A variational model in image processing with focal points

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Abstract

We propose a model for segmentation problems involving an energy concentrated on the vertices of an unknown polyhedral set, where the contours of the images to be recovered have preferred directions and focal points. We prove that such an energy is obtained as a Γ-limit of functionals defined on sets with smooth boundary that involve curvature terms of the boundary. The minimizers of the limit functional are polygons with edges either parallel to some prescribed directions or pointing to some fixed points, that can also be taken as unknown of the problem.

Keywords: Γ-convergence, curvature functionals, segmentation problems, image processing.

1 Introduction

Within the theories of Image Processing, variational methods are frequently used, in particular to ensure existence and approximability for *segmentation problems*, which consist in recovering some image contours from a given input [23, 4]. The mathematical approach for such problems can be summarized as follows: given an input image $g \in L^\infty(\Omega)$ (where Ω denotes the image domain) solve a minimum problem of the form:

$$\min \left\{ \alpha \int_\Omega |u(x) - g(x)|^2 dx + E(u, K) \right\} , \tag{1}$$

for a suitable energy E, where $K \subset \Omega$ is union of C^1-curves and u ranges over a suitable space of functions defined in $\Omega \backslash K$. The solution (u, K) of (1) is a pair where K represents the relevant contours of the image and u is a regularization of g. The first term $\int_\Omega |u(x) - g(x)|^2 dx$ is a *fidelity term* to the input image, while the second one $E(u, K)$ can be seen as an energy which penalizes large variations of u and forces the gradient of u to not oscillate too much outside K (typically there are large oscillations of ∇u when noise is present in the image) while at the same it penalizes over-segmentation; α is a parameter which balances the two terms. One of the first such functionals has been introduced by Mumford and Shah [24, 25, 26] taking E in the following form:

$$E(u, K) = \beta_1 \int_{\Omega \backslash K} |\nabla u(x)|^2 dx + \beta_2 \mathcal{H}^1(K) , \tag{2}$$

where $\mathcal{H}^1(K)$ denotes the one-dimensional *Hausdorff measure* of the set K. Within the Calculus of Variations (1) is called a *free-discontinuity problem* and, denoted by F the functional $F(u, K) = \alpha \int_\Omega |u(x) - g(x)|^2 dx + E(u, K)$ and by $\mathcal{C}$ the family of all the subsets of $\mathbb{R}^2$ which are union of C^1-curves, problem (1) becomes:

$$\min \left\{ F(u, K) : u \in W^{1,2}(\Omega \backslash K), K \in \mathcal{C} \right\}, \tag{3}$$

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where $W^{1,2}(\Omega \setminus K)$ is the usual Sobolev space. To obtain existence, uniqueness, regularity and approximations of solutions for this problem, it is convenient to rewrite (1) in a weaker form, as

$$\min_{u \in SBV(\Omega)} \left\{ \alpha \int_\Omega |u(x) - g(x)|^2 dx + \mathcal{E}(u) \right\}, \tag{4}$$

where $\mathcal{E}(u) = \beta_1 \int_{\Omega \setminus S_u} |\nabla u(x)|^2 dx + \beta_2 \mathcal{H}^1(S_u)$ and $SBV(\Omega)$ denotes the space of special functions with bounded variations on Ω introduced by De Giorgi and Ambrosio (see [2, 10]). A key point for the application of the direct method of the Calculus of Variations is the lower semicontinuity of the functional $\mathcal{E}$ with respect to the L^1-convergence, with respect to which the problems in (4) are coercive.

From the numerical point of view, the computation of solutions is rather difficult due to the presence of the term $\mathcal{H}^1(S_u)$, and a central issue is the approximation of energies such as (2) by more handy energies [10]. An underlying feature of such approximation is that the jump set S_u is 'blurred' so that it becomes a two-dimensional set, to which various approximation procedures can be applied. In a recent paper by Braides, Chambolle and Solci [11] it is shown how we can approximate $\mathcal{E}$ by functionals defined on pairs function-set

$$\widetilde{\mathcal{E}}_\varepsilon(u, A) = \beta_1 \int_\Omega |\nabla u(x)|^2 dx + \frac{1}{2} \beta_2 \mathcal{H}^1(\partial A), \qquad |A| \leq \varepsilon;$$

the introduction of the *perimeter* functional $\mathcal{H}^1(\partial A)$ allows then to use 'classical' approximation results as in the seminal paper by Modica and Mortola [22]. This is a formalization of an argument that is present for example in the classical approximation by elliptic functionals by Ambrosio and Tortorelli [3] or in the finite-element schemes by Chambolle and Dal Maso [14, 15, 16], that actually is immediately derived from [11].

In this paper we investigate the possibility of adding to the procedure above the constraint that the jump set S_u be composed of curves pointing in some fixed directions. We have in mind the practical problem of image reconstruction in an urban environment, where the relevant contours either are parallel to some fixed directions (as for edges of buildings, walls, etc.) or point towards some 'focal point' (as for street pavements, etc.). We will concentrate on the case corresponding to the 'perimeter functional'; *i.e.*, when the unknown u in (3) is a characteristic function, or, equivalently the unknown is a set A and the functional $\mathcal{E}$ reduces to $\beta_2 \mathcal{H}^1(\partial A)$. In this case the simplest constraint that we may add is prescribing the possible directions of the edges of the unknown set A. Unfortunately, this constraint is not closed under L^1 convergence. It may be seen [1] that minimizing sequences may develop increasingly wiggly boundary, and converge to a solution of a problem where the constraint is lost and the usual perimeter is substituted by a different anisotropic perimeter. This over-segmentation of the boundary of A can be avoided by introducing an additional term accounting for the cardinality of the set $\widehat{A}$ of end-points of the segments composing ∂A. In this direction, a simple functional is

$$\beta_2 \mathcal{H}^1(\partial A) + \beta_3 \sharp(\widehat{A}) \tag{5}$$

with the constraint that the tangent τ to ∂A belongs to a fixed finite set of directions. Clearly, the introduction of the second term greatly simplifies the problem from the viewpoint of the Calculus of Variations, as it provides a bound on the number of segments composing ∂A, but it also complicates numerical issues as it introduces a zero-dimensional unknown set.

The functional in (5) (but without the constraints on ν) can also be seen as a particular form of functionals $E(u, K)$ taking into account the curvature of the unknown contour K of the following form:

$$E(u, K) = \beta_1 \int_{\Omega \setminus K} |\nabla u|^2 dx + \beta_2 \mathcal{H}^1(K) + \beta_3 \sharp(\widehat{K}) + \beta_4 \int_K \kappa^2(x) \, d\mathcal{H}^1(x) \,, \tag{6}$$

where $\kappa(x)$ denotes the curvature of K (which is supposed to be a finite union of C^2-curves) in a generic point x, $\widehat{K}$ is the set of endpoints of curves of K and $\sharp(\widehat{K})$ is the number of points in $\widehat{K}$. Analysis

of properties of functionals involving curvature terms can be found for instance in [6, 7, 17, 20]. We refer to [26] for a description of applications to image processing of such a kind of functionals, which are useful for instance to reconstruct overlapping shapes and in particular to recover the edges of regions with occlusion. If we take $\beta_4 = +\infty$ the curvature term is replaced by the constraint that the competitors K in (1) (with $E(u, K)$ given by (6)) be union of segments. It is usual to refer to this kind of problem as an *edge-detection* problem. The approximation of such problems is a much harder issue than for the Mumford-Shah functional and has been addressed by Braides and March [13].

In the present paper we focus our attention on edge-detection functionals depending only on characteristic functions $u = \chi_A$, so that $K = \partial A$ as in (5), and with the constraint that the contours be *parallel to some preferred directions or pointing to some focal points*. Mathematically speaking, this can be phrased as follows: let $\nu_1, ..., \nu_N \in S^1$ be some fixed directions and let $p_1, ..., p_M \in \mathbb{R}^2$ be some fixed points, and define $D_x(x) := \{\nu_i\}_{i=1,...,N} \cup \{x - p_j\}_{j=1,...,M}$; we want to solve (1) with $E(u, K) = E(K)$ given by (5) under the additional constraint that, if $\tau(x)$ is the tangent vector at a point x of a curve of K, then $\tau(x)$ must be parallel to an element of $D_x(x)$. It is interesting to note that in these kinds of problems the directions ν_i and the points p_i can also be considered as an unknown, and a further minimization is possible to determine their optimal choices. Again, this functional is numerically hard to treat due to the term $\sharp(\widehat{K})$.

Scope of this paper is to present a class of functionals of the type above defined as limits of curvature functionals defined on boundary of sets, the underlying idea being that the counting functional can be considered as a degenerate curvature functional. With this approximation the zero-dimensional unknown set $\widehat{K}$ is substituted by a one-dimensional unknown set, to which approximation arguments by elliptic functionals in the spirit of the Modica and Mortola result can be applied (see [12, 22]). We prove a result of Γ-convergence which generalizes that given by Braides and Malchiodi in [12], where the case without focal points is considered. We refer to [12] and the references quoted there also for other possible applications (for instance elastic crystals) of such a kind of functionals.

In the sequel for simplicity we first assume Ω to be equal to the whole plane $\mathbb{R}^2$ and then we show how it is possible to generalize the results to a bounded domain. Let $\varphi : \mathbb{R}^2 \times S^1 \to \mathbb{R}$ be a function such that $\varphi(x, \cdot)$ vanishes only on $D_x(x)$ and, given a function $g : \mathbb{R}^2 \to \mathbb{R}$ and $\varepsilon > 0$, let us consider the family of minimum problems

$$\min \left\{ \int_{\partial A} \left(\frac{\varphi(x, \tau(x))}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) + \int_{\mathbb{R}^2} |\chi_A - g|^2 dx \right\}. \tag{7}$$

where the minimum is taken among subsets of $\mathbb{R}^2$ with C^2-boundary and $\kappa(x)$ and $\tau(x)$ denote respectively the curvature and the unit vector of ∂A at x; χ_A is the characteristic function of the set A. We want to rigorously characterize the properties of the limit of minimizers under the assumption of equi-boundedness on their perimeters, showing that they also solve a minimum problem. To compute the form of the limit energy with respect to the L^1-convergence (of characteristic functions), we use the tools of Γ-convergence. Since the second term in (7) is seen to be a L^1-continuous perturbation, it is enough to prove the Γ-convergence of the first term. Therefore we focus our attention on:

$$G_\varepsilon(E) := \int_{\partial E} \left(\frac{\varphi(x, \tau(x))}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) .$$

Loosely speaking we expect that for ε small the minimizers have the sides either parallel to the directions ν_i or pointing to the focal points. In particular we prove that the minimizers, up to subsequence, converge to the solutions of a minimum problem for a functional defined on a suitable class of polygons; for a polygon P with Lipschitz boundary (that means at each vertex only two segments meet) and with a finite number of vertices (we shall call "*regular*" this kind of polygons), the functional has the following form:

$$\mathcal{G}(P) = \sum_{v \in V(P)} g(p, v^+(p), v^-(p)) , \tag{8}$$

3

where $V(P)$ denotes the set of all vertices of P, $v^-(p)$, $v^+(p)$ are the direction of the two segments meeting at p and g is a continuous function such that $g(\cdot, v_1, v_2)$ vanishes in $\{p_j\}_{j=1,\dots,M}$ (explicitly computed from φ, see (16)). On general P polygons, the functional is defined on P as the lower semi-continuous extension of $\mathcal{G}$:

$$G(P) = \inf\left\{\liminf_{n\to+\infty} \mathcal{G}(P_n) : \ P_n \to P, \ P_n \text{ regular } \right\}. \tag{9}$$

Differently from the problem studied in [12], since $g(\cdot, v_1, v_2)$ tends to 0 as p goes to some p_j (in particular we have that $g(p, v_1, v_2) = O(|p-p_j|)$), this functional can be finite also on polygons with a countable set of vertices accumulating at focal points. Moreover, as in [12], the problem is complicated by the usual non-local effects that are characteristic of curvature energies (see also [5, 7]).

Note that we make an explicit choice of φ as a quadratic function of ν_i and p_j (see (14)) so that a possible interpretation of $\{\nu_i\}$ and $\{p_j\}$ as unknowns could be easily implemented. Moreover, general φ can also be considered.

2 Notation and definitions

We denote by $\mathcal{A}$ the family of all the open sets of $\mathbb{R}^2$ with C^2 boundary. For $\delta > 0$ and $q \in \mathbb{R}^2$ we denote by $B_\delta(q)$ the open ball centered in q and of radius δ. Where there is no risk of confusion we identify S^1 with $\mathbb{R}$ modulo 2π. If ν_1 and ν_2 belong to S^1, we denote by $L(\nu_1, \nu_2)$ the length of the shortest arc (in S^1) linking ν_1 and ν_2. Set $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and for $v \in \mathbb{R}^2$ define $v^\perp := Jv$.

For $a, b \in \mathbb{R}$, let $\gamma : [a, b] \to \mathbb{R}^2$ be the parametrization of a piecewise-C^1 curve C. (When there is no risk of confusion we use the term "curve" indifferently for the support C and the parametrization γ.) If $x = \gamma(t) \in C$ is such that γ is differentiable in t and $\dot\gamma(t) \neq 0$, then we say that x is a *regular point* of C and we denote by $\tau(x) := \frac{\dot\gamma(t)}{|\dot\gamma(t)|}$ and $\nu(x) := \tau(x)^\perp$ the unit tangent and normal vectors at x respectively. We use the variable s when we parameterize C by means of the arc length, that is when $|\dot\gamma(s)| = 1$ for all $s \in [0, L(C)]$ (here $L(C)$ is the length of C). If x is a point where γ is not C^1 then we call x a *vertex* of C. We denote by $V(C)$ the set of all vertices of C. If C is C^2 in x, we denote by $\kappa(x)$ the *curvature* of C at x. Recall that, if $x = \gamma(s)$ (and s is the arc length), then $\kappa(x) = |\gamma''(s)|$.

If $\gamma^1 : [a_1, b_1] \to \mathbb{R}^2$ and $\gamma^2 : [a_2, b_2] \to \mathbb{R}^2$ are such that $\gamma^1(b_1) = \gamma^2(a_2)$ then we denote by $\gamma^1 * \gamma^2$ the curve defined on $[a_1, b_1 + b_2 - a_2]$ and parameterized by

$$\gamma^1 * \gamma^2(t) = \begin{cases} \gamma^1(t) & t \in [a_1, b_1] \\ \gamma^2(t - b_1 + a_2) & t \in [b_1, b_1 + b_2 - a_2] \end{cases}.$$

We say that C is a *closed* curve if $\gamma(a) = \gamma(b)$.

Given a sequence of curves C_n, we say that C_n *converge uniformly* if their parameterizations are defined on the same interval $[a, b]$ and $\{\gamma_n\}$ converge uniformly in the usual sense.

2.1 Γ-convergence

In order to describe the convergence of minimum problems we will use the notion of Γ-convergence [10, 18]. We recall its definition and the main properties we will use.

Definition 2.1. *Let X be a topological space and let $F_n, F : X \to \mathbb{R} \cup \{+\infty\}$. Define the Γ-$\liminf$ and the Γ-$\limsup$ of F_n as:*

$$\Gamma\text{-}\liminf_{n\to+\infty} F_n(x) := \inf\left\{\liminf_{n\to+\infty} F_n(x_n) : x_n \to x\right\}$$

$$\Gamma\text{-}\limsup_{n\to+\infty} F_n(x) := \inf\left\{\limsup_{n\to+\infty} F_n(x_n) : x_n \to x\right\}.$$

We say that F_n Γ-converges *to F if for all $x \in X$ we have:*

$$\Gamma\text{-}\liminf_{n \to +\infty} F_n(x) = \Gamma\text{-}\limsup_{n \to +\infty} F_n(x) = F(x). \tag{10}$$

We say that the family of functionals $\{F_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ Γ-converges to F as $\varepsilon \to 0$ if $\{F_{\varepsilon_h}\}$ Γ-converges to F for every subsequence $\{\varepsilon_h\}$ converging to zero as $h \to +\infty$.

Remark 2.2. It is easy to show that (10) is equivalent that for all $x \in X$ the two following requirements hold:

- (Γ-*liminf inequality*) $F(x) \leq \liminf\limits_{n \to +\infty} F_n(x_n)$ for every sequence $\{x_n\}$ converging to x

- (Γ-*limsup inequality*) there exists a sequence $\{x_n\}$ converging to x such that $F(x) \geq \limsup\limits_{n \to +\infty} F_n(x_n)$.

Definition 2.3. *A sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ is said to be* equicoercive *if for any sequence $\{x_n\}$ such that $\sup_n F_n(x_n) < +\infty$ there exists a convergent subsequence.*

Theorem 2.4. *Let $\{F_n\}$ be a sequence of equicoercive functionals defined on X and Γ-converging to F. Then there exists $\min\limits_X F$ and $\min\limits_X F = \lim\limits_{n \to \infty} \inf\limits_X F_n$. Moreover, if x_n is a minimizer of F_n, then every limit of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a minimizer of F.*

Proposition 2.5. *Let $\{F_n\}$ be a sequence of functionals Γ-converging to F and let G be a continuous functional. Then the sequence $\{F_n + G\}$ Γ-converges to $F + G$.*

2.2 Admissible polygons

Let $N \geq 0$ and $M \geq 0$ be such that $N + M \geq 2$; in the whole paper $\tau_1, ..., \tau_N$ are fixed directions belonging to the unit circle S^1 and $p_1, ..., p_M$ are fixed points in $\mathbb{R}^2$ (in the paper we call them *focal points*). For each $x \in \mathbb{R}^2$ we set:

$$D_x := \{\nu_i\}_{i=1,...,N} \cup \{x - p_j\}_{j=1,...,M} . \tag{11}$$

Definition 2.6 (Admissible and regular polygons). *We define $\mathcal{P}$ as the class of all the polygons P such that $\tau(x) \in D_x$ for almost every $x \in \partial P$ (we call* admissible *such a polygon). We say that $P \in \mathcal{P}$ is* regular *if (it is admissible and) $V(P)$ is finite and ∂P is Lipschitz (in other words an admissible polygon is regular if at each vertex of its boundary only two segments meet). We denote by $\mathcal{R}$ the class of all the regular admissible polygons.*

Definition 2.7. *For $P \in \mathcal{P}$ and $v \in V(\partial P)$ let n be the number of the segments which meet at v. (Note that n is an even number.) We define the* multiplicity *of v as $\frac{n}{2}$.*

Remark 2.8. *If P belongs to $\mathcal{R}$, then each vertex of P has multiplicity 1.*

For $P \in \mathcal{P}$ we define a *side* of P to be the closure of a component of $\partial P \backslash V(P)$. If $q_1, q_2 \in \mathbb{R}^2$, we set:

$$[q_1, q_2] := \{z \in \mathbb{R}^2 : z = q_1 + t(q_2 - q_1), \ t \in [0,1]\} \quad \text{and} \quad (q_1, q_2) := \{z \in \mathbb{R}^2 : z = q_1 + t(q_2 - q_1), \ t \in (0,1)\} .$$

Given $q_1, q_2, q_3, q_4 \in \mathbb{R}^2$, we say that the segments $[q_1, q_2]$ and $[q_3, q_4]$ *do not intersect transversally* if $(q_1, q_2) \cap (q_3, q_4) = \emptyset$.

Define $\Upsilon = \{\gamma^1, \ldots, \gamma^k\}$ as an element of the class

$$\mathcal{C} := \left\{ \{\gamma^1, \ldots, \gamma^k\} \text{ such that for all } i = 1, \ldots, k : \begin{array}{l} \gamma^i : [a_i, b_i] \to \mathbb{R}^2 \text{ is Lipschitz} \\[2mm] \frac{\dot{\gamma}^i(t)}{|\dot{\gamma}^i(t)|} \in D_{\gamma(t)} \text{ a.e. } t \in [0,1] \\[2mm] \gamma^i(b_i) = \gamma^{i+1}(a_{i+1}) \end{array} \right\} .$$

For $\Upsilon \in \mathcal{C}$ define the *image* $Im(\Upsilon)$ as the image of the path $\gamma^1 * \cdots * \gamma^k$ and the *support* $S(\Upsilon)$ as the subset of $Im(\Upsilon)$, where a segment, if it is run an equal number of times in opposite ways, is erased. An example of an element of $\mathcal{C}$ (for which the image and the support are different) is shown below: if we consider the two curves γ^1 and γ^2 in Figure 1, $Im(\Upsilon)$ and $S(\Upsilon)$ are drawn in Figure 2.
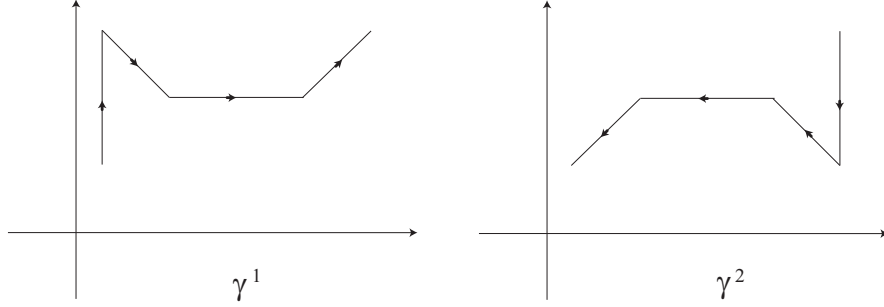

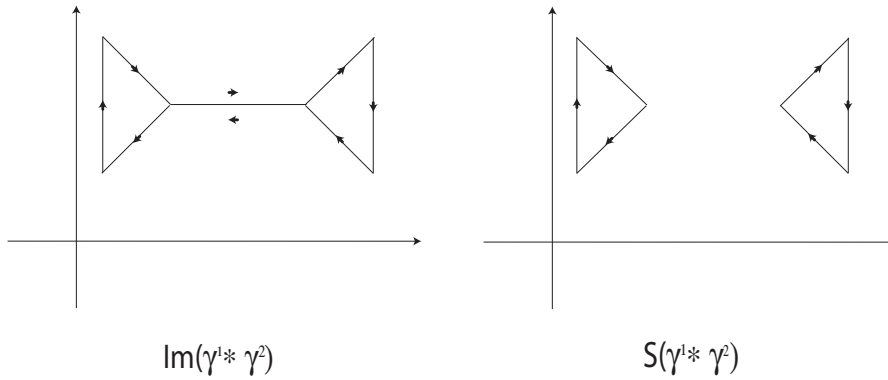Figure 1: here are pictured the two paths γ^1 and γ^2.

Figure 2: fere are pictured the image and the support of $\gamma^1 * \gamma^2$.

We define $\tilde{\mathcal{C}}$ as the class of finite unions of elements of $\mathcal{C}$ and, if $\Upsilon^1, \ldots, \Upsilon^m$ belong to $\mathcal{C}$, we write: $\Upsilon = [\Upsilon^1, \ldots, \Upsilon^m] \in \tilde{\mathcal{C}}$. For $\Upsilon \in \tilde{\mathcal{C}}$ we set:

$$Im(\Upsilon) := \bigcup_{\alpha=1}^{m} Im(\Upsilon^\alpha) \quad \text{and} \quad S(\Upsilon) := \bigcup_{\alpha=1}^{m} S(\Upsilon^\alpha) \ .$$

We say that Υ^α is a *component* of Υ.

Remark 2.9. It is important not to confuse the components of Υ with the connected components of $S(\Upsilon)$.

Lemma 2.10. *For $\delta > 0$ and $q \in \mathbb{R}^2$, if p_j is a focal point, set:*

$$\Sigma_\delta := \bigcup_{j=1}^{M} B_\delta(p_j) \tag{12}$$

and

$$S_\delta = \mathbb{R}^2 \backslash \Sigma_\delta \ . \tag{13}$$

6

Let $P \in \mathcal{P}$ be a polygon such that, for every $\delta > 0$, $V(P \cap S_\delta)$ is finite. Then there exists a polygon $P^\delta \in \mathcal{P}$ such that:

- *$P^\delta \equiv P$ on S_δ*

- *P^δ has a finite number of vertices*

- *the vertices of P^δ in Σ_δ have multiplicity 1.*

For the proof of this Lemma see the Appendix.

Remark 2.11. If $P \in \mathcal{P}$ has a finite number of vertices, then there exists at least a $\Upsilon \in \tilde{\mathcal{C}}$ such that $S(\Upsilon) = \partial P$. Note that even if P has only vertices with multiplicity 1 and is connected, there exist more than one Υ with this property.

Definition 2.12. *For $P \in \mathcal{P}$, let $\Upsilon \in \tilde{\mathcal{C}}$ be such that $S(\Upsilon) = \partial P$. We say that $S(\Upsilon)$ is an* ordered path *of ∂P.*

Remark 2.13. Let P be an admissible polygon with a finite number of vertices and let $\{E_n\} \subset \mathcal{A}$ be a sequence of sets converging to P and such that $\partial E_n \to \partial P$ uniformly.
Then there exists $\Upsilon = [\Upsilon^1, \ldots, \Upsilon^m] \in \tilde{\mathcal{C}}$ (with $\Upsilon^\alpha = \{\gamma^{\alpha,1}, \ldots, \gamma^{\alpha,k}\}$ for $\alpha \in \{1, \ldots, m\}$) such that:

- $S(\Upsilon) = \partial P$

- for each $\alpha \in \{1, \ldots, m\}$ $\gamma^\alpha = \gamma^{\alpha,1} * \cdots * \gamma^{\alpha,k}$ is closed (possibly with vertices of multiplicity greater than 1, but without segments which transversally intersect),

- for all $\alpha \in \{1, \ldots, m\}$ and for all $i \in \{1, \ldots, n\}$ $\gamma^{\alpha,i}$ has vertices only with multiplicity 1 .

Definition 2.14. *Let P be an admissible polygon and Y an element of $\tilde{C}$ with the properties stated in Remark 2.13. We say that $S(\Upsilon)$ is an* ordered path of P induced by $\{E_n\}$.

Remark 2.15. Let P be an admissible polygon with finite number of vertices. If $\Upsilon_1, \Upsilon_2 \in \tilde{\mathcal{C}}$ are such that $S(\Upsilon_1)$ and $S(\Upsilon_2)$ are two ordered paths of P, then we do not necessarily have that $Im(\Upsilon_1) \equiv Im(\Upsilon_2)$.

3   The main result

Remark 3.1. In the sequel we identify $E \subset \mathbb{R}^2$ with its characteristic function χ_E and with a little abuse of notation we say that $\{E_n\}$ converges to E (in $L^1(\mathbb{R}^2)$) if $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^2)$.

Let $\varphi : \mathbb{R}^2 \times S^1 \to \mathbb{R}$ be the function defined as:

$$\varphi(x, v) = \prod_{i=1}^{N} \left(v \cdot \tau_i^\perp \right)^2 \prod_{j=1}^{M} \left(v \cdot (x - p_j)^\perp \right)^2 , \tag{14}$$

where for $v_1, v_2 \in \mathbb{R}^2$ $v_1 \cdot v_2$ denotes the usual scalar product in $\mathbb{R}^2$.

Remark 3.2. For each fixed x, $\varphi(x, \cdot)$ vanishes exactly on vectors parallel to an element of the set D_x defined in (11).

Let $G_\varepsilon : \mathcal{A} \to [0, +\infty)$ be the functional defined as:

$$G_\varepsilon(E) = \int_{\partial E} \left(\frac{\varphi(x, \tau)}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) . \tag{15}$$

For $v_1, v_2 \in S^1$ let $C(v_1, v_2)$ be the shortest of the two arcs in S^1 connecting v_1 and v_2. Let $g : \mathbb{R}^2 \times S^1 \times S^1 \to [0, +\infty)$ be defined as:

$$g(x, v_1, v_2) = 2 \int_{C(v_1, v_2)} \sqrt{\varphi(x, \xi)} d\mathcal{H}^1(\xi) \ . \tag{16}$$

For $P \in \mathcal{P}$ we define the following functional:

$$G_{\mathcal{R}}(P) = \begin{cases} \displaystyle\sum_{p \in V(P) \backslash \{p_1, \ldots, p_M\}} g(p, v^-(p), v^+(p)) & \text{if } P \in \mathcal{R} \\ \\ +\infty & \text{otherwise} \end{cases} \tag{17}$$

where $v^-(p)$ and $v^+(p)$ denote the tangent vectors respectively on both sides of the point p (considered in counterclockwise sense).

We denote by $G(P)$ the lower semicontinuous envelope of $G_{\mathcal{R}}$ with respect to the $L^1(\mathbb{R}^2)$-topology. We have the following formula for G:

$$G(P) := \inf \left\{ \liminf_{n \to +\infty} G_{\mathcal{R}}(P_n) : P_n \to P \text{ in } L^1(\mathbb{R}^2) \right\} \ . \tag{18}$$

Remark 3.3. The set on which the infimum in (18) is performed is not empty. In fact, given $P \in \mathcal{P}$, if for each $\delta > 0$ we consider the polygon P^δ given by Lemma 2.10, to such a kind of polygon we can apply Remark 2.1 in [12], which ensures that there exists a sequence $\{R_k^\delta\} \subset \mathcal{R}$ converging in $L^1(\mathbb{R}^2)$ to P^δ. Then the thesis follows by means of a diagonal argument.

For $\Upsilon = \{\gamma^1, \ldots, \gamma^k\} \in \mathcal{C}$ we define:

$$F(\Upsilon) = \sum_{i=1}^k \sum_{l=1}^{n_i} g(q_l^i, v^-(q_l^i), v^+(q_l^i)) \ , \tag{19}$$

where we are assuming that each γ^i has n_i vertices, denoted by $q_1^i, \ldots, q_{n_i}^i$.

For $\Upsilon = [\Upsilon^1, \ldots, \Upsilon^m] \in \tilde{\mathcal{C}}$, with a little abuse of notation we still denote by F the functional defined as:

$$F(\Upsilon) := \sum_{\alpha=1}^m F(\Upsilon^\alpha) \ .$$

In the sequel we shall need the following lemmata.

Lemma 3.4. Let $P \in \mathcal{P}$ be a polygon with $V(P)$ finite and let $\Upsilon \in \tilde{\mathcal{C}}$ be an ordered path of P induced by a sequence $\{E_n\} \subset \mathcal{A}$. Then there exists a sequence of polygons $\{R_h\} \subset \mathcal{R}$ such that:

$$R_h \to P \quad \text{and} \quad \lim_{h \to +\infty} G_{\mathcal{R}}(R_h) \le F(\Upsilon) \ . \tag{20}$$

Proof. Let us denote by $\{q_i\}_{i=1,\ldots,k}$ the vertices of Υ. We may use Lemma 3.4 in [12] (which gives the desired approximation for P, fixed the set of possible tangent direction of the approximating sets), applied by choosing such direction to be exactly the tangents to P. Hence we deduce that there exists an approximating sequence (of P) $\{R_h\} \subset \mathcal{R}$ such that, if we denote by $\{q_i^h\}_{i=1,\ldots,k_h}$ the vertices of R_h (recall that k_h can be grater or equal than k), for each $i \in \{1, \ldots, k\}$ there exists $l_i \in \mathbb{N}$ (with $\displaystyle\sum_{i=1}^k l_i = k_h$) and $\{q_1^h, \ldots, q_{l_i}^h\} \subset V(R_h)$ such that

$$\lim_{h \to +\infty} q_j^h = q_i \quad \forall j \in \{1, \ldots, l_i\}$$

8

and

$$\sum_{i=1}^{k}\sum_{j=1}^{l_i} g(q_i, v_{j,h}^+, v_{j,h}^-) \le F(\Upsilon) , \tag{21}$$

where $v_{j,h}^+$ and $v_{j,h}^-$ denote the directions of the two segments meeting in q_j^h. Using the continuity of φ and recalling that $G_{\mathcal{R}}(R_h) = \sum_{i=1}^{k}\sum_{j=1}^{l_i} g(q_j^h, v_{j,h}^+, v_{j,h}^-)$, by (21) we argue:

$$\lim_{h\to+\infty} G_{\mathcal{R}}(R_h) = \lim_{h\to+\infty} \sum_{i=1}^{k}\sum_{j=1}^{l_i} \varphi(q_j^h, v_{j,h}^+, v_{j,h}^-) = \sum_{i=1}^{k} g(q_i, v_i^+, v_i^-) \le F(\Upsilon) ,$$

which is the thesis. $\square$

Lemma 3.5. *Let $\{\rho_\varepsilon\} \subset \mathbb{R}$ be a sequence such that $\rho_\varepsilon > 0$ and $\rho_\varepsilon = o(\sqrt{\varepsilon})$. For $q \in \mathbb{R}^2\backslash\{p_1, \ldots, p_M\}$ consider $B_{\rho_\varepsilon}(q)$. For ν_1, $\nu_2 \in S^1$ and $\eta \in \left(0, \frac{L(\nu_1,\nu_2)}{2}\right)$, let $\alpha : [a, b] \to B_{\rho_\varepsilon}(q)$ be a curve of class C^2 parameterized by the arc length and such that $\alpha(a)$, $\alpha(b) \in \partial B_{\rho_\varepsilon}(q)$ and $\dot{\alpha} : [a, b] \to C(\nu_1, \nu_2)$, with $\dot{\alpha}(a) = \nu_1 + \eta$, $\dot{\alpha}(b) = \nu_2 - \eta$. (The value $\dot{\alpha}(a)$ has to be thought as $\lim_{t\to a^+} \dot{\alpha}(t)$, and the same applies for $\dot{\alpha}(b)$.) Then the following inequality holds:*

$$\int_a^b \left(\frac{1}{\varepsilon}\varphi(\alpha(s), \dot{\alpha}(s)) + \varepsilon|\ddot{\alpha}(s)|^2\right) ds \ge g(q, \nu_1, \nu_2) + o_\varepsilon(1) + o_\eta(1) . \tag{22}$$

For the proof of this lemma see the Appendix.

The first theorem we prove is the following Γ-convergence result.

Theorem 3.6 (Γ-convergence). *Let $\widetilde{G}_\varepsilon$, $\widetilde{G} : L^1(\mathbb{R}^2) \to [0, +\infty]$ be the functionals given by:*

$$\widetilde{G}_\varepsilon(u) = \begin{cases} G_\varepsilon(E) & \text{if } u = \chi_E \text{ with } \partial E \text{ of class } C^2 \\ +\infty & \text{otherwise,} \end{cases} \tag{23}$$

$$\widetilde{G}(u) = \begin{cases} G(P) & \text{if } u = \chi_P \text{ with } P \in \mathcal{P} \\ +\infty & \text{otherwise.} \end{cases} \tag{24}$$

Then $\{\widetilde{G}_\varepsilon\}$ Γ-converges to $\widetilde{G}$ with respect to the $L^1(\mathbb{R}^2)$-topology.

Proof. As already focused in Remark 3.1, we recall that for the sake of simplicity we identify functionals defined on sets and functionals defined on characteristic functions of sets. We perform this proof (as well as that of Theorem 3.14) directly taking in account the functionals G_ε and G.

We first prove the Γ-liminf inequality. Let P be an admissible polygon and let $\{E_n\} \subset A$ be a family of sets converging to P in $L^1(\mathbb{R}^n)$ and such that $\liminf_{n\to+\infty} G_{\varepsilon_n}(E_n) < +\infty$. We want to prove that

$$G(P) \le \liminf_{n\to+\infty} G_{\varepsilon_n}(E_n) . \tag{25}$$

Let P_δ be the polygon given by the construction performed in Lemma 2.10 and denote by $\{E_n^\delta\}$ a sequence of subsets of $\mathcal{A}$ such that E_n^δ coincides with E_n in S_δ, E_n^δ converges to P^δ in $L^1(\mathbb{R}^2)$ (the construction of such a kind of sets can be performed as follows: for each pair q_i, q_{i+1} of points belonging to the set $\mathcal{U}$ defined in the proof of Lemma 2.10, consider for instance an extension of P^δ of the type $[q_i, h_i] \cup [h_i, q_{i+1}]$, where h_i is the suitable point given by the same lemma; then we can consider a C^1-curve converging uniformly to $[q_i, h_i] \cup [h_i, q_{i+1}]$ and smoothly joining the curve $E_n \cap S_\delta$

in q_i and q_{i+1}; finally we take as extension of E_n^δ in Σ_δ the union of all these C^1-curves). Let Υ^δ be an ordered path of P^δ induced by $\{E_n^\delta\}$. By Lemma 3.5 we have:

$$F(\Upsilon^\delta \cap S_\delta) \le G_{\varepsilon_n}(E_n) \ , \tag{26}$$

$$F(\Upsilon^\delta) = F(\Upsilon^\delta \cap S_\delta) + \xi_\delta, \qquad \text{with} \quad \xi_\delta = o_\delta(1) \tag{27}$$

and therefore

$$F(\Upsilon^\delta) \le G_{\varepsilon_n}(E_n) + \xi_\delta \ . \tag{28}$$

Let $\{R_h^\delta\} \subset \mathcal{R}$ be the sequence given by Lemma 3.4 for Υ^δ. Recalling the definition of G in (18), we immediately get:

$$G(P^\delta) \le G_{\varepsilon_n}(E_n) + \xi_\delta. \tag{29}$$

If we choose a sequence $\{\delta_n\}$ converging to 0 as n goes to infinity, by construction of P^δ we know that P^{δ_n} converges to P in $L^1(\mathbb{R}^2)$ and, since by definition G is lower semicontinuous, we argue that $G(P) \le \liminf_{n\to+\infty} G(P^\delta)$. Hence, using a diagonal argument and passing to the limit for n going to infinity in (29), we get the desired inequality (25).

We now prove the Γ-limsup inequality, that means: for each admissible polygon P we want to find a sequence $\{E_n\} \subset \mathcal{A}$ converging to P in $L^1(\mathbb{R}^2)$ and such that

$$\limsup_{n\to+\infty} G_{\varepsilon_n}(E_n) \le G(P) \ . \tag{30}$$

We proceed in two steps: we first prove inequality (30) for regular polygons and then we extend the result to a general admissible polygon by means of an approximation lemma.

Let R be a regular polygon. Without loss of generality we can assume that ∂R is connected. Let $\{q_1, \ldots, q_k\}$ be the elements of $V(R)$ and let us assume that they are ordered running through ∂R in counterclockwise sense. Recall that by definition of regular polygon, k is finite. For $q_i \in V(R)$ define the segment $s_i = [q_i, q_{i+1}]$ and let L_i be the length of s_i. If

$$v_1^i := \frac{q_i - q_{i-1}}{|q_i - q_{i-1}|} \qquad \text{and} \qquad v_2^i := \frac{q_{i+1} - q_i}{|q_{i+1} - q_i|}$$

are the two unit vectors of the directions which meet at q_i, define the curve $\gamma^i : \left[-\frac{L_{i-1}}{2}, \frac{L_i}{2}\right] \to \mathbb{R}^2$ in the following way:

$$\gamma^i(t) = \begin{cases} q_i - t\, v_1^i & t \in \left[-\frac{L_{i-1}}{2}, 0\right] \\ q_i + t\, v_2^i & t \in \left[0, \frac{L_i}{2}\right] . \end{cases} \tag{31}$$

Remark 3.7. It can be immediately checked that $\gamma := \gamma^1 * \ldots * \gamma^k$ is a parametrization of ∂P.

Remark 3.8. By construction we have that v_1^i and v_2^i belong to D_{q_i}, that means: $\varphi(q_i, v_1^i) = \varphi(q_i, v_2^i) = 0$.

We construct an approximating sequence $\{\Gamma_n\}$ of ∂P in such a way that γ_n (parametrization of Γ_n) is C^2 and in the form:

$$\gamma_n = \gamma_n^1 * \ldots * \gamma_n^k \ ,$$

with γ_n^i converging to γ^i uniformly. Let E_n be the domain included by Γ_n.

Definition 3.9. Let C be a curve parameterized by $\beta : [a, b] \to \mathbb{R}^2$. For $x_1, x_2 \in C$, we say that "x_1 precedes x_2" (and we write "$x_1 \prec x_2$") if $x_1 = \beta(t_1)$ and $x_2 = \beta(t_2)$ with $t_1 < t_2$.

Definition 3.10. Let $\lambda : \mathbb{R} \to S^1$ be the usual parametrization:

$$\lambda(\vartheta) = (\cos(\vartheta), \sin(\vartheta)) \ ,$$

useful to identify S^1 with $\mathbb{R}$ *modulo* 2π.

Let C be an S^1-valued function. We say that C is S^1-*increasing* if, for every $x_1, x_2 \in C$ with $x_1 \prec x_2$, it holds: $\lambda^{-1}(x_1) < \lambda^{-1}(x_2)$.

Definition 3.11. For $\nu_1, \nu_2 \in S^1$ let $\theta_1, \theta_2 \in \mathbb{R}$ *modulo* 2π be the respective angle. We define $\nu_1 \dotplus \nu_2 := (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2))$.

In the next proposition we construct the optimal profile γ_n^i.

Proposition 3.12. Let Γ^i be the curve parameterized by $\gamma^i : \left[-\frac{L_i}{2}, \frac{L_{i+1}}{2} \right] \to \mathbb{R}^2$ defined as in (31). Then, for every positive sequence $\{\varepsilon_n\}$ converging to 0, there exists a sequence $\{\Gamma_n^i\}$ of curves parameterized by $\gamma_n^i : \left[-\frac{L_i}{2}, \frac{L_{i+1}}{2} \right] \to \mathbb{R}^2$ such that:

1. $\gamma_n^i \in C^2$

2. $\gamma_n^i \left(-\frac{L_i}{2} \right) = \gamma^i \left(-\frac{L_i}{2} \right)$ and $\gamma_n^i \left(\frac{L_{i+1}}{2} \right) = \gamma^i \left(\frac{L_{i+1}}{2} \right)$

3. $\gamma_n^i \to \gamma^i$ uniformly

4. $\displaystyle\int_{\Gamma_n^i} \left[\frac{1}{\varepsilon_n} \varphi(x, \tau(x)) + \varepsilon_n \kappa^2(x) \right] \, d\mathcal{H}^1(x) \leq g(q^i, v_1^i, v_2^i) + o_{\varepsilon_n}(1) \ .$

Proof. (Proposition 3.12) Let $\{\varepsilon_n\} \subset \mathbb{R}$ be a positive sequence converging to 0. For n large enough we can suppose that $2\varepsilon_n$ belongs to the interval $\left(0, \min \left\{ \frac{L_i}{2}, \frac{L_{i+1}}{2} \right\} \right)$. We divide the interval $\left[-\frac{L_i}{2}, \frac{L_{i+1}}{2} \right]$ in the following five intervals:

- $I_1^{i,n} = \left[-\frac{L_i}{2}, -\varepsilon_n - \varepsilon_n^2 \right]$

- $I_2^{i,n} = \left[-\varepsilon_n - \varepsilon_n^2, -\varepsilon_n \right]$

- $I_3^{i,n} = \left[-\varepsilon_n, \varepsilon_n \right]$

- $I_4^{i,n} = \left[\varepsilon_n, \varepsilon_n + \varepsilon_n^2 \right]$

- $I_5^{i,n} = \left[\varepsilon_n + \varepsilon_n^2, \frac{L_{i+1}}{2} \right]$

and we choose a suitable definition of γ_n^i in each $I_j^{i,n}$.

To do this, we consider the problem of finding a solution $u : \mathbb{R} \to S^1$ of the following Cauchy problem:

$$\begin{cases} \dot{u}(t) = \sqrt{\varphi(q_i, u(t))} \, J \, u(t) \\[2mm] u(0) = \dfrac{v_1^i \dotplus v_2^i}{2} \end{cases} , \tag{32}$$

where J is the $\pi/2$-rotation matrix defined in Section 2. By the regularity properties of φ, classical results (on ordinary differential equations) ensure that there exists a C^1 and unique maximal solution of (32). Moreover $|u(t)|$ has to be constant, because $\dfrac{d}{dt}|u(t)|^2 = 2 \, u(t) \cdot \dot{u}(t) = 2\sqrt{\varphi(q_i, u(t))} \, u(t) \cdot u(t)^\perp = 0$ and, since $|u(0)| = 1$, we immediately get:

$$u(t) \in S^1 \qquad \forall \, t \in \mathbb{R} \ . \tag{33}$$

11

If we rewrite equation (32) in the integral form:

$$u(t) = \frac{v_1^i \dotplus v_2^i}{2} + \int_0^t \sqrt{\varphi(q_i, u(\sigma))} \, u(\sigma)^\perp \, d\sigma \, , \tag{34}$$

recalling that we denote by $u(\sigma)^\perp$ a rotation of $\frac{\pi}{2}$ in counterclockwise sense, we get that u is S^1-increasing. We define

$$\beta_n(t) := q_i + \int_0^t u\left(\frac{\sigma}{\varepsilon}\right) \, d\sigma \, , \tag{35}$$

which by construction is C^2 and parameterized by the arc length.

We define $\gamma_n^i : \left[-\frac{L_i}{2}, \frac{L_{i+1}}{2}\right] \to \mathbb{R}^2$ in the following form:

$$\gamma_n^i(t) = \begin{cases} v_1^i & \text{if } t \in I_1^{i,n} \\[2mm] \psi_n^1(t) & \text{if } t \in I_2^{i,n} \\[2mm] \beta_n(t) & \text{if } t \in I_3^{i,n} \\[2mm] \psi_n^2 & \text{if } t \in I_4^{i,n} \\[2mm] v_2^i & \text{if } t \in I_5^{i,n} \, , \end{cases} \tag{36}$$

where $\psi_n^1(t) : I_2^{i,n} \to \mathbb{R}^2$ and $\psi_n^2(t) : I_4^{i,n} \to \mathbb{R}^2$ are C^2 and such that:

$$\begin{cases} \ddot{\psi}_n^1(-\varepsilon_n - \varepsilon_n^2) = 0 \\[2mm] \dot{\psi}_n^1(-\varepsilon_n - \varepsilon_n^2) = \nu_1^i \\[2mm] \dot{\psi}_n^1(-\varepsilon_n) = \beta_n(-\varepsilon_n), \end{cases} \qquad \begin{cases} \ddot{\psi}_n^2(\varepsilon_n + \varepsilon_n^2) = 0 \\[2mm] \dot{\psi}_n^2(\varepsilon_n + \varepsilon_n^2) = \nu_1^i \\[2mm] \dot{\psi}_n^2(\varepsilon_n) = \beta_n(\varepsilon_n) \end{cases} \tag{37}$$

and

$$\begin{cases} |\dot{\psi}_n^2(t)| < 2|\dot{\beta}_n(-\varepsilon_n) - v_1^i| & \forall \, t \, \in I_2^{i,n} \\[2mm] |\dot{\psi}_n^2(t)| < 2|\dot{\beta}_n(-\varepsilon_n) - v_1^i| & \forall \, t \, \in I_2^{i,n}. \end{cases} \tag{38}$$

By construction γ_n^i satisfies properties 1–3. Let us now check property 4. We have:

$$\int_{\Gamma_n^i} \left[\frac{1}{\varepsilon_n}\varphi(x, \tau(x)) + \varepsilon_n \kappa^2(x)\right] \, d\mathcal{H}^1(x)$$

$$= \int_{\Gamma_n^i} \left[\frac{1}{\varepsilon_n}\varphi(q_i, \tau(x)) + \varepsilon_n \kappa^2(x)\right] \, d\mathcal{H}^1(x) + \int_{\Gamma_n^i} \frac{1}{\varepsilon_n}\nabla_x\varphi(q_i, \tau(x)) \cdot (\zeta_x - q_i) \, d\mathcal{H}^1(x) \, , \tag{39}$$

where $\zeta_x \in (\gamma_n^i(x), q_i)$. Let us denote by $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ respectively the first and the second term in (39). We recall that by definition of φ we know that there exists $C_1 > 0$ such that $|\nabla_x\varphi(q_i, \tau(x))| \leq C_1$ for every $x \in \Gamma_n^i$. Moreover by the properties of β_n, ψ_n^1 and ψ_n^2 we have: $|\zeta_x - q_i| \leq C_2\varepsilon_n$, for suitable $C_2 > 0$. Finally, denoted by $\tilde{\Gamma}_n^i$ the support of $\psi_n^1 * \beta_n * \psi_n^2$, then we get:

$$\begin{aligned} |\mathcal{G}_n^2| &= \left|\int_{\Gamma_n^i} \frac{1}{\varepsilon_n}\nabla_x\varphi(q_i, \tau(x)) \cdot (\zeta_x - q_i) \, d\mathcal{H}^1(x)\right| \\[2mm] &\leq \int_{\tilde{\Gamma}_n^i} \frac{1}{\varepsilon_n} |\nabla_x\varphi(q_i, \tau(x))| \, |\zeta_x - q_i| \, d\mathcal{H}^1(x) \leq C_1 C_2 \varepsilon_n \, . \end{aligned} \tag{40}$$

Let us estimate the quantity $\mathcal{G}_n^1$:

$$
\begin{aligned}
\mathcal{G}_n^1 &= \int_{-\frac{L_i}{2}}^{\frac{L_{i+1}}{2}} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\gamma}_n^i(t)\right) + \varepsilon_n \left| \ddot{\gamma}_n^i(t) \right| \right] \, dt \\
&= \int_{I_1^{i,n} \cup I_5^{i,n}} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\gamma}_n^i(t)\right) + \varepsilon_n \left| \ddot{\gamma}_n^i(t) \right| \right] \, dt + \int_{I_2^{i,n} \cup I_4^{i,n}} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\gamma}_n^i(t)\right) + \varepsilon_n \left| \ddot{\gamma}_n^i(t) \right| \right] \, dt \\
&\quad + \int_{I_3^{i,n}} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\gamma}_n^i(t)\right) + \varepsilon_n \left| \ddot{\gamma}_n^i(t) \right| \right] \, dt .
\end{aligned}
\tag{41}
$$

Recalling Remark 3.8 and the definition of γ_n^i, we have that the first term in the latter sum of equation (41) is trivially equal to 0 and we can rewrite:

$$
\begin{aligned}
\mathcal{G}_n^1 &= \int_{-\varepsilon_n - \varepsilon_n^2}^{-\varepsilon_n} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\psi}_n^1(t)\right) + \varepsilon_n \left| \ddot{\psi}_n^1(t) \right| \right] \, dt + \int_{\varepsilon_n}^{\varepsilon_n + \varepsilon_n^2} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\psi}_n^2(t)\right) + \varepsilon_n \left| \ddot{\psi}_n^2(t) \right| \right] \, dt \\
&\quad + \int_{-\varepsilon_n}^{\varepsilon_n} \left[\frac{1}{\varepsilon_n} \varphi\left(q_i, \dot{\beta}_n(t)\right) + \varepsilon_n \left| \ddot{\beta}_n(t) \right| \right] \, dt .
\end{aligned}
\tag{42}
$$

Moreover, by property (38) of ψ_n^1 and ψ_n^2 and recalling the differential property satisfied by the derivative of β_n, for suitable $M > 0$ we get:

$$
|\mathcal{G}_n^1| \le M\varepsilon_n + \mu\varepsilon_n^3 + g(q_i, u(-1), u(1)) \le o(\varepsilon_n) + g(q_i, v_1^i, v_2^i) ,
\tag{43}
$$

where we have set $\mu := |u(1) - v_1^i|^2 + |u(-1) - v_2^i|^2$. By (39), (40) and (43) we finally get the thesis. $\quad\square$

Now we are in position to prove inequality (30) for a regular polygon. In fact, as already said above, we take E_n as the interior of the curve Γ_n parameterized by $\gamma_n := \gamma_n^1 * \ldots * \gamma_n^k$ and we get:

$$
\begin{aligned}
G_{\varepsilon_n}(E_n) &= \sum_{i=1}^{k} \int_{\Gamma_n^i} \left[\frac{1}{\varepsilon_n} \varphi(x, \tau(x)) + \varepsilon_n \kappa^2(x) \right] \, d\mathcal{H}^1(x) \le \sum_{i=1}^{k} g(q^i, v_1^i, v_2^i) + o(\varepsilon_n) \\
&= G_{\mathcal{R}}(P) + o(\varepsilon_n) \equiv G(P) + o(\varepsilon_n)
\end{aligned}
\tag{44}
$$

and, passing to the lim sup for n tending to $+\infty$, we get the thesis.

Now we consider an admissible polygon P such that $G(P) < +\infty$. To prove inequality (30) for this kind of polygon we need the following lemma.

Lemma 3.13. *For each $P \in \mathcal{P}$ such that $G(P) < +\infty$ there exists a sequence $\{R_k\} \subset \mathcal{R}$ converging to P in $L^1(\mathbb{R}^2)$ and such that*

$$
\lim_{k \to +\infty} G_{\mathcal{R}}(R_k) = G(P) .
\tag{45}
$$

To prove this lemma we can take, for each $\delta > 0$, the polygon P^δ given by Lemma 2.10. To this kind of polygon we can apply Remark 2.2 in [12], which ensures (using the continuity of φ) that there exists a sequence $\{R_k^\delta\} \subset \mathcal{R}$ converging to P^δ in $L^1(\mathbb{R}^2)$ and such that (45) holds. Then by means of a diagonal argument we recover an approximating sequence for P. Moreover, since P^δ has no vertices in $\{p_1, \ldots, p_M\}$, we can assume the same property for the approximating sequence R_k, even thought this is not strictly necessary, and the proof could work the same (with some minor modification) also without this requirement.

We can now conclude the proof of Theorem 3.6. Let $\{R_k\}$ be the sequence given by Lemma 3.13. Since R_k is regular, we have already proved that there exists a sequence $\{E_n^k\} \subset \mathcal{A}$ such that:

$$
\limsup_{n \to +\infty} G_{\varepsilon_n}(E_n^k) \le G_{\mathcal{R}}(R_k) .
\tag{46}
$$

Hence by (45) and (46), we can choose a suitable subsequence of $\{E_n^k\}$ converging to P in $L^1(\mathbb{R}^2)$ and with the desired property to fulfill (30). $\quad\square$

In the following theorem we give a compactness result under the additional assumption of equi-boundedness of perimeters.

Theorem 3.14 (Compactness). *Let $\{E_n\} \subset \mathcal{A}$ be such that*

$$\sup_n \mathcal{H}^1(E_n) = M_1 < +\infty \ , \tag{47}$$

$$\sup_n G_{\varepsilon_n}(E_n) = M_2 < +\infty \tag{48}$$

and

$$\text{there exists a bounded } K \subset \mathbb{R}^2 \text{ such that } E_n \subset K \text{ for all } n \ . \tag{49}$$

Then there exist a subsequence $\{E_{n_k}\}$ and $P \in \mathcal{P}$ such that: $E_{n_k} \to P$ in $L^1(\mathbb{R}^2)$ and $G(P) < +\infty$.

Proof. Let K be the number (possibly equal to $+\infty$) of connected components of ∂E_n (passing to a subsequence we can suppose that K is independent of n). We denote by Γ_n^i each connected component of ∂E_n.

Remark 3.15. We assume that the boundary of ∂E_n is oriented in such a way that the outer connected component is oriented in counterclockwise sense.

Let $\{\gamma_n^i\} \subset C^2([0,1]; \mathbb{R}^2)$ be a sequence of parameterizations of Γ_n^i. We suppose that these curves are parameterized proportionally to their arc length, which means there exists $\{c_n^i\} \subset \mathbb{R}$ such that $|\dot{\gamma}_n(t)| = c_n^i \quad \forall t \in [0,1]$. By assumption (47) we immediately get that $\{c_n^i\}$ is bounded and hence for each i we have:

$$\sup_n \int_0^1 |\dot{\gamma}_n^i(t)|^2 dt < +\infty \ . \tag{50}$$

On the other hand we also have:

$$\sup_n \int_0^1 |\gamma_n^i(t)|^2 dt < +\infty \ . \tag{51}$$

This follows by the following inequality:

$$|\gamma_n^i(t)| \leq |\gamma_n^i(0)| + \int_0^t |\dot{\gamma}_n^i(\sigma)| d\sigma \leq |\gamma_n^i(0)| + M_1 \tag{52}$$

and by hypotheses (49) we may assume: $\sup_n |\gamma_n^i(0)| < +\infty$. By (50) and (51) we conclude that $\{\gamma_n^i\}$ is bounded in $W^{1,2}([0,1]; \mathbb{R}^2)$. Since the immersion of $W^{1,2}([0,1]; \mathbb{R}^2)$ in $C^0([0,1]; \mathbb{R}^2)$ is compact, we immediately deduce that there exists a subsequence $\{\gamma_{n_k}^i\}$ uniformly converging to a $\gamma^i \in C^0([0,1]; \mathbb{R}^2)$. We denote by Γ^i the curve parameterized by γ^i.

If Γ is a curve parameterized by $\gamma \in C^2([0,1]; \mathbb{R}^2)$, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \backslash \Gamma$ we denote by $\sigma(\gamma, \xi)$ the *winding number* of γ around ξ defined as:

$$\sigma(\gamma, \xi) := \int_0^1 \left(-\frac{\gamma_2(t) - \xi_2}{(\gamma_1(t) - \xi_1)^2 + (\gamma_2(t) - \xi_2)^2} \dot{\gamma}_1(t) + \frac{\gamma_1(t) - \xi_1}{(\gamma_1(t) - \xi_1)^2 + (\gamma_2(t) - \xi_2)^2} \dot{\gamma}_2(t) \right) dt \ .$$

Without loss of generality we may assume that $\sigma(\gamma_n^i, \xi) \in \{-1, 0, 1\}$ for all $\xi \in \mathbb{R}^2 \backslash \Gamma_n^i$, $n \in \mathbb{N}$, $i \in \{1, \ldots, K\}$. Recalling that $\sigma(\cdot, \xi)$ is continuous with respect to the uniform convergence, we can extend the definition of σ to γ^i, in such a way that: $\sigma(\gamma^i, \xi) = \lim_{n \to +\infty} \sigma(\gamma_n^i, \xi)$. In particular, since the winding number is integer-valued, there exists $\overline{n} \in \mathbb{N}$ such that $\sigma(\gamma^i, \xi) = \sigma(\gamma_n^i, \xi)$ for every $n > \overline{n}$ and for every $\xi \in \mathbb{R}^2 \backslash \cup_{i=1}^K \Gamma^i$. Now we define:

$$P := \left\{ \xi \in \mathbb{R}^2 \backslash \bigcup_{i=1}^K \Gamma^i : \ \sum_{i=1}^K \sigma(\gamma^i, \xi) = 1 \right\} \ .$$

Remark 3.16. The function $\chi(\xi) := \sum_{i=1}^{K} \sigma(\gamma^i, \xi)$ takes only the values 0 or 1. If K is finite, this immediately follows by definition of σ and Remark 3.15. If instead K is equal to $+\infty$, then there exists a focal point p such that $d^i := \sup_{t \in [0,1]} |\gamma^i(t) - p|$ goes to 0 as i tends to $+\infty$. Therefore, if we set $l := |\xi - p|$, we have that there exists $\bar{i} \in \mathbb{N}$ such that $d^i < l$ for every $i > \bar{i}$ and hence $\sigma(\gamma^i, \xi)$ is equal to 0 for every $i > \bar{i}$.

Remark 3.17. By construction we have that: $\partial P = \bigcup_{i=1}^{K} \Gamma^i$.

To conclude the proof, we need to show that P is an admissible polygon. Let $\Gamma^i_{n_k}$ be the curve parameterized by $\gamma^i_{n_k}$. Using (48), we get:

$$\int_{\Gamma^i_{n_k}} \varphi(x, \tau(x)) d\mathcal{H}^1(x) \leq \varepsilon_n M_2 \qquad \forall \varepsilon_n > 0 \tag{53}$$

and hence, for ε_{n_k} going to 0, $\Gamma^i_{n_k}$ converges pointwise to a curve such that its tangent vector $\tau(x)$ belongs to D_x for a.e. $x \in \Gamma^i$.

We now show that the cardinality of $V(P)$ is at most countable and G is finite on P. For $\delta > 0$, consider the set S_δ defined in (13). Recalling the definition of φ given above, for every $\delta > 0$ there exists $c_\delta > 0$ (with $c_\delta = o_\delta(1)$) such that $g(p, v^-(p), v^+(p)) \geq c_\delta$ for all $p \in S_\delta$. Therefore we argue that $\bigcup_{i=1}^{K} \Gamma^i$ must have a finite number of vertices in S_δ. For δ going to 0, by a diagonal argument we get that E_{n_k} tends to an admissible polygon P with (possibly) countable vertices near the focal points.

Note that by (48) and (22) and the definition of G, it immediately follows that G is finite on P. $\square$

3.1 The case of a bounded domain

Until now we have considered our functionals as defined on subsets of the whole $\mathbb{R}^2$. For applications to image processing it is important to consider subsets of a fixed bounded domain. We briefly see how results in Theorem 3.14 and 3.6 can fit this case with some little modification.

Let $\Omega \subset \mathbb{R}^2$ be a bounded star-shaped open set with Lipschitz boundary and set:

$$\begin{aligned}
\mathcal{A}^\Omega &:= \{E \subset \Omega : \partial E \cap \Omega \text{ is } C^2\} \\
\mathcal{R}^\Omega &:= \{P \subset \Omega : \exists \tilde{P} \in \mathcal{R} \text{ such that } \tilde{P} \cap \Omega = P\} \\
\mathcal{P}^\Omega &:= \{P \subset \Omega : \exists \tilde{P} \in \mathcal{P} \text{ such that } \tilde{P} \cap \Omega = P\} .
\end{aligned}$$

For the sake of simplicity we can suppose that Ω is star-shaped with respect to the origin.

Denote by $V(P)$ the set of the vertices of $\partial P \cap \Omega$. Remark that, if $\partial P \cap \partial \Omega$ is not empty, we do not consider as elements of $V(P)$ the end-points of each connected component of $\partial P \cap \Omega$.

We call *efficient boundary* of P the following set: $\tilde{\partial} P := \overline{\partial P \backslash \partial \Omega}$ and we denote by $\tilde{V}(P)$ the set of the vertices of $\tilde{\partial} P$. In general $\tilde{V}(P)$ is greater or equal than $V(P)$. In Figure 3 we picture an example describing the difference between $V(P)$ and $\tilde{V}(P)$.

Analogously to the definitions in (15), (17) and (18), we denote by

$$\mathcal{G}_\varepsilon : \mathcal{A}^\Omega \to [0, +\infty], \quad \mathcal{G}_\mathcal{R} : \mathcal{P}^\Omega \to [0, +\infty] \text{ and } \mathcal{G} : \mathcal{P}^\Omega \to [0, +\infty]$$
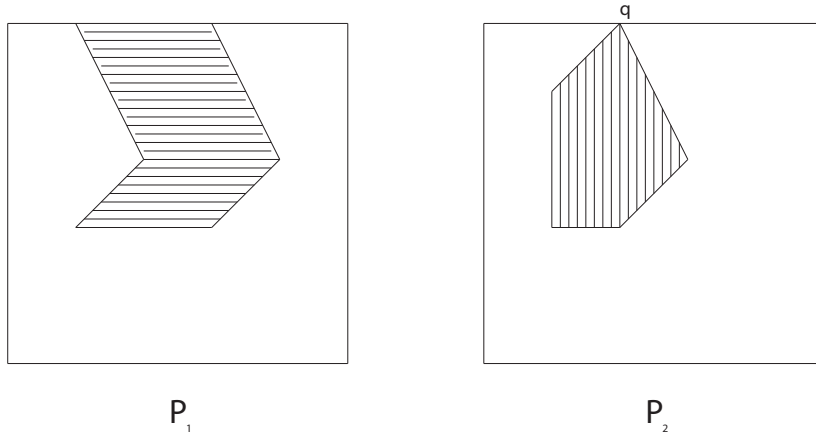
Figure 3: P_1 is such that $V(P_1) = \tilde{V}(P_1)$, while $V(P_2)$ is strictly contained in $\tilde{V}(P_2)$; in fact the point q belongs to $\tilde{V}(P) \backslash V(P)$.

the following functionals:

$$
\begin{aligned}
\mathcal{G}_\varepsilon(E) &= \int_{\partial E \cap \Omega} \left(\frac{\varphi(x, \tau)}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \\
\mathcal{G}_{\mathcal{R}^\Omega}(P) &= \begin{cases} \displaystyle\sum_{p \in V(P) \backslash \{p_1, \ldots, p_M\}} g(p, v^-(p), v^+(p)) & \text{if } P \in \mathcal{R}^\Omega \\[2ex] +\infty & \text{otherwise} \end{cases} \\
\mathcal{G}(P) &= \inf \left\{ \liminf_{n \to +\infty} \mathcal{G}_{\mathcal{R}^\Omega}(P_n) : \ P_n \to P \text{ in } L^1(\Omega) \right\},
\end{aligned}
$$

where g is defined as in (16).

For the family $\{\mathcal{G}_\varepsilon\}$ it can be proved a result of compactness with respect to the $L^1(\Omega)$-topology and the proof essentially follows the same strategy in that of Theorem 3.14.

Moreover the family $\{\mathcal{G}_\varepsilon\}$ Γ-converges to $\mathcal{G}$ with respect to the $L^1(\Omega)$-topology. We do not include the detailed proof in this case, just pointing out the main differences with the case treated in Theorem 3.6. First of all, in the case of a bounded domain, the boundaries of the approximating sets and of the limit sets are union of not necessarily closed curves. However the several definitions in the lemmata used to prove the main results can be rephrased in this framework. Another difference is that the functional $\mathcal{G}$ does not weigh the vertices on the boundary of Ω. In this case for the Γ-liminf inequality we can reason as in the proof of Theorem 3.6 and we note that the lower estimate improves when we remove the terms (in the Γ-limit) related to the vertices on the boundary. To recover the optimal sequence for the Γ-limsup inequality we reason as follows. If P is a polygon such that $\overline{P} \subset \Omega$ (and hence $\partial P \cap \partial \Omega = \emptyset$) or if $\partial P \cap \partial \Omega \neq \emptyset$ and $V(P) \equiv \tilde{V}(P)$, then we consider the sequence $\{E_\varepsilon\}$ given by Theorem 3.6 (for the case when the ambient space is $\mathbb{R}^2$) and we can take as optimal sequence the family $\{E_\varepsilon \cap \Omega\}$. If instead $V(P)$ is strictly contained in $\tilde{V}(P)$, then we consider the dilation $T^\eta : \mathbb{R}^2 \to \mathbb{R}^2$, defined for $\eta > 0$ as: $T^\eta(x) = \eta\, x$. For η small enough we define $P^\eta := T^\eta(P)$ and we choose η in such a way that $V(P^\eta) \equiv \tilde{V}(P^\eta)$. In general P^η could not belong to $\mathcal{P}^\Omega$: this happen if there are segments of the boundary of P pointing to a focal point. In this case however, for every $\varepsilon_\eta > 0$, we can take a polygon $\tilde{P}^\eta \in \mathcal{P}^\Omega$ such that $||\chi_{\tilde{P}^\eta} - \chi_{P^\eta}||_{L^1(\Omega)} < \varepsilon_\eta$. Now, since $\tilde{P}^\eta$ belongs to $\mathcal{P}^\Omega$, we can extend it to a polygon $\overline{P}^\eta$ belonging to $\mathcal{P}$. Let $\{E_\varepsilon^\eta\}$ be the optimal sequence given

by Theorem 3.6 for $\overline{P}^\eta$ and let us consider the family of sets $\{E^\eta_\varepsilon \cap \Omega\}$. By a diagonal argument we recover an optimal approximating sequence for P in such a way that $\mathcal{G}$ does not weigh the vertices of P lying on the boundary of Ω.

Remark 3.18. If Ω is a bounded domain and all the focal points belong to $\mathbb{R}^2 \backslash \overline{\Omega}$, we can slightly modify the definition of G considering the following functional:

$$F(P) := \inf\left\{\liminf_{n \to +\infty} G_\mathcal{R}(P_n) : P_n \to P \text{ in } L^1(\Omega), \ \sup_n \mathcal{H}^1(\partial P_n) < +\infty\right\} , \qquad (54)$$

which in this case is lower semicontinuous with respect to the $L^1(\Omega)$-topology. We have that the family $\{G_\varepsilon\}$ Γ-converges to F with respect to the $L^1(\Omega)$-topology with uniform bounds of the perimeters, that means:

- for every $P \in \mathcal{P}$ and every sequence $\{E_n\} \subset \mathcal{A}$ (with $\sup_n \mathcal{H}^1(\partial E_n) < +\infty$) converging to P
  $\Rightarrow \quad G(P) \leq \liminf_{n \to +\infty} G_{\varepsilon_n}(E_n)$

- for every $P \in \mathcal{P}$ there exists a sequence $\{E_n\} \subset \mathcal{A}$ (with $\sup_n \mathcal{H}^1(\partial E_n) < +\infty$) converging to P such that $G(P) \geq \limsup_{n \to +\infty} G_{\varepsilon_n}(E_n)$.

By Theorem 2.4 and Proposition 2.5 we deduce that, up to a subsequence, the solutions of problem

$$\min\left\{\int_{\partial E}\left(\frac{\varphi(x, \tau(x))}{\varepsilon} + \varepsilon\kappa^2(x)\right) d\mathcal{H}^1(x) + \int_{\mathbb{R}^2} |\chi_E - \chi_A|^2 dx\right\}$$

converge to solutions of problem

$$\min_{P \in \mathcal{P}}\left\{F(P) + \int_{\mathbb{R}^2} |\chi_P - \chi_A|^2 dx\right\} .$$

4 Appendix

4.1 Proof of Lemma 2.10

We perform the construction for a single $B_\delta(p_j)$, which be iterated for each $j = 1, \ldots, M$ (we can suppose that $\partial B_\delta(p_i) \cap \partial B_\delta(p_j)$ for $i \neq j$). Set $\mathcal{U} := \partial B_\delta(p_j) \cap \partial P$. By assumption on P we have that the number of points of $\mathcal{U}$ is finite and we set $\mathcal{U} := \{q_1, \ldots, q_l\}$. Moreover, without loss of generality, we can assume that $V(P) \cap \partial B_\delta(p_j) = \emptyset$ and then l is even, because ∂P is closed. Let us consider the points $\{p_1, \ldots, p_l\}$ ordered in the natural way induced by the identification of $B_\delta(p_j)$ with $\mathbb{R}$ *modulo* 2π. With a little abuse of notation we denote this ordering by the usual symbol "$<$" and for points $y_1, y_2 \in B_\delta(p_j)$ with $y_1 < y_2$ we define $C_\delta(y_1, y_2)$ as the closed arc connecting y_1 and y_2 in $B_\delta(p_j)$. For $x \in \partial B_\delta(p_j)$ let $\nu(x)$ be the outer normal unit vector and define $\psi(x) = \tau(x) \cdot \nu(x)$. By the geometrical properties stated before for P, it is easy to see that $\psi(q_i) \cdot \psi(q_{i+1}) < 0$. Let us now modify P in $B_\delta(p_j)$. For each q_i, with i odd, consider the segments $L_i := [q_i, p_j]$ and $L_{i+1} := [q_{i+1}, p_j]$ and distinguish the two following cases.

Case 1: there exists at least a vector $v \in \{\tau_1, \ldots, \tau_N\}$ such that $\delta v + p_j \notin C_\delta(q_1, q_2) \cup C_\delta(-q_2, -q_1)$. In this case let e_1 and e_2 be the two orthogonal unit vectors defined as:

$$e_2 := \frac{q_1 + q_2 - 2p_j}{|q_1 + q_2 - 2p_j|} , \quad e_1 := -e_2^\perp .$$

Now if

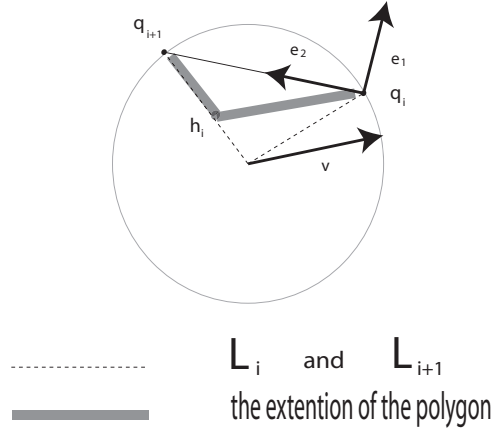$$\frac{v \cdot e_2}{v \cdot e_1} < 0 , \qquad (55)$$

Figure 4: in this figure is shown a possible extension of P in $B_\delta(p_j)$ when there exists at least a vector $v \in \{\tau_1, \ldots, \tau_N\}$ such that $\delta v + p_j \notin C_\delta(q_1, q_2) \cup C_\delta(-q_2, -q_1)$.

then let r_i be the straight line passing thought q_i and parallel to v and set $h_i := L_{i+1} \cap r_i$ (condition (55) ensures that such h_i exists). The required extension of P in $B_\delta(p_j)$ will be the union of the two segments $[q_i, h_i]$ and $[h_i, q_{i+1}]$ (see Figure 4).

Moreover, if $\psi(q_i)$ is negative, we will take a parametrization of $[q_i, h_i] \cup [h_i, q_{i+1}]$ in such a way to go from q_i to q_{i+1}, while in the case $\psi(q_i) > 0$ we consider the path going from q_{i+1} to q_i. If instead

$$\frac{v \cdot e_2}{v \cdot e_1} > 0 \ , \tag{56}$$

we can repeat the previous steps but taking

$$h_i = L_i \cap r_{i+1} \ ,$$

where r_{i+1} is the straight line passing though q_{i+1} and parallel to v.

Case 2: all the fixed directions $\tau_1, \ldots, \tau_N$ belong to $C_\delta(q_i, q_{i+1}) \cup C_\delta(-q_{i+1}, -q_i)$. Let $v, w \in \{\pm\tau_1, \ldots, \pm\tau_N\}$ such that:

$$v \cdot w > 0 \qquad \text{and} \qquad \frac{v \cdot e_2}{v \cdot e_1} < \frac{w \cdot e_2}{w \cdot e_1} \ .$$

For $h_i^1 \in [q_i, p_j]$ and $h_i^2 \in [q_{i+1}, p_j]$ let L_i be the straight line passing through h_i^1 and parallel to w and L_{i+1} be the straight line passing through h_i^2 and parallel to v. We chose h_i^1 and h_i^2 in such a way that $h_i^3 = L_i \cap L_{i+1}$ belongs to $B_\delta(p_j)$. The required extension of P in $B_\delta(p_j)$ in this case is obtained taking the set $[q_i, h_i^1] \cup [h_i^1, h_i^3] \cup [h_i^3, h_i^2] \cup [h_i^2, q_{i+1}]$, paying attention also in this case to the versus of the parametrization according to the sign of $\psi(q_i)$ (see Figure 5).

4.2 Proof of Lemma 3.5

By the regularity of φ we can write

$$\int_a^b \left(\frac{\varphi(\alpha(s), \dot{\alpha}(s))}{\varepsilon} + \varepsilon|\ddot{\alpha}(s)|^2 \right) ds = \int_a^b \left(\frac{\varphi(q, \dot{\alpha}(s))}{\varepsilon} + \varepsilon|\ddot{\alpha}(s)|^2 \right) ds + \int_a^b \frac{\nabla_x \varphi(\zeta, \dot{\alpha}(s)) \cdot (\alpha(s) - q)}{\varepsilon} ds \tag{57}$$
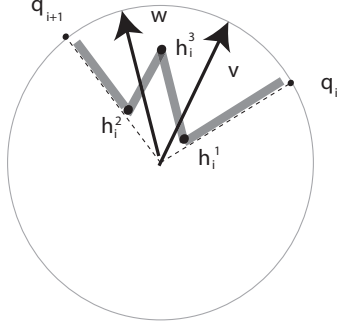
Figure 5: in this figure is shown a possible extension of P in $B_\delta(p_j)$ when all the fixed directions $\tau_1, \ldots, \tau_N$ belong to $C_\delta(q_i, q_{i+1}) \cup C_\delta(-q_{i+1}, -q_i)$.

for suitable $\zeta \in (\alpha(s), q)$. For the first term in the sum in (57) we have:

$$\int_a^b \left(\frac{\varphi(q, \dot\alpha(s))}{\varepsilon} + \varepsilon |\ddot\alpha(s)|^2 \right) ds \geq 2 \int_a^b \sqrt{\varphi(q, \beta(s))} |\dot\beta(s)| ds \tag{58}$$

$$\geq 2 \int_{C(\nu_1+\eta, \nu_2-\eta)} \sqrt{\varphi(q, \xi)} d\mathcal{H}^1(\xi)$$

$$\geq 2 \int_{C(\nu_1, \nu_2)} \sqrt{\varphi(q, \xi)} d\mathcal{H}^1(\xi) + o_\eta(1) = g(q, \nu_1, \nu_2) + o_\eta(1)$$

where we have set $\beta(s) := \dot\alpha(s)$. For the second term in the sum in (57) we have:

$$\int_a^b \frac{\nabla_x \varphi(\zeta, \dot\alpha(s)) \cdot (\alpha(s) - q)}{\varepsilon} ds = o_\varepsilon(1) . \tag{59}$$

In fact by the definition of φ in (14) it follows that there exists $c_1, c_2 > 0$ such that:

$$|\nabla_x \varphi(x, \tau)| \leq c_1 \rho_\varepsilon + c_2 \rho_\varepsilon^2 \qquad \forall x \in B_{\rho_\varepsilon}, \ \forall \tau \in S^1 . \tag{60}$$

By (60) and (47) we get:

$$\left| \int_a^b \frac{\nabla_x \varphi(\zeta, \dot\alpha(s)) \cdot (\alpha(s) - q)}{\varepsilon} ds \right| \leq \frac{1}{\varepsilon} \int_a^b |\nabla_x \varphi(\zeta, \dot\alpha(s))| \cdot |(\alpha(s) - q)| ds \leq 2M_1 \frac{\rho_\varepsilon^2}{\varepsilon}(c_1 + c_2 \rho_\varepsilon) , \tag{61}$$

which is the thesis, choosing $\rho_\varepsilon = o(\sqrt\varepsilon)$.

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