CLASSIFICATION OF BLOW-UP LIMITS FOR THE SINH-GORDON EQUATION

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Abstract. Aim of the paper is to use a selection process and a careful study of the interaction of bubbling solutions to show a classification result for the blow-up values of the elliptic sinh-Gordon equation

$$\Delta u + h_1 e^u - h_2 e^{-u} = 0 \qquad \text{in } B_1 \subset \mathbb{R}^2.$$

In particular we get that the blow-up values are multiple of 8π. It generalizes the result of Jost, Wang, Ye and Zhou [20] where the extra assumption $h_1 = h_2$ is crucially used.

1. Introduction

In this paper we mainly focus on the weak limit of the energy sequence for the following equation

$$\Delta u + h_1 e^u - h_2 e^{-u} = 0 \text{ in } B_1 \subset \mathbb{R}^2, \tag{1.1}$$

where h_1, h_2 are smooth positive functions and B_1 is the unit ball in $\mathbb{R}^2$.

Equation (1.1) arises in the study of the equilibrium turbulence with arbitrarily signed vortices [11, 30, 28, 31], and was first proposed by Onsager [34], Joyce and Montgomery [21] from different statistical arguments. When the nonlinear term e^{-u} in (1.1) is replaced by $\tau e^{-\gamma u}$ with $\tau, \gamma > 0$, the equation (1.1) describes a more general type of equation which arises in the context of the statistical mechanics description of 2D-turbulence. For the recent developments of such equation, we refer the readers to [36, 37, 38] and the references therein. Moreover, it plays also a very important role in the study of the construction of constant mean curvature surfaces initiated by Wente, see [20, 46] and the references therein.

When $h_2 \equiv 0$ the equation (1.1) reduces to the classic Liouville equation

$$\Delta u + h e^u = 0 \qquad \text{in } B_1 \subset \mathbb{R}^2. \tag{1.2}$$

Equation (1.2) is important in the geometry of manifolds as it rules the change of Gaussian curvature under conformal deformation of the metric, see [1, 7, 8, 23, 41]. Another motivation for the study of (1.2) is in mathematical physics as it models the mean field of Euler flows, see [6] and [22]. This equation has been very much studied in the literature; there are by now many results regarding existence, compactness

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of solutions, bubbling behavior, etc. We refer the interested reader to the reviews [29] and [44].

Wente's work [46] on the constant mean curvature surfaces and the work of Sacks-Uhlenbeck [39] concerning harmonic maps led to investigate the blow-up phenomena for variational problems that possess a lack of compactness. Later, in a series work of Steffen [42], Struwe [43] and Brezis, Coron [4], the program of understanding the blow-up for constant mean curvature surfaces was completed.

As many geometric problems, also (1.1) (and (1.2)) presents loss of compactness phenomena, as its solutions might blow-up. Concerning (1.2) it was proved in [5, 24, 25] that for a sequence of blow-up solutions u_k to (1.2) (relatively to h^k) with blow-up point $\bar{x}$ it holds

$$\lim_{\delta\to 0}\lim_{k\to\infty}\int_{B_\delta(\bar{x})} h^k e^{u_k} = 8\pi. \tag{1.3}$$

Somehow, each blow-up point has a quantized local mass.

On the other hand, the blow-up behavior of solutions of equation (1.1) is not yet developed in full generality; this analysis was carried out in [32, 33] and [20] under the assumption that $h_1 = h_2$ or h_1, h_2 are constants. In particular, by using the deep connection of the sinh-Gordon equation and differential geometry, in [20] Jost, Wang, Ye and Zhou proved an analogous quantization property as the one in (1.3), namely that the blow-up limits are multiple of 8π, see Theorem 1.1, Corollary 1.2 and Remark 4.5 in the latter paper. The latter blow-up situation may indeed occur, see [12] and [13]. We point out that the assumption $h_1 = h_2$ (or h_1, h_2 constants) in [20] is crucially used in order to provide a geometric interpretation of equation (1.1) in terms of constant mean curvature surfaces and harmonic maps (see also [46]). In this way they transfer the problem into a blow-up phenomenon for harmonic maps. The core of the argument is then to apply a result about no loss of energy during bubbling off for a sequence of harmonic maps, which was proved in [18, 35].

The study of the blow-up limits is interesting by itself. However, it yields also important results: we point out here the compactness property of the following sequence of solutions to a variant of (1.1):

$$\Delta u_k + \rho_1^k \frac{H_1 e^{u_k}}{\int_\Omega H_1 e^{u_k}} - \rho_2^k \frac{H_2 e^{-u_k}}{\int_\Omega H_2 e^{-u_k}} = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad u_k = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, ρ_1^k, ρ_2^k are non-negative real parameters and H_1, H_2 two *fixed* positive smooth functions (see [2, 15, 16, 17, 47] and the references therein). In fact, from the local quantization result in [20] and some standard analysis (see [3, 5, 26]) one finds the following global compactness result.

Theorem 1.1. *Suppose ρ_1^k, ρ_2^k are two fixed non-negative real numbers and both are not equal to $8\pi\mathbb{N}$. Then the set of solutions to (1.4) are uniformly bounded.*

The latter property is a key ingredient in proving both existence and multiplicity results of (1.4), see for example [2, 15, 16, 17].

We return now to the topic of this paper. We shall study here the same subject of [20] in a more general case (i.e., h_1, h_2 are two different positive C^3 functions) by using pure analytic method. The argument is interesting by itself and for the first time it is used for this class of equations.

Let u_k be a sequence of blow-up solutions

$$\Delta u_k + h_1^k e^{u_k} - h_2^k e^{-u_k} = 0, \tag{1.5}$$

with 0 being its only blow-up point in B_1, i.e.:

$$\max_{K \subset\subset B_1 \setminus \{0\}} |u_k| \leq C(K), \quad \max_{x \in B_1} \{|u_k(x)|\} \to \infty. \tag{1.6}$$

Throughout the paper we will call $\int_{B_1} h_1^k e^{u_k}$ the energy of u_k (analogously is defined the energy of $-u_k$). We assume moreover

$$\frac{1}{C} \leq h_i^k(x) \leq C, \ \|h_i^k(x)\|_{C^3(B_1)} \leq C, \qquad \forall x \in B_1, \ i = 1, 2 \tag{1.7}$$

for some positive constant C and we suppose that u_k has bounded oscillation on ∂B_1 and a uniform bound on its energy:

$$\begin{aligned} &|u_k(x) - u_k(y)| \leq C, \qquad \forall\, x, y \in \partial B_1, \\ &\int_{B_1} h_1^k e^{u_k} \leq C, \qquad \int_{B_1} h_2^k e^{-u_k} \leq C, \end{aligned} \tag{1.8}$$

where C is independent of k.

Our main result is concerned with the limit energy of u_k. Let

$$\begin{aligned} \sigma_1 &= \lim_{\delta \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B_\delta} h_1^k e^{u_k}, \\ \sigma_2 &= \lim_{\delta \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B_\delta} h_2^k e^{-u_k}. \end{aligned} \tag{1.9}$$

Let Σ be the following finite set of points:

$$\Sigma = \Big\{ (\sigma_1, \sigma_2) \ = \big(2m(m+1), 2m(m-1)\big) \text{ or } \big(2m(m-1), 2m(m+1)\big), \quad m \in \mathbb{N} \Big\}. \tag{1.10}$$

Theorem 1.2. *Let σ_i and Σ be defined as in (1.9) and* (1.10)*, respectively. Suppose u_k satisfies (1.5), (1.6), (1.8) and h_i^k satisfy (1.7). Then $(\sigma_1, \sigma_2) \in \Sigma$.*

Remark 1.1. *Theorem 1.2 yields an improvement of the compactness result in Theorem 1.1, which holds now for arbitrary functions H_1, H_2. As a byproduct we get an improvement of both existence and multiplicity results concerning* (1.4) *in* [2, 16, 17]*. Moreover, it will be crucially used in a forthcoming paper about the Leray-Schauder topological degree associated to* (1.4)*.*

Remark 1.2. *Observe that differently from the Liouville equation* (1.2) *and the systems of n equations in* [27]*, where the blow-up limits (see for example* (1.3)*) could assume a finite number of possibilities, we obtain here an infinite number of possibilities for* (1.9)*, see* (1.10)*. The reason for this fact is the different form of the Pohozaev identity associated to the blow-up limits* (1.9)*, see Proposition 3.1.*

The first step in the proof of Theorem 1.2 is to introduce a selection process for describing the situations when blow-up of solutions to (1.5) occurs. This argument has been widely used in the framework of prescribed curvature problems, see for example [9, 23, 40]. It was later modified by Lin, Wei and Zhang in dealing with general systems of n equations to locate the bubbling area which consists of a finite number of disks, see [27]. Roughly speaking, the idea is that in each disk the blow-up solution have the energy of a globally defined system. We use the same technique for equation (1.1). Next we prove that in each bubbling disk the energy of at least one of u_k and $-u_k$ is multiple of 4. Combining then areas closed to each other we deduce that the energy limit of at least one component of u_k and $-u_k$ is multiple of 4. In this procedure we use the same terminology "group" introduced in [27] to describe bubbling disks closest to each other and relatively far away from other disks. Then, Theorem 1.2 is a direct consequence of a global Pohozaev identity.

The organization of this paper is as follows. In Section 2 we introduce the selection process for the class of equations as in (1.1), in Section 3 we prove a Pohozaev identity which is the key element in proving Theorem 1.2, in Section 4 we study the asymptotic behavior of the solutions around the blow-up area and in Section 5 we finally prove Theorem 1.2 by a suitable combination of the bubbling areas.

Notation

The symbol $B_r(p)$ stands for the open metric ball of radius r and center p. To simplify the notation we will write B_r for balls which are centered at 0. We will use the notation $a \sim b$ for two comparable quantities a and b.

Throughout the paper the letter C will stand for large constants which are allowed to vary among different formulas or even within the same lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc. We will write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$; we will similarly use the symbol $O_\alpha(1)$ for bounded quantities.

2. A selection process for the Sinh-Gordon equation

In this section we introduce a selection process for the Sinh-Gordon equation (1.1). In particular, we will select a finite number of bubbling areas. This will be the first tool to be used in the proof of the main Theorem 1.2.

Proposition 2.1. *Let u_k be a sequence of blow-up solutions to (1.5) that satisfy (1.6) and (1.8), and suppose that h_i^k satisfy (1.7). Then, there exist finite sequences of points $\Sigma_k := \{x_1^k, \dots, x_m^k\}$ (all $x_j^k \to 0,\ j = 1, \dots, m$) and positive numbers $l_1^k, \dots, l_m^k \to 0$ such that*

(1) $|u_k|(x_j^k) = \max\limits_{B_{l_j^k}(x_j^k)} \{|u_k|\}$ *for* $j = 1, \dots, m$.

(2) $\exp\left(\frac{1}{2}|u_k|(x_j^k)\right) l_j^k \to \infty$ *for* $j = 1, \dots, m$.

(3) *Let* $\varepsilon_k = e^{-\frac{1}{2}M_k}$, *where* $M_k = \max_{B_{l_j^k}(x_j^k)} |u_k|$. *In each* $B_{l_j^k}(x_j^k)$ *we define the dilated functions*

$$\begin{aligned} v_1^k(y) &= u_k(\varepsilon_k y + x_k^j) + 2\log\varepsilon_k, \\ v_2^k(y) &= -u_k(\varepsilon_k y + x_k^j) + 2\log\varepsilon_k. \end{aligned} \tag{2.1}$$

Then it holds that one of the v_1^k, v_2^k *converges to a function* v *in* $C^2_{loc}(\mathbb{R}^2)$ *which satisfies the Liouville equation* (1.2)*, while the other one tends to minus infinity over all compact subsets of* $\mathbb{R}^2$.

(4) *There exits a constant* $C_1 > 0$ *independent of* k *such that*

$$|u_k|(x) + 2\log \mathrm{dist}(x, \Sigma_k) \le C_1, \qquad \forall x \in B_1.$$

Proof. Without loss of generality we may assume that

$$u_k(x_1^k) = \max_{x \in B_1} |u_k|(x).$$

By assumption we clearly have $x_1^k \to 0$. Let (v_1^k, v_2^k) be defined as in (2.1) with x_j^k, M_k replaced by x_1^k and $u_k(x_1^k)$ respectively. Observe that by construction we have $v_i^k \le 0, i = 1, 2$. Therefore, exploiting the equation (1.1) we can easily see that $|\Delta v_i^k|$ is bounded. By standard elliptic estimate, $|v_i^k(z) - v_i^k(0)|$ is uniformly bounded in any compact subset of $\mathbb{R}^2$. By construction $v_1^k(0) = 0$ and hence v_1^k converges in $C^2_{loc}(\mathbb{R}^2)$ to a function v_1, while the other component is forced to satisfy $v_2^k \to -\infty$ over all compact subsets of $\mathbb{R}^2$. The limit of v_1^k satisfies the following equation:

$$\Delta v_1 + h_1 e^{v_1} = 0 \qquad \text{in } \mathbb{R}^2, \tag{2.2}$$

where $h_i = \lim_{k\to+\infty} h_i^k(x_1^k)$. From (1.8), we have

$$\int_{\mathbb{R}^2} h_1 e^{v_1} < C.$$

By the classification result due to Chen and Li [10] it follows that

$$\int_{\mathbb{R}^2} h_1 e^{v_1} = 8\pi \quad \text{and} \quad v_1(x) = -4\log|x| + O(1),\ |x| > 2.$$

Clearly we can take $R_k \to \infty$ such that

$$v_1^k(y) + 2\log|y| \le C, \qquad |y| \le R_k. \tag{2.3}$$

In other words we can find $l_1^k \to 0$ such that

$$u_k(x) + 2\log|x - x_1^k| \le C, \qquad |x - x_1^k| \le l_1^k,$$

and

$$e^{\frac{1}{2}u_1^k(x_1^k)} l_1^k \to \infty, \qquad \text{as } k \to \infty.$$

Consider now the function

$$|u_k(x)| + 2\log|x - x_1^k|$$

and let q_k be the point where $\max_{|x|\le 1}\big(|u_k(x)| + 2\log|x - x_1^k|\big)$ is achieved. Suppose that

$$\max_{|x|\le 1}\big(|u_k(x)| + 2\log|x - x_1^k|\big) \to \infty. \tag{2.4}$$

Then we define $d_k = \frac{1}{2}|q_k - x_1^k|$ and

$$\begin{aligned} S_1^k(x) &= u_k(x) + 2\log\big(d_k - |x-q_k|\big), \\ S_2^k(x) &= -u_k(x) + 2\log\big(d_k - |x-q_k|\big), \end{aligned} \qquad \text{in } B_{d_k}(q_k).$$

By construction we observe that $S_i^k(x) \to -\infty$ as x approaches $\partial B_{d_k}(q_k)$ while

$$\max\{S_1^k(q_k), S_2^k(q_k)\} = |u_k(q_k)| + 2\log d_k \to \infty$$

by assumption (2.4). Let p_k be where $\max_{x\in\overline{B}_{d_k}(q_k)}\{S_1^k, S_2^k\}$ is attained. Without loss of generality, we assume that $S_2^k(p_k) = \max_{x\in\overline{B}_{d_k}(q_k)}\{S_1^k, S_2^k\}$. Then

$$-u_k(p_k) + 2\log\big(d_k - |p_k - q_k|\big) \geq \max\{S_1^k(q_k), S_2^k(q_k)\} \to \infty. \tag{2.5}$$

Let $l_k = \frac{1}{2}(d_k - |p_k - q_k|)$. By the definition of p_k and l_k we observe that, for $y \in B_{l_k}(p_k)$ it holds

$$\begin{aligned} |u_k(y)| + 2\log\big(d_k - |y-q_k|\big) &\leq -u_k(p_k) + 2\log(2l_k), \\ d_k - |y-q_k| \geq d_k - |p_k - q_k| &- |y - p_k| \geq l_k. \end{aligned}$$

Therefore we get

$$|u_k(y)| \leq -u_k(p_k) + 2\log 2, \quad \forall\, y \in B_{l_k}(p_k). \tag{2.6}$$

Now let $R_k = e^{-\frac{1}{2}u_k(p_k)} l_k$ and define the following functions:

$$\begin{aligned} \hat{v}_1^k(y) &= u_k(p_k + e^{\frac{1}{2}u_k(p_k)}y) + u_k(p_k), \\ \hat{v}_2^k(y) &= -u_k(p_k + e^{\frac{1}{2}u_k(p_k)}y) + u_k(p_k). \end{aligned}$$

Observe that $R_k \to \infty$ by (2.5). Moreover, $|\Delta \hat{v}_i^k|$ is bounded in $B_{R_k}(0)$. Similarly as before $\hat{v}_2^k(y)$ converges to a function v_2 such that

$$\Delta v_2 + h_2(p_k)\, e^{v_2} = 0.$$

On the other hand, $\hat{v}_1^k(y)$ converges uniformly to $-\infty$ over all compact subsets of $\mathbb{R}^2$. Consider now $u_k, -u_k$ in $B_{l_k}(p_k)$ and suppose x_2^k is the point where $\max_{B_{l_k}(p_k)}|u_k|$ is obtained: it is not difficult to see that $-u_k(x_2^k) = \max_{B_{l_k}(p_k)}|u_k|$. Moreover, we can find l_2^k such that

$$|u_k(x)| + 2\log|x - x_2^k| \leq C, \qquad \text{for } |x - x_2^k| \leq l_2^k.$$

By (2.6) we have $-u_k(x_2^k) + u_k(p_k) \leq 2\log 2$ and we observe that

$$\hat{v}_2\Big(e^{-\frac{1}{2}u_k(p_k)}(x_2^k - p_k)\Big) - \hat{v}_2(0) = -u_k(x_2^k) + u_k(p_k) \leq 2\log 2.$$

Therefore we deduce that $e^{-\frac{1}{2}u_k(p_k)}|x_2^k - p_k| = O(1)$. It follows that we can choose $l_2^k \leq \frac{1}{2}l_k$ such that $e^{-\frac{1}{2}u_k(x_2^k)}l_2^k \to \infty$. Then we re-scale $u_k, -u_k$ around x_2^k and let v_i^k defined in (2.1) which will satisfy (1) and (2) in Proposition 2.1. Moreover, it is easy to see that $B_{l_1^k}(x_1^k) \cap B_{l_2^k}(x_2^k) = \emptyset$.

In this way we have defined the selection process. To continue it, we let $\Sigma_{k,2} := \{x_1^k, x_2^k\}$ and consider

$$\max_{x\in B_1} |u_k(x)| + 2\log \operatorname{dist}(x, \Sigma_{k,2}).$$

If there exists a subsequence such that the quantity above tends to infinity we use the same argument to get x_3^k and l_3^k. Since each bubble area contributes a positive energy, the process stops after finite steps due to the bound on the energy (1.8). Finally we get

$$\Sigma_k = \{x_1^k, \ldots, x_m^k\}$$

and it holds

$$|u_k(x)| + 2\log \operatorname{dist}(x, \Sigma_k) \le C, \tag{2.7}$$

which conclude the proof. $\square$

Lemma 2.1. *Let $\Sigma_k = \{x_1^k, \cdots, x_m^k\}$ be the blow-up set obtained in Proposition 2.1. Then for all $x \in B_1 \setminus \Sigma_k$, there exists a constant C independent of x and k such that*

$$|u_k(x_1) - u_k(x_2)| \le C, \qquad \forall\ x_1, x_2 \in B\big(x, d(x, \Sigma_k)/2\big).$$

Proof. Using the Green's representation formula it is not difficult to prove that the oscillation of u_k on $B_1 \setminus B_{\frac{1}{10}}$ is finite. Hence we can assume $|x_i| \le \frac{1}{10}, i = 1, 2$. Let

$$G(x, \eta) = -\frac{1}{2\pi}\log|x - \eta| + H(x, \eta)$$

be the Green's function on B_1 with respect to Dirichlet boundary condition. Let $x_0 \in B_1 \setminus \Sigma_k$ and $x_1, x_2 \in B(x_0, d(x_0, \Sigma_k)/2)$. By using the fact u_k has bounded oscillation on ∂B_1, We have

$$u_k(x_1) - u_k(x_2) = \int_{B_1} \Big(G(x_1, \eta) - G(x_2, \eta)\Big)\Big(h_1^k(\eta)e^{u_k(\eta)} - h_2^k(\eta)e^{-u_k(\eta)}\Big)\mathrm{d}\eta + O(1).$$

Since $|x_i| \le \frac{1}{10}, i = 1, 2$ and $\Delta H = 0$ in B_1, we can use the bound on the energy (1.8) to get

$$\int_{B_1} \Big(H(x_1, \eta) - H(x_2, \eta)\Big)\Big(h_1^k(\eta)e^{u_k(\eta)} - h_2^k(\eta)e^{-u_k(\eta)}\Big)\mathrm{d}\eta = O(1).$$

Therefore, we are left with proving

$$\int_{B_1} \log\frac{|x_1 - \eta|}{|x_2 - \eta|}\Big(h_1^k(\eta)e^{u_k(\eta)} - h_2^k(\eta)e^{-u_k(\eta)}\Big)\mathrm{d}\eta = O(1).$$

Let r_k be the distance between x_0 and Σ_k. We distinguish between two cases. Suppose first $\eta \in B_1 \setminus B_{\frac{3}{4}r_k}(x_0)$. Then

$$\log\frac{|x_1 - \eta|}{|x_2 - \eta|} = O(1)$$

and the integration in this region is bounded.

Consider now $\eta \in B_{\frac{3}{4}r_k}(x_0)$ and let

$$v_1^k(y) = u_k(x_0 + r_k y) + 2\log r_k,$$
$$v_2^k(y) = -u_k(x_0 + r_k y) + 2\log r_k,$$

for $y \in B_{3/4}$. Letting y_1, y_2 be the images of x_1, x_2 after scaling, namely $x_i = x_0 + r_k y_i, i = 1, 2$, we have to prove that

$$\int_{B_{3/4}} \log\frac{|y_1 - \eta|}{|y_2 - \eta|}\Big(h_1^k(x_0 + r_k\eta)e^{v_1^k(\eta)} - h_2^k(x_0 + r_k\eta)e^{v_2^k(\eta)}\Big)\mathrm{d}\eta = O(1).$$

Without loss of generality we may assume that $e_1 = (1, 0)$ is the image after scaling of the blow-up point in Σ_k closest to x_0. By Proposition 2.1 it holds

$$v_i^k(\eta) + 2\log|\eta - e_1| \le C.$$

Therefore

$$e^{v_i^k(\eta)} \le C|\eta - e_1|^{-2}.$$

Moreover, we notice that $|\eta - e_1| \ge C > 0$ for $\eta \in B_{\frac{3}{4}}$. Then for $i, j = 1, 2$, we get

$$\int_{B_{\frac{3}{4}}} \log|y_j - \eta| h_i^k(x_0 + r_k\eta) e^{v_i^k(\eta)} \mathrm{d}\eta \le C \int_{B_{\frac{3}{4}}} \frac{\log|y_j - \eta|}{|\eta - e_1|^2} \mathrm{d}\eta \le C$$

and we are done. □

3. Pohozaev identity and related estimates on the energy

We establish here a Pohozaev-type identity for the class of equations we are considering. The latter will be a crucial tool in proving the quantization result of Theorem 1.2.

We start with some observations and terminology. By Lemma 2.1 one can see that the behavior of blowup solutions away from the bubbling area can be described just by its spherical average in a neighborhood of a point in Σ_k. Moreover, the behavior of the solution on a boundary of a ball, say $\partial B_r(x_0)$, will play a crucial role in the forthcoming arguments, see for example Remark 3.1. Throughout the paper we will say u_k has fast decay on $\partial B_r(x_0)$ if

$$u_k(x) \le -2\log|x| - N_k, \qquad \text{for } x \in \partial B_r(x_0),$$

for some $N_k \to +\infty$. If instead there exists $C > 0$ independent of k such that

$$u_k(x) \ge -2\log|x| - C, \qquad \text{for } x \in \partial B_r(x_0),$$

we say u_k has slow decay on $\partial B_r(x_0)$. The same terminology will be used for $-u_k$.

For a sequence of bubbling solutions u_k of (1.5) recall the definition of local blow-up masses given in (1.9). The main result is the following.

Proposition 3.1. *Let u_k satisfy (1.5), (1.6), (1.8) and h_i^k satisfy (1.7). Then we have*

$$4(\sigma_1 + \sigma_2) = (\sigma_1 - \sigma_2)^2.$$

Before we give a proof of Proposition 3.1, we first establish the following auxiliary lemma.

Lemma 3.1. *For all $\varepsilon_k \to 0$ such that $\Sigma_k \subset B_{\varepsilon_k/2}(0)$, there exists $l_k \to 0$ such that $l_k \ge 2\varepsilon_k$ and*

$$|\overline{u}_k(l_k)| + 2\log l_k \to -\infty, \tag{3.1}$$

where $\overline{u}_k(r) := \frac{1}{2\pi r}\int_{\partial B_r} u_k$.

Proof. Given $\varepsilon_{k,1} \geq \varepsilon_k$ such that $\varepsilon_{k,1} \to 0$, there exist $r_{k,1}, r_{k,2} \geq \varepsilon_{k,1}$ with the following property:

$$\begin{aligned} u_k(x) + 2\log r_{k,1} &\to -\infty, \qquad \forall x \in \partial B_{r_{k,1}}, \\ -u_k(x) + 2\log r_{k,2} &\to -\infty, \qquad \forall x \in \partial B_{r_{k,2}}. \end{aligned} \tag{3.2}$$

Let us focus for example on u_k. If the above property is not satisfied, we have some $\varepsilon_{k,1} \to 0$ with $\varepsilon_{k,1} \geq \varepsilon_k$ such that for all $r \geq \varepsilon_{k,1}$,

$$\sup_{x \in \partial B_r} \big(u_k(x) + 2\log|x| \big) \geq -C,$$

for some $C > 0$. But $u_k(x)$ has bounded oscillation on each ∂B_r by Lemma 2.1. It follows that

$$u_k(x) + 2\log|x| \geq -C$$

for some C and all $x \in \partial B_r,\ r \geq \varepsilon_{k,1}$. This means that

$$e^{u_k(x)} \geq C|x|^{-2}, \qquad \varepsilon_{k,1} \leq |x| \leq 1.$$

Integrating e^{u_k} on $B_1 \setminus B_{\varepsilon_{k,1}}$ we get a contradiction on the uniform energy bound of $\int_{B_1} h_1^k e^{u_k}$ given by (1.8). This proves (3.2).

We start now by taking $\tilde{r}_k \geq \varepsilon_k$ so that

$$\overline{u}_k(\tilde{r}_k) + 2\log \tilde{r}_k \to -\infty.$$

Suppose $\tilde{r}_k$ is not tending to 0. Then by Lemma 2.1 there exists $\hat{r}_k \to 0$ such that

$$\overline{u}_k(r) + 2\log r \to -\infty, \qquad \text{for } \hat{r}_k \leq r \leq \tilde{r}_k. \tag{3.3}$$

To prove this we observe that

$$u_k(x) + 2\log|x| \leq -N_k, \qquad |x| = \tilde{r}_k$$

for some $N_k \to \infty$. Then, for any fixed C, by Lemma 2.1 we obtain

$$u_k(x) + 2\log|x| \leq -N_k + C_0, \qquad \tilde{r}_k/C < |x| < \tilde{r}_k,$$

Therefore, it is not difficult to prove that $\hat{r}_k$ can be found so that $\frac{\hat{r}_k}{\tilde{r}_k} \to 0$ and (3.3) holds.

Suppose now $\tilde{r}_k \to 0$. Similarly as before we can exploit Lemma 2.1 to find $s_k > \tilde{r}_k$ with $s_k \to 0$ and $\frac{s_k}{\tilde{r}_k} \to \infty$ such that

$$\overline{u}_k(r) + 2\log r \to -\infty, \qquad \text{for } \tilde{r}_k \leq r \leq s_k.$$

In both two alternatives we can find r_k with $r_k \in [\hat{r}_k, \tilde{r}_k]$ in the first case, or $r_k \in [\tilde{r}_k, s_k]$ in the second case, such that

$$-\overline{u}_k(r_k) + 2\log r_k \to -\infty.$$

In fact, otherwise we would have

$$-\overline{u}_k(r) + 2\log r \geq -C, \qquad \text{for } \hat{r}_k \leq r \leq \tilde{r}_k \quad \text{or} \quad \tilde{r}_k \leq r \leq s_k.$$

By the same reason, since by construction $\tilde{r}_k/\hat{r}_k \to \infty$ or $s_k/\tilde{r}_k \to \infty$ in each case we get a contradiction to the uniform bound on the energy (1.8). Lemma 3.1 is proved. □

Proof of Proposition 3.1. We start by observing that there exists $l_k \to 0$ such that $\Sigma_k \subset B_{l_k/2}(0)$, (3.1) holds and

$$\begin{aligned} \frac{1}{2\pi}\int_{B_{l_k}} h_1^k e^{u_k} &= \sigma_1 + o(1), \\ \frac{1}{2\pi}\int_{B_{l_k}} h_2^k e^{-u_k} &= \sigma_2 + o(1). \end{aligned} \tag{3.4}$$

In fact, one can first choose l_k so that the property (3.4) is satisfied and then by Lemma 3.1 we can further assume that (3.1) holds true. Let

$$\begin{aligned} v_1^k(y) &= u_k(l_k y) + 2\log l_k, \\ v_2^k(y) &= -u_k(l_k y) + 2\log l_k, \end{aligned}$$

which satisfy

$$\begin{cases} \Delta v_1^k(y) + H_1^k(y)\, e^{v_1^k} - H_2^k(y)\, e^{v_2^k} = 0, \quad |y| \le 1/l_k, \\ \overline{v}_i(1)^k \to -\infty, \quad i = 1,2, \end{cases} \tag{3.5}$$

where

$$H_i^k(y) = h_i^k(l_k y), \qquad i = 1,2.$$

A modification of the Pohozaev-type identity gives us

$$\begin{aligned} &\sum_{i=1}^{2}\int_{B_{\sqrt{R_k}}} (y\cdot\nabla H_i^k)\, e^{v_i^k} + 2\sum_{i=1}^{2}\int_{B_{\sqrt{R_k}}} H_i^k e^{v_i^k} = \\ =&\sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}}\sum_{i=1}^{2} H_i^k e^{v_i^k} + \sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}}\left(|\partial_\nu v_1^k|^2 - \frac{1}{2}|\nabla v_1^k|^2\right), \end{aligned} \tag{3.6}$$

where we used $\nabla v_1^k = -\nabla v_2^k$ and $R_k \to \infty$ will be chosen later. We rewrite the above formula as

$$\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,$$

where the notation is easily understood. First we choose $R_k \to \infty$ sufficiently smaller than l_k^{-1} so that $\mathcal{L}_1 = o(1)$ by $l_k \to 0$. Now we consider $\mathcal{L}_2$. Observe that by Lemma 2.1, $v_i^k(y) \to -\infty$ over all compact subsets of $\mathbb{R}^2 \setminus B_{1/2}$. Thus we can choose R_k so that

$$\int_{B_{R_k}\setminus B_1} H_i^k e^{v_i^k} = o(1). \tag{3.7}$$

Moreover, by the choice of l_k we have

$$\begin{aligned} \frac{1}{2\pi}\int_{B_1} H_1^k e^{v_1^k} &= \frac{1}{2\pi}\int_{B_{l_k}} h_1^k e^{u_k} = \sigma_1 + o(1), \\ \frac{1}{2\pi}\int_{B_1} H_2^k e^{v_2^k} &= \frac{1}{2\pi}\int_{B_{l_k}} h_2^k e^{-u_k} = \sigma_1 + o(1). \end{aligned} \tag{3.8}$$

Therefore, by (3.7) we obtain

$$\mathcal{L}_2 = 4\pi\sum_{i=1}^{2}\sigma_i + o(1).$$

To estimate $\mathcal{R}_1$ we notice that by (3.5) and Lemma 2.1

$$v_i^k(y) + 2\log|y| \to -\infty, \qquad \text{uniformly in } 1 < |y| \le \sqrt{R_k}. \tag{3.9}$$

It follows that $\mathcal{R}_1 = o(1)$.

Next, we shall estimate the terms $\mathcal{R}_2$ and $\mathcal{R}_3$. To do this we have to estimate ∇v_i^k on $\partial B_{\sqrt{R_k}}$. Let

$$G_k(y,\eta) = -\frac{1}{2\pi}\log|y-\eta| + H_k(y,\eta)$$

be the Green's function on $B_{l_k^{-1}}$ with respect to Dirichlet boundary condition. The regular part is expressed as

$$H_k(y,\eta) = \frac{1}{2\pi}\log\frac{|y|}{l_k^{-1}}\left|\frac{l_k^{-2}y}{|y|^2} - \eta\right|$$

and it holds

$$\nabla_y H_k(y,\eta) = O(l_k), \qquad \text{for } y \in \partial B_{\sqrt{R_k}},\ \eta \in B_{l_k^{-1}}. \tag{3.10}$$

We start by estimating ∇v_1^k on $\partial B_{\sqrt{R_k}}$. By the Green's representation formula,

$$v_1^k(y) = \int_{B_{l_k^{-1}}} G(y,\eta)\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big) + F_k,$$

where F_k, which is the boundary term, is a harmonic function satisfying $F_k = v_i^k$ on $\partial B_{l_k^{-1}}$. In particular F_k has bounded oscillation on $\partial B_{l_k^{-1}}$. It follows that $F_k - C_k = O(1)$ for some C_k, which yields $|\nabla F_k(y)| = O(l_k)$.

$$\begin{aligned}\nabla v_1^k(y) &= \int_{B_{l_k^{-1}}} \nabla_y G(y,\eta)\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big)\mathrm{d}\eta + \nabla F_k(y)\\ &= -\frac{1}{2\pi}\int_{B_{l_k^{-1}}} \frac{y-\eta}{|y-\eta|^2}\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big)\mathrm{d}\eta + O(l_k).\end{aligned} \tag{3.11}$$

In order to estimate the integral of (3.11) we divide the domain into few regions. We first observe that

$$\frac{1}{|y-\eta|} \sim \frac{1}{|\eta|} \le o\left(R_k^{-\frac{1}{2}}\right), \qquad \text{for } y \in \partial B_{\sqrt{R_k}}, \quad \eta \in B_{l_k^{-1}} \setminus B_{R_k^{2/3}}.$$

Hence, using the bound of the energy (1.8), the integral over $B_{l_k^{-1}} \setminus B_{R_k^{2/3}}$ is $o(1)R_k^{-\frac{1}{2}}$.

Consider now the integral over B_1: we have

$$\frac{y-\eta}{|y-\eta|^2} = \frac{y}{|y|^2} + O\left(\frac{1}{|y|^2}\right), \qquad \text{for } y \in \partial B_{\sqrt{R_k}}, \quad \eta \in B_1,$$

which, recalling (3.8), yields

$$-\frac{1}{2\pi}\int_{B_1} \frac{y-\eta}{|y-\eta|^2}\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big) = \left(-\frac{y}{|y|^2} + O\left(\frac{1}{|y|^2}\right)\right)\Big(\sigma_1 - \sigma_2 + o(1)\Big).$$

As we will see this will be the leading term.

For the integral over the region $B_{\sqrt{R_k}/2} \setminus B_1$ we observe that

$$\frac{1}{|y-\eta|} \sim \frac{1}{|y|}, \qquad \text{for } y \in \partial B_{\sqrt{R_k}}, \quad \eta \in B_{\sqrt{R_k}/2} \setminus B_1.$$

By the latter estimate and by (3.7) we get

$$\int_{B_{\sqrt{R_k}/2}\setminus B_1} \frac{y-\eta}{|y-\eta|^2}\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big) = o(1)|y|^{-1}.$$

Similarly one gets

$$\int_{B_{R_k^{2/3}}\setminus\big(B_1\cup B_{\frac{|y|}{2}}(y)\big)} \frac{y-\eta}{|y-\eta|^2}\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big) = o(1)|y|^{-1}.$$

Moreover, for the integral in $B_{\frac{|y|}{2}}(y)$ we use $e^{v_i^k(\eta)} = o(1)|\eta|^{-2}$ to get

$$\int_{B_{\frac{|y|}{2}}(y)} \frac{y-\eta}{|y-\eta|^2}\Big(H_1^k e^{v_1^k} - H_2^k e^{v_2^k}\Big) = o(1)|y|^{-1}.$$

Finally, combing all the estimates we deduce

$$\nabla v_1^k(y) = \left(-\frac{y}{|y|^2}\right)\Big(\sigma_1 - \sigma_2 + o(1)\Big) + o\left(|y|^{-1}\right), \qquad \text{for } y \in \partial B_{\sqrt{R_k}}.$$

Exploiting the latter formula in $\mathcal{R}_2$ and $\mathcal{R}_3$ we get

$$\mathcal{R}_2 + \mathcal{R}_3 = \pi(\sigma_1 - \sigma_2)^2 + o(1).$$

Therefore, we end up with

$$4(\sigma_1 + \sigma_2) = (\sigma_1 - \sigma_2)^2 + o(1).$$

Hence, Proposition 3.1 is proved.

□

Remark 3.1. *By the proof of Proposition 3.1 one observes the following fact: the fast decay property is crucial in evaluating the Pohozaev identity, more precisely the term* $\mathcal{R}_1$. *Moreover, letting* $\Sigma_k' \subseteq \Sigma_k$ *suppose that*

$$\operatorname{dist}\big(\Sigma_k', \partial B_{l_k}(p_k)\big) = o(1)\operatorname{dist}\big(\Sigma_k \setminus \Sigma_k', \partial B_{l_k}(p_k)\big).$$

Suppose moreover both components $u_k, -u_k$ *have fast decay on* $\partial B_{l_k}(p_k)$*, namely*

$$|u_k(x)| \le -2\log|x| - N_k, \qquad \text{for } x \in \partial B_{l_k}(p_k),$$

for some $N_k \to +\infty$. *Then, we can evaluate a local Pohozaev identity and get*

$$\big(\tilde\sigma_1^k(l_k) - \tilde\sigma_2^k(l_k)\big)^2 = 4\big(\tilde\sigma_1^k(l_k) + \tilde\sigma_2^k(l_k)\big) + o(1),$$

where

$$\tilde\sigma_1^k(l_k) = \frac{1}{2\pi}\int_{B_{l_k}(p_k)} h_1^k e^{u_k}$$
$$\tilde\sigma_2^k(l_k) = \frac{1}{2\pi}\int_{B_{l_k}(p_k)} h_2^k e^{-u_k}.$$

Observe that if $B_{l_k}(p_k) \cap \Sigma_k = \emptyset$ *then* $\tilde\sigma_i^k(l_k) = o(1), i = 1, 2$ *and the above formula clearly holds.*

This fact will be used in the forthcoming arguments.

4. Asymptotic behavior of solutions around each blow-up point

The goal in this section is to get some energy classification in each blow-up area. We will see in the sequel how the fast decaying property of the solutions plays a crucial role in determining the local energy. Once we obtain the classification around each blow-up point, in Section 5 we combine them together.

By considering suitable translated functions we may assume without loss of generality that $0 \in \Sigma_k$ for any k. Let $\tau_k = \frac{1}{2}\mathrm{dist}(0, \Sigma_k \setminus \{0\})$ we consider the energy limits of $h_1^k e^{u_k}$ and $h_2^k e^{-u_k}$ in B_{τ_k}. Define

$$\begin{aligned} v_1^k &= u_k(\delta_k y) + 2\log\delta_k, \\ v_2^k &= -u_k(\delta_k y) + 2\log\delta_k, \end{aligned} \qquad |y| \le \tau_k/\delta_k, \tag{4.1}$$

where $-2\log\delta_k = \max_{x\in B(0,\tau_k)} |u_k|$. Thus the equation for v_i^k is

$$\Delta v_1^k(y) + h_1^k(\delta_k y)\, e^{v_1^k(y)} - h_2^k(\delta_k y)\, e^{v_2^k(y)} = 0, \qquad |y| \le \tau_k/\delta_k.$$

By the definition of the selection process we have $\tau_k/\delta_k \to \infty$, see Proposition 2.1.

Observe moreover that

$$\begin{aligned} \int_{B_{\tau_k}(0)} h_1^k(x)\, e^{u_k(x)}\mathrm{d}x &= \int_{B_{\tau_k/\delta_k}(0)} h_1^k(\delta_k y)\, e^{v_1^k(y)}\mathrm{d}y, \\ \int_{B_{\tau_k}(0)} h_2^k(x)\, e^{-u_k(x)}\mathrm{d}x &= \int_{B_{\tau_k/\delta_k}(0)} h_2^k(\delta_k y)\, e^{v_2^k(y)}\mathrm{d}y. \end{aligned}$$

Therefore

$$\int_{B_{\tau_k}(0)} h_1^k(x)e^{u_k(x)}\mathrm{d}x = O(1)e^{\overline{v}_1^k(\partial B_{\tau_k/\delta_k}(0))}, \quad \int_{B_{\tau_k}(0)} h_1^k(x)e^{-u_k(x)}\mathrm{d}x = O(1)e^{\overline{v}_2^k(\partial B_{\tau_k/\delta_k}(0))}, \tag{4.2}$$

Define the following local masses:

$$\begin{aligned} \sigma_1^k(r) &= \frac{1}{2\pi}\int_{B_r} h_1^k e^{u_k}, \\ \sigma_2^k(r) &= \frac{1}{2\pi}\int_{B_r} h_2^k e^{-u_k}. \end{aligned} \tag{4.3}$$

The main result of this section is the following.

Proposition 4.1. *Suppose (1.5)-(1.8) hold for u_k, h_i^k and recall the definition in (4.3). For any $s_k \in (0, \tau_k)$ such that both $u_k, -u_k$ have fast decay on ∂B_{s_k}, i.e.*

$$|u_k(x)| \le -2\log|x| - N_k, \qquad \text{for } |x| = s_k \text{ and some } N_k \to \infty, \tag{4.4}$$

we have that $(\sigma_1^k(s_k), \sigma_2^k(s_k))$ is a $o(1)$ perturbation of one of the two following types:

$$\big(2m(m+1), 2m(m-1)\big) \quad \text{or} \quad \big(2m(m-1), 2m(m+1)\big),$$

for some $m \in \mathbb{N}$. In particular, they are both multiple of $4 + o(1)$.

On ∂B_{τ_k}, either both $u_k, -u_k$ have fast decay as in (4.4) and the conclusion is as before, or one component has fast decay while the other one is not fast decay. Suppose for example $-u_k$ has not the fast decay property, i.e.

$$-u_k(x) + 2\log|x| \ge -C, \qquad \text{for } |x| = \tau_k \text{ and some } C > 0,$$

while for u_k it holds

$$u_k(x) \le -2\log|x| - N_k, \qquad \text{for } |x| = s_k \text{ and some } N_k \to \infty.$$

Then $\sigma_1^k(\tau_k) \in 4\pi\mathbb{N} + o(1)$.

In particular, in any case at least one of the two components $u_k, -u_k$ has the local energy in B_{τ_k} multiple of $4 + o(1)$.

Proof. Let v_i^k be defined in (4.1). Observe that by construction one of the v_i^k's converges while the other one goes to minus infinity over all compact subsets of $\mathbb{R}^2$ (see the argument in Proposition 2.1), namely we have just a partially blown-up situation. Without loss of generality we assume that v_1^k converges to v_1 in $C^2_{loc}(\mathbb{R}^2)$ and v_2^k tends to minus infinity over any compact subset of $\mathbb{R}^2$. The equation for v_1 is

$$\Delta v_1 + h_1 e^{v_1} = 0 \text{ in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} h_1 e^{v_1} < \infty,$$

where $h_1 = \lim_{k\to\infty} h_1^k(0)$. By the classification result of Chen-Li [10], we have

$$\int_{\mathbb{R}^2} h_1 e^{v_1} = 8\pi$$

and

$$v_1(y) = -4\log|y| + O(1), \qquad |y| > 1.$$

Therefore, we can take $R_k \to \infty$ (we assume $R_k = o(1)\tau_k/\delta_k$) such that

$$\frac{1}{2\pi}\int_{B_{R_k}} h_1^k(\delta_k y)\, e^{v_1^k} = 4 + o(1), \tag{4.5}$$

and

$$\frac{1}{2\pi}\int_{B_{R_k}} h_2^k(\delta_k y)\, e^{v_2^k} = o(1). \tag{4.6}$$

For $r \ge R_k$ we clearly have

$$\sigma_i^k(\delta_k r) = \frac{1}{2\pi}\int_{B_r} h_i^k(\delta_k y)\, e^{v_i^k}.$$

Up to now we get by (4.5) and (4.6) that

$$\sigma_1^k(\delta_k R_k) = 4 + o(1), \qquad \sigma_2^k(\delta_k R_k) = o(1).$$

Let $\overline{v}_i^k(r)$ be the average of v_i^k on ∂B_r, $i = 1, 2$. It will be important to study $\frac{d}{dr}\overline{v}_i^k(r), i = 1, 2$. In fact if

$$\frac{d}{dr}\left(\overline{v}_i^k(r) + 2\log r\right) > 0, \qquad \text{for some } i,$$

there is a possibility that for some larger radius s, v_i^k becomes a slow decay component on ∂B_s.

The key fact is to observe that

$$\begin{aligned} \frac{d}{dr}\overline{v}_1^k(r) &= \frac{-\sigma_1^k(\delta_k r) + \sigma_2^k(\delta_k r)}{r}, \\ \frac{d}{dr}\overline{v}_2^k(r) &= \frac{\sigma_1^k(\delta_k r) - \sigma_2^k(\delta_k r)}{r}, \end{aligned} \qquad R_k \le r \le \tau_k/\delta_k. \tag{4.7}$$

Clearly we have

$$R_k \frac{d}{dr}\overline{v}_1^k(R_k) = -4 + o(1), \qquad R_k \frac{d}{dr}\overline{v}_2^k(R_k) = 4 + o(1).$$

To continue the proof of Proposition 4.1 we prove now two auxiliary lemmas.

Lemma 4.1. *Suppose there exists $L_k \in (R_k, \tau_k/\delta_k)$ such that*

$$v_i^k(y) \le -2\log|y| - N_k, \qquad \text{for } R_k \le |y| \le L_k,\ i = 1,2 \tag{4.8}$$

for some $N_k \to \infty$. Then σ_i^k does not change much from $\delta_k R_k$ to $\delta_k L_k$: more precisely we have

$$\sigma_i^k(\delta_k L_k) = \sigma_i^k(\delta_k R_k) + o(1), \qquad i = 1, 2.$$

Proof. Suppose the statement is false: then there exists i such that $\sigma_i^k(\delta_k L_k) > \sigma_i^k(\delta_k R_k) + \delta$ for some $\delta > 0$. Let us choose $\tilde{L}_k \in (R_k, L_k)$ such that

$$\max_{i=1,2} \Big(\sigma_i^k(\delta_k \tilde{L}_k) - \sigma_i^k(\delta_k R_k) \Big) = \varepsilon, \tag{4.9}$$

where $\varepsilon > 0$ is taken sufficiently small. Then by using (4.7) we have

$$\frac{d}{dr}\overline{v}_1^k(r) \le \frac{-4 + \varepsilon + o(1)}{r} \le -\frac{2+\varepsilon}{r}, \qquad R_k \le r \le \tilde{L}_k. \tag{4.10}$$

By Lemma 2.1 we observe that

$$v_i^k(x) = \overline{v}_i^k(|x|) + O(1), \qquad x \in B_{\tau_k/\delta_k},$$

where $\overline{v}_i^k(|x|)$ is the average of v_i on $\partial B_{|x|}$. Reasoning as above and exploiting (4.10) jointly with (4.8) it is not difficult to show that

$$\int_{B_{\tilde{L}_k} \setminus B_{R_k}} e^{v_1^k} = O(1) \int_{B_{\tilde{L}_k} \setminus B_{R_k}} e^{\overline{v}_1^k(\tilde{L}_k)} = o(1).$$

In other words we have $\sigma_1^k(\delta_k \tilde{L}_k) = \sigma_1^k(\delta_k R_k) + o(1)$.

It follows that the maximum in (4.9) is attained for $i = 2$, i.e.

$$\sigma_2^k(\delta_k \tilde{L}_k) = \sigma_2^k(\delta_k R_k) + \varepsilon. \tag{4.11}$$

On the other hand, since (4.8) holds, as observed in Remark 3.1 we get

$$\big(\sigma_1^k(\delta_k \tilde{L}_k) - \sigma_2^k(\delta_k \tilde{L}_k)\big)^2 = 4\big(\sigma_1^k(\delta_k \tilde{L}_k) + \sigma_2^k(\delta_k \tilde{L}_k)\big) + o(1).$$

Now we observe that

$$\sigma_1^k(\delta_k \tilde{L}_k) = \sigma_1^k(\delta_k R_k) + o(1) = 4 + o(1),$$

where we used (4.5). Therefore we deduce that

$$\sigma_2^k(\delta_k \tilde{L}_k) = o(1) \qquad \text{or} \qquad \sigma_2^k(\delta_k \tilde{L}_k) = 12 + o(1),$$

which contradicts

$$\sigma_2^k(\delta_k \tilde{L}_k) = \sigma_2^k(\delta_k R_k) + \varepsilon = \varepsilon + o(1),$$

see (4.6) and (4.11). Thus Lemma 4.1 is established. □

By the argument in Lemma 4.1 we observe the following fact: for $r \geq R_k$ either both v_1, v_2 have fast decay up to $\partial B_{\tau_k/\delta_k}$, namely

$$v_i^k(y) \leq -2\log|y| - N_k, \qquad R_k \leq |y| \leq \tau_k/\delta_k,\ i=1,2, \tag{4.12}$$

for some $N_k \to +\infty$, or there exists $L_k \in (R_k, \tau_k/\delta_k)$ such that v_2 has the following slow decay

$$v_2^k(y) \geq -2\log L_k - C, \qquad |y| = L_k, \tag{4.13}$$

for some $C > 0$, while

$$v_1^k(y) \leq -2\log|y| - N_k, \qquad R_k \leq |y| \leq L_k, \tag{4.14}$$

for some $N_k \to +\infty$. Indeed, we have noticed in Lemma 4.1 that if the local energy changes, σ_2^k has to change first. Moreover, we have seen that L_k can be chosen so that $\sigma_2^k(\delta_k L_k) - \sigma_2^k(\delta_k R_k) = \varepsilon$ for some $\varepsilon > 0$ small. The following lemma treats the latter situation.

Lemma 4.2. *Suppose there exists $L_k \geq R_k$ such that (4.13) and (4.14) hold. We assume moreover that $L_k = o(1)\tau_k/\delta_k$. Then there exists $\tilde{L}_k$ such that $\tilde{L}_k/L_k \to \infty$ and $\tilde{L}_k = o(1)\tau_k/\delta_k$ with the following property:*

$$v_i^k(y) \leq -2\log|y| - N_k, \qquad |y| = \tilde{L}_k,\ i=1,2, \tag{4.15}$$

for some $N_k \to \infty$. Moreover

$$\sigma_1^k(\delta_k \tilde{L}_k) = 4 + o(1), \qquad \sigma_2^k(\delta_k \tilde{L}_k) = 12 + o(1). \tag{4.16}$$

Proof. First we observe that by the choice of L_k and being $\sigma_2^k(\delta_k R_k) = o(1)$ we can assume that $\sigma_2^k(\delta_k L_k) \leq \varepsilon$ for some $\varepsilon > 0$ small. By the same reason in Lemma 4.1 we have

$$\frac{d}{dr}\overline{v}_1^k(r) \leq \frac{-4+\varepsilon+o(1)}{r}, \qquad R_k \leq r \leq L_k.$$

Now we claim there exists $N > 1$ such that

$$\sigma_2^k(\delta_k(NL_k)) \geq 8 + o(1). \tag{4.17}$$

Suppose this does not hold. Then there exist $\varepsilon_0 > 0$ and $\tilde{R}_k \to \infty$ such that

$$\sigma_2^k(\delta_k \tilde{R}_k L_k) \leq 8 - \varepsilon_0. \tag{4.18}$$

Moreover, $\tilde{R}_k$ can be chosen to tend to infinity slowly so that by Lemma 2.1 and (4.14) we get

$$v_1^k(y) \leq -2\log|y| - N_k, \qquad L_k \leq |y| \leq \tilde{R}_k L_k. \tag{4.19}$$

By Lemma 4.1, (4.19) implies $\sigma_1^k(\delta_k L_k) = \sigma_1^k(\delta_k \tilde{R}_k L_k) + o(1)$. Thus by (4.18)

$$\frac{d}{dr}\overline{v}_2^k(r) \geq \frac{-4+\varepsilon+o(1)}{r}. \tag{4.20}$$

From (4.20) and (4.13) it is not difficult to show

$$\int_{B_{L_k \tilde{R}_k} \setminus B_{L_k}} e^{v_2^k} \to \infty,$$

which contradicts the energy bound in (1.8). Therefore (4.17) holds.

By Lemma 2.1 we have

$$v_i^k(y) + 2\log NL_k = \overline{v}_i^k(NL_k) + 2\log(NL_k) + O(1), \qquad i = 1, 2, \ |y| = NL_k.$$

Therefore, by the assumptions we get

$$\overline{v}_1^k(NL_k) \leq -2\log(NL_k) - N_k,$$
$$\overline{v}_2^k(NL_k) \geq -2\log(NL_k) - C \geq -2\log(L_k) - C.$$

Furthermore we can assert that

$$\overline{v}_2^k((N+1)L_k) \geq -2\log L_k - C,$$

which, jointly with (4.17), yields

$$\int_{B_{(N+1)L_k}} h_2^k(\delta_k y)\, e^{v_2^k(y)} dy \geq 8 + \varepsilon_0,$$

for some $\varepsilon_0 > 0$.

By the latter estimate we get

$$\frac{d}{dr}\overline{v}_2^k(r) \leq -\frac{2+\varepsilon_0}{r}, \qquad \text{for } r = (N+1)L_k.$$

Therefore we can take $\tilde{R}_k \to \infty$ slowly such that $\tilde{R}_k L_k = o(1)\tau_k/\delta_k$ and

$$v_2^k(y) \leq (-2-\varepsilon_0)\log|y| - N_k, \qquad |y| = \tilde{R}_k L_k,$$
$$v_1^k(y) \leq -2)\log|y| - N_k, \qquad L_k \leq |y| \leq \tilde{R}_k L_k,$$

where we have used also Lemma 2.1. By Lemma 4.1 and (4.5) we have

$$\sigma_1^k(\delta_k \tilde{R}_k L_k) = \sigma_1^k(\delta_k L_k) + o(1) = 4 + o(1).$$

Moreover, on $\partial B_{\tilde{R}_k L_k}$ both components have fast decay. Thus as in Remark 3.1 we can compute the Pohozaev identity and observing (4.17) holds we get

$$\sigma_2^k(\delta_k \tilde{R}_k L_k) = 12 + o(1).$$

Letting $\tilde{L}_k = \tilde{R}_k L_k$ we conclude the proof. □

Returning to the proof of Proposition 4.1, we are left with the region $\tilde{L}_k \leq |y| \leq \tau_k/\delta_k$. We distinguish between two cases.

Suppose first $L_k = O(1)\tau_k/\delta_k$. Then by Lemma 2.1 we directly conclude that one component has fast decay while the other one has slow decay, see for example (4.13) and (4.14). Moreover, we have observed that the energy in B_{τ_k} of the fast decaying component is a small perturbation of 4. This is exactly the second alternative of Proposition 4.1 and therefore the proof is concluded.

Suppose now $L_k = o(1)\tau_k/\delta_k$. In this case $\tilde{L}_k$ can chosen so that $o(1)\tau_k/\delta_k$. By using the local energies given by Lemma 4.2 we have

$$\frac{d}{dr}\overline{v}_1^k(r) = \frac{8+o(1)}{r}, \quad \frac{d}{dr}\overline{v}_2^k(r) = -\frac{8+o(1)}{r}, \qquad \text{for } r = \tilde{L}_k.$$

It follows that

$$\frac{d}{dr}\overline{v}_2^k(r) \leq -\frac{2+\varepsilon}{r}, \qquad r = \tilde{L}_k,$$

for some $\varepsilon > 0$. As in Lemma 4.1 we conclude that $\sigma_2^k(r)$ does not change for $r \geq \tilde{L}_k$ unless σ_1^k changes. By the same ideas of Lemmas 4.1, 4.2 and by the argument of the first case $L_k = O(1)\tau_k/\delta_k$, either v_1^k has slow decay up to B_{τ_k/δ_k} or there is $\hat{L}_k = o(1)\tau_k/\delta_k$ such that

$$\sigma_1^k(\delta \hat{L}_k) = 24 + o(1), \qquad \sigma_2^k(\delta \hat{L}_k) = 12 + o(1).$$

By the latter local energies we deduce

$$\frac{d}{dr}\overline{v}_1^k(r) = -\frac{12 + o(1)}{r}, \quad \frac{d}{dr}\overline{v}_2^k(r) = \frac{12 + o(1)}{r}, \qquad \text{for } r = \hat{L}_k.$$

Thus as before

$$\frac{d}{dr}\overline{v}_1^k(r) \leq -\frac{2 + \varepsilon}{r}, \ r = \hat{L}_k,$$

for some $\varepsilon > 0$. Now $\sigma_1^k(r)$ does not change for $r \geq \hat{L}_k$ unless σ_2^k changes. By repeating the argument we get either v_2^k has slow decay up to B_{τ_k/δ_k} or there is $\overline{L}_k = o(1)\tau_k/\delta_k$ such that

$$\sigma_1^k(\delta \overline{L}_k) = 24 + o(1), \qquad \sigma_2^k(\delta \overline{L}_k) = 40 + o(1).$$

Since after each step one of the local masses changes by a positive number, using the uniform bound on the energy (1.8) the process stops after finite steps. Eventually we get Proposition 4.1. □

5. Combination of bubbling areas and proof of Theorem 1.2

In this section we present an argument for combining the blow-up areas. This strategy has been already used in several frameworks, see the Introduction for more details. The idea is the following: we start by considering blow-up points which are close to each other and we get a quantization property for each group, see the definition of group below. In particular, in each group the local energy of at least one component is a small perturbation of $4n$, for some $n \in \mathbb{N}$. Similarly, we combine the groups and we get the total energy of at least one component is a small perturbation of $4n$, for some $n \in \mathbb{N}$. Then, the conclusion follows by applying a global Pohozaev identity.

Definition. Let $Q_k = \{p_1^k, \cdots, p_q^k\}$ be a subset of Σ_k with more than one point in it. Q_k is called a group if

(1) $\text{dist}(p_i^k, p_j^k) \sim \text{dist}(p_s^k, p_t^k)$,
where $p_i^k, p_j^k, p_s^k, p_t^k$ are any points in Q_k such that $p_i^k \neq p_j^k$ and $p_t^k \neq p_s^k$.

(2) $\dfrac{\text{dist}(p_i^k, p_j^k)}{\text{dist}(p_i^k, p_k)} \to 0,$
for any $p_k \in \Sigma_k \setminus Q_k$ and for all $p_i^k, p_j^k \in Q_k$ with $p_i^k \neq p_j^k$.

Proof of Theorem 1.2: As in Section 4, by considering suitable translated functions we may assume without loss of generality that $0 \in \Sigma_k$ for any k. Let $2\tau_k$ be the distance between 0 and $\Sigma_k \setminus \{0\}$. To describe the group G_0 that contains 0 we proceed in the following way: if for any $z_k \in \Sigma_k \cap \partial B(0, 2\tau_k)$ we have $dist(z_k, \Sigma_k \setminus \{z_k\}) \sim \tau_k$, then G_0 contains at least two points. On the other hand, if there exists $z_k' \in \partial B(0, 2\tau_k) \cap \Sigma_k$ such that $\tau_k / dist(z_k', \Sigma_k \setminus z_k') \to \infty$ we let G_0 be 0 itself. By the definition of group, all points of G_0 are in $B(0, N\tau_k)$ for some N independent of k. Let

$$\begin{aligned} \tilde{v}_1^k(y) &= u_k(\tau_k y) + 2\log \tau_k, \\ \tilde{v}_2^k(y) &= -u_k(\tau_k y) + 2\log \tau_k, \end{aligned} \qquad |y| \le \tau_k^{-1},$$

which satisfy

$$\Delta \tilde{v}_1^k(y) + h_1^k(\tau_k y)\, e^{\tilde{v}_1^k(y)} - h_2^k(\tau_k y)\, e^{\tilde{v}_2^k(y)} = 0, \qquad |y| \le \tau_k^{-1}. \tag{5.1}$$

Let $0, Q_1, \cdots, Q_s$ be the images of members of G_0 after scaling from y to $\tau_k y$. We observe that $Q_i \in B_N$. By Proposition 4.1 at least one of $\tilde{v}_i^k$ decays fast on ∂B_1. Without loss of generality we assume

$$\tilde{v}_1^k \le -2\log|y| - N_k, \qquad \text{on } \partial B_1,$$

for some $N_k \to \infty$. Moreover, we know by Proposition 4.1 that

$$\sigma_1^k(\tau_k) = 4\tilde{n} + o(1),$$

for some $\tilde{n} \in \mathbb{N}$.

Furthermore, by Lemma 2.1 we can assert

$$\tilde{v}_1^k \le -2\log|y| - N_k, \qquad \text{on } \partial B_1(Q_j), \quad j = 1, \ldots, s.$$

Still by using Proposition 4.1 we conclude that the the local energies around Q_j are

$$\frac{1}{2\pi}\int_{B_1(Q_j)} h_1(\tau_k y)\, e^{\tilde{v}_1^k} = 4n_j + o(1), \qquad j = 1, \ldots, s,$$

for some $n_j \in \mathbb{N}, j = 1, \ldots, s$. Let $2\tau_k L_k$ be the distance from 0 to the nearest group from G_0. By the definition of group we have $L_k \to \infty$. By using Lemma 2.1 and Lemma 4.1, using the same reason in Lemma 3.1 we can find $\tilde{L}_k \le L_k,\ \tilde{L}_k \to \infty$ slowly such that the energy of $\tilde{v}_1^k$ in $B_{\tilde{L}_k}(0)$ does not change so much and such that $\tilde{v}_2^k$ has fast decay on $\partial B_{\tilde{L}_k}(0)$:

$$\sigma_1^k(\tau_k \tilde{L}_k) = 4n + o(1), \tag{5.2}$$

for some $n \in \mathbb{N}$ and

$$\tilde{v}_2^k(y) \le -2\log \tilde{L}_k - N_k, \qquad \text{for } |y| = \tilde{L}_k, \tag{5.3}$$

for some $N_k \to +\infty$.

Since on $\partial B_{\tilde{L}_k}$ both components $\tilde{v}_1^k, \tilde{v}_2^k$ have fast decay we apply the argument of Remark 3.1 and compute the Pohozaev identity. Since (5.3) holds we get also

$$\sigma_2^k(\tau_k \tilde{L}_k) = 4\bar{n} + o(1),$$

for some $\bar{n} \in \mathbb{N}$. Putting together the Pohozaev identity with the fact that both local masses are a small perturbation of a multiple of 4 we conclude $(\sigma_1^k(\tau_k \tilde{L}_k), \sigma_2^k(\tau_k \tilde{L}_k))$ is a $o(1)$ perturbation of one of the two following types:

$$\big(2\tilde{m}(\tilde{m}+1), 2\tilde{m}(\tilde{m}-1)\big) \qquad \text{or} \qquad \big(2\tilde{m}(\tilde{m}-1), 2\tilde{m}(\tilde{m}+1)\big), \tag{5.4}$$

for some $\tilde{m} \in \mathbb{N}$. Without loss of generality we assume the former happens. As in the proof of Proposition 4.1 we have

$$\begin{aligned} &\overline{u}_k(\tau_k \tilde{L}_k) \le -2\log(\tau_k \tilde{L}_k) - N_k, \\ &\frac{d}{dr}\overline{u}_k < -\frac{2+\varepsilon}{r}, \end{aligned} \qquad \text{for } r = \tau_k \tilde{L}_k,$$

for some $\varepsilon > 0$. Now, following the steps in the proof of Proposition 4.1, as r grows from $\tau_k \tilde{L}_k$ to $\tau_k L_k$, the following three cases may happen:

Case 1. Both u_k and $-u_k$ have fast decay up to $|x| = \tau_k L_k$:

$$|u_k(x)| \le -2\log|x| - N_k, \qquad \tau_k \tilde{L}_k \le |x| \le \tau_k L_k,$$

for some $N_k \to +\infty$. In this case, by Lemma 4.1 we have

$$\sigma_i^k(\tau_k L_k) = \sigma_i^k(\tau_k \tilde{L}_k) + o(1), \quad i = 1, 2.$$

Case 2. There exists $L_{1,k} \in (\tilde{L}_k, L_k)$, $L_{1,k} = o(1)L_k$ such that

$$-u_k(x) \ge -2\log L_{1,k} - C, \qquad \text{for } |x| = \tau_k L_{1,k}.$$

By the argument of Lemma 4.2 we can find a suitable $L_{2,k} \ge L_{1,k}$ such that

$$|u_k(x)| \le -2\log L_{2,k} - N_k, \qquad \text{for } |x| = \tau_k L_{2,k},$$

for some $N_k \to +\infty$ and $\big(\sigma_1^k(\tau_k L_{2,k}), \sigma_2^k(\tau_k L_{2,k})\big)$ is a $o(1)$ perturbation of

$$\big(2\bar{m}(\bar{m}-1), 2\bar{m}(\bar{m}+1)\big),$$

for some $\bar{m} \in \mathbb{N}$.

Case 3. $-u_k$ has slow decay for $|x| = \tau_k L_k$, i.e.

$$-u_k(x) \ge -2\log \tau_k L_k - C, \qquad |x| = \tau_k L_k,$$

for some $C > 0$ and

$$\sigma_1^k(\tau_k L_k) = \sigma_1^k(\tau_k \tilde{L}_k) + o(1) = 4\bar{n} + o(1).$$

Moreover, on $\partial B_{\tau_k L_k}(0)$, u_k is still the fast decaying component.

The only region we have still to analyze is $B_{\tau_k L_k}(0) \setminus B_{\tau_k L_{2,k}}(0)$ when the second case above happens. However, the argument here is the same as before. At the end, in any case on $\partial B_{\tau_k L_k}(0)$ at least one of the two component $u_k, -u_k$ has fast decay and its energy is a small perturbation of a multiple of 4.

Finally, we have to combine the groups. The procedure is very similar to the combination of bubbling disks as we have done before. For example, we start by considering groups which are close to each other: take $B_{\varepsilon_k}(0)$ for some $\varepsilon_k \to 0$ such that all the groups in $B_{2\varepsilon_k}(0)$, say $G_0, G_1, \cdots, G_t$, (namely $(\Sigma_k \setminus (\cup_{i=0}^t G_i)) \cap B(0, 2\varepsilon_k) = \emptyset$) satisfy

$$\begin{aligned} &\text{dist}(G_i, G_j) \sim \text{dist}(G_l, G_q), \qquad \forall i \ne j, l \ne q, \\ &\text{dist}(G_i, G_j) = o(1)\varepsilon_k, \qquad \forall i, j = 0, \cdots, t,\ i \ne j. \end{aligned}$$

The second property implies that the groups outside $B_{2\varepsilon_k}(0)$ are far away from the groups inside the ball. By the above assumptions, letting $\varepsilon_{1,k} = dist(G_0, G_j)$, for some $j \in \{1, \ldots, t\}$ we have that all $G_0, \cdots, G_t$ are in $B_{N\varepsilon_{1,k}}(0)$ for some $N > 0$

independent of k. Without loss of generality let u_k be the fast decaying component on $\partial B_{N\varepsilon_{1,k}}(0)$. Then we have

$$\sigma_1^k(N\varepsilon_{1,k}) = \sigma_1^k(\tau_k L_k) + 4\hat{m} + o(1),$$

for some $\hat{m} \in \mathbb{N}$ because by Lemma 2.1 u_k is also a fast decaying component for $G_0, \cdots, G_t$.

Now, as before we have three possible cases. If $-u_k$ also has fast decay on $\partial B_{N\varepsilon_{1,k}}(0)$, $\sigma_2^k(N\varepsilon_{1,k})$ is also a small perturbation of a multiple of 4 and we get the quantization as in (5.4).

If instead

$$-u_k(x) \geq -2\log N\varepsilon_{1,k} - C, \qquad |x| = N\varepsilon_{1,k},$$

then as before we can find $\varepsilon_{2,k}$ in $(N\varepsilon_{1,k}, \varepsilon_k)$ such that

$$|u_k(x)| \leq -2\log \varepsilon_{2,k} - N_k, \qquad |x| = \varepsilon_{2,k},$$

for some $N_k \to \infty$. Moreover

$$\sigma_1^k(\varepsilon_{2,k}) = \sigma_1^k(N\varepsilon_{1,k}) + o(1).$$

Thus, by the usual argument we get the quantization as in (5.4).

The last possibility is

$$\sigma_1^k(\varepsilon_k) = \sigma_1^k(N\varepsilon_{1,k}) + o(1) = \sigma_1^k(\tau_k L_k) + 4\hat{m} + o(1)$$

and

$$-u_k(x) \geq -\log \varepsilon_k - C, \qquad |x| = \varepsilon_k,$$

for some $C > 0$. In this case u_k is the fast decaying component on $\partial B_{\varepsilon_k}(0)$.

Observe that at the end, in any case on $\partial B_{\varepsilon_k}(0)$ at least one of the two component $u_k, -u_k$ has fast decay and its energy is a small perturbation of a multiple of 4.

With this argument we continue to include groups further away from G_0. Since by construction we have only finite blow-up disks this procedure only needs to be applied finite times. Finally, reasoning as in Lemma 3.1 we can take $s_k \to 0$ such that $\Sigma_k \subset B_{s_k}(0)$ and both component $u_k, -u_k$ have fast decay on $\partial B_{s_k}(0)$:

$$|u_k(x)| \leq -2\log s_k - N_k, \qquad \text{for } |x| = s_k,$$

for some $N_k \to \infty$. Therefore we have that $(\sigma_1^k(s_k), \sigma_2^k(s_k)$ is a $o(1)$ perturbation of one of the two following types:

$$\big(2m(m+1), 2m(m-1)\big) \qquad \text{or} \qquad \big(2m(m-1), 2m(m+1)\big),$$

for some $m \in \mathbb{N}$. On the other hand, notice that by definition

$$\sigma_i = \lim_{k\to\infty} \lim_{s_k \to 0} \sigma_i^k(s_k), \qquad i = 1, 2.$$

It follows that σ_1, σ_2 satisfy the quantization property of Theorem 1.2 and the proof is completed.

□

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