MULTIPLICITY OF CRITICAL POINTS FOR THE FRACTIONAL ALLEN-CAHN ENERGY

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Abstract. In this paper we study the fractional analogue of the Allen-Cahn energy in bounded domains, and we show that it admits a number of critical points which goes to infinity as the perturbation parameter tends to zero.

1. Introduction

The problems involving fractional operators attracted great attention during the last years. Indeed these problems appear in many areas such as optimization, finance, crystal dislocation, minimal surfaces, water waves, fractional diffusion, (see for example [8], [6], [3], [4], [7], [19], [18]). In particular, from a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a Lévy process, see e.g. [2].

In this paper we present some existence and multiplicity results for critical points of functionals of the form

$$F_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u)\,dx, \qquad \text{if } s\in(0,1/2), \tag{1}$$

$$F_\varepsilon(u) = \frac{1}{|\log\varepsilon|}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,dxdy + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u)\,dx, \qquad \text{if } s=1/2, \tag{2}$$

$$F_\varepsilon(u) = \frac{\varepsilon^{2s-1}}{2}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx, \qquad \text{if } s\in(1/2,1), \tag{3}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, $u \in H^s(\Omega;\mathbb{R})$, $W \in C^2(\mathbb{R};\mathbb{R}^+)$ is the well known double well potential (see Section 2), and $\varepsilon \in \mathbb{R}^+$.

F_ε is the fractional energy of the Allen-Cahn equation. It is the fractional counterpart of the functionals studied by Modica-Mortola in [14], [15] where they proved the Γ-convergence of the energy to De Giorgi's perimeter. In the same way, functionals (1), (2), (3) have been also considered by Valdinoci-Savin in [17], where it is discussed their Γ-convergence.

Moreover, as proved in [13] for the functional

$$\int_\Omega [\varepsilon|Du|^2 + \varepsilon^{-1}(u^2-1)^2]\,dx,$$

we expect that the solutions have interesting geometric properties related to the interface minimality.

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Some autors investigated multiplicity results of nontrivial solution for

$$
\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = h(u) \quad \text{in } \Omega \\ u > 0 \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}
$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n > 2s$, and $h(u)$ has a subcritical growth (see [12]), or for

$$
\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(z)u = f(u) \quad \text{in } \mathbb{R}^n,\ n > 2s \\ u \in H^s(\mathbb{R}^n) \\ u(z) > 0 \quad z \in \mathbb{R}^n \end{cases} \tag{5}
$$

where the potential $V : \mathbb{R}^n \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfy suitable assumptions (see [11]).

Then Cabré and Sire in [5] studied the equation

$$
(-\Delta)^s u + G'(u) = 0 \quad \text{in } \mathbb{R}^n
$$

where G denotes the potential associated to a nonlinearity f, and they proved existence, uniqness and qualitative properties of solutions.

Indeed, Passaseo in [16] studied the analogue of our functional, with the classical Laplacian instead of the fractional one, i.e

$$
f_\varepsilon(u) = \varepsilon \int_\Omega |Du|^2\, dx + \frac{1}{\varepsilon} \int_\Omega G(u)\, dx \tag{6}
$$

where Ω is a bounded domain of $\mathbb{R}^n$, $u \in H^{1,2}(\Omega; \mathbb{R})$, $G \in C^2(\mathbb{R}; \mathbb{R}^+)$ is a nonnegative function having exactly two zeros, α and β, and ε is a positive parameter: he proved that the number of critical points for f_ε goes to ∞ as $\varepsilon \to 0$.

Passaseo was motivated by De Giorgi's idea, contained in [9], i.e. if $u_\varepsilon \to u_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and $\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) < \infty$, then the function $U_\varepsilon(t)$, defined as steepest descent curves for f_ε starting from u_ε, converge to a curve $U_0(t)$ in $L^1(\Omega)$ such that $U_0(t)$ is a function with values in $\{\alpha, \beta\}$ for every $t \geq 0$ and the interface between the sets $E_t = \{x \in \Omega : U_0(t)(x) = \alpha\}$ and $\Omega \setminus E_t$ moves by mean curvature. As a consequence the critical points u_ε of f_ε which satisfy

$$
\liminf_{\varepsilon\to\infty} f_\varepsilon(u_\varepsilon) < +\infty \tag{7}
$$

converge in $L^1(\Omega)$ to a function u_0 taking values in $\{\alpha, \beta\}$. De Giorgi considered also the problem of existence and multiplicity for nontrivial critical points of f_ε with the property (7), and Passaseo's critical points verify this property and

$$
\liminf_{\varepsilon\to\infty} f_\varepsilon(u_\varepsilon) > 0,
$$

so he can say that u_0 is nontrivial.

In this paper we want to extend Passaseo's results by replacing the function G in (6) with the double well potential W, and Passaseo's functional f_ε with its fractional counterpart.

The paper is organized as follow: in the Section 2 we give some preliminaries definitions and results. In the Section 3, we define suitable functions and sets, then most of the work is dedicated to prove nonlocal estimates needful to obtain the bound from above of F_ε, (see Lemma 3.5), and the (PS)-condition, Lemma 3.7. In fact in particular for the first of these results, we had to split the domain in two types of regions and estimate F_ε in the three possible interactions.

Finally, after recalling a technical result, Lemma 3.6, we can apply a classical Krasnoselsii's genus tool to show the existence and multiplicity results for solutions.

Hence, knowing that minimizers of F_ε Γ-converge to minimizers of the area functional, we hope that also min-max solutions can pass to the limit as $\varepsilon \to 0$ in a suitable sense, producing critical points of positive index for local, if $s \in [1/2, 1)$, or nonlocal, if $s \in (0, 1/2)$, area functional.

2. Notation and preliminary results

In this section we introduce the framework that we will be used throughout the paper.

Let Ω be a bounded domain of $\mathbb{R}^n$, denote by $|\Omega|$ its Lebesgue measure and consider W the double well potential, that is

$$\begin{gathered} W : \mathbb{R} \to [0, +\infty) \qquad W \in C^2(\mathbb{R}, \mathbb{R}^+) \qquad W(\pm 1) = 0 \\ W > 0 \text{ in } (-1, 1) \qquad W'(\pm 1) = 0 \qquad W''(\pm 1) > 0. \end{gathered} \tag{8}$$

Now we fix the fractional exponent $s \in (0, 1)$. For any $p \in [1, +\infty)$, we define

$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{s+n/p}} \in L^p(\Omega \times \Omega) \Big\};$$

i.e. an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \Big(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx dy \Big)^{1/p}.$$

If $p = 2$ we define $W^{s,2}(\Omega) = H^s(\Omega)$ and it is a Hilbert space. Now let $\mathscr{S}'(\mathbb{R}^n)$ be the set of all temperated distributions, that is the topological dual of $\mathscr{S}(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we denote by

$$\mathscr{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x) \, dx$$

the Fourier transform of φ and we recall that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$. At this point we can define, for any $u \in \mathscr{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, the fractional Laplacian operator as

$$(-\Delta)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy.$$

Here P.V. stands for the Cauchy principal value and $C(n, s)$ is a normalizing constant (see [10] for more details). It is easy to prove that this definition is equivalent to the following two:

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy \quad \forall x \in \mathbb{R}^n,$$

and

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\xi|^{2s} (\mathscr{F} u)) \quad \forall \xi \in \mathbb{R}^n.$$

Now we remember some embedding's results for the fractional spaces:

Proposition 2.1. [10] *Let $p \in [1, +\infty)$ and $0 < s \leq s' \leq 1$. Let Ω be an open set of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$, denoted by $W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, and the following inequality holds*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$.

Moreover, if also Ω is an open set of $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary, then $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and we have

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$.

Definition 2.2. [10] For any $s \in (0, 1)$ and any $p \in [1, +\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n, p, s, \Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$.

Theorem 2.3. [10] *Let $s \in (0, 1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Let $q \in [1, p^*)$, where $p^* = p^*(n, s) = np/(n - sp)$ is the so-called "fractional critical exponent". Let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\mathscr{I}$ be a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{f \in \mathscr{I}} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dxdy < \infty.$$

Then $\mathscr{I}$ is pre-compact in $L^q(\Omega)$.

We remind also the notion of Krasnoselskii's genus, useful in the sequel:

Definition 2.4. [1] Let H be a Hilbert space and E be a closed subset of $H \setminus \{0\}$, symmetric with respect to 0 (i.e. $E = -E$).

We call genus of E in H, indicated with $\mathrm{gen}_H(E)$, the least integer m such that there exists $\phi \in C(H; \mathbb{R})$ such that ϕ is odd and $\phi(x) \neq 0$ for all $x \in E$.

We set $\mathrm{gen}_H(E) = +\infty$ if there are no integer with the above property and $\mathrm{gen}_H(\emptyset) = 0$.

It is well known that $\mathrm{gen}_H(S^k) = k+1$ if S^k is a k-dimensional sphere of H with centre in zero.

Finally we recall a well known result:

Theorem 2.5. [1] *Let H be a Hilbert space and $f : H \to \mathbb{R}$ be an even C^2-functional satisfying the following Palais-Smale condition: given a sequence $(u_i)_i$ in H such that the sequence $(f(u_i))_i$ is bounded and $f'(u_i) \to 0$, $(u_i)_i$ is relatively compact in H.*

Set $f^c = \{u \in H : f(u) \leq c\}$, $\forall\, c \in \mathbb{R}$. Then, $\forall\, c_1, c_2 \in \mathbb{R}$, such that $c_1 \leq c_2 < f(0)$, we have

$$gen_H(f^{c_2}) \leq gen_H(f^{c_1}) + \#\{(-u_i, u_i) : c_1 \leq f(u_i) \leq c_2,\ f'(u_i) = 0\}, \tag{9}$$

where, if A is a set, we indicate with $\#A$ the cardinality of A.

From now on we consider $H^s(\Omega)$ as Hilbert space and we shall write simply $\mathrm{gen}(E)$ instead of $\mathrm{gen}_{H^s(\Omega)}(E)$; then we refer to the Palais-Smale condition with the symbol (PS)-condition.

3. Multiplicity of critical points

Let us start enouncing the fundamental result of the paper:

Theorem 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and W be a function verifying condition* (8)*. Then there exist two sequences of positive numbers $(\varepsilon_k)_k$, $(c_k)_k$ such that for every $\varepsilon \in (0, \varepsilon_k)$, the functional F_ε has at least k pairs*

$$(-u_{1,\varepsilon}, u_{1,\varepsilon}), \dots, (-u_{k,\varepsilon}, u_{k,\varepsilon})$$

of critical points, all of them different from the costant pair $(-1, 1)$ satisfying

$$-1 \le u_{i,\varepsilon}(x) \le 1 \quad \forall\, x \in \Omega, \quad \forall\, \varepsilon \in (0, \varepsilon_k), \quad \forall\, i = 1, \dots k,$$

and

$$F_\varepsilon(u_{i,\varepsilon}) \le c_k \quad \forall\, \varepsilon \in (0, \varepsilon_k), \quad \forall\, i = 1, \dots, k.$$

Moreover, $\forall\, \varepsilon \in (0, \varepsilon_k)$ and $\forall\, i = 1, \dots, k$ we have

$$F_\varepsilon(u) \ge \min\left\{F_\varepsilon(u) : u \in H^s(\Omega), -1 \le u(x) \le 1 \quad \forall\, x \in \Omega, \int_\Omega u\, dx = 0\right\}. \tag{10}$$

Remark 3.2. The constant function $u \equiv 0$ is obviously a critical point for the functional F_ε for every $\varepsilon > 0$ but it is not included among the ones given by Theorem 3.1. Instead if $s \in (1/2, 1)$, but for the other cases it is similar,

$$F_\varepsilon(0) = \frac{1}{\varepsilon} W(0)|\Omega| \to +\infty \quad \text{as } \varepsilon \to 0.$$

Moreover, since $\inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0$, then one can say that the critical points given by Theorem 3.1 are not constant functions.

In fact, if $u_\varepsilon = c_\varepsilon$ is a costant critical point for F_ε (distinct from -1 and 1), it must be $W'(c_\varepsilon) = 0$ and $-1 < c_\varepsilon < 1$, and therefore

$$W(c_\varepsilon) \ge \inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0 \tag{11}$$

and so, for example by considering the functional related to $s \in (1/2, 1)$, but the other cases are similar,

$$F_\varepsilon(c_\varepsilon) = \frac{1}{\varepsilon} W(c_\varepsilon)|\Omega| \to +\infty \quad \text{as } \varepsilon \to 0, \tag{12}$$

in contradiction with $F_\varepsilon(c_\varepsilon) \le c_k\ \forall\, \varepsilon \in (0, \varepsilon_k)$.

Notice that $\forall \varepsilon > 0$,

$$\min\left\{F_\varepsilon(u) : u \in H^s(\Omega), -1 \le u(x) \le 1 \quad \forall x \in \Omega, \int_\Omega u\, dx = 0\right\} > 0 \tag{13}$$

if we assume, without loss of generality, that Ω is a connected domain.

In fact, let $\bar{u}$ be a minimizing function; if we assume $F_\varepsilon(\bar{u}) = 0$, then

$$\int_\Omega \int_\Omega \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{n+2s}}\, dxdy \equiv 0$$

and $W(\bar{u}) \equiv 0$. Therefore we must have $\bar{u} \equiv 0$ in contradiction with $W(0) > 0$.

Definition 3.3. Let k be a fixed positive integer; for every $\lambda = (\lambda^{(0)}, \dots, \lambda^{(k)}) \in \mathbb{R}^{k+1}$ define the function $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\lambda(t) = \sum_{m=0}^{k} \lambda^{(m)} \cos(mt).$$

For every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$ and $\varepsilon > 0$, let $L_\varepsilon(\varphi_\lambda) : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$L_\varepsilon(\varphi_\lambda)(t) = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\varphi_\lambda(\tau)}{|\varphi_\lambda(\tau)|} \, d\tau;$$

notice that $L_\varepsilon(\varphi_\lambda)$ is well defined because φ_λ has only isolated zeros $\forall \lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$.

For every $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ we consider the projection onto the first component, $P_1(x) = x_1$, and the set

$$S_\varepsilon^k = \{L_\varepsilon(\varphi_\lambda) \circ P_1 : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}.$$

Lemma 3.4. *Let us fix $a, b \in \mathbb{R}$ with $a < b$ and set*

$$\chi(\varphi_\lambda) = \#\{t \in [a,b] : \varphi_\lambda(t) = 0\}$$

for every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$. Then for every $k \in \mathbb{N}$ it results:

$$\sup\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\} < +\infty.$$

Proof. Let us remark that for every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$, the function φ_λ can have only isolated zeros, because, otherwise, it should be identically zero in $\mathbb{R}$, which is impossible since the functions $cos(mt)$ $(m = 0, 1, \dots, k)$ are linearly independent.

Therefore

$$\chi(\varphi_\lambda) < +\infty \quad \forall \lambda \in \mathbb{R}^{k+1} \quad \text{with} \quad |\lambda|_{\mathbb{R}^{k+1}} = 1.$$

Now suppose by contradiction that

$$\sup\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\} = +\infty,$$

i.e. there exists a sequence $(\lambda_i)_i$ in $\mathbb{R}^{k+1}$, with $|\lambda_i|_{\mathbb{R}^{k+1}} = 1 \ \forall i \in \mathbb{N}$, such that $\lim_{i\to\infty} \chi(\varphi_{\lambda_i}) = +\infty$. We can assume that (up to subsequences) $\lambda_i \to \tilde{\lambda}$ in $\mathbb{R}^{k+1}$; so $\varphi_{\lambda_i} \to \varphi_{\tilde{\lambda}}$ uniformly in $\mathbb{R}$. Set

$$\tilde{Z} = \{t \in [a,b] : \varphi_{\tilde{\lambda}}(t) = 0\}$$

and, for every $\varepsilon > 0$,

$$\tilde{Z}_\varepsilon = \{t \in \mathbb{R} : \text{dist}(t, \tilde{Z}) < \varepsilon\}.$$

Since $\min\{|\varphi_{\tilde{\lambda}}(t)| : t \in ([a,b] \setminus \tilde{Z}_\varepsilon)\} > 0 \ \forall \varepsilon > 0$ and $\varphi_{\lambda_i} \to \varphi_{\tilde{\lambda}}$ uniformly in $\mathbb{R}$, for every $\varepsilon > 0$ there exists $j(\varepsilon) \in \mathbb{N}$ such that

$$\min\{|\varphi_{\lambda_i}(t)| : t \in ([a,b] \setminus \tilde{Z}_\varepsilon)\} > 0 \quad \forall i > j(\varepsilon),$$

that is

$$\{t \in [a,b] : \varphi_{\lambda_i}(t) = 0\} \subset \tilde{Z}_\varepsilon \qquad \forall i > j(\varepsilon).$$

Since $\chi(\varphi_{\tilde\lambda}) < +\infty$ and $\lim_{i\to\infty} \chi(\varphi_{\lambda_i}) = +\infty$, we deduce that there exists $\tilde t \in \tilde Z$ such that (up to subsequences) we have:

$$\lim_{i\to\infty} \#\{t \in [\tilde t - \varepsilon, \tilde t + \varepsilon] : \varphi_{\lambda_i} = 0\} = +\infty.$$

It follows that there exists a sequence $(j_h)_h$ in $\mathbb{N}$ such that for every $i > j_h$ there exists $\tilde t^{(h)}_{i,\varepsilon} \in [\tilde t - \varepsilon, \tilde t + \varepsilon]$ which satisfies $D^h \varphi_{\lambda_i}(\tilde t^{(h)}_{i,\varepsilon}) = 0$, where D^h denote the h-order derivative.

We can assume that (up to subsequences) $\lim_{i\to\infty} \tilde t^{(h)}_{i,\varepsilon} = \tilde t^{(h)}_{\varepsilon}$. Hence, $\forall\, \varepsilon > 0$ and $\forall\, h \in \mathbb{N}$ there exists $\tilde t^{(h)}_{\varepsilon} \in [\tilde t - \varepsilon, \tilde t + \varepsilon]$ such that $D^h \varphi_{\tilde\lambda}(\tilde t^{(h)}_{\varepsilon}) = \lim_{i\to\infty} D^h \varphi_{\lambda_i}(\tilde t^{(h)}_{i,\varepsilon}) = 0$ and so, as $\varepsilon \to 0$, we obtain

$$D^h \varphi_{\tilde\lambda}(\tilde t) = 0 \qquad \forall\, h \in \mathbb{N}$$

which implies that $\varphi_{\tilde\lambda} \equiv 0$. But this is a contradiction because $|\tilde\lambda|_{\mathbb{R}^{k+1}} = 1$. □

Lemma 3.5. *Let Ω be a bounded domain of $\mathbb{R}^n$ and W be a function verifying* (8)*. Then, for every $k \in \mathbb{N}$ there exists a positive constant c_k such that*

$$\max F_\varepsilon(f) \le c_k \quad \forall f \in S^k_\varepsilon. \tag{14}$$

Proof. Let $u_{\lambda,\varepsilon} = L_\varepsilon(\varphi_\lambda) \circ P_1 \in S^k_\varepsilon$ and set

$$\begin{aligned} a &= \inf P_1(\Omega),\\ b &= \sup P_1(\Omega),\\ Z_\lambda &= \{t \in [a,b] : \varphi_\lambda(t) = 0\},\\ Z_{\lambda,\varepsilon} &= \{t \in \mathbb{R} : \mathrm{dist}(t, Z_\lambda) < \varepsilon\}. \end{aligned}$$

Notice that

(i) If $P_1(x) \notin Z_{\lambda,\varepsilon}$, then $|u_{\lambda,\varepsilon}(x)| = 1$ and $Du_{\lambda,\varepsilon}(x) = 0$, while

(ii) if $P_1(x) \in Z_{\lambda,\varepsilon}$, then $|u_{\lambda,\varepsilon}(x)| \le 1$ and $|Du_{\lambda,\varepsilon}(x)| \le \dfrac{1}{\varepsilon}$.

Since Ω is bounded, we can suppose it is included in a cube Q of side large enough. We will denote with $Y_{\lambda,\varepsilon} = Z^C_{\lambda,\varepsilon}$ the complement to $Z_{\lambda,\varepsilon}$, then we have to distinguish three cases:

a) if $x \in Y_{\lambda,\varepsilon}$ and $y \in Y_{\lambda,\varepsilon}$;

b) if $x \in Z_{\lambda,\varepsilon}$ and $y \in Y_{\lambda,\varepsilon}$;

c) if $x \in Z_{\lambda,\varepsilon}$ and $y \in Z_{\lambda,\varepsilon}$.

We set $k = \max\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}$, then

$$Z_{\lambda,\varepsilon} = \sum_{i=1}^{k} Z^i_{\lambda,\varepsilon} \quad \text{and} \quad Y_{\lambda,\varepsilon} = \sum_{i=1}^{k} Y^i_{\lambda,\varepsilon}.$$

Now we call $\check Z_{\lambda,\varepsilon} = P_1^{-1}(Z_{\lambda,\varepsilon}) \cap \Omega$, $\check Y_{\lambda,\varepsilon} = P_1^{-1}(Y_{\lambda,\varepsilon}) \cap \Omega$ and we observe that

$$\int_{\check Y_{\lambda,\varepsilon}} W(u_{\lambda,\varepsilon})\, dx = 0, \tag{15}$$

while if we set $\rho = \sup\{|x| : x \in \Omega\}$, $M = \max\{W(t) : |t| \le 1\}$ and denote by ω_{n-1} the $(n-1)$-dimensional measure of the unit sphere of $\mathbb{R}^{n-1}$, it results

$$\int_{\check{Z}_{\lambda,\varepsilon}} W(u_{\lambda,\varepsilon})\,dx \le M|\check{Z}_{\lambda,\varepsilon}| \le 2\varepsilon M\omega_{n-1}\rho^{n-1}. \tag{16}$$

At this point it remains to analyze $\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy$. We split it in three cases:

Case a). We have

(17)
$$\int_{\check{Y}_{\lambda,\varepsilon}}\int_{\check{Y}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy = \sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Y}^i_{\lambda,\varepsilon}}\int_{\check{Y}^j_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy$$

We indicate with $Q_- = Q \cap P_1^{-1}(\{x_1 < 0\})$, $Q_+ = Q \cap P_1^{-1}(\{y_1 > 2\varepsilon\})$ and we split Q_- in N strips of width ε, with N of order $1/\varepsilon$, so we obtain

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Y}^i_{\lambda,\varepsilon}}\int_{\check{Y}^j_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy &\le k^2\int_{Q_-}\int_{Q_+}\frac{4}{|x-y|^{n+2s}}\,dxdy\\ \le 4Nk^2\int_{-\varepsilon}^{-2\varepsilon}\int_{-2x_1}^{+\infty} r^{-2s-1}\,drdx_1 &= \frac{2}{s}Nk^2\int_{-\varepsilon}^{-2\varepsilon}(-2x_1)^{-2s}\,dx_1.\end{aligned} \tag{18}$$

Now we have to distinguish two cases:

j) if $s \neq 1/2$, we have

$$\frac{2}{s}Nk^2\int_{-\varepsilon}^{-2\varepsilon}(-2x_1)^{-2s}\,dx_1 = \frac{2^{1-2s}Nk^2}{s(1-2s)}\varepsilon^{1-2s}(2^{1-2s}-1); \tag{19}$$

jj) while, if $s = 1/2$,

$$\frac{2}{s}Nk^2\int_{-\varepsilon}^{-2\varepsilon}(-2x_1)^{-2s}\,dx_1 = \frac{2^{1-2s}}{s}Nk^2\log 2. \tag{20}$$

Case b). We note that $\check{Y}^i_{\lambda,\varepsilon} \subseteq Q \setminus \check{Z}^i_{\lambda,\varepsilon}$, so

$$
\begin{aligned}
&\int_{\check{Z}_{\lambda,\varepsilon}}\int_{\check{Y}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy \\
&\qquad \le \sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy \\
&\qquad \le 2\omega_{n-1}\rho^{n-1}\varepsilon\sum_{i=1}^{k}\sup_{x\in\check{Z}^i_{\lambda,\varepsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\varepsilon}} \frac{\min\{1/\varepsilon^2|x-y|^2,4\}}{|x-y|^{n+2s}}\,dy \qquad (21)\\
&\qquad \le 2k\varepsilon\omega_{n-1}\rho^{n-1}\Big(\int_0^{2\varepsilon}\frac{1}{\varepsilon^2}r^{1-2s}\,dr+\int_{2\varepsilon}^{+\infty}4r^{-1-2s}\,dr\Big)\\
&\qquad = k\Big(\frac{2}{\varepsilon}\frac{r^{2-2s}}{2-2s}\Big|_0^{2\varepsilon}+8\varepsilon\frac{r^{-2s}}{-2s}\Big|_{2\varepsilon}^{+\infty}\Big)\omega_{n-1}\rho^{n-1}\\
&\qquad = k\varepsilon^{1-2s}\Big(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}\Big)\omega_{n-1}\rho^{n-1}.
\end{aligned}
$$

Case c). It results

(22)

$$
\begin{aligned}
\int_{\check{Z}_{\lambda,\varepsilon}}\int_{\check{Z}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy &= \sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Z}^j_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy.
\end{aligned}
$$

Concerning the first term of the right hand side, we have

$$
\begin{aligned}
(23)\quad &\sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy \\
&\qquad \le \frac{1}{\varepsilon^2}\sum_{i=1}^{k}|\check{Z}^i_{\lambda,\varepsilon}|\int_0^{2\varepsilon} r^{1-2s}\,dr \le k\omega_{n-1}\rho^{n-1}\frac{2^{2-2s}}{1-s}\varepsilon^{1-2s}.
\end{aligned}
$$

The other term is estimated as in *Case b*).

So we can obtain the estimates for the functionals F_ε. In fact, by (18), (19), (20), (21) and (23), if $s \in (0,1/2)$ we have

$$
\begin{aligned}
F_\varepsilon(u_{\lambda,\varepsilon}) &= \int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u_{\lambda,\varepsilon})\,dx \\
&\le 2k\omega_{n-1}\rho^{n-1}\Big(\frac{2^{2-2s}}{1-s}\varepsilon^{1-2s} + \frac{2^{2-2s}}{s}\varepsilon^{1-2s}\Big) \\
&\quad + \varepsilon^{1-2s}\frac{2^{2-2s}}{1-s}k\omega_{n-1}\rho^{n-1} + \frac{kM}{\varepsilon^{2s}}2\varepsilon\omega_{n-1}\rho^{n-1} \\
&\quad + \frac{2^{1-2s}Nk^2}{s(1-2s)}\varepsilon^{1-2s}(2^{1-2s}-1) \\
&\le k\Big(\frac{2^{3-2s}}{1-s} + \frac{2^{3-2s}}{s} + \frac{2^{2-2s}}{1-s} + 2M\Big)\omega_{n-1}\rho^{n-1} \\
&\quad + \frac{2^{1-2s}Nk^2}{s(1-2s)}(2^{1-2s}-1);
\end{aligned}
\tag{24}
$$

if $s = 1/2$ it results

$$
\begin{aligned}
F_\varepsilon(u_\lambda,\varepsilon) &= \frac{1}{|\log\varepsilon|}\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+1}}\,dxdy + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u_{\lambda,\varepsilon})\,dx \\
&\le \left(\frac{20k}{|\log\varepsilon|} + \frac{2kM}{|\log\varepsilon|}\right)\omega_{n-1}\rho^{n-1} + \frac{1}{s|\log\varepsilon|}Nk^2\log 2 \\
&\le k(20+2M)\omega_{n-1}\rho^{n-1} + \frac{1}{s}Nk^2\log 2;
\end{aligned}
\tag{25}
$$

and, if $s \in (1/2,1)$ we get

$$
\begin{aligned}
F_\varepsilon(u_\lambda,\varepsilon) &= \frac{\varepsilon^{2s-1}}{2}\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{n+2s}}\,dxdy + \frac{1}{\varepsilon}\int_\Omega W(u_{\lambda,\varepsilon})\,dx \\
&\le k\Big(\frac{2^{2-2s}}{1-s} + \frac{2^{2-2s}}{s} + \frac{2^{1-2s}}{1-s} + 2M\Big)\omega_{n-1}\rho^{n-1} \\
&\quad + \frac{2^{-2s}Nk^2}{s(1-2s)}(2^{1-2s}-1).
\end{aligned}
\tag{26}
$$

□

Now we show a technical lemma, that we will use to prove our main result:

Lemma 3.6. *For every $\varepsilon > 0$ and $k \in \mathbb{N}$ the set S_ε^k verifies the following properties:*

a) *S_ε^k is a compact subset of $H^s(\Omega)$;*
b) *$S_\varepsilon^k = -S_\varepsilon^k$;*
c) *$\forall\, k \in \mathbb{N}$ there exists $\bar\varepsilon_k > 0$ such that $0 \notin S_\varepsilon^k \quad \forall\,\varepsilon \in (0,\bar\varepsilon_k)$;*
d) *$\forall\, k \in \mathbb{N}$ and $\forall\,\varepsilon > 0$ such that $0 \notin S_\varepsilon^k$, it results $gen(S_\varepsilon^k) \ge k+1$.*

Proof. The points *b*), *c*) and *d*) are proved in [16]. For *a*) we use Lemma 2.8 of [16] and the continuous embedding of $H^1(\Omega)$ in $H^s(\Omega)$ $\forall\, s \in (0,1)$, see Proposition 2.1. □

Before proving the main theorem of this work, we point out a useful property of F_ε:

Lemma 3.7. *Functionals (1), (2), (3) satisfy the (PS)-condition.*

Proof. We will prove the lemma for $s \in (1/2, 1)$ being the other cases analogue.

If W is quadratic, in particular there exist α, $\beta > 0$ such that

$$W(u) \geq \alpha u + \beta \quad \forall\, u \in \mathbb{R}. \tag{27}$$

Since $(F_\varepsilon(u_n))_n$ is bounded, (27) implies that also $\|u_n\|_{H^s(\Omega)}$ is bounded, hence $u_n \rightharpoonup u$ in $H^s(\Omega)$, $u_n \to u$ in L^q, $\forall\, q \in \left[1, 2^* = \dfrac{2n}{n-2s}\right)$ from Theorem 2.3, therefore $u_n \to u$ a.e. $x \in \Omega$.

We claim that u is a critical point of F_ε. In fact $\forall\, v \in H^s(\Omega)$

$$\begin{aligned}
F'_\varepsilon(u)(v) &= \varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} (v(x) - v(y))\, dxdy \\
&\quad + \frac{1}{\varepsilon} \int_\Omega W'(u) v\, dx \\
&= \varepsilon^{2s-1} \lim_{n\to\infty} \int_\Omega \int_\Omega \frac{u_n(x) - u_n(y)}{|x-y|^{n+2s}} (v(x) - v(y))\, dxdy \\
&\quad + \frac{1}{\varepsilon} \lim_{n\to\infty} \int_\Omega W'(u_n) v\, dx
\end{aligned} \tag{28}$$

since $u_n \rightharpoonup u$ in $H^s(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and, by hypothesis, $F'_\varepsilon(u_n) \to 0$.

This implies that $F'_\varepsilon(u_n)(u_n - u) + F'_\varepsilon(u)(u_n - u) \to 0$, but on the other hand

$$\begin{aligned}
&F'_\varepsilon(u_n)(u_n - u) + F'_\varepsilon(u)(u_n - u) = \\
&= \varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{u_n(x) - u_n(y)}{|x-y|^{n+2s}} (u_n(x) - u(x) - u_n(y) + u(y))\, dxdy \\
&\quad - \varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} (u_n(x) - u(x) - u_n(y) + u(y))\, dxdy \\
&\quad + \frac{1}{\varepsilon} \int_\Omega [W'(u_n) - W'(u)(u_n - u)]\, dx,
\end{aligned} \tag{29}$$

and the second term on the right hand side goes to 0. In particular we obtain

$$\int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{n+2s}}\, dxdy \longrightarrow \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dxdy.$$

Hence $\|u_n\|_{H^s(\Omega)} \to \|u\|_{H^s(\Omega)}$ and since $u_n \rightharpoonup u$ in $H^s(\Omega)$, we have the thesis. □

We are now able to prove our main result

Proof of Theorem 3.1. As usual we will prove the theorem only for $s \in (1/2, 1)$.

Consider $\overline{W} \in C^2(\mathbb{R}; \mathbb{R}^+)$ another even function, which satisfies the following properties:

$$\overline{W} = W \quad \forall t \in [-1, 1]; \quad t\overline{W}'(t) > 0 \quad \text{for } |t| > 1$$

and with an asymptotic behaviour guaranteeing that

$$\overline{F}_\varepsilon(u) = \frac{\varepsilon^{2s-1}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dxdy + \frac{1}{\varepsilon} \int_\Omega \overline{W}(u)\, dx$$

is a C^2-functional verifying the (PS)-condition.

We prove now that for every critical point $\overline{u} \in H^s(\Omega)$ which is a critical point for the functional $\overline{F}_\varepsilon$, it results $|\overline{u}(x)| \leq 1 \ \forall x \in \Omega$, and so $\overline{u}$ is a critical point for the functional F_ε too: indeed we have $\forall\, v \in H^s(\Omega)$

$$\varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x-y|^{n+2s}} (v(x) - v(y)) dxdy + \frac{1}{\varepsilon} \int_\Omega \overline{W}'(\overline{u}) v \, dx = 0.$$

In particular, if we set $\hat{u} = \max\{\min\{\overline{u}, 1\}, -1\}$, by choosing $v = \overline{u} - \hat{u}$

$$\varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x-y|^{n+2s}} (\overline{u}(x) - \hat{u}(x) - \overline{u}(y) + \hat{u}(y)) \, dxdy + \frac{1}{\varepsilon} \int_\Omega \overline{W}'(\overline{u})(\overline{u} - \hat{u}) \, dx = 0 \tag{30}$$

with

$$\int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x-y|^{n+2s}} (\overline{u}(x) - \hat{u}(x) - \overline{u}(y) + \hat{u}(y)) \, dxdy = \int_\Omega \int_\Omega \frac{|\overline{u}(x) - \overline{u}(y)|^2}{|x-y|^{n+2s}} \, dxdy \geq 0 \tag{31}$$

and

$$\int_\Omega \overline{W}'(\overline{u})(\overline{u} - \hat{u}) \, dx > 0 \quad \text{ if } \ \overline{u} - \hat{u} \not\equiv 0 \quad \text{ in } \ \Omega$$

since $t\overline{W}'(t) > 0$ for $|t| > 1$. It follows that $\overline{u} = \hat{u}$, i.e. $|\overline{u}(x)| \leq 1$ for almost every $x \in \Omega$.

Let $\varepsilon_k > 0$ be such that $\varepsilon_k < \dfrac{1}{c_k} W(0)|\Omega|$, where c_k is the constant introduced in Lemma 3.5. Then, for every $\varepsilon \in (0, \varepsilon_k)$ we can apply Theorem 2.5 to the functional $\overline{F}_\varepsilon$ with $\overline{c}_1 < 0$ and $c_2 = c_k$, because $\overline{F}_\varepsilon(0) = \dfrac{1}{\varepsilon} W(0)|\Omega| > c_k \ \forall\, \varepsilon \in (0, \varepsilon_k)$. In this way we can prove that for every $\varepsilon \in (0, \varepsilon_k)$, $\overline{F}_\varepsilon$ has at least $(k+1)$ pairs $(-u_{0,\varepsilon}, u_{0,\varepsilon}), \ldots, (-u_{k,\varepsilon}, u_{k,\varepsilon})$ of critical points with $\overline{F}_\varepsilon(u_{i,\varepsilon}) \leq c_k \ \forall\, i = 0, 1, \ldots, k$. In fact gen$(\overline{F}_\varepsilon^{\overline{c}_1})$ =gen$(\emptyset) = 0$, while gen$(\overline{F}_\varepsilon^{\overline{c}_k}) \geq$gen$(S_\varepsilon^k) \geq k+1$ because $S_\varepsilon^k \subseteq \overline{F}_\varepsilon^{\overline{c}_k} \subseteq H^s(\Omega) \setminus \{0\}$.

Notice that these $(k+1)$ pairs of critical points include also the one constitued by the minimizers ± 1; so we can assume that $(-u_{0,\varepsilon}, u_{0,\varepsilon}) = (-1, +1)$.

On the contrary, the other solutions are not minimizers for the functional $\overline{F}_\varepsilon$ if Ω is a connected domain. Indeed it results

$$\overline{F}_\varepsilon(u_{i,\varepsilon}) > 0 \qquad \forall\, \varepsilon \in (0, \varepsilon_k) \text{ and } \forall\, i = 0, 1, \ldots, k$$

because if $F_\varepsilon(u_{i,\varepsilon}) = \overline{F}_\varepsilon(u_{i,\varepsilon}) = 0$, then we should have

$$\int_\Omega \int_\Omega \frac{|u_{i,\varepsilon}(x) - u_{i,\varepsilon}(y)|^2}{|x-y|^{n+2s}} \, dxdy = 0 \quad \text{ and } \quad \overline{W}(u_{i,\varepsilon}) \equiv 0 \quad \text{ in } \Omega$$

and so $u_{i,\varepsilon}$ should be a constant function with value $+1$ or -1.

Moreover let us remark that $\forall\, \varepsilon \in (0, \varepsilon_k)$ and $\forall\, i = 1, \ldots k$ we have

$$F_\varepsilon(u_{i,\varepsilon}) \geq \min \left\{ \overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u \, dx = 0 \right\}. \tag{32}$$

In fact, assume that

$$\min\left\{\overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\,dx = 0\right\} > 0,$$

otherwise (32) would be obvious. Then, for every $\bar{c}_1 > 0$ such that

$$\bar{c}_1 < \min\left\{\overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\,dx = 0\right\},$$

we would have clearly $\mathrm{gen}(\overline{F}_\varepsilon^{\bar{c}_1}) = 1$ because below c_1 the mean is non zero and we can use it as odd function into $\mathbb{R}^1$ in the genus definition, see Definition 2.4; thus, if (32) was false, the solutions would belong to a set of genus one, in contradiction with respect their costruction in Theorem 2.5.

Now, in order to prove (10), let us replace the function $\overline{W}$ appearing in the definition of functional $\overline{F}_\varepsilon$ by a sequence of functions $(\overline{W}_j)_j$ and denote by $(\overline{F}_\varepsilon^j)_j$ the corresponding sequence of new functionals. Assume moreover that the functions $\overline{W}_j$ satisfy the same properties as $\overline{W}$ for all $j \in \mathbb{N}$ and that

$$\lim_{j\to\infty} \overline{W}_j(t) = +\infty \quad \text{for} \quad |t| > 1. \tag{33}$$

Then property (32) holds for the higher critical values of the functional $\overline{F}_\varepsilon^j$ for all $j \in \mathbb{N}$ and so (10) follows for j large enough, taking into account that

$$\begin{aligned}\lim_{j\to\infty} \min&\left\{\overline{F}_\varepsilon^j(u) : u \in H^s(\Omega), \int_\Omega u\,dx = 0\right\}\\ &= \min\left\{F_\varepsilon(u) : u \in H^s(\Omega), |u(x)| \le 1\ \forall x \in \Omega, \int_\Omega u\,dx = 0\right\}\end{aligned}$$

because of (33).

□

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