Curvature-dependent energies: the elastic case

Emilio Acerbi and Domenico Mucci

Abstract. We continue our analysis on functionals depending on the curvature of graphs of curves in high codimension Euclidean space. We deal with the "elastic" case, corresponding to a superlinear dependence on the pointwise curvature. We introduce the corresponding relaxed energy functional and prove an explicit representation formula. Different phenomena w.r.t. the "plastic" case, i.e. to the relaxation of the total curvature functional, are observed. A p-curvature functional is well-defined on continuous curves with finite relaxed energy, and the relaxed energy is given by the length plus the p-curvature. The wider class of graphs of one-dimensional BV-functions is treated.

1 Introduction

In this paper we continue our analysis begun in [1] concerning the curvature functional for non-smooth Cartesian curves in codimension higher than one. In the mathematical literature, energies depending on second order derivatives have recently been applied e.g. in image restoration processes, in order to overcome some drawbacks typical of approaches based on first order functionals, as the total variation. One instance is the approach by Chan-Marquina-Mulet [4] who proposed to consider regularizing terms given by second order functionals of the type

$$\int_\Omega |\nabla u|\,dx + \int_\Omega \psi(|\nabla u|)\,h(\Delta u)\,dx$$

for scalar-valued functions u defined in two-dimensional domains, where the function ψ satisfies suitable conditions at infinity in order to allow jumps.

The downscaled one-dimensional version of the above functional is given by

$$\int_a^b |\dot u|\,dt + \int_a^b \psi(|\dot u|)\,|\ddot u|^p\,dt\,, \qquad p\geq 1\,, \qquad u:[a,b]\to\mathbb{R} \tag{1.1}$$

and it has been thoroughly studied in [5], where Dal Maso-Fonseca-Leoni-Morini proved an explicit formula for the relaxed energy, under suitable assumptions on the function ψ.

The prototypical example is the *curvature energy functional*, given by choosing

$$\psi_p(t) := \frac{1}{(1+t^2)^{(3p-1)/2}}\,.$$

In this case, in fact, the above functional takes the form

$$\mathcal{E}_p(u) := \int_a^b |\dot u|\,dt + \int_a^b \sqrt{1+\dot u(t)^2}\cdot k_u(t)^p\,dt\,, \qquad k_u(t) := \frac{|\ddot u(t)|}{(1+\dot u(t)^2)^{3/2}}\,.$$

Therefore, in the smooth case, considering the *Cartesian curve* $c_u(t) := (t,u(t))$, since $k_u(t)$ is the curvature at the point $c_u(t)$, and replacing the first term with the integral of $\sqrt{1+\dot u^2}$, by the area formula one obtains an intrinsic formulation on the graph curve c_u as

$$\mathcal{E}_p(u) = \mathcal{L}(c_u) + \int_{c_u} k_u^p\,d\mathcal{H}^1\,, \tag{1.2}$$

where $\mathcal{L}$ is the length.

Formula (1.2) can be taken in higher codimension $N \geq 2$ as the definition of our energy functional. In fact, the curvature of a smooth Cartesian curve $c_u(t) = (t, u^1(t), \ldots, u^N(t))$ is given at the point $c_u(t)$ by the formula

$$k_u = \mathbf{k}_{c_u} = \frac{|\dot c_u \wedge \ddot c_u|}{|\dot c_u|^3} = \frac{|\dot u \wedge \ddot u|}{|\dot c_u|^3} = \frac{\big((1+|\dot u|^2)|\ddot u|^2 - (\dot u \bullet \ddot u)^2\big)^{1/2}}{(1+|\dot u|^2)^{3/2}}\,.$$

THE RELAXED ENERGY. In this paper, we shall prove in high codimension a complete explicit formula for the relaxed energy. More precisely, we define

$$\overline{\mathcal{E}}_p(u) := \inf\big\{\liminf_{h\to\infty} \mathcal{E}_p(u_h) \mid \{u_h\} \subset C^2(I, \mathbb{R}^N)\,,\ u_h \to u \text{ in } L^1\big\}$$

for any summable function $u \in L^1(I, \mathbb{R}^N)$, where $I = [a, b]$.

In our recent paper [1] we treated the linear case $p = 1$, which corresponds to "plasticity". We analyze here the "elastic" case $p > 1$, where different phenomena appear, as we now briefly illustrate.

The superlinear growth of the curvature term implies that in the relaxation process the energy does not remain bounded if the curvature radius goes to zero at some point, see Example 3.7, as it instead happens in the plastic case. In codimension $N = 1$, one then obtains that a Cartesian curve with finite relaxed energy cannot have creases or edges, compare [5].

However, in the higher codimension case, corner points (only of a special type, say "vertical") are allowed. This phenomenon is illustrated in Example 6.2. Roughly speaking, there is sufficient freedom in the "vertical" directions in order that a sequence of smooth Cartesian curves approaches a curve with a corner point, by producing a "twist" in order to keep the curvature radius greater than some positive threshold, depending on $p > 1$.

Finding the optimal twist at some corner point seems to be a great challenge. This is due to the difficulties in solving the Euler equation satisfied by energy minimizing loops, compare (5.8).

Notwithstanding, differently to the case $p = 1$, the set of corner points is always finite. This follows from the fact that the contribution of the relaxed energy at any corner point is at least $\pi/2$, as soon as the exponent $p > 1$, compare Theorem 6.3.

THE GAUSS MAP. We shall take advantage of several results that we proved in [1]. In fact, it turns out that a function u with finite relaxed energy $\overline{\mathcal{E}}_p(u)$ for some $p > 1$ also satisfies $\overline{\mathcal{E}}_1(u) < \infty$.

As a consequence, the function u has bounded variation, with distributional derivative decomposed as usual by $Du = \dot u\,\mathcal{L}^1 + D^C u + D^J u$. A crucial role is played by the *Gauss map* $\tau_u : I \to \mathbb{S}^N$ that is defined a.e. in I by means of the *approximate gradient* $\dot u$, namely

$$\tau_u = \frac{\dot c_u}{|\dot c_u|}\,, \qquad \dot c_u = (1, \dot u^1, \ldots, \dot u^N)\,. \tag{1.3}$$

Using that $\overline{\mathcal{E}}_1(u) < \infty$, in [1, Thm. 4.7] we proved that also the Gauss map τ_u is a function with bounded variation. In Theorem 8.1, we shall see that for $p > 1$ actually τ_u is a *special function with bounded variation*, i.e., its distributional derivative has no Cantor part: $D^C\tau_u = 0$.

Therefore, in the particular case of continuous functions u with finite relaxed energy and no corner points, we readily infer that τ_u is a Sobolev function in $W^{1,1}(I, \mathbb{S}^N)$.

Notice moreover that Theorem 8.1 is false for $p = 1$, even in codimension $N = 1$. A counterexample is given by the primitive $u(t) := \int_0^t v(s)\,ds$ of the classical Cantor-Vitali function $v : [0, 1] \to \mathbb{R}$ associated to the "middle thirds" Cantor set. We in fact have $\overline{\mathcal{E}}_1(u) < \infty$ and $D^C\tau_u = \dot f(v)\, D^C v$, where $f(t) = (1, t)/\sqrt{1 + t^2}$, but $\overline{\mathcal{E}}_p(u) = \infty$ for every exponent $p > 1$, see Remark 8.3.

THE TOTAL CURVATURE. For our purposes, we recall that the *total curvature* $\mathrm{TC}(c)$ of a curve c has been defined by Milnor [10] as the supremum of the total curvature (i.e. the sum of the turning angles) of the polygons inscribed in c. A curve with finite total curvature is rectifiable, and hence it admits a Lipschitz parameterization. Therefore, it is well defined the *tantrix* (or tangent indicatrix), that assigns to a.e. point the oriented unit tangent vector $\mathfrak{t}_c$. Moreover, the total curvature agrees with the *essential total variation* of the tantrix.

For smooth Cartesian curves c_u the tantrix $\mathbf{t}_{c_u}$ agrees with the Gauss map τ_u, whence $|\dot{c}_u|\, k_u = |\dot{\tau}_u|$ and the total curvature $\mathrm{TC}(c_u)$ is equal to the total variation of τ_u, namely:

$$\mathrm{TC}(c_u) = \int_{c_u} \mathbf{k}_{c_u}\, d\mathcal{H}^1 = \int_I |\dot{\tau}_u|\, dt\,.$$

For continuous functions such that $\overline{\mathcal{E}}_1(u) < \infty$, in [1] we proved that the relaxed energy is the sum of the length and of the total curvature of the Cartesian curve c_u, together with its representation as the total variation of the distributional derivative of the Gauss map:

$$\overline{\mathcal{E}}_1(u) = \mathcal{L}(c_u) + \mathrm{TC}(c_u)\,, \qquad \mathrm{TC}(c_u) = |D\tau_u|(I)\,.$$

THE p-CURVATURE FUNCTIONAL. In this paper, we shall define the *p-curvature functional* of smooth Cartesian curves as

$$\mathrm{TC}_p(c_u) := \int_{c_u} \mathbf{k}_{c_u}^p\, d\mathcal{H}^1\,, \quad p > 1\,. \tag{1.4}$$

Actually, using that $|\dot{c}_u|\, k_u = |\dot{\tau}_u|$, by the area formula we get

$$\mathrm{TC}_p(c_u) = \int_I |\dot{c}_u|^{1-p}\, |\dot{\tau}_u|^p\, dt\,, \quad p > 1\,. \tag{1.5}$$

We wish to extend our definition to the non-smooth case of continuous functions u with relaxed energy and with no corner points. To this purpose, in the proof of Theorem 7.1 we shall see that the arc-length parameterization $I_L \ni s \mapsto \gamma(s)$ of the curve c_u is a Sobolev function in $W^{2,p}$, where $I_L := [0, L]$ and $L = \mathcal{L}(c_u)$. We can thus define the p-curvature functional by means of (1.4), where the curvature of c_u at the point $\gamma(s)$ is given a.e. by

$$\mathbf{k}_{c_u}(\gamma(s)) := \frac{|\dot{\gamma}(s) \wedge \ddot{\gamma}(s)|}{|\dot{\gamma}(s)|^3}\,, \quad s \in I_L\,.$$

In fact, see Proposition 9.6, formula (1.5) continues to hold, and actually the p-curvature functional agrees with the integral of the p-power of the second derivative of the arc length parameterization:

$$\mathrm{TC}_p(c_u) = \int_{I_L} |\ddot{\gamma}(s)|^p\, ds < \infty\,.$$

MAIN RESULTS. For continuous functions with finite relaxed energy and no corner points, in Corollary 9.7 we shall obtain a nice geometric formula for the relaxed energy:

$$\overline{\mathcal{E}}_p(u) = \mathcal{L}(c_u) + \mathrm{TC}_p(c_u)\,. \tag{1.6}$$

The above expression extends to the wider class of continuous functions with finite relaxed energy. In fact, using that the set $J_{\dot{u}}$ of corner points is finite we shall introduce, Definition 9.8, the *generalized p-curvature functional* by setting

$$\mathrm{TC}_p(c_u) := \int_I |\dot{c}_u|^{1-p}\, |\dot{\tau}_u|^p\, dt + \sum_{t \in J_{\dot{u}}} \mathcal{E}_p^0(\Gamma_t^p)\,.$$

In the above formula, roughly speaking, the energy contribution $\mathcal{E}_p^0(\Gamma_t^p)$ is the integral of the p-power of the curvature of the optimal "vertical" curve that allows to smoothly connect the two edges on the graph of u at the corner $c_u(t)$, i.e., the optimal twist that we previously described.

The more general expression of the relaxed energy (without assuming continuity of u) is :

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot{c}_u|^{1-p} \left(|\dot{c}_u|^p + |\dot{\tau}_u|^p\right) dt + |D^C u|(I) + \sum_{t \in J_{\Phi_u}} \mathcal{E}_p^0(\Gamma_t^p)\,.$$

This formula reduces to (1.6) in case of continuous functions. Here, we denote by J_{Φ_u} the discontinuity set of the map $\Phi_u(t) := (c_u(t), \tau_u(t))$, and $\mathcal{E}_p^0(\Gamma_t^p)$ is the energy of a minimizer among all "vertical" curves in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N$ with end points given by the right and left limits $\Phi_u(t_\pm)$. For this reason, using tools from geometric measure theory, we need to extend our energy functional to 1-dimensional currents with vertical parts, i.e, to Cartesian currents.

PLAN OF THE PAPER. In Sec. 2, we introduce the energy functional and recall from [1] the definition of Gauss graph of Cartesian curves, whereas in Sec. 3, we deal with the energy functional $\mathcal{E}_p^0$ on currents, following the approach by Giaquinta-Modica-Souček [9].

In Sec. 4, we report from [1] the structure properties of the class of currents that naturally arise in the relaxation process. In Sec. 5, we then introduce a class of minimal currents associated to our relaxation problem. The following energy lower bound holds: the relaxed energy of u is greater than the energy of the corresponding minimal current with underlying function u.

In Sec. 6, we shall outline some features concerning functions with finite relaxed energy. In particular, we shall prove that (in high codimension) the set of corner points is always finite, Theorem 6.3.

The energy upper bound is then obtained in Sec. 7, by means of a suitable approximation result, Theorem 7.1. As a consequence, in Sec. 8 we shall prove that the Gauss map τ_u of a function with finite relaxed energy is a BV-function with no Cantor part, Theorem 8.1. Therefore, if u is continuous and with no corner points, then τ_u is a Sobolev function.

Finally, Sec. 9 collects the main results that we previously described.

Acknowledgements. This research was supported by PRIN 2010-2011, "Calculus of Variations". The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

2 Notation and preliminary results

In this section we introduce the energy functional and recall from [1] the definition of Gauss graph of Cartesian maps.

Consider a C^2-function $u : I \to \mathbb{R}^N$ where $I = [a, b]$ and $N \geq 1$, so that the corresponding Cartesian curve is $c_u : I \to \mathbb{R}^{N+1}$ defined by $c_u(t) := (t, u(t))$, where $u = (u^1, \ldots, u^N)$ in components. Any smooth Cartesian curve is automatically regular, as $\dot{c}_u(t) = (1, \dot{u}(t))$ for each t. For simplicity of notation we shall correspondingly denote by τ_u and k_u the tantrix and curvature of c_u, respectively. We thus have

$$\tau_u = \frac{\dot{c}_u}{|\dot{c}_u|}, \qquad \dot{c}_u = (1, \dot{u}^1, \ldots, \dot{u}^N), \qquad |\dot{c}_u| = \sqrt{1 + |\dot{u}|^2}.$$

In codimension $N = 1$, i.e. for $c_u(t) = (t, u(t)) : I \to \mathbb{R}^2$, the curvature of c_u at the point $c_u(t)$ is

$$k_u(t) = \frac{|\ddot{u}(t)|}{\big(1 + \dot{u}(t)^2\big)^{3/2}} \tag{2.1}$$

so that $|\dot{c}_u(t)|\, k_u(t) = |\dot{v}(t)|$, where $v(t) := \arctan \dot{u}(t)$.

In general, denoting by $v_1 \wedge v_2$ and $v_1 \bullet v_2$ the wedge and scalar product of vectors in $\mathbb{R}^n$, respectively, the area of the parallelogram generated by v_1 and v_2 is

$$\text{area}[v_1, v_2] = |v_1 \wedge v_2| = \{|v_1|^2 |v_2|^2 - (v_1 \bullet v_2)^2\}^{1/2}.$$

Therefore, the curvature of a smooth regular curve c in $\mathbb{R}^n$ is

$$\mathbf{k}_c := \frac{\text{area}[\dot{c}, \ddot{c}]}{|\dot{c}^3|} = \frac{|\dot{c} \wedge \ddot{c}|}{|\dot{c}^3|} = \frac{\{|\dot{c}|^2 |\ddot{c}|^2 - (\dot{c} \bullet \ddot{c})^2\}^{1/2}}{|\dot{c}|^3}.$$

In particular, for a smooth Cartesian curve c_u, where $c_u(t) = (t, u(t)) : I \to \mathbb{R}^{N+1}$, the curvature at the point $c_u(t)$ is

$$k_u := \mathbf{k}_{c_u} = \frac{|\dot{c}_u \wedge \ddot{c}_u|}{|\dot{c}_u|^3} = \frac{|\dot{u} \wedge \ddot{u}|}{|\dot{c}_u|^3} = \frac{\big((1 + |\dot{u}|^2)|\ddot{u}|^2 - (\dot{u} \bullet \ddot{u})^2\big)^{1/2}}{(1 + |\dot{u}|^2)^{3/2}}. \tag{2.2}$$

Energy functional. Let $p \geq 1$ a real exponent. For $u \in C^2(I, \mathbb{R}^N)$ we denote

$$\mathcal{E}_p(u) := \mathcal{L}(c_u) + \int_{c_u} k_u^p \, d\mathcal{H}^1 .$$

Using that

$$|\dot{\tau}_u| = \frac{|\dot{u} \wedge \ddot{u}|}{|\dot{c}_u|^2} , \qquad |\dot{u} \wedge \ddot{u}| = |\dot{c}_u \wedge \ddot{c}_u| = \big((1 + |\dot{u}|^2)|\ddot{u}|^2 - (\dot{u} \bullet \ddot{u})^2\big)^{1/2} , \tag{2.3}$$

we have $|\dot{c}_u| \, k_u = |\dot{\tau}_u|$. Whence by the area formula we get

$$\mathcal{E}_p(u) = \int_I |\dot{c}_u|\big(1 + k_u^p\big) \, dt = \int_I |\dot{c}_u|^{1-p} \left(|\dot{c}_u|^p + |\dot{\tau}_u|^p\right) dt .$$

We define

$$\overline{\mathcal{E}}_p(u) := \inf\{\liminf_{h\to\infty} \mathcal{E}_p(u_h) \mid \{u_h\} \subset C^2(I, \mathbb{R}^N) , \; u_h \to u \text{ in } L^1\}$$

for any summable function $u \in L^1(I, \mathbb{R}^N)$, and correspondingly

$$\mathcal{E}_p(I, \mathbb{R}^N) := \{u \in L^1(I, \mathbb{R}^N) \mid \overline{\mathcal{E}}_p(u) < \infty\} . \tag{2.4}$$

Currents. Let $U := \mathring{I} \times \mathbb{R}^N$. We deal with currents $\Sigma \in \mathcal{D}_1(U \times \mathbb{S}^N)$ of the type $\Sigma = [\![\mathcal{M}, \theta, \xi]\!]$. This means that Σ acts on compactly supported and smooth 1-forms $\omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$ as

$$\langle \Sigma, \omega \rangle = \int_{\mathcal{M}} \langle \omega, \xi \rangle \, \theta \, d\mathcal{H}^1 \qquad \forall \, \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$$

where $\mathcal{M}$ is a countably 1-rectifiable set, $\xi : \mathcal{M} \to \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$ is an $\mathcal{H}^1 \llcorner \mathcal{M}$-measurable unit vector field and the multiplicity $\theta : \mathcal{M} \to \mathbb{R}$ is a non-negative $\mathcal{H}^1 \llcorner \mathcal{M}$-measurable function. Whence the *mass*[1] of Σ is given by the formula $\mathbf{M}(\Sigma) = \int_{\mathcal{M}} \theta \, d\mathcal{H}^1$.

In particular, the current $\Sigma = [\![\mathcal{M}, \theta, \xi]\!]$ is said to be an *integer multiplicity* (say i.m.) *rectifiable current* in $\mathcal{R}_1(U \times \mathbb{S}^N)$ if it has finite mass, the multiplicity function θ is integer-valued, and ξ is orienting the approximate tangent 1-space to $\mathcal{M}$ at $\mathcal{H}^1 \llcorner \mathcal{M}$-a.e. point. In this case, $\mathcal{H}^1(\mathcal{M}) \leq \mathbf{M}(\Sigma) < \infty$ and hence $\mathcal{M}$ is 1-rectifiable.

Example 2.1 For smooth functions $u \in C^2(I, \mathbb{R}^N)$, in [1] we defined the *Gauss-graph* by the current $GG_u := \Phi_{u\#}[\![I]\!]$, where $\Phi_u(t) := (c_u(t), \tau_u(t))$. Therefore, GG_u is a 1-current in $U \times \mathbb{S}^N$, and setting

$$\xi_u(t) := \frac{\dot{\Phi}_u(t)}{|\dot{\Phi}_u(t)|} , \quad \text{where} \quad |\dot{\Phi}_u(t)| = |\dot{c}_u| \sqrt{1 + k_u^2} \tag{2.5}$$

by the area formula for every $\omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$ we have

$$\langle GG_u, \omega \rangle := \int_I \Phi_u^{\#} \omega = \int_I |\xi_u(t)| \, \langle \omega(\Phi_u(t), \xi_u(t)) \rangle \, dt = \int_{\mathcal{GG}_u} \langle \omega, \xi_u \rangle \, d\mathcal{H}^1$$

where $\mathcal{GG}_u := \{\Phi_u(t) \mid t \in I\}$ is the Gauss graph of u. Whence $GG_u = [\![\mathcal{GG}_u, 1, \xi_u]\!]$ is an i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$ with finite mass

$$\mathbf{M}(GG_u) = \int_I |\dot{\Phi}_u(t)| \, dt < \infty .$$

[1] The mass $\mathbf{M}(\Sigma)$ of a current $\Sigma \in \mathcal{D}_1(U \times \mathbb{S}^N)$ is defined by

$$\mathbf{M}(\Sigma) := \sup\{\langle \Sigma, \omega \rangle \mid \omega \in \mathcal{D}^1(U \times \mathbb{S}^N) , \; |\omega| \leq 1\} .$$

More precisely, we shall denote by $(dx^0, dx^1, \ldots, dx^N)$ and $(dy^0, dy^1, \ldots, dy^N)$ the canonical bases of 1-forms in $\mathbb{R}^{N+1}_x$ and $\mathbb{R}^{N+1}_y$, respectively. For any $g \in C^\infty_c(U \times \mathbb{S}^N)$ we thus obtain:

$$\begin{aligned} \langle GG_u, g(x,y)\,dx^0\rangle &= \int_I g(\Phi_u(t))\,dt\,, \\ \langle GG_u, g(x,y)\,dx^j\rangle &= \int_I g(\Phi_u(t))\,\dot{u}^j(t)\,dt\,, \quad j=1,\ldots,N\,, \\ \langle GG_u, g(x,y)\,dy^j\rangle &= \int_I g(\Phi_u(t))\,\dot{\tau}^j_u(t)\,dt\,, \quad j=0,1,\ldots,N\,. \end{aligned} \tag{2.6}$$

Furthermore the current GG_u has null-boundary[2] (in $U \times \mathbb{S}^N$) as by Stokes theorem

$$\langle \partial GG_u, f\rangle := \langle GG_u, df\rangle = \int_{\mathcal{GG}_u} df = \int_{\partial\mathcal{GG}_u} f = 0 \qquad \forall\, f \in C^\infty_c(U \times \mathbb{S}^N)\,.$$

3 Energy functional on currents

In this section we extend the energy functional to currents, following the approach by Giaquinta-Modica-Souček [9]. We prove a *regularity* property: if a current has finite p-energy, then it has finite mass, too. We then show that the converse implication is false, in general. Finally, we see that a sequential weak closure property holds.

We shall denote by Π_x and Π_y the canonical projections of $\mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$ onto the first and second factor, respectively. For any $\xi \in \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$, we let $\xi^{(x)} := \Pi_x(\xi)$ and $\xi^{(y)} := \Pi_x(\xi)$, so that $\xi = (\xi^{(x)}, \xi^{(y)})$. If $u \in C^2(I, \mathbb{R}^N)$, setting

$$\mathcal{E}^0_p(GG_u) := \int_{\mathcal{GG}_u} |\xi^{(x)}_u|^{1-p}\big(|\xi^{(x)}_u|^p + |\xi^{(y)}_u|^p\big)\,d\mathcal{H}^1\,, \qquad |\xi^{(x)}_u| = \frac{|\dot{c}_u|}{|\dot{\Phi}_u|}\,, \qquad |\xi^{(y)}_u| = \frac{|\dot{\tau}_u|}{|\dot{\Phi}_u|}$$

where $|\dot{\tau}_u| = |\dot{c}_u|\,k_u$, by the area formula we have

$$\mathcal{E}^0_p(GG_u) = \int_I |\dot{\Phi}_u|\Big(\frac{|\dot{c}_u|}{|\dot{\Phi}_u|} + \frac{|\dot{c}_u|\,k^p_u}{|\dot{\Phi}_u|}\Big)\,dt = \int_I |\dot{c}_u|\big(1 + k^p_u\big)\,dt = \mathcal{E}_p(u)\,. \tag{3.1}$$

This suggests to introduce for smooth functions $(u,v) : I \to \mathbb{R}^N \times \mathbb{R}^{N+1}$, with $|v| \equiv 1$, the energy integrand

$$\mathcal{F}_p(u,v) := \int_I f_p(\dot{c}_u, \dot{v})\,dt\,, \qquad f_p(\dot{c}_u, \dot{v}) := |\dot{c}_u|^{1-p}\left(|\dot{c}_u|^p + |\dot{v}|^p\right)$$

so that $\mathcal{F}_p(u, \tau_u) = \mathcal{E}^0_p(u)$, and to give the following:

Definition 3.1 *We denote by* $F_p : \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y \to [0, +\infty]$ *the parametric convex l.s.c. extension of the integrand* f_p in the sense of [9].

More precisely, denoting by $\xi^0 \in \mathbb{R}$ the first component of a vector $\xi \in \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$, we set

$$f_p(\xi) := |\xi^{(x)}|^{1-p}\left(|\xi^{(x)}|^p + |\xi^{(y)}|^p\right) \qquad \text{if} \quad \xi^0 = 1$$

we extend by homogeneity

$$\underline{f}_p(\xi) := \xi^0 \cdot f_p\Big(\frac{\xi}{\xi^0}\Big) \qquad \text{if} \quad \xi^0 > 0$$

and we define $\overline{f}_p : \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y \to [0, +\infty]$ by

$$\overline{f}_p(\xi) := \begin{cases} \underline{f}_p(\xi) & \text{if} \quad \xi^0 > 0 \\ +\infty & \text{if} \quad \xi^0 \le 0\,. \end{cases}$$

[2]The boundary $\partial\Sigma$ of a current $\Sigma \in \mathcal{D}_1(U \times \mathbb{S}^N)$ is the 0-current (or distribution) in $\mathcal{D}_0(U \times \mathbb{S}^N)$ defined by

$$\langle \partial\Sigma, f\rangle := \langle \Sigma, df\rangle \qquad \forall\, f \in C^\infty_c(U \times \mathbb{S}^N)\,.$$

Then by definition F_p is *the greatest convex and lower semicontinuous function that is lower than or equal to* $\overline{f}_p$.

Proposition 3.2 *For every* $p > 1$ *one has:*

$$F_p(\xi) = \begin{cases} |\xi^{(x)}| + |\xi^{(x)}|^{1-p}|\xi^{(y)}|^p & \textit{if} \quad |\xi^{(x)}| > 0 \ \textit{and} \ \xi^0 \geq 0 \\ +\infty & \textit{otherwise}. \end{cases} \tag{3.2}$$

Proof: We denote by $\xi^{\overline{0}}$ the complementary components to ξ^0 of any vector $\xi \in \mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1}$, so that $\xi = (\xi^0, \xi^{\overline{0}})$. By convexity arguments one has

$$F_p(\xi) = \sup\{a + b \cdot \xi^{\overline{0}} \mid a \in \mathbb{R}\,,\ b \in \mathbb{R}^{2N+1}\,,\ a + b \bullet G \leq f_p(1, G) \ \ \forall\, G \in \mathbb{R}^{2N+1}\}\,. \tag{3.3}$$

By convexity of $\underline{f_p}$ one has $F_p(\xi) = \underline{f_p}(\xi)$ if $\xi^0 > 0$, and $F_p(\xi) = |\xi^{(x)}| + |\xi^{(x)}|^{1-p}|\xi^{(y)}|^p$ if $\xi^0 \geq 0$ provided that $|\xi^{(x)}| > 0$. By homogeneity, for each $\lambda > 0$ and $G \in \mathbb{R}^{2N+1}$ this yields

$$\underline{f_p}(\lambda, \lambda G) = |\Pi_x(\lambda, G)| + \frac{|\Pi_y(\lambda, G)|^p}{|\Pi_x(\lambda, G)|^{p-1}}\,.$$

Choosing G so that $\Pi_x(0, G) = 0$ and $\Pi_y(0, G) \neq 0$, we get $\underline{f_p}(\lambda, \lambda G) \to +\infty$ as $\lambda \to 0^+$. By using (3.3), this implies that $F_p(\xi) = +\infty$ if $|\xi^{(x)}| = 0$. Finally, one similarly obtains that $F_p(\xi) = +\infty$ if $\xi^0 < 0$. □

Remark 3.3 If $p = 1$, instead, one clearly has

$$F_1(\xi) = \begin{cases} |\xi^{(x)}| + |\xi^{(y)}| & \text{if } \xi^0 \geq 0 \\ +\infty & \text{otherwise}. \end{cases}$$

Definition 3.4 The *p-curvature energy functional* is given on currents $\Sigma = [\![\, \mathcal{M}, \theta, \xi \,]\!]$ by

$$\mathcal{E}_p^0(\Sigma) := \int_{\mathcal{M}} F_p(\xi)\, \theta\, d\mathcal{H}^1\,.$$

Therefore, in (3.1) we have just shown that in the smooth case $\mathcal{E}_p^0(GG_u) = \mathcal{E}_p(u)$.

Regularity. Let $\Sigma = [\![\, \mathcal{M}, \theta, \xi \,]\!]$. For our purposes we shall tacitly assume w.l.g. that $\xi^0 \geq 0$. We now see that the functional $\mathcal{E}_p^0$ is *regular*, i.e., every current $\Sigma = [\![\, \mathcal{M}, \theta, \xi \,]\!]$ with finite energy has finite mass, too. For $p = 1$, since $F_1(\xi) = |\xi^{(x)}| + |\xi^{(y)}| \geq |\xi|$ we trivially have

$$\mathbf{M}(\Sigma) = \int_{\mathcal{M}} |\xi|\, \theta\, d\mathcal{H}^1 \leq \int_{\mathcal{M}} F_1(\xi)\, \theta\, d\mathcal{H}^1 = \mathcal{E}_1^0(\Sigma)\,,$$

yielding the regularity property.

If $p > 1$, we observe that $1 + t^p \geq 2^{-1}(1 + t)$ for any $t \geq 0$. We now show the regularity property.

Proposition 3.5 *If* $\Sigma = [\![\, \mathcal{M}, \theta, \xi \,]\!]$*, then* $\mathbf{M}(\Sigma) \leq 2 \cdot \mathcal{E}_p^0(\Sigma)$ *for all exponents* $p > 1$.

Proof: For any $\xi \in \mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1}$ we get

$$F_p(\xi) \geq 2^{-1}\, F_1(\xi)\,.$$

In fact, the above inequality is trivial if $|\xi^{(x)}| = 0$, as $F_p(\xi) = +\infty$. Assuming instead $|\xi^{(x)}| > 0$, we estimate

$$F_p(\xi) = |\xi^{(x)}| \cdot \Big(1 + \frac{|\xi^{(y)}|^p}{|\xi^{(x)}|^p}\Big) \geq |\xi^{(x)}| \cdot 2^{-1} \cdot \Big(1 + \frac{|\xi^{(y)}|}{|\xi^{(x)}|}\Big) = 2^{-1}\,(|\xi^{(x)}| + |\xi^{(y)}|) = 2^{-1}\, F_1(\xi)\,.$$

Since $F_1(\xi) \geq |\xi|$, we deduce that $2 \cdot F_p(\xi) \geq |\xi|$, whence the mass estimate readily follows. □

Remark 3.6 For $p = 1$ we also have $F_1(\xi) \le 2^{1/2}\,|\xi|$, whence

$$2^{-1/2}\,\mathcal{E}^0_1(\Sigma) \le \mathbf{M}(\Sigma) \le \mathcal{E}^0_1(\Sigma)\,.$$

Therefore, for currents $\Sigma = [\![\mathcal{M},\theta,\xi]\!]$ such that $\xi^0 \ge 0$ $\mathcal{H}^1$-a.e., it turns out that Σ has finite mass if and only if it has finite energy $\mathcal{E}^0_1(\Sigma)$. However, as the following example shows, it may happen that a sequence $\{\Sigma_h\}$ has equibounded masses but $\sup_h \mathcal{E}^0_p(\Sigma_h) = \infty$ for every $p > 1$.

Example 3.7 Let $I = [-1,1]$ and $N = 1$. For $0 < R < 1$, let $\gamma_R : [-1, 1+\pi/2] \to U \times \mathbb{S}^1$ given by

$$\gamma_R(\theta) := \begin{cases} (\theta, -R, 1, 0) & \text{if } -1 \le \theta \le 0 \\ (R\sin\theta, -R\cos\theta, \cos\theta, \sin\theta) & \text{if } 0 \le \theta \le \pi/2 \\ (R, \theta-\pi/2, 0, 1) & \text{if } \pi/2 \le \theta \le 1+\pi/2 \end{cases}$$

and define $\Sigma_R := \gamma_{R\#}[\![-1, 1+\pi/2]\!] \in \mathcal{R}_1(U\times\mathbb{S}^1)$, so that $\Sigma_R = [\![\mathcal{M}_R, 1, \xi_R]\!]$ where $\mathcal{M}_R = \gamma_R([-1, 1+\pi/2])$ and $\xi_R(z) = \dot\gamma_R(\theta)/|\dot\gamma_R(\theta)|$ if $z = \gamma_R(\theta)$. Since

$$\dot\gamma_R(\theta) = \begin{cases} (1,0,0,0) & \text{if } -1 \le \theta < 0 \\ (R\cos\theta, R\sin\theta, -\sin\theta, \cos\theta) & \text{if } 0 < \theta < \pi/2 \\ (0,1,0,0) & \text{if } \pi/2 < \theta \le 1+\pi/2 \end{cases}$$

we compute

$$|\dot\gamma_R(\theta)| = \begin{cases} 1 & \text{if } -1 \le \theta < 0 \\ \sqrt{1+R^2} & \text{if } 0 < \theta < \pi/2 \\ 1 & \text{if } \pi/2 < \theta \le 1+\pi/2 \end{cases}$$

and hence $|\xi_R^{(x)}| > 0$ on $\mathcal{M}_R$, as

$$\xi_R(\gamma_R(\theta)) = \begin{cases} (1,0,0,0) & \text{if } -1 \le \theta < 0 \\ (1+R^2)^{-1/2}(R\cos\theta, R\sin\theta, -\sin\theta, \cos\theta) & \text{if } 0 < \theta < \pi/2 \\ (0,1,0,0) & \text{if } \pi/2 < \theta \le 1+\pi/2\,. \end{cases}$$

Now, the mass of Σ_R is equal to the length of the simple curve γ_R, and by the area formula

$$\mathbf{M}(\Sigma_R) = \int_{[-1,1+\pi/2]} |\dot\gamma_R(\theta)|\,d\theta = 2 + \frac{\pi}{2}\sqrt{1+R^2}\,.$$

As before, we let $\gamma_R^{(x)} := \Pi_x \circ \gamma_R$ and $\gamma_R^{(y)} := \Pi_x \circ \gamma_R$, so that $\gamma_R = (\gamma_R^{(x)}, \gamma_R^{(y)})$. Using that

$$|\dot\gamma_R^{(x)}(\theta)| = \begin{cases} 1 & \text{if } -1 \le \theta < 0 \\ R & \text{if } 0 < \theta < \pi/2 \\ 1 & \text{if } \pi/2 < \theta \le 1+\pi/2 \end{cases} \qquad |\dot\gamma_R^{(y)}(\theta)| = \begin{cases} 0 & \text{if } -1 \le \theta < 0 \\ 1 & \text{if } 0 < \theta < \pi/2 \\ 0 & \text{if } \pi/2 < \theta \le 1+\pi/2 \end{cases}$$

for every $p > 1$, by Proposition 3.2 and by the area formula we compute

$$\begin{aligned} \mathcal{E}^0_p(\Sigma_R) &= \int_{[-1,1+\pi/2]} |\dot\gamma_R(\theta)| \Big(\frac{|\dot\gamma_R^{(x)}(\theta)|}{|\dot\gamma_R(\theta)|} + \frac{|\dot\gamma_R^{(x)}(\theta)|^{1-p}}{|\dot\gamma_R(\theta)|^{1-p}} \cdot \frac{|\dot\gamma_R^{(x)}(\theta)|^p}{|\dot\gamma_R(\theta)|^p} \Big)\,d\theta \\ &= \int_{[-1,1+\pi/2]} \big(|\dot\gamma_R^{(x)}(\theta)| + |\dot\gamma_R^{(x)}(\theta)|^{1-p} |\dot\gamma_R^{(y)}(\theta)|^p \big)\,d\theta \end{aligned}$$

where

$$\int_{[-1,1+\pi/2]} |\dot\gamma_R^{(x)}(\theta)|\,d\theta = 2 + \frac{\pi}{2}\,R$$

and

$$\int_{[-1,1+\pi/2]} |\dot\gamma_R^{(x)}(\theta)|^{1-p} |\dot\gamma_R^{(y)}(\theta)|^p\,d\theta = \frac{\pi}{2}\,R^{1-p}\,,$$

whence

$$\mathcal{E}_p^0(\Sigma_R) = 2 + \frac{\pi}{2}\,R + \frac{\pi}{2}\,R^{1-p}\,.$$

Similarly, one has

$$\mathcal{E}_1^0(\Sigma_R) = 2 + \frac{\pi}{2}\,(R+1)\,,$$

whence we recover the estimate

$$2^{-1/2}\,\mathcal{E}_1^0(\Sigma_R) \le \mathbf{M}(\Sigma_R) \le \mathcal{E}_1^0(\Sigma_R)\,, \qquad 2\cdot\mathcal{E}_p^0(\Sigma_R) \ge \mathcal{E}_1^0(\Sigma_R) \quad \forall\, p>1\,.$$

In particular, we have

$$\lim_{R\to 0^+} \mathbf{M}(\Sigma_R) = 2 + \frac{\pi}{2}\,, \qquad \lim_{R\to 0^+} \mathcal{E}_1^0(\Sigma_R) = 2 + \frac{\pi}{2}\,, \qquad \lim_{R\to 0^+} \mathcal{E}_p^0(\Sigma_R) = +\infty \quad \forall\, p>1\,.$$

The cited example is given by $\Sigma_h = \Sigma_{R_h}$ for a sequence of radii $R_h \searrow 0$.

Closure property. By definition it turns out that the functional $\Sigma \mapsto \mathcal{E}_p^0(\Sigma)$ is lower semicontinuous w.r.t. the *weak convergence* in $\mathcal{D}_1(U\times\mathbb{S}^N)$.[3]

We now point out a sequential weak closure property. Fix $p>1$ and choose an i.m. rectifiable current $\Sigma_0 = [\![\,\mathcal{M},\theta,\xi\,]\!] \in \mathcal{R}_1(U\times\mathbb{S}^N)$ with finite energy, $\mathcal{E}_p^0(\Sigma_0)<\infty$. Setting $\Gamma := \partial\Sigma_0$, we introduce the non-empty class

$$\mathcal{F}_\Gamma := \{\Sigma\in\mathcal{R}_1(U\times\mathbb{S}^N) \mid \partial\Sigma = \Gamma\,, \quad \mathcal{E}_p^0(\Sigma)<\infty\}\,.$$

Proposition 3.8 *The class $\mathcal{F}_\Gamma$ is closed along weakly converging sequences with equibounded energies. Therefore, the minimum of the problem $\inf\{\mathcal{E}_p^0(\Sigma)\mid\Sigma\in\mathcal{F}_\Gamma\}$ is attained.*

Proof: If $\{\Sigma_h\}_h\subset\mathcal{F}_\Gamma$ satisfies $\sup_h\mathcal{E}_p^0(\Sigma_h)<\infty$ and $\Sigma_h\rightharpoonup\Sigma$, by the regularity property we have $\sup_h\mathbf{M}(\Sigma_h)<\infty$. We thus may apply Federer-Fleming's closure theorem [7] to deduce that $\Sigma\in\mathcal{R}_1(U\times\mathbb{S}^N)$. The weak convergence conserving the boundary condition, we also have $\partial\Sigma=\Gamma$. Finally, by lower-semicontinuity $\mathcal{E}_p^0(\Sigma)\le\liminf_h\mathcal{E}_p^0(\Sigma_h)<\infty$, whence $\Sigma\in\mathcal{F}_\Gamma$. Choosing an energy minimizing sequence $\{\Sigma_h\}_h\subset\mathcal{F}_\Gamma$, possibly passing to a subsequence we have that $\Sigma_h\rightharpoonup\Sigma$ for some $\Sigma\in\mathcal{D}_1(U\times\mathbb{S}^N)$. The second assertion follows as $\Sigma\in\mathcal{F}_\Gamma$. □

4 Gauss graphs with finite energy

In this section we report from [1] the structure properties of the class of currents that naturally arise in the relaxation process.

The class Gcart. For C^2-functions u we have $\mathcal{E}_p(u)\ge 2^{-1}\,\mathcal{E}_1(u)$. As a consequence, see (2.4), if $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ then $u\in\mathcal{E}_1(I,\mathbb{R}^N)$, and we may use some results proved in [1] for the case $p=1$. We thus recall from [1] the class $\mathrm{Gcart}(U\times\mathbb{S}^N)$, defined by

$$\begin{aligned}\mathrm{Gcart}(U\times\mathbb{S}^N) := \{\Sigma\in\mathcal{D}_1(U\times\mathbb{S}^N) \quad | \quad & \exists\,\{u_h\}\subset C^2(I,\mathbb{R}^N) \text{ such that} \\ & GG_{u_h}\rightharpoonup\Sigma \text{ in } \mathcal{D}_1(U\times\mathbb{S}^N)\,,\ \sup\nolimits_h\mathbf{M}(GG_{u_h})<\infty\}\,.\end{aligned} \tag{4.1}$$

Federer-Fleming's closure theorem [7] yields that Σ is an i.m. rectifiable current in $\mathcal{R}_1(U\times\mathbb{S}^N)$ satisfying the null-boundary condition

$$\partial\Sigma = 0 \quad \text{on} \quad C_c^\infty(U\times\mathbb{S}^N)\,.$$

[3] The weak convergence $\Sigma_h\rightharpoonup\Sigma$ of currents in $\mathcal{D}_1(U\times\mathbb{S}^N)$ is defined by duality as

$$\langle\Sigma_h,\omega\rangle\to\langle\Sigma,\omega\rangle \qquad \forall\,\omega\in\mathcal{D}^1(U\times\mathbb{S}^N)\,.$$

The weak convergence with equibounded masses implies the convergence $u_k \to u$ strongly in L^1 to some function $u \in \mathrm{BV}(I, \mathbb{R}^N)$, that will be sometimes denoted by u_Σ. Moreover, by Remark 3.6 we infer that actually $u \in \mathcal{E}_1(I, \mathbb{R}^N)$. Therefore, each current Σ in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ decomposes as

$$\Sigma = GG_u^a + GG_u^C + \widetilde{\Sigma}\,, \qquad u = u_\Sigma \tag{4.2}$$

and we correspondingly have

$$\mathbf{M}(\Sigma) = \mathbf{M}(GG_u^a) + \mathbf{M}(GG_u^C) + \mathbf{M}(\widetilde{\Sigma}) < \infty\,.$$

The *absolute continuous* and *Cantor* components only depend on $u = u_\Sigma$, and their definition makes sense because in [1], see also Sec. 8 below, we proved that the mapping $\Phi_u(t) := (c_u(t), \tau_u(t))$ satisfies

$$\Phi_u \in \mathrm{BV}(I, \overline{U} \times \mathbb{S}^N) \qquad \forall\, u \in \mathcal{E}_1(I, \mathbb{R}^N)$$

where $\overline{U} = I \times \mathbb{R}^N$. More precisely, if $u \in \mathcal{E}_1(I, \mathbb{R}^N)$, not only $u \in \mathrm{BV}(I, \mathbb{R}^N)$, but also the Gauss map $\tau_u := \dot{c}_u/|\dot{c}_u|$ belongs to $\mathrm{BV}(I, \mathbb{S}^N)$, provided that it is defined in terms of the approximate gradient $\dot{u}$.

Remark 4.1 Since the first component τ_u^0 of the tantrix of a smooth Cartesian curve c_u is positive, by weak BV-convergence we infer that the first component of τ_u is a.e. non-negative for each $u \in \mathcal{E}_1(I, \mathbb{R}^N)$, whence τ_u takes values into the half-sphere

$$\mathbb{S}^N_+ := \{y = (y_0, y_1, \ldots, y_N) \in \mathbb{R}^{N+1}_y \,:\, |y| = 1\,,\ y_0 \geq 0\}\,.$$

At any point $t \in I$ one has $\tau_u^0(t_\pm) = 0$ if and only if $|\dot{u}(t_\pm)| = +\infty$. We thus correspondingly define

$$\mathbb{S}_0^{N-1} := \{y \in \mathbb{S}^N_+ \mid y_0 = 0\} \tag{4.3}$$

so that $\tau_u^0(t_\pm) = 0 \iff \tau_u(t_\pm) \in \mathbb{S}_0^{N-1}$. In a similar way we deduce that the *support* $\operatorname{spt}\Sigma \subset \overline{U} \times \mathbb{S}^N_+$ for every $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$.

A GEOMETRIC PROPERTY. In [1] we proved that currents in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ preserve the geometry of Gauss graphs: indeed, for a regular Gauss graph (c_u, τ_u) the second component τ_u is the normalization of the derivative of the first component c_u. More precisely, we proved that when the first component of the tangent vector to Σ at a point $z = (x, y) \in U \times \mathbb{S}^N_+$ is non zero, then it has to be parallel (and with the same verse) to the second component y of the point z, see (4.4) below.

In fact, for every $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ we showed the existence of a Lipschitz function $\Psi \in \mathrm{Lip}(I_{\widetilde{L}}, \overline{U} \times \mathbb{S}^N)$, where $I_{\widetilde{L}} := [0, \widetilde{L}]$, such that the image current $\Psi_\# [\![I_{\widetilde{L}}]\!]$ agrees with Σ. Moreover, *for a.e.* $s \in I_{\widetilde{L}}$ *such that* $|\Pi_x(\dot{\Psi}(s))| \neq 0$*, we have*

$$\frac{\Pi_x(\dot{\Psi}(s))}{|\Pi_x(\dot{\Psi}(s))|} = \Pi_y(\Psi(s)) \in \mathbb{S}^N_+\,. \tag{4.4}$$

THE A.C. COMPONENT. According to Example 2.1, the current $GG_u^a \in \mathcal{D}_1(U \times \mathbb{S}^N)$ is given by

$$\langle GG_u^a, \omega \rangle := \int_I \langle \omega(\Phi_u(t)), \dot{\Phi}_u(t) \rangle \, dt \qquad \forall\, \omega \in \mathcal{D}^1(U \times \mathbb{S}^N) \tag{4.5}$$

so that $GG_u^a := \Phi_{u\#} [\![I]\!]$ as in the smooth case, but this time the pull-back is defined a.e. in terms of the approximate gradient of the BV-function Φ_u, whence the formulas (2.6) hold with $GG_u = GG_u^a$.

THE CANTOR COMPONENT. The current $GG_u^C \in \mathcal{D}_1(U \times \mathbb{S}^N)$ is defined by linear extension of its action on basic forms. For any $g \in C_c^\infty(U \times \mathbb{S}^N)$ we set:

i) $\langle GG_u^C, g(x, y)\, dx^0 \rangle := 0$

ii) $\langle GG_u^C, g(x, y)\, dx^j \rangle := \displaystyle\int_I g(\Phi_{u+})\, dD^C u^j\,,\ j = 1, \ldots, N$

iii) $\langle GG_u^C, g(x,y)\,dy^j\rangle := \int_I g(\Phi_{u+})\,dD^C\tau_u^j\,,\ j=0,1,\ldots,N\,.$

We shall see, Theorem 8.1, that the Gauss map τ_u of a function u with finite relaxed energy has no Cantor part, $D^C\tau_u=0$, and hence it is a *special function with bounded variation.* This will allow us to prove that the energy contribution of the Cantor component GG_u^C reduces to the total variation $|D^C u|(I)$, see Remark 8.2.

BOUNDARY. We have:

$$\partial(GG_u^a+GG_u^C) = -\sum_{t\in J_{\Phi_u}} \big(\delta_{\Phi_u(t_+)}-\delta_{\Phi_u(t_-)}\big) \quad \text{on } C_c^\infty(U\times\mathbb{S}^N)$$

where δ_P is the unit Dirac mass at the point P. In fact, for each $f\in C_c^\infty(U\times\mathbb{S}^N)$ we compute

$$\begin{aligned}\langle\partial GG_u^a, f\rangle &:= \langle GG_u^a, df\rangle = -\int_I \nabla f(\Phi_{u+})\bullet dD^C\Phi_u - \sum_{t\in J_{\Phi_u}}\big(f(\Phi_u(t_+))-f(\Phi_u(t_-))\big)\\ \langle\partial GG_u^C, f\rangle &:= \langle GG_u^C, df\rangle = \int_I \nabla f(\Phi_{u+})\bullet dD^C\Phi_u\,.\end{aligned}$$

Since the composition function $f\circ\Phi_u$ belongs to $BV(I)$, by definition of distributional derivative we deduce that $\int_I D(f\circ\Phi_u)=0$, whereas by choosing $\Phi_{u+}(t)=\Phi_u(t_+)$ as a precise representative, by the chain-rule formula we get

$$D(f\circ\Phi_u) = \nabla f(\Phi_u)\bullet\dot\Phi_u\,dt + \nabla f(\Phi_{u+})\bullet D^C\Phi_u + \big(f(\Phi_{u+})-f(\Phi_{u-})\big)\,\mathcal{H}^0 \llcorner J_{\Phi_u}$$

and hence

$$\langle\partial(GG_u^a+GG_u^C), f\rangle = -\big(f(\Phi_{u+})-f(\Phi_{u-})\big)\,\mathcal{H}^0\llcorner J_{\Phi_u}\,.$$

JUMP-CORNER AND CORNER COMPONENTS. As a consequence, the third component $\widetilde\Sigma$ in (4.2) satisfies the *verticality condition*

$$\langle\widetilde\Sigma, g(x,y)\,dx^0\rangle = 0 \qquad \forall\, g\in C_c^\infty(U\times\mathbb{S}^N)$$

and the *boundary condition*

$$\partial\widetilde\Sigma = \sum_{t\in J_{\Phi_u}}\big(\delta_{\Phi_u(t_+)}-\delta_{\Phi_u(t_-)}\big) \quad \text{on } C_c^\infty(U\times\mathbb{S}^N)\,.$$

Also, the following *decomposition in mass* holds:

$$\widetilde\Sigma = \widehat\Sigma + \sum_{t\in J_{\Phi_u}}\Gamma_{t,\Sigma}\,, \qquad \mathbf{M}(\widetilde\Sigma) = \mathbf{M}(\widehat\Sigma) + \sum_{t\in J_{\Phi_u}}\mathbf{M}(\Gamma_{t,\Sigma})\,, \tag{4.6}$$

where the current $\widehat\Sigma\in\mathcal{R}_1(U\times\mathbb{S}^N)$ satisfies the null-boundary condition $\partial\widehat\Sigma=0$, and $\Gamma_{t,\Sigma}$ is for each $t\in J_{\Phi_u}$ an *a-cyclic* i.m. rectifiable current in $\mathcal{R}_1(U\times\mathbb{S}^N)$, supported in $\{t\}\times\mathbb{R}^N\times\mathbb{S}^N_+$, and with boundary

$$\partial\Gamma_{t,\Sigma} = \delta_{\Phi_u(t_+)}-\delta_{\Phi_u(t_-)}\,. \tag{4.7}$$

Recalling that $J_{\Phi_u}=J_u\cup(J_{\dot u}\setminus J_u)$, we denote by

$$\Sigma^{Jc} := \sum_{t\in J_u}\Gamma_{t,\Sigma}\,, \qquad \Sigma^c := \sum_{t\in J_{\dot u}\setminus J_u}\Gamma_{t,\Sigma} \tag{4.8}$$

the *jump-corner* and *corner* components of a current Σ in $\text{Gcart}(U\times\mathbb{S}^N)$, respectively. We shall correspondingly call $J_{\dot u}\setminus J_u$ the set of *corner points* of u.

RELATION WITH CARTESIAN CURRENTS. For each $\Sigma\in\text{Gcart}(U\times\mathbb{S}^N)$, the projection

$$T = T(\Sigma) := \Pi_{x\#}\Sigma$$

is a Cartesian current in $\mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$ in the sense of Giaquinta-Modica-Souček [9]. More precisely, according to (4.2), (4.6), and (4.8) we can write

$$T = T_u^a + T_u^C + T^J + T^s \,, \qquad u = u_\Sigma$$

where one respectively has

$$\Pi_{x\#} GG_u^a = T_u^a \,, \quad \Pi_{x\#} GG_u^C = T_u^C \,, \quad \Pi_{x\#} \Sigma^{Jc} = T^J \,, \quad \Pi_{x\#} \Sigma^c = 0 \,, \quad \Pi_{x\#} \widetilde{\Sigma} = T^s \,.$$

The absolute continuous component $T_u^a =: G_u$ is given by $T_u^a = c_{u\#}[\![I]\!]$, so that

$$\langle T_u^a, \phi(x)\, dx^0 \rangle = \int_I \Phi(c_u(t))\, dt \,, \quad \langle T_u^a, \phi(x)\, dx^j \rangle = \int_I \Phi(c_u(t))\, \dot{u}^j(t)\, dt \,, \quad j = 1, \ldots N$$

for every $\phi \in C_c^\infty(\mathring{I} \times \mathbb{R}^N)$. The Cantor component T_u^C satisfies

$$\langle T_u^C, \phi(x)\, dx^0 \rangle = 0 \,, \quad \langle T_u^C, \phi(x)\, dx^j \rangle = \langle D^C u^j, \phi \circ c_{u+} \rangle \,, \quad j = 1, \ldots N$$

The Jump component T_u^J is given by

$$\langle T_u^J, \phi(x)\, dx^0 \rangle = 0 \,, \quad \langle T_u^J, \phi(x)\, dx^j \rangle = \sum_{t \in J_u} \int_{\gamma_t(T)} \phi(x)\, dx^j \,, \quad j = 1, \ldots N$$

where $\gamma_t(T)$ is for every $t \in J_u$ an oriented, simple, and rectifiable arc in $\{t\} \times \mathbb{R}^N$ with end points given by the one sided limits $c_u(t_\pm)$, so that $\partial[\![\gamma_t(T)]\!] = \delta_{c_u(t_+)} - \delta_{c_u(t_-)}$. We remark that property $\Pi_{x\#}\Sigma^c = 0$ follows from the fact that $\Gamma_{t,\Sigma}$ is for each $t \in J_{\dot{u}} \setminus J_u$ an a-cyclic i.m. rectifiable current supported in $\{c_u(t)\} \times \mathbb{S}_+^N$. Arguing as before, one obtains the null-boundary condition

$$\partial(T_u^a + T_u^C + T^J) = 0 \,.$$

As a consequence, the singular component T^s is a (vertical) i.m. rectifiable current such that $\partial T^s = 0$. Finally, a decomposition in mass holds, i.e.

$$\mathbf{M}(T) = \mathbf{M}(T_u^a) + \mathbf{M}(T_u^C) + \mathbf{M}(T^J) + \mathbf{M}(T^s)$$

where one has

$$\mathbf{M}(T_u^a) = \int_I |\dot{c}_u|\, dt \,, \quad \mathbf{M}(T_u^C) = |D^C u|(I) \,, \quad \mathbf{M}(T^J) = \sum_{t \in J_u} \mathbf{M}([\![\gamma_t(T)]\!]) \,.$$

5 Energy lower bound

In this section we introduce a class of minimal currents associated to our relaxation problem. In fact, we prove an energy lower bound: the relaxed energy is greater than the energy of the corresponding minimal current.

THE CASE $p = 1$. According to (4.2), for every $u \in \mathrm{BV}(I, \mathbb{R}^N)$ we define

$$\mathrm{Gcart}_u := \{\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N) \mid u_\Sigma = u\} \,, \qquad u \in \mathrm{BV}(I, \mathbb{R}^N) \,. \tag{5.1}$$

In the case $p = 1$, in [1] we in fact proved the following

Proposition 5.1 *Let $u \in L^1(I, \mathbb{R}^N)$, where $N \geq 1$. Then*

$$u \in \mathcal{E}_1(I, \mathbb{R}^N) \iff \mathrm{Gcart}_u \neq \emptyset \,.$$

In this case, moreover, we have

$$\overline{\mathcal{E}}_1(u) = \min\{\mathcal{E}_1^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\} \,.$$

Minimal currents. In order to obtain a similar formula in the case $p > 1$, for every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we now introduce an optimal current Σ_u^p of the type :

$$\Sigma_u^p := GG_u^a + GG_u^C + S_u^p \,, \tag{5.2}$$

where the third component $S_u^p \in \mathcal{R}_1(U \times \mathbb{S}^N)$ is given by

$$S_u^p := \sum_{t \in J_{\Phi_u}} \Gamma_t^p$$

for suitable minimal currents Γ_t^p that we now describe, see Definition 5.4 below.

For any point $t \in J_{\Phi_u}$ we first introduce a suitable class $\mathcal{F}(u,t)$ of a-cyclic i.m. rectifiable currents Γ in $\mathcal{R}_1(U \times \mathbb{S}^N)$, supported in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$, with boundary

$$\partial \Gamma = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)} \tag{5.3}$$

and such that $\mathcal{E}_p^0(\Gamma) < \infty$. By Federer's structure theorem [6, 4.2.25], for any such Γ we find a Lipschitz function $\gamma = \gamma_\Gamma : [-1/2, 1/2] \to U \times \mathbb{S}^N$ with constant velocity $|\dot\gamma(s)| = \mathbf{M}(\Gamma)$ for a.e. s, and $\gamma_\# [\![-1/2, 1/2]\!] = \Gamma$. Therefore, we deduce that $|\dot\gamma^{(x)}(s)| > 0$ a.e., and as in Example 3.7 we obtain that

$$\mathcal{E}_p^0(\Gamma) = \int_{[-1/2,1/2]} \big(|\dot\gamma^{(x)}(s)| + |\dot\gamma^{(x)}(s)|^{1-p} |\dot\gamma^{(y)}(s)|^p \big)\, ds \,. \tag{5.4}$$

According to the geometric property (4.4), *we say that* $\Gamma \in \mathcal{F}(u,t)$ *if in addition one has*

$$\frac{\dot\gamma^{(x)}(s)}{|\dot\gamma^{(x)}(s)|} = \gamma^{(y)}(s) \qquad \text{for a.e. } s \in [-1/2, 1/2] \,, \qquad \gamma = \gamma_\Gamma \,. \tag{5.5}$$

Remark 5.2 Let $\Gamma \in \mathcal{F}(u,t)$ for some $t \in J_{\Phi_u}$. By property (5.5) we deduce that the second component $\gamma^{(y)} := \Pi_y(\gamma_\Gamma)$ is the tantrix of the first component $\gamma^{(x)} := \Pi_x(\gamma_\Gamma)$, and that the curve $\gamma^{(x)}$ has positive length, as $|\dot\gamma^{(x)}(s)| > 0$ a.e. Therefore, formula (5.4) yields that the p-energy of Γ is equal to the *p-curvature functional* of the curve $\Pi_x(\gamma_\Gamma)$, i.e.

$$\mathcal{E}_p^0(\Gamma) = \int_c (1 + k_c^p)\, d\mathcal{H}^1 \,, \qquad c := \Pi_x(\gamma_\Gamma) \,. \tag{5.6}$$

Finally, notice that the vertical cycle $\widetilde\Sigma$ defined in Example 6.2 below satisfies the above property (5.5), whence $\widetilde\Sigma \in \mathcal{F}(u,t)$, with $t = 0 \in J_{\dot u}$.

Existence. We have:

Proposition 5.3 *For every* $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ *and* $t \in J_{\Phi_u}$*, the minimum of the problem*

$$\inf\{\mathcal{E}_p^0(\Gamma) \mid \Gamma \in \mathcal{F}(u,t)\}$$

is attained.

Proof: In fact, by (4.7) and the structure property (4.4), we deduce that the class $\mathcal{F}(u,t)$ is non-empty for each $t \in J_{\Phi_u}$. We then choose a minimizing sequence $\{\Gamma_h\}_h$ in the class $\mathcal{F}(u,t)$. On account of Proposition 3.8, and using that $\sup_h \mathbf{M}(\Gamma_h) \le 2 \cdot \sup_h \mathcal{E}_p^0(\Gamma_h)$, by Federer-Fleming's theorem we deduce that (up to a subsequence) $\{\Gamma_h\}_h$ weakly converges to a current Γ_∞ in $\mathcal{R}_1(U \times \mathbb{S}^N)$, supported in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$, and satisfying the boundary-condition (5.3). By lower-semicontinuity one has $\mathcal{E}_p^0(\Gamma_\infty) = \inf\{\mathcal{E}_p^0(\Gamma) \mid \Gamma \in \mathcal{F}(u,t)\}$. Therefore, denoting $\gamma_\infty = \gamma_{\Gamma_\infty} : [-1/2, 1/2] \to U \times \mathbb{S}^N$ the parameterization of Γ_∞ as above, we only have to show that the geometric property (5.5) holds for $\gamma = \gamma_\infty$. Since $\mathcal{E}_p^0(\Gamma_\infty) < \infty$, we deduce as above that $|\dot\gamma_\infty^{(x)}(s)| > 0$ for a.e. s. Moreover, by assumption we know that (5.5) holds for $\gamma_h := \gamma_{\Gamma_h}$ for each h, and we thus have to prove that such a geometric property is preserved by weak convergence. This property has been proved in [1, Thm. 6.3], and for completeness we report here the same argument.

Setting $a(s) := \dot\gamma^{(x)}_\infty(s)$ and $b(s) := \gamma^{(y)}_\infty(s)$ we prove that the two vectors $a(s)$ and $b(s)$ are parallel and pointing the same way, for a.e. $s \in [-1/2, 1/2]$.

Now, given two vectors $a, b \in \mathbb{R}^{N+1}$, with $a \neq 0$ and $|b| = 1$, they are parallel and pointing the same way if and only if $a/|a| = b$, that is equivalent to $a \bullet b = |a|$, since

$$|(a/|a|) - b|^2 = 1 + 1 - 2\frac{a \bullet b}{|a|}\,.$$

Since $\mathbf{M}(\Gamma_h) \leq 2 \cdot \mathcal{E}^0_p(\Gamma_h)$, and $|\dot\gamma_h(s)| = \mathbf{M}(\Gamma_h)$ for a.e. $s \in [-1/2, 1/2]$, by Ascoli's theorem we may assume that $\{\gamma_h\}_h$ uniformly converges to the Lipschitz function γ_∞. Therefore, letting $a_h(s) := \dot\gamma^{(x)}_h(s)$ and $b_h(s) := \gamma^{(y)}_h(s)$, we infer that $b_h(s) \to b(s) \in \mathbb{S}^N$ uniformly. Since moreover $\dot\gamma_h \rightharpoonup \dot\gamma_\infty$ weakly-* in L^∞, we deduce that $a_h(s) \rightharpoonup a(s)$ weakly-* in L^∞, whence

$$\lim_{h\to\infty} \int_{[-1/2,1/2]} a_h(s) \bullet b_h(s)\, ds = \int_{[-1/2,1/2]} a(s) \bullet b(s)\, ds\,.$$

Since moreover $a_h(s) \rightharpoonup a(s)$ weakly in L^1, the lower semicontinuity

$$\int_{[-1/2,1/2]} |a(s)|\, ds \leq \liminf_{h\to\infty} \int_{[-1/2,1/2]} |a_h(s)|\, ds$$

holds, and hence

$$\int_{[-1/2,1/2]} \big(|a(s)| - a(s) \bullet b(s)\big)\, ds \leq \liminf_{h\to\infty} \int_{[-1/2,1/2]} \big(|a_h(s)| - a_h(s) \bullet b_h(s)\big)\, ds\,.$$

Since (5.5) holds for $\gamma = \gamma_h$, we have seen that $|b_h(s)| = 1$, $|a_h(s)| \neq 0$, and $a_h(s)/|a_h(s)| = b_h(s)$ for a.e. s and for all h. Therefore, we obtain

$$\int_{[-1/2,1/2]} \big(|a(s)| - a(s) \bullet b(s)\big)\, ds \leq 0\,.$$

The integrand being non-negative by the Schwartz inequality $a \bullet b \leq |a||b| = |a|$, we deduce that $|a(s)| - a(s) \bullet b(s) = 0$ for a.e. $s \in [-1/2, 1/2]$, whence the two vectors $a(s)$ and $b(s)$ are parallel and pointing the same way, as required. □

Definition 5.4 *For each $t \in J_{\Phi_u}$ we denote by Γ^p_t a minimum point for $\mathcal{E}^0_p$ in the class $\mathcal{F}(u, t)$.*

In particular, Γ^p_t is an a-cyclic current in $\mathcal{R}_1(U \times \mathbb{S}^N)$. From our definition (5.2) we thus obtain:

$$\mathcal{E}^0_p(\Sigma^p_u) := \mathcal{E}^0_p(GG^a_u) + \mathcal{E}^0_p(GG^C_u) + \mathcal{E}^0_p(S^p_u)\,, \qquad \mathcal{E}^0_p(S^p_u) = \sum_{t\in J_{\Phi_u}} \mathcal{E}^0_p(\Gamma^p_t)\,. \tag{5.7}$$

Remark 5.5 Examples 6.2 and 6.5 will show that in high codimension $N \geq 2$ in general $\dot c_u(t_-) \neq \dot c_u(t_+)$ at some point $t \in J_{\Phi_u}$, even if we always have $|\dot c_u(t_\pm)| = \infty$. As a consequence, differently to what happens in the case $p = 1$, it turns out that if $t \in J_{\Phi_u}$ the optimal current Γ^p_t is not supported on a straight line-segment, in general.

Euler equation. For the sake of completeness, we report here the result of the computation of the corresponding *Euler-Lagrange equation*.

Proposition 5.6 *For any non-negative smooth function f of the curvature $\mathbf{k}$ of curves γ, the Euler equation of the functional $\mathcal{F}(\gamma) := \int_\gamma f(\mathbf{k})\, d\mathcal{H}^1$ is*

$$\frac{\ddot f(\mathbf{k})\, \ddot{\mathbf{k}}}{|\dot c|} + \frac{\dddot f(\mathbf{k})\, \dot{\mathbf{k}}^2}{|\dot c|} - \frac{\ddot f(\mathbf{k})\, \dot{\mathbf{k}}\, (\dot c \bullet \ddot c)}{|\dot c|^3} + \mathbf{k}\left\{\mathbf{k}\, \dot f(\mathbf{k}) - f(\mathbf{k})\right\} |\dot c| = 0\,. \tag{5.8}$$

In particular, for $f(\mathbf{k}) = 1 + \mathbf{k}^2$, so that $\mathcal{F}(\gamma) = \mathcal{L}(\gamma) + \int_\gamma \mathbf{k}^2 \, d\mathcal{H}^1$, the above equation takes the simpler form:

$$\frac{2\ddot{\mathbf{k}}}{|\dot{c}|} - \frac{2\dot{\mathbf{k}}\,(\dot{c} \bullet \ddot{c})}{|\dot{c}|^3} + \mathbf{k}\,(\mathbf{k}^2 - 1)\,|\dot{c}| = 0\,.$$

Therefore, even for $p = 2$ it seems a difficult task to write explicitly the above minima Γ_t^p.

Notice that by choosing the arc-length parameterization, one has $|\dot{c}| = 1$ and $(\dot{c} \bullet \ddot{c}) = 0$ a.e., whence the above equation (5.8) reduces to the classical one

$$\dot{f}(\mathbf{k})\,\ddot{\mathbf{k}} + \ddot{f}(\mathbf{k})\,\dot{\mathbf{k}}^2 + \mathbf{k}\,\{\mathbf{k}\,\dot{f}(\mathbf{k}) - f(\mathbf{k})\} = 0$$

compare [8, Ch. 1, Sec. 5]. For $f(\mathbf{k}) = 1 + |\mathbf{k}|^p$, where $p > 1$, so that $\mathcal{F}(\gamma) = \mathcal{L}(\gamma) + \int_\gamma |\mathbf{k}|^p \, d\mathcal{H}^1$, the above equation takes the simpler form:

$$p\,|\mathbf{k}|^{p-2}\,\ddot{\mathbf{k}} + p(p-2)\,|\mathbf{k}|^{p-4}\,\mathbf{k}\,\dot{\mathbf{k}}^2 + \mathbf{k}\,|\mathbf{k}|^p - \frac{\mathbf{k}}{(p-1)} = 0\,.$$

Now, searching for smooth closed planar curves with constant curvature, i.e. for minimal circles of radius R, since $\mathbf{k} \equiv R^{-1}$ we deduce that the above equation is solved when $R = (p-1)^{1/p}$. More generally, even in presence of first order boundary conditions, minimizing curves in general depend on the choice of the exponent $p > 1$.

Energy lower bound. As a consequence, we readily obtain the following energy lower bound:

Corollary 5.7 *Let $p > 1$. For every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we have*

$$\overline{\mathcal{E}}_p(u) \geq \mathcal{E}_p^0(GG_u^a) + \mathcal{E}_p^0(GG_u^C) + \mathcal{E}_p^0(S_u^p)$$

where

$$\mathcal{E}_p^0(S_u^p) = \sum_{t \in J_{\Phi_u}} \mathcal{E}_p^0(\Gamma_t^p)\,.$$

Proof: By lower semicontinuity we get

$$\overline{\mathcal{E}}_p(u) \geq \inf\{\mathcal{E}_p^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)\,,\ u_\Sigma = u\}$$

see (4.2), where

$$\mathcal{E}_p^0(\Sigma) = \mathcal{E}_p^0(GG_u^a) + \mathcal{E}_p^0(GG_u^C) + \mathcal{E}_p^0(\widetilde{\Sigma})$$

and according to (4.6)

$$\mathcal{E}_p^0(\widetilde{\Sigma}) = \mathcal{E}_p^0(\widehat{\Sigma}) + \sum_{t \in J_{\Phi_u}} \mathcal{E}_p^0(\Gamma_{t,\Sigma})\,.$$

The energy lower bound follows as by minimality $\mathcal{E}_p^0(\Gamma_{t,\Sigma}) \geq \mathcal{E}_p^0(\Gamma_t^p)$ for each $t \in J_{\Phi_u}$. □

6 Functions with finite relaxed energy

In this section we outline some features concerning functions with finite relaxed energy. In particular, differently to what happens for $N = 1$, we see that corner points may occur in high codimension. However, we prove that the set of corner points of a function with finite relaxed energy is always finite, Theorem 6.3.

The Gauss map. We first point out a property of the Gauss map at the Jump set J_{Φ_u}. Notice that the following proposition is false in the case $p = 1$, compare [1] and Remark 6.4 below.

Proposition 6.1 *Let $N \geq 1$ and $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ for some $p > 1$. Then $\tau_u^0(t_\pm) = 0$ for every $t \in J_{\Phi_u}$.*

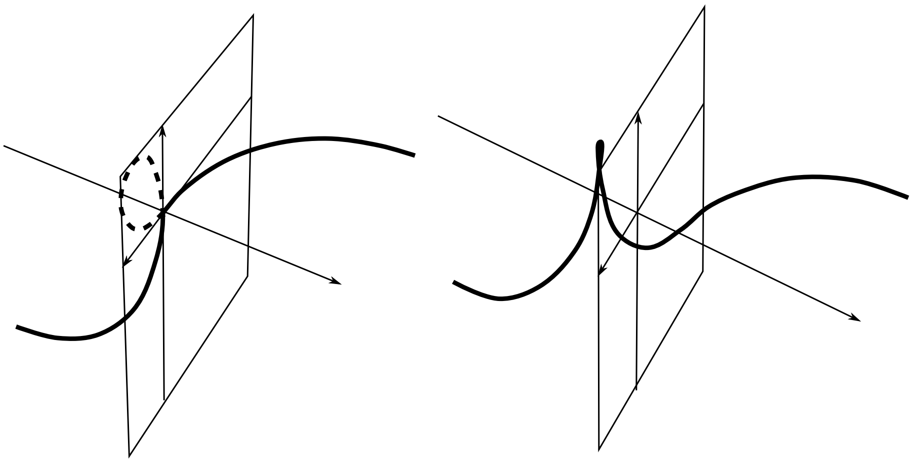

Figure 1: To the left, the curve c_u and (dashed) the current $\widetilde{\Sigma}$ which lies in the vertical plane $\{t=0\}$; to the right, one term of the smooth approximating sequence c_{u_h}.

PROOF: Let $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ be such that $\sup_h \mathcal{E}^0_p(u_h) < \infty$ and $u_h \to u$ strongly in L^1. Using that $\sup_h \mathcal{E}^0_1(u_h) < \infty$, possibly passing to a subsequence we find the existence of a current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ such that $GG_{u_h} \rightharpoonup \Sigma$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$, see (4.1). By lower semicontinuity we have $\mathcal{E}^0_p(\Sigma) < \infty$.

Since $\mathcal{E}^0_p(\Sigma) = \mathcal{E}^0_p(GG^a_u) + \mathcal{E}^0_p(GG^C_u) + \mathcal{E}^0_p(\widetilde{\Sigma})$, according to the decomposition (4.6) we have $\mathcal{E}^0_p(\Gamma_{t,\Sigma}) < \infty$ for every $t \in J_{\Phi_u}$. Assume that $\tau^0_u(t_\pm) > 0$ for some point $t \in J_{\Phi_u}$. Since $\Gamma_{t,\Sigma}$ is an a-cyclic i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$, supported in $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$, the boundary condition (4.7) joined with the structure property (4.4) imply that the unit tangent vector ξ has zero component $\xi^{(x)}$ at a subset of points in $\mathrm{set}(\Gamma_{t,\Sigma})$ with positive $\mathcal{H}^1$-measure. By the structure (3.2) of the energy integrand, this yields that $\mathcal{E}^0_p(\Gamma_{t,\Sigma}) = +\infty$, a contradiction. □

CORNER POINTS. Condition $\tau^0_u(t_\pm) = 0$ is equivalent to $|\dot{u}(t_\pm)| = +\infty$. Therefore, in case of codimension $N = 1$, Proposition 6.1 implies that $\dot{u}(t_+) = \dot{u}(t_-) \in \{\pm\infty\}$ for each $t \in J_{\Phi_u}$, otherwise one would obtain as above that $\mathcal{E}^0_p(\Gamma_{t,\Sigma}) = +\infty$, a contradiction. This implies that $J_{\dot{u}} = \emptyset$ and hence, compare (4.8), that the corner component $\Sigma^c = 0$ if $N = 1$, as already shown in [5].

Example 3.7 may suggest the absence of corner points for Cartesian curves c_u of continuous functions $u \in \mathcal{E}_p(I, \mathbb{R}^N)$. We now see that this is not the case in high codimension $N \geq 2$.

Example 6.2 For $I = [-1, 1]$ and $N = 2$, consider the continuous $W^{1,1}$-function $u : I \to \mathbb{R}^2$ given by

$$u(t) := \begin{cases} (0, -\sqrt{-t^2 - 2t}) & \text{if } t \leq 0 \\ (\sqrt{-t^2 + 2t}, 0) & \text{if } t \geq 0 \end{cases}$$

so that the graph curve c_u is the union of two quarters of unit circles meeting at the point $0_{\mathbb{R}^3}$, centered at the points $(-1, 0, 0)$ and $(1, 0, 0)$ and lying in the hyperplanes $x_1 = 0$ and $x_2 = 0$, respectively. Since $\dot{c}_u(0_-) = (1, 0, +\infty)$ and $\dot{c}_u(0_+) = (1, +\infty, 0)$, we have $\tau_u(0_-) = (0, 0, 1)$ and $\tau_u(0_+) = (0, 1, 0)$, thus a corner point with turning angle equal to $\pi/2$ appears at the point $0_{\mathbb{R}^3}$, whence $J_{\dot{u}} \setminus J_u = \{0\}$.

It is not difficult to check that $u \in \mathcal{E}^0_p(I, \mathbb{R}^2)$ for each $p > 1$. In fact, one may approximate u by a smooth sequence $\{u_h\}$ such that the Gauss graphs GG_{u_h} weakly converge to a current Σ in $\mathrm{Gcart}(U \times \mathbb{S}^2)$ given by $\Sigma = GG^a_u + \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the 1-current integration of a smooth curve in $\{0\} \times \mathbb{R}^2 \times \mathbb{S}^1_0$, see (4.3), with end points $\Phi_u(0_\pm) = (0_{\mathbb{R}^3}, \tau_u(0_\pm))$ and satisfying the geometric property (4.4). Since $\sup_h \mathcal{E}^0_p(u_h) < \infty$, we deduce that $u \in \mathcal{E}^0_p(I, \mathbb{R}^2)$.

When $p = 1$, one simply has to smoothen the angle by means of an arc with small curvature radius. For $p > 1$, instead, good approximations (i.e., with small energy contribution) are performed by means of vertical arcs which have curvature radius greater than a positive constant, depending on p, see Figure 1 on the left.

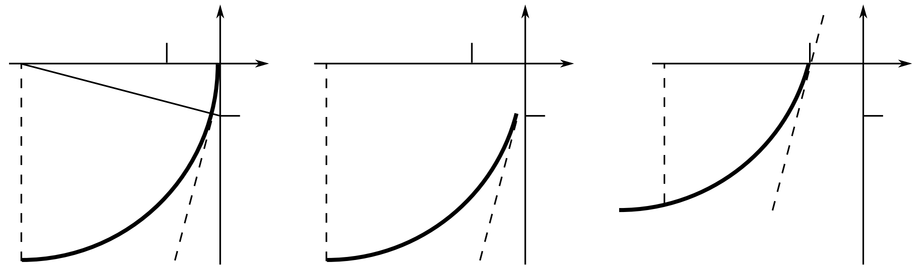

Figure 2: The small notches on the t and x_2 axes are $1/h$ away from the origin; the angled lines thus have slopes $1/h$ and h.

Differently to the case $p=1$ analyzed in [1], the vertical current $\widetilde{\Sigma}$ is not supported in $\{c_u(0)\}\times\mathbb{S}^1_0$, compare (4.3). An example is $\widetilde{\Sigma}:=\gamma_\#[\![\pi/2,\pi]\!]$, where $\gamma:[\pi/2,\pi]\to\mathbb{R}^3_x\times\mathbb{R}^3_y$ is the regular curve defined in components $\gamma^{(x)}:=\Pi_x\circ\gamma$ and $\gamma^{(y)}:=\Pi_y\circ\gamma$ by $\gamma^{(x)}(\theta):=(0,\gamma_x(\theta))$ and $\gamma^{(y)}(\theta):=(0,\gamma_y(\theta))$, where

$$\gamma_x(\theta):=\big(-\sin\theta\cos^2\theta,-\cos\theta\sin^2\theta\big)\,,\qquad \gamma_y(\theta):=\frac{\dot\gamma^{(x)}(\theta)}{|\dot\gamma^{(x)}(\theta)|}\,,$$

so that the geometric property (4.4) holds true. More explicitly, one has

$$\gamma^{(y)}(\theta)=\frac{\big(\cos\theta(3\sin^2\theta-1),\sin\theta(1-3\cos^2\theta)\big)}{(3\sin^4\theta-3\sin^2\theta+1)^{1/2}}\qquad\forall\,\theta\in[\pi/2,\pi]$$

whence $\gamma(\pi/2)=(0_{\mathbb{R}^3},\tau_u(0_-))=\Phi_u(0_-)$ and $\gamma(\pi)=(0_{\mathbb{R}^3},\tau_u(0_+))=\Phi_u(0_+)$, and according to (4.7)

$$\partial\widetilde{\Sigma}=\partial\gamma_\#[\![\pi/2,\pi]\!]=\delta_{\gamma(\pi)}-\delta_{\gamma(\pi/2)}=\delta_{\Phi_u(0_+)}-\delta_{\Phi_u(0_-)}\,.$$

An approximating sequence $u_h:[-1,1]\to\mathbb{R}^2$ satisfying $\sup_h\mathcal{E}^0_p(u_h)<\infty$ is defined by widening the base of the vertical arc so that the loop is made on the interval $-1/h\le t\le 1/h$; the resulting arc does no longer begin and end vertically, but with slope h, so we cannot simply glue the two parts of the graph of u to it (conveniently separated); the problem is easily solved by cutting away a suitable (and small) terminal portion of each arc and translating it so that the three pieces fit, see Figure 2. The resulting curve is defined on a very slightly larger interval than $[-1,1]$ but a reparameterization does the trick, see Figure 1 on the right. Since $\{u_h\}_h\subset W^{1,1}(I,\mathbb{R}^N)$, by applying Step 2 of Theorem 7.1 below to each u_h, a diagonal argument yields the existence of a smooth sequence $\{\widetilde u_h\}_h\subset C^1(I,\mathbb{R}^N)$ with $\sup_h\mathcal{E}^0_p(\widetilde u_h)<\infty$ and $\widetilde u_h\to u$ strongly in L^1. We omit any further detail.

FINITENESS OF CORNER POINTS. We now see that *for every $p>1$, the set $J_{\dot u}\setminus J_u$ of corner points of a function $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ with finite relaxed energy is always finite.* On account of Definition 5.4 and Corollary 5.7, it clearly suffices to prove the following

Theorem 6.3 *Let $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ for some $p>1$. For every $t\in J_{\dot u}\setminus J_u$ we have $\mathcal{E}^0_p(\Gamma^p_t)\ge\pi/2$.*

PROOF: Let $\Gamma\in\mathcal{F}(u,t)$ and consider the curve $c:=\Pi_x(\gamma_\Gamma)$ where $\gamma_\Gamma:[-1/2,1/2]\to U\times\mathbb{S}^N$ is the Lipschitz function with constant velocity such that $\gamma_{\Gamma\#}[\![-1/2,1/2]\!]=\Gamma$. The boundary condition (5.3), where $\Phi_u(t_\pm)=(c_u(t),\tau_u(t_\pm))$ by the continuity of u at the point t, yields that c is a closed curve with initial and final velocities parallel to the unit vectors $\tau_u(t_\pm)\in\mathbb{S}^{N-1}_0$, and $\tau_u(t_-)\neq\tau_u(t_+)$. By (5.6) we estimate

$$\mathcal{E}^0_p(\Gamma)=\int_c(1+k_c^p)\,d\mathcal{H}^1\ge\frac12\int_c(1+k_c)\,d\mathcal{H}^1\ge\frac12\int_c k_c\,d\mathcal{H}^1\,.$$

Therefore, it suffices to observe that the *total curvature* $\int_c k_c\,d\mathcal{H}^1$ of any closed arc with positive length is at least π, compare [10] and also [3, Lemma 3.2]. $\square$

Remark 6.4 Theorem 6.3 fails to hold in the case $p=1$. It suffices to consider a piecewise affine and continuous function $u:[0,1]\to\mathbb{R}$ with a countable set of corner points $J_{\dot u}=\{t_j:=1-2^{-j}\mid j\in\mathbb{N}^+\}$ such that (setting $t_0=0$) the slope of u at the interval $]t_{j-1},t_j[$ is equal to 2^{-j} for every j. In fact, the length of the Cartesian curve c_u is lower than 2, and its total curvature is equal to $\pi/4$, whence $u\in\mathcal{E}_1^0(I,\mathbb{R})$. Notice that by Proposition 6.1 we get $\overline{\mathcal{E}}_p(u)=+\infty$ for every $p>1$.

Discontinuity points. In general a function with finite relaxed energy may have a non-trivial jump set J_u. This was already observed in [5] when $N=1$, and we give here an example for $N=2$.

Example 6.5 Similarly to Example 6.2, consider the BV-function $u:[-1,1]\to\mathbb{R}^2$ given by

$$u(t):=\begin{cases}(0,-\sqrt{-t^2-2t}) & \text{if } t\le 0\\ (\sqrt{-t^2+2t},3) & \text{if } t>0\end{cases}$$

so that u has a jump point at the origin, as $u(0_-)=(0,0)$ and $u(0_+)=(0,3)$. This time the graph of u is the union of two quarters of unit circles centered at the points $(-1,0,0)$ and $(1,1,3)$ and lying in the hyperplanes $x_1=0$ and $x_2=3$, respectively. We again have $\dot c_u(0_-)=(1,0,+\infty)$ and $\dot c_u(0_+)=(1,+\infty,0)$, so that $\tau_u(0_-)=(0,0,1)$ and $\tau_u(0_+)=(0,1,0)$. Again, it is not difficult to check that $u\in\mathcal{E}_p^0(I,\mathbb{R}^2)$ for each $p>1$. In fact, one may approximate u by a smooth sequence $\{u_h\}$ such that the Gauss graphs GG_{u_h} weakly converge to a current Σ in $\text{Gcart}(U\times\mathbb{S}^2)$ given by $\Sigma=GG_u^a+\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the 1-current integration of a smooth curve in $\{0\}\times\mathbb{R}^2\times\mathbb{S}_0^1$ with end points $\Phi_u(0_-)=((0,0,0),(0,0,1))$ and $\Phi_u(0_+)=((0,0,3),(0,1,0))$ and satisfying the geometric property (4.4). Since $\sup_h\mathcal{E}_p^0(u_h)<\infty$, we deduce again that $u\in\mathcal{E}_p(I,\mathbb{R}^2)$.

An explicit formula for $\widetilde{\Sigma}$ can be obtained by slightly modifying the definition of γ from Example 6.2, this time defining a regular curve $\gamma:[\pi/2,\pi]\to\mathbb{R}_x^3\times\mathbb{R}_y^3$ such that

$$\gamma_y(\theta)=\frac{\dot\gamma^{(x)}(\theta)}{|\dot\gamma^{(x)}(\theta)|}\qquad\forall\,\theta\in[\pi/2,\pi]$$

whereas $\gamma(\pi/2)=((0,0,0),\tau_u(0_-))=\Phi_u(0_-)$ and $\gamma(\pi)=((0,0,3),\tau_u(0_+))=\Phi_u(0_+)$. Whence the geometric property (4.4) is satisfied and according to (4.7) we again have $\partial\widetilde{\Sigma}=\delta_{\Phi_u(0_+)}-\delta_{\Phi_u(0_-)}$.

Countable discontinuity set. The lower bound in Theorem 6.3 is false if $t\in J_u$, i.e. on the Jump component Σ_u^J. As a consequence, in general the Jump set J_u of a function $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ is countable, for any $p>1$. We sketch here an example in codimension $N=2$.

Consider $u:[0,1]\to\mathbb{R}^2$ whose components u^j are increasing and bounded functions that are discontinuous on the countable set $J_u=\{t_i=1-2^{-i}\mid i\in\mathbb{N}^+\}$ and smooth outside J_u, in such a way that the integral $\int_I|\dot c_u|^{p-1}(|\dot c_u|^p+|\tau_u|^p)\,dt$ is finite, the series $\sum_i|u(t_i+)-u(t_i-)|$ is convergent, and u is a BV-function with no Cantor-part, $|D^Cu|(I)=0$. We define u with unbounded right and left derivative at the discontinuity set, i.e. $|\dot u(t_i\pm)|=+\infty$ for each i, and the Gauss map $\tau_u:I\to\mathbb{S}^1$ is a BV-function with no Cantor part, $D^C\tau_u=0$, and discontinuity set $J_{\tau_u}=J_u$, i.e. u has no corner points. Then $\tau_u(t_i\pm)\in\mathbb{S}_0^1$ for each i, see Remark 4.1. Denoting by d_i the geodesic distance between $\tau_u(t_i-)$ and $\tau_u(t_i+)$ in $\mathbb{S}_0^1$, we can define u in such a way that the series $\sum_i d_i$ is convergent. Finally, denoting by $\Gamma_{t_i}^p$ a minimum point for $\mathcal{E}_p^0$ in the class $\mathcal{F}(u,t_i)$, see Definition 5.4, we can define u in such a way that also

$$\mathcal{E}_p^0(\Gamma_{t_i}^p)\le c_p\,(|u(t_i+)-u(t_i-)|+d_i)\qquad\forall\,i$$

for some constant $c_p>0$ not depending on i. By our relaxation result, Proposition 9.2 below, we have

$$\overline{\mathcal{E}}_p(u)=\int_I|\dot c_u|^{1-p}\left(|\dot c_u|^p+|\dot\tau_u|^p\right)dt+\sum_{i=1}^\infty\mathcal{E}_p^0(\Gamma_{t_i}^p)<\infty$$

and hence $u\in\mathcal{E}_p^0(I,\mathbb{R}^2)$, but $\mathcal{H}^0(J_u)=+\infty$. We omit any further detail.

7 Energy upper bound

Let $p > 1$ and $N \geq 1$. Recalling that the optimal current Σ_u^p is defined by (5.2), in this section we prove the following density result.

Theorem 7.1 (Energy upper bound). *For every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$, there exists a sequence of smooth functions $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ strongly in L^1, $GG_{u_h} \rightharpoonup \Sigma_u^p$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ and $\mathcal{E}_p(u_h) \to \mathcal{E}_p^0(\Sigma_u^p)$ as $h \to \infty$.*

CONVOLUTION ARGUMENT. As a preliminary step, we report here a density argument from Step 1 in [1, Thm. 8.1], readapted to the case $p > 1$.

Proposition 7.2 *Let $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ be a Sobolev function in $W^{1,1}(I, \mathbb{R}^N)$, with $\dot{u} \in L^\infty(I, \mathbb{R}^N)$. Then $\tau_u \in W^{1,p}(I, \mathbb{S}^N)$. Moreover, there exists a smooth sequence $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ strongly in $W^{1,1}$ and*

$$\lim_{h\to\infty} \int_I |\dot{c}_{u_h}|^{1-p} \left(|\dot{c}_{u_h}|^p + |\dot{\tau}_{u_h}|^p \right) dt = \int_I |\dot{c}_u|^{1-p} \left(|\dot{c}_u|^p + |\dot{\tau}_u|^p \right) dt \,. \tag{7.1}$$

PROOF: Since $u \in \mathcal{E}_1(I, \mathbb{R}^N)$, we know that $\Phi_u = (c_u, \tau_u)$ is a BV-function, where $c_u(t) = (t, u(t))$ and

$$|\dot{c}_u| = \sqrt{1 + |\dot{u}|^2}\,, \qquad \tau_u^0 := \frac{1}{|\dot{c}_u|}\,, \qquad \tau_u^j := \frac{\dot{u}^j}{|\dot{c}_u|}\,, \qquad j = 1, \ldots, N\,.$$

Moreover, if $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ is such that $u_h \to u$ in L^1 and $\sup_h \mathcal{E}_p(u_h) < \infty$, since $\sup_h \mathcal{E}_1(u_h) < \infty$, in [1] we showed that (possibly passing to a subsequence) $\dot{u}_h \to \dot{u}$ a.e. in I. If $u \in W^{1,\infty}(I, \mathbb{R}^N)$, we may and do assume that $\sup_h(\|u_h\|_\infty + \|\dot{u}_h\|_\infty) < \infty$. Therefore, using that $\sup_h \|\dot{c}_{u_h}\|_\infty < \infty$ and $\sup_h \int_I |\dot{c}_{u_h}|^{1-p} |\dot{\tau}_{u_h}|^p \, dt < \infty$, we deduce that $\sup_h \int_I |\dot{\tau}_{u_h}|^p \, dt < \infty$. Therefore, possibly passing to a subsequence we conclude that $\tau_{u_h} \rightharpoonup \tau_u$ weakly in $W^{1,p}$ and hence $\tau_u \in W^{1,p}(I, \mathbb{S}^N)$.

For each $j \geq 1$ we choose a sequence $\{v_h^j\} \subset C^\infty(I)$ that converges strongly in $W^{1,p}$ to τ_u^j. Since moreover $\|\dot{u}\|_\infty < \infty$, we may assume that $\|v_h^j\|_\infty \leq \|\tau_u^j\|_\infty < 1$ for each h and j. Denoting $v_h = (v_h^1, \ldots, v_h^N)$, and $v_h^0 := \sqrt{1 - |v_h|^2}$, we compute

$$\dot{v}_h^0 = -\frac{v_h \bullet \dot{v}_h}{\sqrt{1 - |v_h|^2}}\,.$$

Therefore, by dominated convergence we deduce that the sequence $\{v_h^0\} \subset C^\infty(I)$ converges strongly in $W^{1,p}$ to τ_u^0. Setting then for $j = 1, \ldots, N$

$$w_h^j(t) := \frac{v_h^j(t)}{\sqrt{1 - |v_h(t)|^2}}\,, \qquad u_h^j(t) := u^j(a) + \int_a^t w_h^j(s)\, ds\,, \qquad t \in I$$

we now check the following convergences as $h \to \infty$:

i) $w_h^j \to \dot{u}^j$ strongly in L^1, for each j;

ii) $u_h^j(t) \to u^j(a) + \int_a^t \dot{u}^j(s)\, ds = u^j(t)$ a.e. and strongly in $L^1(I)$;

iii) $\int_I \sqrt{1 + |\dot{u}_h|^2}\, dt \to \int_I \sqrt{1 + |\dot{u}|^2}\, dt$, hence $u_h \to u$ in $W^{1,1}(I, \mathbb{R}^N)$;

iv) $\int_I \left(|\dot{v}_h|^2 + (\dot{v}_h^0)^2 \right)^{p/2} dt \to \int_I |\dot{\tau}_u|^p \, dt$.

Differently to the cited case $p = 1$, we point out that property iv) holds true as the sequence $(v_h^0, v_h) : I \to \mathbb{S}^N$ converges to τ_u strongly in $W^{1,p}$. In Step 1 from [1, Thm. 8.1], we also computed

$$|\dot{\tau}_{u_h}| = |\dot{c}_{u_h}|\, k_{u_h} = \left\{ |\dot{v}_h|^2 + \frac{(v_h \bullet \dot{v}_h)^2}{1 - |v_h|^2} \right\}^{1/2} = \sqrt{|\dot{v}_h|^2 + (\dot{v}_h^0)^2}\,.$$

By property iv) this implies that

$$\lim_{h\to\infty}\int_I |\dot\tau_{u_h}|^p\,dt = \int_I |\dot\tau_u|^p\,dt\,.$$

Using that $\sup_h \|\dot c_{u_h}\|_\infty^{1-p}<\infty$, by dominated convergence we get

$$\lim_{h\to\infty}\int_I |\dot c_{u_h}|^{1-p}|\dot\tau_{u_h}|^p\,dt = \int_I |\dot c_u|^{1-p}|\dot\tau_u|^p\,dt$$

and the claim follows. □

Remark 7.3 If $u\in\mathcal{E}_p(I,\mathbb{R}^N)\cap W^{1,\infty}$, then $\Sigma_u^p = GG_u^a$, whence $\mathcal{E}_p^0(\Sigma_u^p)=\mathcal{E}_p^0(GG_u^a)$. Moreover, arguing as in the smooth case, compare (3.1), we deduce that

$$\mathcal{E}_p^0(GG_u^a) = \int_I |\dot c_u|^{1-p}\left(|\dot c_u|^p + |\dot\tau_u|^p\right)dt \qquad \forall\, u\in\mathcal{E}_p(I,\mathbb{R}^N)\,. \tag{7.2}$$

Therefore, Proposition 7.2 yields the validity of Theorem 7.1 for the subclass of Lipschitz functions u in $W^{1,\infty}(I,\mathbb{R}^N)$.

Proof of Theorem 7.1: According to the previous remark, by a diagonal argument it suffices to find an approximating sequence $\{u_h\}$ in $W^{1,\infty}(I,\mathbb{R}^N)$.

Let $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ and write $\Sigma_u^p = [\![\mathcal{M},\theta,\xi]\!]$, so that the multiplicity $\theta\equiv 1$. By Corollary 5.7 we have $\mathcal{E}_p^0(\Sigma_u^p)<\infty$. On account of (4.4), Definition 5.4, and (3.2), we deduce that for $\mathcal{H}^1$-a.e. $z\in\mathcal{M}$ the following geometric property holds:

$$\frac{\xi^{(x)}}{|\xi^{(x)}|}(z) = y\,,\quad z=(x,y)\in U\times\mathbb{S}^N\,.$$

Let $T_u^p := \Pi_{x\#}\Sigma_u^p$, so that $T_u^p\in\mathrm{cart}(\mathring{I}\times\mathbb{R}^N)$. By Federer's structure theorem [6, 4.2.25], we find a Lipschitz and one-to-one function $\gamma: I_L\to\overline{U}$, where $I_L:=[0,L]$, such that $\gamma_\#[\![I_L]\!] = T_u^p$ and $|\dot\gamma(s)|=1$ a.e. in I_L, so that by the area formula

$$L = \mathcal{L}(\gamma) = \mathbf{M}(T_u^p)\,.$$

The above geometric property says that (up to $\mathcal{H}^1$-null sets) there is a one-to-one correspondence between parameters $s\in I_L$ and points $z\in\mathcal{M}$ such that

$$\dot\gamma(s) = \frac{\xi^{(x)}}{|\xi^{(x)}|}(\gamma(s)) = \Pi_y(z) \quad\text{if}\quad \Pi_x(z)=\gamma(s)\,.$$

In particular, we have $\Sigma_u^p = (\gamma,\dot\gamma)_\#[\![I_L]\!]$ and hence in the above correspondence

$$\xi(z) = \frac{(\dot\gamma(s),\ddot\gamma(s))}{|(\dot\gamma(s),\ddot\gamma(s))|}\,,\quad \xi^{(x)}(z) = \frac{\dot\gamma(s)}{|(\dot\gamma(s),\ddot\gamma(s))|}\,,\quad \xi^{(y)}(z) = \frac{\ddot\gamma(s)}{|(\dot\gamma(s),\ddot\gamma(s))|}\,.$$

By (3.2) and the area formula we thus have

$$\begin{aligned}\mathcal{E}_p^0(\Sigma_u^p) &:= \int_{\mathcal{M}} F_p(\xi)\,d\mathcal{H}^1 = \int_{\mathcal{M}}\left(|\xi^{(x)}| + |\xi^{(x)}|^{1-p}|\xi^{(y)}|^p\right)d\mathcal{H}^1\\ &= \int_{I_L} |(\dot\gamma,\ddot\gamma)|\Big(|(\dot\gamma,\ddot\gamma)|^{-1} + |(\dot\gamma,\ddot\gamma)|^{p-1}\frac{|\ddot\gamma|^p}{|(\dot\gamma,\ddot\gamma)|^p}\Big)\,ds\\ &= \int_{I_L}(1+|\ddot\gamma(s)|^p)\,ds\end{aligned} \tag{7.3}$$

so that

$$\mathcal{L}(\gamma) + \int_{I_L}|\ddot\gamma(s)|^p\,ds = \mathcal{E}_p^0(\Sigma_u^p) < \infty\,.$$

Therefore, it turns out that *the arc length parameterization* γ *is a Lipschitz function in* $W^{2,p}(I_L,\overline{U})$, whence *the tantrix* $\mathfrak{t}_\gamma$ *of the curve* γ *is a Sobolev map in* $W^{1,p}(I_L,\mathbb{S}^N)$.

Since T_u^p is a Cartesian current, we know that the first component $\gamma^0 : I_L \to I$ is a Lipschitz-continuous and surjective function with $\dot\gamma^0(s) \geq 0$ a.e., whence γ^0 is non-decreasing.

We now wish to modify for $\varepsilon > 0$ small the curve γ is such a way that the new curve $\gamma_\varepsilon : I_L \to \mathbb{R}^{N+1}$ satisfies the following properties:

a) γ_ε is Lipschitz continuous with $\dot\gamma_\varepsilon \in W^{1,p}$ and $\|\dot\gamma_\varepsilon\|_\infty \leq 2$;

b) the first component $\psi_\varepsilon := \gamma_\varepsilon^0$ satisfies $\dot\psi_\varepsilon \geq \varepsilon$ a.e. in I_L;

c) the length of γ_ε converges to the length of γ as $\varepsilon \searrow 0$;

d) the support of the curve γ_ε is the graph of a Lipschitz function $v_\varepsilon : I_\varepsilon \to \mathbb{R}^N$ such that $\dot v_\varepsilon$ is differentiable a.e., where $I_\varepsilon := [a, b + L\varepsilon]$ if $I = [a,b]$;

e) following the notation in (2.3), we have
$$\lim_{\varepsilon\to 0}\int_{I_\varepsilon} |\dot c_{v_\varepsilon}|^{1-p}\left(|\dot c_{v_\varepsilon}|^p + |\dot\tau_{v_\varepsilon}|^p\right)d\lambda = \int_{I_L}(1+|\ddot\gamma(s)|^p)\,ds\,.$$

For this purpose, we denote $\gamma = (\gamma^0, \gamma^{\overline{0}})$, so that $\gamma^{\overline{0}} = (\gamma^1,\dots,\gamma^N)$, and define
$$\psi_\varepsilon(s) := \gamma^0(s) + \varepsilon\, s\,, \qquad \gamma_\varepsilon(s) := (\psi_\varepsilon(s), \gamma^{\overline{0}}(s))\,, \quad s \in I_L\,.$$
Using that $\dot\gamma^0 \geq 0$ and $|\dot\gamma| = 1$ a.e., we have $\varepsilon \leq \dot\psi_\varepsilon(s) \leq 1+\varepsilon$ for a.e. $s \in I_L$. Therefore, the inverse function $\phi_\varepsilon := {\psi_\varepsilon}^{-1} : I_\varepsilon \to I_L$ satisfies $(1+\varepsilon)^{-1} \leq \dot\phi_\varepsilon(\lambda) \leq \varepsilon^{-1}$ for a.e. $\lambda \in I_\varepsilon$. We thus define $v_\varepsilon(\lambda) := (v_\varepsilon^1(\lambda),\dots,v_\varepsilon^N(\lambda))$, where
$$v_\varepsilon^j(\lambda) := \gamma^j(\phi_\varepsilon(\lambda))\,, \quad \lambda \in I_\varepsilon\,, \quad j = 1,\dots,N\,.$$
It is readily checked that for a.e. $\lambda \in I_\varepsilon$
$$\dot c_{v_\varepsilon}(\lambda) = \frac{1}{\dot\psi_\varepsilon(s)}\big(\dot\psi_\varepsilon(s), \dot\gamma^{\overline{0}}(s)\big) = \frac{\dot\gamma_\varepsilon(s)}{\dot\psi_\varepsilon(s)}\,, \qquad \tau_{v_\varepsilon}(\lambda) = \frac{\big(\dot\psi_\varepsilon(s), \dot\gamma^{\overline{0}}(s)\big)}{|\big(\dot\psi_\varepsilon(s), \dot\gamma^{\overline{0}}(s)\big)|} = \frac{\dot\gamma_\varepsilon(s)}{|\dot\gamma_\varepsilon(s)|}$$
where $s = \phi_\varepsilon(\lambda)$ and, we recall, $\dot\psi_\varepsilon(s) = \dot\gamma^0(s) + \varepsilon$. In particular v_ε is Lipschitz-continuous. By changing variable $\lambda = \psi_\varepsilon(s)$, we also have:
$$\int_{I_\varepsilon} |\dot c_{v_\varepsilon}(\lambda)|\,d\lambda = \int_{I_L} |\big(\varepsilon + \dot\gamma^0(s), \dot\gamma^{\overline{0}}(s)\big)|\,ds = \mathcal{L}(\gamma_\varepsilon)$$
so that by dominated convergence
$$\lim_{\varepsilon\to 0}\mathcal{L}(\gamma_\varepsilon) = \int_{I_L} |\dot\gamma(s)|\,ds = \mathcal{L}(\gamma)\,.$$
Using that
$$|\dot\tau_{v_\varepsilon}(\lambda)| = \left|\frac{d}{ds}\frac{\dot\gamma_\varepsilon(s)}{|\dot\gamma_\varepsilon(s)|}\right| \cdot \Phi'(\lambda) = \frac{|\dot\gamma_\varepsilon(s) \wedge \ddot\gamma_\varepsilon(s)|}{|\dot\gamma_\varepsilon(s)|^3} \cdot \frac{1}{\dot\psi_\varepsilon(s)}$$
the same change of variable yields that
$$\int_{I_\varepsilon} |\dot c_{v_\varepsilon}|^{1-p}\,|\dot\tau_{v_\varepsilon}|^p\,d\lambda = \int_{I_L} |\dot\gamma_\varepsilon(s)|^{1-3p}\,|\dot\gamma_\varepsilon(s) \wedge \ddot\gamma_\varepsilon(s)|^p\,ds =: \mathcal{F}_\varepsilon\,.$$
Since $\ddot\gamma_\varepsilon(s) = \ddot\gamma(s)$, we have
$$\dot\gamma_\varepsilon(s) \wedge \ddot\gamma_\varepsilon(s) = \dot\gamma(s) \wedge \ddot\gamma(s) + (\varepsilon, 0_{\mathbb{R}^N}) \wedge \ddot\gamma(s)\,.$$

Also, condition $|\dot\gamma|=1$ gives that $\dot\gamma(s)\bullet\ddot\gamma(s)=0$ a.e., whence

$$|\dot\gamma_\varepsilon(s)\wedge\ddot\gamma_\varepsilon(s)|^2=(1+\varepsilon^2)|\ddot\gamma(s)|^2+2\varepsilon\,\dot\gamma^0(s)\,|\ddot\gamma(s)|^2-\varepsilon^2\dot\gamma^0(s)^2$$

whereas

$$|\dot\gamma_\varepsilon(s)|^2=|\dot\gamma(s)|^2+\varepsilon^2+2\varepsilon\,\dot\gamma^0(s)\,.$$

We thus have $\mathcal{F}_\varepsilon=\int_{I_L}f_\varepsilon(s)\,ds$, where

$$f_\varepsilon(s):=\Big(|\dot\gamma(s)|^2+\varepsilon^2+2\varepsilon\,\dot\gamma^0(s)\Big)^{(1-3p)/2}\Big((1+\varepsilon^2)|\ddot\gamma(s)|^2+2\varepsilon\,\dot\gamma^0(s)\,|\ddot\gamma(s)|^2-\varepsilon^2\dot\gamma^0(s)^2\Big)^{p/2}\,.$$

For $\varepsilon>0$ small we have $f_\varepsilon(s)\le 2|\ddot\gamma(s)|^p$ a.e. on I_L. Since $|\ddot\gamma(s)|\in L^p(I_L)$, by dominated convergence one has $\int_{I_L}f_\varepsilon(s)\,ds\to\int_{I_L}|\ddot\gamma(s)|^p\,ds$, whence the limit in e) readily follows.

The only minor fault of v_ε is that it is defined in $I_\varepsilon=[a,b+L\varepsilon]$ instead of $I=[a,b]$. Denoting by g_ε the affine and increasing function mapping I onto I_ε, the function $u_\varepsilon=v_\varepsilon\circ g_\varepsilon$ does the trick. In fact, clearly $\mathcal{E}_p(u_\varepsilon)\to\mathcal{E}^0_p(\Sigma^p_u)$, whereas the *flat convergence* of $(\gamma_\varepsilon,\dot\gamma_\varepsilon)_\#[\![I_L]\!]$ to $(\gamma,\dot\gamma)_\#[\![I_L]\!]$ yields the weak convergence of GG_{u_ε} to Σ^p_u, along a sequence $\varepsilon_h\searrow 0$. □

8 The Gauss map

In this section we prove that the Gauss map τ_u of a function $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ is a BV-function with no Cantor part, Theorem 8.1. If in particular u is continuous and with no corner points, then τ_u is a Sobolev function, Remark 8.3.

We first recall that in the proof of Theorem 7.1 we defined for each $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ a Lipschitz function $\gamma:I_L\to\overline{U}$ satisfying the following properties:

i) $|\dot\gamma(s)|=1$ with $\dot\gamma^0(s)\ge 0$ for a.e. $s\in I_L$;

ii) $\dot\gamma\in W^{1,p}(I,\mathbb{S}^N)$ and $(\dot\gamma,\ddot\gamma)_\#[\![I_L]\!]=\Sigma^p_u$;

iii) $\mathcal{E}^0_p(\Sigma^p_u)=\mathcal{L}(\gamma)+\displaystyle\int_L|\ddot\gamma(s)|^p\,ds$;

iv) since $\gamma_\#[\![I_L]\!]=\Pi_{x\#}\Sigma^p_u\in\text{cart}(\mathring{I}\times\mathbb{R}^N)$, and τ_u is the orienting vector at the points in T^a_u, we have

$$\dot\gamma(s)=\tau_u(\gamma^0(s))\quad\text{for }\mathcal{L}^1\text{-a.e. }s\in\widetilde{I}_L \tag{8.1}$$

where $\widetilde{I}_L$ denotes the open set

$$\widetilde{I}_L:=\{s\in\mathring{I}_L\mid\dot\gamma^0(s)>0\}\,. \tag{8.2}$$

We already know that $c_u\in\text{BV}(I,\mathbb{R}^N)$. Since $u\in\mathcal{E}_1(I,\mathbb{R}^N)$, in [1, Thm. 4.7] we proved that also the Gauss map $\tau_u:I\to\mathbb{S}^N$ is a function with bounded variation. We now show that τ_u is a *special function with bounded variation.*

Theorem 8.1 *Let $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ for some $p>1$. Then the Gauss map τ_u has no Cantor part, i.e. $D^C\tau_u=0$.*

Proof: By changing variable $t=\gamma^0(s)$, for every test function $\varphi\in C^\infty_c(I,\mathbb{R}^{N+1})$ we have

$$\langle D\tau_u,\varphi\rangle:=-\int_I\tau_u(t)\bullet\dot\varphi(t)\,dt=-\int_{I_L}\tau_u(\gamma^0(s))\bullet\dot\varphi(\gamma^0(s))\,\dot\gamma^0(s)\,ds$$

and hence, denoting $\phi(s):=\varphi\circ\gamma^0(s)$, and recalling that $\phi\in W^{2,p}$, by (8.1) and (8.2) we have

$$\langle D\tau_u,\varphi\rangle=-\int_{\widetilde{I}_L}\dot\gamma(s)\bullet\dot\phi(s)\,ds\,.$$

Moreover, using that

$$\ddot{\gamma}(s) = \dot{\tau}_u(\gamma^0(s))\,\dot{\gamma}^0(s) \qquad \text{for a.e. } s \in \widetilde{I}_L \tag{8.3}$$

the absolute continuous component is

$$\langle D^a\tau_u, \varphi\rangle := \int_I \dot{\tau}_u(t) \bullet \varphi(t)\,dt = \int_{\widetilde{I}_L} \ddot{\gamma}(s) \bullet \phi(s)\,ds\,.$$

Therefore, the singular part writes as

$$\langle D^s\tau_u, \varphi\rangle := \langle D\tau_u, \varphi\rangle - \langle D^a\tau_u, \varphi\rangle = -\int_{\widetilde{I}_L} \big(\dot{\gamma}(s) \bullet \dot{\phi}(s) + \ddot{\gamma}(s) \bullet \phi(s)\big)\,ds\,.$$

The open set $\widetilde{I}_L$ is the union of an at most countable disjoint family of open intervals $\{I_l\}$, whence

$$\langle D^s\tau_u, \varphi\rangle = -\sum_{l=1}^{\infty} \int_{I_l} \big(\dot{\gamma}(s) \bullet \dot{\phi}(s) + \ddot{\gamma}(s) \bullet \phi(s)\big)\,ds\,.$$

Since the function $s \mapsto \dot{\gamma}(s) \bullet \phi(s)$ is $W^{1,p}$ with derivative $\big(\dot{\gamma}(s) \bullet \dot{\phi}(s) + \ddot{\gamma}(s) \bullet \phi(s)\big)$, denoting $I_l =]a_l, b_l[$ and integrating by parts we have

$$\langle D^s\tau_u, \varphi\rangle = \sum_{l=1}^{\infty} \big(\dot{\gamma}(a_l) \bullet \phi(a_l) - \dot{\gamma}(b_l) \bullet \phi(b_l)\big)\,.$$

This implies that the singular part $D^s\tau_u$ is concentrated on a countable set, whence $D^C\tau_u = 0$, as required. $\square$

Remark 8.2 Recalling from Sec. 4 the definition iii) of Cantor component GG_u^C, we deduce that for every $g \in C_c^\infty(U \times \mathbb{S}^N)$

$$\langle GG_u^C, g(x,y)\,dy^j\rangle := \int_I g(\Phi_{u+})\,dD^C\tau_u^j = 0 \qquad \forall\, j = 0, 1, \ldots, N\,.$$

As a consequence, writing as usual $GG_u^C = [\![\mathcal{M}, 1, \xi]\!]$, the tangent unit vector ξ has zero y-component $\xi^{(y)} := \Pi_y(\xi)$ at $\mathcal{H}^1 \llcorner \mathcal{M}$-a.e. point. By (3.2) and Definition 3.4, we thus deduce that

$$\mathcal{E}_p^0(GG_u^C) = \int_{\mathcal{M}} |\xi^{(x)}|\,d\mathcal{H}^1 = \mathbf{M}(T_u^C) = |D^C u|(I) \tag{8.4}$$

where $T_u^C = \Pi_{x\#} GG_u^C$.

Remark 8.3 In the case of continuous functions u with finite relaxed energy and no corner points, Theorem 8.1 yields that τ_u is a Sobolev function in $W^{1,1}(I, \mathbb{S}^N)$.

Notice moreover that Theorem 8.1 is false for $p = 1$, even in codimension $N = 1$. In fact, if $I = [0,1]$ and $u(t) := \int_0^t v(s)\,ds$, where $v : I \to \mathbb{R}$ is the classical Cantor-Vitali function associated to the "middle thirds" Cantor set, one has $u \in C^1(I, \mathbb{R})$, the Gauss map $\tau_u \in \mathrm{BV}(I, \mathbb{S}^1)$ is continuous, whence $J_{\Phi_u} = \emptyset$, but $D^C\tau_u \neq 0$, as $D^C\tau_u = \dot{f}(v)\,D^C v$ with $f(t) = (1,t)/\sqrt{1+t^2}$, whereas $\dot{\tau}_u = \dot{f}(v)\,\dot{v} = 0$. Following [1] or also [5], and denoting by $\mathrm{TC}(c_u)$ the *total curvature* of the Cartesian curve c_u, one has

$$\overline{\mathcal{E}}_1(u) = \mathcal{L}^1(c_u) + \mathrm{TC}(c_u) < \infty\,, \quad \mathcal{L}^1(c_u) = \int_I \sqrt{1+v^2}\,dt\,, \quad \mathrm{TC}(c_u) = |D^C\tau_u|(I) = \frac{\pi}{4}\,.$$

On the other hand, the arc-length parameterization $c : I_L \to I \times \mathbb{R}$ is $c(s) = (t(s), u(t(s)))$, where $t(s)$ is the inverse of the function $s(t) := \int_0^t \sqrt{1+v^2(\lambda)}\,d\lambda$, whence $\widetilde{I}_L = \mathring{I}_L$ in definition (8.2), and also $\dot{\gamma}(s) = \dot{t}(s)(1, v(t(s)))$, with $\dot{t}(s) = (1+v^2(t(s)))^{-1/2}$. Therefore, the function $\gamma(s)$ is not $W^{1,1}$ and hence the argument in the proof of Theorem 8.1 fails to hold, as $\dot{\gamma} \bullet \dot{\phi} + \ddot{\gamma} \bullet \phi$ is only the approximate gradient and not the full derivative of $\dot{\gamma} \bullet \phi$. In particular we deduce that the function u satisfies $\overline{\mathcal{E}}_p(u) = +\infty$ for every $p > 1$.

9 Main results

In this final section we collect our main results. Proposition 9.2 deals with the general case and corresponds to Proposition 5.1 for the case $p = 1$. We then restrict to the subclass of continuous functions with no corner points. In this case we are able to give a geometric meaning to the energy term $\mathcal{E}_p^0(GG_u^a)$, Corollary 9.7. In fact, we see that such an energy term agrees with the *p-curvature functional*, see Definition 9.4 and Proposition 9.6. Finally, for continuous functions with finite relaxed energy, we introduce a suitable *generalized p-curvature* functional, Definition 9.8, that comes into play in the explicit formula of the relaxed energy, Corollary 9.9.

THE CLASS Gcart_u^p. We first recall that Gcart_u denotes the subclass of currents in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ such that $u_\Sigma = u$, see (4.1) and (5.1).

Definition 9.1 *For any function $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we denote by Gcart_u^p the class of currents Σ in Gcart_u such that $\mathcal{E}_p^0(\Sigma) < \infty$.*

Proposition 9.2 *Let $p > 1$ and $u \in L^1(I, \mathbb{R}^N)$. Then*

$$u \in \mathcal{E}_p(I, \mathbb{R}^N) \iff \mathrm{Gcart}_u^p \neq \emptyset \,.$$

Moreover, if $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we have

$$\overline{\mathcal{E}}_p(u) = \min\{\mathcal{E}_p^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u^p\}$$

and the Gauss map τ_u is a function of bounded variation in $\mathrm{BV}(I, \mathbb{S}^N)$ whose derivative has no Cantor part, i.e., $D^C \tau_u = 0$. More explicitly, for every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we have

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot{c}_u|^{1-p} \left(|\dot{c}_u|^p + |\dot{\tau}_u|^p \right) dt + |D^C u|(I) + \sum_{t \in J_{\Phi_u}} \mathcal{E}_p^0(\Gamma_t^p)$$

where Γ_t^p denotes for every $t \in J_{\Phi_u}$ a minimum point for $\mathcal{E}_p^0$ in the class $\mathcal{F}(u, t)$, see Definition 5.4.

PROOF: The first equivalence follows from our definitions. If $u \in \mathcal{E}_p(I, \mathbb{R}^N)$, the lower bound "$\geq$" is trivial, whereas the upper bound is a consequence of Theorem 7.1. More precisely, the density theorem implies that the minimal current Σ_u^p defined in (5.2) belongs to the class Gcart_u^p. We thus have:

$$\forall\, u \in \mathcal{E}_p(I, \mathbb{R}^N)\,, \qquad \overline{\mathcal{E}}_p(u) = \mathcal{E}_p^0(\Sigma_u^p)\,.$$

Finally, Theorem 8.1 says that $D^C \tau_u = 0$, and the explicit formula follows from the structure (5.7) of the energy $\mathcal{E}_p^0(\Sigma_u^p)$, on account of (7.2) and (8.4). □

Remark 9.3 The lack of precise knowledge of the minimal arcs Γ_t^p prevents further explicitation of the relaxed energy $\overline{\mathcal{E}}_p(u)$, which is instead possible when u is continuous either if $N = 1$, see Remark 9.10, or (in high codimension) if we assume that u has no corner points, so that no such arcs appear, compare Corollary 9.7 below.

p-CURVATURE FUNCTIONAL. Assume now that $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ is continuous and with no corner points, i.e. $J_{\Phi_u} = \emptyset$. Consider the Lipschitz parameterization $\gamma : I_L \to \overline{U}$ of c_u defined in the proof of Theorem 7.1. Since γ is in $W^{2,p}$, we define the curvature of c_u at the point $\gamma(s)$ by

$$\mathbf{k}_{c_u}(\gamma(s)) := \frac{\mathrm{area}[\dot{\gamma}(s), \ddot{\gamma}(s)]}{|\dot{\gamma}(s)|^3}\,, \qquad s \in I_L\,.$$

Definition 9.4 *Let $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ be such that $J_{\Phi_u} = \emptyset$. The p-*curvature functional *is defined by*

$$\mathrm{TC}_p(c_u) := \int_{c_u} \mathbf{k}_{c_u}^p \, d\mathcal{H}^1\,.$$

Remark 9.5 By the area formula, using (2.3) we have

$$\mathrm{TC}_p(c_u) = \int_{I_L} |\dot\gamma(s)| \Big(\frac{\mathrm{area}[\dot\gamma(s),\ddot\gamma(s)]}{|\dot\gamma(s)|^3}\Big)^p\, ds = \int_{I_L} |\dot\gamma(s)|^{1-3p}\, |\dot\gamma(s)\wedge\ddot\gamma(s)|^p\, ds\,.$$

Since in particular $|\dot\gamma|=1$ a.e., the function $s\mapsto\gamma(s)$ being the arc-length parameterization of c_u, the vector $\ddot\gamma(s)$ is perpendicular to $\dot\gamma(s)$ and hence $|\dot\gamma(s)\wedge\ddot\gamma(s)|=|\ddot\gamma(s)|$, whence $\mathbf{k}_{c_u}(\gamma(s))=|\ddot\gamma(s)|$ for a.e. $s\in I_L$. We thus have:

$$\mathrm{TC}_p(c_u) = \int_{I_L} |\ddot\gamma(s)|^p\, ds\,.$$

The above definition is motivated by the following:

Proposition 9.6 *If $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ is continuous and with no corner points, we have*

$$\mathrm{TC}_p(c_u) = \int_I |\dot c_u|^{1-p}\, |\dot\tau_u|^p\, dt\,.$$

Proof: By (8.1) and (8.2) we deduce that $\dot\gamma^0(s)=0$ if $s\in I_L\setminus\widetilde I_L$, where $|I_L\setminus\widetilde I_L|=0$, the function u being continuous, and $\dot\gamma^0(s)=|\dot c_u(t)|^{-1}$ if $s\in\widetilde I_L$, with $t=\gamma^0(s)$. By changing variable $t=\gamma^0(s)$, so that $\dot\gamma^0(s)\,ds=dt$, and using (8.3) we can write

$$\int_{I_L}|\ddot\gamma(s)|^p\,ds = \int_{\widetilde I_L}|\ddot\gamma(s)|^p\,ds = \int_I \dot\gamma^0(s(t))^{p-1}|\dot\tau_u(t)|^p\,dt$$

where $s(t)=(\gamma^0)^{-1}(t)$, so that $\dot\gamma^0(s(t))^{p-1}=|\dot c_u(t)|^{1-p}$ for a.e. $t\in I$, and hence

$$\int_I |\dot c_u|^{1-p}\,|\dot\tau_u|^p\,dt = \int_{I_L}|\ddot\gamma(s)|^p\,ds\,.$$

The claim follows from Remark 9.5. □

For continuous functions with no corner points, we thus have a more geometric formula:

Corollary 9.7 *Let $p>1$ and $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ such that $J_{\Phi_u}=\emptyset$. Then $\tau_u\in W^{1,1}(I,\mathbb{S}^N)$ and*

$$\overline{\mathcal{E}}_p(u) = \mathcal{L}(c_u) + \mathrm{TC}_p(c_u)$$

where $\mathcal{L}(c_u)$ is the length of the Cartesian curve c_u and $\mathrm{TC}_p(c_u)$ is the p-curvature functional of u, see Definition 9.4.

Proof: We already know that τ_u is a Sobolev function, see Remark 8.3. Moreover, using that $J_{\Phi_u}=\emptyset$, in Proposition 9.2 we have:

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot c_u|^{1-p}\big(|\dot c_u|^p + |\dot\tau_u|^p\big)\,dt + |D^C u|(I)$$

and hence

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot c_u|\,dt + |D^C c_u|(I) + \int_I |\dot c_u|^{1-p}|\dot\tau_u|^p\,dt\,.$$

Since u is continuous, one has

$$\mathcal{L}(c_u) = \int_I |\dot c_u|\,dt + |D^C c_u|(I)$$

whereas the third term agrees with $\mathrm{TC}_p(c_u)$, by Proposition 9.6. □

Generalized p-curvature functional. We assume now more generally that $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ is only continuous. By Theorem 6.3 we deduced that the set $J_{\dot u}$ of corner points is finite. Motivated by Proposition 9.2, we are led to extend the previous definition as follows.

Definition 9.8 *Let* $u \in \mathcal{E}_p(I,\mathbb{R}^N)$ *be continuous. The* generalized p-curvature functional *is defined by*

$$\mathrm{TC}_p(c_u) := \int_I |\dot c_u|^{1-p}\,|\dot\tau_u|^p\,dt + \sum_{t\in J_{\dot u}} \mathcal{E}^0_p(\Gamma^p_t)$$

where for any corner point $t \in J_{\dot u}$ *we have denoted by* Γ^p_t *is a minimum of* $\mathcal{E}^0_p$ *in the class* $\mathcal{F}(u,t)$*, see Definition* 5.4.

In fact, we finally obtain:

Corollary 9.9 *Let* $p > 1$*. If* $u \in \mathcal{E}_p(I,\mathbb{R}^N)$ *is continuous, the set of corner points* $J_{\dot u}$ *is finite and*

$$\overline{\mathcal{E}}_p(u) = \mathcal{L}(c_u) + \mathrm{TC}_p(c_u)$$

where $\mathcal{L}(c_u)$ *is the length of the Cartesian curve* c_u *and* $\mathrm{TC}_p(c_u)$ *is given by Definition* 9.8.

Proof: This time in Proposition 9.2 we have

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot c_u|\,dt + |D^C c_u|(I) + \int_I |\dot c_u|^{1-p}|\dot\tau_u|^p\,dt + \sum_{t\in J_{\dot u}} \mathcal{E}^0_p(\Gamma^p_t)$$

where the sum in $J_{\dot u}$ is finite, as a consequence of Theorem 6.3. □

Remark 9.10 Finally, differently to the case $N \geq 2$, see Example 6.2, we recall that in codimension $N = 1$ no corner points appear when $p > 1$, as already observed in [5].

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Dipartimento di Matematica e Informatica, Università di Parma,
Parco Area delle Scienze 53/A, I-43124 Parma, Italy.
E-mail: emilio.acerbi@unipr.it, domenico.mucci@unipr.it