THE ROBIN-LAPLACIAN PROBLEM ON VARYING DOMAINS

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Abstract. We prove a stability result for elliptic equations under general Dirichlet-Robin boundary conditions with respect to the variation of the domain under the Hausdorff complementary topology. As a by-product, under the additional assumption of the convergence of the perimeters, we obtain a stability result for the classical Robin-Laplacian.

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1. Introduction

In the present paper we will study the effect of domain perturbation on elliptic problems with general Dirichlet-Robin boundary conditions. To be precise, we will focus on problems of the form

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \mu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, ν denoting the exterior normal, $f \in L^2(\Omega)$, and μ is a Borel measure with support on $\partial\Omega$. The weak formulation of the problem is as usual

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} uv \, d\mu = \int_\Omega fu \, dx$$

for every admissible test function v. The natural functional framework for (1.1) is given by the space

$$H^1(\Omega) \cap L^2(\partial\Omega;\mu) := \{u \in H^1(\Omega) \,:\, \text{trace}(u) \in L^2(\partial\Omega;\mu)\}.$$

In order for the space to be well defined and for the problem to be well posed (see Section 3) we require that the measure μ is such that

$$\mu \text{ is absolutely continuous with respect to } c_2\text{-capacity}$$

and

$$\mu \geq c\mathcal{H}^{N-1}\lfloor\partial\Omega,$$

where $c > 0$ and $\mathcal{H}^{N-1}\lfloor\partial\Omega$ denotes the restriction of $(N-1)$-Hausdorff measure to $\partial\Omega$, that is the usual area measure on the boundary (see Section 2 for a precise definition of c_2-capacity).

The choice of the measure μ can lead to Robin as well as Dirichlet boundary conditions. Indeed, if we require μ to be of the form

$$\mu := \beta\mathcal{H}^{N-1}\lfloor\partial\Omega$$

with $\beta > 0$, the functional space involved is simply $H^1(\Omega)$ and we obtain the classical Robin boundary condition (called also Fourier condition)

$$\frac{\partial u}{\partial \nu}(x) + \beta u(x) = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega. \tag{1.2}$$

Variable Robin boundary conditions can be obtained by replacing β with a function defined on $\partial\Omega$. If we choose μ to be infinite on $\partial\Omega$ and zero outside, the functional space reduces to $H^1_0(\Omega)$ and the problem involves thus zero boundary conditions. Both Dirichlet and Robin conditions on different parts of the boundary can be handled under appropriate choices for μ.

Our main concern will be the asymptotic behaviour of solutions u_n of the elliptic problems

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n \\ \frac{\partial u_n}{\partial \nu} + \mu_n u_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{1.3}$$

when the Lipschitz domains Ω_n, contained in a fixed open and bounded box $D \subset \mathbb{R}^N$, approach in a suitable sense a limit domain Ω: here $f \in L^2(D)$ while, as above, μ_n is a Borel measure supported on $\partial\Omega_n$. More precisely, we will be interested in *stability* results, that is if a limit measure μ on $\partial\Omega$ is determined in such that u_n approaches the solution of the corresponding elliptic problem (1.1).

The case of Dirichlet boundary conditions has been extensively studied in the literature. The asymptotic behaviour of the solutions of (1.3) is captured by a *relaxed Dirichlet problem* as described in [8, 9]. If the limit problem is of Dirichlet type on the limit domain Ω (but this is not the case in general), the Ω_n are said to approach Ω in the sense of γ-convergence of domains: this kind of convergence is compatible, from a geometrical point of view, with highly singular perturbations of the limit domain. We refer the reader to [3, Chapter 4] for a survey on this topic.

The case of classical Robin boundary conditions, i.e., for $\mu_n = \beta \mathcal{H}^{N-1} \lfloor \partial\Omega_n$, has been addressed in [10] (see as well [12, Chapter 1, Section 6] for a survey of the topic): in that paper, the authors consider the case in which $\partial\Omega_n$ can be parametrized locally on $\partial\Omega$, the convergence of the boundaries being uniform, and prove that u_n approaches the solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta g(x) u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where the function g is connected to the limit on $\partial\Omega$ of the area measures on $\partial\Omega_n$, that is, from a geometrical point of view, on the *oscillations* of $\partial\Omega_n$. They show that g can be infinite for very rough oscillations, that is the limit problem can exhibit Dirichlet boundary conditions.

In the present paper we will consider (see Section 2 for precise definitions)

$$\Omega_n \to \Omega \quad \text{in the Hausdorff complementary topology,} \qquad |\Omega_n| \to |\Omega|, \tag{1.5}$$

and associated Borel measures μ_n such that

$$\begin{cases} \mu_n \text{ is concentrated on } \partial\Omega_n \text{ and absolutely continuous with respect to } c_2\text{-capacity,} \\ \mu_n \geq c\mathcal{H}^{N-1} \lfloor \partial\Omega_n \end{cases} \tag{1.6}$$

with c independent of n. From a geometrical point of view, approximation (1.5) does not require the "equi-graph" condition of [10], and it is compatible with singular perturbations like for examples the formations of holes concentrating at the boundary or that of sharp spikes with vanishing volume. Moreover condition (1.6) takes into account possible variations of the Dirichlet-Robin conditions on the approximating domains.

Our main result (Theorem 3.1) states that problems (1.3) are stable: we show that there exists a limit measure μ supported on $\partial\Omega$ satisfying the analogue of (1.6) such that u_n approaches the solution u of (1.1) (the type of convergence takes into account the fact that the functions are defined on different domains).

As a by-product of this stability result, we obtain a stability result for classical Robin boundary conditions: indeed we show in Theorem 3.3 that assuming in addition that

$$\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega), \tag{1.7}$$

then the limit measure is given by $\mu = \beta\mathcal{H}^{N-1}\lfloor\partial\Omega$, i.e., the limit problem is of Robin type. From a geometrical point of view, condition (1.7) prohibits oscillations of the boundaries in the convergence $\Omega_n \to \Omega$, which are responsible for the variation of the Robin conditions in agreement with the results of [10].

The main stability results are complemented by the convergence of the associated *resolvent operators* (see Theorem 3.2 and Corollary 3.4): in particular we show that under the same assumptions we have convergence of the whole spectrum of eigenvalues.

Concerning the proofs of the results, our approach is based on the study of the functionals

$$J_n(v) := \int_{\Omega_n} |\nabla v|^2\, dx + \int_{\partial\Omega_n} v^2\, d\mu_n - 2\int_{\Omega_n} fv\, dx \tag{1.8}$$

naturally associated to problem (1.3): the functions u_n are indeed minimizers of J_n such that (Lemma 4.4 and Remark 4.5)

$$\int_{\Omega_n} [|\nabla u_n|^2 + u_n^2]\, dx + \int_{\partial\Omega_n} u_n^2\, d\mathcal{H}^{N-1} \le C \tag{1.9}$$

for some C independent of n.

In order to compare the J_n in a unique functional framework, we extend u_n to the entire box D by setting them equal to zero outside Ω_n, and denote it by $u_n 1_{\Omega_n}$. In this way the function is simply an element of $L^2(D)$ since jumps can appear across $\partial\Omega_n$. It turns out (Lemma 4.2) that up to a subsequence

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{1.10}$$

for some $u \in H^1(\Omega)$.

In view of (1.10), we study the asymptotic behaviour of the functionals in the sense of Γ-convergence (see Section 2) under the L^2-topology. More precisely we will consider a localized version of (1.8)

$$F_n : L^2(D) \times \mathcal{A}(D) \to [0, +\infty]$$

given by

$$F_n(u, A) = \begin{cases} \int_{\Omega_n \cap A} |\nabla u|^2 dx + \int_{\partial\Omega_n \cap A} u^2 d\mu_n & \text{if } u \in H^1(\Omega_n \cap A), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.11}$$

In Theorem 3.6 we show that the variational limit F of (1.11) (the precise notion adapted to *localized* functionals is that of $\overline{\Gamma}$-convergence, see Section 2) can be described by means of a Borel measure μ supported on $\partial\Omega$, in such a way that (thanks to (1.10)) u is a minimizer of

$$J(v) := \int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} v^2\, d\mu - 2\int_\Omega fv\, dx,$$

i.e., it is the solution of (1.1). The convergence of the volume measures in (1.5) is crucial to study the structure of F, in particular, to prove that the measure μ is supported on $\partial\Omega$.

Let us remark that the $L^2(D)$-compactness (1.10) cannot be obtained using extension operators from $H^1(\Omega_n)$ to $H^1(D)$: indeed this would be the case under the assumption that Ω_n are "equi-Lipschitz", which is not entailed by (1.5). The key observation to infer the compactness is that, in view of inequality (1.9), $u_n^2 1_{\Omega_n}$ belongs to the space $BV(D)$ of functions of bounded variation (see Section 2), with a natural uniform bound on the associated norm. Convergence (1.10) is thus a consequence of the compact embedding properties of BV into Lebesgue spaces.

The connection between Robin boundary conditions and spaces of functions of bounded variation (and with suitable free discontinuity functionals) has been exploited recently in [2, 4] to develop a variational approach to the proof of the Faber-Krahn inequality for the Robin-Laplacian, and in [5] to study shape optimization problems under Robin conditions.

The paper is organized as follows. After some preliminaries collected in Section 2, we state the main results in Section 3. The proofs of the stability results are contained in Section 4, while Section 5 is devoted to the proof of Theorem 3.6 concerning the asymptotic behaviour of the functionals F_n which is pivotal for our analysis.

2. Notation and Preliminaries

In this section we introduce the basic notation and recall some notions employed in the rest of the paper. If $E \subseteq \mathbb{R}^N$, we will denote with $|E|$ its N-dimensional Lebesgue measure, and by $\mathcal{H}^{N-1}(E)$ its $(N-1)$-dimensional Hausdorff measure: we refer to [14, Chapter 2] for a precise definition, recalling that for sufficiently regular sets $\mathcal{H}^{N-1}$ coincides with the usual area measure. Moreover, we denote by E^c the complementary set of E, and by 1_E its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise.

If $A \subseteq \mathbb{R}^N$ is open and $1 \le p \le +\infty$, we denote by $L^p(A)$ the usual space of p-summable functions on A with norm indicated by $\|\cdot\|_p$. $H^1(A)$ will stand for the Sobolev space of functions in $L^2(A)$ whose gradient in the sense of distributions belongs to $L^2(A, \mathbb{R}^N)$. Finally $\mathcal{M}_b(A; \mathbb{R}^N)$ will denote the space of $\mathbb{R}^N$-valued Radon measures on A, which can be identified with the dual of $\mathbb{R}^N$-valued continuous functions on A vanishing at the boundary.

Hausdorff complementary topology on open sets. The family $\mathcal{K}(\mathbb{R}^N)$ of closed sets in $\mathbb{R}^N$ can be endowed with the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{\sup_{x\in K_1} \mathrm{dist}(x, K_2), \sup_{y\in K_2} \mathrm{dist}(y, K_1)\right\}$$

with the conventions $\mathrm{dist}(x, \emptyset) = +\infty$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = +\infty$ if $K \neq \emptyset$.

In order to study the behaviour of Robin problems under general domain variations, we will use the *Hausdorff complementary topology* on the family of open sets of $\mathbb{R}^N$ which is defined as follows. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^N$. We say that Ω_n converges to the open set $\Omega \subseteq \mathbb{R}^N$ in the Hausdorff complementary topology and write

$$\Omega_n \overset{\mathcal{H}^c}{\to} \Omega$$

if for every closed ball $B \subseteq \mathbb{R}^N$ we have

$$B \cap \Omega_n^c \to B \cap \Omega^c \quad \text{in the Hausdorff metric on } \mathcal{K}(\mathbb{R}^N).$$

Γ-convergence. Let us recall the definition of De Giorgi's Γ-*convergence* in metric spaces: we refer the reader to [6] for an exhaustive treatment of this subject. Let (X, d) be a metric space. We say that $F_n : X \to \overline{\mathbb{R}}$ Γ-converges to $F : X \to \overline{\mathbb{R}}$ (as $n \to +\infty$) if for all $u \in X$ the following items hold true.

(i) (Γ-*liminf inequality*) For every sequence $(u_n)_{n\in\mathbb{N}}$ converging to u in X,

$$\liminf_n F_n(u_n) \ge F(u).$$

(ii) (Γ-*limsup inequality*) There exists a sequence $(u_n)_{n\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_n F_n(u_n) \le F(u).$$

The function F is called the Γ-limit of the sequence $(F_n)_{n\in\mathbb{N}}$ (with respect to d), and we write $F = \Gamma\!-\lim_n F_n$. The following result holds true (see [6, Corollary 7.20 and Theorem 8.5])

Theorem 2.1. *Let X be a separable and metric space, and let $F_n : X \to \overline{\mathbb{R}}$.*

(a) *There exist $F : X \to \overline{\mathbb{R}}$ and a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ such that*

$$F_{n_k} \xrightarrow{\Gamma} F.$$

(b) *If $x_n \in X$ is a minimizer of F_n such that $x_n \to x \in X$, then x is a minimizer of F.*

Let $D \subseteq \mathbb{R}^N$ be open, and let $\mathcal{A}(D)$ denote the family of open subsets of D. We say that $F : X \times \mathcal{A}(D) \to \overline{\mathbb{R}}$ is increasing if

$$F(u, A) \le F(u, B)$$

for every $u \in X$ and $A, B \in \mathcal{A}(D)$ with $A \subseteq B$.

The variational convergence for this kind of functionals is called $\bar{\Gamma}$-convergence [6, Definitions 16.2 and 15.5].

Definition 2.2 (**$\overline{\Gamma}$-convergence**). *Let $F_n : X \times \mathcal{A}(D) \to \overline{\mathbb{R}}$ be an increasing functional defined on $X \times \mathcal{A}(D)$. We say that the sequence $(F_n)_{n\in\mathbb{N}}$ $\overline{\Gamma}$-converges to F (as $n \to +\infty$) if*

$$F(\cdot, A) = \sup_B F'(\cdot, B) = \sup_B F''(\cdot, B),$$

where the supremum is taken over the family of sets $B \in \mathcal{A}(D)$ such that $\overline{B} \subseteq A$, and

$$F'(u, B) := \inf\{\liminf_n F_n(u_n, B) \,:\, u_n \to u\} \text{ and } F''(u, B) := \inf\{\limsup_n F_n(u_n, B) \,:\, u_n \to u\}.$$

Notice that the $\overline{\Gamma}$-limit F is by definition increasing (with respect to the set inclusion) and lower semicontinuous. The following compactness result holds true [6, Theorem 16.9].

Proposition 2.3. *Let X be a separable metric space. Then every sequence $(F_n)_{n\in\mathbb{N}}$ of increasing functionals from $X \times \mathcal{A}(D)$ into $\overline{\mathbb{R}}$ has a $\overline{\Gamma}$-convergent subsequence.*

It turns out that $\overline{\Gamma}$-convergence is equivalent to the existence of a *rich set* $\mathcal{R} \subseteq \mathcal{A}(D)$ such that

$$\forall B \in \mathcal{R} \,:\, F_n(\cdot, B) \xrightarrow{\Gamma} F(\cdot, B). \tag{2.1}$$

Recall that a rich family $\mathcal{R} \subseteq \mathcal{A}(D)$ is such that for every $A_1, A_2 \in \mathcal{A}(D)$ with $A_1 \subset\subset A_2$, there exists $B \in \mathcal{R}$ with $A_1 \subset\subset B \subset\subset A_2$. In particular we have

$$F(u, A) = \sup_{B\subset\subset A, B\in\mathcal{R}} F(u, B). \tag{2.2}$$

Capacity. Let $E \subseteq \mathbb{R}^N$. We set

$$c_2(E) := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx \,:\, u \in H^1(\mathbb{R}^N), u \geq 1 \text{ a.e. on } E\right\}.$$

For the properties of c_2-capacity, and its relevance in the theory of Sobolev spaces, we refer the reader to [14].

We say that a property $\mathcal{P}(x)$ holds c_2-quasi everywhere (abbreviated c_2-q.e.) on a set $E \subseteq \mathbb{R}^N$ if it holds for every $x \in E$ except a subset B of E such that $c_2(B) = 0$. A Borel measure μ is said to be absolutely continuous with respect to c_2-capacity if $\mu(B) = 0$ for every Borel set B such that $c_2(B) = 0$.

If $A \subseteq \mathbb{R}^N$ is open, every function $u \in H^1(A)$ admits a *quasicontinuous* representative, i.e., a representative $\tilde{u}$ such that for every $\varepsilon > 0$ there exists an open set B_ε with $c_2(B_\varepsilon) < \varepsilon$ and $\tilde{u}_{|A\setminus B_\varepsilon}$ is continuous. The following fact holds true: if $u_n \to u$ strongly in $H^1(A)$, we have that up to a subsequence $\tilde{u}_n \to \tilde{u}$ c_2-q.e. on A.

Functions of bounded variation. If $A \subseteq \mathbb{R}^N$ is open, we say that $u \in BV(A)$ if $u \in L^1(A)$ and its derivative in the sense of distributions is a finite Radon measure on A, i.e., $Du \in \mathcal{M}_b(A; \mathbb{R}^N)$. $BV(A)$ is called the space of *functions of bounded variation* on A. $BV(A)$ is a Banach space under the norm $\|u\|_{BV(A)} := \|u\|_{L^1(A)} + \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^d)}$. We call $|Du|(A) := \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^d)}$ the *total variation* of u. We refer the reader to [1] for an exhaustive treatment of the space BV.

If $u \in BV(A)$, then the measure Du can be decomposed canonically (and uniquely) as

$$Du = D^a u + D^j u + D^c u.$$

The measure $D^a u$ is the absolutely continuous part (with respect to the Lebesgue measure) of the derivative: the associated density is denoted by $\nabla u \in L^1(A; \mathbb{R}^N)$. The measure $D^j u$ is the *jump part* of the derivative and it turns out that

$$D^j u = (u^+ - u^-) \otimes \nu \, \mathcal{H}^{N-1} \lfloor J_u.$$

Here J_u is the *jump set* of u, ν is the normal to J_u, while $u^\pm$ are the two traces of u on the jump set. Finally $D^c u$ is called the *Cantor part* of the derivative, and it vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$.

We will use the following result.

Theorem 2.4. *Let $A \subseteq \mathbb{R}^N$ be open, bounded and with a Lipschitz boundary. The following items hold true.*

(a) Compact embedding. *The space $BV(A)$ is embedded in $L^p(A)$ for every $1 \leq p \leq \frac{N}{N-1}$. The immersion is compact if $p < \frac{N}{N-1}$.*
(b) Lower semicontinuity of the total variation. *If $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(A)$ and $u_n \to u$ strongly in $L^1(A)$, then*
$$|Du|(A) \leq \liminf_n |Du_n|(A).$$

3. The main results

Let us consider $\Omega \subseteq \mathbb{R}^N$ open, bounded and with a Lipschitz boundary, and let μ be a Borel measure such that

$$\begin{cases} \mu \text{ is concentrated on } \partial\Omega \text{ and absolutely continuous with respect to } c_2\text{-capacity,} \\ \mu \geq c\mathcal{H}^{N-1}\lfloor\partial\Omega\,, \end{cases} \tag{3.1}$$

for some $c > 0$. We are interested in the following elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + u\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $f \in L^2(\Omega)$. The weak formulation of the problem involves the Hilbert space

$$H^1(\Omega) \cap L^2(\partial\Omega;\mu) := \{u \in H^1(\Omega) \,:\, u_{|\partial\Omega} \in L^2(\partial\Omega;\mu)\}$$

which is well defined since the measure μ vanishes on sets with zero c_2-capacity. As usual, we say that $u \in H^1(\Omega) \cap L^2(\partial\Omega;\mu)$ is a solution of the problem if for every $v \in H^1(\Omega) \cap L^2(\partial\Omega;\mu)$ we have

$$\int_\Omega \nabla u \cdot \nabla v\,dx + \int_{\partial\Omega} uv\,d\mu = \int_\Omega fv\,dx.$$

In view of the second property of μ in (3.1), it is readily seen that the elliptic problem admits a unique solution thanks to the Lax-Milgram lemma: indeed the associated bilinear form is continuous and such that

$$\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u^2\,d\mu \geq \int_\Omega |\nabla u|^2 dx + c\int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1} \geq C\|u\|^2_{H^1(\Omega)}$$

for some constant $C > 0$ (the middle term given a norm equivalent to the standard one on $H^1(\Omega)$): hence it is also coercive on $H^1(\Omega) \cap L^2(\partial\Omega;\mu)$.

We are interested in the behaviour of the solutions of problem (3.2) under the variation of the domain Ω and of the measure μ.

Let us fix $D \subset \mathbb{R}^N$ open and bounded, and let $\Omega_n, \Omega \subset\subset D$ be open and Lipschitz domains such that

$$\Omega_n \xrightarrow{\mathcal{H}^c} \Omega \qquad \text{and} \qquad |\Omega_n| \to |\Omega|. \tag{3.3}$$

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures in D such that

$$\begin{cases} \mu_n \text{ is concentrated on } \partial\Omega_n \text{ and absolutely continuous with respect to } c_2\text{-capacity,} \\ \mu_n \geq c\mathcal{H}^{N-1}\lfloor\partial\Omega_n \end{cases} \tag{3.4}$$

with c independent of n.

The main result of the papers is the following: recall that for u defined on $E \subseteq D$, we denote with $u1_E$ the function on D extended to zero outside E.

Theorem 3.1 (**Domain perturbation for generalized Robin problems).** *Let $D \subset \mathbb{R}^N$ be open and bounded, and let $\Omega_n, \Omega \subset\subset D$ be open and Lipschitz domains satisfying* (3.3)*. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures on D satisfying* (3.4)*.*

There exists a subsequence (still denoted with the same index) and a Borel measure μ on D satisfying (3.1) *such that if for $f \in L^2(D)$ we let $u_n \in H^1(\Omega_n)$ be the solution to*

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n \\ \frac{\partial u_n}{\partial \nu} + u_n \mu_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.5}$$

then

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{3.6}$$

and

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(D;\mathbb{R}^N), \tag{3.7}$$

where $u \in H^1(\Omega)$ is the solution to (3.2)*. Moreover*

$$\lim_n \left[\int_{\Omega_n} |\nabla u_n|^2 \, dx + \int_{\partial\Omega_n} u_n^2 \, d\mu_n \right] = \int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u^2 \, d\mu. \tag{3.8}$$

The previous stability result is fundamental to obtain the convergence of the *resolvent operators* and then of the associated spectra. Let us consider the operator

$$R_{\Omega_n,\mu_n} : L^2(D) \to L^2(D) \tag{3.9}$$

defined by

$$R_{\Omega_n,\mu_n}(f) := u_n 1_{\Omega_n}, \tag{3.10}$$

where $u_n \in H^1(\Omega_n)$ is the solution of the elliptic problem (3.5). Recall that the complete spectrum of the resolvent operators R_{Ω_n,μ_n} is defined as

$$\lambda_k(\Omega_n,\mu_n) := \inf_{H \in \mathcal{S}_k^n} \max_{u \in H} \frac{\int_{\Omega_n} |\nabla u|^2 \, dx + \int_{\partial\Omega_n} u^2 \, d\mu_n}{\int_{\Omega_n} u^2 \, dx}, \tag{3.11}$$

where $\mathcal{S}_k^n \subseteq H^1(\Omega_n) \cap L^2(\partial\Omega_n;\mu_n)$ denotes the family of k-dimensional subspaces, $k \geq 1$.

The following result holds true.

Theorem 3.2 (**Convergence of the resolvent operators**)**.** *Under the assumptions of Theorem 3.1*

$$R_{\Omega_n,\mu_n} \to R_{\Omega,\mu} \qquad \text{in the operator norm.} \tag{3.12}$$

In particular, for every $k \geq 1$ we have

$$\lim_n \lambda_k(\Omega_n,\mu_n) = \lambda_k(\Omega,\mu).$$

Given $\beta > 0$, the choice

$$\mu_n := \beta \mathcal{H}^{N-1} \lfloor \partial\Omega_n \tag{3.13}$$

leads to an elliptic problem under classical Robin boundary conditions with coefficient β. The second fundamental result of the paper is the following.

Theorem 3.3 (**Stability of classical Robin problems**)**.** *Let $D \subset \mathbb{R}^N$ be open and bounded, and let $\Omega_n, \Omega \subset\subset D$ be open and Lipschitz domains satisfying* (3.3) *and*

$$\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega). \tag{3.14}$$

Given $f \in L^2(D)$ and $\beta > 0$, let $u_n \in H^1(\Omega_n)$ be the solution of

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n, \\ \frac{\partial u_n}{\partial \nu} + \beta u_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.15}$$

Then

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{3.16}$$

and

$$\nabla u_n 1_{\Omega_n} \to \nabla u 1_\Omega \qquad \text{strongly in } L^2(D;\mathbb{R}^N), \tag{3.17}$$

where $u \in H^1(\Omega)$ is the solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

Finally

$$\lim_n \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1} = \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}. \tag{3.19}$$

Let us denote by $R_{\Omega_n,\beta}$ and $\lambda_k(\Omega_n, \beta)$ the resolvent operators and the eigenvalues associated to (3.15) according to (3.10) and (3.11) with the choice (3.13) for μ_n. We have the following result.

Corollary 3.4. *Under the assumptions of Theorem 3.3*

$$R_{\Omega_n,\beta} \to R_{\Omega,\beta} \qquad \text{in the operator norm.} \tag{3.20}$$

In particular, for every $k \geq 1$ we have

$$\lim_n \lambda_k(\Omega_n, \beta) = \lambda_k(\Omega, \beta).$$

In order to prove the previous stability results, and more precisely to deal with the general variation of the domains given by the convergence in the Hausdorff complementary topology, we employ a variational approach based on Γ-convergence.

Let $\mathcal{A}(D)$ be the class of all open subsets of D. We will consider the localized functionals

$$F_n : L^2(D) \times \mathcal{A}(D) \to [0, +\infty]$$

given by

$$F_n(u, A) = \begin{cases} \int_{\Omega_n \cap A} |\nabla u|^2 dx + \int_{\partial\Omega_n \cap A} u^2 d\mu_n & \text{if } u \in H^1(\Omega_n \cap A), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.21}$$

Remark 3.5. The trace term in the previous expression is defined in the following way.

(a) Since Ω_n is Lipschitz regular, for every $\bar{x} \in \partial\Omega_n \cap A$ we can find $r > 0$ such that

$$B_r(\overline{x}) \cap \Omega_n = B_r(\overline{x}) \cap E_{g_n}$$

where $E_{g_n} = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \leq g_n(x')\}$ for a suitable Lipschitz function $g_n : \mathbb{R}^{N-1} \to \mathbb{R}$. Reducing r if necessary, we may assume $B_r(\overline{x}) \subset A$. Hence $u \in H^1(\Omega_n \cap A)$ entails that $u \in H^1(B_r(\overline{x}) \cap E_{g_n})$.

(b) In view of point (a), it turns out that a value of u on $\partial\Omega_n \cap A$ is well defined up to sets negligible with respect to c_2-capacity: it is sufficient to multiply u by a smooth cut-off functions in $B_r(\bar{x})$, extend to $H^1(\mathbb{R}^N)$ and consider the restriction to $\partial\Omega_n \cap B_r(\bar{x})$ near $\bar{x}$ of the associated c_2-quasicontinuous representative.

Notice that if μ_n is in addition absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor\partial\Omega_n$ (as in the case of classical Robin problems), the value of u involved is the usual one in the sense of traces (defined again locally thanks to point (a)). Notice finally that for $u \in H^1(\Omega_n \cap A)$ the trace term in (3.21) is not necessarily finite.

The following result will be pivotal for our analysis: it shows that the limit of the energies F_n in the variational sense of Γ-convergence maintains the same structure at least on the subspace $H^1(D)$: to deal with the localized functionals, we employ the notion of $\bar{\Gamma}$-convergence (see Section 2).

Theorem 3.6 (**The $\bar{\Gamma}$-limit of the energies**). *Let $D \subset \mathbb{R}^N$ be open and bounded, let $\Omega_n, \Omega \subset\subset D$ be open and Lipschitz domains satisfying* (3.3). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures on D satisfying* (3.4)*, and let F_n be the functionals defined in* (3.21).

There exist $F : L^2(D) \times \mathcal{A}(D) \to [0, +\infty]$ (local with respect to the second variable) and a Borel measure μ satisfying (3.1) *such that up to a subsequence (still denoted with the same index)*

$$F_n \xrightarrow{\bar{\Gamma}} F \qquad \text{in the strong topology of } L^2(D),$$

with

$$F(u,A)=\int_{\Omega\cap A}|\nabla u|^2\,dx+\int_{\partial\Omega\cap A}u^2d\mu$$

for every $u\in H^1(D)$ and $A\in\mathcal{A}(D)$ (the trace value of u is defined according to Remark 3.5).

The proof of the results proceeds as follows. On the basis of Theorem 3.6, we will prove the stability results for the generalized and classical Robin problems in Section 4. We complete the analysis with the proof of Theorem 3.6 in Section 5.

4. Proof of the stability results

This section is devoted to the proof of the main stability results of the paper. More precisely, we collect some technical lemmas in Subsection 4.1. The proofs of Theorem 3.1 and of Theorem 3.2 are given in Subsection 4.2, while the stability for the classical Robin problems is studied in Subsection 4.3.

The starting point of our analysis is given by Theorem 3.6, whose proof is postponed to Section 5. Let us thus consider the functionals

$$F_n : L^2(D)\times\mathcal{A}(D)\to[0,+\infty]$$

given by (3.21). Thanks to Theorem 3.6 we have up to a subsequence (still denoted with the same index)

$$F_n\xrightarrow{\overline{\Gamma}}F\quad\text{in the strong topology of } L^2(D), \tag{4.1}$$

where $F : L^2(D)\times\mathcal{A}(D)\to[0,+\infty]$ can be represented as

$$F(u,A)=\int_{A\cap\Omega}|\nabla u|^2\,dx+\int_{\partial\Omega\cap A}u^2\,d\mu \tag{4.2}$$

for every $u\in H^1(D)$ and $A\in\mathcal{A}(D)$. Here μ is a Borel measure supported on $\partial\Omega$ which satisfies (3.1). Moreover for every $u,v\in L^2(D)$ and $A\in\mathcal{A}(D)$

$$u=v\text{ a.e. on }A\implies F(u,A)=F(v,A). \tag{4.3}$$

4.1. **Some technical lemmas.**

The following result will be used several times.

Lemma 4.1. *Let $D\subset\mathbb{R}^N$ be open and bounded and let $\Omega,\Omega_n\subset\subset D$ be open with*

$$\Omega_n\xrightarrow{\mathcal{H}^c}\Omega\qquad\textit{and}\qquad|\Omega_n|\to|\Omega|.$$

The following items hold true.

(a) *We have*

$$1_{\Omega_n}\to 1_\Omega\qquad\textit{strongly in } L^1(D). \tag{4.4}$$

(b) *Let $u_n\in H^1(\Omega_n)$ be such that*

$$\int_{\Omega_n}[|\nabla u_n|^2+u_n^2]\,dx\le C$$

for some C independent of n. Then, up to a subsequence, there exists $u\in H^1(\Omega)$ such that

$$u_n1_{\Omega_n}\rightharpoonup u1_\Omega\qquad\textit{weakly in } L^2(D)$$

and

$$\nabla u_n1_{\Omega_n}\rightharpoonup\nabla u1_\Omega\qquad\textit{weakly in } L^2(D;\mathbb{R}^N).$$

Proof. Let us start with point (a). It suffices to prove that

$$1_{\Omega_n}\to 1_\Omega\qquad\text{strongly in } L^2(D). \tag{4.5}$$

There exists a subsequence such that

$$1_{\Omega_{n_k}}\rightharpoonup f\qquad\text{weakly in } L^2(D),$$

where $f \in L^2(D)$ is such that $f \geq 0$. We have $f = 1$ on Ω. Indeed, if $\varphi \in C_c(\Omega)$, then its support is contained in Ω_n for n large enough in view of the convergence in the Hausdorff complementary topology, so that

$$\int_\Omega f\varphi\,dx = \int_D f\varphi\,dx = \lim_k \int_{\Omega_{n_k}} \varphi\,dx = \int_\Omega \varphi\,dx.$$

We can thus write $f = 1_\Omega + g1_{D\setminus\Omega}$. From the lower semicontinuity of the norms, and in view of the convergence of the measures we may write

$$|\Omega| + \int_{D\setminus\Omega} g^2\,dx = \|f\|^2_{L^2(D)} \leq \liminf_k |\Omega_{n_k}| = |\Omega|$$

so that $f = 1_\Omega$. Since f is independent of the chosen subsequence, convergence (4.5) follows.

Let us come to point (b). Up to a subsequence, there exist $v \in L^2(D)$ and $\Phi \in L^2(D;\mathbb{R}^N)$ such that

$$u_n 1_{\Omega_n} \rightharpoonup v \qquad \text{weakly in } L^2(D)$$

and

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \Phi \qquad \text{weakly in } L^2(D;\mathbb{R}^N).$$

The result follows if we prove that $v = u1_\Omega$ for some $u \in H^1(\Omega)$, with $\Phi = \nabla u 1_\Omega$.

In view of point (a) for every $g \in L^2(D)$ we have

$$\int_D vg\,dx = \lim_n \int_D u_n 1_{\Omega_n} g 1_{\Omega_n}\,dx = \int_D vg1_\Omega\,dx$$

so that $v = v1_\Omega$. The same argument shows that $\Phi = \Phi 1_\Omega$.

Let us call u the restriction of v to Ω, and let $\varphi \in C^1_c(\Omega)$. Since $\varphi \in C^1_c(\Omega_n)$ for n large, we may write for $i = 1, \dots, N$

$$\int_\Omega u\partial_i\varphi\,dx = \lim_n \int_{\Omega_n} u_n\partial_i\varphi\,dx = -\lim_n \int_{\Omega_n} \partial_i u_n \cdot \varphi\,dx = -\int_\Omega \Phi_i\varphi\,dx.$$

We conclude that $\nabla u = \Phi$ on Ω: this yields that $u \in H^1(\Omega)$, and the proof of point (b) is concluded. □

We will need the following stronger version of the previous lemma involving a control on some energies of Robin type at the boundary of the domains Ω_n: in order for these energies to be well defined, we assume the domains to have a Lipschitz boundary.

Lemma 4.2. *Let $D \subset \mathbb{R}^N$ be open and bounded, and let $\Omega, \Omega_n \subset\subset D$ be open, Lipschitz with*

$$\Omega_n \xrightarrow{\mathcal{H}^c} \Omega \qquad \text{and} \qquad |\Omega_n| \to |\Omega|.$$

Let $u_n \in H^1(\Omega_n)$ be such that

$$\int_{\Omega_n} (|\nabla u_n|^2 + u_n^2)\,dx + \int_{\partial\Omega_n} u_n^2\,d\mathcal{H}^{N-1} \leq C \tag{4.6}$$

for some C independent of n. Then, up to a subsequence, there exists $u \in H^1(\Omega)$ such that

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D),$$

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(D;\mathbb{R}^N)$$

and

$$\int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1} \leq \liminf_n \int_{\partial\Omega_n} u_n^2\,d\mathcal{H}^{N-1}.$$

Proof. By Lemma 4.1 up to a subsequence

$$u_n 1_{\Omega_n} \rightharpoonup u 1_\Omega \qquad \text{weakly in } L^2(D)$$

and

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(D;\mathbb{R}^N), \tag{4.7}$$

for a some $u \in H^1(\Omega)$. By the compact embedding of H^1 into L^2, we have that for $A \subset\subset \Omega$

$$u_n \to u \qquad \text{strongly in } L^2(A)$$

since $A \subset\subset \Omega_n$ for n large, thanks to the convergence in the Hausdorff complementary topology. Taking into account (4.4), we may assume that, up to a further subsequence,

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{a.e. in } D. \tag{4.8}$$

Let us consider the functions given by

$$v_n := u_n^2 1_{\Omega_n} \qquad \text{and} \qquad v := u^2 1_\Omega.$$

In view of [1, Theorem 3.87] we have that v_n and v belong to $BV(D)$ with derivatives given by the measures (notice that jumps can appear at the boundary of Ω_n and Ω)

$$Dv_n = D^a v_n + D^j v_n = 2u_n \nabla u_n 1_{\Omega_n}\, dx - u_n^2 \nu_{\partial\Omega_n} \mathcal{H}^{N-1} \lfloor \partial\Omega_n$$

and

$$Dv = D^a v + D^j v = 2u \nabla u 1_\Omega \, dx - u^2 \nu_{\partial\Omega} \mathcal{H}^{N-1} \lfloor \partial\Omega$$

respectively. In particular we have

$$|Dv_n|(D) = \int_{\Omega_n} 2|u_n \nabla u_n|\, dx + \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1} \le \|u_n\|^2_{H^1(\Omega_n)} + \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1} \le C \tag{4.9}$$

thanks to inequality (4.6).

Up to reducing D, it is not restrictive to assume that it has Lipschitz boundary. In view of the compact embedding of $BV(D)$ into $L^1(D)$ we deduce that up to a further subsequence

$$v_n \to w \qquad \text{strongly in } L^1(D) \tag{4.10}$$

for some $w \in BV(D)$, the convergence being also almost everywhere. Recalling (4.8), we get $w = v = u^2 1_\Omega$ so that there is no need of a further subsequence and

$$u_n^2 1_{\Omega_n} \to u^2 1_\Omega \qquad \text{strongly in } L^1(D). \tag{4.11}$$

We immediately deduce that

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D). \tag{4.12}$$

Indeed we can use dominated convergence to infer

$$\lim_n \int_D |u_n 1_{\Omega_n} - u 1_\Omega|^2 \, dx = 0$$

thanks to (4.8) and (4.11).

In view of (4.9) and of (4.10) (since $w = v = u^2 1_\Omega$) we deduce

$$Dv_n \overset{*}{\rightharpoonup} Dv \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(D; \mathbb{R}^N).$$

By (4.12) and (4.7) we have

$$u_n \nabla u_n 1_{\Omega_n} \rightharpoonup u \nabla u 1_\Omega \qquad \text{weakly in } L^1(D; \mathbb{R}^N),$$

which means $D^a v_n \overset{*}{\rightharpoonup} D^a v$ weakly* in $\mathcal{M}_b(D; \mathbb{R}^N)$. As a consequence

$$D^j v_n \overset{*}{\rightharpoonup} D^j v \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(D; \mathbb{R}^N),$$

so that, from the lower semicontinuity of the associated total variations, we get

$$\int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1} \le \liminf_n \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1},$$

and the proof of the proposition is complete. $\square$

4.2. **The stability results for the general problem.** Let us start with the following variational characterization of the solution of the elliptic problem(3.2).

Lemma 4.3. *The solution of the elliptic problem* (3.2) *is the minimizer of the functional* $J : H^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *given by*

$$J(v) := \int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} v^2\, d\mu - 2\int_\Omega fv\, dx. \tag{4.13}$$

Proof. The solution is readily seen to be a minimizer of the functional J. Viceversa, if u minimizes J on $H^1(\Omega)$, by comparing with a function $\varphi \in C^1_c(\Omega)$ we see that $J(u) < +\infty$, so that the trace on $\partial\Omega$ belongs to $L^2(\partial\Omega;\mu)$: hence $u \in H^1(\Omega) \cap L^2(\partial\Omega;\mu)$. Then the Euler-Lagrange equation for variations in $H^1(\Omega) \cap L^2(\partial\Omega;\mu)$ provides the weak formulation of (3.2). □

The following lemma will be useful.

Lemma 4.4. *Under the assumption of Theorem 3.1 we have*

$$\sup_n \left[\int_{\Omega_n} (|\nabla u_n|^2 + u_n^2)\, dx + \int_{\partial\Omega_n} u_n^2\, d\mu_n\right] < +\infty.$$

Proof. The proof is a consequence of the Faber-Krahn's inequality for the first eigenvalue of the Robin-Laplacian [11]: if we set for every open bounded set with Lipschitz boundary $A \subset \mathbb{R}^N$

$$\lambda_1^R(A) := \min_{v\in H^1(A)} \frac{\int_A |\nabla v|^2\, dx + \int_{\partial A} v^2\, d\mathcal{H}^{N-1}}{\int_A v^2\, dx},$$

then

$$\lambda_1^R(A) \geq \lambda_1^R(B) \tag{4.14}$$

where B is a ball such that $|A| = |B|$.

Indeed, taking u_n as a test function in problem (3.5), we first get

$$\int_{\Omega_n} |\nabla u_n|^2 dx + \int_{\partial\Omega_n} u_n^2 d\mu_n \;=\; \int_{\Omega_n} fu_n\, dx \;=\; \int_D fu_n 1_{\Omega_n} dx \;\leq\; \|f\|_{L^2(D)}\|u_n\|_{L^2(\Omega_n)}. \tag{4.15}$$

By assumption (3.4) on μ_n and the Faber-Krahn inequality (4.14) it follows

$$\begin{aligned}\int_{\Omega_n} |\nabla u_n|^2\, dx + \int_{\partial\Omega_n} |u_n|^2\, d\mu_n &\geq \int_{\Omega_n} |\nabla u_n|^2\, dx + c\int_{\partial\Omega_n} |u_n|^2\, d\mathcal{H}^{N-1}\\ &\geq \lambda_1^R(\Omega_n, c)\int_{\Omega_n} |u_n|^2\, dx \geq \lambda_1^R(B_n, c)\int_{\Omega_n} |u_n|^2\, dx,\end{aligned}$$

where B_n is a ball with the same measure of Ω_n. Since $|B_n| \to |\Omega|$ there exists $\bar\lambda > 0$ such that $\lambda_1^R(B_n, c) \geq \bar\lambda$, so that

$$\bar\lambda \int_{\Omega_n} u_n^2\, dx \leq \int_{\Omega_n} |\nabla u_n|^2\, dx + \int_{\partial\Omega_n} |u_n|^2\, d\mu_n. \tag{4.16}$$

Gathering (4.15) and (4.16) we obtain

$$\int_{\Omega_n} |\nabla u_n|^2 dx + \int_{\partial\Omega_n} u_n^2 d\mu_n \leq C$$

for some $C > 0$. The conclusion follows using again (4.16). □

Remark 4.5. In view of the previous proof, there exists a constant C independent of n such that

$$\int_{\Omega_n} [|\nabla u_n|^2 + u_n^2]\, dx + \int_{\partial\Omega_n} u_n^2 d\mathcal{H}^{N-1} \leq C\|f\|_{L^2(D)}\|u_n\|_{L^2(\Omega_n)}.$$

We are now in a position to prove the first stability result.

Proof of Theorem 3.1. By Lemma 4.4 and assumption (3.4) on μ_n we deduce

$$\sup_n \left[\int_{\Omega_n} (|\nabla u_n|^2 + u_n^2)\, dx + c \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1} \right] \leq \sup_n \left[\int_{\Omega_n} (|\nabla u_n|^2 + u_n^2)\, dx + \int_{\partial\Omega_n} u_n^2 \, d\mu_n \right] < +\infty.$$

By Lemma 4.2 there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} 1_{\Omega_{n_k}} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{4.17}$$

and

$$\nabla u_{n_k} 1_{\Omega_{n_k}} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(D;\mathbb{R}^N),$$

for some $u \in H^1(\Omega)$.

In view of (4.1), we can choose $A \subset D$ open with $\Omega \subset\subset A$ such that

$$F_n(v, A) - 2\int_{\Omega_n} fv\, dx \xrightarrow{\Gamma} F(v, A) - 2\int_\Omega fv\, dx,$$

since the integral term involving f is a continuous (with respect to the strong topology of $L^2(D)$) perturbation of $F_n(\cdot, A)$, so that the Γ-convergence is preserved. Thanks to (4.17), the general properties of Γ-convergence entail that $u1_\Omega$ is a minimizer of

$$v \mapsto F(v, A) - 2\int_\Omega fv\, dx.$$

Since Ω is Lipschitz, we can extend u to $\hat{u} \in H^1(D)$. Since $u1_\Omega$ and $\hat{u}$ coincide on $\Omega \cap A = \Omega$, thanks to the locality property (4.3) we can write

$$F(u, A) = F(\hat{u}, A),$$

so that using the representation (4.2) we get

$$F(u, A) = \int_\Omega |\nabla \hat{u}|^2\, dx + \int_{\partial\Omega} \hat{u}^2\, d\mu = \int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u^2\, d\mu.$$

We conclude that $u \in H^1(\Omega)$ is a minimizer of

$$v \mapsto \int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} v^2\, d\mu - 2\int_\Omega fv\, dx$$

so that by Lemma 4.3 it is the solution of problem (3.2), hence it is uniquely determined. In particular the entire sequence $(u_n)_{n\in\mathbb{N}}$ satisfies (3.6) and (3.7). Finally the Γ-convergence ensures also that

$$\lim_n \left[\int_{\Omega_n} |\nabla u_n|^2\, dx + \int_{\partial\Omega_n} u_n^2\, d\mu_n - 2\int_{\Omega_n} fu_n\, dx \right] = \int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u^2\, d\mu - 2\int_\Omega fu\, dx.$$

Since

$$\lim_n \int_{\Omega_n} fu_n\, dx = \int_\Omega fu\, dx,$$

equation (3.8) holds true, and the proof is concluded. □

Let us address now the stability result for the resolvent operators given by Theorem 3.2. The following result holds true.

Lemma 4.6. *The resolvent operator $R_{\Omega,\mu}$ is compact with respect to the weak convergence in $L^2(D)$, i.e. for every $g_n \rightharpoonup g$ weakly in $L^2(D)$ then*

$$R_{\Omega,\mu}(g_n) \to R_{\Omega,\mu}(g) \qquad \textit{strongly in } L^2(D).$$

Proof. Let us set
$$u_n := R_{\Omega,\mu}(g_n) \qquad \text{and} \qquad u := R_{\Omega,\mu}(f).$$
We have testing the equation with u_n
$$\int_\Omega |\nabla u_n|^2 dx + \int_{\partial\Omega} u_n^2 d\mu = \int_\Omega g_n u_n \, dx = \int_D g_n u_n 1_\Omega dx \le \|g_n\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)}.$$
By property (3.1) of μ we have
$$\int_\Omega |\nabla u_n|^2 \, dx + \int_{\partial\Omega} u_n^2 \, d\mu \ge \int_\Omega |\nabla u_n|^2 \, dx + c \int_{\partial\Omega} u_n^2 \, d\mathcal{H}^{N-1} \ge C\|u_n\|^2_{H^1(\Omega)}$$
for some $C > 0$ (the middle term gives a norm equivalent to the usual one on H^1). We thus obtain
$$\|u_n\|_{H^1(\Omega)} \le M$$
with $M > 0$ constant. Thus there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in H^1(\Omega)$ such that
$$u_{n_k} \rightharpoonup u \qquad \text{weakly in } H^1(\Omega)$$
so that in particular, being Ω Lipschitz,
$$u_{n_k} \to u \qquad \text{strongly in } L^2(\Omega). \tag{4.18}$$
Let us see that u is the solution of (3.2). Notice that the functionals $J_n : H^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$
$$J_n(v) := \int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} v^2 d\mu - 2\int_\Omega g_n v \, dx$$
are readily seen to Γ-converge in the strong topology of $L^2(\Omega)$ to the functional J given in (4.13) (indeed, we have a *continuous convergence*). We thus infer that u is the minimizer of J, that is, thanks to Lemma 4.3, u is the solution of (3.2). By uniqueness, there is no need to pass to a subsequence in (4.18), so that the result follows. □

The conclusion of Theorem 3.1 can be rephrased as
$$R_{\Omega_n,\mu_n}(f) \to R_{\Omega,\mu}(f) \qquad \text{strongly in } L^2(D)$$
for every $f \in L^2(D)$. The result can be strengthened in the following way.

Lemma 4.7. *Under the assumptions of Theorem 3.1, let $f_n, f \in L^2(D)$ be such that*
$$f_n \rightharpoonup f \qquad \text{weakly in } L^2(D).$$
Then along the same subsequence given by Theorem 3.1
$$R_{\Omega_n,\mu_n}(f_n) \to R_{\Omega,\mu}(f) \qquad \text{strongly in } L^2(D).$$

Proof. In view of Remark 4.5 and of Lemma 4.2, we have that there exists a subsequence such that
$$R_{\Omega_{n_k},\mu_{n_k}}(f_{n_k}) = u_{n_k} 1_{\Omega_{n_k}} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{4.19}$$
for some $u \in H^1(\Omega)$. In view of Lemma 4.3, in order to conclude it is sufficient to prove that u is the minimizer of $J : H^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by (4.13).

Thanks to (4.1) we can find $A \subseteq D$ open with $\Omega \subset\subset A$ and such that
$$F_n(v, A) - 2\int_{\Omega_n} f_n v \, dx \xrightarrow{\Gamma} F(v, A) - 2\int_\Omega f v \, dx,$$
since the integral term involving f_n is a continuous (with respect to the strong topology of $L^2(D)$) perturbation of that containing f, so that the Γ-convergence is preserved. Thanks to (4.19), the general properties of Γ-convergence entail that $u1_\Omega$ is a minimizer of
$$v \mapsto F(v, A) - 2\int_\Omega f v \, dx = J(v),$$
so that the conclusion follows. □

We can now give the proof of Theorem 3.2.

Proof of Theorem 3.2. The convergence in the operator norm is equivalent to the following relation

$$\sup_{f\in L^2(D),\|f\|_{L^2(D)}\le 1} \|R_{\Omega_n,\mu_n}(f)-R_{\Omega,\mu}(f)\|_{L^2(D)} \to 0. \tag{4.20}$$

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $\|f_n\|_{L^2(D)}\le 1$ and

$$\|R_{\Omega_n,\mu_n}(f_n)-R_{\Omega,\mu}(f_n)\|_{L^2(D)} = \sup_{f\in L^2(D),\|f\|_{L^2(D)}\le 1} \|R_{\Omega_n,\mu_n}(f)-R_{\Omega,\mu}(f)\|_{L^2(D)}.$$

The existence of f_n is guaranteed by Lemma 4.6. Since f_n is bounded in $L^2(D)$, we may assume that up to a further subsequence there exists $f\in L^2(D)$, such that

$$f_n \rightharpoonup f \qquad \text{weakly in } L^2(D).$$

From Lemma 4.6 we have

$$\|R_{\Omega,\mu}(f_n)-R_{\Omega,\mu}(f)\|_{L^2(D)}\to 0,$$

while Lemma 4.7 entails

$$\|R_{\Omega_n,\mu_n}(f_n)-R_{\Omega,\mu}(f)\|_{L^2(D)}\to 0.$$

Since

$$\|R_{\Omega_n,\mu_n}(f_n)-R_{\Omega,\mu}(f_n)\|_{L^2(D)} \le \|R_{\Omega_n,\mu_n}(f_n)-R_{\Omega,\mu}(f)\|_{L^2(D)} + \|R_{\Omega,\mu}(f_n)-R_{\Omega,\mu}(f)\|_{L^2(D)}$$

relation (3.12) easily follows.

Since the resolvent operators are compact in view of Lemma 4.6, the convergence of the eigenvalues is a standard consequence of their convergence [13, Lemma XI.9.5], so that the proof is concluded. □

4.3. **Proof of the stability result for the classical Robin problems.** In this subsection we prove the stability results for classical Robin problems given by Theorem 3.3 and Corollary 3.4. The setting is precisely that of the general stability result with the choice

$$\mu_n := \beta\mathcal{H}^{N-1}\lfloor\partial\Omega_n.$$

The results thus follow from Theorem 3.1 and Theorem 3.2 if we show that the associated measure μ is given by

$$\mu = \beta\mathcal{H}^{N-1}\lfloor\partial\Omega. \tag{4.21}$$

Indeed, if it is the case, being μ uniquely determined, there is no need to pass to a subsequence in the convergence results of Theorem 3.1 so that

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(D) \tag{4.22}$$

and

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(D;\mathbb{R}^N), \tag{4.23}$$

together with

$$\lim_n \left[\int_{\Omega_n} |\nabla u_n|^2\,dx + \beta\int_{\partial\Omega_n} u_n^2\,d\mathcal{H}^{N-1}\right] = \int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1}.$$

Using (4.23) and Lemma 4.2 the previous convergence entails

$$\lim_n \int_{\Omega_n} |\nabla u_n|^2\,dx = \int_\Omega |\nabla u|^2\,dx \quad \text{and} \quad \lim_n \int_{\partial\Omega_n} u_n^2\,d\mathcal{H}^{N-1} = \int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1},$$

so that convergences (3.17) and (3.19) follow.

In order to conclude the proof, we have to show that (4.21) holds true. We claim that for every $\varphi\in H^1(D)\cap L^\infty(D)$ we have

$$\lim_n \int_{\partial\Omega_n} \varphi^2 d\mathcal{H}^{N-1} = \int_{\partial\Omega} \varphi^2 d\mathcal{H}^{N-1}. \tag{4.24}$$

Notice that thanks to assumption (3.14) on the perimeters

$$\int_{\Omega_n} (|\nabla\varphi|^2 + \varphi^2)\,dx + \int_{\partial\Omega_n} \varphi^2\,d\mathcal{H}^{N-1} \le \|\varphi\|^2_{H^1(D)} + \|\varphi\|^2_\infty \mathcal{H}^{N-1}(\partial\Omega_n) \le C$$

for some C independent of n. By Lemma 4.2 applied to the functions $\varphi 1_{\Omega_n}$ and $\varphi 1_\Omega$, we obtain

$$\int_{\partial\Omega} \varphi^2\,d\mathcal{H}^{N-1} \le \liminf_n \int_{\partial\Omega_n} \varphi^2\,d\mathcal{H}^{N-1}. \tag{4.25}$$

On the other hand, applying the same arguments to the functions

$$w_n := (\|\varphi\|^2_\infty - \varphi^2)1_{\Omega_n} \qquad \text{and} \qquad w := (\|\varphi\|^2_\infty - \varphi^2)1_\Omega$$

we deduce

$$\int_{\partial\Omega} (\|\varphi\|^2_\infty - \varphi^2)\,d\mathcal{H}^{N-1} \le \liminf_n \int_{\partial\Omega_n} (\|\varphi\|^2_\infty - \varphi^2)\,d\mathcal{H}^{N-1}.$$

Using again (3.14) we get

$$\int_{\partial\Omega} \varphi^2\,d\mathcal{H}^{N-1} \ge \limsup_n \int_{\partial\Omega_n} \varphi^2\,d\mathcal{H}^{N-1},$$

which together with (4.25) yields claim (4.24).

In view of (4.1), we can choose $A \subset D$ open with $\Omega \subset\subset A$ such that

$$F_n(v, A) \xrightarrow{\Gamma} F(v, A).$$

We may thus write

$$F(\varphi, A) \le \liminf_n F_n(\varphi, A)$$

which gives in this case, for the choice of A and since $1_{\Omega_n} \to 1_\Omega$ strongly in $L^1(D)$

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2 dx + \int_{\partial\Omega} |\varphi|^2 d\mu &\le \liminf_n \int_{\Omega_n} |\nabla\varphi|^2 dx + \beta \int_{\partial\Omega_n} |\varphi|^2 d\mathcal{H}^{N-1} \\ &= \int_\Omega |\nabla\varphi|^2 dx + \liminf_n \beta \int_{\partial\Omega_n} |\varphi|^2 d\mathcal{H}^{N-1}.\end{aligned}$$

This implies using (4.24)

$$\int_{\partial\Omega} \varphi^2 d\mu \le \liminf_n \beta \int_{\partial\Omega_n} \varphi^2 d\mathcal{H}^{N-1} = \beta \int_{\partial\Omega} \varphi^2 d\mathcal{H}^{N-1}.$$

Notice that Theorem 3.1 ensures that $\mu \ge \beta\mathcal{H}^{N-1}\lfloor\partial\Omega$: as a consequence

$$\int_{\partial\Omega} \varphi^2 d\mu = \beta \int_{\partial\Omega} \varphi^2 d\mathcal{H}^{N-1}.$$

Let now $A \in \mathcal{A}(D)$. Choosing $\varphi \in C^1_c(A)$ such that $\varphi \nearrow 1_A$, by the previous equality we deduce since μ is supported on $\partial\Omega$

$$\mu(A) = \mu(\partial\Omega \cap A) = \beta\mathcal{H}^{N-1}(\partial\Omega \cap A),$$

which yields $\mu = \beta\mathcal{H}^{N-1}\lfloor\partial\Omega$, so that relation (4.21) is proved.

5. Proof of the Γ-convergence result

This section is devoted to the proof of Theorem 3.6 on which the analysis leading to the main results of the paper are based.

Let us start noticing that by definition the functional F_n defined in (3.21) is such that $F_n(\cdot, A)$ is lower-semicontinuous (with respect to the strong topology of $L^2(D)$) for all $A \in \mathcal{A}(D)$ and $F_n(u, \cdot)$ is monotone with respect to the set inclusion, for all $u \in L^2(D)$.

The following compactness result holds true.

Proposition 5.1 (**Compactness for the energies**). *There exists a functional $F : L^2(D) \times \mathcal{A}(D) \to [0, +\infty]$, such that up to a subsequence (still denoted with the same index)*

$$F_n \overset{\overline{\Gamma}}{\to} F \tag{5.1}$$

in the strong topology of $L^2(D)$. In particular $F(u, \cdot)$ is inner regular on $\mathcal{A}(D)$ for every $u \in L^2(D)$.

Proof. In view of the monotonicity of $F_n(u, \cdot)$ with respect to set inclusion, the result follows by applying [6, Theorem 16.9] with $X = L^2(D)$ (see also Proposition 2.3). □

We have immediately the following locality property for F.

Lemma 5.2 (**Locality**). *Let $A \in \mathcal{A}(D)$ and $u, v \in L^2(D)$ such that $u = v$ a.e. on $\Omega \cap A$. Then $F(u, A) = F(v, A)$.*

Proof. We follow [6, Proposition 16.15]. According to (5.1) and to (2.1), let $B \subset\subset A$ be such that

$$F_n(\cdot, B) \overset{\Gamma}{\to} F(\cdot, B).$$

Let $v_n \in L^2(D)$ such that $v_n \to v$ strongly in $L^2(D)$ and

$$F_n(v_n, B) \to F(v, B).$$

Then if we set

$$u_n := \begin{cases} v_n & \text{on } \Omega_n \cap B \\ u & \text{otherwise in } D \end{cases}$$

we get $u_n \to u$ strongly in $L^2(D)$ (since $1_{\Omega_n} \to 1_\Omega$ strongly in $L^1(D)$ thanks to Lemma 4.1) so that

$$F(u, B) \le \liminf_n F_n(u_n, B) = \liminf_n F_n(v_n, B) = F(v, B) \le F(v, A).$$

Letting B invade A we obtain according to (2.2) the inequality $F(u, A) \le F(v, A)$. The opposite one comes by interchanging the roles of u and v. □

In order to derive a representation formula for $F(u, A)$, we start with the following lemma.

Lemma 5.3. *For every $(u, A) \in L^2(D) \times \mathcal{A}(D)$ with $F(u, A) < +\infty$ we get that $u_{|\Omega\cap A} \in H^1(\Omega\cap A)$ and*

$$F(u, A) \ge \int_{\Omega\cap A} |\nabla u|^2 dx. \tag{5.2}$$

Proof. According to (5.1) and to (2.1), let $B \subset\subset A \cap \Omega$ be such that

$$F_n(\cdot, B) \xrightarrow{\Gamma} F(\cdot, B) \quad \text{ in the strong topology of } L^2(D).$$

From the definition of Γ-convergence, there exists $u_n \in L^2(D)$ such that $u_n \to u$ strongly in $L^2(D)$ and

$$\lim_n F_n(u_n, B) = F(u, B).$$

The convergence $\Omega_n \overset{\mathcal{H}^c}{\to} \Omega$ yields that $B \subseteq \Omega_n$ for n large enough, so that

$$F(u, B) = \lim_n F_n(u_n, B) \ge \liminf_n \int_{\Omega_n \cap B} |\nabla u_n|^2 \, dx = \liminf_n \int_B |\nabla u_n|^2 \, dx.$$

Since $F(u, B) < +\infty$, we get that the restriction of u_n to B is bounded in $H^1(B)$, hence also the restriction of u to B belongs to $H^1(B)$. We infer by lower-semicontinuity

$$F(u, B) \ge \liminf_n \int_B |\nabla u_n|^2 \, dx \ge \int_B |\nabla u|^2 \, dx. \tag{5.3}$$

Letting B invade $A \cap \Omega$, we obtain according to (2.2)

$$F(u, A) \ge \int_{\Omega\cap A} |\nabla u|^2 \, dx.$$

□

In view of the preceding lemma, we may write for all $(u, A) \in L^2(D) \times \mathcal{A}(D)$

$$F(u, A) = \int_{\Omega \cap A} |\nabla u|^2 dx + G(u, A), \tag{5.4}$$

where

$$G : L^2(D) \times \mathcal{A}(D) \to [0, +\infty],$$

adopting the convention that $G(u, A) = +\infty$ if $F(u, A) = +\infty$ (in this case, also the first term on the right could be infinite, i.e., $u_{|(\Omega \cap A)} \notin H^1(\Omega \cap A)$).

We want to recover an integral representation for G, at least on $H^1(D) \times \mathcal{A}(D)$: this will be done using the results contained in [7]. With this aim, the following result holds true.

Proposition 5.4 (**Properties of** G). *The restriction of G to $H^1(D) \times \mathcal{A}(\Omega)$ satisfies the following properties.*

(P1) *Lower semicontinuity: for every $A \in A(D)$ the mapping $u \to G(u, A)$ is lower semicontinuous with respect to the strong topology of $H^1(D)$.*

(P2) *Measure property: for every $u \in H^1(D)$ the mapping*

$$A \to G(u, A)$$

is the trace of a Borel measure on D.

(P3) *Locality property: for every $A \in \mathcal{A}(D)$ and $u, v \in H^1(D)$ such that $u_{|A} = v_{|A}$ a.e., then $G(u, A) = G(v, A)$.*

(P4) *C^1-convexity: the following items hold true.*

(a) *For every $A \in \mathcal{A}(D)$ the mapping $u \to G(u, A)$ is convex in $H^1(D)$.*

(b) *For every $u, v \in H^1(D)$ and $\varphi \in C^1(D) \cap W^{1,\infty}(D)$ with $0 \le \varphi \le 1$, we have*

$$G(\varphi u + (1 - \varphi)v, A) \le G(u, A) + G(v, A).$$

(P5) *Quadraticity: for every $A \in A(D)$, the mapping*

$$u \to G(u, A)$$

is quadratic on $H^1(D)$, i.e. there exists a linear subspace $Y_A \subseteq H^1(D)$ and a symmetric bilinear form $B_A : H^1(D) \times H^1(D) \to \mathbb{R}$ such that

$$G(u, A) = \begin{cases} B_A(u, u) & \text{for all } u \in Y_A, \\ +\infty & \text{for all } u \in H^1(D) \setminus Y_A. \end{cases}$$

Proof. Let us deal with the several properties separately.

Proof of (P1). Let $u_n \to u$ strongly in $H^1(D)$. We have to prove that

$$G(u, A) \le \liminf_{n \to +\infty} G(u_n, A) \tag{5.5}$$

for every $A \in \mathcal{A}(D)$. First of all we claim that

$$F(u, A) \le \liminf_{n \to +\infty} F(u_n, A). \tag{5.6}$$

Indeed, if according to (2.1) $B \subset\subset A$ is such that

$$F_n(\cdot, B) \xrightarrow{\Gamma} F(\cdot, B)$$

in the strong topology of $L^2(D)$, we have

$$F(u, B) \le \liminf_{n \to +\infty} F(u_n, B) \le \liminf_{n \to +\infty} F(u_n, A)$$

so that thanks to (2.2) claim (5.6) follows.

In view of (4.4) we have

$$\nabla u_n 1_{\Omega_n} \to \nabla u 1_{\Omega} \qquad \text{strongly in } L^2(D; \mathbb{R}^N),$$

so that

$$\lim_n \int_{\Omega_n \cap A} |\nabla u_n|^2 \, dx = \lim_n \int_A |\nabla u_n 1_{\Omega_n}|^2 \, dx = \int_A |\nabla u 1_\Omega|^2 \, dx = \int_{A \cap \Omega} |\nabla u|^2 \, dx.$$

Coming back to (5.6) we obtain

$$\int_{A\cap\Omega} |\nabla u|^2\,dx + G(u,A) \le \liminf_n F(u_n,A) = \int_{A\cap\Omega} |\nabla u|^2\,dx + \liminf_n G(u_n,A)$$

from which (5.5) follows.

Proof of (P2). Since

$$G(u,A) = F(u,A) - \int_{\Omega\cap A} |\nabla u|^2\,dx$$

it suffices to prove that $F(u,\cdot)$ is the trace on $\mathcal{A}(D)$ of a Borel measure on D. Let us first observe that for each $n \in \mathbb{N}$ the functional

$$F_n(u,A) = \int_{\Omega_n\cap A} |\nabla u|^2\,dx + \int_{\partial\Omega_n\cap A} u^2 d\mu_n$$

is the trace on $\mathcal{A}(D)$ of a Borel measure on D. According to [6, Theorem 18.5], in order to transfer this property to $F(u,\cdot)$, it is sufficient to prove that the functionals F_n satisfy the following *uniform fundamental estimate*: for every $\delta > 0$ and for every $A', A'', B \in \mathcal{A}(D)$, with $\overline{A'} \subseteq A''$, there exists a constant $M > 0$ with the property that for every $u, v \in H^1(D)$ there exists a cut-off function φ between A' and A'', such that

$$\begin{aligned} F(\varphi u + (1-\varphi)v, A'\cup B) &\le (1+\delta)\left\{F(u,A'') + F(v,B)\right\} + \delta\left\{\|u\|^2_{L^2(S)} + \|v\|^2_{L^2(S)} + 1\right\} \\ &\quad + M\|u-v\|^2_{L^2(S)}, \end{aligned}$$

where $S := (A'' \setminus A') \cap B$.

Let $A', A'', B \in \mathcal{A}(D)$ with $\overline{A}' \subseteq A''$, and let $u, v \in H^1(D)$. If φ is a cut-off function between A' and A'', let us estimate

$$\begin{aligned} &F_n(\varphi u + (1-\varphi)v, A'\cup B) \\ &\quad = \int\limits_{\Omega_n\cap(A'\cup B)} |\nabla(\varphi u + (1-\varphi)v)|^2\,dx + \int\limits_{\partial\Omega_n\cap(A'\cup B)} [\varphi u + (1-\varphi)v]^2 d\mu_n =: I_1 + I_2. \end{aligned}$$

Let us start with I_2. We find

$$\begin{aligned} I_2 &= \int\limits_{\partial\Omega_n\cap(A'\cup B)} [\varphi u + (1-\varphi)v]^2 d\mu_n \le \int\limits_{\partial\Omega_n\cap(A'\cup B)} \left(\varphi u^2 + (1-\varphi)v^2\right) d\mu_n \\ &\le \int\limits_{\partial\Omega_n\cap A''} \varphi u^2 d\mu_n + \int\limits_{\partial\Omega_n\cap B} (1-\varphi)v^2 d\mu_n \qquad (5.7) \\ &\le \int\limits_{\partial\Omega_n\cap A''} u^2 d\mu_n + \int\limits_{\partial\Omega_n\cap B} v^2 d\mu_n. \end{aligned}$$

Consider now I_1. For every $\delta > 0$, there exists a constant $C_\delta > 0$ such that we may write

$$\begin{aligned} I_1 &= \int\limits_{\Omega_n\cap(A'\cup B)} |\nabla(\varphi u + (1-\varphi)v)|^2\,dx \\ &= \int\limits_{\Omega_n\cap(A'\cup B)} |\varphi\nabla u + (1-\varphi)\nabla v + \nabla\varphi(u-v)|^2\,dx \\ &\le (1+\delta) \int\limits_{\Omega_n\cap(A'\cup B)} |\varphi\nabla u + (1-\varphi)\nabla v|^2\,dx + C_\delta \int\limits_{\Omega_n\cap(A'\cup B)} |\nabla\varphi|^2|u-v|^2\,dx \qquad (5.8) \\ &\le (1+\delta)\left(\int\limits_{\Omega_n\cap A''} |\nabla u|^2\,dx + \int\limits_{\Omega_n\cap B} |\nabla v|^2\,dx\right) + M \int\limits_{\Omega_n\cap(A''\setminus A')\cap B} |u-v|^2\,dx \end{aligned}$$

with $M := C_\delta \max|\nabla\varphi|^2$.

Combining (5.8) and (5.7) we obtain that for every $\delta > 0$ and every cut-off function φ there exists $M > 0$ such that

$$F_n(\varphi u + (1-\varphi)v, A' \cup B) \le (1+\delta)[F_n(u, A'') + F_n(v, B)] + M\|u - v\|^2_{L^2(\Omega_n \cap (A'' \setminus A') \cap B)},$$

i.e., F_n satisfy the uniform fundamental estimate.

Proof of (P3). We proved a stronger locality property in Lemma 5.2.

Proof of (P4). Let $u, v \in H^1(D)$ and $\varphi \in C^1(D)$ with $0 \le \varphi \le 1$. Without loss of generality, we may assume that $G(u, A) < +\infty$ and $G(v, A) < +\infty$. According to (2.1), let $B \subset\subset A$ be such that

$$F_n(\cdot, B) \xrightarrow{\Gamma} F(\cdot, B).$$

There exist $u_n \in L^2(D)$ and $v_n \in L^2(D)$ such that

$$u_n \to u, \quad v_n \to v \quad \text{strongly in } L^2(D) \tag{5.9}$$

with

$$F(u, B) = \lim_n F_n(u_n, B) \quad \text{and} \qquad F(v, B) = \lim_n F_n(v_n, B).$$

By assumption (3.3) on the Hausdorff convergence and since $1_{\Omega_n} \to 1_\Omega$ strongly in $L^1(D)$ thanks to Lemma 4.1, we infer

$$\Omega_n \cap B \xrightarrow{\mathcal{H}^c} \Omega \cap B \qquad \text{and} \qquad |\Omega_n \cap B| \to |\Omega \cap B|.$$

Using again the convergence of the characteristic functions we get

$$\nabla u_n 1_{\Omega_n \cap B} \rightharpoonup \nabla u 1_{\Omega \cap B}, \quad \nabla v_n 1_{\Omega_n \cap B} \rightharpoonup \nabla v 1_{\Omega \cap B} \qquad \text{weakly in } L^2(D; \mathbb{R}^N). \tag{5.10}$$

We may write

$$\begin{aligned}
G(\varphi u + (1-\varphi)v, B) &= F(\varphi u + (1-\varphi)v, B) - \int_{\Omega \cap B} |\nabla(\varphi u + (1-\varphi)v)|^2\,dx \\
&\le \lim_n F_n(\varphi u_n + (1-\varphi)v_n, B) - \int_{\Omega \cap B} |\nabla(\varphi u + (1-\varphi)v)|^2\,dx \\
&= \lim_n \left[\int_{\Omega_n \cap B} |\nabla(\varphi u_n + (1-\varphi)v_n)|^2\,dx + \int_{\partial\Omega_n \cap B} |\varphi u_n + (1-\varphi)v_n|^2\,d\mu_n\right] \\
&\qquad - \int_{\Omega \cap B} |\nabla(\varphi u + (1-\varphi)v)|^2\,dx \\
&= \lim_n \left(\int_{\Omega_n \cap B} |\varphi\nabla(u_n - u) + (1-\varphi)\nabla(v_n - v)|^2\,dx + \int_{\partial\Omega_n \cap B} |\varphi u_n + (1-\varphi)v_n|^2\,d\mu_n\right),
\end{aligned} \tag{5.11}$$

where the last equality follows by a direct calculation taking into account (5.9) and (5.10).

Let us prove point (a) of **(P4)**, i.e., the convexity property. So let us assume that φ reduces to a constant $\lambda \in]0,1[$. We can write

$$
\begin{aligned}
&G(\lambda u+(1-\lambda)v,B)\\
&\quad\le \liminf_n \left(\int_{\Omega_n\cap B} |\lambda\nabla(u_n-u)+(1-\lambda)\nabla(v_n-v)|^2\,dx+\int_{\partial\Omega_n\cap B}|\lambda u_n+(1-\lambda)v_n|^2\,d\mu_n\right)\\
&\qquad\le \liminf_n \left(\lambda\int_{\Omega_n\cap B}|\nabla(u_n-u)|^2\,dx+(1-\lambda)\int_{\Omega_n\cap B}|\nabla(v_n-v)|^2\,dx\right.\\
&\qquad\qquad\left.+\lambda\int_{\partial\Omega_n\cap B}u_n^2\,d\mu_n+(1-\lambda)\int_{\partial\Omega_n\cap B}v_n^2\,d\mu_n\right)\\
&\qquad= \liminf_n \left(\lambda\int_{\Omega_n\cap B}|\nabla u_n|^2\,dx+(1-\lambda)\int_{\Omega_n\cap B}|\nabla v_n|^2\,dx\right.\\
&\quad\left.+\lambda\int_{\partial\Omega_n\cap B}u_n^2\,d\mu_n+(1-\lambda)\int_{\partial\Omega_n\cap B}v_n^2\,d\mu_n\right)-\lambda\int_{\Omega\cap B}|\nabla u|^2\,dx-(1-\lambda)\int_{\Omega\cap B}|\nabla v|^2\,dx\\
&\qquad=\lambda F(u,B)+(1-\lambda)F(v,B)-\lambda\int_{\Omega\cap B}|\nabla u|^2\,dx-(1-\lambda)\int_{\Omega\cap B}|\nabla v|^2\,dx\\
&\qquad\qquad=\lambda G(u,B)+(1-\lambda)G(v,B).
\end{aligned}
$$

Notice that the third equality is again obtained by a direct calculation taking into account (5.10). Letting B invade A we get according to (2.2)

$$G(\lambda u+(1-\lambda)v,A)\le\lambda G(u,A)+(1-\lambda)G(v,A).$$

Let us pass to point (b). Coming back to (5.11) we may write with similar arguments

$$
\begin{aligned}
&G(\varphi u+(1-\varphi)v,B)\\
&\qquad\le\liminf_n\left(\int_{\Omega_n\cap B}\left[|\nabla(u_n-u)|^2+|\nabla(v_n-v)|^2\right]\,dx+\int_{\partial\Omega_n\cap B}\left(u_n^2+v_n^2\right)\,d\mu_n\right)\\
&\qquad=\liminf_n\left(\int_{\Omega_n\cap B}\left[|\nabla u_n|^2+|\nabla v_n|\right]\,dx+\int_{\partial\Omega_n\cap B}\left(u_n^2+v_n^2\right)\,d\mu_n\right)\\
&\qquad\qquad-\int_{\Omega\cap B}|\nabla u|^2\,dx-\int_{\Omega\cap B}|\nabla v|^2\,dx\\
&\qquad=F(u,B)+F(v,B)-\int_{\Omega\cap B}|\nabla u|^2\,dx-\int_{\Omega\cap B}|\nabla v|^2\,dx=G(u,B)+G(v,B).
\end{aligned}
$$

The result follows again letting B invade A.

Proof of (P5). According to (2.1), let $B\in\mathcal{A}(\Omega)$ be such that

$$F_n(\cdot,B)\xrightarrow{\Gamma}F(\cdot,B).$$

Being $F_n(\cdot,B)$ quadratic functionals on $H^1(D)$, we get thanks to [6, Theorem 11.10]) that also $F(\cdot,B)$ is quadratic on $H^1(D)$. In particular we have $F(0,B)=0$ and that for every $u,v\in H^1(D)$ and $t\neq 0$

$$F(tu,B)=t^2F(u,B),\qquad F(u+v,B)+F(u-v,B)\le 2F(u,B)+2F(v,B).$$

These properties hold true replacing B by an arbitrary $A\in\mathcal{A}(D)$ using the inner regularity according to (2.2). This yields that also $F(\cdot,A)$ is quadratic (see [6, Proposition 11.9]). The quadraticity of $G(\cdot,A)$ follows now from the equality

$$G(u,A)=F(u,A)-\int\limits_{\Omega\cap A}|\nabla u|^2\,dx.$$

□

We are in a position to prove an integral representation formula for G.

Proposition 5.5 (**Integral representation of G on $H^1(D) \times \mathcal{A}(D)$**). *There exists a Borel measure μ on D with support on $\partial\Omega$, absolutely continuous with respect to c_2-capacity and such that for every $u \in H^1(D)$ and $A \in \mathcal{A}(D)$*

$$G(u, A) = \int_{\partial\Omega \cap A} u^2 d\mu, \tag{5.12}$$

where the trace value of u is defined according to Remark 3.5.

Proof. Since G satisfies **(P1)**–**(P4)**, according to [7, Theorem 7.5], there exist a finite Borel measure η on D, absolutely continuous with respect to c_2-capacity and a Borel function $f : \Omega \times \mathbb{R} \to [0, +\infty]$ such that for every $u \in H^1(D)$ and $A \in \mathcal{A}(\Omega)$

$$G(u, A) = \int_A f(x, \tilde{u}(x))\, d\eta(x),$$

where $\tilde{u}$ is the c_2-quasi-continuous representative of u. Thanks to the quadratic behaviour given by **(P5)** we have that

$$f(x, \xi) = a(x)\xi^2$$

where $a : D \to [0, +\infty]$ given by $a(x) := f(x, 1)$ is Borel. Set $\mu := a(x)\eta$: then μ is a Borel measure on D, possibly not finite (since $a(x)$ can be infinite somewhere) such that

$$G(u, A) = \int_A \tilde{u}^2(x) d\mu.$$

According to the definition of the trace value of u in (5.12), in order to conclude it suffices to show that μ is supported on $\partial\Omega$.

Let us start by proving that $\mu(A) = 0$ for all $A \subset\subset \Omega$. According to (2.2) we may assume without loss of generality that $F_n(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$. If $\varphi \in C^1_c(A)$, we can write

$$F(\varphi, A) \le \liminf_n F_n(\varphi, A).$$

Since $\Omega_n \xrightarrow{\mathcal{H}^c} \Omega$, we have that $A \subset\subset \Omega_n$ for n large enough, so that

$$\liminf_n F_n(\varphi, A) = \liminf_n \left(\int_{\Omega_n \cap A} |\nabla\varphi|^2\, dx + \int_{\partial\Omega_n \cap A} \varphi^2\, d\mu_n \right) = \int_A |\nabla\varphi|^2\, dx.$$

We infer

$$\int_A |\nabla\varphi|^2\, dx + \int_A \varphi^2\, d\mu \le \int_A |\nabla\varphi|^2\, dx,$$

so that

$$\int_A \varphi^2\, d\mu \le 0.$$

Letting $\varphi \nearrow 1_A$, we conclude that $\mu(A) = 0$.

We prove now that $\mu(A) = 0$ for every $A \subset\subset D \setminus \overline{\Omega}$. As above we may assume $F_n(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A)$. Let

$$K_n := \overline{A} \cap \overline{\Omega}_n.$$

Thanks to (4.4) we have that $|K_n| \to 0$. Let B_n be open with $K_n \subset\subset B_n \subset\subset \mathrm{int}(D \setminus \overline{\Omega})$, $|B_n| \to 0$, and let $\psi_n \in H^1(D)$, with $0 \le \psi_n \le 1$ on D, $\psi_n = 0$ on K_n and $\psi_n = 1$ outside of B_n. We have

$$\psi_n \to 1 \qquad \text{strongly in } L^2(D).$$

Let $\varphi \in C^1_c(A)$: then

$$\varphi_n := \varphi\psi_n \to \varphi \qquad \text{strongly in } L^2(D),$$

and consequently

$$F(\varphi, A) \le \liminf_n F_n(\varphi_n, A).$$

Being $A \cap \Omega = \emptyset$, the left-hand side reduces to

$$F(\varphi, A) = \int_{A \cap \Omega} |\nabla\varphi|^2\, dx + G(\varphi, A) = G(\varphi, A) = \int_A \varphi^2\, d\mu. \tag{5.13}$$

Since $\varphi_n = 0$ on $A \cap \overline{\Omega}_n$, for the right-hand side we have

$$F_n(\varphi_n, A) = 0.$$

We thus conclude

$$\int_A \varphi^2 \, d\mu \leq 0.$$

Letting again $\varphi \nearrow 1_A$, we deduce $\mu(A) = 0$, and the result follows.

The properties above show that μ is concentrated on $\partial\Omega$, and the proof is concluded. □

We are now in a position to prove Theorem 3.6.

Proof of Theorem 3.6. In view of Proposition 5.1, Lemma 5.2, relation (5.4) and Proposition 5.5, we need only to prove that

$$\mu \geq c\mathcal{H}^{N-1} \lfloor \partial\Omega, \tag{5.14}$$

where c is the constant appearing in assumption (3.4) for μ_n.

Let us show that for every $\varphi \in H^1(D)$ we have

$$c \int_{\partial\Omega} \varphi^2 \, d\mathcal{H}^{N-1} \leq \int_{\partial\Omega} \varphi^2 \, d\mu. \tag{5.15}$$

It is not restrictive to assume that the right-hand side is finite. According to (2.1), let $A \in \mathcal{A}(D)$ be such that $\Omega \subset\subset A$ and

$$F_n(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A).$$

Then by the Hausdorff convergence (3.3) we have also $\Omega_n \subset\subset A$ for n large enough. Let $\varphi_n \in L^2(D)$ be a recovery sequence for φ on A. Using a truncation argument we may assume that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$. In view of the assumption on A we have

$$\int_\Omega |\nabla\varphi|^2 dx + \int_{\partial\Omega} \varphi^2 d\mu = \lim_n \left\{ \int_{\Omega_n} |\nabla\varphi_n|^2 dx + \int_{\partial\Omega_n} \varphi_n^2 \, d\mu_n \right\}. \tag{5.16}$$

Thanks to assumption (3.4) on μ_n we obtain that $(\varphi_n)_{|\Omega_n} \in H^1(\Omega_n)$ is such that

$$\int_{\Omega_n} (|\nabla\varphi_n|^2 + \varphi_n^2) \, dx + \int_{\partial\Omega_n} \varphi_n^2 \, d\mathcal{H}^{N-1} \leq C$$

for some C independent of n. By Lemma 4.2 we infer

$$\varphi_n 1_{\Omega_n} \to \varphi 1_\Omega \qquad \text{strongly in } L^2(D),$$

$$\nabla\varphi_n 1_{\Omega_n} \rightharpoonup \nabla\varphi 1_\Omega \qquad \text{weakly in } L^2(D; R^N),$$

and

$$\int_{\partial\Omega} \varphi^2 d\mathcal{H}^{N-1} \leq \liminf_n \int_{\partial\Omega_n} \varphi_n^2 d\mathcal{H}^{N-1}.$$

Taking into account (5.16) we thus deduce

$$\begin{aligned}
\int_\Omega |\nabla\varphi|^2 dx + \int_{\partial\Omega} \varphi^2 d\mu &= \lim_n \left\{ \int_{\Omega_n} |\nabla\varphi_n|^2 dx + \int_{\partial\Omega_n} \varphi_n^2 \, d\mu_n \right\} \\
&\geq \liminf_n \left\{ \int_{\Omega_n} |\nabla\varphi_n|^2 dx + c\int_{\partial\Omega_n} \varphi_n^2 \, d\mathcal{H}^{N-1} \right\} \geq \int_\Omega |\nabla\varphi|^2 dx + c\int_{\partial\Omega} \varphi^2 d\mathcal{H}^{N-1}.
\end{aligned}$$

which yields (5.15).

If $B \in \mathcal{A}(D)$ and $\varphi \in C_c^1(B)$ with $\varphi \nearrow 1_B$, we deduce from (5.15) that

$$c\mathcal{H}^{N-1}(\partial\Omega \cap B) \leq \mu(B \cap \partial\Omega) = \mu(B)$$

from which we get (5.14). The proof is now concluded. □

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