \in S^{n-1}$ there exists the limit*

$$f_{\rm hom}(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{\int_{Q^\nu_T\cap\partial^* E} f(y, \nu_E) d\mathcal{H}^{n-1} : E \setminus Q^\nu_T = H^\nu \setminus Q^\nu_T\Big\} \tag{23}$$

and the functionals F_ε above Γ-converge to the energy

$$F_{\rm hom}(E) = \int_{\Omega\cap\partial^* E} f_{\rm hom}(\nu_E) d\mathcal{H}^{n-1}. \tag{24}$$

3.1 Auxiliary results

In parallel with the proof of the Nonlinear Homogenization Theorem, we include here two auxiliary results. The first one again regards the possibility of fixing boundary conditions.

Lemma 3.2 *Let E_j be sets of equi-bounded perimeter converging to some E_0 in Ω. Then there exists a sequence $\widetilde{E}_j \to E_0$ such that $\widetilde{E}_j = E_0$ in a neighbourhood of $\partial\Omega$ and*

$$\lim_j \mathcal{H}^{n-1}(\Omega \cap (\partial^* \widetilde{E}_j \triangle \partial^* E_j)) = 0.$$

Proof. The lemma can be proved by using the coarea formula: by the convergence of E_j to E_0 we have for all $\delta > 0$ (d denotes the distance from $\mathbb{R}^n \setminus \Omega$, and $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$)

$$\begin{aligned} 0 &= \lim_j |\Omega_\delta \cap (E_j \triangle E_0)| \\ &= \lim_j \int_{\Omega_\delta \cap (E_j \triangle E_0)} |\nabla d|\, dx \\ &= \lim_j \int_0^\delta \mathcal{H}^{n-1}(\{d(x) = t\} \cap (E_j \triangle E_0))\, dt \end{aligned}$$

By suitably choosing $\delta = \delta_j \to 0$ we can then find $t_j \in (0, \delta_j)$ such that $\mathcal{H}^{n-1}(\{d(x) = t_j\} \cap (E_j \triangle E_0)) \to 0$. We then define

$$\widetilde{E}_j = \begin{cases} E_j & \text{on } \Omega \setminus \Omega_{t_j} \\ E_0 & \text{on } \Omega_{t_j}. \end{cases}$$

Since

$$\Omega \cap (\partial^* \widetilde{E}_j \triangle \partial^* E_j) = (\Omega_{t_j} \cap \partial^* E_0) \cup (\{d(x) = t_j\} \cap (E_j \triangle E_0))$$

we obtain the thesis.

The second result concerns the existence of the energy density f_{hom}.

Proposition 3.3 (homogenization formula) *Let (x_T) be a family of points in $\mathbb{R}^n$; then for all $\nu \in S^{n-1}$ there exists the limit*

$$\lim_{T \to +\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{Q_T^\nu(x_T) \cap \partial^* E} f(y, \nu_E) d\mathcal{H}^{n-1} : E \setminus Q_T^\nu = H^\nu(x_T) \setminus Q_T^\nu(x_T) \Big\} \qquad (25)$$

independent of (x_T)

Proof. The proof uses a slightly more complex construction than for the bulk case, since in general Π^ν does not intersect the set of periods $\mathbb{Z}^n$, but it follows the same line. Let $S >> T$. Fix $L > n$, and consider a $(n-1)$-dimensional cubic lattice $\mathcal{L}$ in $\Pi^\nu(x_S)$ isometrically congruent to $(T + L)\mathbb{Z}^{n-1}$. For each point z_j in that lattice consider a point $y_j \in x_T + \mathbb{Z}^n$ (i.e., differing from x_T by a period) such that $|y_j - z_j| \le \sqrt{n}$.

Let E_T be a test set for the minimum problem

$$m_T = \inf\Big\{ \int_{Q_T^\nu(x_T) \cap \partial^* E} f(y, \nu_E) d\mathcal{H}^{n-1} : E \setminus Q_T^\nu = H^\nu(x_T) \setminus Q_T^\nu(x_T) \Big\},$$

and define

$$E_S = \begin{cases} y_j + E_T & \text{if } y_j + Q_T^\nu \subset Q_S^\nu \\ H^\nu(x_S) & \text{otherwise.} \end{cases}$$

By using this test set in the definition of m_S we easily obtain

$$m_S \le m_T + r(S, T), \qquad \text{with} \qquad \limsup_{T \to +\infty} \limsup_{S \to +\infty} r(S, T) = 0,$$

from which we immediately deduce the existence of the limit above.

3.2 Proof of the lower bound by blow up

Let (ε_j) be a positive infinitesimal sequence. Let $E_j \to E$. It is not restrictive to assume that $\liminf_j F_{\varepsilon_j}(E_j)$ is finite, so that we may also suppose that E is a set of finite perimeter.

For all $j \in \mathbb{N}$ and for all A Borel subset of $\mathbb{R}^n$ we define the localized functional

$$F_{\varepsilon_j}(E_j, A) = \int_{A\cap\partial^* E_j} f\Big(\frac{x}{\varepsilon_j}, \nu_{E_j}\Big)\, d\mathcal{H}^{n-1}, \tag{26}$$

and the corresponding positive measures μ_j defined as follows

$$\mu_j = f\Big(\frac{x}{\varepsilon_j}, \nu_{E_j}\Big)\mathcal{H}^{n-1} \llcorner \partial^* E_j; \tag{27}$$

i.e., $\mu_j(A) = F_{\varepsilon_j}(E_j, A)$ for all A.

Step 1. Definition of a limit measure. Again, the measures (μ_j) are equi-bounded so that there exists a positive measure μ on Ω such that $\mu_j \rightharpoonup \mu$, upon possibly passing to subsequences. We consider the Radon-Nikodym decomposition of the limit measure μ with respect to the $(n-1)$-dimensional Hausdorff measure restricted to $\partial^* E$, $\mathcal{H}^{n-1} \llcorner \partial^* E$:

$$\mu = \frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}\mathcal{H}^{n-1} \llcorner \partial^* E + \mu^s, \tag{28}$$

where $\mu^s \perp (\mathcal{H}^{n-1} \llcorner \partial^* E)$ is a positive measure as well.

Step 2. Local analysis. Let $x_0 \in \Omega \cap \partial^* E$ be a Lebesgue point for μ with respect to $\mathcal{H}^{n-1} \llcorner \partial^* E$. We can now write

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q^\nu_\rho(x_0))}{\mathcal{H}^{n-1}(Q^\nu_\rho(x_0) \cap \partial^* E)} = \lim_{\rho\to 0^+} \frac{\mu(Q^\nu_\rho(x_0))}{\rho^{n-1}}. \tag{29}$$

where

$$\nu = \nu_E(x_0).$$

The Besicovitch Derivation Theorem ensures that $\mathcal{H}^{n-1}$-almost every $x \in \Omega \cap \partial^* E$ is a Lebesgue point for μ with respect to $\mathcal{H}^{n-1} \llcorner \partial^* E$. By the definition of reduced boundary, we can assume that in addition x_0 satisfies the following condition:

$$\frac{1}{\rho}(E - x_0) \to H^\nu \text{ as } \rho \to 0^+ \tag{30}$$

in every bounded open subset of $\mathbb{R}^n$. Since μ is finite, we have $\mu(\partial Q^\nu_\rho(x_0)) = 0$ for all $\rho > 0$ but a countable set. For all such ρ we have

$$\mu(Q^\nu_\rho(x_0)) = \lim_j \mu_j(Q^\nu_\rho(x_0)). \tag{31}$$

Step 3. Blow-up. The arguments of Step 2 imply that for $\mathcal{H}^{n-1}$-almost every $x_0 \in \Omega \cap \partial^* E$ there exists a positive infinitesimal sequence (ρ_j) such that

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) = \lim_j \frac{\mu_j(Q^\nu_{\rho_j}(x_0))}{\mathcal{H}^{n-1}(Q_{\rho_j}(x_0) \cap \partial^* E)}. \tag{32}$$

By a diagonalization argument on (29) and (31), we can extract a sequence (ρ_j) satisfying (32). Recalling the expression of μ_j we can note that (32) is equivalent to

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) = \lim_j \frac{1}{\rho_j^{n-1}} \int_{Q^\nu_{\rho_j}(x_0) \cap \partial^* E_{\varepsilon_j}} f\Big(\frac{x}{\varepsilon_j}, \nu_{E_j}(x)\Big)\, d\mathcal{H}^{n-1}. \tag{33}$$

Now we modify E_{ε_j} in order to define a sequence (E'_j) converging to the half space H^ν in Q^ν, by setting

$$E'_j = \frac{1}{\rho_j}(E_{\varepsilon_j} - x_0).$$

In fact

$$|(E'_j \triangle H^\nu) \cap Q^\nu| \leq \Big| Q^\nu \cap \Big(\frac{1}{\rho_j}(E - x_0) \triangle H^\nu\Big)\Big| + \Big| Q^\nu \cap \Big(\frac{1}{\rho_j}(E_{\varepsilon_j} \triangle E_0)\Big)\Big|,$$

which tends to 0, upon an additional requirement in the choice of ρ_j, since $|E_{\varepsilon_j} \triangle E_0| \to 0$. By a scaling argument in (33) we get

$$\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) = \lim_j \int_{Q^\nu \cap E'_j} f\Big(\frac{x_0 + \rho_j y}{\varepsilon_j}, \nu_{E'_j}(y)\Big)\, d\mathcal{H}^{n-1}.$$

By Lemma 3.2 we can modify (E'_j) to get a sequence $(\widetilde{E}'_j)$ such that $\widetilde{E}'_j \setminus Q^\nu = H^\nu \setminus Q^\nu$ and $\mathcal{H}^{n-1}(\partial^* \widetilde{E}'_j \cap \partial Q^\nu) = 0$, and

$$\int_{Q^\nu \cap \partial^* \widetilde{E}'_j} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nu_{\widetilde{E}'_j}(y)\Big)\, d\mathcal{H}^{n-1} \leq \int_{Q^\nu \cap \partial^* E'_j} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nu_{E'_j}(y)\Big)\, d\mathcal{H}^{n-1} + o(1)$$

as $j \to +\infty$. Since $\partial^* \widetilde{E}'_j \cap Q^\nu = \partial^* \widetilde{E}'_j \cap \overline{Q^\nu}$ up to a $\mathcal{H}^{n-1}$-negligible set, there follows that

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) &\geq \liminf_j \int_{Q^\nu \cap \partial^* \widetilde{E}'_j} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nu_{\widetilde{E}'_j}(y)\Big)\, d\mathcal{H}^{n-1} \\ &\geq \liminf_j \inf \Big\{ \int_{\overline{Q^\nu} \cap \partial^* A} f\Big(\frac{\rho_j y + x_0}{\varepsilon_j}, \nu_A(y)\Big)\, d\mathcal{H}^{n-1} : A \setminus Q^\nu = H^\nu \setminus Q^\nu \Big\}.\end{aligned}$$

Scaling back the variables $z = \frac{\rho_j y + x_0}{\varepsilon_j}$ in the previous inequality, we get

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{n-1} \llcorner \partial^* E}(x_0) \;\geq\; \liminf_j \inf \Big\{ &\int_{Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j}) \cap \partial^* A} f(z, \nu_A)\, d\mathcal{H}^{n-1} : \\ &A \setminus Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j}) = H^\nu \setminus Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j}) \Big\}.\end{aligned}$$

Step 4. Local estimates. We can apply Proposition 3.3 with T and x_T replaced by ρ_j/ε_j and x_0/ε_j respectively. Thus we get that there exists the limit

$$\liminf_j \Big\{ \int_{Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j})\cap\partial^* A} f(z,\nu_A)\,d\mathcal{H}^{n-1} : A\setminus Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j}) = H^\nu \setminus Q^\nu_{\frac{\rho_j}{\varepsilon_j}}(\frac{x_0}{\varepsilon_j})\Big\}$$
$$= f_{\rm hom}(\nu_E(x_0)).$$

Note that the lower estimate is independent of the sequences (ε_j) and (ρ_j). To sum up: for $\mathcal{H}^{n-1}$-almost every $x_0\in\Omega\cap\partial^*E$ we have

$$\frac{d\mu}{dx}(x_0) \geq f_{\rm hom}(\nu_E(x_0)). \tag{34}$$

Step 5. Global estimates. The conclusion follows from integrating the local estimates on $\Omega\cap\partial^*E$. Taking into account (28) and (34) we get

$$\mu(\Omega) \geq \int_{\Omega\cap\partial^*E} \frac{d\mu}{d\mathcal{H}^{n-1}\llcorner\partial^*E}\,d\mathcal{H}^{n-1} \geq \int_{\Omega\cap\partial^*E} f_{\rm hom}(\nu_E)\,d\mathcal{H}^{n-1}.$$

Since $\mu_j \rightharpoonup \mu$ we have $\liminf_j \mu_j(\Omega)\geq\mu(\Omega)$, and hence

$$\liminf_j F_{\varepsilon_j}(E_j) = \liminf_{j\to+\infty}\mu_j(\Omega) \geq \mu(\Omega) \geq \int_{\Omega\cap\partial^*E} f_{\rm hom}(\nu_E)\,d\mathcal{H}^{n-1} = F_{\rm hom}(E),$$

as desired.

3.3 Proof of the Γ-limsup inequality

We can now complete the proof by exhibiting the construction of a recovery sequence. Again, it suffices to show this construction for a dense family; in this case a family of *polyhedral sets*.

(i) We first treat the case of a half space: $E = H^\nu(x_0)$ satisfying the condition $\mathcal{H}^{n-1}(H^\nu(x_0)\cap\partial\Omega) = 0$. Upon a translation argument it is not restrictive to suppose $x_0 = 0$.

In this case, we can repeat the construction outlined in the proof of Proposition 3.3: with fixed $L>0$ large enough but independent of T we find points $x_j\in\mathbb{Z}^n$ at a distance at most $\sqrt{n}$ from a $(n-1)$-dimensional cubic lattice $\mathcal{L}$ on Π^ν of lattice spacing $T+L$ (j is an index running in such lattice).

We then choose a set E_T (almost-)minimizing the problem in m_T in (23), define

$$E'_T = \begin{cases} E_T + x_j & \text{in } x_j + Q^\nu_T \\ H^\nu & \text{elsewhere,}\end{cases}$$

and set $E_\varepsilon = \varepsilon E'_T$. We have $E_\varepsilon\to H^\nu$ in Ω, and

$$\limsup_{\varepsilon\to0^+} F_\varepsilon(E_\varepsilon) \leq m_T + o(1)$$

as $T\to+\infty$. Note that with this construction the boundaries of the sets E_ε are contained in $\{|\langle y,\nu\rangle| < \varepsilon(T+\sqrt{n})\}$.

(ii) Let E be a polyhedral set whose boundary is contained in an union of hyperplanes $\Pi^{\nu_k}(x_k)$, each such that $\mathcal{H}^{n-1}(\Pi^{\nu_k}(x_k)\cap\partial\Omega)=0$. For each of the corresponding half spaces $H^{\nu_k}(x_k)$ we may construct sets E^k_ε as in the previous step. With fixed $\eta>0$ and k we can apply Lemma 3.2 with

$$\Omega_k := \Big(\Omega\cap\{y : |\langle y-x_k,\nu\rangle|<\eta\}\Big)\setminus\bigcup_{j\neq k}\{y : |\langle y-x_k,\nu\rangle|\leq\eta\}$$

in place of Ω, to obtain sets $\widetilde{E}^k_\varepsilon$ converging to $H^\nu(x_k)$ in Ω_k, and with $\widetilde{E}^k_\varepsilon = H^\nu(x_k)$ in a neighbourhood of $\partial\Omega_k$. We then define

$$E_\varepsilon = \begin{cases}\widetilde{E}^k_\varepsilon & \text{in } \Omega_k\\ E & \text{in } \Omega\setminus\bigcup_k\Omega_k.\end{cases}$$

We then have

$$\limsup_{\varepsilon\to0^+} F_\varepsilon(E_\varepsilon)\leq F_{\rm hom}(E)+c\eta,$$

which gives the desired inequality by the arbitrariness of η.

(iii) Finally we deduce the desired inequality on all sets of finite perimeter by the density of the class of polyhedral sets as above. To this end we must remark that the function $f_{\rm hom}$ is continuous (this is easily seen by using test sets for $f_{\rm hom}(\nu)$ in the computation of $f_{\rm hom}(\nu'))$ and with values in $[\alpha,\beta]$. This implies that for each E we can find a sequence of polyhedral sets such that $F_{\rm hom}(E_\varepsilon)\to F_{\rm hom}(E)$, and hence we can proceed as done in the case of Sobolev energies.

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