A NOTE ON FRACTIONAL p-LAPLACIAN PROBLEMS WITH SINGULAR WEIGHTS

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Dedicated with admiration to Paul Rabinowitz, a master in Nonlinear Analysis

Abstract. We study a class of fractional p-Laplacian problems with weights which are possibly singular on the boundary of the domain. We provide existence and multiplicity results as well as characterizations of critical groups and related applications.

1. Introduction

Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider also a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Recently, the following semi-linear problem involving the fractional Laplacian has been the subject of various investigations

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The nonlocal operator $(-\Delta)^s$ naturally arises in various fields, such as continuum mechanics, phase transition phenomena, population dynamics, game theory and financial mathematics [1, 4]. Existence [25, 26], non-existence [23] and regularity [5, 22] have been studied intensively. In this paper, we consider quasi-linear problems

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

containing a nonlocal nonlinear operator known as the fractional p-Laplacian, which represents a natural extension of fractional Laplacian. For $p \in (1,\infty)$ and u smooth enough,

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))}{|x-y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N.$$

We refer to [4] for the motivations that lead to the introduction of this operator. So far, existence and regularity for the above problem have been investigated [3, 13–15, 17, 18] under the assumption that the function f is L^∞-Carathéodory, namely *nonsingular* in the x-dependence on $\partial\Omega$. Our goal in this paper is to get existence results for (1.1) when f involves *singular weights*. It is worth noticing that this is new even for the plain fractional Laplacian. This paper is motivated by [21] where the p-Laplace equation $-\Delta_p u = f(x,u)$ was investigated with singular weights, namely $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and f satisfies the subcritical growth condition

$$|f(x,t)| \le h_1(x)\,|t|^{q_1-1} + \cdots + h_n(x)\,|t|^{q_n-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{1.2}$$

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for some $q_i \in [1, p^*)$ with $p^* := Np/(N-p)$ and measurable weights $h_i \geq 0$ which are possibly singular along the boundary $\partial\Omega$. Admissible classes of weights are introduced in [21], which are appropriate in order to use Hölder and Hardy inequalities to show that the functional associated with (1.1) is well-defined and the Palais-Smale condition holds at any energy level, allowing to obtain several existence results. Previously, the semi-linear case was considered e.g. in [7,8,24,27].

By introducing a suitable class of weights we will get results about existence, multiplicity and characterization of critical groups for any $s \in (0,1)$. More precisely, in Theorem 3.4, we get a multiplicity result for $f(x,u) = h(x)|u|^{q-2}u$ with $q \neq p$. Theorems 4.1, 4.2 and 4.8 are about the computation of critical groups at zero of the functional associated with (1.1) when $f(x,u)$ is a sum of terms with singular weight enjoying proper summability. Moreover, in Proposition 3.2 we prove the boundedness of solutions for $f(x,u) = \lambda h(x)|u|^{p-2}u$ with h belonging to a suitable class. In Theorem 5.5, nontrivial solutions are found under various conditions on f. Finally, in Theorem 6.2, we establish the existence of infinitely many solutions when $f(x,u)$ is odd in u.

The paper is organized as follows. In Section 2, we provide a suitable functional framework for problem (1.1) and prove some preliminary results. In Section 3, we consider related eigenvalue problems. In Section 4, we compute the critical groups of the functional associated with (1.1). Section 5 is devoted to show existence of nontrivial solutions via cohomological local splitting and critical groups. Finally, in Section 6, we obtain the existence of infinitely many solutions.

2. Functional framework and preliminaries

Throughout the paper we will assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with $N \geq 2$. In this section, we provide the variational setting on a suitable function space for (1.1), jointly with some preliminary results. We consider, for any $p \in (1,\infty)$ and $s \in (0,1)$, the space

$$W_0^{s,p}(\Omega) := \big\{ u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega \big\},$$

endowed with the standard Gagliardo norm

$$\|u\| := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}. \tag{2.1}$$

We observe that, as it can be readily seen, this norm is equivalent to the full norm

$$u \mapsto \left(\int_{\mathbb{R}^N} |u(x)|^p dx + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p},$$

namely a Poincaré inequality holds in $W_0^{s,p}(\Omega)$. Let

$$p_s^* := \frac{Np}{N-sp},$$

with the agreement that $p_s^* = \infty$ if $N \leq sp$. It is well-known that $W_0^{s,p}(\Omega)$ is a uniformly convex reflexive Banach space, continuously embedded into $L^q(\Omega)$ for all $q \in [1, p_s^*]$ if $N > sp$, for all $1 \leq q < \infty$ if $N = sp$ and into $L^\infty(\Omega)$ for $N < sp$. It is also compactly injected in $L^q(\Omega)$ for any $q \in [1, p_s^*)$ if $N \geq sp$ and into $L^\infty(\Omega)$ for $N < sp$. Furthermore, $C_0^\infty(\Omega)$ is a dense subspace of $W_0^{s,p}(\Omega)$ with respect to the norm (2.1). In particular, restrictions to Ω of functions in $W_0^{s,p}(\Omega)$ belong to the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$, i.e. with respect to the localized norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}.$$

This closure is often denoted with the same symbol $W_0^{s,p}(\Omega)$. Notice that for the seminorm localized on $\Omega \times \Omega$ there is no Poincaré inequality with $\int_\Omega |u|^p dx$ if $sp \leq 1$, cf. [3, Remark 2.4].

Next we state some Hardy-type inequalities from [6, Theorem 1.1 and formula (17)], for the cases $sp \neq 1$ and [16, Theorem 6.5], for the particular case $sp = 1$.

Theorem 2.1 (Hardy inequality)**.** *For any $p \in (1, \infty)$ and $s \in (0, 1)$ the following facts hold:*

- *if $sp > 1$, then for any $u \in W_0^{s,p}(\Omega)$ we have*
$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}} dx \leq C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy,$$
where C is a positive constant depending only on Ω, N, p and s.
- *if $sp < 1$, then for any $u \in W_0^{s,p}(\Omega)$ we have*
$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}} dx \leq C' \left(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right),$$
where C' is a positive constant depending only on Ω, N, p and s.
- *if $sp = 1$, then for any $u \in W_0^{s,p}(\Omega)$ we have*
$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}} dx \leq C'' \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy,$$
where C'' is a positive constant depending only on Ω, N, p and s.

As a by-product of the previous result, in any case, by (2.1) and the related Poincaré inequality,
$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}} dx \leq C\|u\|^p, \qquad \text{for any } u \in W_0^{s,p}(\Omega),$$
where C is a positive constant depending only on Ω, N, p and s. Let us denote by
$$\rho(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega,$$
the distance from $x \in \Omega$ to $\partial\Omega$ and by $|\cdot|_p$ the usual norm in the space $L^p(\Omega)$.

We consider the following classes of singular weights.

Definition 2.2 (Class of weights $\mathscr{A}_q$)**.** *For $q \in [1, p_s^*)$, let $\mathscr{A}_q$ denote the class of measurable functions h such that $h \in L^r(\Omega)$ for some $r \in (1, \infty)$ satisfying $\frac{1}{r} + \frac{q}{p_s^*} < 1$.*

Definition 2.3 (Class of weights $\mathscr{B}_q$)**.** *For $q \in [1, p_s^*)$, let $\mathscr{B}_q$ denote the class of measurable functions h such that $h\rho^{sa} \in L^r(\Omega)$ for some $a \in [0, q-1]$ and $r \in (1, \infty)$ satisfying $\frac{1}{r} + \frac{a}{p} + \frac{q-a}{p_s^*} < 1$.*

Clearly, $\mathscr{A}_q \subset \mathscr{B}_q$, by simply choosing $a = 0$. Explicitly, h belongs to the above classes provided that there exist $r > 1$ and $0 \leq a \leq q - 1$ with
$$\int_\Omega |h(x)|^r dx < \infty, \quad \text{with} \quad \frac{1}{r} + \frac{q}{p_s^*} < 1, \qquad (h \in \mathscr{A}_q),$$
$$\int_\Omega |h(x)|^r \rho(x)^{sar} dx < \infty, \quad \text{with} \quad \frac{1}{r} + \frac{a}{p} + \frac{q-a}{p_s^*} < 1, \qquad (h \in \mathscr{B}_q).$$

If, for instance, we consider
$$h(x) = (1 - |x|)^{-\beta}, \qquad \Omega = B(0, 1),$$
then $h \in \mathscr{B}_q$ if $\beta < sa + r^{-1}$ for some $r > 1$ and $0 \leq a \leq q - 1$ with $1/r + a/p + (q - a)/p_s^* < 1$.

The following lemma will be used frequently.

Lemma 2.4. *Let $h \in \mathscr{B}_q$. Then there holds*

$$\int_\Omega |h(x)||u|^{q-1}|v|dx \le C\|u\|^{q-1}|v|_b, \qquad \text{for every } u, v \in W_0^{s,p}(\Omega),$$

where $b \in (1, p_s^*)$ is such that $\frac{1}{r} + \frac{a}{p} + \frac{q-a}{b} = 1$.

Proof. If $h \in \mathscr{B}_q$, we have

$$\int_\Omega |h(x)||u|^{q-1}|v|dx = \int_\Omega |h\rho^{sa}| \left|\frac{u}{\rho^s}\right|^a |u|^{q-1-a}|v|dx \le |h\rho^{sa}|_r \left|\frac{u}{\rho^s}\right|_p^a |u|_b^{q-1-a}\, |v|_b.$$

In light of Theorem 2.1 and $W_0^{s,p}(\Omega) \hookrightarrow L^b(\Omega)$ we get the conclusion. □

We now define the operator $A : W_0^{s,p}(\Omega) \to W_0^{-s,p'}(\Omega)$ as

$$\langle A(u), v\rangle := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy, \quad \text{for all } u, v \in W_0^{s,p}(\Omega).$$

A weak solution of problem (1.1) is a function $u \in W_0^{s,p}(\Omega)$ such that

$$\langle A(u), v\rangle = \int_\Omega f(x,u)vdx, \quad \text{for all } u, v \in W_0^{s,p}(\Omega).$$

It is easy to see that A satisfies the following compactness condition [15,20]:

($\mathcal{S}$) If $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ is such that $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and $\langle A(u_n), u_n - u\rangle \to 0$ as $n \to \infty$, then $u_n \to u$ in $W_0^{s,p}(\Omega)$ as $n \to \infty$.

Assuming that f satisfies condition (1.2) for some exponents $q_i \in [1, p_s^*)$ and $h_i \in \mathscr{B}_{q_i}$, in light of Lemma 2.4, there exists a constant $C > 0$ such that, for all $u, v \in W_0^{s,p}(\Omega)$,

$$\left|\int_\Omega f(x,u)v\,dx\right| \le \sum_{i=1}^n \int_\Omega h_i(x)|u|^{q_i-1}|v|dx \le C\sum_{i=1}^n \|u\|^{q_i-1}|v|_{b_i} \le C_u\|v\|,$$

so that $f(x,u) \in W_0^{-s,p'}(\Omega)$. Weak solutions of (1.1) are thus critical points of $\Phi : W_0^{s,p}(\Omega) \to \mathbb{R}$,

$$\Phi(u) := \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx, \qquad F(x,t) := \int_0^t f(x,s)ds.$$

By the property ($\mathcal{S}$), we easily obtain the next lemma.

Lemma 2.5 (Palais-Smale condition)**.** *Assume that f satisfies (1.2) for some $q_i \in [1, p_s^*)$ and $h_i \in \mathscr{B}_{q_i}$. Then any bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ such that $\Phi'(u_n) \to 0$ has a convergent subsequence. In particular, bounded Palais-Smale sequences of Φ are precompact in $W_0^{s,p}(\Omega)$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{s,p}(\Omega)$, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and $|u_n - u|_{b_i} \to 0$ as $n \to \infty$ for $i = 1, \dots, n$, since $b_i \in (1, p_s^*)$. Then, in light of Lemma 2.4, we get

$$\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| \le \sum_{i=1}^n \int_\Omega |h_i(x)||u_n|^{q_i-1}|u_n-u|dx \le C\sum_{i=1}^n \|u_n\|^{q_i-1}|u_n-u|_{b_i},$$

for some positive constant C. This yields

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx = 0,$$

via the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{s,p}(\Omega)$ and $u_n \to u$ in $L^{b_i}(\Omega)$ for any i. Thus,

$$\langle A(u_n), u_n-u\rangle = \langle \Phi'(u_n), u_n-u\rangle + \int_\Omega f(x,u_n)(u_n-u)dx \to 0.$$

Hence, $u_n \to u$ in $W_0^{s,p}(\Omega)$ as $n \to \infty$, by means of $(\mathcal{S})$. $\square$

3. Eigenvalue problems

We consider the eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

where $h \in \mathscr{A}_p$ is possibly sign-changing with $|\{x \in \Omega : h(x) > 0\}| > 0$ and λ is a real number. We consider the first eigenvalue, defined as follows

$$\mu_1 := \inf\left\{ \frac{1}{p}\|u\|^p : u \in W_0^{s,p}(\Omega),\ \int_\Omega \frac{h(x)}{p}|u|^p\, dx = 1 \right\}.$$

We have the following

Theorem 3.1. *Let $h \in \mathscr{B}_p$. Then μ_1 is attained by some nonnegative $\phi_1 \in W_0^{s,p}(\Omega)$. Furthermore, $\phi_1 > 0$ a.a. if $h \geq 0$, and any two first eigenfunctions are proportional.*

Proof. Since $h \in \mathscr{B}_p$, a standard argument yields the existence of an eigenfunction $\phi_1 \geq 0$. In fact, notice that, if $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ is weakly convergent to some u, we have

$$\begin{aligned} \left| \int_\Omega h(x)|u_n|^p dx - \int_\Omega h(x)|u|^p dx \right| &\leq C \int_\Omega h(x)|u_n|^{p-1}|u_n - u| dx + C \int_\Omega h(x)|u_n|^{p-1}|u_n - u| dx \\ &\leq C(\|u_n\|^{p-1} + \|u\|^{p-1})|u_n - u|_b \leq C|u_n - u|_b = o_n(1), \end{aligned}$$

by Lemma 2.4 and since $|u_n - u|_b \to 0$ up to a subsequence, where $b \in (1, p_s^*)$ is such that $1/r + a/p + (p-a)/b = 1$, for some $a \in [0, p-1]$. It follows from [2, Theorem A.1] that $\phi_1 > 0$, provided $h \geq 0$. The simplicity follows as in [12]. $\square$

Next we show the boundedness of weak solutions by modifying the argument in [12].

Proposition 3.2. *Let u be any eigenfunction of (3.1). Then $u \in L^\infty(\mathbb{R}^N)$.*

Proof. The proof follows the line of [12]. We shall provide the details for the sake of completeness. Denoting $u_+ := \max\{u, 0\}$, it suffices to show that, for any weak solution $u \in W_0^{s,p}(\Omega)$,

$$|u|_\infty \leq 1 \text{ provided that } |u_+|_q \leq \delta, \quad q := \frac{pr}{r-1} \in (1, p_s^*), \tag{3.2}$$

for some $\delta > 0$. For each $k \in \mathbb{N} \cup \{0\}$, set $w_k := (u - (1 - 1/2^k))_+$. Then $w_k \in W_0^{s,p}(\Omega)$ and

$$w_{k+1}(x) \leq w_k(x) \quad \text{a.a.,} \quad u(x) < (2^{k+1} - 1)w_k(x) \quad \text{a.a. } x \in \{w_{k+1} > 0\},$$

and $\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\}$. Moreover, for a measurable v, we have the inequality

$$|v(x) - v(y)|^{p-2}(v_+(x) - v_+(y))(v(x) - v(y)) \geq |v_+(x) - v_+(y)|^p \tag{3.3}$$

for a.a. $x, y \in \mathbb{R}^N$. Applying (3.3) for $v = u - (1 - 1/2^{k+1})$, we have $v_+ = w_{k+1}$ and

$$\begin{aligned} \|w_{k+1}\|^p &\leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)(w_{k+1}(x) - w_{k+1}(y))}{|x - y|^{N+sp}} dxdy \\ &= \lambda \int_{\{w_{k+1}>0\}} h(x)|u(x)|^{p-2}u(x)w_{k+1}(x)dx. \end{aligned}$$

Whence, by the above stated properties, we get

$$\|w_{k+1}\|^p \leq |\lambda|(2^{k+1} - 1)^{p-1} \int_{\{w_{k+1}>0\}} |h(x)||w_k(x)|^p dx.$$

By the Hölder inequality, we then obtain

$$\|w_{k+1}\|^p \le |\lambda|(2^{k+1}-1)^{p-1}|h|_r U_k, \quad U_k := |w_k|_q^p.$$

Let $\bar{q}$ be such that $q < \bar{q} < p_s^*$. Using again Hölder inequality, we easily get

$$U_{k+1} \le C\|w_{k+1}\|^p |\{w_{k+1} > 0\}|^{\frac{p(\bar{q}-q)}{q\bar{q}}}$$

by the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\bar{q}}(\Omega)$. On the other hand, Chebychev's inequality entails

$$|\{w_{k+1} > 0\}| \le |\{w_k > 2^{-(k+1)}\}| \le 2^{q(k+1)} U_k^{q/p}.$$

Combining the previous inequality yields $U_{k+1} \le C|\lambda|(2^{k+1}-1)^{p-1}|h|_r 2^{\frac{p(\bar{q}-q)(k+1)}{\bar{q}}} U_k^{1+\frac{\bar{q}-q}{\bar{q}}}$, namely

$$U_{k+1} \le C_0 b^k U_k^{1+\alpha}, \qquad k \in \mathbb{N},\ C_0 > 0,\ \alpha > 0,\ b > 1.$$

By [9, Lemma 4.7, Ch. II], this yields $U_k \to 0$ as $k \to \infty$ if $U_0 \le {C_0}^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$. Hence, if we choose

$$|u_+|_q \le {C_0}^{-\frac{1}{p\alpha}} b^{-\frac{1}{p\alpha^2}} =: \delta,$$

which is made possible by a simple scaling argument due to the homogeneity of the problem, we conclude $U_k \to 0$ as $k \to \infty$, namely $|u|_\infty \le 1$, via Fatou's Lemma, concluding the proof. □

Next we consider the eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda h(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.4}$$

where $q \in [1, p_s^*)$ and $h \in \mathscr{B}_q$ with $|\{x \in \Omega : h(x) > 0\}| > 0$. We shall produce a sequence of eigenvalues for problem (3.4), following the argument in [21]. Let

$$J(u) := \int_\Omega \frac{h(x)}{q}|u|^q \, dx, \quad u \in W_0^{s,p}(\Omega),$$

and set (we use the notations of [20, Chapter 4])

$$\Psi(u) := \frac{1}{J(u)}, \quad u \in \mathcal{M}, \qquad \mathcal{M} := \left\{ u \in W_0^{s,p}(\Omega) : \frac{1}{p}\|u\|^p = 1 \text{ and } J(u) > 0 \right\}.$$

Then $\mathcal{M}$ is nonempty and positive eigenvalues and associated eigenfunctions of (3.4) on $\mathcal{M}$ coincide with critical values and critical points of Ψ, respectively. By Lemma 2.4, we have

$$0 < J(u) \le C\|u\|^q \le C, \quad \text{for all } u \in \mathcal{M},$$

for some constant $C > 0$, and hence $\lambda_1 := \inf_{u \in \mathcal{M}} \Psi(u) > 0$. A slight variant of Lemma 2.5 yields the following

Lemma 3.3. *For all $c \in \mathbb{R}$, Ψ satisfies the Palais-Smale condition, namely every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ with $\Psi(u_n) \to c$ and $\Psi'(u_n) \to 0$ has a subsequence converging to some $u \in \mathcal{M}$.*

Although one can obtain an increasing and unbounded sequence of critical values of Ψ via standard minimax schemes, we prefer to use a cohomological index as in Perera [19], which provides additional topological information about the associated critical points. Let us recall the definition of the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [10]. Let W be a Banach space. For a symmetric subset M of $W \setminus \{0\}$, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \overline{M} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{M}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{M})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup \left\{ m \geq 1 : f^*(\omega^{m-1}) \neq 0 \right\}, & M \neq \emptyset, \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$, is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 1.$$

Then $\{\lambda_k\}_{k \in \mathbb{N}}$ is a sequence of positive eigenvalues of (3.4), $\lambda_k \nearrow +\infty$, and

$$i(\{u \in \mathcal{M} : \Psi(u) \leq \lambda_k\}) = i(\{u \in \mathcal{M} : \Psi(u) < \lambda_{k+1}\}) = k \tag{3.5}$$

if $\lambda_k < \lambda_{k+1}$, see [20, Propositions 3.52 and 3.53]. As a simple application, we consider the problem

$$\begin{cases} (-\Delta)_p^s u = h(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.6}$$

By arguing as in the proof of [21, Theorem 3.2], we have the following

Theorem 3.4. *Let $q \in [1, p_s^*) \setminus \{p\}$ and $h \in \mathscr{B}_q$ with $|\{x \in \Omega : h(x) > 0\}| > 0$. Then problem (3.6) admits a sequence of nontrivial weak solutions $\{u_n\}_{n \in \mathbb{N}} \subset W_0^{s,p}(\Omega)$ such that*

i) *if $q < p$, then $\|u_n\| \to 0$ as $n \to \infty$,*
ii) *if $q > p$, then $\|u_n\| \to \infty$ as $n \to \infty$.*

4. Critical groups

In this section we compute the critical groups at zero of the functional

$$\Phi(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega h(x)|u|^p dx - \int_\Omega G(x,u)dx, \quad u \in W_0^{s,p}(\Omega),$$

where $G(x,t) = \int_0^t g(x,\tau)\, d\tau$, which is associated with the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda h(x)|u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

where $\lambda \geq 0$ is a parameter, $h \in \mathscr{B}_p$ with $|\{x \in \Omega : h(x) > 0\}| > 0$ and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical p-superlinear growth condition

$$|g(x,t)| \leq \sum_{i=1}^n K_i(x)|t|^{q_i - 1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{4.2}$$

for some $q_i \in (p, p_s^*)$ and $K_i \in \mathscr{B}_{q_i}$. The critical groups of Φ at zero are given by

$$C^q(\Phi, 0) := H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \geq 0, \tag{4.3}$$

where $\Phi^0 := \Phi^{-1}((-\infty, 0])$, U is any neighborhood of 0, and H denotes the Alexander-Spanier cohomology with $\mathbb{Z}_2$-coefficients. Following the steps in [21], we can obtain

Theorem 4.1 (Critical groups I)**.** *Assume that $h \in \mathscr{B}_p$ with $|\{x \in \Omega : h(x) > 0\}| > 0$, g satisfies (4.2) and 0 is an isolated critical point of Φ. Then we have*

(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ *and* $C^q(\Phi, 0) = 0$ *for* $q \geq 1$ *in the following cases:*
  (a) $0 \leq \lambda < \lambda_1$*;*
  (b) $\lambda = \lambda_1$ *and* $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*.*

(2) $C^k(\Phi, 0) \neq 0$ *in the following cases:*
  (a) $\lambda_k < \lambda < \lambda_{k+1}$*;*
  (b) $\lambda = \lambda_k < \lambda_{k+1}$ *and* $G(x, t) \geq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*
  (c) $\lambda_k < \lambda_{k+1} = \lambda$ *and* $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*.*

In the absence of a direct sum decomposition, the main technical tool we use to get an estimate of the critical groups is the notion of a cohomological local splitting introduced in [20]. When $sp > N$, it suffices to assume the sign conditions on G in Theorem 4.1 for small $|t|$ by the imbedding $W_0^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Theorem 4.2 (Critical groups II)**.** *Assume that* $sp > N$, $h \in \mathscr{B}_p$ *with* $|\{x \in \Omega : h(x) > 0\}| > 0$, g *satisfies* (4.2) *with* $q_i > p$ *and* 0 *is an isolated critical point of* Φ*. Then we have*

(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ *and* $C^q(\Phi, 0) = 0$ *for* $q \geq 1$ *if* $\lambda = \lambda_1$ *and, for some* $\delta > 0$, $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*.*

(2) $C^k(\Phi, 0) \neq 0$ *in the following cases:*
  (a) $\lambda = \lambda_k < \lambda_{k+1}$ *and, for some* $\delta > 0$, $G(x, t) \geq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*;*
  (b) $\lambda_k < \lambda_{k+1} = \lambda$ *and, for some* $\delta > 0$, $G(x, t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*.*

As we will show next, the conclusions of Theorem 4.2 hold for $sp \leq N$ when the weights h and K_i belong to suitable strengthened subclasses of $\mathscr{B}_p$ and $\mathscr{B}_{q_i}$, respectively.

Definition 4.3 (Class of weights $\widetilde{\mathscr{B}}_q$)**.** *For* $sp \leq N$ *and* $q \in [1, p_s^*)$*, we denote by* $\widetilde{\mathscr{B}}_q$ *the class of functions* K *with* $K\rho^{sa} \in L^r(\Omega)$ *for some* $a \in [0, q-1]$ *and* $r \in (1, \infty)$ *satisfying*

$$\frac{1}{r} + \frac{a}{p} + \frac{q-1-a}{p_s^*} < \frac{sp}{N}.$$

Remark 4.4. Note that $\widetilde{\mathscr{B}}_q = \mathscr{B}_q$ when $sp = N$ and $\widetilde{\mathscr{B}}_q \subset \mathscr{B}_q$ when $sp < N$ since

$$\frac{1}{p_s^*} + \frac{sp}{N} < 1.$$

We have the following

Lemma 4.5. *Let* $sp \leq N$, $q \in [1, p_s^*)$ *and* $K \in \widetilde{\mathscr{B}}_q$*. Then there exists* $\tau > N/sp$ *such that*

$$\left|K(x)\,|u|^{q-1}\right|_\tau \leq C\|u\|^{q-1}$$

for all $u \in W_0^{s,p}(\Omega)$*, for some constant* $C > 0$*.*

Proof. Let $0 \leq a \leq q-1$ and $r > 1$ be as in Definition 4.3. Then, there exists $b < p_s^*$ sufficiently close to p_s^* such that

$$\frac{1}{r} + \frac{a}{p} + \frac{q-1-a}{b} < \frac{sp}{N}.$$

Then, choosing

$$\tau := \left(\frac{1}{r} + \frac{a}{p} + \frac{q-1-a}{b}\right)^{-1} > \frac{N}{sp},$$

by the Hölder inequality, we obtain

$$\int_\Omega |K(x)|^\tau\,|u|^{(q-1)\,\tau}\,dx = \int_\Omega |K\rho^{sa}|^\tau \left|\frac{u}{\rho^s}\right|^{a\tau} |u|^{(q-1-a)\,\tau}\,dx \leq |K\rho^{sa}|_r^\tau \left|\frac{u}{\rho^s}\right|_p^{a\tau} |u|_b^{(q-1-a)\,\tau}.$$

In light of Theorem 2.1 and $|u|_b \leq C\|u\|$, the conclusion follows. $\square$

Solutions of $(-\Delta)_p^s u = f(x)$ enjoy the useful L^q-estimate given next (cf. [18, Lemma 2.3]).

Lemma 4.6 (Summability lemma)**.** *Let $f \in L^q(\Omega)$ for some $1 < q \leq \infty$ and assume that $u \in W_0^{s,p}(\Omega)$ is a weak solution of the equation $(-\Delta)_p^s u = f(x)$ in Ω. Then*

$$|u|_r \leq C\,|f|_q^{1/(p-1)}, \tag{4.4}$$

where

$$r := \begin{cases} \dfrac{N\,(p-1)\,q}{N-spq}, & 1 < q < \dfrac{N}{sp}, \\[2mm] \infty, & \dfrac{N}{sp} < q \leq \infty, \end{cases}$$

and $C = C(N, \Omega, p, s, q) > 0$.

Assume that $sp \leq N$, $h \in \widetilde{\mathscr{B}}_p$ and $K_i \in \widetilde{\mathscr{B}}_{q_i}$. First we show that the critical groups of Φ at zero depend only on the values of $g(x,t)$ for small $|t|$.

Lemma 4.7. *Let $\delta > 0$ and let $\vartheta : \mathbb{R} \to [-\delta, \delta]$ be a smooth nondecreasing function such that $\vartheta(t) = -\delta$ for $t \leq -\delta$, $\vartheta(t) = t$ for $-\delta/2 \leq t \leq \delta/2$ and $\vartheta(t) = \delta$ for $t \geq \delta$. Set*

$$\Phi_1(u) := \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega h(x)|u|^p dx - \int_\Omega G(x, \vartheta(u))dx, \quad u \in W_0^{s,p}(\Omega).$$

If 0 is an isolated critical point of Φ, then it is also an isolated critical point of Φ_1 and

$$C^q(\Phi, 0) \approx C^q(\Phi_1, 0) \quad \forall q.$$

Proof. By applying [21, Proposition 4.1] to the family of functionals

$$\Phi_\tau(u) := \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega h(x)|u|^p dx - \int_\Omega G(x, (1-\tau)\,u + \tau\,\vartheta(u))dx, \quad u \in W_0^{s,p}(\Omega),\ \tau \in [0,1]$$

in a small ball $B_\varepsilon(0) = \{u \in W_0^{s,p}(\Omega) : \|u\| \leq \varepsilon\}$, observing that each Φ_τ satisfies the Palais-Smale condition over $B_\varepsilon(0)$ in light of Lemma 2.5 and that the map $[0,1] \ni \tau \to \Phi_\tau \in C^1(B_\varepsilon(0), \mathbb{R})$ is continuous, we need to show that for sufficiently small ε, $B_\varepsilon(0)$ contains no critical point of any Φ_τ other than 0. Suppose $u_j \to 0$ in $W_0^{s,p}(\Omega)$, $\Phi'_{\tau_j}(u_j) = 0$, $\tau_j \in [0,1]$ and $u_j \neq 0$. Then

$$\begin{cases} (-\Delta)_p^s u_j = \lambda h(x)|u_j|^{p-2}u_j + g_j(x, u_j) & \text{in } \Omega, \\ u_j = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$g_j(x,t) := (1 - \tau_j + \tau_j\,\vartheta'(t))\,g(x, (1-\tau_j)\,t + \tau_j\,\vartheta(t)).$$

Following the proof of [21, Lemma 4.8] where we use Lemma 4.5 several times, we obtain that $u_j \in L^\infty(\Omega)$ and $u_j \to 0$ in $L^\infty(\Omega)$ by means of Lemma 4.6. Then, for sufficiently large $j \in \mathbb{N}$, $|u_j(x)| \leq \delta/2$ for a.a. $x \in \Omega$ and hence $\Phi'(u_j) = \Phi'_{\tau_j}(u_j) = 0$, contradicting the assumption that 0 was an isolated critical point of Φ. □

Lemma 4.7 and Theorem 4.1 immediately give

Theorem 4.8 (Critical groups III)**.** *Assume that $sp \leq N$, $h \in \widetilde{\mathscr{B}}_p$ with $|\{x \in \Omega : h(x) > 0\}| > 0$, g satisfies condition (4.2) with $q_i \in (p, p_s^*)$ and $K_i \in \widetilde{\mathscr{B}}_{q_i}$, and 0 is an isolated critical point of Φ. Then we have*

(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ and $C^q(\Phi, 0) = 0$ for $q \geq 1$ if $\lambda = \lambda_1$ and, for some $\delta > 0$, $G(x,t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$.

(2) $C^k(\Phi, 0) \neq 0$ in the following cases:

(a) $\lambda = \lambda_k < \lambda_{k+1}$ and, for some $\delta > 0$, $G(x,t) \geq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;

(b) $\lambda_k < \lambda_{k+1} = \lambda$ *and, for some* $\delta > 0$, $G(x,t) \le 0$ *for a.a.* $x \in \Omega$ *and* $|t| \le \delta$.

5. Nontrivial solutions

We now investigate the existence of nontrivial solutions of the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda h(x)|u|^{p-2}u + K(x)|u|^{q-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1}$$

where $q \in (p, p_s^*)$, $K \in \mathscr{B}_q$ satisfies

$$\inf_{x\in\Omega} K(x) > 0, \tag{5.2}$$

and g satisfies (4.2) with each $q_i \in (p,q)$. Once again, we will first find suitable subclasses of $\mathscr{B}_p$ and $\mathscr{B}_{q_i}$ for the weights h and K_i, respectively, to ensure the Palais-Smale condition.

Definition 5.1 (Class of weights $\mathscr{B}_t^q$)**.** *For* $q \in (1, p_s^*)$ *and* $t \in [1,q)$, *let* $\mathscr{B}_t^q$ *denote the class of functions* K *such that* $K\rho^{sa} \in L^r(\Omega)$ *for some* $a \in [0, t-1]$ *and* $r \in (1,\infty)$ *satisfying*

$$\frac{1}{r} + \frac{a}{p} + \frac{t-a}{q} \le 1. \tag{5.3}$$

Clearly, $\mathscr{B}_t^q \subset \mathscr{B}_t$.

Lemma 5.2. *Let* $q \in [1, p_s^*)$, $t \in [1,q)$ *and* $K \in \mathscr{B}_t^q$. *Then there exist* $0 \le m < p$ *and, for any* $\varepsilon > 0$, *a constant* $C(\varepsilon) > 0$ *such that*

$$\int_\Omega |K(x)|\,|u|^t\,dx \le C(\varepsilon)\|u\|^m + \varepsilon|u|_q^q \qquad \textit{for every } u \in W_0^{s,p}(\Omega).$$

Proof. Denoting a and r as in Definition 5.1, by the Hölder inequality, we have

$$\int_\Omega |K(x)|\,|u|^t\,dx = \int_\Omega |K\rho^{sa}| \left|\frac{u}{\rho^s}\right|^a |u|^{t-a}\,dx \le |K\rho^{sa}|_r \left|\frac{u}{\rho^s}\right|_p^a |u|_b^{t-a},$$

where $1/r + a/p + (t-a)/b = 1$ and hence $b \le q$ by (5.3). The last expression is less than or equal to $C\|u\|^a|u|_q^{t-a}$ due to $|u|_b \le C|u|_q$ and Theorem 2.1. The Young inequality implies that the latter is less than or equal to $C(\varepsilon)\|u\|^m + \varepsilon|u|_q^q$, where $a/m + (t-a)/q = 1$ when $a > 0$ and $m = 0$ when $a = 0$. Hence $0 \le m < p$ by means of (5.3). ∎

Assuming that $h \in \mathscr{B}_p^q$ and $K_i \in \mathscr{B}_{q_i}^q$, modifying the argument in the proof of [21, Lemma 5.3], the associated functional

$$\Phi(u) := \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\int_\Omega h(x)|u|^p dx - \frac{1}{q}\int_\Omega K(x)|u|^q dx - \int_\Omega G(x,u)dx, \qquad u \in W_0^{s,p}(\Omega),$$

where $G(x,t) := \int_0^t g(x,\tau)\,d\tau$, satisfies the Palais-Smale condition.

Lemma 5.3. *Any sequence* $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ *such that* $\{\Phi(u_n)\}_{n\in\mathbb{N}}$ *is bounded and* $\Phi'(u_n) \to 0$ *admits a convergent subsequence in* $W_0^{s,p}(\Omega)$.

Concerning the structure of the sublevel sets $\Phi^\alpha := \{u \in W_0^{s,p}(\Omega) : \Phi(u) \le \alpha\}$ for $\alpha < 0$ with $|\alpha|$ large, one can follow the proofs of [21, Lemma 5.4 and 5.5] and get

Lemma 5.4. *There exists* $\alpha < 0$ *such that* Φ^α *is contractible in itself.*

We now state the main existence result of this section.

Theorem 5.5. *Assume that* $\lambda \ge 0$, $q \in (p, p_s^*)$, $h \in \mathscr{B}_p^q$ *with* $|\{x \in \Omega : h(x) > 0\}| > 0$, $K \in \mathscr{B}_q$ *satisfies* (5.2), *and* g *satisfies* (4.2) *with* $K_i \in \mathscr{B}_{q_i}^q$ *for* $i = 1, \ldots, n$. *Then problem* (5.1) *has a nontrivial weak solution in each of the following cases:*

(1) $\lambda \notin \{\lambda_k : k \geq 1\}$*;*
(2) $G(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*
(3) $G(x,t) \leq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*
(4) $sp > N$ *and, for some* $\delta > 0$, $G(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*;*
(5) $sp > N$ *and, for some* $\delta > 0$, $G(x,t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*;*
(6) $sp \leq N$, $h \in \widetilde{\mathscr{B}}_p$, $K_i \in \widetilde{\mathscr{B}}_{q_i}$ *and, for some* $\delta > 0$, $G(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*;*
(7) $sp \leq N$, $h \in \widetilde{\mathscr{B}}_p$, $K_i \in \widetilde{\mathscr{B}}_{q_i}$ *and, for some* $\delta > 0$, $G(x,t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$.

Proof. Suppose that 0 is the only critical point of Φ. Taking $U = W_0^{s,p}(\Omega)$ in (4.3), we have

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}).$$

Let $\alpha < 0$ be as in Lemma 5.4. Since Φ has no other critical points and satisfies the Palais-Smale condition by Lemma 5.3, Φ^0 is a deformation retract of $W_0^{s,p}(\Omega)$ and Φ^α is a deformation retract of $\Phi^0 \setminus \{0\}$ by the second deformation lemma. So

$$C^q(\Phi, 0) \approx H^q(W_0^{s,p}(\Omega), \Phi^\alpha) = 0 \quad \forall q$$

since Φ^α is contractible in itself, contradicting Theorem 4.1, Theorem 4.2, or Theorem 4.8. □

6. Multiplicity

In this section we show the existence of infinitely many solutions via the Fountain Theorem. Since $W_0^{s,p}(\Omega)$ is separable, there exist $\{e_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset W_0^{-s,p'}(\Omega)$ with

$$W_0^{s,p}(\Omega) = \overline{\operatorname{span}\{e_n\}_{n=1}^\infty}, \qquad W_0^{-s,p'}(\Omega) = \overline{\operatorname{span}\{f_n\}_{n=1}^\infty}, \qquad \langle f_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W_0^{-s,p'}(\Omega)$ and $W_0^{s,p}(\Omega)$ (see [29, Section 17]). Let

$$X_n = \operatorname{span}\{e_n\}, \qquad Y_n = \bigoplus_{k=1}^n X_k, \qquad Z_n = \overline{\bigoplus_{k=n}^\infty X_k}.$$

With X_n, Y_n, Z_n taken as the above, we have [28, Fountain Theorem]

Theorem 6.1. *Let* $\Phi \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$ *be even and suppose that there exist* $\rho_n > \gamma_n > 0$ *such that*

$(\mathcal{H}_1)$ $a_n = \inf\limits_{u \in Z_n,\, \|u\| = \gamma_n} \Phi(u) \to +\infty$ *as* $n \to \infty$*;*
$(\mathcal{H}_2)$ $b_n = \max\limits_{u \in Y_n,\, \|u\| = \rho_n} \Phi(u) \leq 0$*;*
$(\mathcal{H}_3)$ Φ *satisfies the Palais-Smale condition at the level* c *for all* $c > 0$.

Then Φ *has a sequence of critical values tending to* $+\infty$.

Invoking this theorem, we obtain the existence of infinitely many solutions for problem (5.1).

Theorem 6.2. *Assume that* $q \in (p, p_s^*)$, $h \in \mathscr{B}_p^q$ *with* $|\{x \in \Omega : h(x) > 0\}| > 0$, $K \in \mathscr{B}_q$ *satisfies* (5.2)*,* g *satisfies* (4.2) *with* $K_i \in \mathscr{B}_{q_i}^q$ *and* $g(x, -u) = -g(x, u)$*. Then problem* (5.1) *admits a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ *of solutions such that* $\Phi(u_n) \to +\infty$ *as* $n \to \infty$.

Proof. Due to Lemma 5.3, we only need to verify conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. For $(\mathcal{H}_1)$, let

$$\beta_n' := \sup\{|u|_q : u \in Z_n, \|u\| = 1\}, \qquad \beta_n'' := \sup\{|u|_b : u \in Z_n, \|u\| = 1\},$$

where b satisfies $1/r + a/p + (q-a)/b = 1$, where a, r are as in $\mathscr{B}_q$. Set $\beta_n = \max\{\beta_n', \beta_n''\}$. Due to the compact injections $W_0^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^b(\Omega)$ and $W_0^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ we have that $\beta_n \to 0$ as $n \to \infty$ in view of the abstract result [11, Lemma 3.3]. Then there exists $n_0 \in \mathbb{N}$ such that

$\beta_n \leq 1$ for all $n \geq n_0$. For each $n \in \mathbb{N}$, define γ_n by $\gamma_n := 1/\sqrt{\beta_n}$ and hence, $\gamma_n \geq 1$ for all $n \geq n_0$ and $\gamma_n \to \infty$ as $n \to \infty$. For $u \in Z_n$ with $\|u\| = \gamma_n$, we have

$$\Phi(u) \geq \frac{1}{p}\|u\|^p - \frac{|\lambda|}{p}\int_\Omega |h||u|^p dx - \frac{1}{q}\int_\Omega K|u|^q dx - \sum_{i=1}^n \frac{1}{q_i}\int_\Omega |K_i||u|^{q_i} dx. \tag{6.1}$$

By Lemma 5.2 for any $\varepsilon > 0$ there exist $0 \leq m_i < p$ and $C_i(\varepsilon) > 0$, $i = 0, \cdots, n$, such that

$$\int_\Omega |h||u|^p dx \leq \varepsilon\|u\|^{m_0} + C_0(\varepsilon)|u|_q^q, \qquad \int_\Omega |K_i||u|^{q_i} dx \leq \varepsilon\|u\|^{m_i} + C_i(\varepsilon)|u|_q^q,$$

and, see the proof of Lemma 2.4,

$$\int_\Omega K|u|^q dx \leq C\|u\|^a|u|_b^{q-a}.$$

Applying these estimates with $\varepsilon < \frac{1}{2(n+1)p}$ we deduce from (6.1) that, for any $n \geq n_0$,

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{p}\|u\|^p - \frac{C}{q}\|u\|^a|u|_b^{q-a} - \varepsilon\sum_{i=0}^n \|u\|^{m_i} - \Big(\sum_{i=0}^n C_i(\varepsilon)\Big)|u|_q^q \\
&\geq \frac{1}{p}\|u\|^p - \frac{C}{q}\|u\|^a|u|_b^{q-a} - (n+1)\varepsilon\|u\|^p - \Big(\sum_{i=0}^n C_i(\varepsilon)\Big)|u|_q^q \\
&\geq \frac{1}{2p}\|u\|^p - \frac{C}{q}\|u\|^a|u|_b^{q-a} - \Big(\sum_{i=0}^n C_i(\varepsilon)\Big)|u|_q^q \\
&\geq \frac{\beta_n^{-\frac{p}{2}}}{2p} - \frac{C}{q}\beta_n^{\frac{q}{2}-a} - \Big(\sum_{i=0}^n C_i(\varepsilon)\Big)\beta_n^{\frac{q}{2}} \to \infty,
\end{aligned}$$

as $n \to \infty$ since $0 \leq a < \frac{p+q}{2}$ yielding $(\mathcal{H}_1)$. For $(\mathcal{H}_2)$, again by Lemma 5.2, for $u \in Y_n$ we have

$$\begin{aligned}
\Phi(u) &\leq \frac{1}{p}\|u\|^p + \frac{|\lambda|}{p}\int_\Omega |h||u|^p dx - \frac{1}{q}\int_\Omega K|u|^q dx + \sum_{i=1}^n \frac{1}{q_i}\int_\Omega |K_i||u|^{q_i} dx \\
&\leq \frac{1}{p}\|u\|^p + \frac{|\lambda|}{q}\left(\widehat{C}_0(\varepsilon)\|u\|^{m_0} + \varepsilon|u|_q^q\right) + \sum_{i=1}^n \frac{1}{q_i}\left(\widehat{C}_i(\varepsilon)\|u\|^{m_i} + \varepsilon|u|_q^q\right) - \frac{1}{q}|u|^q_{L^q(K,\Omega)}.
\end{aligned}$$

Since $\dim(Y_n) < \infty$, the norms $\|\cdot\|$, $|\cdot|_q$ and $|\cdot|_{L^q(K,\Omega)}$ are equivalent. As $p, m_i < q$, choosing ε small enough the last estimate yields $\Phi(u) \leq 0$ for all $u \in Y_n$ with $\|u\|$ large enough. This completes $(\mathcal{H}_2)$. The proof is now complete. □

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