Coupled Klein–Gordon and Born–Infeld type equations: looking for solitary waves

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The existence of infinitely many nontrivial radially symmetric solitary waves for the nonlinear Klein-Gordon equation, coupled with a Born-Infeld type equation, is established under general assumptions.

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1. Introduction

Let us consider the following nonlinear Klein-Gordon equation:

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \tag{1.1}$$

where $\psi = \psi(t,x) \in \mathbb{C}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $m \in \mathbb{R}$ and $2 < p < 6$.

In the last years a wide interest was born about solitary waves of (1.1), i.e. solutions of the form

$$\psi(x,t) = u(x)e^{i\omega t}, \tag{1.2}$$

where u is a real function and $\omega \in \mathbb{R}$. If one looks for solutions of (1.1) having the form (1.2), the nonlinear Klein-Gordon equation reduces to a semilinear elliptic equation, as well as if one looks for solitary waves of nonlinear Schrödinger equation (see [10], [12] and the papers quoted therein). Many existence results have been established for solutions u of such a semilinear equation, both in the case in which u is radially symmetric and real or non-radially symmetric and complex (e.g., see [6], [7], [15]).

From equation (1.1) it is possible to develop the theory of electrically charged fields (see [13]) and study the interaction of ψ with an assigned electromagnetic field (see [1], [2], [9]). On the other hand, it is also possible to study the interaction of ψ with its own electromagnetic field (see [3], [4], [5], [10]), which is not assigned, but is an unknown of the problem. More precisely, if the electromagnetic field is described by the gauge potentials $(\phi, \mathbf{A})$

$$\phi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad \mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3,$$

then, by Maxwell equations, the electric field is given by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

and the magnetic induction field by

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Of course, equation (1.1) is the Euler-Lagrange equation associated to the Lagrangian density

$$\mathcal{L}_{KG} = \frac{1}{2}\left[\left|\frac{\partial \psi}{\partial t}\right|^2 - |\nabla \psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p. \tag{1.3}$$

The interaction of ψ with the electromagnetic field is described by the minimal coupling rule, that is the formal substitution

$$\frac{\partial}{\partial t} \longmapsto \frac{\partial}{\partial t} + ie\phi, \qquad \nabla \longmapsto \nabla - ie\mathbf{A}, \tag{1.4}$$

where e is the electric charge.

By (1.4) the Lagrangian density (1.3) takes the form

$$\mathcal{L}_{KGM} = \frac{1}{2}\left[\left|\frac{\partial \psi}{\partial t} + ie\psi\phi\right|^2 - |\nabla \psi - ie\mathbf{A}\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

The total action of the system is thus given by

$$\mathcal{S} = \int\int (\mathcal{L}_{KGM} + \mathcal{L}_{\text{emf}})\, dx\, dt,$$

where $\mathcal{L}_{\text{emf}}$ denotes the Lagrangian density of the electromagnetic field. In the classical Maxwell theory, $\mathcal{L}_{\text{emf}}$ can be written as

$$\mathcal{L}_{\text{emf}} = \mathcal{L}_{\text{M}} = \frac{1}{8\pi}(|\mathbf{E}|^2 - |\mathbf{B}|^2).$$

If, as in [4], we look for ψ of the form

$$\psi(x,t) = u(x,t)e^{iS(x,t)}$$

for some $u, S : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$, then $\mathcal{S} = \mathcal{S}(u, S, \phi, \mathbf{A})$. If $u = u(x)$, $S = \omega t$, $\phi = \phi(x)$ and $\mathbf{A} = \mathbf{0}$, then the existence of infinitely many solutions for the Euler-Lagrange equation of

$$\mathcal{S} = \int\int (\mathcal{L}_{KGM} + \mathcal{L}_{\text{M}})\, dx\, dt$$

has been proved in [4] for $4 < p < 6$ and in [10] for $2 < p < 6$.

Unfortunately, the theory of Maxwell exhibits some difficulties because the electromagnetic field corresponding to "some" charge distributions (point charge) have infinite energy (see [14]).

Born and Infeld suggested a way to overcome such difficulties in [8], introducing the Lagrangian density

$$\mathcal{L}_{\text{BI}} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{1}{b^2}(|\mathbf{E}|^2 - |\mathbf{B}|^2)}\right),$$

where $b \gg 1$ is the so-called *Born-Infeld parameter*. It is clear that the first order expansion of $\mathcal{L}_{\rm BI}$ coincides with $\mathcal{L}_{\rm M}$, but using $\mathcal{L}_{\rm BI}$ the electrostatic case of a point charge has finite energy (see [14]).

As done in [14] and [11], we set

$$\beta := \frac{1}{2b^2}$$

and consider the second order expansion of $\mathcal{L}_{\rm BI}$ for $\beta \to 0^+$. In this way the Lagrangian density takes the form

$$\mathcal{L}_{\rm BI'} = \frac{1}{4\pi}\left[\frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) + \frac{\beta}{4}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right)^2\right].$$

In [14] the existence of electrostatic solutions with finite energy associated to $\mathcal{L}_{\rm BI'}$ was proved, while in [11] the authors considered the total action given by

$$\mathcal{S} = \int\int (\mathcal{L}_{\rm KGM} + \mathcal{L}_{\rm BI'})\, dx\, dt.$$

In this case the authors proved that the associated Euler-Lagrange equation with $u = u(x)$, $S = \omega t$, $\phi = \phi(x)$ and $\mathbf{A} = \mathbf{0}$,

$$-\Delta u + [m^2 - (\omega + e\phi)^2]u - |u|^{p-2}u = 0 \quad (1.5)$$

$$\Delta\phi + \beta\Delta_4\phi = 4\pi e(\omega + e\phi)u^2, \quad (1.6)$$

has infinitely many solutions, provided that $4 < p < 6$ and $0 < |\omega| < |m|$, where $\Delta_4 u = \mathrm{div}(|Du|^2 Du)$.

By a suitable L^∞–*a priori* estimate for ϕ, such a result is extended here in the following way.

Theorem 1.1. *If* $0 < \omega < \sqrt{\frac{p}{2} - 1}\,|m|$ *and* $2 < p < 4$, *or* $0 < \omega < |m|$ *and* $4 \le p < 6$, *the system* (1.5) − (1.6) *has infinitely many radially symmetric solution* (u, ϕ) *with* $u \in H^1(\mathbb{R}^3)$, $\phi \in L^6(\mathbb{R}^3)$ *and* $|\nabla\phi| \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$.

Remark 1.1. The assumption $\omega < (\frac{p}{2} - 1)|m|$ is merely technical, since it is needed only to prove the Palais–Smale condition if $p < 4$ (see §3). Of course it implies the assumption $\omega < |m|$ if $p \le 4$.

2. Proof of Theorem 1.1

Let us choose $e = 1$ in (1.5)–(1.6), so that the system reduces to

$$-\Delta u + [m^2 - (\omega + \phi)^2]u - |u|^{p-2}u = 0 \quad (2.1)$$

$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2. \quad (2.2)$$

By $H^1 \equiv H^1(\mathbb{R}^3)$ we denote the usual Sobolev space endowed with the norm

$$\|u\|_{H^1} \equiv \left(\int_{\mathbb{R}^3}\left(|Du|^2 + |u|^2\right) dx\right)^{1/2}$$

and by $D(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the norm

$$\|\phi\|_{D(\mathbb{R}^3)} = \|D\phi\|_{L^2} + \|D\phi\|_{L^4}.$$

Denoting by $D^{1,2}(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R})$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |Du|^2\,dx\right)^{1/2},$$

it is clear that $D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. Moreover $D^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6(\mathbb{R}^3)$ by Sobolev inequality and $D(\mathbb{R}^3)$ is continuously embedded in $L^\infty(\mathbb{R}^3)$ by Proposition 8 in [14]. We remark that $q=6$ is the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$, so that equation (2.1) has a *subcritical* nonlinearity.

Theorem 1.1 is then proved if we show the existence of solutions (u,ϕ) with $u\in H^1(\mathbb{R}^3)$ and $\phi\in D(\mathbb{R}^3)$, so that

$$\int_{\mathbb{R}^3} |Du|^2\,dx + \int_{\mathbb{R}^3} |u|^2\,dx < +\infty \tag{2.3}$$

$$\int_{\mathbb{R}^3} |D\phi|^2\,dx + \int_{\mathbb{R}^3} |D\phi|^4\,dx < +\infty \tag{2.4}$$

For the sake of simplicity, from now on, the integration domain is intended to be $\mathbb{R}^3$, if not explicitly stated otherwise.

Consider the functional $F: H^1(\mathbb{R}^3)\times D(\mathbb{R}^3)\longrightarrow\mathbb{R}$ defined by

$$F(u,\phi) = \int\left\{\frac{1}{2}|Du|^2 - \frac{1}{8\pi}|D\phi|^2 + \frac{1}{2}\Big(m^2-(\omega+\phi)^2\Big)u^2 \right.$$
$$\left. - \frac{\beta}{16\pi}|D\phi|^4 - \frac{1}{p}|u|^p\right\}dx,$$

whose Euler-Lagrange equation is precisely (2.1)-(2.2).

It is readily seen that the following result holds.

Proposition 2.1. *The functional F is of class C^1 on $H^1(\mathbb{R}^3)\times D(\mathbb{R}^3)$ and its critical points solve* (2.1) *and* (2.2) *under the conditions* (2.3)–(2.4)*.*

It seems quite hard to find critical points for F directly, since it depends on two variables and since it is strongly indefinite, i.e. unbounded both from above and below, even modulo compact perturbations. Therefore, following the approach of [4], [10], [11], we introduce a new functional J depending only on u in such a way that critical points of J give rise to critical points of F.

First of all we need the following Lemma.

Lemma 2.2. *For every $u\in H^1(\mathbb{R}^3)$ there exists a unique $\phi=\Phi[u]\in D(\mathbb{R}^3)$ which solves* (2.2)*. If u is radially symmetric, then $\Phi[u]$ is radially symmetric.*

The first result is proved in [11, Lemma 3], while the second one, though not explicitly stated, is proved in [11, Lemma 5].

A deeper look at solutions of (2.2) leads to the following L^∞–*a priori* estimate on $\Phi[u]$, which is a fundamental tool to prove Theorem 1.1.

Lemma 2.3. *For any $u \in H^1(\mathbb{R}^3)$, it results $\Phi[u] \leq 0$. Moreover $\Phi[u](x) \geq -\omega$ if $u(x) \neq 0$.*

Proof. Multiplying (2.2) by $\Phi^+ = \max\{\Phi[u], 0\}$, we get

$$-\int |D\Phi^+|^2 \, dx - \beta \int |D\Phi^+|^4 \, dx = 4\pi\omega \int u^2 \Phi^+ \, dx + 4\pi \int u^2 (\Phi^+)^2 \, dx \geq 0,$$

so that $\Phi^+ \equiv 0$.

If we multiply (2.2) by $(\omega + \Phi[u])^-$, which is an admissible test function since $\omega > 0$, we get

$$\int_{\{x:\Phi[u]<-\omega\}} |D\Phi[u]|^2 \, dx + \beta \int_{\{x:\Phi[u]<-\omega\}} |D\Phi[u]|^4 \, dx$$
$$= -4\pi \int_{\{x:\Phi[u]<-\omega\}} (\omega + \Phi[u])^2 u^2 \, dx,$$

so that $(\omega + \Phi[u])^- = 0$ where $u \neq 0$. □

In view of Lemma 2.2, we consider the map $\Phi : H^1(\mathbb{R}^3) \longrightarrow D$, $u \mapsto \Phi[u]$. From standard arguments, $\Phi \in C^1(H^1, D^{1,2})$ and from the very definition of Φ we get

$$F'_\phi(u, \Phi[u]) = 0 \quad \forall\, u \in H^1.$$

Now consider the functional $J : H^1(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined by

$$J(u) = F(u, \Phi[u]),$$

that is

$$J(u) = \int \left\{ \frac{1}{2}|Du|^2 - \frac{1}{8\pi}|D\Phi[u]|^2 + \frac{1}{2}\Big(m^2 - (\omega + \Phi[u])^2\Big)u^2 - \frac{\beta}{16\pi}|D\Phi[u]|^4 - \frac{1}{p}|u|^p \right\} dx. \tag{2.5}$$

Of course $J \in C^1(H^1, \mathbb{R})$, since both F and Φ are C^1.

The next lemma states a relationship between the critical points of the functionals F and J (the proof can be found in [4, Proposition 3.5]).

Lemma 2.4. *The following statements are equivalent:*

i) $(u, \Phi) \in H^1 \times D^{1,2}$ *is a critical point of* F,

ii) u *is a critical point of* J *and* $\Phi = \Phi[u]$.

Then, in order to get solutions of (2.1)–(2.2), we look for critical points of J. Assume the following result holds.

Theorem 2.5. *Assume $2 < p < 6$ and $0 < \omega < (\frac{p}{2} - 1)|m|$. Then the functional J has infinitely many critical points having a radial symmetry.*

It is now immediate to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Theorem 2.5 establishes the existence of infinitely many nontrivial critical points u_n of J. Lemma 2.4 implies that $(u_n, \Phi[u_n])$ are critical points for F. Finally Proposition 2.1 guarantees that $(u_n, \Phi[u_n])$ solve (2.1)–(2.2) under the conditions (2.3)–(2.4). □

3. Proof of Theorem 2.5

Since J is invariant under the group of translations, there is an evident lack of compactness. In order to overcome this difficulty, we constrain J on the space of radially symmetric functions. More precisely we introduce the subspace

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \,|\, u(x) = u(|x|)\},$$

and we look for critical points of J constrained on $H_r^1(\mathbb{R}^3)$. In order to prove Theorem 2.5, this is enough in view of the following result (see [11] for the proof).

Lemma 3.1. *Any critical point $u \in H_r^1$ of $J\big|_{H_r^1(\mathbb{R}^3)}$ is also a critical point of J.*

The choice of $H_r^1(\mathbb{R}^3)$ lets us recover a compactness property: indeed the embedding of $H_r^1 \hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $q \in (2, 6)$ (see [6]).

We recall that a C^1 functional f defined on a Banach space B satisfies the *Palais–Smale condition*, (PS) for short, if

any sequence $(u_n)_n$ such that $f'(u_n) \to 0$ in B' and $f(u_n)$ is bounded, has a converging subsequence.

In [11] the Palais–Smale condition, as well as the existence of infinitely many solutions to (2.1)–(2.2), was proved for the case $4 < p < 6$. By Lemma 2.3 we are able to extend the result to the general case $2 < p < 6$.

Proposition 3.2. *If $2 < p < 6$, then $J_{|H_r^1(\mathbb{R}^3)}$ satisfies the Palais–Smale condition.*

Proof. From now on, let us write Ω in place of $m^2 - \omega^2$. Moreover, according to what we said above, it is enough to assume $2 < p \le 4$.

Suppose $(u_n)_n$ in $H_r^1(\mathbb{R}^3)$ is such that

$$J'_{|H_r^1(\mathbb{R}^3)}(u_n) \to 0 \quad \text{and} \quad |J(u_n)| \le M,$$

for a positive M.

From equation (2.2) we get

$$-\frac{1}{8\pi}\int |D\Phi[u]|^2\,dx - \frac{\beta}{8\pi}\int |D\Phi[u]|^4\,dx = \frac{\omega}{2}\int u^2\Phi[u]\,dx + \frac{1}{2}\int u^2\Phi^2[u]\,dx.$$

Substituting in (2.5), we obtain the following form of J:

$$J(u) = \frac{1}{2}\int \left\{|Du|^2 + \Omega u^2 - \omega u^2\Phi[u]\right\}\,dx + \frac{\beta}{16\pi}\int |D\Phi[u]|^4\,dx - \frac{1}{p}\int |u|^p\,dx. \tag{3.1}$$

Then

$$pJ(u_n) - J'(u_n)(u_n) = \left(\frac{p}{2} - 1\right)\left\{\int |Du_n|^2\,dx + \Omega\int u_n^2\,dx\right\}$$

$$+\int u_n^2\Phi^2[u_n]\,dx - \omega\left(\frac{p}{2} - 2\right)\int u_n^2\Phi[u_n]\,dx + \frac{p\,\beta}{16\pi}\int |D\Phi[u_n]|^4\,dx$$

$$\ge \left(\frac{p}{2} - 1\right)\left\{\int |Du_n|^2\,dx + m^2\int u_n^2\,dx\right\} - \omega^2\left(\frac{p}{2} - 1\right)\int u_n^2\,dx$$

$$-\omega\left(\frac{p}{2}-2\right)\int u_n^2\Phi[u_n]\,dx$$

and by Lemma 2.3, since $2 < p \leq 4$,

$$\geq \left(\frac{p}{2}-1\right)\left\{\int |Du_n|^2\,dx + m^2\int u_n^2\,dx\right\} - \omega^2\int u_n^2\,dx.$$

Therefore, since $(\frac{p}{2}-1)m^2 > \omega^2$ for any $p \in (2,4]$, we get

$$pJ(u_n) - J'(u_n)(u_n) \geq C\|u_n\|^2$$

for a positive constant C independent of n.

On the other hand, by hypotheses,

$$pJ(u_n) - J'(u_n)(u_n) \leq pM + \varepsilon\|u_n\|,$$

where $\|J'(u_n)\|_{(H^1)'} \leq \varepsilon$.

In this way $(u_n)_n$ is bounded.

Then, up to a subsequence,

$$u_n \rightharpoonup u \qquad \text{in } H_r^1(\mathbb{R}^3).$$

Moreover from equation (2.2) we have

$$\int |D\Phi[u_n]|^2\,dx + \int |D\Phi[u_n]|^4\,dx = -4\pi\omega\int u_n^2\Phi[u_n]\,dx$$

$$-4\pi\int u_n^2\Phi^2[u_n]\,dx \leq 4\pi\omega\left(\int \Phi^6[u_n]\,dx\right)^{1/6}\left(\int |u_n|^{12/5}\,dx\right)^{5/6}$$

$$\leq C\|\Phi[u_n]\|\cdot\|u_n\|^2,$$

and so also $(\Phi[u_n])_n$ is bounded. This fact and the compact embedding of $H_r^1 \hookrightarrow L^q(\mathbb{R}^3)$ for any $q \in (2,6)$ ([6]), imply that

$$u_n \to u \text{ strongly in } H_r^1(\mathbb{R}^3)$$

in a standard fashion (see, for example, [11]). □

Now we show that $J_{\big|H_r^1(\mathbb{R}^3)}$ satisfies the three geometrical hypothesis of the following $\mathbb{Z}_2$ version of the mountain pass theorem (see [16, Theorem 9.12]).

Theorem 3.3 ($\mathbb{Z}_2$–Mountain Pass). *Let E be a Banach space with $dim(E) = \infty$, and $I \in C^1(E,\mathbb{R})$ be even, satisfy (PS) and $I(0) = 0$. Assume*

- *$\exists\,\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha$ $\forall\, u$ with $\|u\| = \rho$;*
- *for every finite dimensional subspace X of E there exists $R = R(X)$ such that $I(u) \leq 0$ if $\|u\| \geq R$.*

Then I has an unbounded sequence of critical values.

First of all we observe that $J(0) = 0$. Moreover the hypotheses on the coefficients imply that there exists $\rho > 0$ and small enough such that

$$\inf_{\|u\|=\rho} J(u) > 0.$$

In fact, from (2.2) we get

$$-\omega \int u^2\Phi[u]\,dx = \frac{1}{4\pi}\int |D\Phi[u]|^2\,dx + \frac{\beta}{4\pi}\int |D\Phi[u]|^4\,dx + \int u^2\Phi[u]^2\,dx,$$

and substituting this quantity in the expression (2.5) of J, we get

$$J(u) = \int \left\{ \frac{1}{2}|Du|^2 + \frac{1}{8\pi}|D\Phi[u]|^2 + \frac{\Omega}{2}u^2 + \frac{1}{2}u^2\Phi^2[u] + \frac{3\beta}{16\pi}|D\Phi[u]|^4 - \frac{1}{p}|u|^p \right\} dx.$$

Since $p > 2$, the claim follows.

Finally, let $X \subset H^1(\mathbb{R}^3)$ be a finite–dimensional subspace. From (2.2) we get

$$-\frac{\beta}{16\pi}\int |D\Phi[u]|^4\,dx = \frac{1}{16\pi}\int |D\Phi[u]|^2\,dx + \frac{\omega}{4}\int u^2\Phi[u]\,dx + \frac{1}{4}\int u^2\Phi^2[u]\,dx.$$

Substituting in (2.5), we obtain the following form for J:

$$\begin{aligned} J(u) = {} & \frac{1}{2}\int |Du|^2\,dx + \frac{\Omega}{2}\int u^2\,dx - \frac{1}{4}\int u^2\Phi^2[u]\,dx \\ & -\frac{3}{4}\omega\int u^2\Phi[u]\,dx - \frac{1}{16\pi}\int |D\Phi[u]|^2\,dx - \frac{1}{p}\int |u|^p\,dx. \end{aligned} \tag{3.2}$$

If $u \in X$, using (3.2), we get

$$J(u) \le \frac{1}{2}\int |Du|^2\,dx + \frac{\Omega}{2}\int u^2\,dx - \frac{3}{4}\omega\int u^2\Phi[u]\,dx - \frac{1}{p}\int |u|^p\,dx.$$

By Lemma 2.3,

$$J(u) \le \frac{1}{2}\int |Du|^2\,dx + \frac{\Omega}{2}\int u^2\,dx + \frac{3}{4}\omega^2\int_{\{x:u(x)\neq 0\}} u^2\,dx - \frac{1}{p}\int |u|^p\,dx.$$

Therefore, if R is big enough and $\|u\| \ge R$, then $J(u) \le 0$, since all norms are equivalent in X.

We have thus verified all the conditions of Theorem 3.3, proving that J has an unbounded sequence of critical values.

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