An L^2 gradient flow and its quasi-static limit in phase-field fracture by alternate minimization

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Abstract. We consider an evolution in phase field fracture which combines, in a system of PDEs, an irreversible gradient-flow for the phase-field variable with the equilibrium equations for the displacement field. We employ a discretization in time and an alternate minimization scheme with a quadratic penalty in the phase-field variable (i.e. an "alternate minimizing movement"). First, we prove that discrete solutions converge to a solution of our system of PDEs. Then, we show that the vanishing viscosity limit is a quasi-static BV-evolution. Both our convergence results are formulated in terms integral characterizations, à la De Giorgi, for gradient flows and parametrized BV-evolutions, from which the PDEs follow.

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1 Introduction

Phase field models are widely employed to simulate crack propagation; nowadays, even within the linear-elastic setting, the literature comprises a huge number of phase field models with different potentials and evolution laws for both the elastic and the phase field variable, e.g. [2, 4, 7, 8, 16, 21, 23, 29, 31]. Among the many, our interest is restricted to phase field energies of the form

$$\mathcal{E}(u,v) = \int_\Omega (v^2 + \eta_\delta) W(Du)\, dx, \qquad G_c\, \mathcal{L}(v) = \tfrac{1}{2}\, G_c \int_\Omega (4\delta)^{-1}(v-1)^2 + \delta |\nabla v|^2\, dx,$$

as approximations of the linear-elastic and brittle energy respectively. Here, u is the displacement field, $W(Du)$ is the linear elastic energy density, v is the phase field variable, G_c is toughness while $\delta > 0$ and $\eta_\delta > 0$ are regularization parameters. There are practical and theoretical reasons behind this approximation: for static problems, it is well known [3, 9, 12] that for $\delta \to 0$ and $\eta_\delta = o(\delta)$ the Γ-limit of the energy $\mathcal{F}(u,v) = \mathcal{E}(u,v) + G_c\mathcal{L}(v)$ takes the form

$$\mathcal{F}(u) = \int_\Omega W(Du)\, dx + G_c \mathcal{H}^{n-1}(J_u),$$

where the set J_u of discontinuity points of u represents the crack. This is clearly a strong argument in favour of this approach, however it is fair to remember that Γ-convergence yields only the convergence of global minimizers, for applications it would be more useful to have also convergence of critical points and energy release; these interesting questions are technically hard, some partial answers can be found in [5, 13, 26].

Let us turn our attention toward evolutions problems considering in particular discrete schemes, used in many applications, and based on the separate convexity of the energy $\mathcal{F}(u,v) = \mathcal{E}(u,v) + \mathcal{D}(v)$. We start with the quasi-static variational evolution of [8]. Consider on $\partial\Omega$ a boundary condition $u = g(t)$ for t in the time interval $[0,T]$ and the initial conditions u_0 and v_0 (for $t = 0$).

For $\tau > 0$ let $t_n = n\tau \in [0,T]$ for $n \in \mathbb{N}$. Given an equilibrium configuration (u_{n-1}, v_{n-1}) (at time t_{n-1}) the equilibrium configuration (u_n, v_n) (at time t_n) is obtained by means of a sequence $(u_{n,k}, v_{n,k})$ of intermediate states. More precisely, $u_n = \lim_k u_{n,k}$ and $v_n = \lim_k v_{n,k}$ where the sequences $u_{n,k}$ and $v_{n,k}$ are defined inductively by the alternate minimization scheme

$$\begin{cases} u_{n,k} \in \operatorname{argmin}\{\mathcal{F}(t_n, u, v_{n,k-1}) : u = g(t_n) \text{ on } \partial\Omega\} \\ v_{n,k} \in \operatorname{argmin}\{\mathcal{F}(t_n, u_{n,k}, v) : v \le v_{n,k-1}\}. \end{cases}$$

Intuitively, the sequence $(u_{n,k}, v_{n,k})$ provides a "zig-zagging" evolution in the time interval $[t_{n-1}, t_n]$ connecting the equilibrium configurations (u_{n-1}, v_{n-1}) and (u_n, v_n) with the separately stable configurations $(u_{n,k}, v_{n,k})$. This is by definition a discrete-in-time scheme; it's time-continuous limit (as $\tau \to 0$) has been fully identified in [18]. Mathematically the limit is characterized by equilibrium and energy balance in the framework of parametrized (quasi-static) BV-evolutions [11, 24, 27], with respect to a suitable family of intrinsic norms. Keeping aside the technical details, here it is important to remark that BV-evolutions may present both a stable and an unstable, discontinuous propagation regime. For this reason it is convenient to introduce an "arc-length" parametrization $s \mapsto (t(s), u(s), v(s))$ in which stable and unstable regimes correspond respectively to $t'(s) > 0$ and $t'(s) = 0$. From the mechanical point of view, it is remarkable that in the stable points the BV-limit satisfies a phase-field Griffith's criterion, indeed, if $t'(s) = 0$ it holds

$$\mathcal{G}(t(s), v(s)) \le G_c, \qquad (\mathcal{G}(t(s), v(s)) - G_c)\, \mathcal{L}'(v(s)) = 0,$$

where $\mathcal{G}(t,v)$ is the phase-field energy release rate while $\mathcal{L}'(s)$ denotes the derivative of $\mathcal{L}(s)$ with respect to the parametrization variable (cf. [18] for further details).

In this work we will consider an alternate minimizing movement scheme [30, 31] which provides a discretization of a system of PDEs employed in [21]. Let $\varepsilon > 0$ denote a "mobility parameter" or "viscosity". Given (u_{n-1}, v_{n-1}) (at time t_{n-1}) the configuration at time t_n is obtained by solving

$$\begin{cases} u_n \in \operatorname{argmin}\{\mathcal{F}(t_n, u, v_{n-1}) : u = g(t_n) \text{ on } \partial\Omega\}, \\ v_n \in \operatorname{argmin}\{\mathcal{F}(t_n, u_n, v) + \tfrac{1}{2}\varepsilon\tau^{-1}\|v - v_{n-1}\|^2 : v \le v_{n-1}\}, \end{cases}$$

where $\tau > 0$ is again the discrete time increment. Note that here the updated configuration is determined by a single iteration. First, we will see that (as $\tau \to 0$) the time continuous limit $t \mapsto (u(t), v(t))$ for every $t \in [0,T]$ satisfies: $u(t) \in \operatorname{argmin}\{\mathcal{E}(t, u, v(t)) : u = g(t) \text{ on } \partial\Omega\}$ together with the energy balance

$$\begin{aligned} \mathcal{F}(t, u(t), v(t)) = {} & \mathcal{F}(0, u_0, v_0) - \tfrac{1}{2}\int_0^t \|\varepsilon\dot{v}(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r, u(r), v(r))|^2_{L^2}\, dr \; + \\ & + \int_0^t \partial_t \mathcal{F}(r, u(r), v(r))\, dr, \end{aligned} \tag{1}$$

where the unilateral slope $|\partial_v^- \mathcal{F}(t,u,v)|_{L^2} = \sup\{-\partial_v \mathcal{F}(t,u,v)[\xi] : \xi \le 0, \|\xi\|_{L^2} \le 1\}$ takes into account the irreversibility constraint. This is De Giorgi's integral characterization of gradient flows. Then, after improving time regularity estimates, we show that the limit evolution solves for a.e. time the system of PDEs

$$\begin{cases} \varepsilon\dot{v}(t) = -\big[v(t)W(Du(t)) + G_c(4\delta)^{-1}(v(t) - 1) - G_c\delta\Delta v(t)\big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{v(t)}(u(t))\big) = 0 \end{cases} \tag{2}$$

where $v(t)W(Du(t)) + G_c(4\delta)^{-1}(v(t) - 1) - G_c\delta\Delta v(t)$ is a Radon measure in Ω with positive part $[\cdot]^+$ in $L^2(\Omega)$ while $\boldsymbol{\sigma}_v(u) = (v^2 + \eta_\delta)\,\boldsymbol{\sigma}(u)$ is the phase-field stress. This system, proposed in [21], is one of the "Ginzburg-Landau" models employed in phase-filed fracture with many variations, see for instance [1, 16].

Finally, we consider (as $\varepsilon \to 0$) the quasi-static (vanishing viscosity) limit, characterized again in the parametrized setting. More precisely, the limit, represented by $s \mapsto (t(s), w(s), z(s))$, satisfies $w(s) \in \operatorname{argmin} \{\mathcal{E}(t(s), w, z(s)) : w = g(s) \text{ on } \partial\Omega\}$ together with the energy balance

$$\mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \tag{3}$$

In analogy with the gradient flow, this balance together with the Lipschitz continuity of parametrized solutions implies the main differential properties of BV-evolutions [24], including a characterization of stable and unstable regimes in terms of PDEs (cf. § 5.4).

Let us turn to some technical considerations about our convergence results. First of all, the proof of (1) relies essentially on the separate convexity of the energy $\mathcal{F}_\varepsilon(t, \cdot, \cdot)$, the lower semi-continuity of the slope and an upper gradient inequality; these ingredients allow to work with evolutions of class $W^{1,2}(0, T; L^2)$ which is a "minimal" regularity assumption for (1) to make sense. Then, in order to prove (2) it is necessary (at least in our proof) to employ the chain rule, which in turn requires evolutions of class $W^{1,2}(0, T; H^1)$; proving time regularity in H^1, instead of L^2, is quite delicate and follows from a discrete Gronwall argument. Finally, in order to prove (3) from (1), it is very convenient to employ a parametrized setting [11, 27] for both the gradient flow and the quasi-static evolutions; this is based on the fact that the length of the parametrized curve is finite in L^2, a delicate property which shares several points in common with the estimate in $W^{1,2}(0, T; H^1)$.

To conclude, let us also mention that an alternate minimization scheme has been employed, in different ways, also in a dynamic visco-elastic setting [22] and in another gradient flow setting [5]. Different approaches which provide existence of solutions without alternate minimization can be found in several publications, see e.g. [19, 20, 27].

Contents

2 Setting, energy and its derivatives

Assume that Ω is an open, bounded, connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Deformations are assumed to be of the form $\tilde{u} = u + g(t)$ for $u \in \mathcal{U} = H^1_0(\Omega, \mathbb{R}^2)$ and $g \in C^1([0,T]; W^{1,\bar{p}}(\Omega, \mathbb{R}^2))$ for $\bar{p} > 2$. The phase-field "space" $\mathcal{V}$ is $H^1(\Omega, [0,1])$. In the sequel, we will say that $v_n \rightharpoonup v$ in $\mathcal{V}$ if $v_n \rightharpoonup v$ in $H^1(\Omega)$.

The potential energy $\mathcal{F} : [0,T] \times \mathcal{U} \times \mathcal{V} \to [0, +\infty)$ is given by the following [3, 8] phase field energy for brittle fracture

$$\mathcal{F}(t,u,v) = \tfrac{1}{2} \int_\Omega (v^2 + \eta) W(D\tilde{u}(t))\, dx + \tfrac{1}{2}\, G_c \int_\Omega (v-1)^2 + |\nabla v|^2 \, dx, \tag{4}$$

where $\tilde{u}(t) = u + g(t)$ and $W(D\tilde{u}) = D\tilde{u} : \boldsymbol{C} D\tilde{u} = \boldsymbol{\varepsilon}(\tilde{u}) : \boldsymbol{\sigma}(\tilde{u})$ is the linear elastic energy density, $G_c > 0$ is the toughness while $\eta > 0$ is a (small) regularization parameter. For convenience of notation, let

$$\mathcal{E}(t,u,v) = \tfrac{1}{2} \int_\Omega (v^2 + \eta) W(D\tilde{u}(t))\, dx, \qquad \mathcal{D}(v) = \tfrac{1}{2}\, G_c \int_\Omega (v-1)^2 + |\nabla v|^2 \, dx$$

denote respectively the elastic and the fracture (dissipated) phase-field energy.

Lemma 2.1 *If $t_n \to t$, $u_n \rightharpoonup u$ in $\mathcal{U}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$\mathcal{F}(t,u,v) \le \liminf_{n\to+\infty} \mathcal{F}(t_n, u_n, v_n).$$

Proof. Since $v_n \rightharpoonup v$ in $\mathcal{V}$ it is clear that $\mathcal{D}(v) \le \liminf_{n\to+\infty} \mathcal{D}(v_n)$. It is thus enough to show that

$$\mathcal{E}(t,u,v) \le \liminf_{n\to+\infty} \mathcal{E}(t_n, u_n, v_n).$$

First, extract a subsequence (not relabelled) such that $\liminf_n \mathcal{E}(t_n,u_n,v_n) = \lim_n \mathcal{E}(t_n,u_n,v_n)$. Since v_n is bounded, we can extract a further subsequence (again not relabelled) such that $v_n \to v$ a.e. in Ω. By Egorov's Theorem, for every $\varepsilon \ll 1$ there exists Ω_ε with $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ such that $v_n \to v$ uniformly in Ω_ε. Hence for $\delta \ll 1$ and $n \gg 1$ it holds $0 \le (v^2 + \eta) - \delta \le (v_n^2 + \eta)$. Then

$$\tfrac{1}{2} \int_\Omega (v_n^2 + \eta) W(D\tilde{u}_n(t_n))\, dx \ge \tfrac{1}{2} \int_{\Omega_\varepsilon} (v^2 + \eta - \delta) W(D\tilde{u}_n(t_n))\, dx.$$

By the weak lower semi-continuity of the right hand side

$$\liminf_{n\to+\infty} \mathcal{E}(t_n,u_n,v_n) \ge \tfrac{1}{2} \int_{\Omega_\varepsilon} (v^2 + \eta - \delta) W(D\tilde{u}(t))\, dx.$$

To conclude, it is sufficient to take first the supremum for $\delta \searrow 0$ and then the supremum for $\varepsilon \searrow 0$. ■

In the sequel, both the gradient flow and the quasi-static evolution will be defined by means of a suitable unilateral slope of $\mathcal{F}(t,u,\cdot)$. If the displacement field u is sufficiently regular (and this is the case for our evolutions) variations of energy take a simple form; more precisely, if $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ then by Lemma A.5 the energy $\mathcal{F}(t,u,\cdot)$ is differentiable with

$$\partial_v \mathcal{F}(t,u,v)[\xi] = \int_\Omega v\xi\, W(D\tilde{u}(t))\, dx + G_c \int_\Omega (v-1)\xi + \nabla v \cdot \nabla \xi \, dx \quad \forall\, \xi \in H^1(\Omega). \tag{5}$$

Due to the irreversibility constraint in the evolution we will consider only negative variations, i.e. $\xi \le 0$; thus if $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ for some $p > 2$ the unilateral slope is defined by

$$|\partial_v^- \mathcal{F}(t,u,v)|_{L^2} = |\inf\{\partial_v \mathcal{F}(t,u,v)[\xi] : \xi \in H^1(\Omega),\ \xi \le 0,\ \|\xi\|_{L^2} \le 1\}|. \tag{6}$$

For future convenience denote $\Xi = \{\xi \in H^1(\Omega),\ \xi \le 0,\ \|\xi\|_{L^2} \le 1\}$. Note that the slope could be equivalently defined as

$$|\partial_v^- \mathcal{F}(t,u,v)|_{L^2} = \sup\{-\partial_v \mathcal{F}(t,u,v)[\xi] : \xi \in \Xi\}. \tag{7}$$

Lemma 2.2 *If $t_n \to t$, $u_n \to u$ in $W^{1,p}(\Omega, \mathbb{R}^2)$ for $p > 2$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$|\partial_v^- \mathcal{F}(t, u, v)|_{L^2} \le \liminf_{n\to+\infty} |\partial_v^- \mathcal{F}(t_n, u_n, v_n)|_{L^2}. \tag{8}$$

Proof. First, we show that for every $\xi \in \Xi$ we have

$$\lim_{n\to+\infty} \partial_v \mathcal{F}(t_n, u_n, v_n)[\xi] = \partial_v \mathcal{F}(t, u, v)[\xi]. \tag{9}$$

By weak convergence in $H^1(\Omega)$

$$\lim_{n\to+\infty} \int_\Omega (v_n - 1)\xi + \nabla v_n \cdot \nabla \xi \, dx = \int_\Omega (v - 1)\xi + \nabla v \cdot \nabla \xi \, dx,$$

while

$$\lim_{n\to+\infty} \int_\Omega (v_n^2 + \eta) W(D\tilde{u}_n(t_n)) \, dx = \int_\Omega (v^2 + \eta) W(D\tilde{u}(t)) \, dx$$

because $v_n \to v$ in $L^q(\Omega)$ for every $q < \infty$ (by compact embedding) while

$$D\tilde{u}(t_n) = Du_n + Dg(t_n) \to D\tilde{u}(t) = Du + Dg(t) \text{ in } L^r(\Omega, \mathbb{R}^{2\times 2}) \text{ for } r = p \wedge \bar{p}$$

and thus $W(D\tilde{u}_n(t_n)) \to W(D\tilde{u}(t))$ in $L^{r/2}$ for $r/2 > 1$.

By (7) for every $\xi \in \Xi$

$$|\partial_v^- \mathcal{F}(t_n, u_n, v_n)|_{L^2} \ge -\partial_v \mathcal{F}(t_n, u_n, v_n)[\xi]$$

and hence by (9) for every $\xi \in \Xi$ we get

$$\liminf_{n\to+\infty} |\partial_v^- \mathcal{F}(t_n, u_n, v_n)|_{L^2} \ge -\lim_{n\to+\infty} \partial_v \mathcal{F}(t_n, u_n, v_n)[\xi] = -\partial_v \mathcal{F}(t, u, v)[\xi].$$

Taking the supremum with respect to $\xi \in \Xi$ concludes the proof. ∎

By the regularity in time of g the partial time derivative takes the form

$$\partial_t \mathcal{F}(t, u, v) = \int_\Omega (v^2 + \eta) \boldsymbol{\varepsilon}(\tilde{u}(t)) : \boldsymbol{\sigma}(\dot{g}(t)) \, dx. \tag{10}$$

In particular, for $v \in \mathcal{V}$ by continuity and coercivity of the elastic energy we have

$$|\partial_t \mathcal{F}(t, u, v)| \le C \|\boldsymbol{\varepsilon}(\tilde{u}(t))\|_{L^2} \le C' \left(\|\boldsymbol{\varepsilon}(\tilde{u}(t))\|_{L^2}^2 + 1 \right) \le C''(\mathcal{F}(t, u, v) + 1). \tag{11}$$

Lemma 2.3 *If $t_n \to t$, $u_n \rightharpoonup u$ in $\mathcal{U}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ then*

$$\lim_{n\to+\infty} \partial_t \mathcal{F}(t_n, u_n, v_n) = \partial_t \mathcal{F}(t, u, v). \tag{12}$$

Proof. It is sufficient to pass to the limit in (10) using the fact that $D\dot{g}(t_n)(v_n^2 + \eta) \to D\dot{g}(t)(v^2 + \eta)$ in $L^2(\Omega, \mathbb{R}^{2\times 2})$. ∎

3 Gradient flows

3.1 Incremental problems

Our gradient flow will be defined as an "alternate minimizing movement", i.e. as the limit of an alternate implicit Euler discretization. For sake of simplicity we will assume that the initial configuration u_0, v_0 (at time $t = 0$) is in equilibrium, i.e. that

$$\partial_u \mathcal{F}(0, u_0, v_0) = 0, \qquad \partial_v \mathcal{F}(0, u_0, v_0)[\xi] \geq 0 \quad \forall \xi \in \Xi.$$

Since the energy $\mathcal{F}(t, \cdot, \cdot)$ is separately quadratic, equilibrium is equivalent to separate minimality, i.e.

$$u_0 \in \operatorname{argmin} \{\mathcal{E}(0, v_0, \cdot) : u \in \mathcal{U}\}, \qquad v_0 \in \operatorname{argmin} \{\mathcal{F}(0, \cdot, u_0) : v \leq v_0 \, , v \in \mathcal{V}\}.$$

Fix $\tau = T/m > 0$ (for some $m \in \mathbb{N}$ with $m > 0$) and for $k = 0, ..., m$ consider the discrete times $t_k = k\tau \in [0, T]$. In the sequel, for sake of simplicity, we will drop the dependence on τ in the notation. Given $u_{k-1} = u(t_{k-1})$ and $v_{k-1} = v(t_{k-1})$ the irreversible alternate minimizing movement is defined by

$$\begin{cases} v_k \in \operatorname{argmin} \{\mathcal{F}(t_k, u_{k-1}, v) + \frac{1}{2\tau} \|v - v_{k-1}\|^2_{L^2} : v \leq v_{k-1} \, , v \in \mathcal{V}\} \\ u_k \in \operatorname{argmin} \{\mathcal{F}(t_k, u, v_k) : u \in \mathcal{U}\} = \operatorname{argmin} \{\mathcal{E}(t_k, u, v_k) : u \in \mathcal{U}\}. \end{cases} \tag{13}$$

By Lemma A.4 we get the regularity and the continuous dependence of u_k stated in the next Lemma.

Lemma 3.1 *There exists $p \in (2, \bar{p})$ such that $u_k \in W^{1,p}(\Omega, \mathbb{R}^2)$ for every $k \in \mathbb{N}$. Moreover, there exists $C > 0$, independent of τ and k, such that*

$$\|u_k - u_{k-1}\|_{W^{1,p}} \leq C\big(|t_k - t_{k-1}| + \|v_k - v_{k-1}\|_{L^q}\big) \tag{14}$$

for $1/q = 1/p - 1/\bar{p}$ and for every $k \geq 1$. In particular, u_k is bounded in $W^{1,p}(\Omega, \mathbb{R}^2)$ uniformly with respect to k and τ.

The next two lemmas provide the main ingredients in the proof of the convergence Theorem 3.6.

Lemma 3.2 *For every $k \geq 1$ let $\dot{v}_k = (v_k - v_{k-1})/(t_k - t_{k-1})$, then*

$$\|\dot{v}_k\|^2_{L^2} = |\partial^-_v \mathcal{F}(t_k, u_{k-1}, v_k)|_{L^2} \, \|\dot{v}_k\|_{L^2} = -\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\dot{v}_k].$$

Proof. First of all, by a standard truncation argument we know that we can replace $\mathcal{V}$ with the whole $H^1(\Omega)$ in (13), hence

$$v_k \in \operatorname{argmin} \{\mathcal{F}(t_k, u_{k-1}, v) + \tfrac{1}{2\tau} \|v - v_{k-1}\|^2_{L^2} : v \leq v_{k-1} \, , v \in H^1(\Omega)\}.$$

By minimality, v_k solves the variational inequality

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[w - v_k] + \langle \dot{v}_k, w - v_k \rangle_{L^2} \geq 0 \tag{15}$$

for every $w \in H^1(\Omega)$ with $w \leq v_{k-1}$. Choosing $w - v_k = \pm \tau \dot{v}_k$ (corresponding to $w = v_{k-1}$ and $w = 2v_k - v_{k-1}$) provides

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\dot{v}_k] + \|\dot{v}_k\|^2_{L^2} = 0. \tag{16}$$

Next, by (7) and (15) with $w - v_k = \xi \in \Xi$ we get

$$\begin{aligned} |\partial^-_v \mathcal{F}(t_k, u_{k-1}, v_k)|_{L^2} &= \sup\{-\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\xi] : \xi \in \Xi\} \\ &\leq \sup\{\langle \dot{v}_k, \xi \rangle_{L^2} : \xi \in \Xi\} = \|\dot{v}_k\|_{L^2}. \end{aligned} \tag{17}$$

If $\dot{v}_k = 0$ there is nothing else to prove. Otherwise, $\xi = \dot{v}_k / \|\dot{v}_k\|_{L^2}$ is an admissible variation and by (16) the inequality in (17) becomes an equality, thus $|\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)|_{L^2} = \|\dot{v}_k\|_{L^2}$. ■

Lemma 3.3 *For every $k \geq 1$ it holds the following energy estimate*

$$\mathcal{F}(t_k, u_k, v_k) \le \mathcal{F}(t_{k-1}, u_{k-1}, v_{k-1}) - \tfrac{1}{2} \int_{t_{k-1}}^{t_k} \|\dot v_k\|^2_{L^2} + |\partial^-_v \mathcal{F}(t_k, u_{k-1}, v_k)|^2_{L^2} \, dt \, + \\ + \int_{t_{k-1}}^{t_k} \partial_t \mathcal{F}(t, u_{k-1}, v_{k-1}) \, dt. \qquad (18)$$

Proof. By minimality and convexity of $\mathcal{F}(t_k, u_{k-1}, \cdot)$ we get

$$\mathcal{F}(t_k, u_k, v_k) \le \mathcal{F}(t_k, u_{k-1}, v_k) \le \mathcal{F}(t_k, u_{k-1}, v_{k-1}) - \partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[v_{k-1} - v_k]. \qquad (19)$$

By Lemma 3.2

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[v_{k-1} - v_k] = \tau \|\dot v_k\|^2_{L^2} = \tau \tfrac{1}{2} \big(\|\dot v_k\|^2_{L^2} + |\partial^-_v \mathcal{F}(t_k, u_{k-1}, v_k)|^2_{L^2} \big)$$

and hence by (19)

$$\mathcal{F}(t_k, u_k, v_k) \le \mathcal{F}(t_k, u_{k-1}, v_{k-1}) - \tfrac{1}{2} \int_{t_{k-1}}^{t_k} \|\dot v_k\|^2_{L^2} + |\partial^-_v \mathcal{F}(t_k, u_{k-1}, v_k)|^2_{L^2} \, dt.$$

Finally,

$$\mathcal{F}(t_k, u_{k-1}, v_{k-1}) = \mathcal{F}(t_{k-1}, u_{k-1}, v_{k-1}) + \int_{t_{k-1}}^{t_k} \partial_t \mathcal{F}(t, u_{k-1}, v_{k-1}) \, dt$$

and the proof is concluded. ∎

Lemma 3.4 *There exists $C > 0$, independent of τ, such that $\mathcal{F}(t_k, u_k, v_k) \le C(\mathcal{F}(t_0, u_0, v_0) + 1)$ for every index k.*

Proof. By minimality of u_k and v_k

$$\mathcal{F}(t_k, u_k, v_k) \le \mathcal{F}(t_k, u_{k-1}, v_k) \le \mathcal{F}(t_k, u_{k-1}, v_k) + \tfrac{1}{2\tau} \|v_k - v_{k-1}\|^2_{L^2} \le \mathcal{F}(t_k, u_{k-1}, v_{k-1}).$$

Further,

$$\mathcal{F}(t_k, u_{k-1}, v_{k-1}) - \mathcal{F}(t_{k-1}, u_{k-1}, v_{k-1}) = \tfrac{1}{2} \int_\Omega (v^2_{k-1} + \eta)[W(Du_{k-1} + Dg_k) - W(Du_{k-1} + Dg_{k-1})] \, dx$$

Since W is quadratic

$$W(Du_{k-1} + Dg_k) - W(Du_{k-1} + Dg_{k-1}) = [2\,\boldsymbol{\varepsilon}(u_{k-1} + g_{k-1}) + \boldsymbol{\varepsilon}(g_k - g_{k-1})] : \boldsymbol{\sigma}(g_k - g_{k-1})$$

where $\|\boldsymbol{\sigma}(g_k - g_{k-1})\|_{L^2} \le C\tau$ while

$$\|\boldsymbol{\varepsilon}(u_{k-1} + g_{k-1})\|_{L^2} \le C(\|\boldsymbol{\varepsilon}(u_{k-1} + g_{k-1})\|^2_{L^2} + 1) \le C'(\mathcal{F}(t_{k-1}, u_{k-1}, v_{k-1}) + 1).$$

In summary, we can write

$$(\mathcal{F}(t_k, u_k, v_k) + 1) \le (1 + C\tau)(\mathcal{F}(t_{k-1}, u_{k-1}, v_{k-1}) + 1).$$

It follows that $\mathcal{F}(t_k, u_k, v_k) \le (1 + C\tau)^k (\mathcal{F}(t_0, u_0, v_0) + 1)$ and then, since $\tau \le T/k$,

$$\mathcal{F}(t_k, u_k, v_k) \le (1 + CT/k)^k (\mathcal{F}(t_0, u_0, v_0) + 1),$$

since $(1 + CT/k)^k \to e^{CT}$ the required estimate follows. ∎

3.2 Compactness and convergence

In order to find the time continuous limit it is convenient to introduce a sequence $\tau_m = T/m$ of discrete time increments together with the corresponding discrete times $t_{m,k} = k\tau_m$ for $k = 0, ..., m$. Moreover, let us denote by $u_m : [0,T] \to \mathcal{U}$ and $v_m : [0,T] \to \mathcal{V}$ the corresponding piecewise affine evolutions, i.e. the interpolation of $u_{m,k} = u_m(t_{m,k})$ and $v_{m,k} = v_m(t_{m,k})$ obtained by the alternate minimization scheme (13).

Lemma 3.5 *The sequence v_m is bounded in $L^\infty(0,T;H^1(\Omega))$ and in $H^1(0,T;L^2(\Omega))$ and thus, upon extracting a (non-relabelled) subsequence, $v_m \rightharpoonup v$ in $H^1(0,T;L^2(\Omega))$.*

Moreover, if $t_m \to t$ then $v_m(t_m) \rightharpoonup v(t)$ in $H^1(\Omega)$ and $u_m(t_m) \to u(t)$ in $W^{1,p}(\Omega,\mathbb{R}^2)$, for some $p > 2$, where $u(t) \in \mathrm{argmin}\,\{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$.

Proof. From Lemma 3.4 we known that $\mathcal{F}(t_{m,k}, u_{m,k}, v_{m.k})$ is uniformly bounded and thus v_m is bounded in $L^\infty(0,T;H^1(\Omega))$ while u_m is bounded in $L^\infty(0,T;H^1_0(\Omega,\mathbb{R}^2))$ by Korn's inequality. Then from (18) and (11) we get

$$\mathcal{F}(t_{m,k}, u_{m,k}, v_{m,k}) \le \mathcal{F}(t_{m,k-1}, u_{m,k-1}, v_{m,k-1}) - \tfrac{1}{2}\int_{t_{m,k-1}}^{t_{m,k}} \|\dot v_{m,k}\|^2_{L^2}\,dt + C\tau_m.$$

By induction we get

$$\tfrac{1}{2}\int_0^T \|\dot v_{m,k}\|^2_{L^2}\,dt \le \mathcal{F}(0,u_0,v_0) + CT$$

and thus v_m is bounded in $H^1(0,T;L^2(\Omega))$. As a consequence, (up to subsequences) $v_m \rightharpoonup v$ in $H^1(0,T;L^2(\Omega))$ and $v_m(t_m) \rightharpoonup v(t)$ in $L^2(\Omega)$ for $t_m \to t$; since $v_m(t_m)$ is bounded in $H^1(\Omega)$ it turns out that $v_m(t_m) \rightharpoonup v(t)$ in $H^1(\Omega)$.

Being $u_m(t_m) \in \mathrm{argmin}\,\{\mathcal{E}(t_m,u,v_m(t_m)) : u \in \mathcal{U}\}$ we have

$$\int_\Omega (v_m^2(t_m)+1) D\tilde u_m(t_m) : \mathbf{C} D\phi\,dx = 0 \qquad \forall \phi \in \mathcal{U}.$$

Since $u_m(t_m)$ is bounded in $H^1_0(\Omega,\mathbb{R}^2)$ there exists a subquences $u_{m_j}(t_{m_j})$ weakly converging to some $u_\infty \in H^1_0(\Omega,\mathbb{R}^2)$. Clearly

$$Du_{m_j}(t_{m_j}) + Dg(t_{m_j}) \rightharpoonup Du_\infty + Dg(t) \quad \text{and} \quad (v^2_{m_j}(t_{m_j})+1)D\phi \to (v^2(t)+1)D\phi$$

in $L^2(\Omega,\mathbb{R}^{2\times 2})$, thus

$$\int_\Omega (v^2(t)+1) D\tilde u_\infty(t) : \mathbf{C} D\phi\,dx = 0 \qquad \forall \phi \in \mathcal{U}$$

and $u_\infty \in \mathrm{argmin}\,\{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$. Since the limit is uniquely determined the whole sequence converges.

By compact embedding $v_m(t_m) \to v(t)$ in $L^q(\Omega)$ for every $q < +\infty$. Then, by Lemma A.4 we get the strong convergence of displacements in $W^{1,p}(\Omega,\mathbb{R}^2)$. ■

Theorem 3.6 *Let v be a limit of v_m (as in Lemma 3.5) and let u be the corresponding (pointwise) limit of u_m; then for every $t \in [0,T]$ it holds*

$$\mathcal{F}(t,u(t),v(t)) = \mathcal{F}(0,u_0,v_0) - \tfrac{1}{2}\int_{t_0}^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\,dr + \qquad (20)$$
$$+ \int_{t_0}^t \partial_t \mathcal{F}(r,u(r),v(r))\,dr.$$

Moreover for every $t \in [0,T]$ we have $u(t) \in \mathrm{argmin}\,\{\mathcal{E}(t,u,v(t)) : u \in \mathcal{U}\}$ and

$$\|\dot v(t)\|_{L^2} = |\partial_v^- \mathcal{F}(t,u(t),v(t))|_{L^2}. \qquad (21)$$

For sake clarity the proof of the previous Theorem will be split into a couple of Propositions.

Proposition 3.7 *Under the hypotheses of Theorem 3.6 for every $t \in [0,T]$ it holds*

$$u(t) \in \operatorname{argmin} \{\mathcal{E}(t, u, v(t)) : u \in \mathcal{U}\},$$

$$\mathcal{F}(t, u(t), v(t)) \le \mathcal{F}(0, u_0, v_0) - \tfrac{1}{2} \int_0^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r, u(r), v(r))|^2_{L^2} \, dr + \tag{22}$$
$$+ \int_0^t \partial_t \mathcal{F}(r, u(r), v(r)) \, dr.$$

Proof. Given $t \in [0,T]$ let $1 \le k_m \le m$ such that $t_{m,k_m} \to t$. Then, by induction (18) provides

$$\begin{aligned} \mathcal{F}(t_{m,k_m}, u(t_{m,k_m}), v(t_{m,k_m})) + \tfrac{1}{2} \int_0^{t_{m,k_m}} \|\dot v_m(r)\|^2_{L^2} \, dr + \\ + \tfrac{1}{2} \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|^2_{L^2} \, dr \le \\ \le \mathcal{F}(0, u_0, v_0) + \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} \partial_t \mathcal{F}(r, u_{m,k-1}, v_{m,k-1}) \, dr. \end{aligned} \tag{23}$$

We know that $t_{m,k_m} \to t$, then by Lemma 3.5 $v_m(t_{m,k_m}) \rightharpoonup v(t)$ in $H^1(\Omega)$ and $u_m(t_{m,k_m}) \to u(t)$ in $W^{1,p}(\Omega, \mathbb{R}^2)$. Then by Lemma 2.1

$$\mathcal{F}(t, u(t), v(t)) \le \liminf_{m \to \infty} \mathcal{F}(t_{m,k_m}, u(t_{m,k_m}), v(t_{m,k_m})).$$

Since $v_m \rightharpoonup v$ in $H^1(0,T; L^2(\Omega))$ we have

$$\int_0^t \|\dot v\|^2_{L^2} \, dr \le \liminf_{m \to \infty} \int_0^{t_{m,k_m}} \|\dot v_m\|^2_{L^2} \, dr.$$

Next, given $r \in (0,t)$ let $r \in [t_{m,k'_m}, t_{m,k'_m+1})$ for $k'_m \le k_m$. Clearly, both $t_{m,k'_m} \to r$ and $t_{m,k'_m-1} \to r$. By Lemma 3.5 we know that $u_m(t_{m,k'_m-1}) \to u(r)$ strongly in $W^{1,p}(\Omega, \mathbb{R}^2)$ (for some $p > 2$) while $v_m(t_{m,k'_m}) \rightharpoonup v(r)$ in $H^1(\Omega)$ and then by Lemma 2.2 we get

$$|\partial_v^- \mathcal{F}(r, u(r), v(r))|_{L^2} \le \liminf_{m \to +\infty} |\partial_v^- \mathcal{F}(t_{m,k'_m}, u_m(t_{m,k'_m-1}), v_m(t_{m,k'_m-1}))|_{L^2}.$$

By Fatou's Lemma we conclude that

$$\int_0^t |\partial_v^- \mathcal{F}(r, u(r), v(r))|^2_{L^2} \, dr \le \liminf_{m \to +\infty} \sum_{k=0}^{k_m-1} \int_{t_{m,k}}^{t_{m,k+1}} |\partial_v^- \mathcal{F}(t_{m,k}, u_{m,k-1}, v_{m,k})|^2_{L^2} \, dr.$$

By Lemma 2.3 and (11) we get, by dominated convergence,

$$\limsup_{m \to +\infty} \sum_{k=0}^{k_m-1} \int_{t_{m,k-1}}^{t_{m,k}} \partial_t \mathcal{F}(r, u_{m,k-1}, v_{m,k-1}) \, dr \le \int_0^t \partial_t \mathcal{F}(r, u(r), v(r)) \, dr.$$

Taking respectively the liminf on the left hand side and the limsup on the right hand side of (23) we get the energy inequality

$$\begin{aligned} \mathcal{F}(t, u(t), v(t)) + \tfrac{1}{2} \int_0^t \|\dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r, u(r), v(r))|^2_{L^2} \, dr \le \\ \le \mathcal{F}(0, u_0, v_0) + \int_0^t \partial_t \mathcal{F}(r, u(r), v(r)) \, dr, \end{aligned}$$

which conclude the proof. ■

There are different ways to prove the "upper gradient inequality"

$$\begin{aligned}\mathcal{F}(0,u_0,v_0)-\mathcal{F}(t,u(t),v(t)) &\le \int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|_{L^2}\ \|\dot v(r)\|_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr \\ &\le \tfrac12\int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|^2_{L^2}+\|\dot v(r)\|^2_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr.\end{aligned}$$

For instance, the estimate will follow from the chain rule Lemma A.6 once we will know (from Theorem 3.9) that the limit evolution v actually belongs to $H^1(0,T;H^1(\Omega))$. Proposition 3.8 below provides instead a proof in $H^1(0,T;L^2(\Omega))$ based only on measure theory and separate convexity.

Proposition 3.8 *Let $v\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;\mathcal{V})$ and $u(t)\in\operatorname{argmin}\{\mathcal{F}(t,v(t),u): u\in\mathcal{U}\}$ such that $t\mapsto|\partial_v^-\mathcal{F}(t,u(t),v(t))|$ belongs to $L^2(0,T)$. Then for every $t\in(0,T)$*

$$\begin{aligned}\mathcal{F}(0,u_0,v_0)-\mathcal{F}(t,u(t),v(t)) &\le \int_0^t |\partial_v^-\mathcal{F}(r,u(r),v(r))|_{L^2}\ \|\dot v(r)\|_{L^2}\,dr\ + \\ &\quad -\int_0^t \partial_t\mathcal{F}(r,u(r),v(r))\,dr. \qquad (24)\end{aligned}$$

Proof. Step I. Since the slope belongs to $L^2(0,t)$ there exists a sequence of finite subdivisions $t_{j,i}$ (for $j\in\mathbb{N}$ and $i=0,...,I_j$) of the time interval $[0,t]$ with $0=t_{j,0}<...<t_{j,i}<t_{j,i+1}<...<t_{j,I_j}=t$ and with $\lim_{j\to+\infty}\Delta t_j=0$, for $\Delta t_j=\max_i|t_{j,i+1}-t_{j,i}|$, such that the piecewise constant functions

$$F_j(t)=\sum_{i=0}^{I_j-1}\chi_{(t_{j,i},\,t_{j,i+1})}(t)\,|\partial_v^-\mathcal{F}(t_{j,i},u(t_{j,i}),v(t_{j,i}))|_{L^2}$$

converge to $|\partial_v^-\mathcal{F}|$ strongly in $L^2(0,t)$ (cf. Theorem 4.12 in [10] or [15]).

Denote for simplicity $u_{j,i}=u(t_{j,i})$ and $\chi_{j,i}=\chi_{(t_{j,i},\,t_{j,i+1})}$ etc. For each $j\in\mathbb{N}$ and $i=0,...,I_j$ write

$$\begin{aligned}\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i+1},u_{j,i+1},v_{j,i+1}) =&\,\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})\,+ \\ &+\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})\,+ \\ &+\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i+1},v_{j,i+1}).\end{aligned}$$

We will consider the three lines above separately, starting with the first. If $\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})>\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})$ by convexity of $\mathcal{F}(t_{j,i},u_{j,i},\cdot)$ we get

$$\begin{aligned}\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1}) &\le \partial_v\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})[v_{j,i+1}-v_{j,i}] \\ &\le |\partial_v^-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \|v_{j,i+1}-v_{j,i}\|_{L^2} \\ &\le \int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^-\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \ \|\dot v_{j,i+1}\|_{L^2}\ dr\end{aligned}$$

where $\dot v_{j,i}=(v_{j,i+1}-v_{j,i})/(t_{j,i+1}-t_{j,i})$ denotes the "discrete" velocity. If $\mathcal{F}(t_{j,i},u_{j,i},v_{j,i})\le\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})$ the same estimate holds since the right hand side is non-negative. For the second term it is sufficient to write

$$\mathcal{F}(t_{j,i},u_{j,i},v_{j,i+1})-\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1})=-\int_{t_{j,i}}^{t_{j,i+1}}\partial_t\mathcal{F}(r,u_{j,i},v_{j,i+1})\,dr.$$

For the third term, remember that by minimality

$$\int_\Omega (v_{j,i+1}^2+\eta)\,\boldsymbol{\sigma}(u_{j,i+1}+g_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i}-u_{j,i+1})\,dx = 0$$

and that $\mathcal{F}(t_{j,i+1},\cdot,v_{j,i+1})$ is quadratic, then

$$\begin{aligned}
\mathcal{F}(t_{j,i+1},u_{j,i},v_{j,i+1}) - \mathcal{F}(t_{j,i+1},u_{j,i+1},v_{j,i+1}) = \\
&= \tfrac{1}{2}\int_\Omega (v_{j,i+1}^2+\eta)\big(W(Du_{j,i}+Dg_{j,i+1}) - W(Du_{j,i+1}+Dg_{j,i+1})\big)\,dx \\
&= \tfrac{1}{2}\int_\Omega (v_{j,i+1}^2+\eta)\,\boldsymbol{\sigma}(u_{j,i}+u_{j,i+1}+2g_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i}-u_{j,i+1})\,dx \\
&= \tfrac{1}{2}\int_\Omega (v_{j,i+1}^2+\eta)\,\boldsymbol{\sigma}(u_{j,i}-u_{j,i+1}) : \boldsymbol{\varepsilon}(u_{j,i}-u_{j,i+1})\,dx \\
&\le C\|u_{j,i+1}-u_{j,i}\|_{H^1}^2 = C \fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1}-u_{j,i}\|_{H^1}^2\,dr.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\mathcal{F}(t_{j,i},u_{j,i},v_{j,i}) - \mathcal{F}(t_{j,i+1},\, u_{j,i+1},v_{j,i+1}) \le \\
&\le \int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^- \mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \|\dot v_{j,i+1}\|_{L^2}\ dr\ - \int_{t_{j,i}}^{t_{j,i+1}} \partial_t \mathcal{F}(r,u_{j,i},v_{j,i+1})\,dr\,+ \\
&\quad + C\fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1}-u_{j,i}\|_{H^1}^2\,dr.
\end{aligned}$$

Taking the sum for $i=0,...,I_j$ yields

$$\begin{aligned}
\mathcal{F}(0,u_0,v_0) - \mathcal{F}(t,u(t),v(t)) &\le \sum_{i=0}^{I_j-1}\int_{t_{j,i}}^{t_{j,i+1}} |\partial_v^- \mathcal{F}(t_{j,i},u_{j,i},v_{j,i})|_{L^2}\ \|\dot v_{j,i+1}\|_{L^2}\ dr\,+ \\
&- \sum_{i=0}^{I_j-1}\int_{t_{j,i}}^{t_{j,i+1}} \partial_t \mathcal{F}(r,u_{j,i},v_{j,i+1})\,dr\ + \sum_{i=0}^{I_j-1}\fint_{t_{j,i}}^{t_{j,i+1}} \|u_{j,i+1}-u_{j,i}\|_{H^1}^2\,dr.
\end{aligned} \tag{25}$$

Step II. Let us re-write (25) as

$$\mathcal{F}(0,u_0,v_0) - \mathcal{F}(t,u(t),v(t)) \le \int_0^t F_j(r)V_j(r) - P_j(r) + E_j(r)\,dr$$

in terms of the piecewise-constant functions F_j (defined above) and

$$V_j(r) = \sum_{i=0}^{I_j-1}\chi_{j,i}(r)\ \|\dot v_{j,i+1}\|_{L^2}\,, \qquad P_j(r) = \sum_{i=0}^{I_j-1}\chi_{j,i}(r)\,\partial_t\mathcal{F}(r,u_{j,i},v_{j,i+1}),$$

$$E_j(r) = \sum_{i=0}^{I_j-1}\chi_{j,i}(r)\,|t_{j,i+1}-t_{j,i}|^{-1}\|u_{j,i+1}-u_{j,i}\|_{H^1}^2.$$

Since the above estimate holds for every subdivision $t_{j,i}$ it must hold also

$$\mathcal{F}(0,u_0,v_0) - \mathcal{F}(t,u(t),v(t)) \le \lim_{j\to+\infty}\int_0^t F_j(r)V_j(r) - P_j(r) + E_j(r)\,dr.$$

We will show that

$$\lim_j \int_0^t F_j(r) V_j(r) dr = \int_0^t |\partial_v^- \mathcal{F}(r, u(r), v(r))|_{L^2} \, \|\dot v(r)\|_{L^2} \, dr, \tag{26}$$

$$\lim_j \int_0^t P_j(r)\, dr = \int_0^t \partial_t \mathcal{F}(r, u(r), v(r))\, dr, \tag{27}$$

$$\lim_j \int_0^t E_j(r)\, dr = 0, \tag{28}$$

which will prove (24).

Since F_j converge strongly in $L^2(0,t)$ (by construction) to prove (26) it is enough to see that

$$V_j = \sum_{i=0}^{I_j - 1} \chi_{j,i} \, \|\dot v_{j,i+1}\|_{L^2} \rightharpoonup \|\dot v\|_{L^2} \quad \text{weakly in } L^2(0,t).$$

Note that $V_j \to \|\dot v\|$ a.e. in $[0,t]$ since $v \in W^{1,2}(0,t; L^2)$. Write, by Jensen's inequality

$$\|\dot v_{j,i+1}\|_{L^2} = \left\| \frac{v_{j,i+1} - v_{j,i}}{t_{j,i+1} - t_{j,i}} \right\|_{L^2} = \left\| \fint_{t_i}^{t_{i+1}} \dot v(r)\, dr \right\|_{L^2} \le \fint_{t_i}^{t_{i+1}} \|\dot v(r)\|_{L^2}\, dr \tag{29}$$

so that $0 \le V_j \le C_j$ where

$$C_j = \sum_{i=0}^{I_j - 1} \chi_{j,i} \fint_{t_i}^{t_{i+1}} \|\dot v(r)\|_{L^2}\, dr.$$

Note that $C_j \to \|\dot v\|_{L^2}$ a.e. in $(0,t)$ and that $C_j \in L^1(0,t)$ with

$$\int_0^t C_j(r)\, dr = \sum_{i=0}^{I_j - 1} (t_{i+1} - t_i) \fint_{t_i}^{t_{i+1}} \|\dot v(r)\|_{L^2}\, dr = \int_0^t \|\dot v(r)\|_{L^2}\, dr.$$

Hence, by generalized dominated convergence $V_j \to \|\dot v\|_{L^2}$ in $L^1(0,t)$. Arguing as in (29) we easily get that V_j is bounded in $L^2(0,t)$, thus $V_j \rightharpoonup \|\dot v\|_{L^2}$ in $L^2(0,t)$.

Let us prove (27). Fix $r \in (0,t)$ and let $t_{j,i} \le r < t_{j,i+1}$ (with i depending on j). For a.e. $r \in (0,t)$ we have

$$\partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1}) = \int_\Omega (v_{j,i+1}^2 + \eta) \boldsymbol{\sigma}(u_{j,i} + g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\, dr.$$

Remember that $v \in H^1(0,t; L^2(\Omega)) \cap L^\infty(0,t; \mathcal{V})$, hence $v_{j,i+1} = v(t_{j,i+1}) \to v(r)$ in $L^q(\Omega)$ for every $q < \infty$. Similarly, $v_{j,i} = v(t_{j,i}) \to v(r)$ in $L^q(\Omega)$ and thus $u_{j,i} = u(t_{j,i}) \to u(r)$ in $W^{1,p}(\Omega, \mathbb{R}^2)$ for $p > 2$ (by Lemma 3.1). As a consequence

$$\int_\Omega (v_{j,i+1}^2 + \eta) \boldsymbol{\sigma}(u_{j,i} + g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\, dr \;\to\; \int_\Omega (v^2(r) + \eta)\; \boldsymbol{\sigma}(u(r) + g(r)) : \boldsymbol{\varepsilon}(\dot g(r))\, dr.$$

Therefore $\partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1}) \to \partial_t \mathcal{F}(r, u(r), v(r))$ a.e. in $(0,t)$. Since $v \in L^\infty(0,t; H^1)$ by (11) we get that $|\partial_t \mathcal{F}(r, u_{j,i}, v_{j,i+1})|$ is uniformly bounded and thus (27) follows by dominated convergence.

Finally, let us prove (28). Since $u_{j,i} \in \operatorname{argmin} \{\mathcal{E}(t_{j,i}, \cdot, v_{j,i})\}$ by Lemma 3.1 we known that

$$\|u_{j,i+1} - u_{j,i}\|_{H^1}^2 \le C |t_{j,i+1} - t_{j,i}|^2 + C \|v_{j,i+1} - v_{j,i}\|_{L^q}^2,$$

for some q sufficiently large. Since $v_{j,i} \in L^p(\Omega)$ for every $p < \infty$ (by Sobolev embedding) we can apply the interpolation inequality

$$\|v_{j,i+1} - v_{j,i}\|_{L^q} \le \|v_{j,i+1} - v_{j,i}\|_{L^2}^\alpha \; \|v_{j,i+1} - v_{j,i}\|_{L^{\bar q}}^{1-\alpha}$$

with $1/q = \alpha/2 + (1-\alpha)/\bar{q}$ (for a suitable $\bar{q}$). Hence, for $\alpha = 1/2$ we get

$$|t_{j,i+1} - t_{j,i}|^{-1} \|v_{j,i+1} - v_{j,i}\|^2_{L^q} \le \|\dot{v}_{j,i+1}\|_{L^2} \, \|v_{j,i+1} - v_{j,i}\|_{L^{\bar{q}}}.$$

Then,

$$\begin{aligned} \int_0^t E_j(r)\, dr &\le \sum_{i=0}^{I_j - 1} \int_{t_{j,i}}^{t_{j,i+1}} \ |t_{j,i+1} - t_{j,i}|^{-1} \|u_{j,i+1} - u_{j,i}\|^2_{H^1} \\ &\le \sum_{i=0}^{I_j - 1} \int_{t_{j,i}}^{t_{j,i+1}} C|t_{j,i+1} - t_{j,i}| + C\, \|\dot{v}_{j,i+1}\|_{L^2} \, \|v_{j,i+1} - v_{j,i}\|_{L^{\bar{q}}} \, dr \\ &\le C \int_0^t \Delta t_j + V_j(r) D_j(r)\, dr, \end{aligned} \tag{30}$$

where V_j has been defined before while $D_j(r) = \sum_i \chi_{j,i} \|v_{j,i+1} - v_{j,i}\|_{L^q}$. We have already seen that $V_j \rightharpoonup \|\dot{v}\|_{L^2}$ weakly in $L^2(0,t)$. Moreover, $\|v_{j,i+1} - v_{j,i}\|_{L^{\bar{q}}} \le C$ and

$$\|v_{j,i+1} - v_{j,i}\|_{L^{\bar{q}}} = \|v(t_{j,i+1}) - v(t_{j,i})\|_{L^{\bar{q}}} \to 0 \qquad \text{a.e. in } (0,t).$$

As a consequence $\|v_{j,i+1} - v_{j,i}\|_{L^{\bar{q}}} \to 0$ in $L^2(0,t)$ and from (30) follows (28). ■

3.3 From energy balance to PDEs

In this subsection we will see that the limit evolution v obtained by the sequence v_m (as in Lemma 3.5) is a solution of (2). First, we need to show that v_m is weakly compact in $H^1(0,T;\mathcal{V})$.

Theorem 3.9 *Let v_m, u_m be the discrete evolutions provided by the alternate minimization algorithm* (13)*. Then v_m is bounded in $W^{1,\infty}(0,T;L^2) \cap H^1(0,T;H^1)$ and u_m is bounded in $W^{1,\infty}(0,T;W^{1,p})$ for some $p > 2$ independent of m. Hence, the limit evolutions v and u (cf. Lemma 3.5) belong respectively to $W^{1,\infty}(0,T;L^2) \cap H^1(0,T;H^1)$ and $W^{1,\infty}(0,T;W^{1,p})$.*

Proof. For later use the proof is divided into several steps. Moreover, for sake of simplicity we will drop in the subscript the dependence on m.

Step I. By Lemma 3.2 we have for $k \ge 0$

$$\partial_v \mathcal{F}(t_{k+1}, u_k, v_{k+1})[\dot{v}_{k+1}] + \|\dot{v}_{k+1}\|^2_{L^2} = 0. \tag{31}$$

By a simple truncation argument, v_k is actually a minimizer among $v \in H^1$ (and not only in $\mathcal{V}$) with $v \le v_k$, i.e.

$$v_k \in \operatorname{argmin} \{ \mathcal{F}(t_k, u_{k-1}, v) + \tfrac{1}{2\tau} \|v - v_{k-1}\|^2_{L^2} : v \le v_{k-1}\,, v \in H^1 \}.$$

Thus for $k \ge 1$

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[w - v_k] + \langle \dot{v}_k, w - v_k \rangle_{L^2} \ge 0 \quad \text{for every } w \in H^1(\Omega) \text{ with } w \le v_{k-1}.$$

Choosing $w = v_k + \dot{v}_{k+1}$ provides

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\dot{v}_{k+1}] + \langle \dot{v}_k, \dot{v}_{k+1} \rangle_{L^2} \ge 0. \tag{32}$$

For $k = 0$ we have, by equilibrium,

$$\partial_v \mathcal{F}(0, u_0, v_0)[\dot{v}_1] \ge 0. \tag{33}$$

Hence, from (31) and (32) we obtain, for $k \geq 1$,

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{F}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] \leq \\ \leq \|\dot v_{k+1}\|^2_{L^2} - \langle \dot v_k, \dot v_{k+1}\rangle_{L^2} \leq \tfrac12 \|\dot v_{k+1}\|^2_{L^2} - \tfrac12 \|\dot v_k\|^2_{L^2}. \tag{34}$$

For $k = 0$ from (31) and (33) we get

$$\partial_v \mathcal{F}(0, u_0, v_0)[\dot v_1] - \partial_v \mathcal{F}(t_1, u_0, v_1)[\dot v_1] \geq \|\dot v_1\|^2_{L^2}.$$

Setting $\dot v_0 = 0$ we can also write

$$\partial_v \mathcal{F}(0, u_0, v_0)[\dot v_1] - \partial_v \mathcal{F}(t_1, u_0, v_1)[\dot v_1] \geq \tfrac12 \|\dot v_1\|^2_{L^2} - \tfrac12 \|\dot v_0\|^2_{L^2} . \tag{35}$$

and thus (34) actually holds for every $k \geq 0$.

For the left hand side of (34) we proceed as follows.

$$\partial_v \mathcal{F}(t_k, u_{k-1}, v_k)[\dot v_{k+1}] - \partial_v \mathcal{F}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] = \\ = \partial_v \mathcal{E}(t_k, u_{k-1}, v_k)[\dot v_k] - \partial_v \mathcal{E}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}] + \\ + \partial_v \mathcal{D}(v_k)[\dot v_{k+1}] - \partial_v \mathcal{D}(v_{k+1})[\dot v_{k+1}].$$

Write $\partial_v \mathcal{D}(v_k)[\dot v_{k+1}] - \partial_v \mathcal{D}(v_{k+1})[\dot v_{k+1}]$ as

$$G_c \int_\Omega (v_k - 1)\, \dot v_{k+1} + \nabla v_k \cdot \nabla \dot v_{k+1}\, dx - G_c \int_\Omega (v_{k+1} - 1)\, \dot v_{k+1} + \nabla v_{k+1} \cdot \nabla \dot v_{k+1}\, dx = \\ = G_c \int_\Omega (v_k - v_{k+1})\, \dot v_{k+1} + \nabla (v_k - v_{k+1}) \cdot \nabla \dot v_{k+1}\, dx = -\tau G_c \|\dot v_{k+1}\|^2_{H^1}.$$

We can estimate $\partial_v \mathcal{E}(t_k, u_{k-1}, v_k)[\dot v_k] - \partial_v \mathcal{E}(t_{k+1}, u_k, v_{k+1})[\dot v_{k+1}]$ by

$$\begin{aligned} &\int_\Omega v_k \dot v_{k+1} W(Du_{k-1} + Dg_k)\, dx - \int_\Omega v_{k+1}\, \dot v_{k+1}\, W(Du_k + Dg_{k+1})\, dx \leq \\ &\quad \leq \int_\Omega v_k \dot v_{k+1} W(Du_{k-1} + Dg_k)\, dx - \int_\Omega v_{k+1} \dot v_{k+1} W(Du_{k-1} + Dg_k)\, dx + \\ &\quad\quad + \int_\Omega v_{k+1} \dot v_{k+1} W(Du_{k-1} + Dg_k)\, dx - \int_\Omega v_{k+1} \dot v_{k+1} W(Du_k + Dg_{k+1})\, dx \\ &\quad \leq \int_\Omega (v_k - v_{k+1}) \dot v_{k+1} W(Du_{k-1} + Dg_k)\, dx + \\ &\quad\quad + \int_\Omega v_{k+1} \dot v_{k+1} \big(W(Du_{k-1} + Dg_k) - W(Du_k + Dg_{k+1}) \big)\, dx \\ &\quad \leq \int_\Omega v_{k+1} \dot v_{k+1} \big(W(Du_{k-1} + Dg_k) - W(Du_k + Dg_{k+1}) \big)\, dx, \end{aligned}$$

where last inequality follows from $(v_k - v_{k+1})\, \dot v_{k+1}\, W(Du_k + Dg_k) \leq 0$. For $1/s + 2/p = 1$ we get by Hölder inequality

$$\int_\Omega \dot v_{k+1} v_{k+1} \big(W(Du_{k-1} + Dg_k) - W(Du_k + Dg_{k+1}) \big)\, dx \leq \\ \leq \|\dot v_{k+1}\|_{L^s}\, \|W(Du_{k-1} + Dg_k) - W(Du_k + Dg_{k+1})\|_{L^{p/2}}.$$

Since u_k is uniformly bounded in $W^{1,p}(\Omega, \mathbb{R}^2)$ for $2 < p < \bar p$ (by Lemma A.4) and since $g \in W^{1,\infty}(0, T; W^{1,\bar p}(\Omega, \mathbb{R}^2)$ we get

$$\sup_k \|(Du_{k-1} + Dg_k) + (Du_k + Dg_{k+1})\|_{L^p} < +\infty,$$

$$\begin{aligned} \|(Du_{k-1} + Dg_k) - (Du_k + Dg_{k+1})\|_{L^p} &\leq C(|t_k - t_{k-1}| + \|v_k - v_{k-1}\|_{L^r} + |t_{k+1} - t_k|) \\ &\leq C\tau (1 + \|\dot v_k\|_{L^r}), \end{aligned}$$

where $1/r = 1/p - 1/\bar{p}$. Thus for $q = s \vee r$ we have

$$\int_\Omega \dot{v}_{k+1} v_{k+1} \big(W(Du_{k-1} + Dg_k) - W(Du_k + Dg_{k+1}) \big) \, dx \leq C\tau \|\dot{v}_{k+1}\|_{L^q} (1 + \|\dot{v}_k\|_{L^q}).$$

In conclusion, for $k \geq 0$ we have

$$\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} - \tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} \leq -\tau G_c \|\dot{v}_{k+1}\|^2_{H^1} + C\tau \|\dot{v}_{k+1}\|_{L^q}(1 + \|\dot{v}_k\|_{L^q}). \tag{36}$$

Step II. Let us see that for every $0 < \delta \ll 1$ there exists C_δ such that

$$C\|\dot{v}_{k+1}\|_{L^q}(1 + \|\dot{v}_k\|_{L^q}) \leq C_\delta (1 + \|\dot{v}_{k+1}\|_{L^1} \|\dot{v}_{k+1}\|_{L^2}) + \delta \|\dot{v}_{k+1}\|^2_{H^1} + \delta \|\dot{v}_k\|^2_{H^1}. \tag{37}$$

By Young's inequality, for $0 < \mu \ll 1$ and $C_\mu > 1$

$$\|\dot{v}_{k+1}\|_{L^q}(1 + \|\dot{v}_k\|_{L^q}) \leq \|\dot{v}_{k+1}\|_{L^q} + C_\mu \|\dot{v}_{k+1}\|^2_{L^q} + \mu \|\dot{v}_k\|^2_{L^q} \leq C'_\mu (1 + \|\dot{v}_{k+1}\|^2_{L^q}) + \mu \|\dot{v}_k\|^2_{L^q}.$$

Write $1/q = \alpha + (1-\alpha)/\bar{q}$ for $\alpha \in (0,1)$ and $q < \bar{q} < +\infty$. Then, by interpolation and Young's inequality, with $p = 1/\alpha$, for $0 < \lambda \ll 1$ we have

$$\begin{aligned} \|\dot{v}_{k+1}\|^2_{L^q} &\leq \|\dot{v}_{k+1}\|^{2\alpha}_{L^1} \, \|\dot{v}_{k+1}\|^{2(1-\alpha)}_{L^{\bar{q}}} \leq C_\lambda \|\dot{v}_{k+1}\|^2_{L^1} + \lambda \|\dot{v}_{k+1}\|^2_{L^{\bar{q}}} \\ &\leq C'_\lambda \|\dot{v}_{k+1}\|_{L^1} \|\dot{v}_{k+1}\|_{L^2} + \lambda \|\dot{v}_{k+1}\|^2_{L^{\bar{q}}}. \end{aligned} \tag{38}$$

By embedding $\|\dot{v}_{k+1}\|^2_{L^{\bar{q}}} \leq C \|\dot{v}_{k+1}\|^2_{H^1}$ and $\|\dot{v}_k\|^2_{L^{\bar{q}}} \leq C \|\dot{v}_k\|^2_{H^1}$. Hence, upon choosing μ and λ sufficiently small we can write (37). Joining (36) and (37) yields the estimate

$$\begin{aligned} &\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} - \tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} \leq \\ &\qquad \leq -\tau G_c \|\dot{v}_{k+1}\|^2_{H^1} + \tau\delta \|\dot{v}_{k+1}\|^2_{H^1} + \tau\delta \|\dot{v}_k\|^2_{H^1} + \tau C_\delta \|\dot{v}_{k+1}\|_{L^1} \|\dot{v}_{k+1}\|_{L^2} + \tau C_\delta. \end{aligned} \tag{39}$$

Step III. In order to apply the discrete Gronwall Lemma A.1 we need to re-write (39). First,

$$\begin{aligned} \tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} - \tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} \leq &- \tau (G_c - 2\delta) \|\dot{v}_{k+1}\|^2_{H^1} - \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} + \delta\tau \|\dot{v}_k\|^2_{H^1} \\ &+ \tau C_\delta \|\dot{v}_{k+1}\|_{L^1} \|\dot{v}_{k+1}\|_{L^2} + \tau C_\delta. \end{aligned}$$

Hence,

$$\begin{aligned} &\big(\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} + \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} \big) - \big(\tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} + \delta\tau \|\dot{v}_k\|^2_{H^1} \big) \leq \\ &\qquad - \tau (G_c - 2\delta) \|\dot{v}_{k+1}\|^2_{H^1} + \tau C_\delta \|\dot{v}_{k+1}\|_{L^1} \|\dot{v}_{k+1}\|_{L^2} + \tau C_\delta. \end{aligned}$$

For $\tau, \delta \ll 1$ and $\gamma > 0$ we can write

$$\gamma \left(\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} + \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} \right) \leq (G_c - 2\delta) \|\dot{v}_{k+1}\|^2_{H^1}.$$

Therefore, we get

$$\begin{aligned} &\big(\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} + \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} \big) - \big(\tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} + \delta\tau \|\dot{v}_k\|^2_{H^1} \big) \leq -\gamma\tau \left(\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} + \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} \right) + \\ &\qquad + C'_\delta \tau \|\dot{v}_{k+1}\|_{L^1} \left(\tfrac{1}{2}\|\dot{v}_{k+1}\|^2_{L^2} + \delta\tau \|\dot{v}_{k+1}\|^2_{H^1} \right)^{1/2} + C_\delta \tau. \end{aligned} \tag{40}$$

Define

$$a_k = \left(\tfrac{1}{2}\|\dot{v}_k\|^2_{L^2} + \delta\tau \|\dot{v}_k\|^2_{H^1} \right)^{1/2}, \qquad b_k = C'_\delta \|\dot{v}_k\|_{L^1}, \qquad c_k^2 = C_\delta.$$

Hence (40) reads: for every $0 \leq k \leq m-1$

$$a^2_{k+1} - a^2_k \leq -\tau\gamma a^2_{k+1} + \tau a_{k+1} b_{k+1} + \tau c^2_{k+1}.$$

Then, for $0 < \beta < \gamma/2$ by Lemma A.1 we get

$$a_k \le \Big(\sum_{i=0}^{k} \tau e^{-2\beta(t_k - t_i)} c_i^2 \Big)^{1/2} + \sum_{i=0}^{k} \tau e^{-\beta(t_k - t_i)} b_i .$$

Remembering the definition of a_k, b_k and c_k, the previous estimate gives

$$\begin{aligned} \tfrac12 \|\dot v_k\|_{L^2} &\le \big(\tfrac12 \|\dot v_k\|^2_{L^2} + \delta\tau \|\dot v_k\|^2_{H^1} \big)^{1/2} \le C \Big(\sum_{i=0}^{k} e^{-2\beta(t_k - t_i)} \tau \Big)^{1/2} + C \sum_{i=0}^{k} \tau e^{-\beta(t_k - t_i)} \|\dot v_k\|_{L^1} \\ &\le C \Big(\int_0^{t_k} e^{-2\beta(t_k - r)} \, dr \Big)^{1/2} + C \int_0^{t_k} e^{-\beta(t_k - r)} \|\dot v(r)\|_{L^1} \, dr \\ &\le C/2\beta + C \int_0^{t_k} \|\dot v(r)\|_{L^1} \, dr \le C'(1 + |\Omega|), \end{aligned}$$

where the last inequality follows from monotonicity (in time) and boundedness of v. Hence $v \in W^{1,\infty}(0,T;L^2)$.

Step IV. Let us go back to (39), i.e.

$$\begin{aligned} &\tfrac12 \|\dot v_{k+1}\|^2_{L^2} - \tfrac12 \|\dot v_k\|^2_{L^2} \le \\ &\qquad \le -\tau(G_c - \delta) \|\dot v_{k+1}\|^2_{H^1} + \tau\delta \|\dot v_k\|^2_{H^1} + \tau C_\delta \|\dot v_{k+1}\|_{L^1} \|\dot v_{k+1}\|_{L^2} + \tau C_\delta . \end{aligned}$$

Let $0 < C = G_c - \delta$ for $0 < \delta \ll 1$; being $\|\dot v_k\|_{L^2}$ uniformly bounded (by the previous step) the above estimate can be written as

$$\tau C \|\dot v_{k+1}\|^2_{H^1} \le \big(\tfrac12 \|\dot v_k\|^2_{L^2} - \tfrac12 \|\dot v_{k+1}\|^2_{L^2} \big) + \tau\delta \|\dot v_k\|^2_{H^1} + \tau C'_\delta (1 + \|\dot v_{k+1}\|_{L^1}).$$

Taking the sum for $k = 0, ..., m-1$ gives

$$\begin{aligned} C \int_0^T \|\dot v\|^2_{H^1} \, dt = C \sum_{k=0}^{m-1} \tau \|\dot v_{k+1}\|^2_{H^1} &\le \sum_{k=0}^{m-1} \big(\tfrac12 \|\dot v_k\|^2_{L^2} - \tfrac12 \|\dot v_{k+1}\|^2_{L^2} \big) + \\ &\quad + \delta \sum_{k=0}^{m-1} \tau \|\dot v_k\|^2_{H^1} + C'_\delta \sum_{k=0}^{m-1} \tau (1 + \|\dot v_{k+1}\|_{L^1}) \\ &\le \tfrac12 \|\dot v_0\|^2_{L^2} - \tfrac12 \|\dot v_m\|^2_{L^2} + \delta \int_0^T \|\dot v\|^2_{H^1} \, dt + C'_\delta \int_0^T 1 + \|\dot v_{k+1}\|_{L^1} \, dt \\ &\le \delta \int_0^T \|\dot v\|^2_{H^1} \, dt + C(T + |\Omega|). \end{aligned} \tag{41}$$

Since $0 < \delta \ll 1$ it follows that $v \in H^1(0,T;H^1)$. ∎

Theorem 3.10 *Let v, u be the limit evolution obtained by Lemma 3.5 then for a.e. $t \in [0,T]$ it holds*

$$\begin{cases} \dot v(t) = -\big[v(t) W(D\tilde u(t)) + G_c(v(t) - 1) - G_c \Delta v(t) \big]^+ \\ \operatorname{div}\big(\boldsymbol\sigma_{v(t)}(\tilde u(t)) \big) = 0 \end{cases} \tag{42}$$

where $v(t)W(D\tilde u(t)) + G_c(v(t) - 1) - G_c\Delta v(t)$ is a (Radon) measure in Ω, with positive part in $L^2(\Omega)$, and $\boldsymbol\sigma_v(\tilde u) = (v^2 + \eta)\,\mathbf{C} D\tilde u$ is the phase-field stress.

Proof. By the chain rule, cf. Lemma A.6, for a.e. $t \in [0,T]$ we have

$$\dot{\mathcal F}(t, u(t), v(t)) = \partial_t \mathcal F(t, u(t), v(t)) + \partial_v \mathcal F(t, u(t), v(t)) \, [\dot v(t)] \, .$$

On the other hand, by the energy balance

$$\dot{\mathcal{F}}(t,u(t),v(t)) = -\tfrac{1}{2}\|\dot{v}(t)\|^2_{L^2} - \tfrac{1}{2}|\partial_v^- \mathcal{F}(r,u(t),v(t))|^2_{L^2} + \partial_t \mathcal{F}(t,u(t),v(t)).$$

Hence, by Young's inequality

$$\begin{aligned} -|\partial_v^- \mathcal{F}(t,u(t),v(t))|_{L^2}\,\|\dot{v}(t)\|_{L^2} &\le \partial_v \mathcal{F}(t,u(t),v(t))\,[\dot{v}(t)] \\ &\le -\tfrac{1}{2}\|\dot{v}(t)\|^2_{L^2} - \tfrac{1}{2}|\partial_v^- \mathcal{F}(r,u(t),v(t))|^2_{L^2} \\ &\le -|\partial_v^- \mathcal{F}(r,u(t),v(t))|_{L^2}\,\|\dot{v}(t)\|_{L^2}. \end{aligned}$$

It turna out that all inequalities are equalities and thus $\|\dot{v}(t)\|_{L^2} = |\partial_v^- \mathcal{F}(r,u(t),v(t))|_{L^2}$ and

$$\dot{v}(t) \in \operatorname{argmin}\{\partial_v \mathcal{F}(t,u(t),v(t))[\xi] : \xi \in H^1,\ \xi \le 0,\ \|\xi\|_{L^2} \le R\}, \tag{43}$$

where $R = \|\dot{v}(t)\|_{L^2} = |\partial_v^- \mathcal{F}(t,u(t),v(t))|_{L^2}$. Equivalently,

$$-\dot{v}(t) \in \operatorname{argmax}\{\partial_v \mathcal{F}(t,u(t),v(t))[\xi] : \xi \in H^1,\ \xi \ge 0,\ \|\xi\|_{L^2} \le R\}.$$

If $\zeta = \partial_v \mathcal{F}(t,u(t),v(t)) \neq 0$ we get by Lemma A.3

$$-\dot{v}/R = \zeta^+/\|\zeta^+\|_{L^2} = \zeta^+/R$$

and thus $\dot{v}(t) = [-\partial_v \mathcal{F}(t,u(t),v(t))]^+$ in the sense of distributions and in $L^2(\Omega)$. If $R = 0$ the same equation clearly holds. Hence, for $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} \partial_v \mathcal{F}(t,u(t),v(t))[\phi] &= \int_\Omega v\phi W(D\tilde{u}(t))\,dx + G_c \int_\Omega (v(t)-1)\phi + \nabla v \cdot \nabla \phi\,dx \\ &= \langle vW(D\tilde{u}(t)) + G_c(v(t)-1) - G_c \Delta v(t), \phi\rangle, \end{aligned} \tag{44}$$

where the duality is in the sense of distributions.

For the second equation it is sufficient to write the Euler-Lagrange equation

$$\int_\Omega (v^2+\eta)\mathbf{C} D\tilde{u}(t) : D\phi\,dx = 0 \qquad \text{for every } \phi \in C_0^\infty(\Omega,\mathbb{R}^2)$$

in distributional form. ∎

4 Time rescaling

For $\varepsilon > 0$ let us consider the boundary condition $g_\varepsilon(t) = g(\varepsilon t)$ defined in $[0,T_\varepsilon]$, for $T_\varepsilon = T/\varepsilon$. Clearly g_ε is Lipschitz continuous in $W^{1,\bar{p}}(\Omega,\mathbb{R}^2)$ with

$$\|g_\varepsilon(t_2) - g_\varepsilon(t_1)\|_{W^{1,\bar{p}}} \le \varepsilon C|t_2 - t_1|.$$

Next, we define $\mathcal{F}_\varepsilon : [0,T_\varepsilon] \times \mathcal{U} \times \mathcal{V} \to [0,+\infty)$ by

$$\mathcal{F}_\varepsilon(t,u,v) = \tfrac{1}{2}\int_\Omega (v^2+\eta) W(Du + Dg_\varepsilon(t))\,dx + \tfrac{1}{2}\,G_c \int_\Omega (v-1)^2 + |\nabla v|^2\,dx.$$

As in §3 fix $\tau = T_\varepsilon/m > 0$ and let $t_k = k\tau$ for $k = 0,\dots,m$. Given u_{k-1} and v_{k-1} define by induction

$$\begin{cases} v_k \in \operatorname{argmin}\,\left\{\mathcal{F}_\varepsilon(t_k,u_{k-1},v) + \tfrac{1}{2\tau}\|v - v_{k-1}\|^2_{L^2} : v \le v_{k-1}\,,\ v \in \mathcal{V}\right\} \\ u_k \in \operatorname{argmin}\,\{\mathcal{F}_\varepsilon(t_k,u,v_k) : u \in \mathcal{U}\}\,. \end{cases} \tag{45}$$

Now, consider a sequence $\tau_m = T_\varepsilon/m$, for $m \in \mathbb{N}$ with $m > 0$, and denote by $u_{\varepsilon,m}$ and $v_{\varepsilon,m}$ the corresponding piecewise affine interpolate. By Lemma 3.5 together with Theorem 3.6 we easily get the following convergence result.

Theorem 4.1 *There exists a subsequence (not relabelled) of $v_{\varepsilon,m}$ such that $v_{\varepsilon,m} \rightharpoonup v_\varepsilon$ in $H^1(0,T_\varepsilon;L^2(\Omega))$. Let u_ε be the corresponding pointwise limit. Then, for every $t \in [0,T_\varepsilon]$ it holds*

$$\begin{aligned} \mathcal{F}_\varepsilon(t,u_\varepsilon(t),v_\varepsilon(t)) = \mathcal{F}_\varepsilon(0,u_0,v_0) - \tfrac{1}{2}\int_0^t \|\dot v_\varepsilon(r)\|^2_{L^2} + |\partial^-_v \mathcal{F}_\varepsilon(r,u_\varepsilon(r),v_\varepsilon(r))|^2_{L^2}\,dr\ + \\ + \int_{t_0}^t \partial_t \mathcal{F}_\varepsilon(r,u_\varepsilon(r),v_\varepsilon(r))\,dr, \end{aligned} \tag{46}$$

Moreover for every $t \in [0,T_\varepsilon]$ we have $u_\varepsilon(t) \in \operatorname{argmin}\{\mathcal{E}(t,v_\varepsilon(t),u) : u \in \mathcal{U}\}$ and

$$\|\dot v_\varepsilon(t)\|_{L^2} = |\partial^-_v \mathcal{F}_\varepsilon(t,u_\varepsilon(t),v_\varepsilon(t))|_{L^2}. \tag{47}$$

Corollary 4.2 *For every $\lambda \in [0,1]$ it holds*

$$\begin{aligned} \mathcal{F}_\varepsilon(t,u_\varepsilon(t),v_\varepsilon(t)) = \mathcal{F}_\varepsilon(0,u_0,v_0) + \int_0^t \partial_t \mathcal{F}_\varepsilon(r,u_\varepsilon(r),v_\varepsilon(r))\,dr\ + \\ - \int_0^t \lambda\,\|\dot v_\varepsilon(r)\|^2_{L^2} + (1-\lambda)|\partial_v \mathcal{F}_\varepsilon(r,u_\varepsilon(r),v_\varepsilon(r))|^2_{L^2}\,dr. \end{aligned} \tag{48}$$

Proof. It is sufficient to re-write (46) taking into account (47). ■

We remark that in general (48) does not provide a characterization of the gradient flow, unless it holds for $\lambda = 1/2$. For instance, if $\mathcal{F}_\varepsilon$ is independent of time then $t \mapsto (u_0,v_0)$ is a solution of (48) for $\lambda = 1$ independently of $|\partial_v \mathcal{F}_\varepsilon(u_0,v_0)|^2_{L^2}$.

5 Quasi-static limit of the rescaled evolutions

In this section we will apply the change of variable

$$t \mapsto s^\varepsilon(t) = \varepsilon t + \int_0^t \|\dot v_\varepsilon(r)\|_{L^2}\,dr$$

in order to obtain a parametrization of the evolution v_ε and u_ε (defined for $t \in [0,T/\varepsilon]$) in terms of an arc-length parameter $s \in [0,S_\varepsilon]$. First of all, let us see that s_ε maps the "physical" time interval $[0,T/\varepsilon]$ onto a reference parametrization interval $[0,S_\varepsilon]$ with $S_\varepsilon = s^\varepsilon(T_\varepsilon)$ uniformly bounded with respect to $\varepsilon > 0$. We will prove this property employing again the time discretization scheme.

5.1 Finite length

Theorem 5.1 *The length of the discrete curves $v_{\varepsilon,m}$ is uniformly bounded in L^2, i.e., there exists $C > 0$ (independent of ε and m) such that for τ_m sufficiently small it holds*

$$\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}(t)\|_{L^2}\,dt \le C(\varepsilon T + |\Omega|).$$

Moreover,

$$\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}(t)\|^2_{H^1}\,dt \le C(\varepsilon T + |\Omega|).$$

Proof. The first part follows step by step the proof of Theorem 3.9. For sake of simplicity we will drop the dependence on ε and m in the discrete evolution (45).

Step I. Replacing $\mathcal{F}$ with $\mathcal{F}_\varepsilon$ in (36) yields for every $k \ge 0$

$$\tfrac{1}{2}\|\dot v_{k+1}\|^2_{L^2} - \tfrac{1}{2}\|\dot v_k\|^2_{L^2} \le -\tau G_c \|\dot v_{k+1}\|^2_{H^1} + C\tau\|\dot v_{k+1}\|_{L^q}(\varepsilon + \|\dot v_k\|_{L^q}) \tag{49}$$

where ε in the right hand comes from the Lipschitz continuity of g_ε.

Step II. By Young's inequality for $0 < \mu \ll 1$ we get

$$\|\dot v_{k+1}\|_{L^q}(\varepsilon + \|\dot v_k\|_{L^q}) \le \varepsilon\|\dot v_{k+1}\|_{L^q} + C_\mu\|\dot v_{k+1}\|^2_{L^q} + \mu\|\dot v_k\|^2_{L^q} \le C'_\mu(\varepsilon^2 + \|\dot v_{k+1}\|^2_{L^q}) + \mu\|\dot v_k\|^2_{L^q}.$$

By interpolation and embedding, for $0 < \lambda \ll 1$ we get, as in (38),

$$\|\dot v_{k+1}\|^2_{L^q} \le C'_\lambda\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + C\lambda\|\dot v_{k+1}\|^2_{H^1}.$$

Then, for every $0 < \delta \ll 1$, upon choosing μ and λ sufficiently small, from (49) we get

$$\begin{aligned} \tfrac12\|\dot v_{k+1}\|^2_{L^2} - \tfrac12\|\dot v_k\|^2_{L^2} \le& -\tau G_c\|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_k\|^2_{H^1} + \\ &+\tau C_\delta\|\dot v_{k+1}\|_{L^1}\|\dot v_{k+1}\|_{L^2} + \tau C_\delta\varepsilon^2. \end{aligned} \tag{50}$$

Step III. In order to apply the discrete Gronwall Lemma A.1 we re-write (50) as (cf. (40))

$$\begin{aligned} \left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right) - \left(\tfrac12\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\right) \le -\gamma\tau\left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right) + \\ +C'_\delta\tau\|\dot v_{k+1}\|_{L^1}\left(\tfrac12\|\dot v_{k+1}\|^2_{L^2} + \delta\tau\|\dot v_{k+1}\|^2_{H^1}\right)^{1/2} + C_\delta\tau\varepsilon^2. \end{aligned} \tag{51}$$

Defining

$$a_k = \left(\tfrac12\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\right)^{1/2}, \qquad b_k = C'_\delta\|\dot v_k\|_{L^1}, \qquad c_k^2 = C_\delta\varepsilon^2,$$

(51) becomes

$$a_{k+1}^2 - a_k^2 \le -\tau\gamma a_{k+1}^2 + \tau a_{k+1}b_{k+1} + \tau c_{k+1}^2.$$

By Lemma A.1 we get

$$a_k \le \left(\sum_{i=0}^k \tau e^{-2\beta(t_k-t_i)}c_i^2\right)^{1/2} + \sum_{i=0}^k \tau e^{-\beta(t_k-t_i)}b_i$$

and then

$$\begin{aligned} \tfrac12\|\dot v_k\|_{L^2} \le \left(\tfrac12\|\dot v_k\|^2_{L^2} + \delta\tau\|\dot v_k\|^2_{H^1}\right)^{1/2} &\le C\left(\sum_{i=0}^k e^{-2\beta(t_k-t_i)}\tau\varepsilon^2\right)^{1/2} + C\sum_{i=0}^k \tau e^{-\beta(t_k-t_i)}\|\dot v_k\|_{L^1} \\ &\le C\varepsilon\left(\int_0^{t_k} e^{-2\beta(t_k-r)}\,dr\right)^{1/2} + C\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v(r)\|_{L^1}\,dr \\ &\le C\varepsilon(2\beta)^{-1/2} + C\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v(r)\|_{L^1}\,dr. \end{aligned} \tag{52}$$

In particular $\|\dot v_k\|_{L^2} \le C(\varepsilon + |\Omega|)$. Moreover,

$$\int_0^{T/\varepsilon} \|\dot v_{\varepsilon,m}(t)\|_{L^2}\,dt \le \sum_{k=1}^m \tau\|\dot v_k\|_{L^2} \le C\varepsilon T + C\sum_{k=1}^m \tau\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v(r)\|_{L^1}\,dr.$$

Then, for $t \in [t_k, t_{k+1}]$ we can write

$$\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v(r)\|_{L^1}\,dr \le \int_0^t e^{-\beta(t-\tau-r)}\|\dot v(r)\|_{L^1}\,dr$$

and thus

$$\sum_{k=1}^m \tau\int_0^{t_k} e^{-\beta(t_k-r)}\|\dot v(r)\|_{L^1}\,dr \le \int_0^{T/\varepsilon}\int_0^t e^{-\beta(t-\tau-r)}\|\dot v(r)\|_{L^1}\,dr\,dt.$$

By Fubini's Theorem

$$\int_0^{T/\varepsilon} \int_0^t e^{-\beta(t-\tau-r)} \|\dot v(r)\|_{L^1}\, dr\, dt \le e^{\beta\tau} \int_0^{T/\varepsilon} \|\dot v(r)\|_{L^1} \int_r^{T/\varepsilon} e^{-\beta(t-r)}\, dt\, dr$$
$$\le C \int_0^{T/\varepsilon} \|\dot v(r)\|_{L^1} dr \le C|\Omega|.$$

Step IV. We back to (50), i.e.

$$\tfrac12 \|\dot v_{k+1}\|^2_{L^2} - \tfrac12 \|\dot v_k\|^2_{L^2} \le$$
$$\le -\tau G_c \|\dot v_{k+1}\|^2_{H^1} + \tau\delta \|\dot v_{k+1}\|^2_{H^1} + \tau\delta\|\dot v_k\|^2_{H^1} + \tau C_\delta \|\dot v_{k+1}\|_{L^1} \|\dot v_{k+1}\|_{L^2} + \tau C_\delta \varepsilon^2.$$

For $0 < \delta \ll 1$ by (52) the previous estimate becomes

$$\tau C \|\dot v_{k+1}\|^2_{H^1} \le \big(\tfrac12 \|\dot v_k\|^2_{L^2} - \tfrac12 \|\dot v_{k+1}\|^2_{L^2}\big) + \tau\delta \|\dot v_k\|^2_{H^1} + \tau C'_\delta (\varepsilon^2 + \|\dot v_{k+1}\|_{L^1}).$$

Taking the sum for $k = 1, ..., m$ provides (cf. (41))

$$\int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}\|^2_{H^1}\, dt \le \delta \int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}\|^2_{H^1}\, dt + C T_\varepsilon \varepsilon^2 + C \int_0^{T_\varepsilon} \|\dot v_{\varepsilon,m}\|_{L^1}\, dt,$$

which concludes the proof. ∎

Passing to the limit for $\tau_m \to 0$ we get the following result.

Corollary 5.2 *The limit evolution v_ε (provided by Theorem 4.1) satisfies*

$$\int_0^{T_\varepsilon} \|\dot v_\varepsilon(t)\|_{L^2} + \|\dot v_\varepsilon(t)\|^2_{H^1}\, dt \le C(\varepsilon T + |\Omega|).$$

Hence $S_\varepsilon = s^\varepsilon(T/\varepsilon)$ is uniformly bounded.

5.2 Rescaled parametrized gradient flows

Let us go back to our parametrization

$$t \mapsto s^\varepsilon(t) = \varepsilon t + \int_0^t \|\dot v_\varepsilon(r)\|_{L^2}\, dr \tag{53}$$

from $[0, T_\varepsilon]$ onto $[0, S_\varepsilon]$. The map $t \mapsto s^\varepsilon(t)$ is absolutely continuous and strictly monotone; let $t^\varepsilon(s)$ be its inverse. Denote also

$$t_\varepsilon(s) = \varepsilon t^\varepsilon(s), \qquad z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s), \qquad w_\varepsilon(s) = u_\varepsilon \circ t^\varepsilon(s). \tag{54}$$

Accordingly, let $w_0 = u_0$ and $z_0 = v_0$.

Lemma 5.3 *The functions $s \mapsto t_\varepsilon(s)$ and $s \mapsto z_\varepsilon(s)$ are Lipschitz continuous in $[0, S_\varepsilon]$, more precisely for a.e. $s \in [0, S_\varepsilon]$ it holds*

$$t'_\varepsilon(s) + \|z'_\varepsilon(s)\|_{L^2} = 1, \qquad t'_\varepsilon(s) = \frac{\varepsilon}{\varepsilon + |\partial^-_v \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}}.$$

Proof. As $t \mapsto s^\varepsilon(t)$ is absolutely continuous with $\dot s^\varepsilon(t) \ge \varepsilon$ a.e. in $[0, T_\varepsilon]$ the inverse function $s \mapsto t^\varepsilon(s)$ turns out to be Lipschitz continuous with $(t^\varepsilon)'(s) = 1/\dot s^\varepsilon(t^\varepsilon(s))$ a.e. in $[0, S_\varepsilon]$. Hence, by (53) and (54)

$$1 = \dot s^\varepsilon(t^\varepsilon(s))\, (t^\varepsilon)'(s) = (\varepsilon + \|\dot v_\varepsilon(t^\varepsilon(s))\|_{L^2})\, (t^\varepsilon)'(s) = t'_\varepsilon(s) + \|z'_\varepsilon(s)\|_{L^2}.$$

Moreover, by (21) for a.e. $t \in [0, T_\varepsilon]$ we have

$$\|\dot{v}_\varepsilon(t)\|_{L^2} = |\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2} = |\partial_v^- \mathcal{F}(\varepsilon t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2}.$$

Thus

$$\|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2} = |\partial_v^- \mathcal{F}(\varepsilon t^\varepsilon(s), u_\varepsilon \circ t^\varepsilon(s), v_\varepsilon \circ t^\varepsilon(s))|_{L^2} = |\partial_z^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}.$$

Since sets of measure zero are mapped to sets of measure zero, both by $s \mapsto t^\varepsilon(s)$ and by $t \mapsto s^\varepsilon(t)$, for a.e. $s \in [0, S_\varepsilon]$ we have

$$t'_\varepsilon(s) = \frac{\varepsilon}{\dot{s}^\varepsilon(t^\varepsilon(s))} = \frac{\varepsilon}{\varepsilon + \|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2}} = \frac{\varepsilon}{\varepsilon + |\partial_v \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2}}, \tag{55}$$

which concludes the proof. ■

Lemma 5.4 *For $\varepsilon > 0$, the rescaled parametrized evolutions $(t_\varepsilon, z_\varepsilon)$ are (uniformly) bounded in $W^{1,\infty}(0, S_\varepsilon; [0,T] \times L^2)$ and in $L^\infty(0, S_\varepsilon; [0,T] \times \mathcal{V})$ with $t'_\varepsilon \geq 0$, $z'_\varepsilon \leq 0$ and $t'_\varepsilon + \|z'_\varepsilon\|_{L^2} \leq 1$. Further, for every $s \in [0, S_\varepsilon]$ and every $\lambda \in [0,1]$ the following energy balance holds:*

$$\begin{aligned} \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s)) = {} & \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t_\varepsilon(r), w_\varepsilon(r), z_\varepsilon(r))\, t'_\varepsilon(r)\, dr + \\ & - \int_0^s \lambda \Psi_\varepsilon(\|z'_\varepsilon(r)\|_{L^2}) + (1-\lambda)\Phi_\varepsilon(|\partial_z^- \mathcal{F}(t_\varepsilon(r), w_\varepsilon(r), z_\varepsilon(r))|_{L^2})\, dr, \end{aligned} \tag{56}$$

where

$$\Psi_\varepsilon(\xi) = \begin{cases} \varepsilon \xi^2/(1-\xi) & 0 \leq \xi < 1 \\ +\infty & \xi \geq 1, \end{cases} \qquad \Phi_\varepsilon(\xi) = \xi^2/(\varepsilon + \xi).$$

We consider both Ψ_ε and Φ_ε to be defined in $[0, +\infty)$. Clearly, $w_\varepsilon(s) \in \operatorname{argmin} \{\mathcal{E}(t_\varepsilon(s), w, z_\varepsilon(s)) : w \in \mathcal{U}\}$

Proof. By Corollary 4.2 we known that for every $\bar{t} \in [0, T_\varepsilon]$ it holds

$$\begin{aligned} \mathcal{F}_\varepsilon(\bar{t}, u_\varepsilon(\bar{t}), v_\varepsilon(\bar{t})) = {} & \mathcal{F}_\varepsilon(0, u_0, v_0) + \int_0^{\bar{t}} \partial_t \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))\, dt + \\ & - \int_0^{\bar{t}} \lambda\, \|\dot{v}_\varepsilon(t)\|^2_{L^2} + (1-\lambda)|\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|^2_{L^2}\, dt. \end{aligned}$$

Remember that $\mathcal{F}_\varepsilon(t, u, v) = \mathcal{F}(\varepsilon t, u, v)$ and thus

$$\partial_t \mathcal{F}_\varepsilon(t, u, v) = \varepsilon\, \partial_t \mathcal{F}(\varepsilon t, u, v), \qquad \partial_v \mathcal{F}_\varepsilon(t, u, v) = \partial_v \mathcal{F}(\varepsilon t, u, v).$$

Hence, by the change of variable $t = t^\varepsilon(s) = t_\varepsilon(s)/\varepsilon$, the energy balance in parametrized form reads: for a.e. $\bar{s} \in [0, S_\varepsilon]$ it holds

$$\begin{aligned} \mathcal{F}(t_\varepsilon(\bar{s}), w_\varepsilon(\bar{s}), z_\varepsilon(\bar{s})) = {} & \mathcal{F}(0, w_0, z_0) + \int_0^{\bar{s}} \partial_t \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))\, t'_\varepsilon(s)\, ds + \\ & - \int_0^s \left[\lambda\, \|\dot{v}_\varepsilon(t^\varepsilon(s))\|^2_{L^2} + (1-\lambda)|\partial_v^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|^2_{L^2}\right] (t^\varepsilon)'(s)\, ds. \end{aligned}$$

Since $\|\dot{v}_\varepsilon(t^\varepsilon(s))\|_{L^2}\, (t^\varepsilon)'(s) = \|z'_\varepsilon(s)\|_{L^2}$ and $(t^\varepsilon)'(s) = t'_\varepsilon(s)/\varepsilon$ it follows by Lemma 5.3 that

$$\|\dot{v}_\varepsilon(t_\varepsilon(s))\|^2_{L^2}\, (t^\varepsilon)'(s) = \varepsilon \|z'_\varepsilon(s)\|^2_{L^2}/t'_\varepsilon(s) = \varepsilon \|z'_\varepsilon(s)\|^2_{L^2}/(1 - \|z'_\varepsilon(s)\|_{L^2}) = \Psi_\varepsilon(\|z'_\varepsilon(s)\|_{L^2}).$$

Again by Lemma 5.3, $(t^\varepsilon)'(s) = 1/(\varepsilon + |\partial_z^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2})$. ■

Since S_ε is uniformly bounded, by Corollary 5.2, we have $S = \liminf_\varepsilon S_\varepsilon < +\infty$. For compactness, it will be convenient to consider parametrized evolutions t_ε and z_ε to be defined in $[0, S]$ with a constant extension in $(S_\varepsilon, S]$ (clearly only in the case $S_\varepsilon < S$). In this way all the (possibly extended) evolutions enjoy the compactness properties of the previous Lemma in the parametrization interval $[0, S]$. Note however that, with this simple extension, the energy balance is not true, in general, for $s \in (S_\varepsilon, S]$. Using Lemma 5.3 and Lemma A.4 it is immediate to prove the following compactness property.

Corollary 5.5 *For $\varepsilon_n \to 0$ there exists a subsequence (not relabelled) such that*

$$(t_{\varepsilon_n}, z_{\varepsilon_n}) \overset{*}{\rightharpoonup} (t, z) \text{ in } W^{1,\infty}(0, S; [0, T] \times L^2).$$

Moreover, for a.e. $s \in [0, S]$ we have $z_{\varepsilon_n}(s) \rightharpoonup z(s)$ in H^1 and thus $w_{\varepsilon_n}(s) \to w(s)$ in $W^{1,p}$ (for $p > 2$) where $w(s) \in \operatorname{argmin}\{\mathcal{E}(t(s), w, z(s)) : w \in \mathcal{U}\}$.

5.3 Quasi-static limit

Theorem 5.6 *Every limit evolution obtained by Corollary 5.5 satisfies $z' \le 0$, $t' \ge 0$ and $t' + \|z'\|_{L^2} \le 1$. Moreover, for every $s \in [0, S]$ we have $w(s) \in \operatorname{argmin}\{\mathcal{E}(t(s), w, z(s)) : w \in \mathcal{U}\}$ and the following energy balance*

$$\mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \tag{57}$$

Any such limit is called a parametrized BV-evolution (cf. Proposition 5.10).

Proof. Part I. The proof follows closely that of [27, Theorem 4.4]. If $s < S$ then $s \in [0, S_{\varepsilon_n})$ for $\varepsilon_n \ll 1$; thus (56), with $\lambda = 0$, provides

$$\mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)) = \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(t))\, t'_{\varepsilon_n}(r)\, dr + \\ - \int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{58}$$

By Corollary 5.5 we known that $t_{\varepsilon_n}(s) \to t(s)$, $z_{\varepsilon_n}(s) \rightharpoonup z(s)$ in H^1 and $w_{\varepsilon_n}(s) \to w(s)$ in $W^{1,p}$ (for $p > 2$). As a consequence, by Lemma 2.2

$$\mathcal{F}(t(s), w(s), z(s)) \le \liminf_{\varepsilon_n \to 0} \mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)). \tag{59}$$

Next, taking the $\limsup_{\varepsilon_n \to 0}$ in (58) we get

$$\limsup_{\varepsilon_n \to 0} \mathcal{F}(t_{\varepsilon_n}(s), w_{\varepsilon_n}(s), z_{\varepsilon_n}(s)) \le \mathcal{F}(0, w_0, z_0) + \\ + \limsup_{\varepsilon_n \to 0} \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))\, t'_{\varepsilon_n}(r)\, dr \\ - \liminf_{\varepsilon_n \to 0} \int_0^s \Phi_{\varepsilon_n}(|\partial_z \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{60}$$

First, let us see that

$$\lim_{\varepsilon_n \to 0} \int_0^s \partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))\, t'_{\varepsilon_n}(r)\, dr = \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \tag{61}$$

By Lemma 2.3 we know that $\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(t)) \to \partial_t \mathcal{F}(t(r), w(r), z(r))$ for a.e. $r \in [0, s]$. Moreover $\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))$ is uniformly bounded since

$$|\partial_t \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))| \leq C(\mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r)) + 1) < \bar{C}.$$

Hence $\partial_t \mathcal{F}(t_{\varepsilon_n}(\cdot), w_{\varepsilon_n}(\cdot), z_{\varepsilon_n}(\cdot))$ converge to $\partial_t \mathcal{F}(t(\cdot), w(\cdot), z(\cdot))$ strongly in $L^1(0, s)$ (by dominated convergence). Since $t_{\varepsilon_n} \overset{*}{\rightharpoonup} t$ in $L^\infty(0, s)$ we get (61).

Finally, let us show that

$$\int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr \leq \liminf_{\varepsilon \to 0} \int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr. \tag{62}$$

It is not difficult to check that $\Phi_\varepsilon(\xi) \geq \xi - \varepsilon$ for every $\xi \in [0, +\infty)$. Thus we can write

$$\int_0^s \Phi_{\varepsilon_n}(|\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2})\, dr \geq \int_0^s |\partial_z^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2}\, dr - \varepsilon_n s.$$

By Lemma 2.2

$$|\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2} \leq \liminf_{\varepsilon_n \to 0} |\partial_v^- \mathcal{F}(t_{\varepsilon_n}(r), w_{\varepsilon_n}(r), z_{\varepsilon_n}(r))|_{L^2}$$

and thus (62) follows from Fatou's Lemma. Joining (59)-(62) yields

$$\mathcal{F}(t(s), u(s), v(s)) \leq \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr.$$

Part II. To prove the opposite inequality we employ the "upper gradient inequality" as in Proposition 3.8. In this setting, $t \in W^{1,\infty}(0, s)$, $z \in W^{1,\infty}(0, s; L^2(\Omega)) \cap L^\infty(0, s; \mathcal{V})$, $w(r) \in \operatorname{argmin}\{\mathcal{F}(t(r), u, z(r)) : u \in \mathcal{U}\}$ and $r \mapsto |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}$ belongs to $L^1(0, s)$. Then, following step by step the proof of Proposition 3.8 it is not difficult (but lengthy) to check that

$$\mathcal{F}(0, w_0, z_0) - \mathcal{F}(t(s), w(s), z(s)) \leq \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, \|z'(r)\|_{L^2}\, dr + \\ - \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr. \tag{63}$$

Since $\|z'(r)\|_{L^2} \leq 1$ we get

$$\mathcal{F}(0, w_0, z_0) - \mathcal{F}(t(s), w(s), z(s)) \leq \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ - \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr, \tag{64}$$

which concludes the proof. ■

Corollary 5.7 *If $s_{\varepsilon_n} \to s$ then $\mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \to \mathcal{F}(t(s), w(s), z(s))$ and $z_{\varepsilon_n}(s_{\varepsilon_n}) \to z(s)$ in H^1. Moreover $s \mapsto z(s)$ is continuous from $(0, S)$ to H^1.*

Proof. Following the proof of Theorem 5.6 it is easy to check that

$$\mathcal{F}(t(s), w(s), z(s)) \leq \liminf_{n \to +\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \\ \leq \limsup_{n \to +\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) \leq \mathcal{F}(t(s), w(s), z(s)).$$

Thus $\lim_{n\to+\infty} \mathcal{F}(t_{\varepsilon_n}(s_{\varepsilon_n}), w_{\varepsilon_n}(s_{\varepsilon_n}), z_{\varepsilon_n}(s_{\varepsilon_n})) = \mathcal{F}(t(s), w(s), z(s))$.

If $s_\varepsilon \to s$ then by compactness (cf. Corollary 5.5) $z_{\varepsilon_n}(s_{\varepsilon_n})$ converge to $z(s)$ weakly in H^1 and thus, by compact embedding, strongly in L^q for every $q < +\infty$. Since $t_{\varepsilon_n} \overset{*}{\rightharpoonup} t$ we get $t_{\varepsilon_n}(s_{\varepsilon_n}) \to t(s)$. Then by Lemma A.4 we have $w_\varepsilon(s_\varepsilon) \to w(s)$ in $W^{1,p}$ for some $p > 2$. Hence

$$\int_\Omega (z^2_{\varepsilon_n}(s_{\varepsilon_n}) + \eta) W(D\tilde{u}_{\varepsilon_n}(s_{\varepsilon_n}))\, dx \;\to\; \int_\Omega (z^2(s) + \eta) W(D\tilde{u}(s))\, dx,$$
$$\int_\Omega (z_{\varepsilon_n}(s_{\varepsilon_n}) - 1)^2\, dx \;\to\; \int_\Omega (z(s) - 1)^2\, dx.$$

By convergence of the energy it follows that

$$\int_\Omega |\nabla z_{\varepsilon_n}(s_{\varepsilon_n})|^2\, dx \;\to\; \int_\Omega |\nabla z(s)|^2\, dx,$$

from which follows the strong convergence in H^1. Since $z_{\varepsilon_n}(s_{\varepsilon_n}) \to z(s)$ in H^1 for every sequence $s_{\varepsilon_n} \to s$ we get that $z_{\varepsilon_n} \to z$ (strongly in H^1) locally uniformly in $(0, S)$.

Remember that, by Theorem 3.9, given $\varepsilon > 0$ the evolution v_ε is bounded in $W^{1,\infty}(0, T_\varepsilon; L^2) \cap H^1(0, T_\varepsilon; H^1)$ and thus it is continuous in H^1. As a consequence $s \mapsto z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s)$ is continuous from $[0, S_\varepsilon]$ to H^1. Since z_ε converge to z locally uniformly, its limit z is continuous as well. ■

Remark 5.8 *Using the Legendre transform it is possible to write* (57) *"in gradient flow fashion". Let*

$$\tilde{\Psi}(z) = \begin{cases} 0 & z \le 1 \\ +\infty & z > 1, \end{cases} \qquad \tilde{\Phi}(z) = \begin{cases} +\infty & z < 0 \\ z & z \ge 0\,. \end{cases}$$

Note that $\tilde{\Phi}(z) = \tilde{\Psi}^*(z)$. *With this notation* (57) *reads*

$$\mathcal{F}(t(s), w(s), z(s)) = \mathcal{F}(0, w_0, z_0) + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr \,+ \\ - \int_0^s \tilde{\Psi}(\|z'(r)\|_{L^2}) + \tilde{\Psi}^*(|\partial_v^- \mathcal{F}(t(r), w(r), z(r))|_{L^2})\, dr\,. \tag{65}$$

5.4 From energy balance to PDEs

In this last subsection we provide some properties, in terms of PDEs, of the parametrized evolution characterized by Theorem 5.6. Intuitively such an evolution is an "arc-length" parametrization of a BV-evolution [24, 25].

Remember that quasi-static evolutions for non-convex energies may have discontinuity in time and that characterization of these points makes the difference between different notion of quasi-static evolution, e.g. energetic, BV or local [24, 25]. Remember also that discontinuity points t_d (in time) correspond in the parametric picture to intervals $(s^\flat, s^\sharp)$ with $t(s) = t_d$, $z(s^\flat) = z^-(t_d)$ and $z(s^\sharp) = z^+(t_d)$. "Vice versa" if $t'(s_c) > 0$ then $t_c = t(s_c)$ is a continuity point in time.

Most of the informations are provided by the relationship between the derivative $t'(s)$ and the slope $|\partial_v^- \mathcal{F}(t(s), w(s), z(s))|_{L^2}$, which is the subject of Proposition 5.10 ; its PDEs form is provided in Corollary 5.11. First, in order to employ the chain rule, we prove the following lemma.

Lemma 5.9 *Any limit z, provided by Theorem 5.6, belongs to $W^{1,2}_{loc}(0, S; H^1)$.*

Proof. Remember that $z_\varepsilon(s) = v_\varepsilon \circ t^\varepsilon(s)$ and that $(t^\varepsilon)'(s) = 1/\dot{s}^\varepsilon(t^\varepsilon(s))$ (being t^ε the inverse of

s^ε); then, for $s_1 < s_2$ by the change of variable $s = s^\varepsilon(t)$ we get

$$\begin{aligned}\int_{s_1}^{s_2} \|z'_\varepsilon(s)\|^2_{H^1}\, ds &= \int_{s_1}^{s_2} \|\dot v_\varepsilon(t^\varepsilon(s))\|^2_{H^1} |(t^\varepsilon)'(s)|^2\, ds \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|^2_{H^1}}{\dot s^\varepsilon(t)}\, dt \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|^2_{H^1}}{\varepsilon + \|\dot v_\varepsilon(t)\|_{L^2}}\, dt \\ &= \int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|^2_{H^1}}{\varepsilon + |\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)|_{L^2}}\, dr.\end{aligned}$$

Now, let $\bar s \in [0, S]$ such that $|\partial_v^- \mathcal{F}(t(\bar s), w(\bar s), z(\bar s))|_{L^2} \neq 0$. By the lower semi-continuity of the slope (cf. Lemma 2.2) for $\delta \ll 1$ it holds $|\partial_v^- \mathcal{F}(t, w, z)|_{L^2} \ge C > 0$ for

$$(t, z) \in I_\delta \times B_\delta = \{|t - t(\bar s)| \le \delta\} \times \{\|z - z(\bar s)\|_{H^1} \le \delta\}$$

and $w \in \operatorname{argmin}\{\mathcal{F}(t, \cdot, z)\}$. Since $s \mapsto (t(s), z(s))$ is continuous in $[0, T] \times H^1$ and since t_ε and z_ε converge locally uniformly (cf. Corollary 5.7) there exists $s_1 < s_2$ such that both $(t(s), z(s)) \in I_\delta \times B_\delta$ and $(t_\varepsilon(s), z_\varepsilon(s)) \in I_\delta \times B_\delta$ and for $s \in [s_1, s_2]$. Thus,

$$|\partial_v^- \mathcal{F}(t_\varepsilon(s), w_\varepsilon(s), z_\varepsilon(s))|_{L^2} \ge C > 0 \qquad \text{for } s \in [s_1, s_2].$$

Equivalently, for $t_1^\varepsilon = t^\varepsilon(s_1)$ and $t_2^\varepsilon = t^\varepsilon(s_2)$ we have $s^\varepsilon(t) \in [s_1, s_2]$. Hence, with the change of variable $s = s^\varepsilon(t)$ we get

$$|\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t))|_{L^2} \ge C > 0 \qquad \text{for } t \in [t_1^\varepsilon, t_2^\varepsilon].$$

Hence,

$$\int_{t_1^\varepsilon}^{t_2^\varepsilon} \frac{\|\dot v_\varepsilon(t)\|^2_{H^1}}{\varepsilon + |\partial_v^- \mathcal{F}_\varepsilon(t, u_\varepsilon(t), v_\varepsilon(t)|_{L^2}}\, dr \le \frac{1}{\varepsilon + C} \int_0^{T_\varepsilon} \|\dot v_\varepsilon(t)\|^2_{H^1} \le +\infty.$$

Thus, z_ε and its limit z belong to $W^{1,2}(s_1, s_2; H^1)$. ■

Proposition 5.10 *Let (t, w, z) be a parametrized evolution (provided by Theorem 5.6) then for a.e. $s \in [0, S]$ it holds*

- $\partial_w \mathcal{F}(t(s), w(s), z(s)) = 0$,
- *if* $t'(s) > 0$ *then* $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0$,
- *if* $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \neq 0$ *then* $t'(s) = 0$ *and* $\|z'(s)\|_{L^2} = 1$,
- $z'(s) \in \operatorname{argmin}\{\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi,\ \|\xi\|_{L^2} \le 1\}$.

Proof. Equilibrium for the displacement field follows from the minimality of $w(s)$.

Since $\|z'(s)\|_{L^2} \le 1$ for a.e. $s \in [0, S]$ we can write by (57) and (63)

$$\begin{aligned}\mathcal{F}(t(s), w(s), z(s)) &= \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_z^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, dr + \\ &\quad + \int_0^s \partial_t \mathcal{F}(t(r), w(r), z(r))\, t'(r)\, dr \\ &\le \mathcal{F}(0, w_0, z_0) - \int_0^s |\partial_v^- \mathcal{F}(t(r), w(r), z(r))|_{L^2}\, \|z'(r)\|_{L^2}\, dr + \\ &\quad + \int_0^s \partial_t \mathcal{F}(t(r), u(r), v(r))\, t'(r)\, dr \le \mathcal{F}(t(s), u(s), v(s)).\end{aligned}$$

Hence all inequalities becomes equalities and hold in every subinterval $(s_1, s_2) \subset (0, S)$. In particular, for a.e. $s \in (0, S)$ we have

$$|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \, (1 - \|z'(s)\|_{L^2}) = 0. \tag{66}$$

Hence, if $t'(s) > 0$ then $\|z'(s)\|_{L^2} < 1$ (simply because $t'(s) + \|z'(s)\|_{L^2} \le 1$) and thus

$$|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0.$$

On the contrary, if $|\partial_z^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \neq 0$ then $\|z'(s)\|_{L^2} = 1$ and $t'(s) = 0$.

It remains to show that $z'(s) \in \operatorname{argmin} \{\partial_v \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi, \ \|\xi\|_{L^2} \le 1\}$. Clearly, if $|\partial_v^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0$ there is nothing to prove. In the case $|\partial_v^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} \neq 0$ our proof employs the chain rule (76) which requires the evolution z to be in $W^{1,2}_{loc}(0, S; H^1)$ (cf. Lemma 5.9). Hence,

$$\mathcal{F}'(t(s), w(s), z(s)) = \partial_z \mathcal{F}(t(s), w(s), z(s))[z'(s)] + \partial_t \mathcal{F}(t(s), w(s), z(s)) \, t'(s) \tag{67}$$

for a.e. in $s \in (0, S)$. On the other hand, by Theorem 5.6 for a.e. $s \in [0, S]$ it holds

$$\mathcal{F}'(t(s), w(s), z(s)) = -|\partial_v \mathcal{F}(t(s), w(s), z(s)|_{L^2} + \partial_t \mathcal{F}(t(s), w(s), z(s)) \, t'(s).$$

Hence,

$$\partial_z \mathcal{F}(t(s), w(s), z(s))[z'(s)] = -|\partial_v \mathcal{F}(t(s), w(s), z(s))|_{L^2}.$$

Therefore $z'(s) \in \operatorname{argmin} \{\partial_v \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi, \ \|\xi\|_{L^2} \le 1\}$. ■

Corollary 5.11 *Let (t, w, z) be a parametrized evolution (provided by Theorem 5.6) then for a.e. $s \in [0, S]$ we have*

- *if $t'(s) > 0$ then*

$$\begin{cases} \big[z(s) W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)\big]^+ = 0 \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde{w}(s))\big) = 0, \end{cases} \tag{68}$$

- *if $t(s) = t_d$ in $(s^\flat, s^\sharp)$ then*

$$\begin{cases} \lambda(s) z'(s) = -\big[z(s) W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)\big]^+ \\ \operatorname{div}\big(\boldsymbol{\sigma}_{z(s)}(\tilde{w}(s))\big) = 0, \end{cases} \tag{69}$$

  where $\lambda(s) = \|[z(s) W(D\tilde{w}(s)) + G_c(z(s) - 1) - G_c \Delta z(s)]^+\|_{L^2}$.

Remember that the first case corresponds to a continuity point in time, the second describes instead the "instantaneous evolution" in the discontinuity point t_d.

Proof. If $t'(s) > 0$ then by Proposition 5.10 $|\partial_v^- \mathcal{F}(t(s), w(s), z(s))|_{L^2} = 0$, i.e.

$$\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] \ge 0 \quad \text{for every } \xi \in H^1 \text{ with } \xi \le 0.$$

As a consequence $\partial_z \mathcal{F}(t(s), w(s), z(s))$ (as a distribution) is a negative Radon measure, or equivalently a Radon measure μ with positive part $\mu^+ = 0$. As in (44), writing $\partial_v \mathcal{F}(t(s), w(s), z(s))$ in the sense of distribution yields (68).

By Proposition 5.10 we know that $z'(s) \in \operatorname{argmin} \{\partial_z \mathcal{F}(t(s), w(s), z(s))[\xi] : \xi \in \Xi, \ \|\xi\|_{L^2} \le 1\}$ and thus by Lemma A.3 we get (69). ■

A Some Lemmas

A.1 Discrete Gronwall

First of all let us provide the Gronwall estimate to be used in the proof of Theorem 5.1. It's proof originates from [28] and [19].

Lemma A.1 *Let $\gamma > 0$, $a_k, b_k, c_k \geq 0$ and $a_0 = 0$ such that*

$$a_{k+1}^2 - a_k^2 \leq -\tau\gamma a_{k+1}^2 + \tau a_{k+1}b_{k+1} + \tau c_{k+1}^2 \quad \text{for } k \in \mathbb{N}. \tag{70}$$

Denote $t_k = k\tau$ for $k \in \mathbb{N}$. Then for $0 < \beta < \gamma/2$ and $\tau \ll 1$ it holds

$$a_k \leq \Big(\sum_{i=0}^{k} \tau e^{-2\beta(t_k - t_i)} c_i^2 \Big)^{1/2} + \sum_{i=0}^{k} \tau e^{-\beta(t_k - t_i)} b_i \quad \text{for } k \in \mathbb{N}.$$

Proof. for $\lambda = (1+\tau\gamma)^{1/2}$ and for $\tau < 1$ let us re-write (70) as $\lambda^2 a_{k+1}^2 - a_k^2 - a_{k+1}b_{k+1} \leq c_{k+1}^2$. Denote

$$A_k = \lambda^{-k}(C_k + B_k)\,, \qquad C_k = \Big(\sum_{i=0}^{k} \lambda^{2i} c_i^2 \Big)^{1/2}, \qquad B_k = \sum_{i=0}^{k} \lambda^i b_i\,.$$

Let us show that A_k satisfies

$$\lambda^2 A_{k+1}^2 - A_k^2 - A_{k+1}b_{k+1} \geq c_{k+1}^2 \tag{71}$$

In terms of C_k and B_k, the left hand side reads

$$\lambda^{-2(k+1)+2}\big(C_{k+1}^2 + B_{k+1}^2 + 2C_{k+1}B_{k+1}\big) - \lambda^{-2k}\big(C_k^2 + B_k^2 + 2C_kB_k\big) - \lambda^{-(k+1)}\big(C_{k+1} + B_{k+1}\big)b_{k+1}$$

Let us see that (71) holds. First, since $\lambda > 1$

$$\lambda^{-2k}C_{k+1}^2 - \lambda^{-2k}C_K^2 = \lambda^{-2k}(C_{k+1}^2 - C_k^2) \geq \lambda^{-2k+2(k+1)}c_{k+1}^2 \geq c_{k+1}^2.$$

Next,

$$\begin{aligned}\lambda^{-2k}B_{k+1}^2 \;-\; &\lambda^{-2k}B_k^2 - \lambda^{-(k+1)}B_{k+1}b_{k+1} = \\ &= \lambda^{-2k}(B_k + \lambda^{k+1}b_{k+1})^2 - \lambda^{-2k}B_k^2 - \lambda^{-(k+1)}(B_k + \lambda^{k+1}b_{k+1})b_{k+1} \\ &= \big(\lambda^{-2k+2(k+1)} - 1\big)b_{k+1}^2 + \big(2\lambda^{-2k+(k+1)} - \lambda^{-(k+1)}\big)B_kb_{k+1} \geq 0,\end{aligned}$$

where the last inequality follows again from $\lambda > 1$. Finally,

$$\begin{aligned}2\lambda^{-2k}C_{k+1}\big(B_k + \lambda^{k+1}b_{k+1}\big) \;-\; &2\lambda^{-2k}C_kB_k - \lambda^{-(k+1)}C_{k+1}b_{k+1} = \\ &= 2\lambda^{-2k}\big(C_{k+1} - C_k\big)B_k + \big(2\lambda^{-2k+(k+1)} - \lambda^{-(k+1)}\big)C_{k+1}b_{k+1} \geq 0,\end{aligned}$$

again because $\lambda > 1$.

Since $\lambda^2 a_{k+1}^2 - a_k^2 - a_{k+1}b_{k+1} \leq c_{k+1}^2$ and $a_{k+1} \geq 0$ we get

$$a_{k+1} \leq \frac{1}{2\lambda^2}\left(b_{k+1} + \sqrt{b_{k+1}^2 + 4\lambda^2(a_k^2 + c_{k+1}^2)}\right).$$

In the same way

$$A_{k+1} \geq \frac{1}{2\lambda^2}\left(b_{k+1} + \sqrt{b_{k+1}^2 + 4\lambda^2(A_k^2 + c_{k+1}^2)}\right).$$

Hence by induction $a_k \leq A_k$ for every $k \in \mathbb{N}$, i.e.

$$a_k \leq \Big(\sum_{i=0}^{k} \tau\lambda^{2(i-k)} c_i^2 \Big)^{1/2} + \sum_{i=0}^{k} \tau\lambda^{i-k} b_i\,.$$

Finally, it is not hard to check that for $0 < \beta < \gamma/2$ and $0 < \tau \ll 1$ it holds

$$\lambda^{-1} = (1+\tau\gamma)^{-1/2} \leq 1 - \beta\tau.$$

Hence, for $t_k = k\tau$ we have

$$\lambda^{(i-k)} = \lambda^{-(k-i)} \leq (1-\beta\tau)^{(k-i)} = e^{(k-i)\ln(1-\beta\tau)} \leq e^{-\beta(k-i)\tau} = e^{-\beta(t_k - t_i)}.$$

Then

$$a_k \leq \left(\sum_{i=0}^{k} \tau e^{-2\beta(t_k-t_i)} c_i^2 \right)^{1/2} + \sum_{i=0}^{k} \tau e^{-\beta(t_k-t_i)} b_i \, .$$

which concludes the proof. ■

A.2 Representation of linear functionals

We provide here a couple of representations, to be used in Theorem 3.10 and in Corollary 5.11. A similar result, related to unilateral gradient flows, is already stated (without proof) in [14].

Lemma A.2 *Let $\zeta \in H^{-1}$. If*

$$\sup\left\{ \langle \zeta, \xi \rangle : \xi \in H_0^1,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \right\} < +\infty \tag{72}$$

then ζ (as a distribution) is a Radon measure whose positive part belongs to L^2.

Proof. We introduce the indicator functions $I_B, I_+ : H_0^1 \to [0, +\infty]$ given by

$$I_B(\xi) = \begin{cases} 0 & \|\xi\|_{L^2} \leq 1 \\ +\infty & \text{otherwise,} \end{cases} \qquad I_+(\xi) = \begin{cases} 0 & \xi \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then (72) reads

$$\sup_{\xi \in H_0^1} \langle \zeta, \xi \rangle - \big(I_+(\xi) + I_B(\xi) \big) < +\infty.$$

In other terms, ζ belongs to the proper domain of the Legendre transform $(I_+ + I_B)^*$ in H^{-1}. In order to characterize the proper domain, let us write by inf-convolution, e.g. §15.1 in [6],

$$(I_+ + I_B)^*(\zeta) = \min_{\varphi \in H^{-1}} I_+^*(\varphi) + I_B^*(\zeta - \varphi).$$

Clearly, if $(I_+ + I_B)^*(\zeta) < +\infty$ there exists $\mu \in H^{-1}$ such that $I_B^*(\zeta - \mu) + I_+^*(\mu) < +\infty$. Since

$$I_+^*(\mu) = \sup_{\xi \in H_0^1} \langle \mu, \xi \rangle + I_+(\xi) \geq 0 \quad \text{and} \quad I_B^*(\zeta - \mu) = \sup_{\xi \in H_0^1} \langle \zeta - \mu, \xi \rangle + I_B(\xi) \geq 0$$

both $I_+^*(\mu) < +\infty$ and $I_B^*(\zeta - \mu) < +\infty$. Choosing $\xi = \lambda\hat{\xi}$, for $\lambda \geq 0$ and $\hat{\xi} \geq 0$, yields

$$\lambda \langle \mu, \hat{\xi} \rangle \leq \sup\{ \langle \mu, \xi \rangle : \xi \in H_0^1,\, \xi \geq 0 \} = I_+^*(\mu) < +\infty \quad \text{for every } \lambda \geq 0;$$

hence $\langle \mu, \hat{\xi} \rangle \leq 0$ for every $\hat{\xi} \geq 0$ in H_0^1 and thus μ is a negative Radon measure. Further, since

$$I_B^*(\zeta - \mu) = \sup\{ \langle \zeta - \mu, \xi \rangle : \xi \in H_0^1,\, \|\xi\|_{L^2} \leq 1 \} < +\infty,$$

the functional $\zeta - \mu$ can be extended from H_0^1 to the whole L^2 (by Hahn-Banach Theorem) and thus it can be represented as an element $f \in L^2$ (by Riesz's representation Theorem). In summary, we write $\zeta = \mu + f\mathcal{L}$, where μ is a negative measure, f is an L^2-function and $\mathcal{L}$ is the Lebesgue measure. Write $\mu = \mu_{ac} + \mu_s$ where μ_{ac} and μ_s are, respectively, absolutely continuous and singular with respect to $\mathcal{L}$. Then $\mu_{ac} = -m\mathcal{L}$ (by Radon-Nikodym Theorem) where $m \in L^1$ and $m \geq 0$. Hence

$$\zeta^+ = (f-m)^+\mathcal{L} + \mu_s^+ = (f-m)^+\mathcal{L} = (f-m)\mathcal{L}_{|A}$$

where $A = \{f - m \geq 0\}$. In A we have $f \geq m \geq 0$ and thus $m \in L^2(A)$. It follows that $\zeta^+ \in L^2$. ■

Lemma A.3 *Let $\zeta \in (H^1)^*$ such that there exists*

$$\xi_M \in \operatorname{argmax} \left\{ \langle \zeta, \xi \rangle : \xi \in H^1,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \right\},$$

then ζ (as a distribution) is a Radon measure and $\zeta^+ = \xi_M \, \|\zeta^+\|_{L^2}$.

Proof. Considering $\zeta \in H^{-1}$ (and thus $\xi \in H^1_0$) provides

$$\sup \left\{ \langle \zeta, \xi \rangle : \xi \in H^1_0,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1 \right\} < +\infty.$$

By the previous Lemma, ζ is a Radon measure with positive part in L^2. Thus we can write $\zeta = \zeta^+ - \zeta^-$ where ζ^- is a positive Radon measure while $\zeta^+ = f\mathcal{L}$, for $f \in L^2$ with $f \geq 0$. Since the measures $\zeta^\pm$ are supported on disjoint Borel sets, say $\Omega^\pm$, we have $f = 0$ on Ω^-.

If $f = 0$ there is nothing to prove. Otherwise, since $\xi \geq 0$ we have

$$\begin{aligned} \sup\{\langle \zeta, \xi \rangle : \xi \in H^1_0,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1\} &\leq \sup\{\langle \zeta, \xi \rangle : \xi \in H^1,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1\} \\ &\leq \sup\{\langle \zeta^+, \xi \rangle : \xi \in L^2,\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1\} = \|f\|_{L^2}. \end{aligned}$$

Note that $\xi = f/\|f\|_{L^2}$ is the unique maximizer in L^2. Now, we will show that

$$\sup\{\langle \zeta, \xi \rangle : \xi \in C^\infty_0(\Omega),\, \xi \geq 0,\, \|\xi\|_{L^2} \leq 1\} = \|f\|_{L^2} \tag{73}$$

from which we get $\xi_M = f/\|f\|_{L^2}$ and then the thesis. For sake of simplicity we will consider only the case in which the measure ζ^- is finite. In the case of a locally finite measure it is sufficient to employ an increasing sequence of open sets $\Omega_n \subset\subset \Omega$ whose union is Ω.

Let us choose a (smooth) convolution kernel ρ_n such that $\|f * \rho_n - f\|_{L^2} \leq 1/n$. Clearly $f * \rho_n$ is smooth, bounded and non-negative. Since $\Omega^\pm$ are Borel sets and since both $\mathcal{L}$ and ζ^- are regular, there exists compact set $K^\pm_n \subset \Omega^\pm$ such that

$$\zeta^-(\Omega^- \backslash K^-_n)\|f * \rho_n\|_{L^\infty} \leq 1/n, \qquad \mathcal{L}(\Omega^+ \backslash K^+)\, \|f * \rho_n\|^2_{L^\infty} \leq 1/n^2.$$

For every $n \in \mathbb{N}$ there exists a smooth cut off function η_n with $0 \leq \eta_n \leq 1$ such that $\eta_n = 1$ on K^+ and $\eta_n = 0$ on K^-. Consider the smooth, non-negative function $(f * \rho_n)\, \eta_n$. We have

$$\int_\Omega (f * \rho_n)\, \eta_n \, d\zeta^- \leq \|f * \rho_n\|_{L^\infty} \int_{\Omega^-} \eta_n \, d\zeta^- \leq \|f * \rho_n\|_{L^\infty} \zeta^-(\Omega^- \backslash K^-_n) \leq 1/n. \tag{74}$$

Moreover

$$\begin{aligned} \int_{\Omega^+} |(f * \rho_n)\, \eta_n - f|^2 \, dx &\leq 2 \int_{\Omega^+} |(f * \rho_n)\, (\eta_n - 1)|^2 \, dx + 2 \int_{\Omega^+} |f * \rho_n - f|^2 \, dx \\ &\leq 2\, \mathcal{L}(\Omega^+ \backslash K^+_n)\, \|f * \rho_n\|^2_{L^\infty} + 2 \int_{\Omega^+} |f * \rho_n - f|^2 \, dx \leq 4/n^2. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Omega^+} f(f * \rho_n)\, \eta_n \, dx &= \int_{\Omega^+} f^2 \, dx - \int_{\Omega^+} f(f - (f * \rho_n)\, \eta_n) \, dx \\ &\geq \|f\|^2_{L^2} - \|f\|_{L^2} \|f - (f * \rho_n)\, \eta_n\|_{L^2} \geq \|f\|^2_{L^2} - 2\|f\|_{L^2}/n. \end{aligned} \tag{75}$$

Let $\lambda_n = \|(f * \rho_n)\eta_n\|_{L^2} \to \|f\|$. Then $\xi = \lambda_n (f * \rho_n)\eta_n$ is an admissible test function in (73) and from (74) and (75) we get

$$\langle \zeta, \xi_n \rangle = \lambda_n \int_{\Omega^+} f(f * \rho_n)\eta_n \, dx - \lambda_n \int_{\Omega^-} (f * \rho_n)\eta_n \, d\zeta^- \geq \lambda_n \|f\|^2 - C/n.$$

Thus we have (73). ∎

A.3 Continuous dependence and differentiability

Finally, we collect, for the readers convenience, few results from [19] adapted to our notation and framework; the first follows from a general regularity result proved in [17].

Lemma A.4 *Let $g \in C^1([0,T]; W^{1,\bar p}(\Omega,\mathbb{R}^2))$ for $\bar p > 2$. For $t \in [0,T]$ and $v \in \mathcal{V}$ denote $u(t,v) = \operatorname{argmin}\{\mathcal{F}(t,\cdot,v) : u \in \mathcal{U}\}$. For every $2 < p < \bar p$ there exists $C > 0$ s.t. for every $t_1, t_2 \in [0,T]$ and every $v_1, v_2 \in \mathcal{V}$*

$$\|u(t_2,v_2) - u(t_1,v_1)\|_{W^{1,p}} \le C\|g(t_2) - g(t_1)\|_{L^q} + C\|v_2 - v_1\|_{L^q}$$

where $1/q = 1/p - 1/\bar p$.

Lemma A.5 *If $u \in W^{1,p}(\Omega,\mathbb{R}^2)$ for some $p > 2$ then $\mathcal{F}(t,u,\cdot)$ is Gateaux differentiable and*

$$\partial_v \mathcal{F}(t,u,v)[\xi] = 2\int_\Omega v\xi\, W(Du + Dg(t))\,dx + G_c\int_\Omega (v-1)\xi + \nabla v\cdot\nabla\xi\,dx \qquad \forall\,\xi \in H^1(\Omega).$$

Lemma A.6 *If $v \in W^{1,2}(0,T;H^1)$ and $u(t) \in \operatorname{argmin}\{\mathcal{F}(t,u,v(t)) : u \in \mathcal{U}\}$ then the energy $t \mapsto \mathcal{F}(t,u(t),v(t))$ is a.e. differentiable in $(0,T)$ and the following chain rule holds:*

$$\dot{\mathcal{F}}(t,u(t),v(t)) = \partial_t\mathcal{F}(t,u(t),v(t)) + \partial_v\mathcal{F}(t,u(t),v(t))\,[\dot v(t)]\,. \tag{76}$$

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