Generalised Touching Point Arguments in Geometric Measure Theory

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Abstract

We provide a strengthening of an elementary technique in Geometric Measure Theory. Given an s-set $S \subset \mathbb{R}^n$ with upper and lower density bounds, in the language of tangent measures, this technique establishes the existence of tangent measures to the measure $H^s_{\lfloor S}$ on one side of an $(n-1)$-plane.

If S is purely unrectifiable or of dimension less than $n-1$, our strengthening consists of being able to find tangent measures on one side of an $(n-1)$-plane containing a vector of our choice. We give an application of this result to the problem of characterising removable sets for harmonic functions.

1 Introduction

In this note we give a strengthening of a well known technique in the subfield of Geometric Measure Theory known as "Rectifiability and Densities". Arguments using this technique are usually called "Touching point arguments". Given our set S, touching point arguments (roughly speaking) show that for all most all points $x \in S$, there exists a sequence of radii $r_n \to 0$ such that $B_{r_n}(x) \cap S$ lies mostly on one side of an $(n-1)$ plane intersecting x. This essentially follows from the existence of large empty sub-balls in any ball centered on any point of S, which in turn follows from basic density estimates. The technique originates from work of Besicovitch [1], and has been used in an essential way in [6],[8] and [12].

For those familiar with the language of tangent measures, these arguments establish the existence of a tangent measure to the measure $H^s_{\lfloor S}$ that exists on one side of an $(n-1)$ plane. For sets S that are unrectifiable or of dimension less than $n-1$, our strengthening consists of being able to show that $B_{r_n}(x) \cap S$ lies increasingly on one side of an $(n-1)$ plane containing a vector in S^{n-1} of our choice

This result comes from a reworking of the basic idea using Besicovitch-Federer projection theorem instead of density estimates. The proof is intricate, but in our opinion the method is classical. This work arose from the author's attempts to generalise rectifiability and density theorems outside Euclidean space, [5]. Our main lemma is an n-dimensional version of Lemma 14 from [5]. Touching point arguments have application in a variety of geometrical problems. We hope the results of this note might also be of some utility; by way of example, we apply the theorem to the results of Mattila,

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Paramonov, to give a strengthening (under more restrictive hypotheses) of one of their theorems related to the problem of characterising removable sets for harmonic functions.

First some notation. Let $B_r(z) := \{y \in \mathbb{R}^n : |y-z| < r\}$. Let H^s denote Hausdorff s-measure. Let P_V be the orthogonal projection onto subspace V.

We define the cone $X(x,v,\alpha) = \left\{y \in \mathbb{R}^n : \alpha\left|P_{v^\perp}(y-x)\right| \leq \left|P_{\langle v\rangle}(y-x)\right|\right\}$.
Let $X^+(x,v,\alpha) = \{y \in \mathbb{R}^n : y \in X(x,v,\alpha) \text{ and } (y-x)\cdot v > 0\}$.

Theorem 1 *Given integer $n \geq 3$ and real numbers $s \in (0, n-1]$ and $\rho \in \left(0, \frac{1}{2}\right)$, if $S \subset \mathbb{R}^n$ is a set (purely unrectifiable in the case $s = n-1$) of positive finite H^s measure with the property that for H^s a.e. $x \in S$*

$$\liminf_{r\to 0} \frac{H^s(B_r(x)\cap S)}{r^s} > 0. \tag{1}$$

Then for any $\psi \in S^{n-1}$ we have that for H^s a.e. $x \in S$

$$\liminf_{r\to 0} \frac{H^s(B_r(x)\cap S \cap X^+(x,\phi,\rho))}{r^s} = 0$$

for some $\phi \in \psi^\perp \cap S^{n-1}$.

1.1 Application: Removable sets for harmonic functions

A compact set $E \subset \mathbb{C}$ is said to be *removable* if and only if the following holds true:

If $U \subset \mathbb{C}$ is open, $E \subset U$ and $f : U\backslash E \to \mathbb{C}$ is a bounded analytic function, then f has an analytic extension to U.

The long standing problem of geometrically characterizing removable sets was resolved for sets of finite H^1 measure in [2], building on work of [3] and [10]. It is known and relatively easy to show that (see [9]) sets of zero H^1 measure are removable and sets of Hausdorff dimension greater than 1 are not removable. So the characterisation is needed for sets of dimension 1.

First some notation: An *m-rectifiable* set S in $\mathbb{R}^n$ is a set that can be covered by countably many C^1 submanifolds of dimension m. A *purely m-unrectifiable* set T is a set with the property; $H^m(T\cap M) = 0$ for all C^1 m dimensional submanifolds M.

It was proved in [2] that if S has finite H^1 measure and is not removable then S must have a 1-rectifiable subset. The converse is well known, so sets of finite H^1 measure are removable if and only if they are purely 1-unrectifiable. Analogous questions can be asked about harmonic functions.

Following [11] a subset E of $\mathbb{R}^n$ is called Lip_1-removable for harmonic functions, (abbreviated L_1RH) if for each domain $D \subset \mathbb{R}^n$, every locally Lipschitz function $f : D \to \mathbb{R}$ which is harmonic in $D\backslash E$ is harmonic in D.

It would be reasonable to conjecture that E is Lip_1-removable if and only if it is purely $(n-1)$ rectifiable.

The only results on this conjecture are from [11] (Theorem 5.5) where it is proved;

Theorem 2 (Mattila, Paramonov) *Let $X \subset \mathbb{R}^n$ be a compact set of finite H^{n-1} measure such that for some constant A,*

$$H^{n-1}(X \cap B_r(a)) \leq Ar^{n-1} \tag{2}$$

for $a \in \mathbb{R}^n$ and $r > 0$. Suppose that the following holds at H^{n-1} almost all points $a \in X$: For every $v \in S^{n-1}$ there is a $\delta > 0$ such that

$$\liminf_{r \to 0} r^{1-n} H^{n-1} \left(y \in X \cap B_r(a) : |(y-a) \cdot v| > \delta |y-a| \right) > 0 \tag{3}$$

then X is Lip_1*-removable.*

Given a set S of positive finite H^{n-1} measure, if $a \in S$ has the property that

$$\liminf_{r \to 0} r^{1-n} H^{n-1} \left(y \in S \cap B_r(a) : |(y-a) \cdot v| > \delta |y-a| \right) = 0 \tag{4}$$

for any $\delta > 0$ then we say S has a *weak tangent* $V := v^{\perp}$ at a. Weak tangents are a well known partial result to rectifiability (see [7] where the term was coined); if the lim inf of (4) is replaced by a lim sup, then this condition holding for almost every point $a \in S$ and for any $\delta > 0$ is equivalent to $(n-1)$-rectifiability, see [9].

So Theorem 2 says that any set (with density bound (2)) which is not Lip_1-removable must have a subset B of positive H^{n-1} measure such that for each $a \in B$ there exists $v \in S^1$ with condition (4) holding for every $\delta > 0$. Informally; a non Lip_1 removable set must have weak tangents on some subset of positive measure. This result suggests that these sets should be rectifiable.

Given constants $0 < A_1 < A_2 < \infty$, we say a set $S \subset \mathbb{R}^n$ satisfying

$$A_1 r^{n-1} \leq H^{n-1}\left(S \cap B_r(a)\right) \leq A_2 r^{n-1}$$

for any point $a \in S$ and any $r > 0$ is an A.D. (Alfors-David) $(n-1)$-set.

Suppose the conjecture for harmonic functions is not true even for A.D. sets and so we have an A.D. purely $(n-1)$ unrectifiable set S which is not Lip_1-removable, then by directly inserting the statement of Theorem 1 into the proof of Theorem 5.5 of Mattila Paramonov [11] (instead of their Lemma 5.2) we have the following result

Theorem 3 *Suppose $S \subset \mathbb{R}^n$ is purely unrectifiable A.D. $(n-1)$-set which is not* Lip_1*-removable. Then there must exist a subset $B \subset S$ of positive H^{n-1} measure such that for any $\psi \in S^{n-1}$ we have that for almost all $x \in B$ the set S must have a weak tangent V at x with the property that $\psi \in V$.*

So if there exists a purely unrectifiable A.D. $(n-1)$-set S which is not Lip_1 removable then it must have a subset B of positive measure such that for H^{n-1} almost every point $z \in B$ there exists an infinite collection of $(n-1)$ subspaces $\{V_1^z, V_2^z, \ldots\}$ each of which is a weak tangent to S at z.

This is a strengthening for a smaller class of sets but it shows how Theorem 1 can be used.

2 Background and Notation

First we will need to introduce some more notation. If $x \in \mathbb{R}^n$, $v \in S^{n-1}$, $s > 0$, let $K(x,v,s) := P_{v^{\perp}}^{-1}\left(B_s(x) \cap v^{\perp}\right)$ be the standard definition of a cylinder.

We will also need to define a kind of double cone object, see figure 1. Given $x \in \mathbb{R}^n$, $v \in S^{n-1}$, $s \in (0,1)$ and $r > 0$ define
$\Psi^u(x,v,s,r) := \{z \in B_r(x) : |P_{v^{\perp}}(z-x)| \leq s\,|P_v(z-(x+rv))|\}$.
$\Psi^d(x,v,s,r) := \{z \in B_r(x) : |P_{v^{\perp}}(z-x)| \leq s\,|P_v(z-(x-rv))|\}$.
And define $\Psi(x,v,s,r) = \Psi^u(x,v,s,r) \cap \Psi^d(x,v,s,r)$. Let $A(x,\alpha,\beta) := B_\beta(x) \setminus \overline{B_\alpha(x)}$.

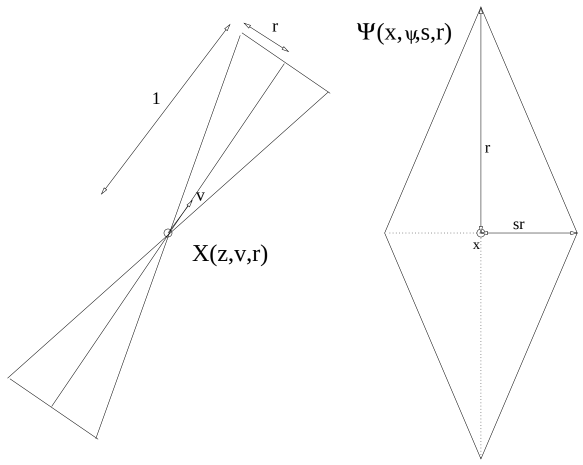

Figure 1:

3 Proof

Proof

Suppose the statement of Theorem 1 was false, then there exists $\psi \in S^{n-1}$ for which we can take some subset $S_0 \subset S$ such that for some $\rho > 0$ and some $\lambda_1 > 0$ we have for all $\phi \in \psi^{\perp} \cap S^{n-1}$

$$\liminf_{r \to 0} \frac{H^s\left(B_r\left(x\right) \cap X^+\left(x, \phi, \rho\right) \cap S\right)}{r^s} > 2\lambda_1,$$

for all $x \in S_0$. Since we have (1), we can take subset S_0 to be such that for some small $\alpha > 0$ we have

$$\liminf_{r \to 0} \frac{H^s\left(B_r\left(x\right) \cap S\right)}{r^s} > \alpha,$$

for all $x \in S_0$.

Now we can take a closed subset $S_1 \subset S_0$ and some small $r_0 > 0$ such that

$$\frac{H^s\left(B_r\left(x\right) \cap X^+\left(x, \phi, \rho\right) \cap S\right)}{r^s} > \lambda_1 \tag{5}$$

for all $\phi \in \psi^{\perp} \cap S^{n-1}$, $x \in S_1$, $r \in (0, r_0)$. And (using Theorem 6.2 [9] for the upper bound on the density)

$$\alpha < \frac{H^s\left(B_r\left(x\right) \cap S\right)}{r^s} < 2 \tag{6}$$

for all $x \in S_1$, $r \in (0, r_0)$.

First we prove the following.

Lemma 1 *Given integer $n \geq 3$ and real number $s \in (0, n)$ if we have a set $S \subset \mathbb{R}^n$ of positive finite H^s measure and a closed subset $S_1 \subset S$ with the following two properties:*

Firstly, there exists a vector $\psi \in S^{n-1}$ and a number $r_0 > 0$ such that for some small $\lambda_1 > 0$ and $\rho > 0$ we have

$$\frac{H^s\left(B_r\left(x\right) \cap X^+\left(x, \phi, \rho\right) \cap S\right)}{r^s} > \lambda_1$$

for all $\phi \in \psi^{\perp} \cap S^{n-1}$ *and all* $x \in S_1$*,* $r \in (0, r_0)$*.*

Secondly, for some small $\alpha > 0$ *we have*

$$\alpha < \frac{H^s\left(B_r\left(x\right) \cap S\right)}{r^s} < 2 \tag{7}$$

for all $x \in S_1$*,* $r \in (0, r_0)$*.*

Then we can find constants $\kappa_{\rho}^{\lambda_1} > 0$ *and* $\vartheta_{\rho}^{\lambda_1} > 0$ *such that the following statement holds true: If* $d \in (0, r_0)$ *is such that*

$$\frac{H^s\left(B_{4d}\left(x\right) \backslash S_1\right)}{d^s} \leq \epsilon \tag{8}$$

then for all $z \in \left(\psi^{\perp} + x\right) \cap K\left(x, \psi, \kappa_{\rho}^{\lambda_1} d\right)$ *we have that*

$$K\left(z, \psi, \epsilon^{\frac{1}{s}} \vartheta_{\rho}^{\lambda_1} d\right) \cap S_1 \cap B_d\left(x\right) \neq \emptyset.$$

Proof Firstly to simplify the expressions, we let $\mu := H^s_{\lfloor S}$. We start by showing the following:

If $a \in \partial K\left(x, \psi, r\right)$ let W_a denote the $(n-1)$ dimensional tangent plane of the boundary of the cylinder $K\left(x, \psi, r\right)$ at point a. Let $n_a \in W_a^{\perp} \cap S^{n-1}$ be the unit normal pointing "inwards" towards the center of cylinder $K\left(x, \psi, r\right)$.

Step 1 : We will show we have

$$X^{+}\left(a, n_a, \rho\right) \cap B_{\frac{2r\rho}{(1+\rho)}}\left(a\right) \subset K\left(x, \psi, r\right). \tag{9}$$

To see this, it makes the calculations a lot easier if we change our orthonormal basis. Let $\varepsilon_1 = n_a$ and $\varepsilon_2 = \psi$ and complete this to get an orthonormal set of vectors $\{\varepsilon_j\}$ where $\{\varepsilon_2, \ldots \varepsilon_n\}$ span W_a. Let $c = a + r\varepsilon_1$. See figure 2. Now in our new basis (using

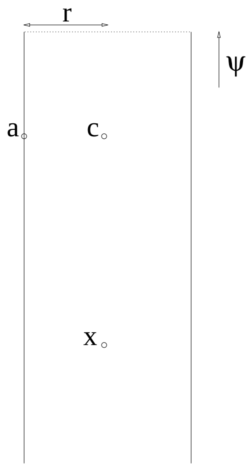

Figure 2:

point c) the definition of the cylinder is as follows

$$K\left(x, \varepsilon_2, r\right) = \left\{ z \in \mathbb{R}^n : \left(\left(z - c\right) \cdot \varepsilon_1\right)^2 + \sum_{j=3}^{n} \left(\left(z - c\right) \cdot \varepsilon_j\right)^2 < r^2 \right\}.$$

And in our basis we have that

$$X^{+}(a,\varepsilon_1,\rho)=\left\{z\in\mathbb{R}^n:(z-a)\cdot\varepsilon_1\geq\rho\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)}\right\}.$$

So if $z\in X^{+}(a,\varepsilon_1,\rho)\cap B_{\frac{2r\rho}{(1+\rho)}}(a)$ we have that $(z-a)\cdot\varepsilon_1\in\left(0,\frac{2r\rho}{(1+\rho)}\right)$ and so as ρ is small $(z-a)\cdot\varepsilon_1\in(0,r)$ and we have

$$\begin{aligned}((z-c)\cdot\varepsilon_1)^2+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2 &= ((c-a)\cdot\varepsilon_1-(z-a)\cdot\varepsilon_1)^2+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2\\ &= (r-(z-a)\cdot\varepsilon_1)^2+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2\\ &\leq \left(r-\rho\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)}\right)^2+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2\\ &\leq r^2-2r\rho\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)}\\ &\quad+\rho^2\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2. \qquad (10)\end{aligned}$$

Now as $a=c-r\varepsilon_1$ we know that $(c-a)\cdot\varepsilon_k=0$ for $k\in\{2,3,\dots n\}$ so

$$\begin{aligned}\sum_{k=3}^{n}((z-a)\cdot\varepsilon_k)^2 &= \sum_{k=3}^{n}((z-c)\cdot\varepsilon_k+(c-a)\cdot\varepsilon_k)^2\\ &= \sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2.\end{aligned}$$

So putting this into (10) we get that

$$\begin{aligned}((z-c)\cdot\varepsilon_1)^2+\sum_{k=3}^{n}((z-c)\cdot\varepsilon_k)^2 &\leq r^2-2r\rho\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)}\\ &\quad+\left(1+\rho^2\right)\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right). \qquad (11)\end{aligned}$$

Thus if

$$2r\rho\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)}\geq\left(1+\rho^2\right)\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right) \qquad (12)$$

then $z\in K(x,\varepsilon_2,r)$. Now (12) is equivalent to

$$2r\rho\geq\left(1+\rho^2\right)\sqrt{\left(\sum_{k=2}^{n}((z-a)\cdot\varepsilon_k)^2\right)} \qquad (13)$$

and as $z \in B_{\frac{2r\rho}{(1+\rho^2)}}(a)$ so (13) obviously holds and so (12) also holds and $z \in K(x, \psi, r)$, we have shown Step 1.

Step 2:
Now suppose $a \in \partial K(x, \psi, r)$ and $n_a \in S^{n-1} \cap \psi^{\perp}$ such that

$$X^{+}(a, n_a, \rho) \cap B_{\frac{r\rho}{(1+\rho^2)}}(a) \subset K(x, \psi, r)$$

and we have that

$$\mu\left(X^{+}(a, n_a, \rho) \cap B_{\frac{r\rho}{(1+\rho^2)}}(a) \cap S_1\right) \geq \frac{\lambda_1}{2}\left(\frac{r\rho}{(1+\rho^2)}\right)^{s}, \tag{14}$$

we will show there exists some number $\zeta_{\rho}^{\lambda_1} > 0$ with the property that

$$X^{+}(a, n_a, \rho) \cap K\left(x, \psi, \left(1 - \zeta_{\rho}^{\lambda_1}\right) r\right) \cap S_1 \neq \emptyset. \tag{15}$$

Now note that for any $\beta > 0$, if $z \in X^{+}(a, n_a, \rho) \cap B_{\beta}(a)$ then $X^{+}(a, n_a, \rho) \cap B_{\beta}(a) \subset B_{2\beta}(z)$. So if $X^{+}(a, n_a, \rho) \cap B_{\beta}(a) \cap S_1 \neq \emptyset$ then by (7)

$$\mu\left(X^{+}(a, n_a, \rho) \cap B_{\beta}(a)\right) \leq 2(2\beta)^{s}$$

hence there exists some small number $\sigma_{\lambda_1} \in \left(0, \frac{\rho}{8}\right)$ such that

$$\mu\left(A\left(a, \sigma_{\lambda_1} r, \frac{r\rho}{(1+\rho^2)}\right) \cap X^{+}(a, n_a, \rho) \cap S_1\right) \geq \frac{\lambda_1}{4}\left(\frac{r\rho}{(1+\rho^2)}\right)^{s}.$$

We are going to use this and (11) to show Step 2. Firstly let $g(p) = -2r\rho p + \left(1+\rho^2\right)p^2$, the two zeros of this quadratic are at 0 and $\frac{2r\rho}{(1+\rho^2)}$. Since σ_{λ_1} is much smaller that ρ

$$\begin{aligned} g(\sigma_{\lambda_1} r) &= -2r^2\rho\sigma_{\lambda_1} + \left(1+\rho^2\right)\sigma_{\lambda_1}^2 r^2 \\ &= \left(\left(1+\rho^2\right)\sigma_{\lambda_1}^2 - 2\rho\sigma_{\lambda_1}\right) r^2 \\ &< 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} g\left(\frac{r\rho}{(1+\rho^2)}\right) &= \frac{-2(r\rho)^2}{(1+\rho^2)} + \left(1+\rho^2\right)\left(\frac{r\rho}{(1+\rho^2)}\right)^2 \\ &= \frac{\left(-2(r\rho)^2 + (r\rho)^2\right)}{(1+\rho^2)} \\ &= \frac{-(r\rho)^2}{(1+\rho^2)}. \end{aligned}$$

If we let

$$\zeta_{\rho}^{\lambda_1} = \frac{1}{2}\min\left\{-\frac{g(\sigma_{\lambda_1} r)}{r^2}, -\frac{g\left(\frac{r\rho}{(1+\rho^2)}\right)}{r^2}\right\}$$

then we have that for $q \in \left(\sigma_{\lambda_1} r, \frac{r\rho}{(1+\rho^2)}\right)$

$$g(q) \leq -2\zeta_\rho^{\lambda_1} r^2.$$

Let $z \in A\left(a, \sigma_{\lambda_1} r, \frac{r\rho}{(1+\rho^2)}\right) \cap X^+(a, n_a, \rho) \cap S_1$. Now in our notation inequality (11) becomes

$$\begin{aligned} ((z-c)\cdot\varepsilon_1)^2 + \sum_{j=3}^{n} ((z-c)\cdot\varepsilon_j)^2 &\leq r^2 + g\left(\sqrt{\sum_{k=2}^{n} ((z-a)\cdot\varepsilon_k)^2}\right) \\ &\leq \left(1 - 2\zeta_\rho^{\lambda_1}\right) r^2 \\ &\leq \left(1 - \zeta_\rho^{\lambda_1}\right)^2 r^2 \end{aligned}$$

so $z \in K\left(x, \psi, \left(1-\zeta_\rho^{\lambda_1}\right) r\right)$ and this establishes Step 2.

Let $\xi_\rho = \frac{\rho}{(1+\rho^2)}$ and let $\kappa_\rho^{\lambda_1} = \frac{\zeta_\rho^{\lambda_1}}{\xi_\rho}$. Given $x \in S_1$ let $d > 0$ such that for some small $\epsilon > 0$

$$\frac{\mu\left(B_{4d}(x) \backslash S_1\right)}{d^s} \leq \epsilon. \tag{17}$$

Let $z \in \psi^\perp + x$ be such that $\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \subset B_{2d}(x)$.

Define

$$H\left(z, \psi^\perp, \alpha\right) := \bigcup_{h \in (-\alpha, \alpha)} z + h\psi + \psi^\perp.$$

Let $\phi_k = \sum_{j=0}^{k} \xi_\rho \kappa_\rho^{\lambda_1} d \left(1-\zeta_\rho^{\lambda_1}\right)^j$ where we let $\phi_{-1} = 0$. Define

$$\Gamma_k := H\left(z, \psi^\perp, \phi_k\right) \cap K\left(z, \psi, \left(1-\zeta_\rho^{\lambda_1}\right)^k \kappa_\rho^{\lambda_1} d\right). \tag{18}$$

Step 3 :

We will show

$$\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \subset \bigcup_{j=0}^{\infty} \Gamma_j. \tag{19}$$

See figure 3. We argue inductively. Firstly recall that $\phi_0 = \xi_\rho \kappa_\rho^{\lambda_1} d$ and note that it is obvious that

$$\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap H\left(z, \psi^\perp, \xi_\rho \kappa_\rho^{\lambda_1} d\right) \subset \Gamma_0.$$

Suppose we have that

$$\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap H\left(z, \psi^\perp, \phi_k\right) \subset \bigcup_{j=0}^{k} \Gamma_j. \tag{20}$$

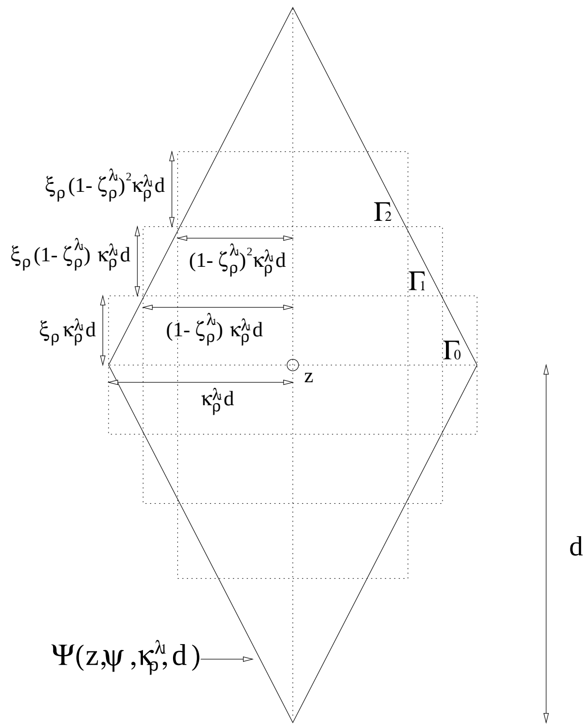

Figure 3:

Let ϑ_{k+1} be the radius of the two congruent spheres given by $\partial\Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)\cap\partial H\left(z,\psi^\perp,\phi_k\right)$. So by definition of $\Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$ we know

$$\begin{aligned}\vartheta_{k+1} &= \kappa_\rho^{\lambda_1}d-\kappa_\rho^{\lambda_1}\phi_k\\ &= \kappa_\rho^{\lambda_1}d-\kappa_\rho^{\lambda_1}\left(\sum_{j=0}^{k}\xi_\rho\kappa_\rho^{\lambda_1}d\left(1-\zeta_\rho^{\lambda_1}\right)^j\right)\\ &= \kappa_\rho^{\lambda_1}d-\xi_\rho\left(\kappa_\rho^{\lambda_1}\right)^2d\left(\sum_{j=0}^{k}\left(1-\zeta_\rho^{\lambda_1}\right)^j\right). \end{aligned}\tag{21}$$

And note

$$\sum_{j=0}^{k}\left(1-\zeta_\rho^{\lambda_1}\right)^j = \frac{1}{\zeta_\rho^{\lambda_1}}\left(1-\left(1-\zeta_\rho^{\lambda_1}\right)^{k+1}\right).$$

Recall $\kappa_\rho^{\lambda_1}=\frac{\zeta_\rho^{\lambda_1}}{\xi_\rho}$ so putting the above expression into (21) we get

$$\vartheta_{k+1} = \kappa_\rho^{\lambda_1}d\left(1-\zeta_\rho^{\lambda_1}\right)^{k+1}$$

which is exactly the width of the cylinder Γ_{k+1}.
Now as ϑ_{k+1} is the biggest radius of the spheres given by $\psi^\perp$ slices of

$$\partial\Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)\cap\left(H\left(z,\psi^\perp,\phi_{k+1}\right)\backslash H\left(z,\psi^\perp,\phi_k\right)\right).$$

So we know that $\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap \left(H\left(z, \psi^\perp, \phi_{k+1}\right) \backslash H\left(z, \psi^\perp, \phi_k\right)\right)$ is "thin" enough to fit into Γ_{k+1}. As it is also, by definition, short enough to fit into Γ_{k+1}. So by inductive assumption (20) we have

$$\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap H\left(z, \psi^\perp, \phi_{k+1}\right) \subset \bigcup_{j=0}^{k+1} \Gamma_j.$$

This establishes Step 3.

Step 4:

Recall $z \in \psi^\perp + x$ such that $\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \subset B_{2d}(x)$.

Suppose $\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap S_1 \neq \emptyset$. Let $k_1 = \max\left\{k : \Gamma_k \cap \Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap S_1 \neq \emptyset\right\}$. We will show k_1 is sufficiently big so that $\lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho \kappa_\rho^{\lambda_1} d\right)^s < 2\epsilon d^s$.

Suppose not and

$$\lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho \kappa_\rho^{\lambda_1} d\right)^s \geq 2\epsilon d^s. \tag{22}$$

We know that $\Gamma_{k_1+1} \cap \Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap S_1 = \emptyset$. Let $y_1 \in \Gamma_{k_1} \cap \Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \cap S_1$, let $h = \left|P_{\psi^\perp}(y_1 - z)\right|$ so we know that

$$h \in \left(\left(1-\zeta_\rho^{\lambda_1}\right)^{k_1+1} \kappa_\rho^{\lambda_1} d, \left(1-\zeta_\rho^{\lambda_1}\right)^{k_1} \kappa_\rho^{\lambda_1} d\right). \tag{23}$$

Now $y_1 \in \Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right) \subset B_{2d}(x)$ and as $h\xi_\rho \leq \xi_\rho\left(1-\zeta_\rho^{\lambda_1}\right)^{k_1} \kappa_\rho^{\lambda_1} d = \zeta_\rho^{\lambda_1}\left(1-\zeta_\rho^{\lambda_1}\right) d$ so

$$X^+(y_1, n_{y_1}, \rho) \cap B_{h\xi_\rho}(y_1) \subset B_d(y_1) \subset B_{4d}(x). \tag{24}$$

Now by the fact that $y_1 \in S_1$ (recall (18)) we know that

$$\mu\left(B_{h\xi_\rho}(y_1) \cap X^+(y_1, n_{y_1}, \rho)\right) \geq \lambda_1(\xi_\rho h)^s.$$

By using (22), (23) we know $\lambda_1(\xi_\rho h)^s \geq 2\epsilon d^s$ and so from (24) and the density estimate (8) we know that we have

$$\begin{aligned} \mu\left(B_{h\xi_\rho}(y_1) \cap X^+(y_1, n_{y_1}, \rho) \cap S_1\right) &\geq \lambda_1(\xi_\rho h)^s - \epsilon d^s \\ &\geq \frac{\lambda_1}{2}(\xi_\rho h)^s \end{aligned}$$

and so by equations (14), (15) (recall $\xi_\rho = \frac{\rho}{1+\rho^2}$) we can pick $y_2 \in X^+(y_1, n_{y_1}, \rho) \cap B_{h\xi_\rho}(y_1) \cap S_1$ such that

$$y_2 \in K\left(z, \psi, \left(1-\zeta_\rho^{\lambda_1}\right) h\right) \subset K\left(z, \psi, \left(1-\zeta_\rho^{\lambda_1}\right)^{k_1+1} \kappa_\rho^{\lambda_1} d\right). \tag{25}$$

Now $y_1 \in \Psi^u\left(z, \psi, \kappa_\rho^{\lambda_1}, d\right)$ which means

$$h = \left|P_{\psi^\perp}(y_1 - z)\right| \leq \kappa_\rho^{\lambda_1}\left|P_{\langle\psi\rangle}(y_1 - (z + d\psi))\right|.$$

So from (25)

$$\left|P_{\psi^\perp}(y_2 - z)\right| \leq \left(1-\zeta_\rho^{\lambda_1}\right) h \leq \left(1-\zeta_\rho^{\lambda_1}\right) \kappa_\rho^{\lambda_1}\left|P_{\langle\psi\rangle}(y_1 - (z + d\psi))\right|.$$

And since $y_2 \in B_{h\xi_\rho}(y_1)$

$$\begin{aligned}
\left|P_{\langle\psi\rangle}(y_2-(z+d\psi))\right| &\geq \left|P_{\langle\psi\rangle}(y_1-(z+d\psi))\right| - h\xi_\rho \\
&\geq \left|P_{\langle\psi\rangle}(y_1-(z+d\psi))\right| - \xi_\rho\kappa_\rho^{\lambda_1}\left|P_{\langle\psi\rangle}(y_1-(z+d\psi))\right| \\
&= \left|P_{\langle\psi\rangle}(y_1-(z+d\psi))\right|\left(1-\zeta_\rho^{\lambda_1}\right).
\end{aligned}$$

Putting these two together we have

$$\left|P_{\psi^\perp}(y_2-z)\right| \leq \kappa_\rho^{\lambda_1}\left|P_{\langle\psi\rangle}(y_2-(z+d\psi))\right|.$$

So $y_2 \in \Psi^u\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$, similarly $y_1 \in \Psi^d\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$ implies $y_2 \in \Psi^d\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$ so we have $y_2 \in \Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$. Now from (19) we know that (y_2-z) is "short" enough and from (25) we know (y_2-z) "thin" enough (recall the definition of Γ_{k+1}, see (18)) for

$$y_2 \in \Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right) \cap \left(\bigcup_{j=k_1+1}^{\infty}\Gamma_j\right) \cap S_1.$$

Contradicting the maximality of k_1. So we have established Step 4 and so we know (recall (22)) that

$$\lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho\kappa_\rho^{\lambda_1}d\right)^s < 2\epsilon d^s. \tag{26}$$

Let $\theta = \left(1-\zeta_\rho^{\lambda_1}\right)^{k_1}\kappa_\rho^{\lambda_1}d$ recall, this is the width of cylinder Γ_{k_1}. So

$$2\epsilon d^s > \lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^{s(k_1+1)}\left(\xi_\rho\kappa_\rho^{\lambda_1}d\right)^s = \lambda_1\left(1-\zeta_\rho^{\lambda_1}\right)^s\xi_\rho^s\theta^s.$$

And so

$$\theta < \frac{(2\epsilon)^{\frac{1}{s}}d}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}.$$

So to summerise what we have proved; we have shown that for any $z \in \psi^\perp + x$ such that $\Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right) \subset B_{2d}(x)$ if we know that

$$\Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right) \cap S_1 \neq \emptyset$$

then

$$K\left(z,\psi,\frac{(2\epsilon)^{\frac{1}{s}}d}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}\right) \cap B_{2d}(x) \cap S_1 \neq \emptyset.$$

As we know that for any $z \in \left(\psi^\perp + x\right) \cap K\left(x,\psi,\kappa_\rho^{\lambda_1}d\right)$ we have

$$x \in \Psi\left(z,\psi,\kappa_\rho^{\lambda_1},d\right)$$

and letting $\vartheta_\rho^{\lambda_1} = \frac{2^{\frac{1}{s}}}{\lambda_1^{\frac{1}{s}}\left(1-\zeta_\rho^{\lambda_1}\right)\xi_\rho}$ completes the proof. $\square$

Proof of Theorem 1 continued.

Let $\epsilon > 0$ be some very small number. Again we simplify expressions by letting $\mu := H^s_{\lfloor S}$. Let $x \in S_1$ be a density point of S_1, let $d \in (0, r_0)$ such that

$$\frac{\mu\left(B_{16d}\left(x\right)\backslash S_1\right)}{d^s} \leq \epsilon. \tag{27}$$

We can partition $[2d, 4d]$ into $M := \left[\frac{2}{4\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1}}\right]$ subintervals. Call them I_1, I_2,... I_M and let c_k be the center of interval I_k. Now for $k \in \{1, 2, \ldots M\}$ we know the following is true: For all $z \in \left(\psi^\perp + x\right) \cap K\left(x, \psi, \kappa_\rho^{\lambda_1}\frac{c_k}{2}\right)$ we have that $\Psi\left(z, \psi, \kappa_\rho^{\lambda_1}, \frac{c_k}{2}\right) \cap S_1 \neq \emptyset$ and so by Lemma 1

$$K\left(z, \psi, \epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1}\frac{c_k}{2}\right) \cap S_1 \cap B_{c_k}\left(x\right) \neq \emptyset. \tag{28}$$

Now fix $k \in \{1, 2, \ldots M\}$. From (28) we have that for every $z \in \left(\psi^\perp + x\right) \cap K\left(x, \psi, \kappa_\rho^{\lambda_1} d\right)$ we have $K\left(z, \psi, 2\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1} d\right) \cap S_1 \cap B_{c_k}\left(x\right) \neq \emptyset$.

Let $E_k = \left(\psi^\perp + x\right) \backslash P_{\psi^\perp}^{-1}\left(P_{\psi^\perp}\left(S_1 \cap \overline{B_{c_k}\left(x\right)}\right)\right)$.

Now since S_1 is purely unrectifiable or of Hausdorff dimension $< n - 1$ we know by the projection theorems of Besicovitch-Federer and Marstrand (Theorems 18.1, 9.4 [9]) that the projection of S_1 onto an $(n-1)$ plane arbitrarily close to $\psi^\perp$ has zero Lebesgue measure. For simplicity we will assume S_1 has zero projection onto $\psi^\perp$ because if this was not true then we can just define everything in terms of a slightly different coordinate system and the arguments remain the same, so

$$L^{n-1}\left(E_k\right) \geq L^{n-1}\left(B_1\left(0\right)\right)\left(\kappa_\rho^{\lambda_1} d\right)^{n-1}.$$

Since S_1 is closed so E_k is open. Thus for each $z \in E_k$ we can find

$$r_z \in \left(0, 2\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1} d\right) \tag{29}$$

such that

- $K\left(z, \psi, r_z\right) \cap B_{c_k}\left(x\right) \cap \left(\psi^\perp + x\right) \cap S_1 = \emptyset$
- $\partial K\left(z, \psi, r_z\right) \cap B_{c_k}\left(x\right) \cap \left(\psi^\perp + x\right) \cap S_1 \neq \emptyset$.

Now by the $5r$ covering Theorem (Theorem 2.1 [9]) from the covering of E_k given by $\{B_{r_z}\left(z\right) : z \in E_k\}$ we can find a disjoint collection of balls

$$\left\{B_{r_m^k}\left(z_m^k\right) : z_m^k \in E_k,\ m = 1, 2, \ldots N_k\right\}$$

which are centered on points of E_k and are such that

$$\sum_{m=1}^{N_k}\left(r_m^k\right)^{n-1} > \frac{\left(\kappa_\rho^{\lambda_1} d\right)^{n-1}}{5^{n-1}}. \tag{30}$$

Now each $m \in \{1, 2, \ldots N_k\}$ we can pick a point

$$x_m^k \in \partial K\left(z_m^k, \psi, r_m^k\right) \cap \overline{B_{c_n}(x)} \cap S_1.$$

The essential point is that by the fact that the points $\left\{x_m^k : m = 1, 2, \ldots N_k\right\}$ satisfy inequality (5) in every direction perpendicular to ψ, unless the points $\left\{x_m^k : m = 1, \ldots N_k\right\}$ are very near the surface of the ball $B_{c_k}(x)$, there will be some fraction of the set S_1 in the interior of the cylinders $\left\{K\left(z_m^k, \psi, r_m^k\right), m = 1, \ldots N_k\right\}$ of Hausdorff measure approximately $\sum_{k=1}^{N_k}\left(r_m^k\right)^s$.

Now since $s \in (0, n-1]$, by the well known inequality ([4] Theorem 19)

$$\begin{aligned} \left(\sum_{m=1}^{N_k}\left(r_m^k\right)^s\right)^{\frac{1}{s}} &\geq \left(\sum_{m=1}^{N_k}\left(r_m^k\right)^{n-1}\right)^{\frac{1}{n-1}} \\ &\geq \frac{\kappa_\rho^{\lambda_1} d}{5} \end{aligned} \tag{31}$$

so we see this can not happen too often, see figure 4. Proceeding formally. For each

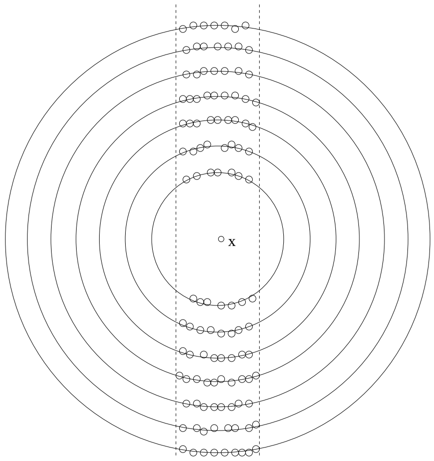

Figure 4:

$k \in \{1, 2, \ldots M\}$, $m \in \{1, 2, \ldots N_k\}$ let $n_{x_m^k}$ denote the inner normal to the boundary of the cylinder $K\left(z_m^k, \psi, r_m^k\right)$ at point x_m^k. Again let us fix $k \in \{1, 2, \ldots M\}$. We know that for each $m \in \{1, \ldots N_k\}$

$$\mu\left(X^+\left(x_m^k, n_{x_m^k}, \rho\right) \cap B_{\xi_\rho r_m^k}\left(x_m^k\right)\right) \geq \lambda_1\left(\xi_\rho r_m^k\right)^s. \tag{32}$$

Let $I_1^k = \left\{m \in \{1, 2, \ldots N_k\} : X^+\left(x_m^k, n_{x_m^k}, \rho\right) \cap B_{\xi_\rho r_m^k}\left(x_m^k\right) \subset B_{c_k}(x)\right\}$. Let $I_2^k = \{1, 2, \ldots N_k\} \backslash I_1^k$. Now firstly for any $m \in I_1^k$ as

$$X^+\left(x_m^k, n_{x_m^k}, \rho\right) \cap B_{\xi_\rho r_m^k}\left(x_m^k\right) \subset K\left(z_m^k, \psi, r_m^k\right) \cap B_{c_k}(x)$$

and so $X^+\left(x_m^k, n_{x_m^k}, \rho\right) \cap B_{\xi_\rho r_m^k}\left(x_m^k\right) \subset B_{c_k}(x) \backslash S_1$, thus we have

$$\mu\left(\bigcup_{m \in I_1^k} X^+\left(x_m^k, n_{x_m^k}, \rho\right) \cap B_{\xi_\rho r_m^k}\left(x_m^k\right)\right) \leq \epsilon d^s.$$

So

$$\sum_{m\in I_1^k}\left(r_m^k\right)^s\leq\frac{\epsilon d^s}{\lambda_1\xi_\rho^s}.$$

Thus given sufficient smallness of ϵ, from (31) we have

$$\sum_{m\in I_2^k}\left(r_m^k\right)^s\geq\frac{1}{5^s}\left(\kappa_\rho^{\lambda_1}d\right)^s-\frac{\epsilon d^s}{\lambda_1\xi_\rho^s}\geq\frac{1}{2\times 5^s}\left(\kappa_\rho^{\lambda_1}d\right)^s. \tag{33}$$

However by definition I_2^k we know that for $k=1,2,\dots M$ (recall (29))

$$\left\{B_{\xi_\rho r_m^k}\left(x_m^k\right),m\in I_2^k\right\}\subset N_{2\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1}d}\left(\partial B_{c_k}\left(x\right)\right).$$

So using the density estimate of subset S_1 (i.e. (6)) and (33)

$$\begin{aligned}
\mu\left(B_{5d}\left(x\right)\right) &\geq \mu\left(\bigcup_{k=1}^{M}N_{2\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1}d}\left(\partial B_{c_k}\left(x\right)\right)\right)\\
&\geq \sum_{k=1}^{M}\sum_{m\in I_2^k}\mu\left(B_{\xi_\rho r_m^k}\left(x_m^k\right)\right)\\
&\geq \frac{\alpha\xi_\rho^s}{2}\left(\frac{M}{5^s}\right)\left(\kappa_\rho^{\lambda_1}d\right)^s
\end{aligned}$$

and since M can be made as large as we like by reducing ϵ (recall $M=\left[\frac{2}{4\epsilon^{\frac{1}{s}}\vartheta_\rho^{\lambda_1}}\right]$) we end up contradicting the density bound (6) and thus Theorem 1 is proved. □

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