

Rectifiability of non Euclidean planar self-contracted curves

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Abstract

We prove that any self-contracted curve in \mathbb{R}^2 endowed with a C^2 and strictly convex norm, has finite length. The proof follows from the study of the curve bisector of two points in \mathbb{R}^2 for a general norm together with an adaptation of the argument used in [2].

1 Introduction

The concept of self-contracted curve was first introduced by Daniilidis, Ley and Sabourau in [5, Definition 1.2]. For a given metric space (E, d) and a possibly unbounded interval I of \mathbb{R} , a map $\gamma : I \rightarrow E$ is called a self-contracted curve, if for every $[a, b] \subset I$, the real-valued function

$$t \mapsto d(\gamma(t), \gamma(b))$$

is non-increasing on $[a, b]$. Sometimes to emphasis on the used distance we will also say d -self-contracted curve or $\|\cdot\|$ -self-contracted curve when the distance comes from a norm.

The origin of this definition comes from the fact that, in the Euclidean space $(\mathbb{R}^n, |\cdot|)$, any solution of the gradient descent of a proper convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, i.e. a solution γ of the gradient system

$$\begin{cases} \gamma'(t) = -\nabla f(\gamma(t)) & t > 0, \\ \gamma(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (1.1)$$

is a self-contracted curve (actually it is enough for f to be quasiconvex, that is, its sublevel sets to be convex).

One of the main question about solutions of (1.1), is whether or not bounded solutions are of finite length. If f is analytic (and not necessarily convex) then

it follows from the famous Lojasiewicz inequality [6], while it fails for general C^∞ functions [8, p. 12]. Now for a general convex function, the Lojasiewicz inequality does not need to hold (see [1, Section 4.3]). However, bounded solutions have finite length.

The latter follows from the main result of [5] (see also [7]), which focuses on the following more general question, which is purely metric:

$$\textit{Does any bounded self-contracted curve have finite length?} \quad (1.2)$$

In [5] it is proved that the answer is yes in a two dimensional Euclidean space. It was then established in higher dimensions in [2], still in the Euclidean setting. However, it is no more true in a general infinite dimensional Hilbert space.

A natural question is whether the answer to (1.2) remains yes in a finite dimensional space, when the metric space is not Euclidean anymore. In the recent paper [3], it is proved that it holds true on a Riemannian manifold.

In this paper we prove that the answer to (1.2) is yes in \mathbb{R}^2 endowed with a strictly convex C^2 norm.

Theorem 1.1. *Let $\|\cdot\|$ be a C^2 and strictly convex norm on \mathbb{R}^2 , let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \rightarrow \mathbb{R}^2$ be a $\|\cdot\|$ -self-contracted curve. Then there exists a constant $C > 0$ depending only on $\|\cdot\|$ such that*

$$\ell(\gamma) \leq C \text{diam}(K(\gamma)) \quad (1.3)$$

where $\ell(\gamma)$ is the length of the curve, $K(\gamma)$ is the closed convex hull of the support of the curve γ , and $\text{diam}(K)$ is the diameter of K . In particular, any bounded $\|\cdot\|$ -self-contracted curve has finite length.

Notice that to establish the inequality (1.3), the norm which is used to compute $\ell(\gamma)$ or $\text{diam}(K)$ does not really matter, since all the norms are equivalent on \mathbb{R}^2 . Actually, we will establish (1.3) with $\ell(\gamma)$ and $\text{diam}(K)$ computed using the Euclidean norm on \mathbb{R}^2 , denoted $|\cdot|$. More precisely, our strategy is to reproduce the Euclidean argument used in [2], in the simple case of dimension 2, and prove that if γ is no more $|\cdot|$ -self-contracted but merely $\|\cdot\|$ -self-contracted, then the proof still works almost by the same way.

However, there are some notable differences. The main one concerns the starting point of the proof, which consists, for any $t_0 < t_1$, of localizing the curve “after” t_1 on one side of the perpendicular bisector of the segment $[\gamma(t_0), \gamma(t_1)]$. This follows directly from the definition of being self-contracted. Indeed, for $t_0 < t_1 < t$, since the function $s \mapsto d(\gamma(s), \gamma(t))$ is non-decreasing on $[t_0, t]$ we have

$$d(\gamma(t_0), \gamma(t)) \leq d(\gamma(t_1), \gamma(t))$$

which means that $\gamma(t)$ is situated on one side of the bisector

$$M(\gamma(t_0), \gamma(t_1)) := \{x \in \mathbb{R}^2 : d(\gamma(t_0), x) = d(\gamma(t_1), x)\}.$$

For the Euclidean distance this yields a “separating line” for the curve after t_0 , whose direction is orthogonal to $\gamma(t_0) - \gamma(t_1)$. If the distance is not Euclidean anymore, the line segment bisector is no longer perpendicular, and not even a line anymore. However, if the distance is coming from a C^2 and strictly convex norm, we prove that the bisector is a curve which is asymptotic to a line at infinity, whose direction is well identified: in a certain sense it is a direction dual to $\gamma(t_0) - \gamma(t_1)$. Moreover, the bisector stays close enough to the middle line having the same direction. After noticing those facts, we are able to adapt the proof of [2] and this is how we prove Theorem 1.1.

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1.1 Notation and terminology

In this paper we will work on \mathbb{R}^2 endowed with the Euclidean norm $|x|$ with scalar product $\langle x, y \rangle$. The Euclidean ball with center x and radius R will be denoted by $B(x, R)$ and \mathbb{S}^{N-1} is the Euclidean unit sphere. For $v \in \mathbb{R}^2$ we will denote by v^\perp the image of v the rotation of angle $\frac{\pi}{2}$ in the anticlockwise direction.

Eventually, on this Euclidean space \mathbb{R}^2 we shall also consider another non Euclidean norm that will be denoted by $\|\cdot\|$. The associated ball will be denoted by $B_{\|\cdot\|}(x, r)$ and the sphere for the norm $\|\cdot\|$ will then be denoted by $\partial B_{\|\cdot\|}(x, r)$.

A curve is a mapping $\gamma : I \rightarrow \mathbb{R}^2$, not necessarily continuous, from some interval $I \subset \mathbb{R}$. The length of a curve is the quantity

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{m-1} |\gamma(t_i) - \gamma(t_{i+1})| \right\},$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \dots < t_m$ that lie in the interval I .

The mean width of a convex $K \subset \mathbb{R}^2$ is the quantity

$$W(K) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \mathcal{H}^1(P_u(K)) du$$

where P_u is the orthogonal projection onto the real line in \mathbb{R}^2 directed by the vector $u \in \mathbb{S}^1$, and \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure. Although we shall

not use it in this paper, let us mention the following nice identity valid for any compact and convex set $K \subset \mathbb{R}^2$,

$$W(K) = \frac{\mathcal{H}^1(\partial K)}{\pi}.$$

It is also clear from the definition that

$$W(K) \leq \text{diam}(K).$$

If $x, y \in \mathbb{R}^2$ we will denote by $[x, y] \subset \mathbb{R}^2$ the segment between x and y .

A norm $\|\cdot\|$ is said to be C^2 if $x \mapsto \|x\|$ is of class C^2 on $\mathbb{R}^2 \setminus \{0\}$. This implies that the sphere $\partial B_{\|\cdot\|}(x, r)$ is a C^2 manifold. A norm is said to be C^2 and strictly convex if, for any $x \in \mathbb{R}^2 \setminus \{0\}$ we have $D^2\|x\| > 0$ (in the sense of quadratic forms). This implies that the ball $B_{\|\cdot\|}(0, 1)$ is strictly convex in the geometrical sense, i.e. for any couple of points $x, y \in \partial B_{\|\cdot\|}(0, 1)$ we have $[x, y] \cap \partial B_{\|\cdot\|}(0, 1) = \{x, y\}$. In other words the segment $[x, y]$ without its extremities x and y lies in the interior of the ball $B_{\|\cdot\|}(0, 1)$. This means that the sphere $\partial B_{\|\cdot\|}(0, 1) = \{x, y\}$ contains no “flat-parts”.

2 Preliminaries about the curve bisector

Let $\|\cdot\|$ be a C^2 and strictly convex norm on \mathbb{R}^2 , and $a, b \in \mathbb{R}^2$ be any given points. Then we focus on the curve bisector defined by

$$M(a, b) := \{z \in \mathbb{R}^2 : \|a - z\| = \|b - z\|\}.$$

If the norm is Euclidean, it is well known that $M(a, b)$ is the line passing through the middle point $\frac{a+b}{2}$ and perpendicular to $b - a$. But for a general norm, the curve bisector may not be perpendicular and may even not be a line. However, it is not difficult to see that $M(a, b)$ is always asymptotically converging to a line at infinity, and the direction of that line is given by the dual vector to $b - a$ (i.e. the vector on the sphere $B_{\|\cdot\|}(0, 1)$ at which the tangent to the sphere has direction $b - a$). Figure 1 represents the curve bisector of two points for the norm $\|\cdot\|_4$ in \mathbb{R}^2 .

To prove this, we shall need to consider the direction defined in the following definition.

Definition 2.1 (Definition of L_x). *Let $\|\cdot\|$ be a C^2 and strictly convex norm on \mathbb{R}^2 . For any $x \in \partial B_{\|\cdot\|}(0, 1)$ we define the vectorial line “dual to x ”, denoted L_x , in the following way: there exists exactly two points $y_1, y_2 \in \partial B_{\|\cdot\|}(0, 1)$ such that the tangent line to $\partial B_{\|\cdot\|}(0, 1)$ at point y_i is directed by x . Then L_x is the line passing through y_1 and y_2 . If $x \neq 0$ and $\|x\| \neq 1$, we denote by L_x the line $L_{x/\|x\|}$.*

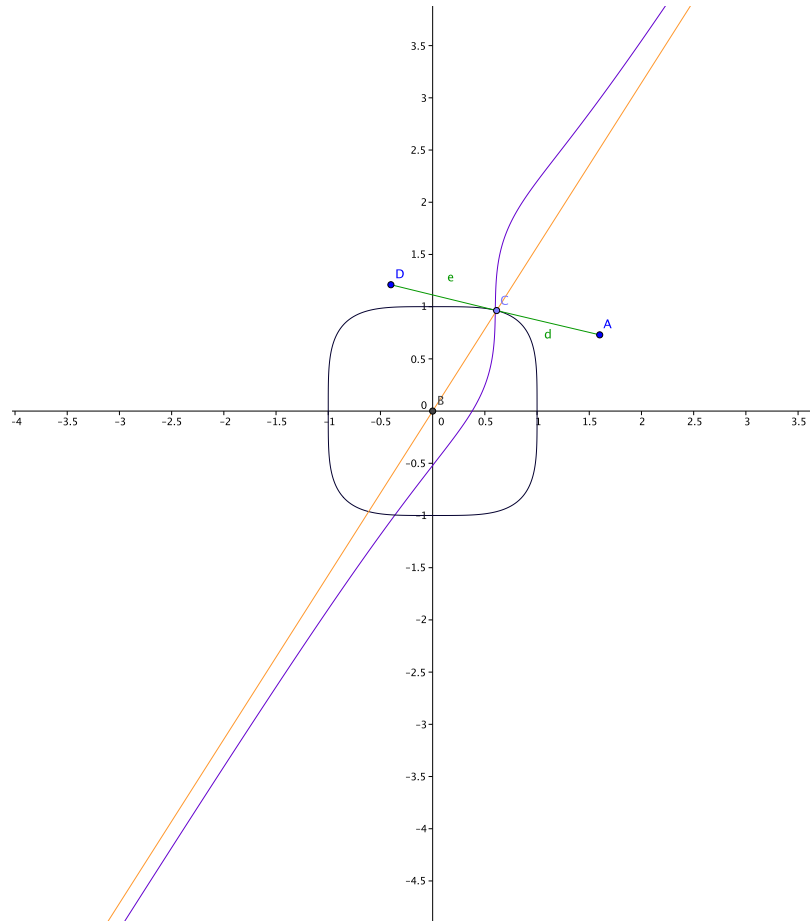


Figure 1: The curve bisector of the segment $[A, D]$ for the norm $\|\cdot\|_4$ is a curve which is asymptotic to a line at infinity, whose direction is given by the point on the sphere $\partial B_{\|\cdot\|_4}(0, 1)$ at which the tangent to the sphere has the same direction as the segment $[A, D]$.

Remark 2.2. A geometrical way to find the y_i is as follows. Let $H_x \subset \mathbb{R}^2 \setminus B_{\|\cdot\|}(0, 1)$ be a line directed by x . Then translate this line in the direction x^\perp until it touches the sphere $\partial B_{\|\cdot\|}(0, 1)$. The touching point is then one of the y_i . Since the norm is C^2 and strictly convex, there exists only two of those points, which from the symmetry of the ball with respect to the origin must satisfy $y_1 = -y_2$. For example, in Figure 1, the line passing through the origin is $L_{D-A} = L_{A-D}$.

Remark 2.3. For all $x \in \partial B_{\|\cdot\|}(0, 1)$, let $\alpha(x) \in [0, \frac{\pi}{2}]$ be the smallest angle between the two lines L_x and $\mathbb{R}x$. Then it is easy to see that $\frac{\pi}{2} \geq \inf_{x \in \partial B_{\|\cdot\|}(0, 1)} \alpha(x) =: \alpha_0 > 0$. Indeed, the infimum is actually a minimum, and it cannot be zero because the origin lies in the interior of the ball. Roughly speaking, the angle α_0 quantifies how far the ball for $\|\cdot\|$ is from being the Euclidean ball, in the sense of how the tangent line to the sphere can be far from making an angle of $\frac{\pi}{2}$ with the radius. Another way to define α_0 is as follows: for every $x \in \partial B_{\|\cdot\|}(0, 1)$ let ν_x be outer unit normal vector to $\partial B_{\|\cdot\|}(0, 1)$ at point x . Then α_0 is the unique real number in $[0, \frac{\pi}{2}]$ solution to

$$\sin(\alpha_0) = \inf_{x \in \partial B_{\|\cdot\|}(0, 1)} \left\langle \nu_x, \frac{x}{|x|} \right\rangle. \quad (2.1)$$

The following proposition says that $M(a, b)$ converges to the line

$$L(a, b) := \frac{b+a}{2} + L_{b-a} \quad (2.2)$$

at infinity.

Proposition 2.4. *Let $\|\cdot\|$ be a C^2 and strictly convex norm on \mathbb{R}^2 . Then for every $\varepsilon > 0$ there exists $R > 0$ such that for all couple of points $a, b \in \mathbb{R}^2$ satisfying $a \neq b$ and $\|b - a\| = 1$,*

$$\sup\{\text{dist}(z, L(a, b)) : z \in M(a, b) \setminus B(a, R)\} \leq \varepsilon,$$

where $L(a, b)$ is defined in (2.2), and dist is the Euclidean distance.

Proof. Let $a, b \in \mathbb{R}^2$ be satisfying $a \neq b$ and $\|b - a\| = 1$, and let $z_R \in M(a, b)$ be such that $\|z_R - a\| = R \rightarrow +\infty$. Assume for simplicity that z_R stays ‘‘above’’ the line directed by $b - a$. Then it is easy to see that $\frac{z_R - a}{R}$ and $\frac{z_R - b}{R}$ both converge to the same direction $w \in \partial B_{\|\cdot\|}(0, 1)$, the one directing the line L_{b-a} . To see this, we let R_n be a subsequence making $w_n := \frac{z_{R_n} - a}{R_n}$ and $w'_n := \frac{z_{R_n} - b}{R_n}$ converging on the compact set $\partial B_{\|\cdot\|}(0, 1)$ to some w and w' . We actually have that $w = w'$ because

$$\|w_n - w'_n\| = \|b - a\|/R_n = 1/R_n \rightarrow 0.$$

Now we claim that w is directing the line L_x , where $x = b - a$. To see this, it is enough showing that the tangent line to $\partial B_{\|\cdot\|}(0, 1)$ at point w is directed by x . This

follows from the fact that, since $\partial B_{\|\cdot\|}(0, 1)$ is a C^2 manifold, the rescaled secants $(w_n - w'_n)/\|w_n - w'_n\|$, which is nothing but $a - b$, converges to the tangent line to $\partial B_{\|\cdot\|}(0, 1)$ at the point w .

Now to prove the proposition, we need to go one order further, namely use a C^2 argument. For this purpose, we adopt the following point of view: let $a, b \in \mathbb{R}^2$ be such that $\|a - b\| = 1$ and let $v = b - a$ the direction of the segment $[a, b]$. For all $t \in \mathbb{R}$, we denote by $H_t := (0, t) + \mathbb{R}v$ the line directed by v in \mathbb{R}^2 , of height t . Let $t_0 > 0$ be the maximum t such that $H_t \cap \partial B_{\|\cdot\|}(0, 1) \neq \emptyset$. Let also a_t and b_t be the two points of $H_t \cap \partial B_{\|\cdot\|}(0, 1)$, for $t \in (-t_0, t_0)$. Then for any $t \in (-t_0, t_0)$, it is clear that the origin lies on the curve bisector $M(a_t, b_t)$ (see Figure 2). Actually, by using the rescaling $(b_t - a_t)/\|b_t - a_t\|$ we get this way a complete parametrization on $(-t_0, t_0)$ of $M(0, v)$, the curve bisector of a the segment of unit length directed by v starting from the origin (which is the same as $M(a, b)$, after translation to the origin).

Moreover, when t converges to $-t_0$ or t_0 , the points a_t and b_t converge to some points $-y$ and y on the sphere $\partial B_{\|\cdot\|}(0, 1)$, the ones at which the tangent line to $\partial B_{\|\cdot\|}(0, 1)$ is directed by v . The line passing through $-y$ and y is exactly L_v .

Next, for any $R \geq \frac{1}{2}$, there exists a unique $t_R \in (-t_0, 0)$ such that $\frac{1}{\|a_{t_R} - b_{t_R}\|} = R$. Therefore, if $z_R \in M(0, v)$ is such that $\|z_R\| = R$,

$$\text{dist}(z_R, L(0, v)) \leq R|m_{t_R} - l_{t_R}|, \quad (2.3)$$

where $m_t = (a_t + b_t)/2$ is the middle point of the segment $[a_t, b_t]$, and l_t is the intersecting point of the two segments $[0, -y]$ and $[a_t, b_t]$ (see Figure 2).

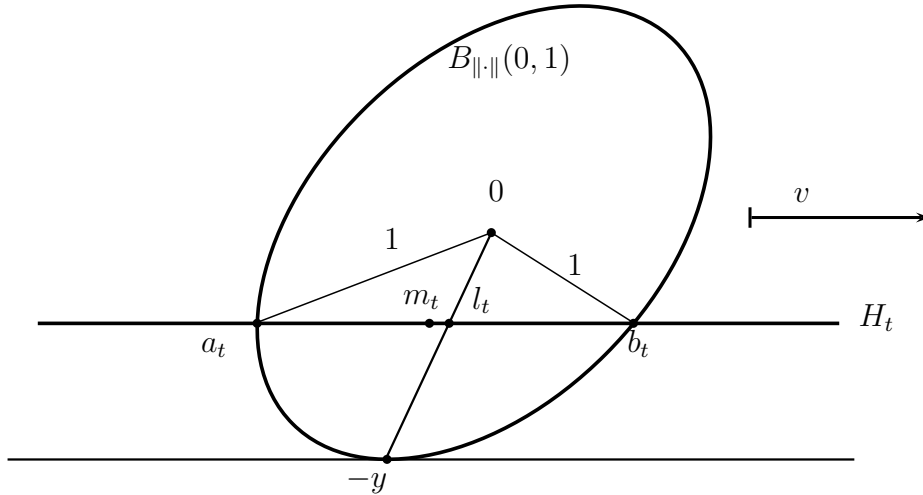


Figure 2: The distance from $M(a_t, b_t)$ to $L(a, b)$ is the same as the distance from m_t to $L(b - a)$ (the line passing through 0 and $-y$), which is less than the distance between m_t and l_t .

Now, we claim that, when $|a_t - b_t| \rightarrow 0$,

$$|m_t - l_t| \leq C|a_t - b_t|^2 + o(|a_t - b_t|^2), \quad (2.4)$$

for some $C > 0$ (depending only on $\|\cdot\|$). This is enough to conclude because returning to (2.3) we get, using that $\|a_{t_R} - b_{t_R}\| = \frac{1}{R}$,

$$\text{dist}(z_R, L(0, v)) \leq R|m_{t_R} - l_{t_R}| \leq C' \frac{1}{R} \xrightarrow{R \rightarrow +\infty} 0.$$

Now to prove (2.4), we work locally around the point $-y \in \partial B_{\|\cdot\|}(0, 1)$. Let T_{-y} be the tangent line at point $-y$. Since the norm is C^2 and strictly convex, in a certain well chosen coordinate system, the sphere $\partial B_{\|\cdot\|}(0, 1)$ almost coincides with the graph of $s \mapsto As^2$ for some $A > 0$, up to some error of second order. Consequently, in that coordinate system, the segments $[a_t - b_t]$ which have the same direction as T_{-y} , almost coincides with the segment $[-x_s, x_s]$ where $x_s = (s, As^2)$, again up to some error of second order. On the other hand the angle θ between the radius $[0, -y]$ of the ball $B_{\|\cdot\|}(0, 1)$ and the vertical line orthogonal to T_{-y} , is at most $\frac{\pi}{2} - \alpha_0$, where α_0 is defined in Remark 2.3 (see Figure 3).

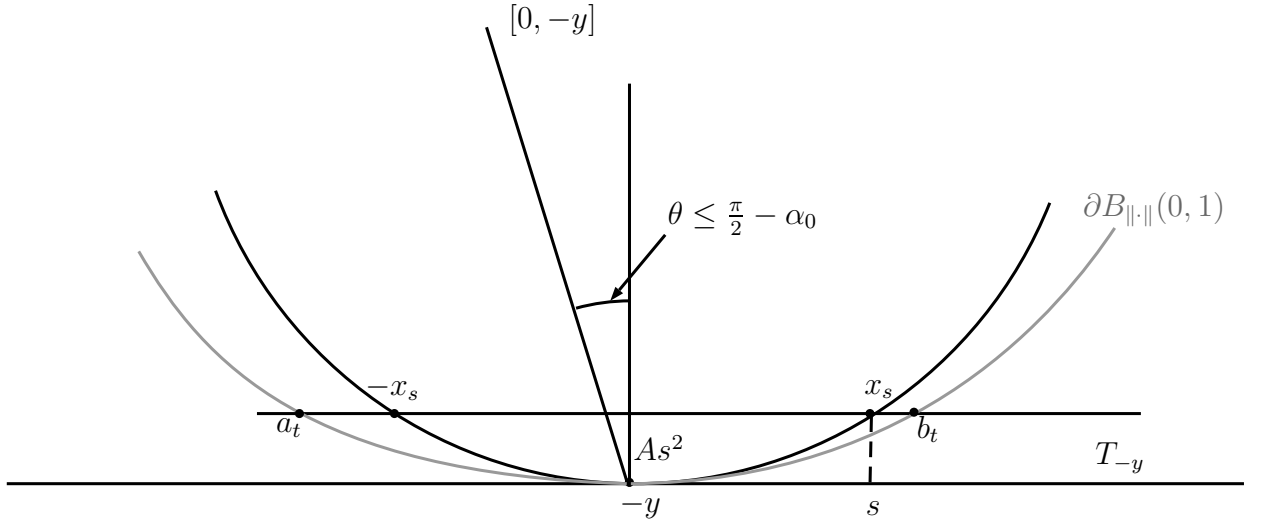


Figure 3: Situation in the proof of (2.4).

We deduce that $|m_t - l_t| \leq |A \sin(\theta)s^2| + o(s^2) = Cs^2 + o(s^2)$, as desired. This achieves the proof of (2.4), and of the proposition. \square

Definition 2.5. For any $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$ with $\|v\| = 1$ we denote by $Q_v(x)$ the projection of $x \in \mathbb{R}^2$ on $\mathbb{R}v$ parallelly to L_v (i.e. the v component of x written in the basis (v, v') where v' is a vector director for L_v).

The next proposition quantifies the maximum distance between $M(a, b)$ and the line $L(a, b)$.

Proposition 2.6. *Let $\|\cdot\|$ be a C^1 and strictly convex norm on \mathbb{R}^2 . There exists $0 < \kappa < 1/2$ depending on the norm $\|\cdot\|$, such that for any given $a, b \in \mathbb{R}^2$, the curve bisector $M(a, b)$ is contained in the strip $S_\kappa(a, b)$ defined as*

$$S_\kappa(a, b) := \{x \in \mathbb{R}^2 : |Q_{\frac{b-a}{\|b-a\|}}(x - \frac{a+b}{2})| \leq \kappa\|b-a\|\}.$$

Proof. We first prove that for all a, b in \mathbb{R}^2 , $M(a, b) \subseteq S_{1/2}(a, b)$. Let a, b be given. We can assume that $\|b-a\| = 1$ and we denote by $v = b-a$. Adopting the same notation a_t, b_t, m_t as in the proof of Proposition 2.4, we see that

$$\sup\{|Q_v(x - v/2)| : x \in M(0, v)\} = \sup_{t \in [-t_0, t_0]} \frac{|Q_v(m_t)|}{\|b_t - a_t\|}.$$

By convexity of the ball $B_{\|\cdot\|}(0, 1)$ we have $L_v \cap [a_t, b_t] \neq \emptyset$ for all $|t| \leq t_0$, from which we find that

$$\sup_{t \in [-t_0, t_0]} \frac{|Q_v(m_t)|}{\|b_t - a_t\|} \leq 1/2,$$

as claimed.

Now to pass from $1/2$ to some $\kappa < 1/2$ we argue by contradiction. Assuming the proposition is false, we can find $a_n := a_{t_n}, b_n := b_{t_n}$, such that

$$\frac{|Q_v(m_n)|}{\|b_n - a_n\|} \xrightarrow{n \rightarrow +\infty} 1/2, \quad (2.5)$$

where $m_n := (a_n + b_n)/2$. According to Proposition 2.4, we can assume that

$$\inf_n \|a_n - b_n\| \geq \delta, \quad (2.6)$$

for some $\delta > 0$. Indeed, let $R > 0$ be given by Proposition 2.4 for $\varepsilon := 1/4$ and let $\delta := 1/R$. Then, for all a_t and b_t satisfying $r := \|a_t - b_t\| \leq \delta$, we have that $\|a_t/r\| = 1/r \geq 1/\delta = R$ thus $(0, 0) \in M(a_t/r, b_t/r) \setminus B(a_t/r, R)$ and therefore the conclusion of Proposition 2.4 says

$$\text{dist}((0, 0), L(a_t/r, b_t/r)) \leq \varepsilon = \frac{1}{4}.$$

In particular this implies $|Q_v(\frac{a_t+b_t}{2})|/\|b_t - a_t\| \leq 1/4$. Hence, our sequence of a_n and b_n satisfying (2.5) must satisfy $\|a_n - b_n\| \geq \delta$ for n large enough. Now by compactness of $\partial B_{\|\cdot\|}(0, 1)$ we can assume that $a_n \rightarrow a = a_{t_0}$ and $b_n \rightarrow b = b_{t_0}$ satisfying $\|a - b\| \geq \delta$. By (2.5) we also have that

$$\frac{|Q_v(m)|}{\|b - a\|} = 1/2,$$

where $m = (a + b)/2$. But this implies that $L_v \cap [a, b]$ is either the point a or the point b , which is a contradiction because, by strict convexity of the ball B , the line L_v must meet the segment $[a, b]$ only by its interior. This finishes the proof of the Proposition. \square

3 Rectifiability of $\|\cdot\|$ -SC-curves

In the sequel if γ is a $\|\cdot\|$ -self-contracted curve, we denote by $\Gamma := \gamma(I)$ the image of the curve, and for all $x \in \Gamma$ we introduce the “piece of curve after x ” namely,

$$\Gamma(x) := \{y \in \Gamma : x \preceq y\},$$

where \preceq denotes the order on the curve given by its parameterization. We will also denote by $\Omega(x)$ the convex hull of $\Gamma(x)$.

We start with a first lemma about the maximum aperture of the angle between $y' - x_0$ and $y - x_0$ when x_0, y, y' are all lying on Γ with y and y' after x_0 . Namely, for an Euclidean self-contracted-curve, it is easy to see that whenever $x_0, y, y' \in \Gamma$ are satisfying $x_0 \preceq y \preceq y'$ we have

$$\langle y - x_0, y' - x_0 \rangle \geq 0.$$

The way this is proved in [2] is as follows. From the self-contracted property we infer that

$$|y' - x_0| \geq |y' - y|.$$

Thus writing $y' - y = (y' - x_0) + x_0 - y$ and squaring the estimates we get

$$|y' - x_0|^2 \geq |y' - x_0|^2 + |y - x_0|^2 - 2\langle y' - x_0, y - x_0 \rangle$$

which yields

$$\langle y' - x_0, y - x_0 \rangle \geq 0.$$

For a general norm which is no more Euclidean, we get a similar estimate from a different argument.

Lemma 3.1. *Let $\|\cdot\|$ be a C^2 and strictly convex norm on \mathbb{R}^2 . Let α_0 be the constant of Remark 2.3 and let $\gamma : I \rightarrow \mathbb{R}^2$ be a $\|\cdot\|$ -self-contracted curve. Then for every $x_0, y, y' \in \Gamma$ satisfying $x_0 \preceq y \preceq y'$ we have*

$$\left\langle \frac{y - x_0}{|y - x_0|}, \frac{y' - x_0}{|y' - x_0|} \right\rangle \geq -\cos(\alpha_0). \quad (3.1)$$

Proof. The argument is purely geometric. Let $x_0, y, y' \in \Gamma$ satisfying $x_0 \preceq y \preceq y'$. The self contracting property yields

$$\|y' - x_0\| \geq \|y' - y\|$$

which means that $x_0 \notin B_{\|\cdot\|}(y', \|y - y'\|)$. Hence, assuming $y' = 0$ and $\|y\| = 1$, to find a lower bound for the left-hand side of (3.1) we can consider the following problem, denoting $B := B_{\|\cdot\|}(0, 1)$, and fixing $y \in \partial B$,

$$\inf_{x \in \mathbb{R}^2 \setminus B} \left\langle \frac{y - x}{|y - x|}, \frac{-x}{|x|} \right\rangle. \quad (3.2)$$

To solve this problem, we first prove that it is equivalent to take the infimum in $x \in \partial B$. Indeed, let us define

$$\varphi(x) := \left\langle \frac{x - y}{|y - x|}, \frac{x}{|x|} \right\rangle$$

and consider the function f of the real variable $t > 0$ defined by $f(t) = \varphi(tx)$. A simple computation shows that $\frac{d}{dt}|y - tx| = \frac{\langle tx - y, x \rangle}{|y - tx|}$ and $\frac{d}{dt}\langle tx - y, x \rangle = |x|^2$ thus

$$f'(t) = \frac{|x|^2|y - tx|^2 - \langle tx - y, x \rangle^2}{|x||y - tx|^3} \geq 0,$$

due to the Cauchy-Schwarz inequality. This implies that $t \mapsto \varphi(tx)$ is non decreasing in t and the infimum in (3.2) is the same as:

$$\inf_{x \in \partial B} \left\langle \frac{x - y}{|y - x|}, \frac{x}{|x|} \right\rangle. \quad (3.3)$$

But now it is easy to conclude using the convexity of B . Indeed, for a given $x \in \partial B$, let ν_x be the unit inner normal vector to ∂B at point x . Then by convexity of B , since $y \in \partial B$ we must have

$$\langle y - x, \nu_x \rangle \geq 0,$$

and by definition of α_0 we deduce that

$$\left\langle \frac{x - y}{|y - x|}, \frac{x}{|x|} \right\rangle \geq -\cos(\alpha_0).$$

□

We are now ready to prove the main result of this paper.

Proof of Theorem 1.1. We use the same notation \prec , Γ , $\Gamma(x)$, $\Omega(x)$ as before. The beginning of the proof follows essentially the proof of [2, Theorem 3.3] which is divided into several steps. Although the first two steps are very close to the Euclidean situation, we write here the full detail.

The first step consists in first noticing that to prove the theorem, it is enough finding some $c_0 > 0$ such that for any pair of points $x, x' \in \Gamma$ with $x' \preceq x$ it holds

$$W(\Omega(x)) + c_0|x - x'| \leq W(\Omega(x')) \quad (3.4)$$

(this is Claim 1 of [2, Theorem 3.3]).

Indeed, letting $t_0 < t_1 \dots < t_m$ be any increasing sequence in I , and set $x_i := \gamma(t_i)$. If (3.4) holds, then

$$\begin{aligned} \sum_{i=0}^{m-1} |\gamma(t_{i+1}) - \gamma(t_i)| &= \sum_{i=0}^{m-1} |x_{i+1} - x_i| \leq \frac{1}{c_0} \sum_{i=0}^{m-1} (W(\Omega(x_i)) - W(\Omega(x_{i+1}))) \\ &= \frac{1}{c_0} (W(\Omega(x_0)) - W(\Omega(x_m))) \leq \frac{1}{c_0} W(\Omega(x_0)), \end{aligned}$$

since the mean width $W(H)$ is a nondecreasing function of H (the variable H is ordered via the set inclusion). Taking the supremum over all choices of $t_0 < t_1 \dots < t_m$ in I we obtain (1.3) for $C = 1/c_0$.

Therefore, the theorem will be proved, if we show that (3.4) holds for some constant $c_0 > 0$. Before we proceed, we introduce some extra notation similar to the ones of [2] but modified with the constant κ . Indeed, let κ be the constant given by Proposition 2.6 depending only on $\|\cdot\|$ and let $\lambda := \frac{1}{2} - \kappa > 0$. Let x, x' be fixed in Γ with $x' \prec x$. We set (see Figure 4)

$$v_0 := \frac{x' - x}{|x' - x|} \quad \text{and} \quad x_0 := x' - v_0 \frac{\lambda}{2} |x - x'|. \quad (3.5)$$

Let us also set

$$\xi_0(y) = \frac{y - x_0}{|y - x_0|} \in \mathbb{S}^1, \quad \text{for any } y \in \Gamma(x). \quad (3.6)$$

Notice that x_0, v_0 and $\xi_0(y)$ depend on the points x, x' , while the desired constant c_0 does not (may depend only on $\|\cdot\|$). To determine c_0 , we shall again transform the problem into another one (similar to Claim 2 of [2, Theorem 3.3]).

Let us assume that there exists some constants $\delta > 0$ and $\tau > 0$ (depending only on $\|\cdot\|$), such that for all x, x' in Γ with $x' \prec x$ (and for x_0, v_0 defined by (3.5)), there exists $\bar{v} \in \mathbb{S}^1$ such that the following two properties hold:

$$\langle \bar{v}, v_0 \rangle \geq \tau \quad (3.7)$$

$$\langle \bar{v}, \xi_0(y) \rangle \leq -\delta \quad \text{for all } y \in \Gamma(x). \quad (3.8)$$

Then (3.4) holds true (and consequently (1.3) follows).

This is a slight modification of Claim 2 of [2, Theorem 3.3]). To prove the latter, assume that such constants δ, τ , and a vector \bar{v} exist, so that (3.7) and (3.8) holds. Up to change δ into $\tau/2$ if necessary, we may assume that $\delta \leq \tau/2$. Set

$$V := \{v \in \mathbb{S}^1; |v - \bar{v}| \leq \delta\}.$$

From (3.8) we get

$$\langle v, y - x_0 \rangle \leq 0, \quad \text{for all } v \in V \text{ and } y \in \Omega(x). \quad (3.9)$$

Let us now assume that x_0 is the origin. Recall that, for $v \in \mathbb{R}^2$, P_v denotes the orthogonal projection onto the line $\mathbb{R}v$. Observe that $\Omega(x')$ contains the convex hull of $\Omega(x) \cup [x, x']$, whence

$$P_v(\Omega(x)) \subset P_v(\Omega(x')).$$

In particular,

$$\mathcal{H}^1(P_v(\Omega(x))) \leq \mathcal{H}^1(P_v(\Omega(x'))) \quad \text{for } v \in \mathbb{S}^{n-1}, \quad (3.10)$$

Then (3.9) says that for all directions v in V we have

$$\sup P_v(\Omega(x)) \leq 0 < P_v(x') \leq \sup P_v(\Omega(x')).$$

Now for every $v \in V$ we have

$$\langle v, v_0 \rangle = \langle \bar{v}, v_0 \rangle + \langle v - \bar{v}, v_0 \rangle \geq \tau - |v_0||v - \bar{v}| \geq \tau/2 \quad (3.11)$$

because $\delta \leq \tau/2$. This gives a lower bound for the length of the projected segment $[x_0, x']$ onto $\mathbb{R}v$. Precisely, recalling that

$$|x_0 - x'| = \frac{\lambda}{2}|x - x'|,$$

we get

$$P_v(x') \geq \frac{\lambda\tau}{4}|x - x'|.$$

This yields

$$\mathcal{H}^1(P_v(\Omega(x))) + \frac{\lambda\tau}{4}|x - x'| \leq \mathcal{H}^1(P_v(\Omega(x'))) \quad \text{for } v \in V. \quad (3.12)$$

Integrating (3.12) for $v \in V$ and (3.10) for $v \in \mathbb{S}^1 \setminus V$, and summing up the resulting inequalities we obtain (3.4).

Consequently, our next goal is to determine $\delta > 0$ and $\tau > 0$ so that (3.7) and (3.8) hold.

Here comes the point where the proof slightly differs from [2, Theorem 3.3] and where the preliminary section about the curve bisector plays a role. First we notice that from the self-contracted property of the curve, the set $\Gamma(x)$ lies only on one side of the curve bisector $M(x, x')$ (the one containing x). On the other hand by applying Proposition 2.6, we know that $M(x, x') \subset S_\kappa(x, x')$ (see Proposition 2.6 for the definition of $S_\kappa(x, x')$). Let w be a vector parallel to L_{v_0} and let $\nu \in \mathbb{S}^1$ the vector orthogonal to it pointing in the opposite direction with respect to $\Gamma(x)$.

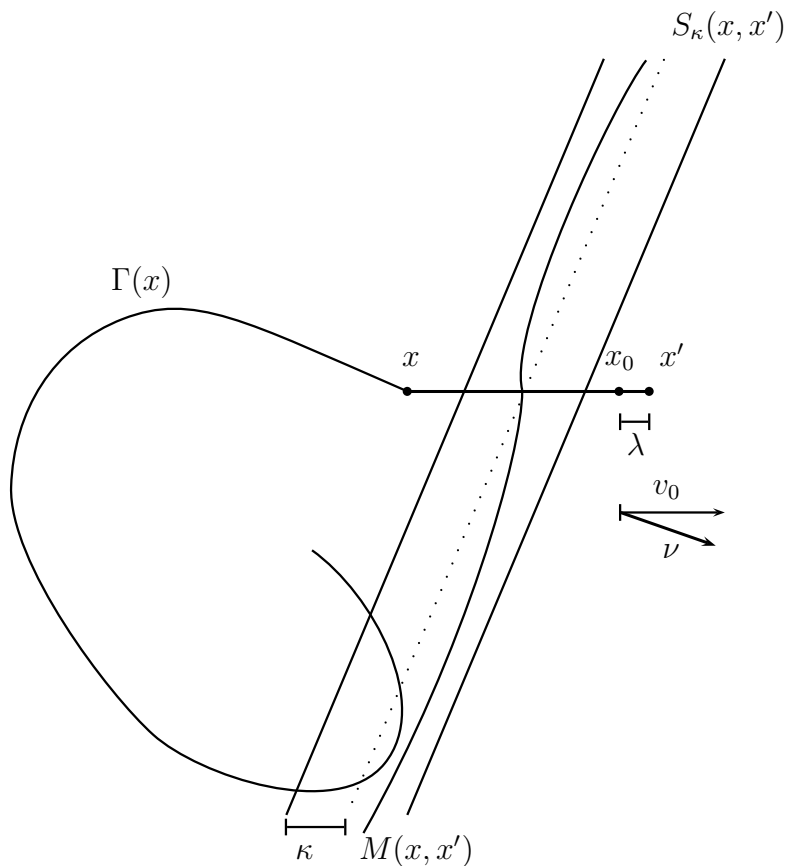


Figure 4: $\Gamma(x)$ lies on one side of the curve bisector $M(x, x')$ which is contained in the strip $S_\kappa(x, x')$.

Then it holds, thanks to our definition of x_0 in correspondance to the width of S_κ ,

$$\langle y - x_0, \nu \rangle \leq -\frac{\lambda}{2} \leq 0 \quad \forall y \in \Gamma(x). \quad (3.13)$$

Moreover, the angle between ν and v_0 can be estimated by the constant α_0 defined in Remark 2.3, which yields

$$\langle \nu, v_0 \rangle \geq \cos\left(\frac{\pi}{2} - \alpha_0\right) =: \tau_1 > 0.$$

The vector ν is very close to satisfy the required conditions (3.7) and (3.8). Actually, as already used in [2], the desired vector \bar{v} will be constructed as a little perturbation of ν , precisely, of the form

$$\nu_\varepsilon := \frac{\nu + \varepsilon\nu^\perp}{|\nu + \varepsilon\nu^\perp|}$$

for some $\varepsilon > 0$ small, to be chosen later.

Let us check first that for any $\varepsilon \leq \tau_1/6$, it holds

$$\langle \nu_\varepsilon, v_0 \rangle \geq \tau_1/2$$

so that condition (3.7) would be satisfied for any of those ν_ε , with $\tau = \tau_1/2$. Indeed, using the inequality $\sqrt{1+t} \leq 1 + \frac{t}{2}$, we obtain

$$|\nu - \nu_\varepsilon|^2 = \frac{1}{1+\varepsilon^2}[(\sqrt{1+\varepsilon^2} - 1)^2 + \varepsilon^2] \leq \frac{1}{2}\varepsilon^4 + \varepsilon^2$$

which yields, for $\varepsilon \leq 1$,

$$|\nu - \nu_\varepsilon| \leq 3\varepsilon. \tag{3.14}$$

Consequently,

$$\langle \nu_\varepsilon, v_0 \rangle = \langle \nu, v_0 \rangle + \langle \nu_\varepsilon - \nu, v_0 \rangle \geq \tau_1 - |\nu - \nu_\varepsilon| \geq \tau_1/2$$

when $3\varepsilon \leq \tau_1/2$.

Now it remains to find $\varepsilon \leq \tau_1/6$ that would moreover make ν_ε satisfying an inequality like (3.8). To do so, we decompose the half space $H := \{y : P_\nu(y) \leq 0\}$ in two different regions. Let

$$H_1 := \{y : \langle \xi_0(y), \nu \rangle \leq -2\mu\}$$

and

$$H_2 := \{y : -2\mu < \langle \xi_0(y), \nu \rangle \leq 0\},$$

for some $\mu > 0$ that will be chosen later. Notice that, by (3.14), if $\varepsilon \leq \min(\tau_1/6, \mu/3)$, then

$$\text{for all } y \in H_1, \quad \langle \xi_0(y), \nu_\varepsilon \rangle \leq \langle \xi_0(y), \nu \rangle + |\nu - \nu_\varepsilon| \leq -\mu, \tag{3.15}$$

as desired, and we only need to take care of H_2 . For this purpose, we furthermore decompose H_2 itself in the two regions,

$$H_\mu := H_2 \cap B(x_0, 1)$$

$$H'_\mu := H_2 \setminus H_\mu.$$

We know from (3.13) that for all $y \in H_\mu$,

$$\langle \xi_0(y), \nu \rangle = \left\langle \frac{y - x_0}{|y - x_0|}, \nu \right\rangle \leq -\frac{\lambda}{2|y - x_0|} \leq -\frac{\lambda}{2}.$$

Therefore, if $\varepsilon \leq \min(\alpha_0/6, \mu/3, \lambda/12)$ then (3.15) holds and moreover,

$$\langle \xi_0(y), \nu_\varepsilon \rangle \leq \langle \xi_0(y), \nu \rangle + |\nu - \nu_\varepsilon| \leq -\frac{\lambda}{4}, \quad \text{for all } y \in H_\mu.$$

It remains finally to consider $y \in H'_\mu$. For this we may assume that $2\mu \leq \frac{1}{2}$ so that H'_μ is divided in two connected components, one denoted H'_μ^+ lying in the upper half-space $\{x : \langle x, v_0^\perp \rangle \geq 0\}$ and another one denoted H'_μ^- lying in the other part $\{x : \langle x, v_0^\perp \rangle \leq 0\}$. Now for $\mu \in (0, \frac{1}{2})$ let us define the quantity

$$\beta_\mu := \sup\{\langle \xi_0(x), \xi_0(y) \rangle : x \in H'_\mu^+ \text{ and } y \in H'_\mu^-\}.$$

It is clear that $\beta_\mu \rightarrow -1$ when μ goes to 0. Therefore, since $\alpha_0 > 0$, there exists $\mu > 0$ small enough (depending only on α_0), such that $\beta_\mu < -\cos(\alpha_0)$. By applying Lemma 3.1, we infer that the following alternative holds

$$\Gamma(x) \cap H'_\mu \subset H'_\mu^+ \quad \text{or} \quad \Gamma(x) \cap H'_\mu \subset H'_\mu^-.$$

Next, we set

$$\varepsilon_0 = \min(\alpha_0/6, \mu/3, \lambda/12).$$

In the first case (i.e. when $\Gamma(x) \cap H'_\mu \subset H'_\mu^+$) we define

$$\bar{\nu} := \nu_{-\varepsilon_0}$$

and in the second case (i.e. when $\Gamma(x) \cap H'_\mu \subset H'_\mu^-$) we define

$$\bar{\nu} := \nu_{\varepsilon_0}.$$

We finally check that this choice suit our purposes. Indeed, it is not difficult to see that

$$\sup_{y \in H'_\mu^+} \left\langle \xi_0(y), \frac{v_0 - \varepsilon_0 v_0^\perp}{|v_0 - \varepsilon_0 v_0^\perp|} \right\rangle = \cos\left(\frac{\pi}{2} + \theta\right),$$

where θ is the angle between ν and $\nu_{-\varepsilon_0}$ (see Figure 5).

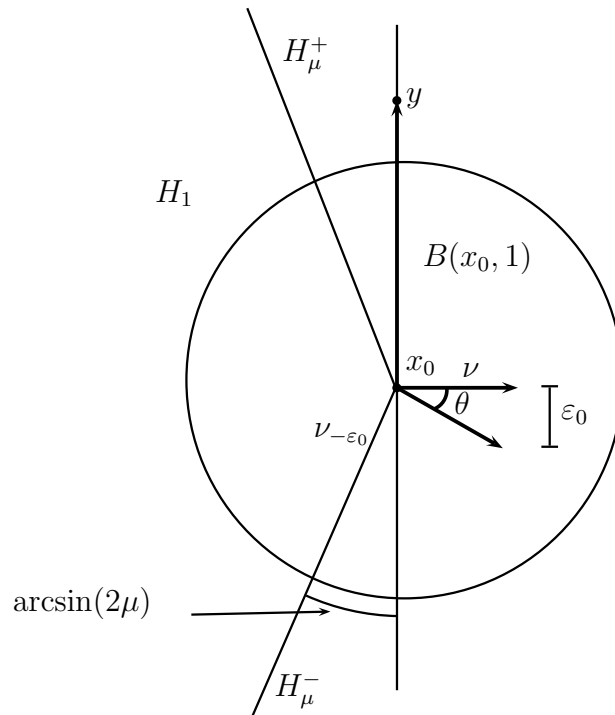


Figure 5: The minimum angle between $y - x_0$ and $\nu_{-\varepsilon_0}$ for $y \in H_\mu^+$ is achieved when $y - x_0 \in \nu^\perp$.

This yields

$$\sup_{y \in H_\mu^+} \left\langle \xi_0(y), \frac{v_0 - \varepsilon_0 v_0^\perp}{|v_0 - \varepsilon_0 v_0^\perp|} \right\rangle = -\frac{\varepsilon_0}{\sqrt{1 + \varepsilon_0}} \leq -\frac{\varepsilon_0}{2},$$

provided that $\varepsilon_0 \leq 1$.

Gathering all the estimates together, we have found some $\bar{\nu}$ satisfying

$$\sup_{y \in \Gamma(x)} \langle \xi_0(y), \bar{\nu} \rangle \leq -\tau,$$

with $\tau = \min(\mu, \frac{\lambda}{4}, \frac{\varepsilon_0}{2})$, and this finishes the proof of the Theorem. \square

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