

On the isoperimetric properties of planar N -clusters

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*Devenere locos ubi nunc ingentia cernes
Moenia surgentemque novae Karthaginis arcem,
Mercatique solum, facti de nomine Byrsam,
Taurino quantum possent circumdare tergo.*
Virgilio, Eneide, Libro I, vv 365-368.

*They came to this spot, where today you can behold the mighty
Battlements and the rising citadel of New Carthage,
And purchased a site, which was named 'Bull's Hide' after the bargain,
By which they should get as much land as they could enclose with a bull's hide.*
Virgil, Aeneid, Book I, vv 365-368.

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Notations

$B_r(x)$	Ball in \mathbb{R}^n centered at x and with radius r
B_r	$= B_r(0)$ ball in \mathbb{R}^n centered at 0 and with radius r
S^{n-1}	$= \partial B_1$, $(n - 1)$ -dimensional sphere in \mathbb{R}^n
$E \Delta F$	$= (E \setminus F) \cup (F \setminus E)$, symmetric difference between E and F
$ E $	Lebesgue measure of the set E
$\mathcal{H}^s(E)$	s -dimensional Hausdorff measure of the set E
\xrightarrow{X}	Convergence wrt the topology induced from the metric space X
$\xrightarrow{*}$	Weak-star convergence
$L^p_{loc}(A; \mathbb{R}^k)$	Space of the functions $f : A \rightarrow \mathbb{R}^k$ with values in \mathbb{R}^k and which are p -summable on every compact set strictly contained in the set $A \subseteq \mathbb{R}^n$
$L^p_{loc}(A)$	$= L^p_{loc}(A; \mathbb{R})$, space of the \mathbb{R} -valued functions which are p -summable on every compact set strictly contained in the set $A \subseteq \mathbb{R}^n$
$L^p(A; \mathbb{R}^k)$	Space of the functions $f : A \rightarrow \mathbb{R}^k$ with values in \mathbb{R}^k which are p -summable on $A \subseteq \mathbb{R}^n$
$L^p(A)$	$= L^p(A; \mathbb{R})$, space of the \mathbb{R} -valued functions which are p -summable on $A \subseteq \mathbb{R}^n$
$C^k_c(A; \mathbb{R}^k)$	Space of the of the k -differentiable functions $f : A \rightarrow \mathbb{R}^k$ with values in \mathbb{R}^k and which are compactly supported on the set A
$C^k_c(A)$	$C^k_c(A; \mathbb{R})$, space of the k -differentiable, \mathbb{R} -valued functions which are compactly supported on the set A
$\mathbb{1}_E(x)$	Characteristic function of the set E
$\operatorname{div}(T)$	Divergence of the vector field T
$P(E; F)$	Relative perimeter of the Borel set E inside F
$P(E)$	$= P(E; \mathbb{R}^n)$, global perimeter of the Borel set E
$ \mu (A)$	Total variation of the Radon measure μ on the set $A \subseteq \mathbb{R}^n$,
$\mu \llcorner E$	Radon measure obtained as the restriction of the Radon measure μ to the Borel set E , $\mu \llcorner E(F) = \mu(E \cap F)$

$\vartheta_n(x, E)$	n -dimensional density of the set E at the point x
$E^{(t)}$	$= \{x \in \mathbb{R}^n \mid \vartheta_n(x, E) = t\}$
$\partial^* E$	Reduced boundary of the set E
$\partial^e E$	$= \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$, essential boundary of the set E
$\nu_E(x)$	Measure-theoretic outer unit normal to E at $x \in \partial^* E$
\mathcal{E}	$= \{\mathcal{E}(i)\}_{i=1}^N$, N -cluster of \mathbb{R}^n
$\mathcal{E}(h, k)$	$= \partial^* \mathcal{E}(k) \cap \partial^* \mathcal{E}(h)$, interface between the chambers $\mathcal{E}(k)$ and $\mathcal{E}(h)$,
$\partial \mathcal{E}$	$= \bigcup_{i=1}^N \partial \mathcal{E}(i)$, topological boundary of the cluster \mathcal{E} ,
$\partial^* \mathcal{E}$	$= \bigcup_{0 \leq h < k \leq N} \mathcal{E}(h, k)$, reduced boundary of the cluster \mathcal{E} ,
$\mathcal{E} \Delta \mathcal{F}$	$= \bigcup_{i=0}^N \mathcal{E}(i) \Delta \mathcal{F}(i)$, symmetric difference between the N -clusters \mathcal{E} and \mathcal{F} ,
\approx	Equal up to an \mathcal{H}^{n-1} -negligible set,
$P(\mathcal{E}; F)$	Relative perimeter of the cluster \mathcal{E} inside F ,

Introduction

Isoperimetric problems have fascinated the human being since the ancient times, starting from the legend of Dido *who left the city of Tyre to escape her brother* and whose problem was to enclose *as much land as could be enclosed with a bull's hide* up to the present days, with Joseph Plateau who experimented with soap films in order to figure out what is the surface with the smallest amount of area among all the surfaces that share a fixed boundary in the three dimensional space. In general a problem can be defined *isoperimetric* whenever we seek for objects attaining the smallest (or the largest) amount of (a suitable notion of) area (or volume) among all those objects satisfying a given constraint. The most famous one (so famous that is called just *the isoperimetric problem*) is the one that Dido solved in the planar case when she built New Carthage with a bull's hide. It can be stated, in modern mathematical language, as follows: what is the n -dimensional object having the smallest perimeter ($(n - 1)$ dimensional area of the boundary) among all the sets with a fixed amount of volume? Or, equivalently: what is the n -dimensional object having the biggest amount of volume among all the sets with a fixed amount of perimeter ($(n - 1)$ dimensional area of the boundary)?

The reasons that led scientists and mathematicians to be attracted by this kind of questions might rely on the fact that the energy needed in a number of physical processes is related with the surface area or the mass. For example it is a well known fact that the shapes of crystals are polyhedral (see [FMP10]) because they solve a variant of the classical isoperimetric problem (let us recall for the sake of completeness that the solution to the classical isoperimetric problem is the n -dimensional euclidean ball, see [DG58]). The techniques and the ideas developed in order to approach this kind of questions turn out to be a useful equipment for the treatment of various type of problems concerning geometry and optimization process, as in the case of image recovery (see [MS89] and [AT90]). That is another reason that explain why these issues have been so fruitfully studied in the past and why they are, still today, a central topic in Calculus of Variations.

This Thesis aims to highlight some isoperimetric questions involving the so-called N -clusters. The term *cluster* has been used in many different areas of mathematics to denote “a family of objects that share a precise property and that are combined and connected in a specific way”. This points out that a *cluster* is not just a *set* but it is somehow an agglomerate. In our context an N -cluster \mathcal{E} is a generic family $\{\mathcal{E}(i)\}_{i=1}^N$ of N sets with disjoint interiors (called chambers) that compete in some variational (isoperimetric) problem as a unique object. We refer to Chapter One where the main definitions and tools are recalled. The main problem leading to define these objects, is the natural extension of the classical isoperimetric problem: the multi-chamber isoperimetric problem. This problem can be easily stated as follows. Among all families of N sets $\mathcal{E}(1), \dots, \mathcal{E}(N) \subset \mathbb{R}^n$ with disjoint interiors and with fixed volume $|\mathcal{E}(i)| = v_i$, what is the family that minimizes

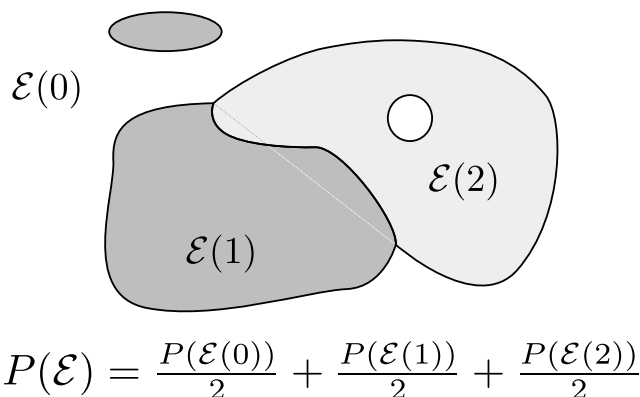


Figure 0.0.1: An example of a 2-cluster in the plane. The introduction of the external chamber $\mathcal{E}(0)$ allows us to define naturally the perimeter of the cluster $P(\mathcal{E})$ as the half-sum of the perimeter of each chamber, so that each piece of boundary is counted just once.

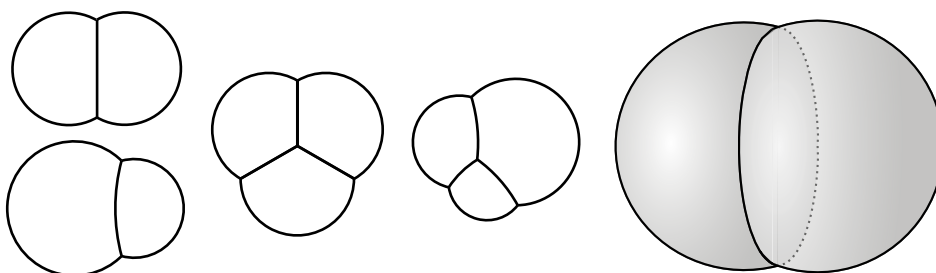


Figure 0.0.2: Some examples of perimeter-minimizing N -clusters for $N = 2, 3$ in dimension $n = 2$ and $n = 3$. The 2-clusters on the left are, respectively, the minimizer for the problem (0.0.1) with equal-volume (equal-area) chambers $v_1 = v_2$ and the minimizer for the problem (0.0.1) when different volumes $v_1 \neq v_2$ have been assigned. The same situation is the central one, for $N = 3$, while the right-hand picture is the perimeter-minimizing 2-cluster for equal-volume chambers in \mathbb{R}^3 .

the $(n - 1)$ dimensional area of the boundary, being careful to count once every possible common boundary between two sets? Technically speaking, given a vector of positive numbers $(v_1, \dots, v_N) \in \mathbb{R}^N$ we look for a family of N -disjoint (up to a negligible set) Borel sets $\mathcal{E}(1), \dots, \mathcal{E}(N)$ such that $|\mathcal{E}(i)| = v_i$ and

$$P(\mathcal{E}) = \inf \{P(\mathcal{F}) \mid \mathcal{F} \text{ is an } N\text{-cluster with } |\mathcal{F}(i)| = v_i\} \quad (0.0.1)$$

where

$$P(\mathcal{E}) := \sum_{i=0}^N \frac{P(\mathcal{E}(i))}{2}, \quad \mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{i=1}^N \mathcal{E}(i)$$

and where $P(\cdot)$ denotes the distributional perimeter (see [DG54]), that here could be intended as the $(n - 1)$ dimensional area of the boundary. The addition of the *external chamber* $\mathcal{E}(0)$ allows us to define the perimeter in a very natural way in order to count once every piece of boundary shared by two different set from the family (see Figure 0.0.1). The existence of such objects (see Figure 0.0.2), called *perimeter-minimizing N -clusters*, was proved by [Alm75], together with a partial regularity theorem (see Chapter One for details). Since the chambers of a perimeter-minimizing N -cluster will try to share as much boundary as possible these objects in general will presents some “angle point” that we call *singularity*. The collection of the singularity of an N -cluster \mathcal{E} is called *singular set* and

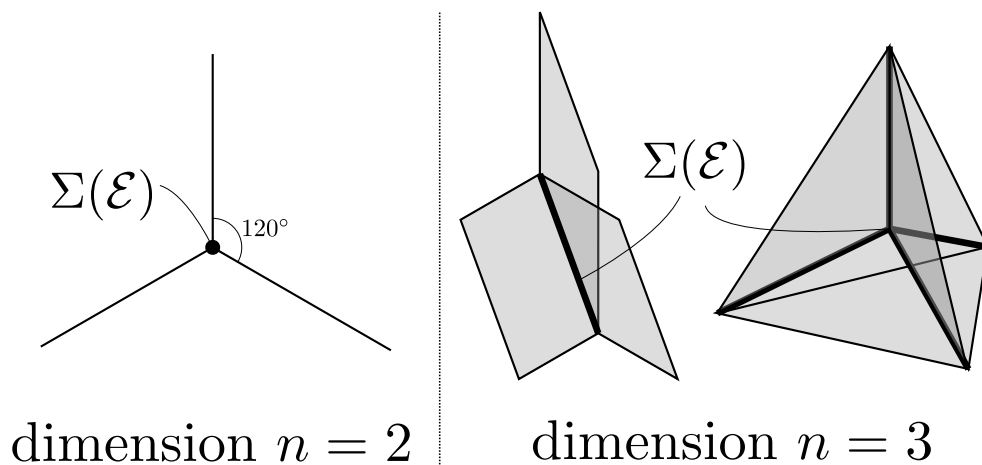
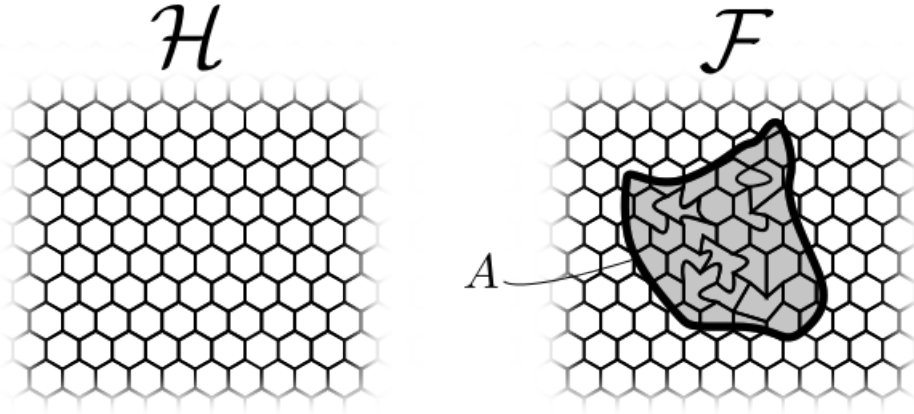


Figure 0.0.3: The singular set of a perimeter-minimizing N -cluster in dimension $n = 2$ and $n = 3$. In dimension $n = 2$ each singular point is an isolated point where three curves meet in three at equal angles. In dimension $n = 3$ the singular set $\Sigma(\mathcal{E})$ consists of Hölder continuous curves along which three sheets of the surface meet in three at equal angles together with isolated points at which four of such curves meet.

is usually denoted by $\Sigma(\mathcal{E})$. It is worth to remark that a complete characterization of the singular set of a perimeter-minimizing N -cluster, so far, is known only in dimension $n = 2$ (see [Mor09]) and $n = 3$ (see [Tay76]), depicted in Figure 0.0.3. Also a precise characterization of the minimizers is well-known only for few values of N . Essentially the ones depicted in Figure 0.0.2 are, so far, the only perimeter-minimizing known. The case $N = 2$ in the plane was solved in 1993 by Foisy, Alfaro, Brock, Hodges, Zimba and Jasonin in [FAB⁺93] while a proof for the case $N = 3$ was obtained by Wichiramala and it appeared first in 2002 in [Wic02]. The case $N = 2$ in the space was solved in 2002 by Hutchings, Morgan, Ritorè and Ros in [HMRR02]. The proof for $N = 2$ in higher dimension was obtained by Reichardt in [Rei08] as a generalization of the proof given by Hutchings, Morgan, Ritorè and Ros for the 3-dimensional case. In every situation listed above it has been proved that the minimizer is unique up to an isometry of the space. In 2002 Cicalese, Leonardi and Maggi in [CLM12] proves what is called a *quantitative inequality* for the case $N = 2, n = 2$. They showed, by exploiting that every standard double bubble \mathcal{B} is the only minimizer for the problem relative to its own volumes (problem (0.0.1) with $v_1 = |\mathcal{B}(1)|, v_2 = |\mathcal{B}(2)|$), that every other 2-cluster \mathcal{E} having the same volumes of \mathcal{B} and perimeter close to $P(\mathcal{B})$ must be diffeomorphic and close (in a suitable sense) to \mathcal{B} .

Let us briefly expose what are the topics treated in each chapter. We do not focus on the details, since every chapter has its own introduction where the main questions are clarified and exposed. We limit to give a brief overview.

In Chapter Two we provide an asymptotic result concerning perimeter-minimizing N -clusters with fixed boundary. Since the detailed study of perimeter-minimizing N -clusters for a fixed value of N seems to be a hard task, it could make sense to approach the problem from an asymptotic point of view, namely: is there some recognizable trend in the structure of these objects as N approaches $+\infty$? In 2001 in [Hal01], Thomas Hales provided a proof of the hexagonal honeycomb conjecture: the regular hexagonal tiling (a tiling can be viewed as an ∞ -cluster) provides the only partition of the plane in equal-area



$$P(\mathcal{H}; A) \leq P(\mathcal{F}; A)$$

Figure 0.0.4: The hexagonal honeycomb theorem (proved in 2001) states that the hexagonal tiling minimizes the localized perimeter among all its own compactly supported and mass preserving perturbations.

chambers having the minimum amount of localized perimeter (see Figure 0.0.4). His result provides an answer for the case $N = \infty$ and it turns out to be a powerful instrument for the study of the asymptotic behavior of perimeter-minimizing planar N -clusters.

Starting from Hales's result, it is natural to expect that the interior chambers of a minimizing planar N -cluster with equal-volume chambers try to get closer and closer to regular hexagons as N increase. However, we are still quite far from proving this fact. Another interesting question involves the external chamber of a minimizing planar N -cluster and it appears in [HM05]: does the boundary of the external chamber try to look like a circle (the isoperimetric profile for the case $N = 1$ in the plane) in order to save perimeter? We postpone this discussion to Chapter Two where we examine in depth questions concerning the global and local shape of perimeter-minimizing N -clusters. In particular, we provide a *uniform distribution-type theorem*, in the spirit of the one obtained in [ACO09], stating that, under some reasonable assumption on the structure of these objects and far away from the boundary of the external chamber the localized perimeter is uniformly distributed. Moreover we show that the localized perimeter is equal to the localized perimeter of the hexagonal tiling, up to a remainder that is a second order term. This seems to suggest that from an energetic point of view, the interior chambers of these objects are close to regular hexagons. This result was obtained in collaboration with prof. Giovanni Alberti.

Chapter Three is devoted to a quantitative version of the hexagonal honeycomb theorem. We show that if \mathcal{E} is a compactly supported and mass preserving perturbation of the hexagonal tiling \mathcal{H} and its localized perimeter is close to the localized perimeter of \mathcal{H} then \mathcal{E} must be diffeomorphic and close (in a suitable sense) to \mathcal{H} . This result is obtained by exploiting the techniques developed by Cicalese, Leonardi and Maggi in [CLM15] (starting from an idea contained in [CL12a]) and used by the authors to prove the sharp quantitative inequality for the planar double-bubble. This result was obtained in collaboration with prof. Francesco Maggi while I was visiting the University of Texas at Austin Fall

2014. This result, if combined with the energetic estimates contained in Chapter two, seems to suggest that the interior chambers of a perimeter-minimizing planar N -cluster with equal-volume chambers are close to regular hexagons, providing an answer to the initial question involving the asymptotic trend of these objects. The main obstacle that arises in developing this argument is that we actually have the strong information involving the shape of the chambers only when we deal with tilings and, in order to apply the quantitative version of the hexagonal honeycomb theorem, we need to be able to “convert” a cluster into a compactly supported perturbation of an hexagonal tiling without adding too much perimeter.

In Chapter Four we move to another type of isoperimetric problem concerning N -clusters that can be viewed as a generalization of the Cheeger problem (see [Che70], [Par11] and [Leo15] for more details about Cheeger problem). Given an open set Ω we look for the solution to the variational problem

$$H_N(\Omega) = \inf \left\{ \sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \mid \mathcal{E} \subseteq \Omega, \mathcal{E} \text{ } N\text{-cluster} \right\}.$$

This variational problem turns out to be related to spectral problem of the Laplacian with Dirichlet boundary conditions. We mainly focus on the regularity of the solution to this kind of problems in order to lay the basis for future investigations (see Chapter Five where some interesting directions of research in these topics are briefly exposed). The structure of these objects is slightly different from the one of perimeter-minimizing N -clusters. The reason is that there is no advantage, in this variational problem, in sharing boundary and thus the chambers will try to separate as much as possible (they are constrained into Ω). However $\frac{P(\cdot)}{|\cdot|}$ is not scale invariant and, in particular, it makes convenient to have chambers as big as possible. Hence there are two factors that compete in opposite directions leading to non-trivial solutions. These facts imply that the boundary of each chamber is locally a C^1 surface inside Ω and that no “angle points” (in the planar case) are attained. After we have discussed in detail the regularity of these objects we move to study their asymptotic behavior as N approaches $+\infty$. It is reasonable to expect some kind of periodicity in the asymptotic trend of these objects as N increases.

In Chapter Five we briefly point out how these topics could be related and we highlight some interesting issues related to these questions.

Let us conclude by saying that, probably, the main reason for which mathematicians are attracted by an isoperimetric problem relies in the wonderful symmetries that arise in seeking a solution. We cannot be emotionless as we become acquainted with a wonderful structure and every human being, rich or poor, quieten his incessant research of perfection standing in front of the symmetry. Probably we will never know whether God made men or not, but what we know for sure is that Symmetry made them equal.

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My Ph.D thesis would have never been completed without the infinite support of all these people or without their friendship.

Chapter 1

Sets of finite perimeter and N -clusters

In this chapter we define the general context of the theory of sets of finite perimeter without entering into the details of the proofs. We briefly recall the basic concepts about sets of finite perimeter we are going to use in the sequel. Here and in the sequel every set $E \subset \mathbb{R}^n$ will always be a Borel set. We denote, as usual, with \bar{E} , $\overset{\circ}{E}$ and ∂E respectively the *interior*, the *closure* and the *topological boundary* of the set E . We write $E \subset\subset F$ and say E is *compactly contained* in F if $\bar{E} \subset F$.

The proof of the results that are recalled in this section, besides more details about sets of finite perimeter and Radon measures, can be found in [Mor09], [Mag12], [Fed69]. The original works [DG54] and [DG55] from Ennio De Giorgi, where the foundational part of the theory of sets of finite perimeter is developed, are in italian. The english versions of such works can be found in the book [DGA06] at pp. 58-78 and 112-127.

1.1 Radon measures

The concept of Radon measures, more precisely of vector-valued Radon measures, plays a key role in the theory of sets of finite perimeter. We do not need to explain in detail what a vector-valued Radon measure is (for a complete overview on such a topic we refer to [Mag12] pp. 1-62) and so we just recall that vector measures can be represented as positive measures multiplied by a (summable) vector-valued density.

1.1.1 Definition of Radon measures

A measure $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathbb{R}^+$ is a **positive Radon measure on \mathbb{R}^n** (or simply a **Radon measure**) if

- a) any Borel set E is a μ -measurable set;
- b) for any set $F \subset \mathbb{R}^n$ there exists a Borel set $E \supseteq F$ such that $\mu(E) = \mu(F)$;
- c) $\mu(K) < +\infty$ for every $K \subset \mathbb{R}^n$ compact.

Property a) is ensuring that the family of all μ -measurable sets will be not trivial. Property b) gives us some sort of regularity, since it allows us to consider just the Borel's algebra, while property c) guarantees the local finiteness of μ .

We say that a measure $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathbb{R}^m$ is an \mathbb{R}^m -valued **Radon measure** (we sometimes simply write vector-valued Radon measure) if there exists a positive Radon measure μ_0 and Borel map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $|f(x)| = 1$ μ_0 -almost everywhere such that

$$\mu(E) = \int_E f(x) d\mu_0(x)$$

for every Borel set $E \subset \mathbb{R}^n$. Given a vector-valued Radon measure μ , the measure μ_0 associated to μ is uniquely identified and the **total variation of μ** :

$$|\mu| := \mu_0, \tag{1.1.1}$$

is well defined. Since also the density f is unique up to a $|\mu|$ -negligible set, in the sequel we are always adopting the notation

$$\mu = f|\mu|.$$

Note that $|\mu| = 0$ implies $\mu = 0$. Given a Radon measure μ we define the **support of μ** as the set

$$\text{spt}(\mu) = \{x \in \mathbb{R}^n \mid |\mu|(B_r(x)) > 0 \text{ for all } r > 0\}. \tag{1.1.2}$$

Every function $h \in L^1_{loc}(\mathbb{R}^n)$ induce a positive Radon measure on \mathbb{R}^n by defining

$$\mu(E) = \int_E |h(x)| dx,$$

for every Borel set E . In this case we write

$$\mu = h\mathcal{L}^n.$$

In particular this implies that, having defined *the characteristic function of a Borel set E* as

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.3}$$

if $h(x) = \mathbb{1}_E(x)$ for some E with $\mathcal{L}^n(E) = |E| < +\infty$, then the measure

$$\mu \llcorner E(F) = \mathcal{L}^n(E \cap F) = (\mathbb{1}_E \mathcal{L}^n)(F)$$

is a positive Radon measure.

In general, if μ is a positive Radon measure and $h \in L^1_{loc}(\mathbb{R}^n, \mu)$ is a function, we have that $\nu = |h|\mu$ is a positive Radon measure. In particular if E is a Borel set with $\mu(E) < +\infty$ we have that *the restriction of μ to E* defined on every Borel set F as

$$\mu \llcorner E(F) = \mu(E \cap F) = (\mathbb{1}_E \mu)(F)$$

is a positive Radon measure.

An example of vector-valued Radon measure can be obtained by setting $|\mu| = \mu_0 = \mathcal{L}^n$ and by choosing a generic Borel vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $|f(x)| = 1$ almost everywhere. Note that if $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a Borel vector field, then $\mu = g\mathcal{L}^n$ is an \mathbb{R}^m -valued Radon measure. Indeed, by defining

$$\begin{aligned} |\mu| &= |g(x)|\mathcal{L}^n, \\ f(x) &= \frac{g(x)}{|g(x)|} \mathbb{1}_{\{g(x) \neq 0\}}(x), \end{aligned}$$

we have that

$$\mu = f|\mu|$$

with $|f(x)| = 1$ for $|\mu|$ -almost every $x \in \mathbb{R}^n$.

1.1.2 Weak-star convergence of Radon measures

In order to speak of compactness and semi-continuity of perimeter we need to briefly introduce the weak-star convergence of Radon measures. A sequence of Radon measures $\{\mu_h\}_{h \in \mathbb{N}}$ on \mathbb{R}^n with values in \mathbb{R}^m is said to be **convergent in the weak-star sense** to a Radon measure μ , and we write $\mu_h \xrightarrow{*} \mu$, if and only if for every $\varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^m)$ it holds

$$\int_{\mathbb{R}^n} \varphi \cdot d\mu = \lim_{h \rightarrow +\infty} \int_{\mathbb{R}^n} \varphi \cdot d\mu_h.$$

The following equivalences about convergence of positive Radon measures are very useful, (see [Mag12, Proposition 4.26] for a detailed proof).

Proposition 1.1.1. *If $\{\mu_h\}_{h \in \mathbb{N}}$ and μ are positive Radon measures on \mathbb{R}^n , then the following three statements are equivalent.*

(i) $\mu_h \xrightarrow{*} \mu$.

(ii) If K is compact and A is open, then

$$\mu(K) \geq \limsup_{h \rightarrow +\infty} \mu_h(K), \quad (1.1.4)$$

$$\mu(A) \leq \liminf_{h \rightarrow +\infty} \mu_h(A). \quad (1.1.5)$$

(iii) If E is a Borel set with $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{h \rightarrow +\infty} \mu_h(E).$$

1.2 Sets of finite perimeter

1.2.1 Hausdorff measures and Hausdorff dimension

For every $s, \delta > 0$ the s -dimensional Hausdorff measure of step δ of a set $E \subset \mathbb{R}^n$ is defined as:

$$\mathcal{H}_\delta^s(E) = \inf_{\mathcal{F}_\delta} \left\{ \sum_{F \in \mathcal{F}_\delta} \omega_s \left(\frac{\text{diam}(F)}{2} \right)^s \right\} \quad (1.2.1)$$

where

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}, \quad \Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

and where the infimum in (1.2.1) is taken among all \mathcal{F}_δ , namely countable coverings of E by Borel sets $F \subset \mathbb{R}^n$ with $\text{diam}(F) \leq \delta$. If $s = k$ is an integer then ω_k is exactly the Lebesgue measure of a k -dimensional ball in \mathbb{R}^k . The **s -dimensional Hausdorff measure of a set $E \subset \mathbb{R}^n$** is then defined as:

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \{\mathcal{H}_\delta^s(E)\} = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(E). \quad (1.2.2)$$

From the definition it follows that the Hausdorff measure \mathcal{H}^s is invariant under isometries and that

$$\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E), \quad \forall \lambda, s > 0, E \subset \mathbb{R}^n.$$

Furthermore the following properties hold:

- 1) $\mathcal{H}^n(E) = \mathcal{L}^n(E)$ for every $E \subset \mathbb{R}^n$;
- 2) $\mathcal{H}^s(E) < +\infty$ implies $\mathcal{H}^t(E) = 0$ for every $t > s$;
- 3) $\mathcal{H}^s(E) > 0$ implies $\mathcal{H}^t(E) = +\infty$ for every $t < s$.

Thanks to property 2) and 3) above it is well defined the **Hausdorff dimension of a Borel set** E as

$$\dim(E) := \inf\{s \in [0, +\infty] \mid \mathcal{H}^s(E) = 0\} \quad (1.2.3)$$

$$= \sup\{s \in [0, +\infty] \mid \mathcal{H}^s(E) = +\infty\}. \quad (1.2.4)$$

We underline that if $1 \leq k \leq n-1$ and M is a k -dimensional C^1 -surface in \mathbb{R}^n then $\mathcal{H}^k(M)$ coincides with the classical k -dimensional area of M and $\dim(M) = k$ (we refer the reader to [Mag12, Chapter 3]). In the sequel, whenever we talk about the *dimension of a set* E we are always meaning the Hausdorff dimension of the set E .

Let us point out that property 1) and 3) tells us that \mathcal{H}^s is not a Radon measure in \mathbb{R}^n unless $s \geq n$ (and in this case, for $s > n$ it is trivial thanks to property 2)). Indeed $\mathcal{H}^s(A) = +\infty$ for every $s < n$ and every open set $A \subset \mathbb{R}^n$. Anyway, if E is such that $\mathcal{H}^s(E) < +\infty$ the measure $\mathcal{H}^s \llcorner E$, given by the restriction of \mathcal{H}^s to E , is a Radon measure on \mathbb{R}^n .

1.2.2 L^1 topology

Given a subset $\Omega \subseteq \mathbb{R}^n$ we need first to specify the topology that we are considering on the Borel's algebra of Ω . The correct one for this framework is the one induced by the L^1_{loc} convergence of the characteristics function. More precisely a sequence of Borel sets $\{E_h\}_{h \in \mathbb{N}}$ is converging to a set E (in L^1_{loc}) if and only if:

$$\mathbb{1}_{E_h} \xrightarrow{L^1_{loc}} \mathbb{1}_E$$

or equivalently if for every compact set $K \subset \Omega$ it holds

$$\lim_{h \rightarrow +\infty} |(E \Delta E_h) \cap K| \rightarrow 0. \quad (1.2.5)$$

Clearly, if the convergence of the characteristic functions is stronger, say L^1 , we speak of L^1 convergence instead of L^1_{loc} and (1.2.5) becomes just

$$\lim_{h \rightarrow +\infty} |E \Delta E_h| \rightarrow 0.$$

1.2.3 Sets of finite perimeter and Gauss-Green measure

A Borel set E of \mathbb{R}^n is said to be a *set of locally finite perimeter* if there exists an \mathbb{R}^n -valued Radon measure μ_E such that:

$$\int_E \operatorname{div}(T) \, dx = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \quad (1.2.6)$$

Notice that (1.2.6) just means that the characteristic function of E admits as distributional derivative the vector-valued Radon measure μ_E . In other words $D\mathbb{1}_E(x) = \mu_E$ in the sense

of distributions. The measure μ_E is also called the *Gauss-Green measure of E* and we define the relative perimeter of E in the Borel set $F \subset \mathbb{R}^n$:

$$P(E; F) = |\mu_E|(F), \quad (1.2.7)$$

where $|\mu_E|(F)$ denotes the total variation of μ_E defined in 1.1.1, formula (1.1.1).

The *perimeter of a set E* is defined as

$$P(E) := P(E; \mathbb{R}^n).$$

The reason why μ_E is called Gauss-Green measure is that whenever E is a set with C^1 boundary, the Gauss-Green Theorem implies

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$$

where ν_E denotes the outer unit normal of ∂E . Notice that in this case $P(E; F) = \mathcal{H}^{n-1}(\partial E \cap F)$, $P(E) = \mathcal{H}^{n-1}(\partial E)$.

By exploiting (1.2.6) and (1.2.7) we reach also the useful alternative definition of relative perimeter

$$P(E; A) = \sup \left\{ \int_E \operatorname{div}(T) \, dx \mid T \in C_c^1(A; B_1) \right\}, \quad (1.2.8)$$

when A is open and

$$P(E; F) = \inf \{ P(E; A) \mid A \text{ open and } F \subseteq A \}, \quad (1.2.9)$$

when F is a generic Borel set.

1.2.4 An equivalent definition of sets of finite perimeter

We sometimes make use of an equivalent definition of sets of finite perimeter introduced first by De Giorgi in [DG54] by exploiting regularizing kernels. More precisely, having defined $E_\varepsilon(x) := \mathbb{1}_E \star \rho_\varepsilon(x)$, where E is a given a Borel set and $\{\rho_\varepsilon(x)\}_{\varepsilon>0}$ is a regularizing kernel, if E has locally finite perimeter then

$$-(\nabla E_\varepsilon) \mathcal{L}^n \stackrel{*}{\rightharpoonup} \mu_E, \quad |\nabla E_\varepsilon| \mathcal{L}^n \stackrel{*}{\rightharpoonup} |\mu_E| \quad (1.2.10)$$

and conversely if E is such that

$$\limsup_{\varepsilon \rightarrow 0} \int_K |\nabla E_\varepsilon(x)| \, dx < \infty \quad \text{for all compact sets } K \quad (1.2.11)$$

then E has locally finite perimeter.

1.2.5 Compactness and semicontinuity with respect to the L^1 topology

In order to ensure existence of solutions in many variational problems we need a suitable compactness property of finite perimeter sets together with the semi-continuity of the functional perimeter (see [Mag12, Proposition 12.15, Theorem 12.26] for detailed proofs).

Theorem 1.2.1 (Compactness theorem for sets of finite perimeter). *Let $\{E_h\}_{h \in \mathbb{N}}$ be a sequence of sets of finite perimeter such that*

- a) $\sup_{h \in \mathbb{N}} \{P(E_h)\} < +\infty$
- b) *there exists $R > 0$ such that $E_h \subset B_R$ for all $h \in \mathbb{N}$.*

Then, there exists a subsequence $\{E_{h_j}\}_{j \in \mathbb{N}} \subseteq \{E_h\}_{h \in \mathbb{N}}$ and a set of finite perimeter $E \subset B_R$ such that

$$E_{h_j} \xrightarrow{L^1} E, \quad \mu_{E_{h_j}} \xrightarrow{*} \mu_E.$$

Theorem 1.2.2 (Lower semicontinuity of the perimeter). *If $\{E_h\}_{h \in \mathbb{N}}$ is a sequence of sets of locally finite perimeter in \mathbb{R}^n such that*

$$E_h \xrightarrow{L^1_{loc}} E, \quad \limsup_{h \rightarrow +\infty} P(E_h; K) < +\infty$$

for every compact set K in \mathbb{R}^n , then E is of locally finite perimeter in \mathbb{R}^n , $\mu_{E_h} \rightharpoonup^ \mu_E$ and, for every open set $A \subset \mathbb{R}^n$ we have*

$$P(E; A) \leq \liminf_{h \rightarrow +\infty} P(E_h; A). \quad (1.2.12)$$

1.2.6 The structure of the Gauss-Green measure

For every set E of locally finite perimeter the **reduced boundary** $\partial^* E$ is defined as the set of points $x \in \text{spt } \mu_E$ such that the limit

$$\lim_{r \rightarrow 0} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} \text{ exists and belongs to } S^{n-1} = \partial B_1. \quad (1.2.13)$$

For every point $x \in \partial^* E$ we set:

$$\nu_E(x) := \lim_{r \rightarrow 0} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))}.$$

The vector field ν_E is called **measure-theoretic outer unit normal to E** and by the Besicovitch-Lebesgue differentiation theorem we have that

$$\mu_E = \nu_E \llcorner |\mu_E| \llcorner \partial^* E.$$

Note that if E is a set of finite perimeter with reduced boundary $\partial^* E$ then

$$\begin{aligned} \partial^* E^c &= \partial^* E, \\ \nu_{E^c}(x) &= -\nu_E(x), \quad \forall x \in \partial^* E^c. \end{aligned}$$

A key tool in the whole theory of sets of finite perimeter is the following theorem due to De Giorgi about the structure of the Gauss-Green measure (see [DG55], [DGA06, pp. 111-127], [Mag12, Theorem 15.5, Theorem 15.9]).

Theorem 1.2.3 (De Giorgi's structure Theorem). *If E is a set of locally finite perimeter in \mathbb{R}^n , then the following properties hold.*

1) The Gauss-Green measure μ_E of E satisfies

$$|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \quad \mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \quad (1.2.14)$$

and the generalized Gauss-Green formula holds true:

$$\int_E \nabla \varphi \, dx = \int_{\partial^* E} \varphi \nu_E \, d\mathcal{H}^{n-1}, \quad \forall \varphi \in C_c^1(\mathbb{R}^n); \quad (1.2.15)$$

2) There exists countably many C^1 -hypersurfaces $\{M_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n$, compact sets $K_h \subset M_h$ and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ such that

$$\partial^* E = F \cup \bigcup_{h \in \mathbb{N}} K_h,$$

and for every $x \in K_h$, $\nu_E(x)^\perp = T_x M_h$ is the tangent space of M_h at x ;

3) For every $x \in \partial^* E$ the sequence of sets $\{E_{x,r} = \frac{E-x}{r}\}_{r>0}$ locally converges, (as $r \rightarrow 0^+$), to the half space

$$H_{\nu_E(x)} := \{y \in \mathbb{R}^n \mid y \cdot \nu_E(x) \leq 0\}$$

and it holds:

$$\mu_{E_{x,r}} \xrightarrow{*} \nu_E(x) \mathcal{H}^{n-1} \llcorner \partial H_{\nu_E(x)}, \quad |\mu_{E_{x,r}}| \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \partial H_{\nu_E(x)}.$$

1.2.7 Essential boundary

The n -dimensional density of a set E at the point x is the quantity

$$\vartheta_n(x, E) = \lim_{r \rightarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}, \quad (1.2.16)$$

whenever it exists. We notice that, thanks to the Besicovitch-Lebesgue differentiation theorem applied to the Radon measure $\mathcal{L}^n \llcorner E$, the limit in (1.2.16) exists for almost every x in \mathbb{R}^n . Given a set E we can define the set of points of \mathbb{R}^n having the same n -dimensional density $t \in [0, 1]$:

$$E^{(t)} = \{x \in \mathbb{R}^n \mid \vartheta_n(x, E) = t\}.$$

Note that $E^{(0)} = (\mathbb{R}^n \setminus E)^{(1)}$.

By denoting with $Q_r(x)$ a cube centered at x and with side-length r , we could have defined the n -dimensional density of a set E at the point x also as the limit

$$\bar{\vartheta}_n(x, E) = \lim_{r \rightarrow 0} \frac{|E \cap Q_r(x)|}{|Q_r(x)|},$$

whenever it exists. This two definitions are equivalent on the points of density 0 and 1. Indeed on every ball $B_r(x)$ it holds

$$Q_{\frac{2r}{\sqrt{n}}}(x) \subset B_r(x) \subset Q_{2r},$$

and thus

$$\vartheta_n(x, E) = 0 \Leftrightarrow \bar{\vartheta}_n(x, E) = 0,$$

$$\vartheta_n(x, E) = 1 \Leftrightarrow \bar{\vartheta}_n(x, E) = 1.$$

However in the sequel, unless it is not specified, we are always making use of Definition (1.2.16) since it is the most common one in literature.

With these notation the **essential boundary** $\partial^e E$ of a Borel set is defined as:

$$\partial^e E := \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)} \right) = \{x \in \mathbb{R}^n \mid 0 < \vartheta_n(x, E) < 1\}. \quad (1.2.17)$$

The following theorem clarifies the relation between the essential boundary and the reduced boundary of a set of finite perimeter E (see [Mag12, Theorem 16.2]).

Theorem 1.2.4. *If E is a set of locally finite perimeter in \mathbb{R}^n then $\partial^* E \subset E^{(\frac{1}{2})} \subset \partial^e E$ and*

$$\mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0. \quad (1.2.18)$$

A useful consequence of Theorem 1.2.4 is the following Lemma 1.2.5. The proof can be obtained as a consequence of [LT02, Theorem 4.1] or [Leo02, Theorem 2.4] on the structures of the Caccioppoli partitions combined with Theorem 1.2.4. Since Lemma 1.2.5 will be repeatedly used in Chapter 4 and since we have not been able to find a direct (and easy) proof of this fact in literature we provide a proof.

Lemma 1.2.5. *If E_1, \dots, E_k are k sets of locally finite perimeter such that*

$$|E_i \cap E_j| = 0 \quad \forall i \neq j,$$

then the following holds:

$$\begin{aligned} \partial^* \left(\bigcup_{i=1}^k E_i \right) &\approx \left(\bigcup_{i=1}^k \partial^* E_i \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k \partial^* E_j \cap \partial^* E_i \right) \\ &= \left[\bigcup_{i=1}^k \partial^* E_i \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^k \partial^* E_j \cap \partial^* E_i \right) \right] \end{aligned} \quad (1.2.19)$$

where the symbol \approx means equal up to an \mathcal{H}^{n-1} -negligible set. In particular for every ball $B_r = B_r(x)$ it holds:

$$P \left(\bigcup_{i=1}^k E_i; B_r \right) = \sum_{i=1}^k P(E_i; B_r) - \sum_{\substack{i,j=1, \\ j \neq i}}^k \mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap B_r). \quad (1.2.20)$$

Proof. Relation (1.2.20) follows straightforwardly from (1.2.19). We recall from Theorem 1.2.4 that $\partial^* E \approx E^{(\frac{1}{2})}$ for every locally finite perimeter set E . Hence, by setting $E_0 = \bigcup_{i=1}^k E_i$, it is enough to prove that there exist two \mathcal{H}^{n-1} -negligible set M_1, M_2 such that

$$E_0^{(\frac{1}{2})} \subseteq M_1 \cup \left[\left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right) \right] \subseteq (E_0^{(\frac{1}{2})} \cup M_2). \quad (1.2.21)$$

Let us also point out that, if E is a set of locally finite perimeter, Theorem 1.2.4 implies that there exists an \mathcal{H}^{n-1} -negligible set R with following property

$$\mathbb{R}^n = E^{(0)} \cup E^{(\frac{1}{2})} \cup E^{(1)} \cup R.$$

Thus, for every $i = 0, \dots, k$, we choose R_i be the \mathcal{H}^{n-1} -negligible set such that

$$\mathbb{R}^n = E_i^{(0)} \cup E_i^{(\frac{1}{2})} \cup E_i^{(1)} \cup R_i, \quad (1.2.22)$$

and we set

$$M_1 := \left(E_0^{(\frac{1}{2})} \cap \bigcup_{i=1}^k R_i \right), \quad M_2 := \bigcup_{i=1}^k R_i.$$

We prove that (1.2.21) holds with this choice of M_1, M_2 (note that $\mathcal{H}^{n-1}(M_1) = \mathcal{H}^{n-1}(M_2) = 0$ is immediate). Let us set, for the sake of brevity

$$F := M_1 \cup \left[\left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right) \right],$$

and divide the proof in two steps.

Step one: $E_0^{(\frac{1}{2})} \subseteq F$. In particular we prove that if $x \notin F$ then $x \notin E_0^{(\frac{1}{2})}$. For $x \notin F$ one of the following must be in force

a) $x \notin M_1$ and $x \in \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \cap \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right)$.

b) $x \notin M_1$ and $x \notin \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right)$ and in this case either:

b.1) $x \notin E_0^{(\frac{1}{2})}$ and $x \in \bigcup_{i=1}^k R_i$ and $x \notin \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right)$;

b.2) $x \notin E_0^{(\frac{1}{2})}$ and $x \notin \bigcup_{i=1}^k R_i$ and $x \notin \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right)$;

b.3) $x \in E_0^{(\frac{1}{2})}$ and $x \notin \bigcup_{i=1}^k R_i$ and $x \notin \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right)$.

If situation a) is in force we immediately have that $x \in E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})}$ for some $i \neq j$ which leads to $x \in E_0^{(1)}$ (since $|E_i \cap E_j| = 0$) and thus $x \notin E_0^{(\frac{1}{2})}$. Since b.1) and b.2) implies straightforwardly $x \notin E_0^{(\frac{1}{2})}$, we need just to verify that situation b.3) cannot be attained. Assume b.3) is in force and note that, for every $i = 1, \dots, k$, thanks to (1.2.22) it must hold $x \in E_i^{(1)} \cup E_i^{(0)}$. If $x \in E_i^{(0)}$ for all i we have $x \in E_0^{(0)}$. If, instead, $x \in E_i^{(1)}$ for some i then $x \in E_0^{(1)}$. In both cases we reach a contradiction because of $x \in E_0^{(\frac{1}{2})}$.

Step two: $F \subseteq (E_0^{(\frac{1}{2})} \cup M_2)$. For every $x \in F$ one of the following must be in force.

a) $x \in M_1$;

b) $x \in \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right)$ and $x \notin M_1$ and in this case either:

b.1) $x \in \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right)$ and $x \notin \bigcup_{i=1}^k R_i$;

b.2) $x \in \left(\bigcup_{i=1}^k E_i^{(\frac{1}{2})} \right) \setminus \left(\bigcup_{\substack{i,j=1 \\ j \neq i}}^k E_i^{(\frac{1}{2})} \cap E_j^{(\frac{1}{2})} \right)$ and $x \notin E_0^{(\frac{1}{2})}$;

If a) is the case, then $x \in M_1 \subset E_0^{(\frac{1}{2})}$ and we are done. If b.1) is in force then there exists exactly one j such that $x \in E_j^{(\frac{1}{2})}$ and $x \in E_i^{(0)}$ for $i \neq 0, j$ since the sets $\{E_h\}_{h=1}^k$ are disjoint up to an \mathcal{L}^n -negligible set. Thus

$$\frac{|(\mathbb{R}^n \setminus E_0) \cap B_r(x)|}{\omega_n r^n} = 1 - \frac{|E_j \cap B_r(x)|}{\omega_n r^n} - \sum_{\substack{i=1, \\ i \neq j}}^k \frac{|E_i \cap B_r(x)|}{\omega_n r^n},$$

which, passing to the limit as r goes to 0^+ implies $x \in (\mathbb{R}^n \setminus E_0)^{(\frac{1}{2})} = E_0^{(\frac{1}{2})}$. Finally, by considering situation b.2) we deduce that there exists exactly one $j \in \{1, \dots, k\}$ such that $x \in E_j^{(\frac{1}{2})}$ and $x \in E_i^{(0)} \cup R_i$ for $i \neq j$. If $x \in E_i^{(0)}$ for all $i \neq 0, j$ then, as above $x \in E_0^{(\frac{1}{2})}$ and this is a contradiction (in this situation we are assuming $x \notin E_0^{(\frac{1}{2})}$). Hence there is an index $i \neq 0$ such that $x \in R_i$ which means $x \in M_2$. The proof is complete. \square

1.2.8 Topological boundary

If A is an open set and E and F are sets of finite perimeter in A with $|(E \Delta F) \cap A| = 0$ then

$$P(E; A) = P(F; A).$$

Indeed considered a generic map $T \in C_c^1(A; B_1)$, by exploiting definition (1.2.8), we have

$$P(E; A) \geq \int_E \operatorname{div}(T) \, dx = \int_{E \cap A} \operatorname{div}(T) \, dx = \int_{F \cap A} \operatorname{div}(T) \, dx = \int_F \operatorname{div}(T) \, dx.$$

By taking the supremum among all $T \in C_c^1(A; B_1)$ we conclude $P(E; A) \geq P(F; A)$. The reverse inequality follows in the same way. Hence the distributional perimeter of a set E depends only on its L^1 equivalence class.

In particular this implies that the L^1 equivalence class of a set of finite perimeter contains a lot of set with very irregular topological boundary. For example if E is a set of finite perimeter, we can always find another set F with $\partial F = \mathbb{R}^n$ and $|E \Delta F| = 0$ so that $P(E) = P(F)$. The following proposition is what we need for select a “good” representative (see [Mag12, Proposition 12.19]).

Proposition 1.2.6. *If E is a set of locally finite perimeter in \mathbb{R}^n , then*

$$\text{spt}(\mu_E) = \{x \in \mathbb{R}^n \mid 0 < |E \cap B_r(x)| < \omega_n r^n, \quad \forall r > 0\} \subset \partial E.$$

Moreover there exists a Borel set F such that

$$|E \Delta F| = 0, \quad \text{spt}(\mu_F) = \partial F. \quad (1.2.23)$$

By (1.2.23) the set F given in Proposition 1.2.6 has perimeter equal to $P(E)$ and has a precise characterization of its topological boundary.

In the sequel, whenever we speak of a set of finite perimeter E , we implicitly assume E to be a representative of its own L^1 equivalence class satisfying $\text{spt}(\mu_E) = \partial E$.

1.2.9 Union, intersection and difference of finite perimeter sets

Let E and F be sets of locally finite perimeter. Then the intersection $E \cap F$, the union $E \cup F$ and the difference $E \setminus F$, $F \setminus E$ are sets of locally finite perimeter and the following properties hold:

$$\begin{aligned} \mu_{E \cap F} &= \mu_{E \setminus F}^{(1)} + \mu_{F \setminus E}^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\} \\ \mu_{E \cup F} &= \mu_{E \setminus F}^{(0)} + \mu_{F \setminus E}^{(0)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\} \\ \mu_{E \setminus F} &= \mu_{E \setminus F}^{(0)} - \mu_{F \setminus E}^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = -\nu_F\}, \end{aligned}$$

where

$$\begin{aligned} \{\nu_E = \nu_F\} &= \{x \in \partial^* E \cap \partial^* F \mid \nu_E(x) = \nu_F(x)\}, \\ \{\nu_E = -\nu_F\} &= \{x \in \partial^* E \cap \partial^* F \mid \nu_E(x) = -\nu_F(x)\}. \end{aligned}$$

Moreover the reduced boundaries satisfy

$$\partial^*(E \cap F) \approx (F^{(1)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = \nu_F\} \quad (1.2.24)$$

$$\partial^*(E \cup F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(0)} \cap \partial^* F) \cup \{\nu_E = \nu_F\} \quad (1.2.25)$$

$$\partial^*(E \setminus F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = -\nu_F\}, \quad (1.2.26)$$

where “ \approx ” means *equal up to an \mathcal{H}^{n-1} -negligible set*. It follows that, for every Borel set $G \subseteq \mathbb{R}^n$, the following hold:

$$P(E \cap F; G) = P(E; F^{(1)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G) \quad (1.2.27)$$

$$P(E \cup F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(0)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G) \quad (1.2.28)$$

$$P(E \setminus F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = -\nu_F\} \cap G). \quad (1.2.29)$$

We refer the reader to [Mag12, Theorem 16.3] for the proof of these assertions.

1.2.10 Indecomposable sets of finite perimeter

The notion of connectedness sets it is not relevant in the context of finite perimeter sets, since, by adding a suitable null set, we can always make a Borel set E connected. The correct notion in this context is that of *indecomposable set*. A set of finite perimeter E

is said to be **decomposable** if there exists two sets $E_1, E_2 \subseteq E$ with $0 < |E_1|, |E_2|$ and $|E_1 \cap E_2| = 0$ such that

$$P(E) = P(E_1) + P(E_2).$$

A set of finite perimeter E is said to be **indecomposable** if it is not decomposable, namely if for every $E_1, E_2 \subset E$ with $|E_1 \cap E_2| = 0$ and such that

$$P(E) = P(E_1) + P(E_2)$$

then either $|E_1| = 0$ or $|E_2| = 0$.

The following theorem allows us to define the indecomposable components of a set of finite perimeter E . We refer the reader to [ACMM01] for a detailed proof.

Theorem 1.2.7. *Let E be a set with finite perimeter in \mathbb{R}^n . Then there exists a unique finite or countable family of pairwise disjoint indecomposable set $\{E_i\}_{i \in \mathbb{N}}$ such that $|E_i| > 0$ and $P(E) = \sum_i P(E_i)$. Moreover*

$$\mathcal{H}^{n-1} \left(E^{(1)} \setminus \bigcup_i E_i^{(1)} \right) = 0$$

and the E_i 's are maximal indecomposable sets, i.e. any indecomposable set $F \subset E$ is contained up to an \mathcal{L}^n -negligible set in some set E_i .

We say that each set E_i given by Theorem 1.2.7 is an *indecomposable component* of E . In particular note that a set E is indecomposable if and only if it has only one indecomposable component.

The set E made by the union of two tangent ball B_1 and B_2 will be decomposable by setting $E_1 = B_1, E_2 = B_2$, since in this way

$$P(E) = P(E_1) + P(E_2).$$

In this case B_1 and B_2 are the indecomposable components of E .

A cube Q in \mathbb{R}^n instead is an example of indecomposable set.

A very useful relation is attained between perimeter and diameter in the class of indecomposable planar sets of finite perimeter.

Proposition 1.2.8. *If $E \subset \mathbb{R}^2$ is an indecomposable set of finite perimeter with $|E| < +\infty$, then*

$$P(E) \geq 2 \operatorname{diam}(E^{(1)}). \tag{1.2.30}$$

The validity of this fact can be deduced as a consequence of [Mag12, Proposition 19.22]. Relation (1.2.30) combined with Theorem 1.2.1 is very useful since gives us the compactness of a sequence of indecomposable sets of finite perimeter $\{E_h\}_{h \in \mathbb{N}}$ whenever a uniform bound on $P(E_h)$ holds.

1.2.11 First variation of perimeter and of the potential energy

A one-parameter family of diffeomorphisms $\{f_t \mid -\varepsilon < t < \varepsilon\}$ of \mathbb{R}^n is a *local variation in an open set A* if

$$\begin{aligned} f_0(x) &= x & \forall x \in \mathbb{R}^n, \\ \{x \in \mathbb{R}^n \mid f_t(x) \neq x\} &\subset\subset A & \forall |t| < \varepsilon. \end{aligned} \quad (1.2.31)$$

A map T is said to be the *initial velocity of a local variation $\{f_t\}_{|t|<\varepsilon}$ in A* if

$$T = \left. \frac{\partial f_t}{\partial t} \right|_{t=0}.$$

The following theorem allows us to compute the first variation of a perimeter for a finite perimeter set E (see [Mag12, Theorem 17.5])

Theorem 1.2.9. *Given an open set A , a set of finite perimeter E and a local variation $\{f_t \mid -\varepsilon < t < \varepsilon\}$ in A , then*

$$P(f_t(E); A) = P(E; A) + t \int_{\partial^* E} \operatorname{div}_E T(x) \, d\mathcal{H}^{n-1}(x) + o(t), \quad (1.2.32)$$

where T is the initial velocity of the local variation and

$$\operatorname{div}_E T(x) = \operatorname{div} T(x) - \nu_E(x) \cdot \nabla T(x) \nu_E(x) \quad \text{for } x \in \partial^* E,$$

is the tangential divergence of T on $\partial^* E$.

The following theorem, instead, is what we need to compute the first variation of a functional defined as $\mathcal{G}(E) = \int_E g \, dx$ for a continuous function g . In particular if $g = 1$ the theorem provides the first variation of the Lebesgue n -dimensional measure (see [Mag12, Theorem 17.8]).

Theorem 1.2.10. *Given an open set A , a set of finite perimeter E , $|E| < +\infty$ a continuous function $g \in C^0(\mathbb{R}^n)$ and a local variation $\{f_t \mid -\varepsilon < t < \varepsilon\}$ in A , then*

$$\int_{f_t(E)} g(x) \, dx = \int_E g(x) \, dx + t \int_{\partial^* E} g(x) (T(x) \cdot \nu_E(x)) \, d\mathcal{H}^{n-1}(x) + o(t), \quad (1.2.33)$$

where T is the initial velocity of the local variation.

Remark 1.2.11. If E is a set of finite perimeter with $|E| < +\infty$ and we apply Theorem 1.2.10 by choosing $g(x) = 1$ we obtain the useful formula

$$|f_t(E)| = |E| + t \int_{\partial^* E} (T(x) \cdot \nu_E(x)) \, d\mathcal{H}^{n-1}(x) + o(t). \quad (1.2.34)$$

1.2.12 Distributional mean curvature of a set of finite perimeter

Let E be a finite perimeter set, A an open set and H_E a function in $L^1(A \cap \partial^* E; \mathcal{H}^{n-1})$. We say that H_E is the **distributional mean curvature of E in A** if it holds:

$$\int_{\partial^* E \cap A} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap A} H_E(x) (T \cdot \nu_E) \, d\mathcal{H}^{n-1} \quad \forall T \in C_c^\infty(A; \mathbb{R}^n). \quad (1.2.35)$$

Note that if E is a finite perimeter set with distributional mean curvature H_E , then the distributional mean curvature of E^c is $-H_E$.

Remark 1.2.12. Let E be a set of finite perimeter with constant distributional mean curvature equal to C in an open set A . Given an initial velocity $T(x) \in C_c(A; \mathbb{R}^n)$ and having defined the local variation $f_t(x) = \{x + tT(x) \mid -\varepsilon < t < \varepsilon\}$, by putting together (1.2.32) and (1.2.35) we obtain the useful formula:

$$\frac{d}{dt}P(f_t(E))\Big|_{t=0} = \int_{\partial^*E \cap A} \operatorname{div}_E T(x) \, d\mathcal{H}^{n-1}(x) = C \int_{\partial^*E \cap A} (T \cdot \nu_E) \, d\mathcal{H}^{n-1}. \quad (1.2.36)$$

1.3 Regularity of perimeter almost minimizing sets

Definition 1.3.1 ((Λ, r_0) -perimeter-minimizing inside Ω , [Mag12] pp. 278-279). We say that a set of finite perimeter E is a (Λ, r_0) -perimeter-minimizing in Ω if for every $B_r \subset \Omega$ with $r < r_0$ and every set F such that $E \Delta F \subset \subset B_r$, it holds

$$P(E; B_r) \leq P(F; B_r) + \Lambda |E \Delta F|.$$

The following theorem clarifies why Definition 1.3.1 is so important (see [Mag12, Chapter 21 and pp. 354, 363-365]).

Theorem 1.3.2. *If Ω is an open set in \mathbb{R}^n , $n \geq 2$ and E is a (Λ, r_0) -perimeter-minimizing in Ω , with $\Lambda r_0 \leq 1$, then for every $\alpha \in (0, 1)$ the set $\Omega \cap \partial^*E$ is a $C^{1,\alpha}$ hypersurface that is relatively open in $\Omega \cap \partial E$, and it is \mathcal{H}^{n-1} equivalent to $\Omega \cap \partial E$. Moreover, setting*

$$\Sigma(E; \Omega) := \Omega \cap (\partial E \setminus \partial^*E),$$

then the following statements hold true:

- (i) if $2 \leq n \leq 7$, then $\Sigma(E; \Omega)$ is empty;
- (ii) if $n = 8$, then $\Sigma(E; \Omega)$ is discrete;
- (iii) if $n \geq 9$, then $\mathcal{H}^s(\Sigma(E; \Omega)) = 0$ for every $s > n - 8$.

The set $\Sigma(E; \Omega)$ is called **singular set**. In every dimension greater than or equal to 8 it is possible to exhibit an example of a (Λ, r_0) -perimeter-minimizing set E with $\mathcal{H}^{n-8}(\Sigma(E)) > 0$ (see [DPP09], [BDGG69], [Mag12, Section 28.6]). Assertion (iii) has been recently improved in [NV15] where the authors show that the singular sets of minimizing hypersurfaces in dimension greater than or equal to 8 is exactly an $(n - 8)$ rectifiable sets with finite $(n - 8)$ -dimensional Hausdorff measure.

1.4 Useful inequalities for sets of finite perimeter

We here recall some useful inequalities holding on the family of sets of finite perimeter.

1.4.1 Isoperimetric inequality

For every set of finite perimeter E with $|E| < +\infty$ it holds

$$P(E) \geq n\omega_n^{1/n} |E|^{1-1/n} = P(B^E), \quad (1.4.1)$$

where B^E is a ball such that $|B^E| = |E|$. Equality is attained if and only if E is (equivalent to) a ball. Note that (1.4.1) states that among all sets of finite perimeter E with the same amount of fixed volume $|E| = v$ the n -dimensional Euclidean ball with radius $r = \left(\frac{v}{\omega_n}\right)^{\frac{1}{n}}$ is the one attaining the smallest perimeter (see [DG58], [DGA06, pp. 185-197], [Mag12, Chapter 14]). Quantitative versions of (1.4.1) are provided, through different methods, in [CL12b], [FMP08] and [FMP10].

1.4.2 Isodiametric inequality

For every Borel set E with $|E| < +\infty$ it holds

$$|E| \leq \left(\frac{\text{diam}(E)}{2} \right)^n = |B_{\frac{\text{diam}(E)}{2}}| \quad (1.4.2)$$

where $B_{\frac{\text{diam}(E)}{2}}$ is a ball of radius $\text{diam}(E)/2$. Equality is attained if and only if E is (equivalent to) a ball. Note that (1.4.2) states that among all the Borel sets E with the same diameter $\text{diam}(E) = d$ the n -dimensional Euclidean ball with radius $r = \frac{d}{2}$ is the one attaining the biggest volume (see [Mag12, Theorem 3.11]). A quantitative version of (1.4.2) is provided in [MP14].

1.4.3 Cheeger inequality for Borel sets

For a bounded Borel set $E \subseteq \mathbb{R}^n$, the **Cheeger constant of E** is defined as

$$h(E) = \inf \left\{ \frac{P(F)}{|F|} \mid F \subset E, \text{ set of finite perimeter} \right\} \quad (1.4.3)$$

Then for every set of finite perimeter $E \subset \mathbb{R}^n$ and it holds

$$h(E) \geq \frac{n\omega_n^{1/n}}{|E|^{1/n}} = \omega_n^{1/n} \frac{h(B)}{|E|^{1/n}}. \quad (1.4.4)$$

where B is a ball of unit-radius. Equality is attained if and only if E is (equivalent to) a ball. Note that (1.4.4) states that among all sets of finite perimeter E with the same amount of fixed volume $|E| = v$ the n -dimensional Euclidean ball with radius $r = \left(\frac{v}{\omega_n} \right)^{\frac{1}{n}}$ is the one attaining the smallest Cheeger constant. The proof of this fact can be obtained as a consequence of the isoperimetric property of the ball (1.4.1). A quantitative version of (1.4.4) is provided in [FMP09].

We refer to Chapter 4, Section 4.1 below where the main properties of the Cheeger constant together with some brief historical notes are recalled.

1.5 N -clusters and tilings

1.5.1 N -clusters of \mathbb{R}^n

By N -cluster we mean a family of N sets of finite perimeter $\mathcal{E} = \{\mathcal{E}(0), \mathcal{E}(1), \dots, \mathcal{E}(N)\} \subset \mathbb{R}^n$, having positive Lebesgue measure and pairwise disjoint up to a set of measure zero. In other words a family of Borel sets $\{\mathcal{E}(i)\}_{i=1}^N$ is called an N -cluster if

- 1) $P(\mathcal{E}(i)) < +\infty$, for $i = 1, \dots, N$;
- 2) $0 < |\mathcal{E}(i)| < +\infty$, for $i = 1, \dots, N$;
- 3) $|\mathcal{E}(i) \cap \mathcal{E}(k)| = 0$ for all $k \neq i$.

We allow in the previous definition also the case $N = +\infty$. In the sequel, unless it is not specified, we are always assuming that $N < +\infty$. We define the **volume vector of \mathcal{E}** , $\text{vol}(\mathcal{E}) \in \mathbb{R}^N$ as:

$$\text{vol}(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|).$$

The **external chamber** $\mathcal{E}(0)$ of the N -cluster \mathcal{E} is the set

$$\mathcal{E}(0) := \mathbb{R}^n \setminus \left(\bigcup_{i=1}^N \mathcal{E}(i) \right). \quad (1.5.1)$$

We define the (h, k) -interface of \mathcal{E} as

$$\mathcal{E}(h, k) := \partial^* \mathcal{E}(h) \cap \partial^* \mathcal{E}(k). \quad (1.5.2)$$

We moreover introduce the boundary of \mathcal{E} and the reduced boundary of \mathcal{E} as

$$\partial \mathcal{E} := \bigcup_{i=1}^N \partial \mathcal{E}(i), \quad \partial^* \mathcal{E} := \bigcup_{0 \leq h < k \leq N} \mathcal{E}(h, k). \quad (1.5.3)$$

With these notations we can easily define the *perimeter of an N -cluster \mathcal{E}* relative to a Borel set F as

$$P(\mathcal{E}; F) := \frac{1}{2} \sum_{i=0}^N P(\mathcal{E}(i); F) = \sum_{0 \leq h < k \leq N} \mathcal{H}^{n-1}(\mathcal{E}(h, k) \cap F). \quad (1.5.4)$$

Note that with the notation (1.5.3) introduced above, on every Borel set F it holds:

$$P(\mathcal{E}; F) = \mathcal{H}^{n-1}(\partial^* \mathcal{E} \cap F).$$

We define the *distance* between two given N -clusters \mathcal{E} and \mathcal{F} as

$$d(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \sum_{i=0}^N |\mathcal{E}(i) \Delta \mathcal{F}(i)|.$$

1.5.2 Planar tilings

A planar tiling is a countable family of sets of finite perimeter in \mathbb{R}^2 , $\mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^{+\infty} \subset \mathbb{R}^2$, such that

- 1) $P(\mathcal{E}(i)) < +\infty$, for all $i \geq 1$;
- 2) $0 < |\mathcal{E}(i)| < +\infty$, for all $i \geq 1$;
- 3) $|\mathcal{E}(i) \cap \mathcal{E}(k)| = 0$ for all $k \neq i$;
- 4) $|\mathcal{E}(0)| = 0$.

A planar tiling is substantially an ∞ -cluster with empty external chamber. In the sequel the *regular hexagonal tiling of \mathbb{R}^2* (or simply the *hexagonal tiling*) is the planar tiling $\mathcal{H} = \left\{ H + \sqrt[4]{12} \left(\frac{k}{\sqrt{3}}, \frac{j}{2} \right) \right\}_{k, j \in \mathbb{Z}}$ where H denotes a unit-area regular hexagon.

1.5.3 The flat torus

Let $v, w \in \mathbb{R}^2$ be two orthogonal vectors. We say that two points $x_1, x_2 \in \mathbb{R}^2$ are equivalent, and we write $x_1 \sim x_2$, if there exists two integers $k_v, k_w \in \mathbb{Z}$ such that

$$x_1 - x_2 = k_v v + k_w w.$$

We define the *flat torus* to be the collection of all the equivalence classes of \mathbb{R}^2 with respect to \sim :

$$\mathcal{T}(v, w) := \mathbb{R}^2 / \sim.$$

Note that

$$Q_{\mathcal{T}} := \{sv \mid s \in (0, 1]\} \times \{tw \mid t \in (0, 1]\} \subset \mathbb{R}^2$$

is a fundamental domain, namely a set containing exactly one representative in each equivalence class. Moreover for any given Borel set $E \subseteq \mathcal{T}(v, w)$ we can always consider the *periodic extension* $\widehat{E} \subset \mathbb{R}^2$. Thus it is well defined the relative perimeter of $E \subset \mathcal{T}(v, w)$ inside $F \subset \mathcal{T}(v, w)$ as

$$P_{\mathcal{T}}(E; F) := P(\widehat{E}; \widehat{F} \cap Q_{\mathcal{T}}).$$

The total perimeter of a set E is then defined as

$$P_{\mathcal{T}}(E) := P(\widehat{E}; Q_{\mathcal{T}}).$$

Note that if $E, F \subseteq Q_{\mathcal{T}}$, by denoting with

$$E' = E / \sim, \quad F' = F / \sim$$

it must hold:

$$P_{\mathcal{T}}(E'; F') = P(E; F).$$

Indeed $\partial^* \widehat{E}' \cap \widehat{F}' \cap Q_{\mathcal{T}} = \partial^* E \cap F$. For this reason in the sequel we avoid the subscript \mathcal{T} and we simply write $P(E; F)$ also to denote the relative perimeter (and the perimeter) of a Borel set $E \subseteq \mathcal{T}(v, w)$ inside $F \subseteq \mathcal{T}(v, w)$. We usually write \mathcal{T} instead of $\mathcal{T}(v, w)$ whenever the role of the vectors v, w is clear from the context.

1.5.4 N -clusters on the torus

Given a flat torus \mathcal{T} we define an N -cluster of the torus as a family of N Borel sets $\{\mathcal{E}(1), \dots, \mathcal{E}(N)\} \subset \mathcal{T}$ with

- 1) $P(\mathcal{E}(i)) < +\infty$, for all $i \geq 1$;
- 2) $0 < |\mathcal{E}(i)|$, for all $i \geq 1$;
- 3) $|\mathcal{E}(i) \cap \mathcal{E}(k)| = 0$ for all $k \neq i$;

Note that, since $|\mathcal{T}| < +\infty$ we do not need to add the request $|\mathcal{E}(i)| < +\infty$ as in the planar case. The external chamber is then defined as

$$\mathcal{E}(0) = \mathcal{T} \setminus \left(\bigcup_{i=1}^N \mathcal{E}(i) \right).$$

The volume of \mathcal{E} is $\text{vol}(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|)$, and the relative perimeter of \mathcal{E} in $A \subset \mathcal{T}$ is given by

$$P(\mathcal{E}; A) = \frac{1}{2} \sum_{h=0}^N P(\mathcal{E}(h); A),$$

while the distance between two N -clusters \mathcal{E} and \mathcal{F} is defined as

$$d(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \sum_{h=0}^N |\mathcal{E}(h) \Delta \mathcal{F}(h)|.$$

1.5.5 N -tilings of the torus

An N -tiling of a two-dimensional flat torus \mathcal{T} is an N -cluster $\mathcal{E} \subseteq \mathcal{T}$ with the additional request

$$|\mathcal{T} \setminus \bigcup_{h=1}^N \mathcal{E}(h)| = |\mathcal{E}(0)| = 0.$$

The volume of \mathcal{E} is $\text{vol}(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|)$, and the relative perimeter of \mathcal{E} in $A \subset \mathcal{T}$ is given by

$$P(\mathcal{E}; A) = \frac{1}{2} \sum_{h=1}^N P(\mathcal{E}(h); A),$$

while the distance between two tilings \mathcal{E} and \mathcal{F} is defined as

$$d(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \sum_{h=1}^N |\mathcal{E}(h) \Delta \mathcal{F}(h)|.$$

We say that \mathcal{E} is a *unit-area tiling* of \mathcal{T} if $|\mathcal{E}(h)| = 1$ for every $h = 1, \dots, N$. (In particular, in that case, it must be $N = |\mathcal{T}|$).

Obviously, every N -cluster is an $(N + 1)$ -tiling and every N -tiling defines an $(N - 1)$ -cluster. Notice also that every N -tiling of a flat torus \mathcal{T} can be viewed as a periodic planar tiling in \mathbb{R}^2 .

1.6 Set operations on Clusters

1.6.1 Union of Clusters

An N -cluster \mathcal{E} and an M -cluster \mathcal{F} are said to be *disjoint* if

$$|\mathcal{E}(i) \cap \mathcal{F}(j)| = 0, \quad \text{for every } i = 1, \dots, N \text{ and } j = 1, \dots, M.$$

In this case we define the $(M + N)$ -Cluster $\mathcal{E} \cup \mathcal{F}$ as

$$\mathcal{E} \cup \mathcal{F} := \{\mathcal{E}(1), \dots, \mathcal{E}(N), \mathcal{F}(1), \dots, \mathcal{F}(M)\}.$$

Note that $(\mathcal{E} \cup \mathcal{F})(0) = \mathcal{E}(0) \cap \mathcal{F}(0)$.

By exploiting formulas (1.2.27), we obtain:

$$P(\mathcal{E} \cup \mathcal{F}) = P(\mathcal{E}) + P(\mathcal{F}) - \mathcal{H}^{n-1}(\partial^* \mathcal{E}(0) \cap \partial^* \mathcal{F}(0)). \quad (1.6.1)$$

1.6.2 Intersection of a Cluster with a Borel set

Given a Borel set F and an N -cluster \mathcal{E} we define the cluster $\mathcal{E} \cap F$ as the family of sets:

$$\mathcal{E} \cap F := \{\mathcal{E}(j) \cap F \text{ for all } j = 1, \dots, N \text{ such that } |\mathcal{E}(j) \cap F| > 0\}.$$

Note that $(\mathcal{E} \cap F)(0) = \mathcal{E}(0) \cup F^c$. The number of chambers k of $\mathcal{E} \cap F$ is given by

$$k = \#\{j \in \{1, \dots, N\} \mid |\mathcal{E}(j) \cap F| > 0\}.$$

By exploiting formulas (1.2.27) and (1.2.28), we obtain:

$$P(\mathcal{E} \cap F) = P(\mathcal{E}; F^{(1)}) + P(F; \mathcal{E}(0)^{(0)}) + \mathcal{H}^{n-1}(\{\nu_{\mathcal{E}(0)} = -\nu_F\}). \quad (1.6.2)$$

1.6.3 Difference between a set and a cluster

Given a Borel set F and an N -cluster \mathcal{E} we define the Borel set

$$F \setminus \mathcal{E} := \bigcap_{i=1}^N (F \setminus \mathcal{E}(i)) = F \cap \mathcal{E}(0).$$

By exploiting formula (1.2.29), we obtain:

$$P(F \setminus \mathcal{E}) = P(F; \mathcal{E}(0)^{(1)}) + P(\mathcal{E}(0); F^{(1)}) + \mathcal{H}^{n-1}(\{\nu_{\mathcal{E}(0)} = \nu_F\}). \quad (1.6.3)$$

We also define the cluster $\mathcal{E} \setminus F$ as

$$\mathcal{E} \setminus F := \{\mathcal{E}(j) \setminus F \mid \text{for all } j = 1, \dots, N \text{ such that } |\mathcal{E}(j) \setminus F| > 0\} = \mathcal{E} \cap F^c.$$

Note that $(\mathcal{E} \setminus F)(0) = \mathcal{E}(0) \cup F$. The number of chambers k of $\mathcal{E} \setminus F$ is given by

$$k = \#\{j \in \{1, \dots, N\} \mid |\mathcal{E}(j) \setminus F| > 0\}.$$

By exploiting formula (1.6.2), we obtain:

$$P(\mathcal{E} \setminus F) = P(\mathcal{E}; F^{(0)}) + P(F; \mathcal{E}(0)^{(0)}) + \mathcal{H}^{n-1}(\{\nu_{\mathcal{E}(0)} = \nu_F\}). \quad (1.6.4)$$

1.6.4 Symmetric difference between clusters

Given two N -clusters \mathcal{E}, \mathcal{F} we define the *symmetric difference between \mathcal{E} and \mathcal{F}* as the set

$$\mathcal{E} \Delta \mathcal{F} := \bigcup_{i=1}^N \mathcal{E}(i) \Delta \mathcal{F}(i).$$

1.7 $C^{k,\alpha}$ N -clusters in \mathbb{R}^2

For a given a closed curve with boundary $\gamma : [a, b] \rightarrow \mathbb{R}^2$ we introduce the notations

$$\gamma = \gamma([a, b]), \quad \text{int}(\gamma) = \gamma((a, b)), \quad \text{bd}(\gamma) = \{\gamma(a), \gamma(b)\}.$$

We say that a family of closed connected curves with boundary $\{\gamma_i\}_{i \in I}$ is a *network* if, having defined $\{p_j\}_{j \in J} = \{\text{bd}(\gamma_i)\}_{i \in I}$, the following properties hold:

- 1) I and J are at most countable;

2) $\{p_j\}_{j \in J}$ and $\{\gamma_i\}_{i \in I}$ are locally finite, in the sense that

$$\#(\{j \in J \mid p_j \in B_r\}) + \#(\{i \in I \mid \gamma_i \cap B_r \neq \emptyset\}) < +\infty \quad \text{for all } r > 0;$$

3) $\text{int}(\gamma_i) \cap \text{int}(\gamma_h) = \emptyset$, for all $i, h \in I$, $i \neq h$;

4) Each p_j is a common end-point to at least three different curves from $\{\gamma_i\}_{i \in I}$.

If each γ_i is also a $C^{k,\alpha}$ -curve we say that the family $\{\gamma_i\}_{i \in I}$ is a $C^{k,\alpha}$ -network.

We say that a cluster $\mathcal{E} \subset \mathbb{R}^2$ is of class $C^{k,\alpha}$ inside an open set Ω if there exists a $C^{k,\alpha}$ -network $\{\gamma_i\}_{i \in I}$ such that

$$\partial\mathcal{E} \cap \Omega = \bigcup_{i \in I} \gamma_i, \quad (1.7.1)$$

$$\partial^*\mathcal{E} \cap \Omega = \bigcup_{i \in I} \text{int}(\gamma_i), \quad (1.7.2)$$

$$\Sigma(\mathcal{E}; \Omega) = \Omega \cap (\partial\mathcal{E} \setminus \partial^*\mathcal{E}) = \Omega \cap \bigcup_{j \in J} \{p_j\}. \quad (1.7.3)$$

If $\Omega = \mathbb{R}^2$ we simply say that \mathcal{E} is of class $C^{k,\alpha}$.

1.8 perimeter-minimizing N -clusters

1.8.1 Definition of perimeter-minimizing N -clusters

An N -cluster \mathcal{E} is said to be a **perimeter-minimizing N -cluster** if for every other N -cluster \mathcal{F} with

$$\text{vol}(\mathcal{E}) = \text{vol}(\mathcal{F})$$

(up to relabeling the chambers) it holds

$$P(\mathcal{E}) \leq P(\mathcal{F}).$$

The existence of perimeter-minimizing N -clusters was proved by Almgren in [Alm76] where also a partial regularity of these objects was discussed.

1.8.2 (Λ, r_0) -perimeter-minimizing N -clusters inside Ω

We say that \mathcal{E} is a (Λ, r_0) -perimeter-minimizing N -cluster inside an open set Ω if for every $B_r(x) \subset \Omega$ with $r < r_0$ and every N -cluster \mathcal{F} with

$$\mathcal{E} \Delta \mathcal{F} \subset \subset B_r(x),$$

it holds

$$P(\mathcal{E}; B_r(x)) \leq P(\mathcal{F}; B_r(x)) + \Lambda d(\mathcal{E}, \mathcal{F}). \quad (1.8.1)$$

It can be shown that each perimeter-minimizing N -cluster \mathcal{E} is a (Λ, r_0) -perimeter-minimizing cluster for a suitable choice of Λ and r_0 and that this fact leads to the regularity given by Theorem 1.3.2 on $\partial^*\mathcal{E}$ and that the singular set

$$\Sigma(\mathcal{E}; A) := (\partial\mathcal{E} \setminus \partial^*\mathcal{E}) \cap A, \quad (1.8.2)$$

is closed and \mathcal{H}^{n-1} -negligible. More precisely the following statement holds true (see [CLM15, Corollary 4.6], [Mag12, Chapter 30])

Theorem 1.8.1. *If $\mathcal{E} \subset \mathbb{R}^n$ is a (Λ, r_0) -perimeter-minimizing cluster in an open set Ω , then $\partial^* \mathcal{E} \cap \Omega$ is a $C^{1,\alpha}$ -hypersurface for every $\alpha \in (0, 1)$, it is relatively open inside $\partial \mathcal{E} \cap \Omega$, and $\mathcal{H}^{n-1}(\Sigma(\mathcal{E}; \Omega)) = 0$. Moreover, if $n = 2$, then we can replace $C^{1,\alpha}$ with $C^{1,1}$.*

As pointed out in Theorem 1.8.1 in the planar case it is possible to improve the regularity of perimeter-minimizing N -clusters (see for example [Mag12, Section 30.3], [Mor09]) thanks to a detailed study of the singular set $\Sigma(\mathcal{E}) = \partial \mathcal{E} \setminus \partial^* \mathcal{E}$. In particular it can be shown that each perimeter-minimizing N -cluster in the plane is a cluster of class $C^{1,1}$ and

$$\partial^* \mathcal{E} = \bigcup_{i \in I} \text{int}(\gamma_i)$$

where each curve γ_i is either a segment or a circular arc. Furthermore any two arcs belonging to the same interface $\mathcal{E}(h, k)$ have the same curvature, the singular set $\Sigma(\mathcal{E})$ is locally finite and each singular point $p_j \in \Sigma(\mathcal{E})$ is a common end-point to exactly three different curves from $\{\gamma_i\}_{i \in I}$, which form three 120 degree angles at p_j .

Remark 1.8.2. In the celebrated work of Taylor [Tay76] the singular set of a 3-dimensional perimeter-minimizing N -cluster is completely characterized. In particular is proved that the singular set $\Sigma(\mathcal{E})$ consists of Hölder continuously differentiable curves along which three sheets of the surface meet at equal angles, together with isolated points at which four such curves meet bringing together six sheets of the surface at equal angles.

So far, except for the general regularity structure given by Theorem 1.3.2, the description of the singular set of a perimeter-minimizing N -cluster in dimension bigger than 3 is still mostly unknown.

1.9 Useful tools from the theory of N -clusters

1.9.1 Hales's Theorem and its consequences

On every flat torus \mathcal{T} the following Theorem due to Hales ([Hal01]) holds true.

Theorem 1.9.1. *If \mathcal{E} is an N -cluster of a torus \mathcal{T} , with $|\mathcal{E}(i)| \leq 1$ for every $i = 1, \dots, N$ then the following estimate holds*

$$P(\mathcal{E}) \geq \frac{P(H)}{2} \left(\min\{|\mathcal{E}(0)|, 1\} + \sum_{i=1}^N |\mathcal{E}(i)| \right), \quad (1.9.1)$$

where H denotes a unit-area regular hexagons. Equality in (1.9.1) is attained if and only if \mathcal{E} is an hexagonal tiling with unit-area chambers.

Theorem 1.9.2 tells us that among all the N -clusters (tilings) of the torus with unit-area chambers, the hexagonal tiling is the one attaining the smallest perimeter.

If \mathcal{E} is a bounded planar N -cluster we can always find two orthogonal vectors v, w such that

$$\mathcal{E} \subset \subset \{sv \mid s \in (0, 1]\} \times \{tw \mid t \in (0, 1]\}.$$

We can consider the $\mathcal{E}' := \{\mathcal{E}(i)/\sim\}_{i=1}^N$ on the flat torus $\mathcal{T} = \mathcal{T}(v, w)$ and apply Hales's Theorem. As a consequence starting from Theorem 1.9.1 it is possible to prove the following Theorem (also appearing in [Hal01]) holding on planar N -clusters.

Theorem 1.9.2. *If \mathcal{E} is a bounded planar N -cluster with $|\mathcal{E}(i)| \leq 1$ for every $i = 1, \dots, N$ then it holds*

$$P(\mathcal{E}) > \frac{P(H)}{2} \sum_{i=1}^N |\mathcal{E}(i)|. \quad (1.9.2)$$

where H denote a unit-area regular hexagons.

It is worth noticing that inequality (1.9.2) is strict. Indeed Theorem 1.9.2 is a consequence of Theorem 1.9.1. Since equality in (1.9.1) is attained if only if $\mathcal{E}' = \mathcal{H}$ and since there is no bounded planar N -cluster \mathcal{E} such that $\mathcal{E}' = (\mathcal{E} / \sim) = \mathcal{H}$ equality in (1.9.2) can never be attained.

1.9.2 The "improved convergence" for planar clusters

We here recall for the sake of completeness (and clarity), the basic concepts and the main theorem we are making use in Chapter 3. All these results can be found in [CLM15].

Let $\gamma : [0, \mathcal{H}^1(\gamma)] \rightarrow \mathbb{R}^2$ be a simple, closed and connected $C^{1,\alpha}$ -curve, parametrized by the arc length and with non empty boundary $\text{bd}(\gamma) \neq \emptyset$. A map $f : \gamma \rightarrow \mathbb{R}^2$ is said to be of class $C^{1,\alpha}(\gamma; \mathbb{R}^2)$ if

$$\begin{aligned} f \circ \gamma &\in C^{1,\alpha}([0, \mathcal{H}^1(\gamma)]; \mathbb{R}^2), \\ \|f\|_{C^{1,\alpha}(\gamma)} &:= \|(f \circ \gamma)\|_{C^1([0, \mathcal{H}^1(\gamma)])} + \|(f \circ \gamma)'\|_{C^{0,\alpha}([0, \mathcal{H}^1(\gamma)])} < +\infty. \end{aligned}$$

Let \mathcal{E} be a $C^{1,\alpha}$ planar N -cluster and let $\{\gamma_i\}_{i \in I}$ be the $C^{1,\alpha}$ -network associated to \mathcal{E} . We say that $f : \partial\mathcal{E} \rightarrow \mathbb{R}^2$ is of class $C^{1,\alpha}(\partial\mathcal{E}; \mathbb{R}^2)$ if f is continuous on $\partial\mathcal{E}$, $f \in C^{1,\alpha}(\gamma_i; \mathbb{R}^2)$ for every $i \in I$ and

$$\|f\|_{C^{1,\alpha}(\partial\mathcal{E})} := \sup_{i \in I} \|f\|_{C^{1,\alpha}(\gamma_i)} < +\infty.$$

We say that f is a $C^{1,\alpha}$ -diffeomorphism between two clusters \mathcal{E} and \mathcal{E}' if f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}'$ with

$$f \in C^{1,\alpha}(\partial\mathcal{E}; \mathbb{R}^2), \quad f^{-1} \in C^{1,\alpha}(\partial\mathcal{E}'; \mathbb{R}^2) \quad \text{and} \quad f(\Sigma(\mathcal{E})) = \Sigma(\mathcal{E}').$$

We define the *tangential component* of a vector field $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with respect to an N -cluster \mathcal{E} as

$$\tau_{\mathcal{E}} f(x) := f(x) - (f(x) \cdot \nu_{\mathcal{E}}(x)) \nu_{\mathcal{E}}(x),$$

where $\nu_{\mathcal{E}} : \partial^* \mathcal{E} \rightarrow \mathbb{R}^2$ is any Borel function such that either $\nu_{\mathcal{E}}(x) = \nu_{\mathcal{E}(h)}(x)$ or $\nu_{\mathcal{E}(k)}(x)$ for every $x \in \mathcal{E}(h, k)$.

Theorem 1.9.3. *[Improved convergence for planar almost-minimizing clusters] Given $\Lambda \geq 0, r_0 > 0$ and \mathcal{E} , a bounded $C^{2,1}$ -cluster in \mathbb{R}^2 , there exist positive constant μ_0 and C_0 (depending on Λ and \mathcal{E}) with the following property. If $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ is a sequence of perimeter (Λ, r_0) -minimizing N -clusters in \mathbb{R}^2 such that $d(\mathcal{E}_k, \mathcal{E}) \rightarrow 0$ (as $k \rightarrow +\infty$), then for every $\mu < \mu_0$ there exists $k(\mu) \in \mathbb{N}$ and a sequence of maps $\{f_k\}_{k \in k(\mu)}$ such that each f_k is a $C^{1,1}$ -diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ with*

$$\|f_k\|_{C^{1,1}(\partial\mathcal{E})} \leq C_0, \quad (1.9.3)$$

$$\lim_{k \rightarrow +\infty} \|f_k - \text{Id}\|_{C^1(\partial\mathcal{E})} = 0, \quad (1.9.4)$$

$$\|\tau_{\mathcal{E}}(f_k - \text{Id})\|_{C^1(\partial^* \mathcal{E})} \leq \frac{C}{\mu} \|f_k - \text{Id}\|_{C^0(\Sigma(\mathcal{E}))}, \quad (1.9.5)$$

$$\tau_{\mathcal{E}}(f_k - \text{Id}) = 0 \quad \text{on} \quad \partial\mathcal{E} \setminus I_{\mu}(\Sigma(\mathcal{E})), \quad (1.9.6)$$

where $Id(x) = x$ and

$$I_\mu(\Sigma(\mathcal{E})) = \{x \in \mathbb{R}^n \mid d(x, \Sigma(\mathcal{E})) < \mu\}.$$

Chapter 2

Uniform distribution of the energy for an isoperimetric partition problem with fixed boundary

2.1 Introduction

A conjecture due to Morgan and Heppes, appeared in [HM05], states that the global shape of perimeter-minimizing planar N -clusters having equal-volume chambers, suitably normalized must converge, in the L^1 -sense, to a ball. The global shape should be intended as $\mathcal{E}(0)^c$, where $\mathcal{E}(0)$ is the external chamber of the cluster \mathcal{E} . So far, no progress has been made in proving this conjecture and the reason could lie in the difficulties arising when we try to understand in which sense the shape of the internal chambers has an influence on the global shape of perimeter-minimizing N -clusters. In 2001 Thomas Hales [Hal01] solved the so-called *hexagonal honeycomb conjecture* providing Theorems 1.9.1 and 1.9.2 that somehow give us information about the internal structure of such perimeter-minimizing clusters. Theorem 1.9.2 combined with a suitable comparison argument tells us that, for N approaching $+\infty$, the perimeter of a perimeter-minimizing planar N -cluster \mathcal{E} with equal volume chambers is asymptotic equivalent to the perimeter of a grid of $N \times N$ hexagons:

$$\frac{P(H)}{2}N \leq P(\mathcal{E}) \leq \frac{P(H)}{2}N + C\sqrt{N}.$$

Hales, in its paper [Hal01], proves more: when we consider the partition problem on the torus (which is a way to consider a periodic tiling of \mathbb{R}^2), the hexagonal tiling is the only minimizer. A new result on this topic is the one contained in Chapter Three (that can also be found in [CM14]) where a stability results of the hexagonal tiling with respect to compactly-supported and mass-preserving perturbations has been proved. Everything suggests that the internal chambers of perimeter-minimizing N -clusters try to get closer and closer to regular hexagons and, so far, it is not clear whether this behavior affects the global shape of perimeter-minimizing N -clusters (and in which sense).

In order to investigate the influence of the boundary on the internal structure of perimeter-minimizing N -clusters it makes sense to consider an isoperimetric problem with fixed boundary on planar N -clusters. Namely for a fixed set Ω with finite perimeter we consider the quantity

$$\rho(N, \Omega) := \inf_{\mathcal{E} \in \text{Cl}(N, \Omega)} \{P(\mathcal{E})\}, \quad (2.1.1)$$

where the infimum is taken among all the N -clusters of Ω :

$$\text{Cl}(N, \Omega) = \left\{ \mathcal{E} \text{ planar } N\text{-cluster with } |\mathcal{E}(j)| = \frac{|\Omega|}{N} \text{ for } j \neq 0 \text{ and } \mathcal{E}(0) = \Omega^c \right\}. \quad (2.1.2)$$

Thanks to the compactness for sets of finite perimeter and the semi-continuity property of the functional $P(\cdot)$ with respect to the L^1 convergence (see Theorems 1.2.1, 1.2.2) we get the existence of minimizers for $\rho(N, \Omega)$ for every set Ω with finite perimeter and for every $N \in \mathbb{N}$. We call such clusters *perimeter-minimizing N -clusters for Ω* or simply *minimizing N -clusters for Ω* . In the following we will not use any regularity property of such clusters, however with the same techniques developed for the perimeter-minimizing N -clusters with free boundary it is possible to show that, if Ω is open, each \mathcal{E} , minimizing N -cluster for Ω , is a (Λ, r_0) -perimeter-minimizing N -cluster inside Ω . In particular \mathcal{E} is of class $C^{1,1}$ inside Ω . This also means that each singular point $p_j \in \Sigma(\mathcal{E}) \cap \Omega$ is a common end-point to three different curves that meet in three 120 degree angles in p_j and that the singular set $\Sigma(\mathcal{E}) \cap \Omega$ is discrete.

Our main purpose here is to better understand the behavior of the localized energy $P(\mathcal{E}; Q_l)$ where Q_l is a square of edge-length l and \mathcal{E} is a minimizing N -cluster for an open set Ω . To describe this behavior we provide two “equidistribution theorems” (see Theorems 2.3.2, 2.4.2) in the spirit of the one obtained by Alberti, Choksi e Otto in [ACO09]. For the sake of clarity, let us state a “heuristic version” of the theorems we are going to prove.

There exists a universal constant C such that for every open bounded set Ω , every $\mathcal{E} \in \text{Cl}(N, \Omega)$ minimizing N -cluster for Ω and every closed cube $Q \subset\subset \Omega$ “far enough” from the boundary and “large enough with respect to the size of the chambers” the following holds:

$$\left| P(\mathcal{E}; Q) - |Q| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} \right| \leq CP(Q). \quad (2.1.3)$$

where H denotes a unit-area regular hexagons.

Remark 2.1.1. From a qualitative point of view estimate (2.1.3) gives us information about the average energy of \mathcal{E} inside the cube Q . If we divide both member of (2.1.3) by $\frac{|Q|}{|\Omega|}N$, which represents the expected number of chambers of \mathcal{E} lying inside Q , we obtain

$$\left| \frac{P(\mathcal{E}; Q)}{\frac{|Q|}{|\Omega|}N} - \frac{P(H)}{2} \sqrt{\frac{|\Omega|}{N}} \right| \leq \frac{CP(Q)}{|Q|} \frac{|\Omega|}{N}.$$

Note that $\frac{P(H)}{2} \sqrt{\frac{|\Omega|}{N}}$ is the average energy of a uniform grid of hexagons H having volume $\frac{|\Omega|}{N}$ (and thus perimeter $P(H) \sqrt{\frac{|\Omega|}{N}}$). Hence we can interpret estimate (2.1.3) as follows: the average energy $P(\mathcal{E}; Q)$ of a minimizing N -cluster for Ω computed on a fixed cube $Q \subset\subset \Omega$ approaches the average energy of a grid of hexagons with area $\frac{|\Omega|}{N}$. Estimate (2.1.3) suggests that, no matter where we are localizing for N sufficiently large the boundary $\partial\Omega$ does not affect the energetic behavior of minimizing N -clusters for Ω . This also indicates that some approximate periodicity in the behavior of internal chambers, at least from an energetic point of view, is attained.

Remark 2.1.2. The term $CP(Q)$ appearing in the right-hand side of (2.1.3) is the optimal one. Indeed, assume for a moment that \mathcal{E} is a perfect hexagonal grid made by hexagons

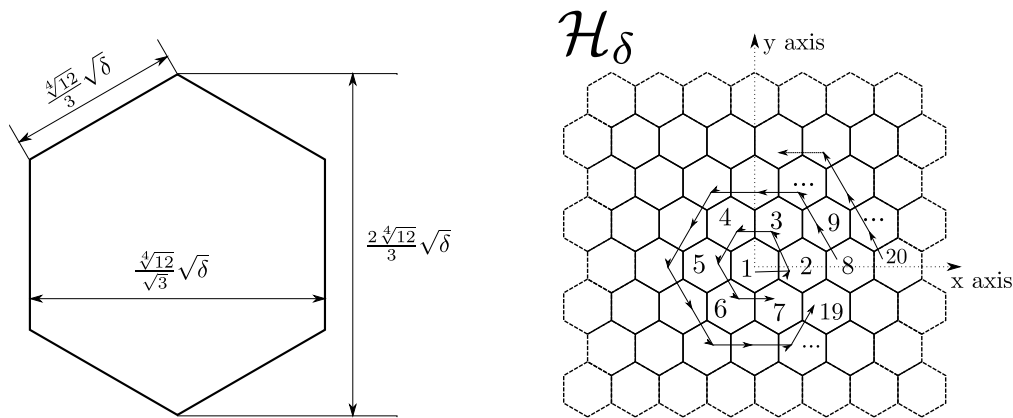


Figure 2.1.1: The reference regular hexagon of area δ and the correspondent reference hexagonal tiling of size δ .

of area $\frac{|\Omega|}{N}$. In this situation, if we compute the localized energy $P(\mathcal{E}; Q)$, we discover that the principal part is just the perimeter of all the hexagons compactly contained inside Q , that is $|Q| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}}$. The contribution of the hexagons intersecting ∂Q will be of order $CP(Q)$ for a universal constant C .

Remark 2.1.3. Let us focus on why we need to be on a cube Q “far enough from the boundary” and “large compared to the size of the chambers”. We cannot expect the estimate (2.1.3) to work on every cube compactly contained in Ω . For example it may happen that the geometry of $\partial\Omega$ can affect the internal energy at least in its proximity and so for all cubes too close to $\partial\Omega$ the localized energy could be very far from the one of the hexagonal tiling. Moreover, if a cube Q is very small (say for example $|Q| < \frac{|\Omega|}{N}$, smaller than the size of the chambers) the theorem will probably be meaningless since the localized energy will be zero or comparable. We are going to quantify in a precise way what “far enough from the boundary” and “large compared to the size of the chambers” mean.

An estimate of the type of (2.1.3) helps us to better understand the relation between the boundary and the internal chambers in the free boundary case. Indeed, it seems that no matter what ambient space Ω we choose, for N sufficiently large we expect to see hexagons inside. This could mean that the behavior of the boundary does not affect the shape of interior chambers. Thus, can the shape of internal chambers affect the behavior of the boundary in perimeter-minimizing N -clusters? We cannot say. We point out here, that it seems that the fixed boundary does not influences the asymptotic trend of internal chambers.

Throughout this chapter we denote with H a unit-area regular hexagon so that $\sqrt{\delta}H$ will be a regular hexagon of area δ . We are sometimes making use of the notation \mathcal{H}_δ meaning the tiling of \mathbb{R}^2 made by regular hexagons of area δ oriented and labeled as in Figure 2.1.1.

2.1.1 Brief sketch of the proof

To prove an estimate of the type (2.1.3) we need to provide two bounds on the localized energy $P(\mathcal{E}; Q)$. A lower bound is easily obtained in Lemma 2.2.1 for a generic N -cluster \mathcal{E}

as a consequence of Hales’s Theorem 1.9.2. On the other hand we obtain an upper bound by comparison, building a competitor through the following geometric construction. We fix a square $Q_l \subset\subset \Omega$ and we exploit the simple idea to substitute in a suitable way the existing cluster with a hexagonal grid (that we know be a heuristic minimizer) inside the square. In order to do this we first suitably enlarge Q_l into Q_{l+d} and then we remove all the chambers compactly contained into Q_{l+d} . After that, we completely cover Q_l with an hexagonal grid (see Figure 2.1.2). We need to make sure that the grid that we have built does not overlap some remaining “long chambers” with some tentacles intersecting the boundary of the bigger square Q_{l+d} . To handle this phenomenon we restrict to two different classes of minimizing N -clusters giving us some control and allowing us to prove two different Theorems: 2.3.2 and 2.4.2. To complete the construction we need to “stitch” with a suitable surgery the grid with the remaining parts of \mathcal{E} (see Figure 2.1.3). The surgery will be the cluster with chambers of area exactly $\frac{|\Omega|}{N}$ provided by Proposition 2.2.2. We thus obtain an N -cluster \mathcal{F} differing from \mathcal{E} only inside Q_{l+d} and by comparison (and up to choose d in a suitable way) we are able to reach the estimate

$$P(\mathcal{E}_N; Q_l) \leq P(\mathcal{F}; Q_l) \leq \frac{P(H)}{2} |Q_l| \sqrt{N} + \text{lower order terms.}$$

The behavior of the *lower order terms* depends on the class of minimizers that we are considering.

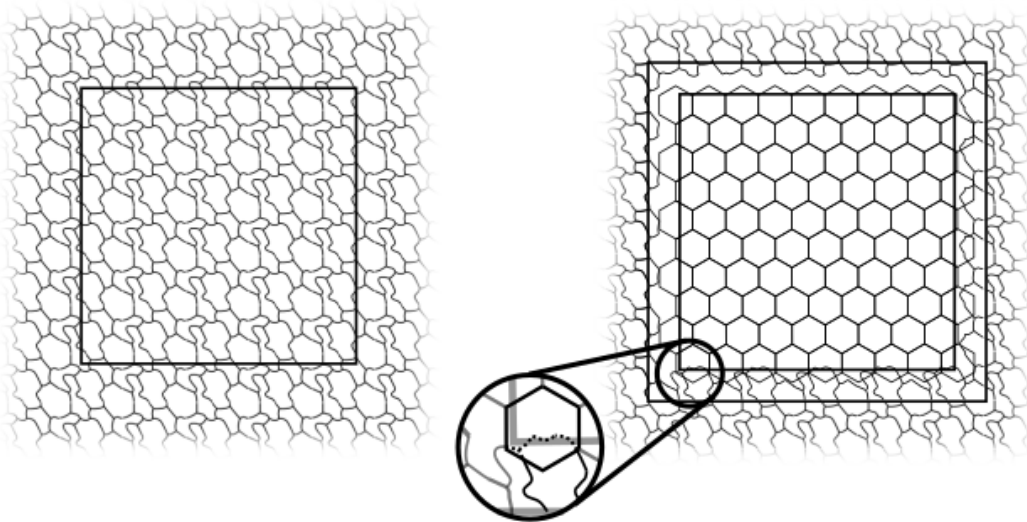


Figure 2.1.2: We consider a square $Q_l \subset\subset \Omega$ and we enlarge it in order to have space to complete the construction. We remove all the chambers of the cluster compactly contained into the bigger square and we cover Q_l with an hexagonal tiling. We need to be sure that the tiling do not overlap some “long chamber”.

The Chapter is organized as follows. In Section 2.2 we prove two technical lemmas: 2.2.1 and 2.2.2. The first one is a consequence of Hales’s Theorem 1.9.2 and gives us a lower bound on the localized energy of planar N -cluster. The second one is a geometric construction for partition in equal-area chambers with a controlled amount of perimeter within a particular class of sets. This second lemma is the one that we need to perform the surgeries. In Sections 2.3 and 2.4 we prove two different Equidistribution-type theorems holding on two different classes of minimizing N -clusters for a given open set Ω , namely the

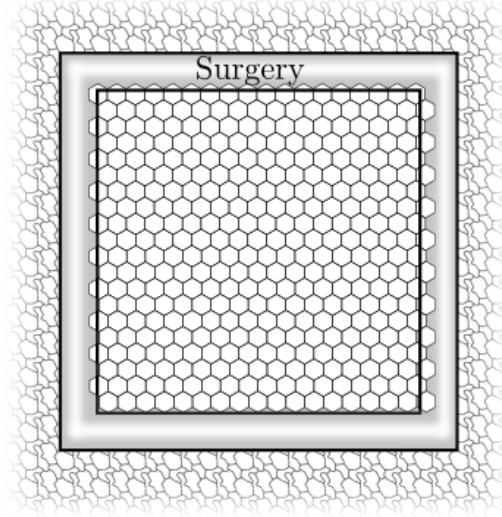


Figure 2.1.3: We build a suitable surgery partition of the remaining part in chambers having the right amount of area.

μ -bounded minimizing N -clusters (see Definition 2.3.1) and the indecomposable minimizing N -clusters (see Definition 2.4.1). Both the sections are mostly devoted to the construction of a suitable competitor (through the idea explained in Subsection 2.1.1) in order to derive an upper bound on the localized energy.

2.2 Technical lemmas

Lemma 2.2.1. *Let Ω be an open bounded set in \mathbb{R}^2 and let $\mathcal{E} \in Cl(N, \Omega)$ be an N -cluster for Ω . Then for every open set $O \subset\subset \Omega$ it holds*

$$P(\mathcal{E}; O) \geq |O| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} - P(O) \quad (2.2.1)$$

Proof. Since $|\mathcal{E}(i) \cap O| \leq \frac{|\Omega|}{N}$, we can apply Theorem 1.9.2 to

$$\mathcal{F} = \sqrt{\frac{N}{|\Omega|}} (\mathcal{E} \cap O)$$

and obtain

$$P(\mathcal{F}) \geq \frac{P(H)}{2} \sum_{i=1}^N \frac{N}{|\Omega|} |\mathcal{E}(i) \cap O| \geq \frac{P(H)}{2} \frac{N}{|\Omega|} |O|.$$

But since, thanks to (1.6.2),

$$P(\mathcal{F}) = \sqrt{\frac{N}{|\Omega|}} P(\mathcal{E} \cap O) = \sqrt{\frac{N}{|\Omega|}} [P(\mathcal{E}; O) + P(O)]$$

we obtain (2.2.1). □

Lemma 2.2.2. *Let $A \subset \mathbb{R}^2$, $0 < |A| < +\infty$ be a set for which there exists two concentric cubes $Q_0 \subset \subset Q_1$ such that*

$$A \subseteq Q_1 \setminus Q_0.$$

Then for every $M \in \mathbb{N}$ there exists an M -cluster \mathcal{E}_M such that $|\mathcal{E}_M(i)| = \frac{|A|}{M}$ for all $i = 1, \dots, M$, $\mathcal{E}_M(0) = A^c$ and for which the following estimate holds:

$$P(\mathcal{E}_M) \leq C|Q_1 \setminus Q_0| \sqrt{\frac{M}{|A|}} + P(A), \quad (2.2.2)$$

for a universal constant $C > 0$.

Proof. Let $M \in \mathbb{N}$ be a fixed number. We want to partition A in chambers of area $\frac{|A|}{M}$. To do that, we first partition A in sectors S_i enclosing the same (suitable) amount of area using lines starting from the baricenter O of the cubes (as in Figure 2.2.1a). Then we divide each sector in chambers of area $\frac{|A|}{M}$ with circular arcs centered at O (as in Figure 2.2.1b). We need to choose the amount of area that we want to allocate in each sector in a coherent way.

Set $d = \frac{1}{2}(\sqrt{|Q_1|} - \sqrt{|Q_0|})$ to be the thickness of the frame $Q_1 \setminus Q_0$. Since we are planning to cover each sector with chambers of measure $\frac{|A|}{M}$ (and thus of diameter of order $\frac{\sqrt{|A|}}{\sqrt{M}}$), it is natural to say that the number of chambers that we expect to be the right one, allocable in each sector, would be $\frac{d}{\frac{\sqrt{|A|}}{\sqrt{M}}} = \frac{d\sqrt{M}}{\sqrt{|A|}}$. Hence we define the integer value

$$s = \left\lceil \frac{d\sqrt{M}}{\sqrt{|A|}} \right\rceil. \quad (2.2.3)$$

We can write,

$$M = sk + r, \quad r < s,$$

for some $k \in \mathbb{N}$. We thus divide A in k sectors S_1, S_2, \dots, S_k in a way that each sector S_i has Lebesgue measure exactly equal to $s\frac{|A|}{M}$, plus an eventual remainder sector $R = A \setminus (\cup_i S_i)$ with measure $|R| < s\frac{|A|}{M}$ (see Figure 2.2.1a).

An upper bound on the value of k can be obtained by exploiting the relation:

$$sk\frac{|A|}{M} \leq |A|,$$

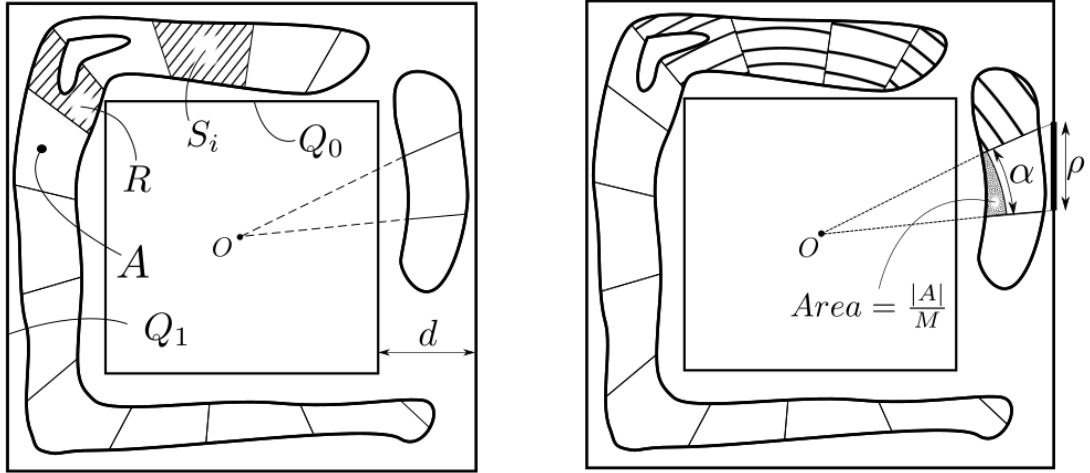
which, thanks to the definition of s in (2.2.3), implies

$$|A| \geq sk\frac{|A|}{M} \geq dk\sqrt{\frac{|A|}{M}},$$

and hence:

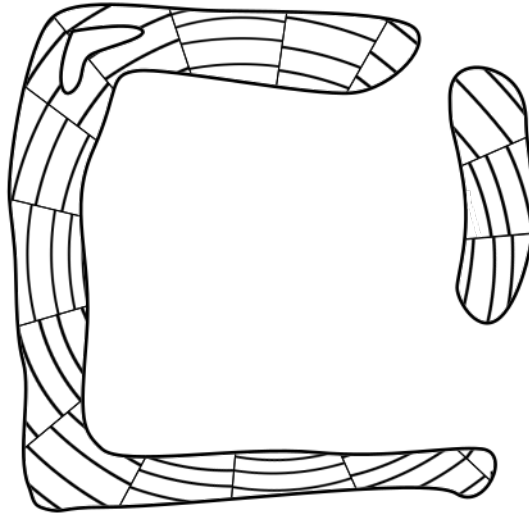
$$k \leq \frac{\sqrt{M|A|}}{d}. \quad (2.2.4)$$

We then divide each sector S_i and the sector R with circular arcs having a suitable radius and centered at O in order to obtain chambers of area exactly $\frac{|A|}{M}$. In this way each sectors but the sector R is containing exactly s chambers (note that the sector R will



(a) We radially partition A in sectors S_i enclosing the same (suitable) amount of area plus an eventual remaining sector R enclosing a possibly smaller area.

(b) We proceed to partition each sector in chambers of area $|A|/M$ with circular arcs centered at O . The length α of each circular arc is always less than the length ρ of its radial projection onto ∂Q_1 .



(c) The M -cluster \mathcal{E}_M .

Figure 2.2.1: The construction of the M -cluster \mathcal{E}_M in the proof of Proposition 2.2.2.

contain $r < s$ chambers). Of course, since each chamber has area $|A|/M$, we end up with exactly M chambers. We thus define \mathcal{E}_M to be the cluster given by this construction (see Figure 2.2.1c).

To build the sectors S_i we make use of k segments of length less than $2d$. Note that two arcs from the same sector have the same radial projection on ∂Q_1 and each circular arc has length less than the length of its radial projection onto ∂Q_1 (see Figure 2.2.1b).

These facts lead us to say that the global contribution of the circular arcs to the perimeter of \mathcal{E}_M will be less than $P(Q_1)s$. Hence, thanks to (2.2.4), the global perimeter

of \mathcal{E}_M inside $A^{(1)}$ is easily estimated by

$$\begin{aligned} P(\mathcal{E}_M; A^{(1)}) &\leq 2kd + P(Q_1)s \\ &\leq 2\sqrt{M|A|} + 2P(Q_1)\frac{d\sqrt{M}}{\sqrt{|A|}} \\ &\leq 2(|A| + P(Q_1)d)\frac{\sqrt{M}}{\sqrt{|A|}}. \end{aligned}$$

By noticing that $P(Q_1)d \leq 4|Q_1 \setminus Q_0|$ and $|A| \leq |Q_1 \setminus Q_0|$ we reach

$$P(\mathcal{E}_M; A^{(1)}) \leq C|Q_1 \setminus Q_0|\frac{\sqrt{M}}{\sqrt{|A|}}, \quad (2.2.5)$$

for a universal constant $C > 0$ and thus

$$P(\mathcal{E}_M) \leq C|Q_1 \setminus Q_0|\frac{\sqrt{M}}{\sqrt{|A|}} + P(A). \quad (2.2.6)$$

□

2.3 Uniform distribution for clusters with equi-bounded diameter.

We are always considering N -clusters from the class $\text{Cl}(N, \Omega)$ defined in (2.1.2) where Ω is an open bounded set with finite perimeter and $N \in \mathbb{N}$ is a natural number.

Definition 2.3.1. Let $\mu > 0$ be a positive constant. An N -Cluster $\mathcal{E} \in \text{Cl}(N, \Omega)$ is said to be a μ -bounded N -Cluster for Ω if

$$\text{diam}(\mathcal{E}(i)) \leq \mu\sqrt{\frac{|\Omega|}{N}} \quad \text{for all } i = 1, \dots, N.$$

If \mathcal{E} is also a minimizing cluster for Ω we call it a μ -bounded minimizing N -cluster.

On this class we are able to prove the following Theorem:

Theorem 2.3.2. Let Ω be an open and bounded set with finite perimeter. There exists a universal constant $C > 0$ with the following property. For every $\mu \geq \text{diam}(H)$, every closed cube $Q_l \subset\subset \Omega$ such that

$$d(\partial Q_l, \partial \Omega) > 4\mu\sqrt{\frac{|\Omega|}{N}}, \quad l \geq 6\mu\sqrt{\frac{|\Omega|}{N}} \quad (2.3.1)$$

and every $\mathcal{E} \in \text{Cl}(N, \Omega)$ μ -bounded minimizing N -cluster the following holds:

$$\left| P(\mathcal{E}; Q_l) - |Q_l|\frac{P(H)}{2}\sqrt{\frac{N}{|\Omega|}} \right| \leq CP(Q_l)\mu. \quad (2.3.2)$$

In this class we can exploit the advantage that the chambers cannot be “too long”. Note that what really matters in Theorem 2.3.2 is how small is the size of the expected chambers ($N^{-\frac{1}{2}}$) compared to the size of the cube Q_l that we are considering.

Remark 2.3.3. It may seem that the restriction $\mu > \text{diam}(H)$ is a disadvantage in all the eventual situations where a very small diameter is attained. But the point is that the class of μ -bounded N -cluster of Ω is empty when μ is too small. Indeed, thanks to the planar isodiametric inequality (1.4.2) we have that

$$\text{diam}(\mathcal{E}(i)) \geq 2\sqrt{\frac{|\mathcal{E}(i)|}{\pi}},$$

so

$$\mu \geq \frac{2}{\sqrt{\pi}}.$$

Thus it is not restrictive to require $\mu > \mu_0$ for some universal constant μ_0 and the choice of $\mu_0 = \text{diam}(H)$ is just the most convenient one.

Remark 2.3.4. Note that each N -cluster is a μ -bounded minimizing N -cluster with

$$\mu := \sqrt{\frac{N}{|\Omega|}} \max \{ \text{diam}(\mathcal{E}(i)) \mid i = 1, \dots, N \}.$$

The fact that μ appears in the right-hand side of (2.3.2) means that, without a good information about μ , at an asymptotic level the estimate is meaningless. For example if we only know that $\mu \leq N^{\frac{1}{2}}$, we get

$$\left| P(\mathcal{E}; Q_l) - |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} \right| \leq CP(Q_l)N^{\frac{1}{2}}$$

which does not carry any information. The optimal situation, when the Theorem becomes sharp, is attained when μ is of order of a constant, meaning that each chamber has diameter really of order $N^{-\frac{1}{2}}$. Let us also point out that, since $\mu\sqrt{\frac{|\Omega|}{N}}$ is the size of each chamber both the restrictions on l appearing in Theorem 2.3.2 are sharp.

We premise the geometric construction of a competitor, working on every μ -bounded N -cluster.

2.3.1 Construction of a competitor

Proposition 2.3.5. *Let Ω be an open and bounded set with finite perimeter. There exists a universal constant C_0 with the following property. For every $\mu \geq \text{diam}(H)$, every closed cube $Q_l \subset \subset \Omega$ with*

$$d(\partial Q_l, \partial \Omega) > 4\mu\sqrt{\frac{|\Omega|}{N}}, \quad l \geq 6\mu\sqrt{\frac{|\Omega|}{N}}$$

every μ -bounded N -cluster $\mathcal{E} \in Cl(N, \Omega)$ there exists an N -cluster \mathcal{F} in $Cl(N, \Omega)$ for which the following estimate holds:

$$P(\mathcal{F}) \leq |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} + P(\mathcal{E}; Q_l^c) + C_0 P(Q_l) \mu. \quad (2.3.3)$$

Proof. Thanks to the assumption on $d(\partial Q_l, \partial \Omega)$, setting $d := 2\mu\sqrt{\frac{|\Omega|}{N}}$, we can consider the cube Q_{l+d} concentric to Q_l and still have $Q_{l+d} \subset \subset \Omega$ (see Figure 2.3.1a). With this choice,

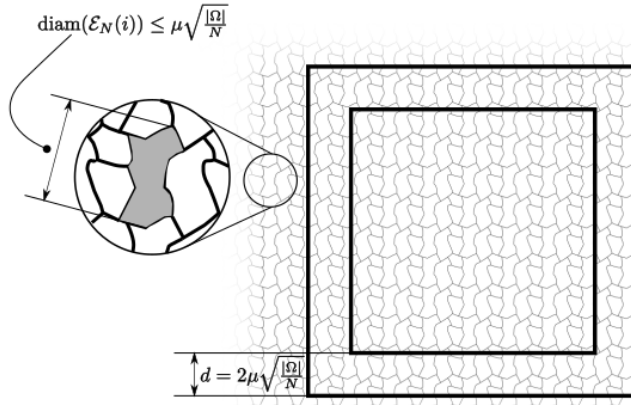
since \mathcal{E} is a μ -bounded N -cluster, every chamber intersecting Q_{l+d}^c does not intersect Q_l . Set

$$\begin{cases} I_{l,d} &= \{i \in \{1, \dots, N\} \mid \mathcal{E}(i) \subset\subset Q_{l+d}\}, \\ k(l,d) &= \#(I_{l,d}), \end{cases} \quad (2.3.4)$$

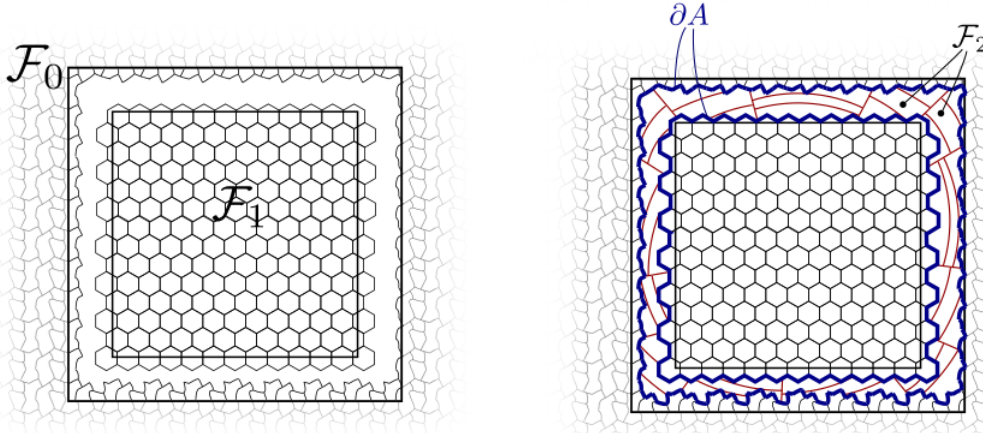
and let us define

$$\mathcal{F}_0 = \{\mathcal{E}(i) \mid \mathcal{E}(i) \cap Q_{l+d}^c \neq \emptyset\}.$$

We now remove all the k chambers $\mathcal{E}(i)$ compactly contained into Q_{l+d} and we completely cover Q_l with an hexagonal grid. We can be sure that this hexagonal grid does not overlap \mathcal{F}_0 since $\mu \geq \text{diam}(H)$.



(a) We place the cube $Q_l \subset\subset \Omega$ and we suitably enlarge it in a cube Q_{l+d} .



(b) We remove all the chambers compactly contained into Q_{l+d} and we cover Q_l with an hexagonal grid. The request $\mu > \text{diam}(H)$ and the choice of d ensure us that the grid do not overlap the remaining chambers

(c) We apply Lemma 2.2.2 to build a partition of the remaining part of $Q_{l+d} \setminus Q_l$.

Figure 2.3.1: The construction of the N -cluster \mathcal{F} in the proof of Proposition 2.3.5.

Denote with h the total number of hexagons needed to completely cover Q_l . We build our covering in a way that each hexagon intersects Q_l (see Figure 2.3.1b). In this way the

hexagonal grid is completely contained into $Q_{l+\text{diam}(H)\sqrt{\frac{|\Omega|}{N}}}$ and so, since

$$l \geq 6\mu\sqrt{\frac{|\Omega|}{N}} \geq 6\text{diam}(H)\sqrt{\frac{|\Omega|}{N}},$$

we obtain

$$h\frac{|\Omega|}{N} \leq \left(l + \text{diam}(H)\sqrt{\frac{|\Omega|}{N}} \right)^2 \leq l^2 + Cl\sqrt{\frac{|\Omega|}{N}}$$

where C is a universal constant. Hence

$$h \leq l^2 \frac{N}{|\Omega|} + Cl\sqrt{\frac{N}{|\Omega|}}. \quad (2.3.5)$$

Denote this hexagonal grid with \mathcal{F}_1 . The perimeter of \mathcal{F}_1 is estimated by

$$P(\mathcal{F}_1) \leq h\frac{P(H)}{2}\sqrt{\frac{|\Omega|}{N}} + P(\mathcal{F}_1(0)).$$

It is straightforward that $P(\mathcal{F}_1(0)) \leq Cl$ for a universal constant C . Thus, thanks to (2.3.5), we reach:

$$P(\mathcal{F}_1) \leq l^2\frac{P(H)}{2}\sqrt{\frac{N}{|\Omega|}} + Cl, \quad (2.3.6)$$

for a universal constant C .

After the construction of \mathcal{F}_1 and \mathcal{F}_0 we are left to partition the open set

$$A = \left(\bigcup_{i \in I_{l,d}} \mathcal{E}(i) \right) \setminus \left(\bigcup_{i=1}^h \mathcal{F}_1(i) \right)$$

(evidenced in blue in Figure 2.3.1c). We here make use of Lemma 2.2.2 and we divide A into $(k-h)$ chambers. Note that

$$|A| = k\frac{|\Omega|}{N} - h\frac{|\Omega|}{N},$$

and thus

$$\frac{|A|}{k-h} = \frac{|\Omega|}{N}. \quad (2.3.7)$$

Since the set A is contained into $Q_{l+d} \setminus Q_l$ we can apply Lemma 2.2.2 with $Q_0 = Q_l, Q_1 = Q_{l+d}$ and discover that there exists a $(k-h)$ -cluster $\mathcal{E}_{k-h} \in \text{Cl}(k-h, A)$ such that

$$P(\mathcal{E}_{k-h}) \leq |Q_{l+d} \setminus Q_l| \sqrt{\frac{(k-h)}{|A|}} + P(A),$$

which, thanks to (2.3.7) and since $d < l$, means

$$P(\mathcal{E}_{k-h}; A) \leq 3ld\sqrt{\frac{N}{|\Omega|}} + P(A). \quad (2.3.8)$$

Setting $\mathcal{F}_2 = \mathcal{E}_{k-h}$ and

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2,$$

clearly $\mathcal{F} \in \text{Cl}(N, \Omega)$. Notice that (see Figures 2.3.1b, 2.3.1c)

$$P(\mathcal{F}) = P(\mathcal{F}_0) + P(\mathcal{F}_1) + P(\mathcal{F}_2) - P(A).$$

Furthermore by exploiting (2.3.6), (2.3.8) we obtain

$$\begin{aligned} P(\mathcal{F}) &= P(\mathcal{F}_0) + P(\mathcal{F}_1) + P(\mathcal{F}_2) - P(A) \\ &\leq P(\mathcal{F}_0) + l^2 \frac{P(H)}{2} \frac{N}{|\Omega|} + 3ld \sqrt{\frac{N}{|\Omega|}} + Cl. \end{aligned}$$

Since, by construction, it holds

$$P(\mathcal{F}_0) \leq P(\mathcal{E}; Q_l^c),$$

by recalling that $d = 2\mu\sqrt{\frac{|\Omega|}{N}}$ we obtain:

$$\begin{aligned} P(\mathcal{F}) &\leq l^2 \frac{P(H)}{2} \frac{N}{|\Omega|} + P(\mathcal{E}; Q_l^c) + 3ld \sqrt{\frac{N}{|\Omega|}} + Cl \\ &\leq l^2 \frac{P(H)}{2} \frac{N}{|\Omega|} + P(\mathcal{E}; Q_l^c) + Cl(\mu + 1) \\ &\leq l^2 \frac{P(H)}{2} \frac{N}{|\Omega|} + P(\mathcal{E}; Q_l^c) + Cl\mu \left(1 + \frac{1}{\text{diam}(H)}\right) \end{aligned}$$

Setting $C_0 = C \left(1 + \frac{1}{\text{diam}(H)}\right)$, we get the thesis (2.3.3). \square

2.3.2 Proof of Theorem 2.3.2

Proof of Theorem 2.3.2. Let C_0 be the constant given by Proposition 2.3.5. Let $\mu > \text{diam}(H)$, $Q_l \subset\subset \Omega$ be a closed cube satisfying (2.3.1) and $\mathcal{E} \in \text{Cl}(N, \Omega)$ be a μ -bounded minimizing N -cluster for Ω . Thanks to Proposition 2.3.5 we can find an N -cluster $\mathcal{F} \in \text{Cl}(N, \Omega)$ for which it holds:

$$P(\mathcal{F}) \leq |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} + C_0 P(Q_l) \mu + P(\mathcal{E}; Q_l^c).$$

By exploiting the minimality of \mathcal{E} we obtain

$$P(\mathcal{E}) \leq P(\mathcal{F})$$

which leads to

$$P(\mathcal{E}; Q_l) \leq |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} + C_0 P(Q_l) \mu. \quad (2.3.9)$$

Proposition 2.2.1 ensures that on Q_l it holds

$$P(\mathcal{E}; Q_l) \geq |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} - P(Q_l),$$

and hence

$$P(\mathcal{E}; Q_l) \geq |Q_l| \frac{P(H)}{2} \sqrt{\frac{N}{|\Omega|}} - \frac{\mu^2}{\text{diam}(H)^2} P(Q_l). \quad (2.3.10)$$

Up to choosing $C = \max \left\{ C_0, \frac{1}{\text{diam}(H)^2} \right\}$, by combining (2.3.10) and (2.3.9) we achieve the proof. \square

2.4 Uniform distribution for indecomposable minimizing clusters

Since getting information about the diameter of the chambers seems to be a very hard task we provide a second result, which applies on a possibly wider class of minimizing N -clusters.

Definition 2.4.1. An N -Cluster $\mathcal{E} \in \text{Cl}(N, \Omega)$ is said to be an *indecomposable N -Cluster* for Ω if each chamber $\mathcal{E}(j)$ is an indecomposable set of finite perimeter. If \mathcal{E} is also a minimizing N -cluster we call it an *indecomposable minimizing N -cluster* for Ω .

On this class the following result holds.

Theorem 2.4.2. Let Ω be an open bounded set with Lipschitz boundary and $0 \leq \beta < \frac{1}{2}$ be a positive real number. Then there exist three positive constant η, λ, C depending only on β and on the shape of Ω with the following property. For every cube $Q_l \subset\subset \Omega$ with

$$d(\partial Q_l, \partial \Omega) > \eta \sqrt{|\Omega|} N^{-\frac{1}{6}}, \quad l \geq \lambda \sqrt{|\Omega|} N^{-\beta} \quad (2.4.1)$$

and for every indecomposable minimizing N -cluster $\mathcal{E} \in \text{Cl}(N, \Omega)$ the following holds

$$\left| P(\mathcal{E}; Q_l) - \frac{P(H)}{2} |Q_l| \sqrt{\frac{N}{|\Omega|}} \right| \leq C P(Q_l)^{\frac{3}{2}} \left(\frac{N}{|\Omega|} \right)^{\frac{1}{4}}. \quad (2.4.2)$$

In this case we follow the simple idea that, the longer is the chamber the bigger will be its contribution to the global perimeter. An a priori estimate (Proposition 2.4.9) on the global energy $P(\mathcal{E})$ allows us to control the number of the bad chambers and leads us to the sought upper bound on $P(\mathcal{E}; Q_l)$.

Remark 2.4.3. We need Ω to have Lipschitz boundary in order to achieve the proof of Proposition 2.4.9 (a key step in the proof of Theorem 2.4.2).

Remark 2.4.4. We are going to explain in Remark 2.4.12 below where the exponent $\frac{1}{6}$ in the hypothesis on the distance between ∂Q_l and $\partial \Omega$ comes from.

Remark 2.4.5. Let us remark that the existence of indecomposable minimizing N -cluster is actually an open problem though, intuitively, there are good reasons that underline that the class defined in (2.4.1) is not empty. In many situations it could happen that the chambers in the proximity of $\partial \Omega$ decide to split in order to compensate the effect of a possibly irregular boundary. As an example consider the case when Ω is an open square Q (with $|Q| = N$) union two disjoint thin open rectangles R_1, R_2 of area $\frac{1}{2}$, height t , length $\frac{1}{2t}$ (see Figure 2.4.1). If we consider \mathcal{E} a minimizing $(N + 1)$ -cluster for Ω it is not clear whether it is convenient for \mathcal{E} to be indecomposable or to have a chamber with two big indecomposable components given by R_1 and R_2 .

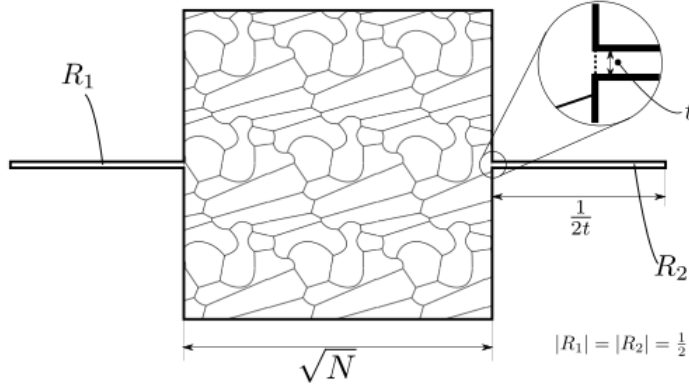


Figure 2.4.1: For this ambient space Ω and for t small enough the $(N + 1)$ -minimizers could have at least an indecomposable chamber.

It is reasonable to expect that on every fixed ambient space Ω , for N sufficiently large this behavior is avoided at least for every chambers far enough from the boundary. We could have enlarged our class a bit more by requiring the indecomposability only for those chambers lying at a distance $d(N)$ (decaying in N) from the boundary of the ambient space, but since our arguments will work in the same way we prefer, for the sake of clarity, not to add this more technical restriction.

2.4.1 Construction of a competitor.

The construction of the competitor in the case of indecomposable N -cluster is a slight modification of the one developed in the proof of Proposition 2.3.5. We first state and prove Proposition 2.4.6 which, for a fixed open set $\Omega' \subseteq \Omega$ and for a fixed (suitable) cube $Q_l \subset \subset \Omega'$, starting from a generic $\mathcal{E} \in \text{Cl}(N, \Omega)$ gives us an N -cluster $\mathcal{F} \in \text{Cl}(N, \Omega)$ having perimeter

$$P(\mathcal{F}) \leq |Q_l| \frac{P(H)}{2} \sqrt{N} + P(\mathcal{E}; Q_l^c) + C_0 \sqrt{P(\mathcal{E}; \Omega')} \sqrt{l},$$

for some constant C_0 .

The presence of the localized perimeter $P(\mathcal{E}; \Omega')$ in the right-hand side of the previous estimate requires some kind of weak control on the perimeter of \mathcal{E} and thus we need to exploit minimality to complete our construction. This is done in Proposition 2.4.11 where a first rough estimate

$$P(\mathcal{E}; Q_l) \leq P_0 |Q_l| \sqrt{N} \tag{2.4.3}$$

for a universal constant P_0 , is obtained for every closed cube $Q_l \subset \subset \Omega$ far enough from $\partial\Omega$. The proof of (2.4.3) is achieved by combining Proposition 2.4.6 with an estimate on the global energy proved in Proposition 2.4.9.

We choose a cube Q_l satisfying the hypothesis of Theorem 2.4.2 and we carefully enlarge it into Q_{l+d} . By setting $\Omega' = Q_{l+d}$ and by applying again Proposition 2.4.6, thanks to the rough estimate (2.4.3) on $P(\mathcal{E}; Q_{l+d})$ (provided a suitable d), we build the competitor with the desired energy in Proposition 2.4.14.

In the following we are always considering an open set Ω with $|\Omega| = 1$. In the end, with a scaling argument, we achieve the proof of Theorem 2.4.2 for a generic set Ω .

Proposition 2.4.6. *Let Ω be an open, bounded set with Lipschitz boundary and $|\Omega| = 1$. There exist universal constants η_0, λ_0, C_0 with the following property. Let $\Omega' \subseteq \Omega$ and let $\mathcal{E} \in \mathcal{C}(N, \Omega)$ be a generic indecomposable N -cluster. Set:*

$$P := P(\mathcal{E}; \Omega') \quad (2.4.4)$$

Then for every closed cube $Q_l \subset \subset \Omega'$ with

$$d(\partial Q_l, \partial \Omega') > \eta_0 \sqrt{\frac{P}{lN}}, \quad l \geq \lambda_0 \left[\left(\frac{P}{N} \right)^{\frac{1}{3}} + \frac{1}{\sqrt{N}} \right] \quad (2.4.5)$$

there exists $\mathcal{F} \in \mathcal{C}(N, \Omega)$ for which the following estimate holds:

$$P(\mathcal{F}) \leq |Q_l| \frac{P(H)}{2} \sqrt{N} + P(\mathcal{E}; Q_l^c) + C_0 \sqrt{Pl}. \quad (2.4.6)$$

Remark 2.4.7. Note that assumption (2.4.5) implies that Proposition 2.4.6 is meaningless whenever the energy of the indecomposable N -cluster for Ω is too much. In particular the restriction on the size of the cube implies that $P(\mathcal{E}; \Omega')$ must be less than or equal to $\frac{l^3}{\lambda_0^3} N$. In particular it could happen that for some “wrong” choice of \mathcal{E} there are no cubes satisfying restrictions (2.4.5). However, we are going to apply Proposition 2.4.6 on the indecomposable minimizing N -clusters for Ω where an upper bound on the global energy is always attained (see Proposition 2.4.9).

Remark 2.4.8. Note that the exponent $1/3$ in the hypothesis (2.4.5) on l cannot be improved. Indeed, assume that we are able to prove Proposition 2.4.6 with

$$l \geq \lambda_0 \left[\left(\frac{P}{N} \right)^\alpha + \frac{1}{\sqrt{N}} \right], \quad (2.4.7)$$

for some $\alpha > 1/3$. As we are going to show below in Proposition 2.4.11, if \mathcal{E} is a perimeter-minimizing N -cluster for Ω then $P = P(\mathcal{E}; Q_l) \approx l^2 \sqrt{N}$ which means that (2.4.7) on l can be wrote as

$$l \geq l^{2\alpha} N^{-\alpha/2}, \quad l \geq N^{\frac{-\alpha}{2-4\alpha}}.$$

Since $l \geq N^{-1/2}$ (it does not make sense to consider cubes smaller than the expected size of the chambers) we are lead to

$$N^{\frac{1}{2} - \frac{\alpha}{2-4\alpha}} \geq 1 \Rightarrow \alpha \leq 1/3.$$

Proof of Proposition 2.4.6. Fix $Q_l \subset \subset \Omega'$ satisfying (2.4.5). Note that, thanks to Proposition (2.2.1), on Q_l it holds

$$P(\mathcal{E}; Q_l) \geq |Q_l| \frac{P(H)}{2} \sqrt{N} - P(Q_l) = l^2 \frac{P(H)}{2} \sqrt{N} - 4l,$$

and since $Q_l \subset \subset \Omega'$:

$$\begin{aligned} P = P(\mathcal{E}; \Omega') &\geq P(\mathcal{E}; Q_l) \\ &\geq l^2 \frac{P(H)}{2} \sqrt{N} - 4l. \end{aligned}$$

Hence, by using (2.4.5) and observing that $l\sqrt{N} \geq \lambda_0$,

$$\frac{P}{l} \geq \left(\frac{P(H)}{2} \lambda_0 - 4 \right).$$

Thus, up to taking λ_0 bigger than a universal constant we can always assume

$$\frac{P}{l} \geq 1. \quad (2.4.8)$$

Let $d \in \mathbb{R}$ be defined as

$$d := \eta \sqrt{\frac{P}{lN}}, \quad (2.4.9)$$

for some η to be chosen (we postpone the choice of λ_0, η_0, η to the end of the proof). Let us set the restriction

$$\frac{l}{100} \geq d$$

in order to be sure that d is much smaller than l . This leads to the restriction:

$$l \geq (100\eta)^{\frac{2}{3}} \left(\frac{P}{N} \right)^{\frac{1}{3}},$$

which becomes immediately a restriction on λ_0

$$\lambda_0 \geq (100\eta)^{\frac{2}{3}}. \quad (2.4.10)$$

In this way the concentric closed boxes

$$Q_l \subset\subset Q_{l+\frac{1}{4}d} \subset\subset Q_{l+\frac{3}{4}d} \subset\subset Q_{l+d},$$

are all compactly contained into Ω' providing $\eta_0 > 2\eta$ (see Figure 2.4.2a).

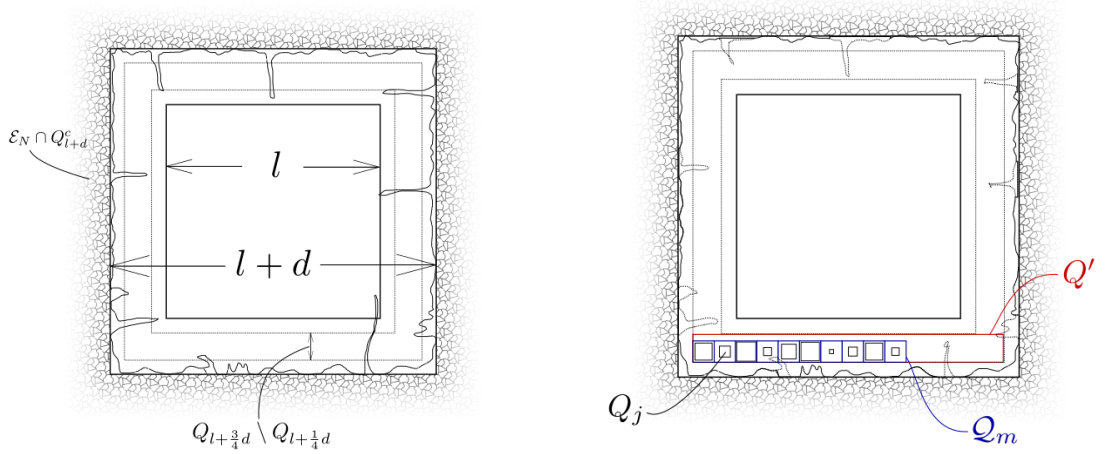
Define the sets:

$$\begin{cases} I_{l,d} &= \{i \in \{1, \dots, N\} \mid \mathcal{E}(i) \subset\subset Q_{l+d}\}, \\ k(l,d) &= \#(I_{l,d}), \\ J_{l,d} &:= \left\{ j \in \{1, \dots, N\} \mid \mathcal{E}(j)^{(1)} \cap \partial Q_{l+d} \neq \emptyset \text{ and } \mathcal{E}(j)^{(1)} \cap \partial Q_{l+\frac{3}{4}d} \neq \emptyset \right\}, \\ m(l,d) &:= \#(J_{l,d}). \end{cases} \quad (2.4.11)$$

We now divide the proof into four steps, for the sake of clarity. In the sequel we always adopt the same letter (namely C , except for η, η_0, λ_0) for the constants though the value of the constants can change from line to line. Let us set $\delta = \frac{1}{N}$.

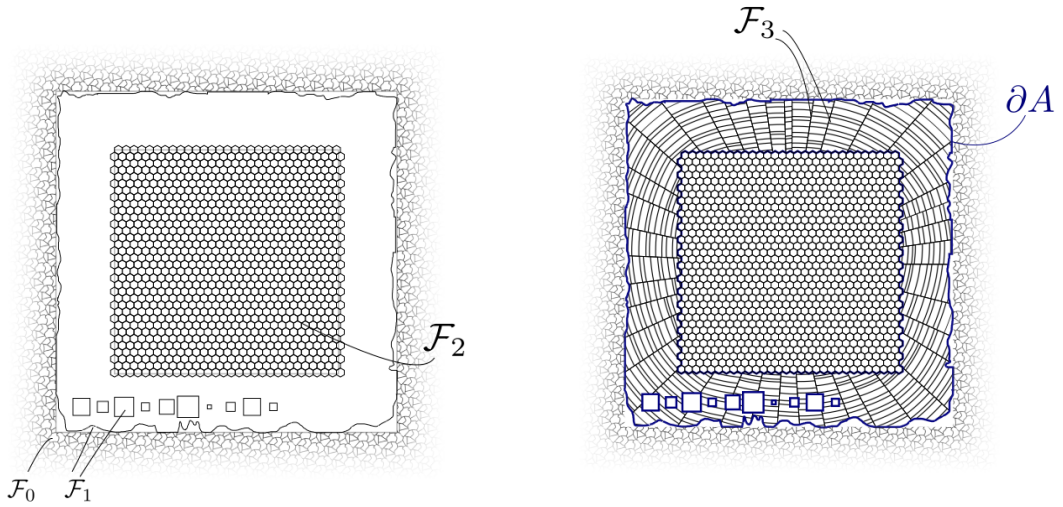
Step one: Figure 2.4.2b. The cluster \mathcal{F}_0 and \mathcal{F}_1 : replacement of the long chambers. In this step we provide a suitable adjustment of all the m chambers $\mathcal{E}(j)$ for $j \in J_{l,d}$ that are too long.

We cut the part of the chambers $\{\mathcal{E}(j)\}_{j \in J_{l,d}}$ lying inside Q_{l+d} . After this operation we need to recover the loss of area. Our aim is to recover the area by placing m small cubes with the right amount of area inside Q' (evidenced in red in Figure 2.4.2b) the lower rectangle of the stripe $Q_{l+\frac{3}{4}d} \setminus Q_{l+\frac{1}{4}d}$. To do that we first place a big grid \mathcal{Q}_m (evidenced in blue in Figure 2.4.2b) of m boxes of area δ (suitably arranged as in Figure 2.4.3) that we



(a) We consider a square $Q_l \subset\subset Q_{l+d} \subset\subset \Omega'$. The square Q_l is where we perform our construction. We remove all the chambers $\mathcal{E}(i)$ for $i \in I_{l,d}$ (defined in (2.4.11)) completely lying inside Q_{l+d} . We save a small substripe $Q_{l+\frac{3}{4}d} \setminus Q_{l+\frac{1}{4}d}$ where we are going to allocate some small squares Q_j needed to recover the area loss during the construction.

(b) We cut away all the tentacles $\mathcal{E}(j) \cap Q_{l+d}$ for $j \in J_{l,d}$ (defined in (2.4.11)) and we substitute them with squares Q_j inside the rectangle $Q' \subset Q_{l+\frac{3}{4}d} \setminus Q_{l+\frac{1}{4}d}$. We allocate each Q_j inside a square of the pre-allocate grid Q_m . The number of such long chambers $m(l,d)$ (and thus of the chambers of the grid Q_m) is controlled by the starting upper bound on the localized energy (2.4.4).



(c) We can now cover the square Q_l with an hexagonal grid, without overlapping the other part of the construction.

(d) We finally get the competitor \mathcal{F} , an N -cluster of equal area chambers agreeing with \mathcal{E} outside of Q_{l+d} and satisfying (2.4.6) by using Lemma 2.2.2 to cover the remaining part of $Q_{l+d} \setminus Q_l$ with chambers having the right amount of area.

Figure 2.4.2: The construction of the N -cluster \mathcal{F} in the proof of Proposition 2.4.6.

are using as skeleton. Inside each box we place a cube Q_j having the right amount of area ($|Q_j| = |\mathcal{E}(j) \cap Q_{l+d}|$ for $j \in J_{l,d}$) and we complete the construction. Clearly, to perform this construction we need to show that there is enough space inside Q' . By making use of an estimate on the number $m(l,d)$ and provided λ_0 and η are big enough, we show that

this is the case.

Note that since \mathcal{E} is an indecomposable N -cluster, for every $j \in J_{l,d}$ it holds

$$\text{diam}(\mathcal{E}(j)^{(1)}) \geq \frac{d}{4}$$

and thus thanks to Proposition 1.2.8 we must have $P(\mathcal{E}(j); Q_{l+d}) \geq \frac{d}{2}$. By the trivial upper bound

$$\sum_{j \in J_{l,d}} P(\mathcal{E}(j); Q_{l+d}) \leq 2P(\mathcal{E}; Q_{l+d}) \leq 2P$$

we obtain

$$m \frac{d}{2} \leq \sum_{j \in J_{l,d}} P(\mathcal{E}(j); Q_{l+d}) \leq 2P$$

and thus

$$m \leq 4 \frac{P}{d}. \quad (2.4.12)$$

The total area of the union of the long chambers inside Q_{l+d} is easily estimated from above by

$$\sum_{j \in J_{l,d}} |\mathcal{E}(j) \cap Q_{l+d}| \leq m\delta,$$

which, combined with (2.4.12), implies:

$$\begin{aligned} \sum_{j \in J_{l,d}} |\mathcal{E}(j) \cap Q_l| &\leq m\delta \\ &\leq 4 \frac{P}{d} \delta. \end{aligned} \quad (2.4.13)$$

Since we want to cut the long chambers and rebuild them into Q' (evidenced in red in Figure 2.4.2b) where $|Q'| = (l - \frac{d}{2}) \frac{d}{2} > \frac{ld}{4}$ (because of $d < \frac{l}{100} < \frac{l}{2}$), it is enough to ensure that

$$\sum_{j \in J_{l,d}} |\mathcal{E}(j) \cap Q_{l+d}| \leq \frac{ld}{4} \quad (2.4.14)$$

which, thanks to (2.4.13), can be obtained as a consequence of

$$4 \frac{P}{d} \delta \leq \frac{ld}{4},$$

or equivalently

$$d \geq 4 \sqrt{\frac{P}{l}} \delta. \quad (2.4.15)$$

Thanks to the definition of d (2.4.9), up to taking η bigger than a universal constant (as well as λ_0 according to (2.4.10)) we can always ensure the validity of (2.4.15) and thus of (2.4.14).

We now show how to place the grid Q_m (see Figure 2.4.3). Choose $v \in \mathbb{N}$ such that

$$v\sqrt{\delta} \leq \frac{d}{2}, \quad \frac{d}{2} \leq (v+1)\sqrt{\delta}.$$

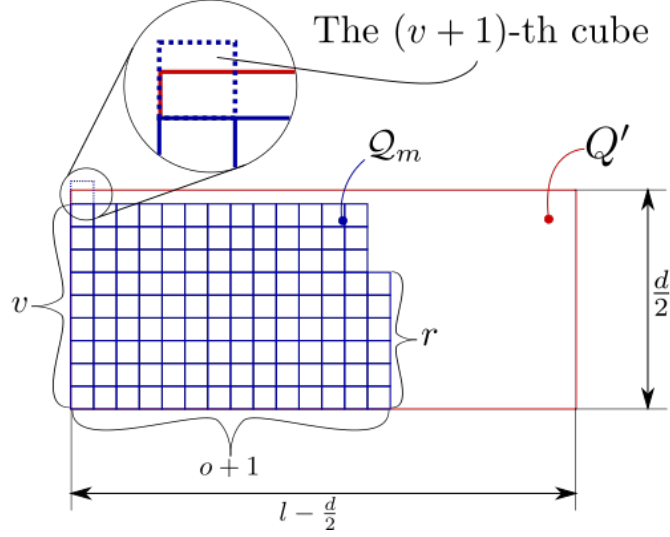


Figure 2.4.3: We place the skeleton grid \mathcal{Q}_m by choosing $v \in \mathbb{N}$ in a way that exactly v vertical cubes of area δ (and no one more) are contained into Q' . Then we choose o, r to be the integer such that $m = vo + r$.

The number v represent the maximum number of cubes of area δ that we can place "vertically" inside Q' (for example, $v = 1$ in Figure 2.4.2b). In particular

$$\frac{d}{2\sqrt{\delta}} - 1 \leq v \leq \frac{d}{2\sqrt{\delta}}. \quad (2.4.16)$$

Let $o, r \in \mathbb{N}$ be such that:

$$m = vo + r, \quad r \leq v.$$

We choose \mathcal{Q}_m to be a grid of $o + 1$ columns of cubes where the first o columns are made by v cubes and the $(o + 1)$ -th column contains exactly r cubes of area δ (see Figure 2.4.3 where a generic situation is represented, or 2.4.2b where $v = 1$). Clearly

$$vo \leq m \leq (v + 1)o,$$

and so

$$\frac{2m\sqrt{\delta}}{d + 2\sqrt{\delta}} \leq o \leq \frac{2m\sqrt{\delta}}{d - 2\sqrt{\delta}}. \quad (2.4.17)$$

In order to be sure that we have enough space inside Q' (to insert the grid \mathcal{Q}_m) we need to check that

$$(o + 1)\sqrt{\delta} \leq l - \frac{d}{2}.$$

Since, thanks to (2.4.17) and to (2.4.12),

$$\begin{aligned} (o + 1)\sqrt{\delta} &\leq \left(\frac{2(m - 1)\sqrt{\delta} + d}{d - 2\sqrt{\delta}} \right) \sqrt{\delta} \\ &\leq \left(\frac{2m\sqrt{\delta} + d}{d - 2\sqrt{\delta}} \right) \sqrt{\delta} \\ &\leq \left(\frac{8P\sqrt{\delta} + d^2}{d^2 - 2d\sqrt{\delta}} \right) \sqrt{\delta}, \end{aligned}$$

it is enough to check

$$\left(\frac{8P\sqrt{\delta} + d^2}{d^2 - 2d\sqrt{\delta}} \right) \sqrt{\delta} \leq \frac{l}{2},$$

which means

$$d^2 \left(\sqrt{\delta} - \frac{l}{2} \right) + dl\sqrt{\delta} + 8P\delta \leq 0,$$

that is satisfied when

$$d \geq \sqrt{\delta} \left(\frac{1 + \sqrt{1 + \frac{32P}{l} \left(\frac{1}{2} - \frac{\sqrt{\delta}}{l} \right)}}{1 - 2\frac{\sqrt{\delta}}{l}} \right).$$

By exploiting $P/l \geq 1$ and up to taking λ_0 and η bigger than a universal constant, by exploiting (2.4.5), we can always ensure that the previous condition is satisfied by d .

Thus we have space to place the grid $\mathcal{Q}_m \subset Q'$. For every $j \in J_{l,d}$ we consider a cube Q_j with the property $|Q_j| = |\mathcal{E}(j) \cap Q_{l+d}|$ and we place it into an empty box of \mathcal{Q}_m . We define the following clusters \mathcal{F}_0 and \mathcal{F}_1 as:

$$\begin{aligned} \mathcal{F}_0(j) &= \mathcal{E}(j) \quad j \notin J_{l,d} \cup I_{l,d} \\ \mathcal{F}_1(j) &= (\mathcal{E}(j) \cap \overline{Q_{l+d}^c}) \cup Q_j \quad j \in J_{l,d}. \end{aligned}$$

By construction each chamber of \mathcal{F}_0 and \mathcal{F}_1 has area $\frac{1}{N}$. Moreover

$$\begin{aligned} P(\mathcal{F}_0 \cup \mathcal{F}_1) &\leq P(\mathcal{E}; Q_{l+d}^c) + 4l + \sum_{j \in J} P(Q_j) \\ &\leq P(\mathcal{E}; Q_{l+d}^c) + C \left(l + \frac{P}{d} \sqrt{\delta} \right) \end{aligned} \quad (2.4.18)$$

for a universal constant C , because of (2.4.12).

Step two: Figure 2.4.2c. The h -cluster \mathcal{F}_2 : the hexagonal tiling. We completely cover Q_l with an hexagonal grid (see Figure 2.4.2c). As in the proof of Proposition 2.3.5 we do not consider the hexagons that do not intersect Q_l . Up to taking λ_0 and η bigger than a universal constant the total number of hexagons h (as in the proof of Proposition 2.3.5) is estimated by:

$$h\delta \leq \left(l + \text{diam}(H)\sqrt{\delta} \right)^2 \leq l^2 + Cl\sqrt{\delta}$$

where C is a universal constant. Hence

$$h \leq \frac{l^2}{\delta} + C \frac{l}{\sqrt{\delta}}, \quad (2.4.19)$$

If we denote with \mathcal{F}_2 such a cluster we obtain:

$$P(\mathcal{F}_2) \leq \frac{P(H)}{2} \frac{l^2}{\sqrt{\delta}} + Cl, \quad (2.4.20)$$

for a universal constant C . Note that up to choose a universal η big enough we can ensure that the cluster \mathcal{F}_1 and \mathcal{F}_2 do not overlap.

Step three: Figure 2.4.2d. The $(k-h)$ -Cluster \mathcal{F}_3 : a link between \mathcal{F}_0 and \mathcal{F}_2 . After the construction of \mathcal{F}_2 and \mathcal{F}_1 in the first two steps we are left to partition the set $A = (Q_{l+d} \setminus [\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2])$ (evidenced in blue in Figure 2.4.2d). We use Lemma 2.2.2 to build an $(k-h)$ -cluster \mathcal{E}_{k-h} with

$$P(\mathcal{E}_{k-h}) \leq C|Q_{l+d} \setminus Q_l| \sqrt{\frac{k-h}{|A|}} + P(A),$$

for a universal constant C . By construction and thanks to the choice of d the following hold

$$\frac{|A|}{k-h} = \frac{1}{N} = \delta, \quad |Q_{l+d} \setminus Q_l| \leq 3ld.$$

We thus set $\mathcal{F}_3 = \mathcal{E}_{k-h}$:

$$P(\mathcal{F}_3) \leq C \frac{ld}{\sqrt{\delta}} + P(A). \quad (2.4.21)$$

Step four: Figure 2.4.2d. The N -cluster \mathcal{F} and estimate (2.4.6). We now consider the N -cluster

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.$$

Notice that (see Figures 2.4.2c, 2.4.2d)

$$P(\mathcal{F}) = P(\mathcal{F}_0 \cup \mathcal{F}_1) + P(\mathcal{F}_2) + P(\mathcal{F}_3) - P(A).$$

Furthermore, by exploiting (2.4.18), (2.4.20), (2.4.21)

$$\begin{aligned} P(\mathcal{F}) &= P(\mathcal{F}_0 \cup \mathcal{F}_1) + P(\mathcal{F}_2) + P(\mathcal{F}_3) - P(A) \\ &\leq \frac{P(H)}{2} \frac{l^2}{\sqrt{\delta}} + P(\mathcal{E}; Q_{l+d}^c) + C \left(l + \frac{P}{d} \sqrt{\delta} + \frac{ld}{\sqrt{\delta}} \right) \end{aligned} \quad (2.4.22)$$

Notice that

$$l + \frac{P}{d} \sqrt{\delta} \leq \frac{ld}{\sqrt{\delta}},$$

when

$$d \geq \sqrt{\delta} \left(1 + \sqrt{1 + 4 \frac{P}{l}} \right). \quad (2.4.23)$$

Up to taking η and λ_0 bigger than a universal constant, thanks to (2.4.8), we can always guarantee that d satisfies (2.4.23). Hence, (2.4.22) leads us to

$$P(\mathcal{F}) \leq \frac{P(H)}{2} \frac{l^2}{\sqrt{\delta}} + P(\mathcal{E}; Q_l^c) + C \frac{ld}{\sqrt{\delta}}. \quad (2.4.24)$$

We now fix a universal η big enough. After we set $\eta_0 = 4\eta$ and we choose λ_0 big enough in dependence on η (and thus universal) satisfying (2.4.10). Since $d = \eta \sqrt{\frac{P}{lN}}$ and $\delta = \frac{1}{N}$ we obtain from (2.4.24):

$$P(\mathcal{F}) \leq \frac{P(H)}{2} l^2 \sqrt{N} + P(\mathcal{E}; Q_l^c) + C\eta \sqrt{Pl}. \quad (2.4.25)$$

By setting $C_0 = C\eta$ we get (2.4.6) from (2.4.25). \square

Condition (2.4.5) needs a starting energy estimate that is provided in the next Proposition. It is obtained by comparison, with a competitor (see Figure 2.4.4) constructed by simply intersecting Ω with the hexagonal tiling \mathcal{H}_δ (in Figure 2.1.1) for $\delta = \frac{1}{N}$.

Proposition 2.4.9. *For every $\Omega \subset \mathbb{R}^2$ open bounded set with Lipschitz boundary having $|\Omega| = 1$ there exists a natural number $M_0 > 0$ depending only on the shape of Ω , and a universal constant C_0 such that*

$$\frac{P(H)}{2}\sqrt{N} \leq \rho(N, \Omega) \leq \frac{P(H)}{2}\sqrt{N} + C_0P(\Omega) \quad \text{for } N \geq M_0. \quad (2.4.26)$$

Proof. By applying Proposition 2.2.1 with $\Omega = A$ we immediately get that for every \mathcal{E} minimizing N -cluster for Ω it holds

$$\rho(N, \Omega) = P(\mathcal{E}) \geq P(\mathcal{E}; \Omega) \geq \frac{P(H)}{2}\sqrt{N}. \quad (2.4.27)$$

Set $\delta = \frac{1}{N}$, consider the planar tiling \mathcal{H}_δ as in Figure 2.1.1 and the sets of indexes

$$\begin{aligned} I(\Omega) &= \{i \in \mathbb{N} \mid \mathcal{H}_\delta(i) \subset\subset \Omega\}, \\ I(\partial\Omega) &= \{i \in \mathbb{N} \mid \mathcal{H}_\delta(i) \cap \partial\Omega \neq \emptyset\} \end{aligned}$$

with $k = \#(I(\Omega))$, $h = \#(I(\partial\Omega))$. Clearly

$$\delta k = \sum_{i \in I(\Omega)} |\mathcal{E}(i)| \leq |\Omega| = 1, \quad (2.4.28)$$

thus $k \leq \frac{1}{\delta} = N$. Moreover, if we introduce

$$(\partial\Omega)_{2\sqrt{\delta}} = \partial\Omega + B_{2\sqrt{\delta}},$$

we notice that $\mathcal{H}_\delta(i) \subset\subset (\partial\Omega)_{2\sqrt{\delta}}$ for $i \in I(\partial\Omega)$ and so

$$\delta h = \sum_{i \in I(\partial\Omega)} |\mathcal{H}_\delta(i)| \leq |(\partial\Omega)_{2\sqrt{\delta}}|. \quad (2.4.29)$$

Since Ω has Lipschitz boundary there exists a $\delta_0 > 0$ depending only on the shape of Ω and a universal constant C such that

$$|(\partial\Omega)_{2\sqrt{\delta}}| \leq C\sqrt{\delta}P(\Omega), \quad \forall \delta < \delta_0. \quad (2.4.30)$$

By asking δ small enough and by plugging (2.4.30) into (2.4.29) we reach

$$h \leq \frac{CP(\Omega)}{\sqrt{\delta}}.$$

Summarizing, for a suitable δ_0 depending only on the shape of Ω only the following bounds hold:

$$k \leq \frac{1}{\delta} \quad \text{and} \quad h \leq \frac{CP(\Omega)}{\sqrt{\delta}}, \quad \text{for } \delta \leq \delta_0, \quad (2.4.31)$$

for a universal constant C . Define the cluster

$$\mathcal{F}_\Omega = \{\mathcal{H}_\delta(i) \mid i \in I(\Omega)\}.$$

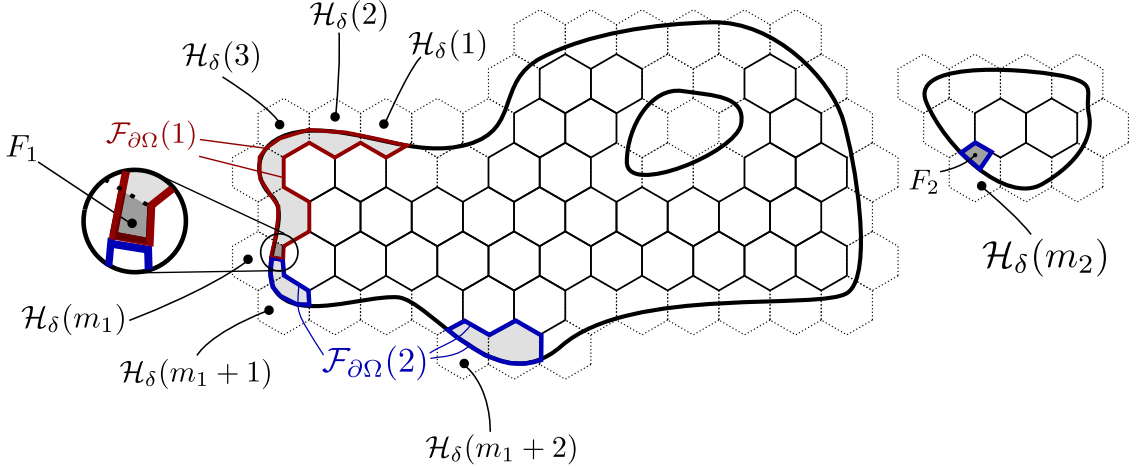


Figure 2.4.4: We consider all the hexagons from the tiling \mathcal{H}_δ that are compactly contained into Ω . Then we re-organize the hexagons intersecting $\partial\Omega$ in order to build chambers of area exactly $\frac{1}{N}$. To do that we suitably put together pieces of those chambers.

We want to re-organize the chambers $\{\mathcal{H}_\delta(i) \cap \Omega\}_{i \in I(\partial\Omega)}$ into a new cluster $\mathcal{F}_{\partial\Omega}$ in a way that every chamber $\mathcal{F}_{\partial\Omega}(j)$ encloses exactly an area δ . To do that consider a relabeling of \mathcal{H}_δ such that $I(\partial\Omega) = \{1, 2, \dots, h\}$ and

$$|\mathcal{H}_\delta(1) \cap \Omega| \leq |\mathcal{H}_\delta(2) \cap \Omega| \leq \dots \leq |\mathcal{H}_\delta(h) \cap \Omega|.$$

Let $1 \leq m_1 \leq h$ be the first index such that

$$\sum_{j=1}^{m_1-1} |\mathcal{H}_\delta(j) \cap \Omega| \leq \delta \leq \sum_{j=1}^{m_1} |\mathcal{H}_\delta(j) \cap \Omega|. \quad (2.4.32)$$

If (2.4.32) is in force we can select $F_1 \subset \mathcal{H}_\delta(m_1) \cap \Omega$ in a way that

$$\sum_{j=1}^{m_1-1} |\mathcal{H}_\delta(j) \cap \Omega| + |F_1| = \delta, \quad (2.4.33)$$

for example by simply tracing a segment (see Figure 2.4.4). We define the first chamber of the cluster $\mathcal{F}_{\partial\Omega}$ as

$$\mathcal{F}_{\partial\Omega}(1) = F_1 \cup \bigcup_{i=1}^{m_1-1} \mathcal{H}_\delta(i) \cap \Omega.$$

We consider now $m_1 \leq m_2 \leq h$, the first index after m_1 such that

$$\sum_{j=m_1+1}^{m_2-1} |\mathcal{H}_\delta(j) \cap \Omega| \leq \delta - \left| (\mathcal{H}_\delta(m_1) \cap \Omega) \setminus F_1 \right| \leq \sum_{j=m_1+1}^{m_2} |\mathcal{H}_\delta(j) \cap \Omega|. \quad (2.4.34)$$

Again, the validity of (2.4.34) ensures that it is possible to cut away a subset $F_2 \subset \mathcal{H}_\delta(m_2) \cap \Omega$ such that

$$\left| (\mathcal{H}_\delta(m_1) \cap \Omega) \setminus F_1 \right| + \sum_{j=m_1+1}^{m_2-1} |\mathcal{H}_\delta(j) \cap \Omega| + |F_2| = \delta,$$

and define

$$\mathcal{F}_{\partial\Omega}(2) = F_2 \cup [(\mathcal{H}_\delta(m_1) \cap \Omega) \setminus F_1] \cup \bigcup_{i=m_1+1}^{m_2-1} \mathcal{H}_\delta(j) \cap \Omega,$$

as clarified from Figure 2.4.4. Since $\delta = \frac{1}{N}$

$$h\delta \geq \sum_{j=1}^h |\mathcal{H}_\delta(j) \cap \Omega| = 1 - k\delta = (N - k)\delta.$$

By iterating the previous argument we end up, in exactly $N - k \leq h$ steps, with an $(N - k)$ -cluster $\mathcal{F}_{\partial\Omega} = \{\mathcal{F}_{\partial\Omega}(j)\}_{j=1}^{N-k}$ having chambers of area δ and partitioning $\Omega \setminus \mathcal{F}_\Omega$. Moreover, by construction we have added almost h hexagons and h segment of length smaller than $2\sqrt{\delta}$, so

$$P(\mathcal{F}_{\partial\Omega}) \leq P(\Omega) + hP(H)\sqrt{\delta} + h2\sqrt{\delta} = P(\Omega) + Ch\sqrt{\delta} \quad (2.4.35)$$

for a universal constant C . If we now combine the cluster $\mathcal{F}_\Omega = \{\mathcal{H}_\delta(i)\}_{i \in I(\Omega)}$ with $\mathcal{F}_{\partial\Omega}$ we obtain a competitor for $\rho(N, \Omega)$ and by construction and (2.4.31),(2.4.35) we get

$$\begin{aligned} \rho(N, \Omega) &\leq P(\mathcal{F}_{\partial\Omega} \cup \mathcal{F}_\Omega) \\ &\leq k \frac{P(H)}{2} \sqrt{\delta} + P(\mathcal{F}_{\partial\Omega}) \\ &\leq k \frac{P(H)}{2} \sqrt{\delta} + Ch\sqrt{\delta} + P(\Omega) \\ &\leq \frac{P(H)}{2} \frac{1}{\sqrt{\delta}} + CP(\Omega) \quad \text{for every } \delta < \delta_0. \end{aligned} \quad (2.4.36)$$

By recalling that $\delta = \frac{1}{N}$ and combining (2.4.27) and (2.4.36) we get (2.4.26) with $C_0 = C$ a universal constant and for every $N > \frac{1}{\delta_0} =: M_0(\Omega)$. \square

Remark 2.4.10. With Proposition 2.4.9 in mind we note that Theorems 2.4.2, 2.3.2 imply that at every scale $L > 0$ and for N big enough, the solution \mathcal{E} attaining $\rho(N, \Omega)$ has the main part of the localized energy close to the main part of $\rho(N, Q_L)$, namely dependent only on L (as in the case of periodic pattern). Therefore:

$$\lim_{N \rightarrow +\infty} \frac{P(\mathcal{E}; Q_L)}{\sqrt{|Q_L|N}} \sqrt{\frac{|\Omega|}{|Q_L|}} = \frac{P(H)}{2} = \lim_{N \rightarrow +\infty} \frac{\rho(N, Q_L)}{\sqrt{|Q_L|N}}.$$

By combining Proposition 2.4.6 with Proposition 2.4.9 we obtain a first (rough) estimate on the energy in a cube Q_l that we are going to refine in the sequel.

Proposition 2.4.11. *Let Ω be an open bounded set with Lipschitz boundary and $|\Omega| = 1$. Let $0 \leq \beta < \frac{1}{2}$ be a real number. Then there exist three positive constants $\bar{\eta}, \bar{\lambda}, \bar{M}$ depending only on β and on the shape of Ω and a universal constant $P_0 \geq 1$ with the following property. For every $N > \bar{M}$, every closed cube $Q_l \subset\subset \Omega$ satisfying*

$$d(Q_l, \partial\Omega) > \bar{\eta}N^{-\frac{1}{6}}, \quad l \geq \bar{\lambda}N^{-\beta} \quad (2.4.37)$$

and every indecomposable minimizing N -cluster $\mathcal{E} \in Cl(N, \Omega)$ the following estimate holds:

$$P(\mathcal{E}; Q_l) \leq P_0|Q_l|\sqrt{N}. \quad (2.4.38)$$

Remark 2.4.12. We are now in the position for explain where the exponent $\frac{1}{6}$, appearing in the hypothesis on the distance between boundaries (2.4.1) in Theorem 2.4.2, comes from. To prove that estimate (2.4.38) is in force on every scale $l \gg N^{-\frac{1}{2}}$ we repeatedly apply Proposition 2.4.6. We argue, essentially, as follows. We start by applying Proposition 2.4.6 with $\Omega' = \Omega$ and by exploiting the global energy estimate given by Proposition 2.4.9. This leads to the validity of (2.4.38) for every cube from a certain family $Q_L \in \mathcal{Q}_1$ of edges bigger than a certain $N^{-\alpha_1}$ and with $d(\partial Q_L, \partial \Omega) > N^{-\frac{1}{6}}$. Now we select a suitable cube $Q_L \in \mathcal{Q}_1$ with $L = N^{-\alpha_1}$ on which (2.4.38) is in force thanks to the previous step and we apply again Proposition 2.4.6 by setting $\Omega' = Q_L$. By exploiting $P \leq P_0 L^2 \sqrt{N}$ we are lead to the validity of (2.4.38) for every cube from a family $Q_l \in \mathcal{Q}_2 \supset \mathcal{Q}_1$ of size bigger than a certain $N^{-\alpha_2}$ but still lying at a distance $d(\partial Q_l, \partial \Omega) > N^{-\frac{1}{6}}$. The iteration of this argument leads to the proof of (2.4.38) for every closed cube Q_l with $l > N^{-\frac{1}{2}}$. At each application we gain the validity of (2.4.38) at a smaller scale but the restriction on $d(\partial Q_l, \partial \Omega)$ cannot be improved by this argument, since we need always to have at least the space for run the first iteration (which is $N^{-\frac{1}{6}}$).

Proof of Proposition 2.4.11. Set

$$\begin{cases} \alpha_{k+1} &= \frac{2}{3}\alpha_k + \frac{1}{6} \\ \alpha_0 &= 0, \end{cases}$$

and let us divide the proof in two steps. Note that

$$\begin{cases} \alpha_k < \frac{1}{2} & \text{for all } k \geq 0, \\ \alpha_k < \alpha_{k+1} & \text{for all } k \geq 0, \\ \lim_{k \rightarrow +\infty} \alpha_k = \frac{1}{2}. \end{cases}$$

Step one. We prove that for every $k \in \mathbb{N}$ there exist positive constants $\tau_{k-1}, \tau_k, \eta_k, M_k$ depending only on the shape of Ω and a universal positive constant $P_0 \geq 1$ such that

$$\tau_0 = \text{diam}(\Omega), \quad \eta_k \geq \eta_{k-1}, \quad M_k \geq M_{k-1} \quad \forall k$$

and with the following property. For every $N \geq M_k$, for every closed cube $Q_l \subset \subset \Omega$ with

$$d(\partial Q_l, \partial \Omega) > \eta_k N^{-\frac{1}{6}} \quad \tau_{k-1} N^{-\alpha_{k-1}} \geq l \geq \tau_k N^{-\alpha_k} \quad (2.4.39)$$

and for every $\mathcal{E} \in \text{Cl}(N, \Omega)$ indecomposable minimizing N -cluster for Ω the following estimate holds

$$P(\mathcal{E}; Q_l) \leq P_0 |Q_l| \sqrt{N}. \quad (2.4.40)$$

We argue by induction on k . We start by proving the validity of our assertion for $k = 1$ by setting $\tau_0 = \text{diam}(\Omega)$. Thanks to Proposition 2.4.9 there exists an M_0 depending on the shape of Ω only such that if $N > M_0$ and $\mathcal{E} \in \text{Cl}(N, \Omega)$ is a minimizing N -cluster for Ω the following estimate holds:

$$P(\mathcal{E}) \leq \frac{P(H)}{2} \sqrt{N} + C_0 P(\Omega), \quad (2.4.41)$$

for a universal constant C_0 . We now want to apply Proposition 2.4.6 with $\Omega' = \Omega$. Let us choose η_1, τ_1 such that

$$\tau_1 \geq 2\lambda_0 P(H)^{\frac{1}{3}}, \quad \eta_1 \geq \frac{\eta_0}{\sqrt{\tau_1}} \sqrt{P(H)},$$

where λ_0, η_0 are the constants given by Proposition 2.4.6. We choose $M_1 \geq M_0$ big enough so that

$$P(\mathcal{E}) \leq \frac{P(H)}{2} \sqrt{N} + C_0 P(\Omega) \leq P(H) \sqrt{N}, \quad (2.4.42)$$

and

$$\tau_0 > \tau_1 N^{-\frac{1}{6}}$$

for all $N \geq M_1$. If $N \geq M_1$ and $Q_l \subset\subset \Omega$ satisfies

$$d(\partial Q_l, \partial \Omega) \geq \eta_1 N^{-\frac{1}{6}}, \quad \tau_0 \geq l \geq \tau_1 N^{-\frac{1}{6}}, \quad (2.4.43)$$

with the previous choice of η_1, τ_1, M_1 , then it satisfies also

$$d(\partial Q_l, \partial \Omega) \geq \eta_0 \sqrt{\frac{P(\mathcal{E})}{Nl}}, \quad l \geq \lambda_0 \left(\left(\frac{P(\mathcal{E})}{N} \right)^{\frac{1}{3}} + N^{-\frac{1}{2}} \right). \quad (2.4.44)$$

Indeed if $N \geq M_1$ and $Q_l \subset\subset \Omega$ is a closed cube for which (2.4.43) holds we see that

$$\frac{N^{-\frac{1}{4}}}{\sqrt{l}} \leq \frac{1}{\sqrt{\tau_1}} N^{-\frac{1}{6}},$$

and since $M_1 \geq M_0$, thanks to (2.4.42) and to the choice of η_1, τ_1 we obtain (2.4.44):

$$\begin{aligned} d(Q_l, \partial \Omega) &> \eta_1 N^{-\frac{1}{6}} \\ &> \frac{\eta_0 \sqrt{P(H)}}{\sqrt{\tau_1}} N^{-\frac{1}{6}} \\ &> \eta_0 \sqrt{P(H)} \frac{N^{-\frac{1}{4}}}{\sqrt{l}} \\ &\geq \eta_0 \sqrt{\frac{P(\mathcal{E})}{Nl}}, \end{aligned} \quad (2.4.45)$$

$$\begin{aligned} l &\geq \tau_1 N^{-\frac{1}{6}} \\ &\geq \frac{\tau_1}{2} N^{-\frac{1}{6}} + \frac{\tau_1}{2} N^{-\frac{1}{6}} \\ &\geq \lambda_0 P(H)^{\frac{1}{3}} N^{-\frac{1}{6}} + \lambda_0 N^{-\frac{1}{2}} \\ &\geq \lambda_0 \left(\left(\frac{P(\mathcal{E})}{N} \right)^{\frac{1}{3}} + N^{-\frac{1}{2}} \right). \end{aligned} \quad (2.4.46)$$

In particular we can apply Proposition 2.4.6 by setting $\Omega' = \Omega$ and find an N -cluster $\mathcal{F} \in \text{Cl}(N, \Omega)$ such that

$$P(\mathcal{F}) \leq l^2 \frac{P(H)}{2} \sqrt{N} + P(\mathcal{E}; Q_l^c) + C_0 \sqrt{P(\mathcal{E})} l. \quad (2.4.47)$$

Estimate on l in (2.4.44) implies also that

$$P(\mathcal{E}) \leq \frac{l^3}{\lambda_0^3} N,$$

which inserted in (2.4.47) leads to

$$P(\mathcal{F}) \leq \left(\frac{P(H)}{2} + \frac{C_0}{\lambda_0^{3/2}} \right) |Q_l| \sqrt{N} + P(\mathcal{E}; Q_l^c). \quad (2.4.48)$$

By minimality we have

$$P(\mathcal{E}) \leq P(\mathcal{F}),$$

which combined with (2.4.48) leads to

$$P(\mathcal{E}; Q_l) \leq P_0 |Q_l| \sqrt{N}, \quad (2.4.49)$$

where we have set $P_0 = \left(\frac{P(H)}{2} + \frac{C_0}{\lambda_0^{3/2}} \right) \geq 1$. This proves the claim for $k = 1$ with the previous choice of universal $\tau_0, \tau_1, \eta_1, M_1$ depending only on the shape of Ω .

Assume now that our assertion holds on α_k for some constants $\tau_{k-1}, \tau_k, \eta_k, M_k$ ($k \geq 2$) and with the same constant P_0 defined above. We can conclude the validity of our assertion on α_{k+1} as a consequence of the following *claim*.

Claim. *There exist constants $\tau_{k+1}, \eta_{k+1}, M_{k+1}$ such that for every $N \geq M_{k+1}$ and every cube $Q_l \subset\subset \Omega$ satisfying*

$$d(\partial Q_l, \partial \Omega) > \eta_{k+1} N^{-\frac{1}{6}} \quad \tau_k N^{-\alpha_k} \geq l \geq \tau_{k+1} N^{-\alpha_{k+1}} \quad (2.4.50)$$

there exists a cube Q_L concentric to Q_l satisfying $Q_l \subset\subset Q_L \subset\subset \Omega$ and

$$d(\partial Q_L, \partial \Omega) > \eta_k N^{-\frac{1}{6}}, \quad \tau_{k-1} N^{-\alpha_{k-1}} \geq L \geq \tau_k N^{-\alpha_k} \quad (2.4.51)$$

for which the following holds on every indecomposable minimizing N -cluster $\mathcal{E} \in Cl(N, \Omega)$:

$$d(\partial Q_l, \partial Q_L) \geq \eta_0 \sqrt{\frac{P(\mathcal{E}; \mathring{Q}_L)}{Nl}}, \quad l \geq \lambda_0 \left(\left(\frac{P(\mathcal{E}; \mathring{Q}_L)}{N} \right)^{\frac{1}{3}} + N^{-\frac{1}{2}} \right). \quad (2.4.52)$$

Indeed, if we assume for a moment the validity of the claim, on every cube Q_l satisfying (2.4.50) we can apply Proposition 2.4.6 with $\Omega' = \mathring{Q}_L$ and find an N -cluster $\mathcal{F} \in Cl(N, \Omega)$ such that

$$P(\mathcal{F}) \leq \frac{P(H)}{2} |Q_l| \sqrt{N} + P(\mathcal{E}; Q_l^c) + C_0 \sqrt{P(\mathcal{E}; \mathring{Q}_L) l}. \quad (2.4.53)$$

Since, thanks to (2.4.52), it holds

$$P(\mathcal{E}; \mathring{Q}_L) \leq \frac{l^3}{\lambda_0^3} N$$

by comparison (2.4.53) is leading to (2.4.38) with the same constant $P_0 = \frac{P(H)}{2} + \frac{C_0}{\lambda_0^{3/2}}$. Hence we can achieve the proof of step one by induction.

Let us focus on the proof of the claim. Set $L = 2\tau_k N^{-\alpha_k}$ and let Q_L be the cube concentric to Q_l . Choose

$$\eta_{k+1} > \eta_k + 2\tau_k, \quad \tau_{k+1} \geq \left\{ 6\lambda_0 P_0^{\frac{1}{3}} \tau_k^{\frac{2}{3}}, 8(\eta_0^2) P_0, 2\lambda_0 \right\} \quad (2.4.54)$$

and $M_{k+1} \geq M_k$ to be such that

$$\tau_{k-1}N^{-\alpha_{k-1}} \geq 2\tau_k N^{-\alpha_k} \quad (2.4.55)$$

for every $N \geq M_{k+1}$. The cube Q_L and the cube Q_l share the center, hence we easily have

$$d(\partial Q_l, \partial Q_L) = \frac{L-l}{\sqrt{2}}$$

that, combined with the triangular inequality, implies

$$d(\partial Q_L, \partial \Omega) > d(\partial Q_l, \partial \Omega) - (L-l). \quad (2.4.56)$$

Since $\alpha_k \geq \frac{1}{6}$ ($k \geq 1$) and $d(\partial Q_l, \partial \Omega) > \eta_{k+1}N^{-\frac{1}{6}}$, relation (2.4.1) leads to

$$\begin{aligned} d(\partial Q_L, \partial \Omega) &> d(\partial Q_l, \partial \Omega) - (L-l) \\ &\geq (\eta_{k+1} - 2\tau_k)N^{-\frac{1}{6}}, \end{aligned}$$

which implies (2.4.51) because of (2.4.54). Moreover by exploiting hypothesis (2.4.50) on Q_l and the fact that on Q_L , by induction (thanks to (2.4.51)), it holds

$$P(\mathcal{E}; Q_L) \leq P_0|Q_L|\sqrt{N}$$

for every $\mathcal{E} \in \text{Cl}(N, \Omega)$ indecomposable minimizing N -cluster for Ω (with $N \geq M_{k+1}$) we obtain (2.4.52):

$$\begin{aligned} d(\partial Q_l, \partial Q_L) &= \frac{L-l}{\sqrt{2}} \\ &\geq \frac{\tau_k N^{-\alpha_k}}{\sqrt{2}} \sqrt{\frac{lN}{P_0|Q_L|\sqrt{N}}} \sqrt{\frac{P_0|Q_L|\sqrt{N}}{Nl}} \\ &\geq \frac{\sqrt{\tau_{k+1}}}{2\sqrt{2}P_0} N^{\frac{1}{4} - \frac{\alpha_{k+1}}{2}} \sqrt{\frac{P(\mathcal{E}; \mathring{Q}_L)}{Nl}} \\ &\geq \eta_0 \sqrt{\frac{P(\mathcal{E}; \mathring{Q}_L)}{Nl}}, \\ l &\geq \frac{\tau_{k+1}}{2} N^{-\alpha_{k+1}} + \frac{\tau_{k+1}}{2} N^{-\alpha_{k+1}} \\ &\geq \frac{\tau_{k+1}}{6\tau_k^{\frac{2}{3}}P_0^{\frac{1}{3}}} N^{\frac{1}{6} + \frac{2}{3}\alpha_k - \alpha_{k+1}} \left(\frac{P_0|Q_L|\sqrt{N}}{N} \right)^{\frac{1}{3}} + \lambda_0 N^{-\frac{1}{2}} \\ &\geq \lambda_0 \left(\left(\frac{P(\mathcal{E}; \mathring{Q}_L)}{N} \right)^{\frac{1}{3}} + N^{-\frac{1}{2}} \right) \end{aligned}$$

and we achieve the proof of the claim and thus of step one.

Step two. Let $\beta \in [0, \frac{1}{2})$ be a positive number and let k be such that $\alpha_{k-1} \leq \beta \leq \alpha_k$. Let $\tau_{k-1}, \tau_k, \eta_k, M_k$ be the constants given by the step one. We now set $\bar{\lambda} = \tau_k$, $\bar{\eta} = \eta_k$ and $\bar{M} = M_k$. Then we argue as follows. Let $Q_l \subset \subset \Omega$ be a closed cube with

$$d(\partial Q_l, \partial \Omega) > \bar{\eta}N^{-\frac{1}{6}}, \quad l > \bar{\lambda}N^{-\beta}.$$

By construction, $l \geq \tau_k N^{-\alpha_k}$. If it holds also $l \leq \tau_{k-1} N^{-\alpha_{k-1}}$ and thus

$$d(\partial Q_l, \partial \Omega) > \bar{\eta} N^{-\frac{1}{6}} = \eta_k N^{-\frac{1}{6}} \quad \tau_{k-1} N^{-\alpha_{k-1}} \geq l \geq \tau_k N^{-\alpha_k},$$

thanks to step one we immediately have that (2.4.38) is in force on Q_l . If it does not hold $l \leq \tau_{k-1} N^{-\alpha_{k-1}}$, then it must hold $l \geq \tau_{k-1} N^{-\alpha_{k-1}}$. As before, if it is true that $l \leq \tau_{k-2} N^{-\alpha_{k-2}}$, since we have chosen $\bar{\eta} = \eta_k \geq \eta_{k-1}$, we get

$$d(\partial Q_l, \partial \Omega) \geq \bar{\eta} N^{-\frac{1}{6}} \geq \eta_{k-1} N^{-\frac{1}{6}} \quad \tau_{k-2} N^{-\alpha_{k-2}} \geq l \geq \tau_{k-1} N^{-\alpha_{k-1}}. \quad (2.4.57)$$

Thus, thanks to step one, (2.4.38) must be in force for every $N \geq M_{k-1}$ hence for every $N \geq \bar{M} = M_k \geq M_{k-1}$. If this is not the case, and thus $l \geq \tau_{k-2} N^{-\alpha_{k-2}}$ we iterate the previous argument and we move to the interval $l \in [\tau_{k-2} N^{-\alpha_{k-2}}, \tau_{k-3} N^{-\alpha_{k-3}}]$ by exploiting $\eta_{k-1} \geq \eta_{k-2}$ and $M_{k-1} \geq M_{k-2}$. This argument will end in exactly k steps since, for sure $l \leq \text{diam}(\Omega) = \tau_0 N^{\alpha_0}$. As a consequence (2.4.38) must be in force for every $N \geq \bar{M}$, $l \geq \bar{\lambda} N^{-\beta}$ and for every cube Q_l lying at a distance bigger than $\bar{\eta} N^{-\frac{1}{6}}$ from $\partial \Omega$ where $\bar{\eta}, \bar{\lambda}, \bar{M}$ depend only on the shape of Ω and on the first α_k bigger than β (and thus on β). \square

Remark 2.4.13. The proof of Proposition (2.4.11) points out that we cannot reach the value $\beta = \frac{1}{2}$ because the constants $\bar{\eta}(\beta), \bar{\tau}(\beta), \bar{M}(\beta)$ are approaching $+\infty$ when β gets closer to $\frac{1}{2}$.

Finally we put together Propositions 2.4.6 and 2.4.11 to get our competitor at every scale $l \gg N^{-\frac{1}{2}}$.

Proposition 2.4.14. *Let Ω be an open bounded set with Lipschitz boundary and $|\Omega| = 1$. Let $0 \leq \beta < \frac{1}{2}$ be a fixed number. Then there exists three positive constants η, λ, M depending only on β and on the shape of Ω , and a universal constant C with the following property. For every $N > M$, every closed cube $Q_l \subset \subset \Omega$ satisfying*

$$d(\partial Q_l, \partial \Omega) > \eta N^{-\frac{1}{6}}, \quad l \geq \lambda N^{-\beta} \quad (2.4.58)$$

and every indecomposable minimizing N -cluster $\mathcal{E} \in Cl(N, \Omega)$ there exists an N -cluster $\mathcal{F} \in Cl(N, \Omega)$ such that:

$$P(\mathcal{F}) \leq \frac{P(H)}{2} |Q_l| \sqrt{N} + P(\mathcal{E}; Q_l^c) + CP(Q_l)^{\frac{3}{2}} N^{\frac{1}{4}}. \quad (2.4.59)$$

Proof. The proof follows easily as a consequence of the following claim.

Claim. *Let $0 \leq \beta < \frac{1}{2}$ be a fixed number. We can find positive constants η, λ, M depending only on β and on the shape of Ω with the following property. For every $N \geq M$, if $Q_l \subset \subset \Omega$ is a closed cube satisfying (2.4.58) there exists a concentric cube Q_{l+d} with $d < l$ such that $Q_l \subset \subset Q_{l+d} \subset \subset \Omega$ and*

$$P(\mathcal{E}; Q_{l+d}) \leq P_0 |Q_{l+d}| \sqrt{N} \quad (2.4.60)$$

where P_0 is the universal constant appearing in (2.4.11). Moreover for every \mathcal{E} indecomposable minimizing N -cluster for Ω it holds

$$d(\partial Q_l, \partial Q_{l+d}) \geq \eta_0 \sqrt{\frac{P(\mathcal{E}; Q_{l+d}^\circ)}{Nl}}, \quad l \geq \lambda_0 \left(\left(\frac{P(\mathcal{E}; Q_{l+d}^\circ)}{N} \right)^{\frac{1}{3}} + N^{-\frac{1}{2}} \right), \quad (2.4.61)$$

where λ_0, η_0 are the constants appearing in Proposition 2.4.6.

Indeed, assume for a moment the validity of the previous fact and choose a cube $Q_l \subset\subset \Omega$ satisfying (2.4.58) with the constants given by the claim. For any given indecomposable minimizing N -cluster $\mathcal{E} \in \text{Cl}(N, \Omega)$, estimate (2.4.61) ensures us that we can apply Proposition 2.4.6 with $\Omega' = Q_{l+d}$ and thus find an N -cluster $\mathcal{F} \in \text{Cl}(N, \Omega)$ such that

$$P(\mathcal{F}) \leq \frac{P(H)}{2} |Q_l| \sqrt{N} + P(\mathcal{E}; Q_l^c) + C_0 \sqrt{lP(\mathcal{E}; Q_{l+d})},$$

for a universal constant C_0 . By combining (2.4.60) with the previous estimate and by recalling that $d < l$ we reach

$$P(\mathcal{F}) \leq \frac{P(H)}{2} |Q_l| \sqrt{N} + P(\mathcal{E}; Q_l^c) + 2C_0 \sqrt{P_0} l^{\frac{3}{2}} N^{\frac{1}{4}},$$

which is (2.4.59) with $C = 2C_0 \sqrt{P_0}$.

Let us focus on the proof of the claim. Let $\bar{\eta}, \bar{\lambda}, \bar{M}$ be the constants given by Proposition 2.4.11. We show that by choosing λ, η such that

$$\eta \geq \max\{2\bar{\eta}, 4\eta_0 \sqrt{2P_0}\}, \quad \lambda \geq \max\left\{\bar{\lambda}, 32\lambda_0^3 P_0, \frac{\eta^2}{4}\right\} \quad (2.4.62)$$

and M such that

$$N^{\frac{1}{12}} \geq 2\sqrt{\text{diam}(\Omega)} + \bar{M}, \quad \text{for all } N \geq M, \quad (2.4.63)$$

the claim holds. Choose a cube $Q_l \subset\subset \Omega$ satisfying (2.4.58) with the previous choice of λ, η, M . Set

$$d = \frac{\eta}{2} \sqrt{l} N^{-\frac{1}{4}},$$

let Q_{l+d} be a cube concentric to Q_l and note that $d < l$ since:

$$\frac{\eta}{2} \sqrt{l} N^{-\frac{1}{4}} \leq \frac{\eta}{2\sqrt{\lambda}} \sqrt{l} \sqrt{\lambda} N^{-\beta} \leq \frac{\eta}{2\sqrt{\lambda}} l \leq l.$$

Thanks to the choice of M , for $N \geq M$ it holds:

$$\begin{aligned} d(\partial Q_{l+d}, \partial \Omega) &\geq d(\partial Q_l, \partial \Omega) - 2d \\ &\geq \eta N^{-\frac{1}{6}} - \eta \sqrt{l} N^{-\frac{1}{4}} \\ &\geq \eta N^{-\frac{1}{6}} (1 - \sqrt{\text{diam}(\Omega)} N^{-\frac{1}{12}}) \\ &\geq \frac{\eta}{2} N^{-\frac{1}{6}} \geq \bar{\eta} N^{-\frac{1}{6}} \end{aligned}$$

$$l + d \geq l \geq \lambda N^{-\beta} \geq \bar{\lambda} N^{-\beta}.$$

But then Q_{l+d} satisfies hypothesis (2.4.37) and thus (2.4.60) must be in force thanks to Proposition 2.4.11. Moreover for $N \geq M$ ($\geq \bar{M}$), thanks to Proposition 2.4.11 and to the validity of (2.4.60), we have that on Q_{l+d} it holds

$$P(\mathcal{E}; Q_{l+d}) \leq P_0 (l + d)^2 \sqrt{N} \leq 4P_0 l^2 \sqrt{N},$$

for every indecomposable minimizing N -cluster $\mathcal{E} \in \text{Cl}(N, \Omega)$. Thus

$$\begin{aligned}
d(\partial Q_l, \partial Q_{l+d}) &= \frac{d}{\sqrt{2}} \\
&= \frac{\eta}{2} \sqrt{l} N^{-\frac{1}{4}} \sqrt{\frac{Nl}{4P_0 l^2 \sqrt{N}}} \sqrt{\frac{4P_0 l^2 \sqrt{N}}{Nl}} \\
&\geq \frac{\eta}{4\sqrt{2}P_0} \sqrt{\frac{P(\mathcal{E}; Q_{l+d})}{Nl}} \\
&\geq \eta_0 \sqrt{\frac{P(\mathcal{E}; Q_{l+d})}{Nl}}.
\end{aligned} \tag{2.4.64}$$

From $\beta < \frac{1}{2}$ it follows

$$\lambda N^{-\frac{1}{2}} \leq l, \quad \Rightarrow \quad \frac{1}{\sqrt{N}} \leq \frac{l}{\lambda}$$

that leads to

$$\frac{P(\mathcal{E}; Q_{l+d})}{N} \lambda_0^3 \leq \frac{P_0}{\sqrt{N}} 4l^2 \lambda_0^3 \leq \frac{4P_0 \lambda_0^3}{\lambda} l^3 \leq \frac{l^3}{8}$$

and so

$$l \geq 2\lambda_0 \left(\frac{P(\mathcal{E}; Q_{l+d})}{N} \right)^{\frac{1}{3}}. \tag{2.4.65}$$

Clearly,

$$l \geq \lambda N^{-\beta} \geq 2\lambda_0 N^{-\frac{1}{2}}, \tag{2.4.66}$$

and thus by combining (2.4.64), (2.4.65) and (2.4.66) we obtain the validity of (2.4.61) and we achieve the proof. \square

2.4.2 Proof of Theorem 2.4.2

Proof of Theorem 2.4.2. Let Ω be a generic open set with Lipschitz boundary and let $\beta < \frac{1}{2}$ be fixed. We choose η, λ, M, C to be the constants given by Proposition 2.4.14. Let $Q_l \subset\subset \Omega$ be a cube satisfying (2.4.1). Up to a roto-translation we can assume that the center of Q_l is the origin. Then we perform the scaling $\Omega_0 = \frac{\Omega}{\sqrt{|\Omega|}}$, $Q = \frac{Q_l}{\sqrt{|\Omega|}}$. This immediately implies that

$$d(\partial Q, \partial \Omega) > \eta N^{-\frac{1}{6}}, \quad \sqrt{|Q|} > \lambda N^{-\beta},$$

since Q_l was satisfying (2.4.1). Thus Q is compactly contained into Ω_0 and thanks to the choice of the constants it satisfies (2.4.58). Moreover, if $\mathcal{E} \in \text{Cl}(N, \Omega)$ indecomposable minimizing N -cluster for Ω then $\frac{\mathcal{E}}{\sqrt{|\Omega|}}$ is an indecomposable minimizing N -cluster for Ω_0 . Thus for every $N \geq M$ and for every $\mathcal{E} \in \text{Cl}(N, \Omega)$ indecomposable minimizing N -cluster for Ω we can apply Proposition 2.4.14 and find an N -cluster $\mathcal{F} \in \text{Cl}(N, \Omega_0)$ such that:

$$P(\mathcal{F}) \leq \frac{P(H)}{2} |Q| \sqrt{N} + C |Q|^{\frac{3}{4}} N^{\frac{1}{4}} + P \left(\frac{\mathcal{E}}{\sqrt{|\Omega|}}; Q^c \right).$$

By comparison, this leads to

$$\begin{aligned}
P\left(\frac{\mathcal{E}}{\sqrt{|\Omega|}}; Q\right) &\leq \frac{P(H)}{2}|Q|\sqrt{N} + C|Q|^{\frac{3}{4}}N^{\frac{1}{4}} \\
\frac{1}{\sqrt{|\Omega|}}P(\mathcal{E}; Q_l) &\leq \frac{P(H)}{2}\frac{|Q_l|}{|\Omega|}\sqrt{N} + C\frac{|Q_l|^{\frac{3}{4}}}{|\Omega|^{\frac{3}{4}}}N^{\frac{1}{4}} \\
P(\mathcal{E}; Q_l) &\leq \frac{P(H)}{2}|Q_l|\sqrt{\frac{N}{|\Omega|}} + CP(Q_l)^{\frac{3}{2}}\left(\frac{N}{|\Omega|}\right)^{\frac{1}{4}}. \tag{2.4.67}
\end{aligned}$$

We have obtained (2.4.67) for every $N > M$ where M is depending only on the shape of Ω . Notice that if $N \leq M$ the restriction on l implies that $l \geq \lambda\sqrt{|\Omega|}M^{-\beta}$. Moreover, we can easily build a competitor with N chambers of the right measure (for example by dividing with N parallel segments the set Ω in chambers of the right amount of area) and obtain the estimate

$$P(\mathcal{E}; Q_l) \leq P(\mathcal{E}) \leq N \operatorname{diam}(\Omega) + P(\Omega) \leq N(\operatorname{diam}(\Omega) + P(\Omega)).$$

For all $N \leq M$ we have:

$$\begin{aligned}
\frac{P(\mathcal{E}; Q_l) - \frac{P(H)}{2}|Q_l|\sqrt{\frac{N}{|\Omega|}}}{P(Q_l)^{\frac{3}{2}}\left(\frac{N}{|\Omega|}\right)^{\frac{1}{4}}} &\leq \frac{N(\operatorname{diam}(\Omega) + P(\Omega))|\Omega|^{\frac{1}{4}}}{P(Q_l)^{\frac{3}{2}}} \\
&\leq \frac{M(\operatorname{diam}(\Omega) + P(\Omega))|\Omega|^{\frac{1}{4}}}{\lambda^{\frac{3}{2}}|\Omega|^{\frac{3}{4}}M^{-\frac{3}{2}\beta}} \\
&\leq \frac{(\operatorname{diam}(\Omega) + P(\Omega))M^{1+\frac{3}{2}\beta}}{\sqrt{|\Omega|}\lambda^{\frac{3}{2}}}.
\end{aligned}$$

Note that the quantity

$$\frac{(\operatorname{diam}(\Omega) + P(\Omega))}{\sqrt{|\Omega|}}$$

is invariant under scaling. So up to increasing the constant C in dependence only on β and the shape of Ω we can provide estimate (2.4.67) for every $N \in \mathbb{N}$.

As in the proof of Theorem 2.4.2, Lemma 2.2.1 gives us

$$P(\mathcal{E}; Q_l) \geq \frac{P(H)}{2}|Q_l|\sqrt{\frac{N}{|\Omega|}} - P(Q_l). \tag{2.4.68}$$

Since $l > \frac{\sqrt{|\Omega|}}{\sqrt{N}}$ we immediately have

$$\left(\frac{N}{|\Omega|}\right)^{\frac{1}{4}}|Q_l|^{\frac{3}{4}} \geq l^{-\frac{1}{2}}l^{\frac{3}{2}} = l$$

which, together with (2.4.68) implies

$$P(\mathcal{E}; Q_l) \geq \frac{P(H)}{2}|Q_l|\sqrt{\frac{N}{|\Omega|}} - 4\left(\frac{N}{|\Omega|}\right)^{\frac{1}{4}}|Q_l|^{\frac{3}{4}}. \tag{2.4.69}$$

By combining (2.4.67) and (2.4.69) we achieve the proof. \square

Chapter 3

A sharp quantitative version of Hales' isoperimetric Honeycomb Theorem

3.1 Introduction

The isoperimetric nature of the planar “honeycomb tiling” has been apparent since antiquity. Referring to [Mor09, Section 15.1] for a brief historical account on this problem, we just recall here that Hales’s isoperimetric Theorem, see inequality (3.1.2) below or Theorem 1.9.1, gives a precise formulation of this intuitive idea. Our goal here is to strengthen Hales’s theorem into a quantitative statement, similarly to what has been done with other isoperimetric theorems in recent years (see, for example, [FMP08, FMP10]).

Let \hat{H} denote the reference unit-area hexagon in \mathbb{R}^2 depicted in Figure 3.1.1, so that $\ell = (12)^{1/4}/3 = P(H)/6$ is the side-length of \hat{H} . Given $\alpha, \beta \in \mathbb{N}$, let us consider the torus $\mathcal{T} = \mathcal{T}(v_\beta, w_\alpha)$ where

$$v_\beta := (\sqrt{3}\beta\ell, 0), \quad w_\alpha := \left(0, \frac{3}{2}\alpha\ell\right).$$

We recall that v_β, w_α defines an equivalence relation \sim on \mathbb{R}^2 (see Subsection 1.5.3 where the flat torus is defined) thus we are allowed to define $H = \hat{H}/\sim \subset \mathcal{T}$.

In order to avoid degenerate situations, *we shall always assume that*

$$\alpha \text{ is even and } \beta \geq 2. \tag{3.1.1}$$

In this way, H is a regular unit-area hexagon (i.e., the vertices of \hat{H} belong to six different equivalence classes) and one obtains a reference unit-area tiling $\mathcal{H} = \{\mathcal{H}(h)\}_{h=1}^N$ of \mathcal{T} consisting of α rows and β columns of regular hexagons by considering translations of H by $(h\sqrt{3}\ell, 3\ell k/2)$ ($h, k \in \mathbb{Z}$); see again Figure 3.1.1. Under this assumption, *Hales’s isoperimetric honeycomb theorem* asserts that

$$P(\mathcal{E}) \geq P(\mathcal{H}), \tag{3.1.2}$$

whenever \mathcal{E} is a unit-area tiling of \mathcal{T} , and that $P(\mathcal{E}) = P(\mathcal{H})$ if and only if (up to a relabeling of the chambers of \mathcal{E}) one has $\mathcal{E}(h) = v + \mathcal{H}(h)$ for every $h = 1, \dots, N$ and for some $v = (t\sqrt{3}\ell, s\ell)$ with $s, t \in [0, 1]$. Our first main result strengthens this isoperimetric theorem in a sharp quantitative way.

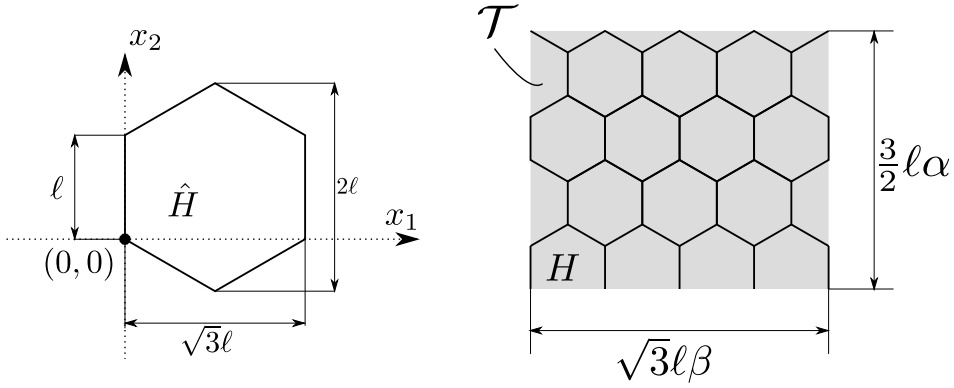


Figure 3.1.1: Throughout the chapter \hat{H} denotes the unit-area regular hexagon in \mathbb{R}^2 depicted on the left and we set $H = \hat{H}/\sim$. Since $|H| = 1$, one has $P(H) = 2(12)^{1/4}$, and the side-length of H is thus $\ell = (12)^{1/4}/3$. On the right, the torus \mathcal{T} (depicted in gray) and the reference unit-area tiling \mathcal{H} of \mathcal{T} (with $\alpha = \beta = 4$). Notice that $N = |\mathcal{T}| = \alpha\beta$. The chambers of \mathcal{H} are enumerated so that $\mathcal{H}(1) = H$, $\{\mathcal{H}(h)\}_{h=1}^\beta$ is the bottom row of hexagons in \mathcal{T} , and, more generally, if $0 \leq k \leq \alpha - 1$, then $\{\mathcal{H}(h)\}_{h=1+k\beta}^{(k+1)\beta}$ is the $(k+1)$ th row of hexagons in \mathcal{T} .

Theorem 3.1.1. *There exists a positive constant κ depending on \mathcal{T} such that*

$$P(\mathcal{E}) \geq P(\mathcal{H}) \left\{ 1 + \kappa \alpha(\mathcal{E})^2 \right\}, \quad (3.1.3)$$

whenever \mathcal{E} is a unit-area tiling of \mathcal{T} and

$$\alpha(\mathcal{E}) = \inf d(\hat{\mathcal{E}}, v + \mathcal{H})$$

where the minimization takes place among all $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0, 1]$, and among all tilings $\hat{\mathcal{E}}$ obtained by setting $\hat{\mathcal{E}}(h) = \mathcal{E}(\sigma(h))$ for a permutation σ of $\{1, \dots, N\}$. (Recall that the chambers of the reference honeycomb \mathcal{H} are enumerated in a specific way, see Figure 3.1.1.)

Remark 3.1.2. We notice that (3.1.3) is sharp in the decay rate of $\alpha(\mathcal{E})$ in terms of $P(\mathcal{E}) - P(\mathcal{H})$. Indeed, if $\omega : (0, \infty) \rightarrow (0, \infty)$ is such that $P(\mathcal{E}) \geq P(\mathcal{H})(1 + \omega(\alpha(\mathcal{E})))$ for every unit-area tiling \mathcal{E} , then, for some $s_0 > 0$, one must have $\omega(s) \leq C s^2$ for $s \in (0, s_0)$. Indeed, one can explicitly construct a one-parameter family $\{\mathcal{E}_t\}_{0 < t < \varepsilon}$ of unit-area tilings of \mathcal{T} (by gently pushing three edges of the grid around a singular point by maintaining the area constraints) such that $P(\mathcal{E}_t) \leq P(\mathcal{H})(1 + C \alpha(\mathcal{E}_t)^2)$ and $\{\alpha(\mathcal{E}_t) : t \in (0, \varepsilon)\} = (0, s_0)$, so that $\omega(s) \leq C s^2$ for every $s \in (0, s_0)$.

In Theorem 3.3.1 below, inequality (3.1.3) is proven in much stronger form for $\partial\mathcal{E}$ in a special class of C^1 -small $C^{1,1}$ -diffeomorphic images of $\partial\mathcal{H}$, see (3.3.3) and (3.3.4). The two main ingredients in the proof of Theorem 3.3.1 are: a quantitative version of the hexagonal isoperimetric inequality, which we deduce from [FRS85, IN14], see Lemma 3.2.1; and a quantitative version of Hales's hexagonal isoperimetric inequality (the key tool behind Hales's proof of (3.1.2)), proved in Lemma 3.3.2. These inequalities allow one to prove that each chamber of the unit-area tiling \mathcal{E} is actually close, in terms of the size of $P(\mathcal{E}) - P(\mathcal{H})$, to some regular unit-area hexagon in \mathcal{T} . These hexagons have no reason to fit nicely into an hexagonal honeycomb of \mathcal{T} (that is, a translation of \mathcal{H}), therefore we need an additional argument to show that, up to translations and rotations of order

$P(\mathcal{E}) - P(\mathcal{H})$, one can achieve this. Having completed the proof of Theorem 3.3.1, we deduce Theorem 3.1.1 by a contradiction argument based on an improved convergence theorem for planar bubble clusters that was recently established in [CLM15], and along the lines of the selection principle method proposed in [CL12b]. Another consequence of Theorem 3.3.1, obtained in a similar vein, is the following result, which gives a precise description of isoperimetric tilings of \mathcal{T} subject to an ‘‘almost unit-area’’ constraint.

Theorem 3.1.3. *There exist positive constants C_0, δ_0 depending on \mathcal{T} with the following property. If $\sum_{h=1}^N m_h = N$ with $m_h > 0$ and $|m_h - 1| < \delta_0$ for every $h = 1, \dots, N$, and if \mathcal{E}_m is an N -tiling of \mathcal{T} which is a minimizer in*

$$\inf\{P(\mathcal{E}) : |\mathcal{E}(h)| = m_h \quad \forall h = 1, \dots, N\} \quad (3.1.4)$$

then, up to a relabeling of the chambers of \mathcal{E}_m , there exists a $C^{1,1}$ -diffeomorphism $f_m : \partial\mathcal{H} \rightarrow \partial\mathcal{E}_m$ such that

$$\|f_m - (v + \text{Id})\|_{C^0(\partial\mathcal{H})}^2 + \|f_m - (v + \text{Id})\|_{C^1(\partial\mathcal{H})}^4 \leq C_0 \sum_{h=1}^N |m_h - 1|, \quad (3.1.5)$$

for some $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0, 1]$.

Next, let us consider the family X of those $\Phi \in C^0(\mathcal{T} \times S^1; (0, \infty))$ such that the positive one-homogeneous extension of $\Phi(x, \cdot)$ to \mathbb{R}^2 is convex, fix $\psi \in C^0(\mathcal{T}; (0, \infty))$, and consider the isoperimetric problem

$$\lambda(\Phi, \psi) = \inf \left\{ \Phi(\mathcal{E}) = \frac{1}{2} \sum_{h=1}^N \Phi(\mathcal{E}(h)) : \int_{\mathcal{E}(h)} \psi = \frac{1}{N} \int_{\mathcal{T}} \psi \quad \forall h = 1, \dots, N \right\}, \quad (3.1.6)$$

where for a set of finite perimeter $E \subset \mathcal{T}$ we have set

$$\Phi(E; A) = \int_{A \cap \partial^* E} \Phi(x, \nu_E(x)) d\mathcal{H}^1(x), \quad \Phi(E) = \Phi(E; \mathcal{T}),$$

provided $\partial^* E$ and $\nu_E : \partial^* E \rightarrow S^1$ denote, respectively, the reduced boundary and the measure-theoretic outer unit normal of E . Notice that although we do not assume Φ to be even, we have nevertheless that $\lambda(\Phi, \psi) = \lambda(\hat{\Phi}, \psi)$ where $\hat{\Phi}(x, \nu) = (\Phi(x, \nu) + \Phi(x, -\nu))/2$. An interesting example is obtained when g is a Riemannian metric on \mathcal{T} and

$$\Phi(x, \nu) = \sqrt{g(x)[\nu^\perp, \nu^\perp]}, \quad \psi = \sqrt{\det(g(x))},$$

where $\nu^\perp = (\nu_2, -\nu_1)$ if $\nu = (\nu_1, \nu_2)$. In this case, (3.1.6) boils down to minimizing the total Riemannian perimeter of a partition of \mathcal{T} into N -regions of equal Riemannian area.

Theorem 3.1.4. *Given $L > 0$ and $\gamma \in (0, 1]$, there exist $C_0, \delta_0 > 0$ (depending on \mathcal{T} , L and γ) with the following property. If \mathcal{E} is a minimizer in (3.1.6) for $\Phi \in X \cap \text{Lip}(\mathcal{T} \times S^1)$ and $\psi \in C^{1,\gamma}(\mathcal{T})$ such that*

$$\begin{aligned} \text{Lip } \Phi + \|\psi\|_{C^{1,\gamma}(\mathcal{T})} &\leq L, \\ \|\Phi - 1\|_{C^0(\mathcal{T} \times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})} &< \delta_0, \end{aligned} \quad (3.1.7)$$

then

$$\inf_{s,t \in [0,1]} \text{hd}(\partial\mathcal{E}, v + \partial\mathcal{H})^4 \leq C_0 \left(\|\Phi - \text{Id}\|_{C^0(\mathcal{T} \times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})} \right), \quad (3.1.8)$$

where $v = (t\sqrt{3}\ell, s\ell)$ and $\text{hd}(S, T)$ denote the Hausdorff distance between the closed sets S and T in \mathcal{T} .

We deduce Theorem 3.1.4 from Theorem 3.1.1 by some comparison arguments and density estimates. Since we are assuming that $\nabla\Phi$ is merely bounded, we do not expect $\partial\mathcal{E}$ to be a C^1 -diffeomorphic image of $\partial\mathcal{H}$. From this point of view, (3.1.8) seems to express a qualitatively sharp control on $\partial\mathcal{E}$. At the same time, when more regular integrands Φ are considered (see, e.g., [DS02] for the kind of assumption one may impose here) one would expect to be able to obtain a control in the spirit of (3.1.5). However a description of singularities of isoperimetric clusters in this kind of setting, although arguably achievable at least in some special cases, is missing at present. In turn, understanding singularities would be essential in order to adapt the improved convergence theorem from [CLM15] to this context (see Theorem 1.9.3 above), and thus to be able to strengthen (3.1.8) into an estimate analogous to (3.1.5).

The chapter is organized as follows. In Section 3.2 we deduce from [FRS85, IN14] a quantitative isoperimetric inequality for polygons of possible independent interest. In Section 3.3 we prove Theorem 3.3.1 on small C^1 -deformations of $\partial\mathcal{H}$ (actually with the Hausdorff distance between $\partial\mathcal{E}$ and $\partial\mathcal{H}$ in place of $d(\mathcal{E}, \mathcal{H})$ on the right-hand side of (3.1.3)). In Section 3.4 we exploit the improved convergence theorem from [CLM15] to deduce Theorem 3.1.1 and Theorem 3.1.3 from Theorem 3.3.1, and, finally, to deduce Theorem 3.1.4 from Theorem 3.1.1.

3.2 A quantitative isoperimetric inequality for polygons

Throughout this section we fix $n \geq 3$. We denote by Π a convex unit-area n -gon, and by Π_0 a reference unit-area regular n -gon. If ℓ and r denote, respectively, the side-length and radius of Π_0 , then one easily finds that

$$P(\Pi_0) = n\ell = 2\sqrt{n \tan\left(\frac{\pi}{n}\right)}, \quad r^{-1} = \sqrt{n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)}.$$

(Notice that in the other sections of the chapter we always assume $n = 6$, so that $\ell = (12)^{1/4}/3$ according to the convention set in the introduction.) The isoperimetric theorem for n -gons asserts that

$$P(\Pi) \geq n\ell, \quad (3.2.1)$$

with equality if and only if $\Pi = \rho(\Pi_0)$ for a rigid motion ρ of \mathbb{R}^2 . A sharp quantitative version of (3.2.1) is proved in [IN14] starting from the main result in [FRS85]. Precisely, let us now denote by ℓ_i and r_i the lengths of the i th edge and the i th radius of Π (labeled so that $\ell_i = \ell_j$ and $r_i = r_j$ if $i = j$ modulo n), and set

$$\bar{\ell} = \frac{1}{n} \sum_{i=1}^n \ell_i, \quad \bar{r} = \frac{1}{n} \sum_{i=1}^n r_i.$$

Then [IN14, Corollary 1.3] asserts that

$$C(n) \left(P(\Pi)^2 - (n\ell)^2 \right) \geq \sum_{i=1}^n (r_i - \bar{r})^2 + \sum_{i=1}^n (\ell_i - \bar{\ell})^2. \quad (3.2.2)$$

The right-hand side of inequality (3.2.2) measures the distance of Π from being a unit-area regular n -gon in the sense that if $r_i = \bar{r}$ and $\ell_i = \bar{\ell}$, then it must be $\bar{r} = r$ and $\bar{\ell} = \ell$ by the area constraint, and thus Π is a regular unit-area n -gon. However, in addressing our problem we shall need (in the case $n = 6$) to control the distance of Π from a specific regular unit-area n -gon by means of $P(\Pi)^2 - (n\ell)^2$. Passing from (3.2.2) to this kind of control is the subject of the following proposition.

Proposition 3.2.1. *There exists a positive constant $C(n)$ with the following property: for every convex unit-area n -gon Π there exists a rigid motion ρ of \mathbb{R}^2 such that*

$$C(n) (P(\Pi)^2 - (n\ell)^2) \geq \text{hd}(\partial\Pi, \partial\rho\Pi_0)^2. \quad (3.2.3)$$

Proof. Up to a translation, we can assume that Π has barycenter at 0. Next, if $P(\Pi) \geq n\ell + \eta P(\Pi)$ for some $\eta > 0$, then $P(\Pi)^2 - (n\ell)^2 \geq \eta P(\Pi)^2$. Since $\text{hd}(\partial\Pi, \partial\rho\Pi_0) < \text{diam}(\Pi) + \text{diam}(\Pi_0) \leq (P(\Pi) + P(\Pi_0))/2 \leq P(\Pi)$ whenever $\partial\rho\Pi_0$ intersects $\partial\Pi$, we conclude that (3.2.3) holds with $C(n) = \eta^{-1}$. In other words, in proving (3.2.3), one can assume without loss of generality that

$$P(\Pi) - n\ell < \eta P(\Pi) \quad (3.2.4)$$

for an arbitrarily small constant $\eta = \eta(n)$. By a trivial compactness argument (on the class of convex n -gons with barycenter at 0), one sees that given $\varepsilon > 0$ there exists $\eta > 0$ such that if (3.2.4) holds, then, up to rigid motions,

$$\text{hd}(\partial\Pi, \partial\Pi_0) < \varepsilon, \quad (3.2.5)$$

where the reference regular unit-area n -gon Π_0 is assumed to have barycenter at 0.

Now let v_i and w_i denote the positions of the vertices of Π and Π_0 respectively: by (3.2.5) and up to a rotation, one can entail that

$$|v_i - w_i| < \varepsilon, \quad \forall i = 1, \dots, n, \quad v_1 = \lambda w_1 \quad \text{for some } \lambda > 0.$$

Let ρ_i denote the rotation around the origin such that $\rho_i(v_i) = \lambda_i w_i$ for some $\lambda_i > 0$ (so that $\rho_1 = \text{Id}$ by $v_1 = \lambda w_1$), and let θ_i denote the angle identifying ρ_i as a counterclockwise rotation; since $\|\rho_i - \text{Id}\| \leq |\theta_i|$ and $|\rho_i(v_i) - w_i| = |r_i - r|$, one has

$$\text{hd}(\partial\Pi, \partial\Pi_0) \leq C \sum_{i=1}^n |v_i - w_i| \leq C \sum_{i=1}^n r_i |\theta_i| + |r_i - r|. \quad (3.2.6)$$

Let us now set $\delta = P(\Pi) - n\ell$: by (3.2.2) and (3.2.4) one finds

$$\max_{1 \leq i \leq n} |r_i - \bar{r}| + |\ell_i - \bar{\ell}| \leq C \sqrt{\delta}. \quad (3.2.7)$$

Since $\bar{\ell} = n^{-1}P(\Pi)$ gives $|\bar{\ell} - \ell| = n^{-1}\delta$, we deduce from $|\ell_i - \bar{\ell}| \leq C\sqrt{\delta}$ that

$$\max_{1 \leq i \leq n} |\ell_i - \ell| \leq C\sqrt{\delta}. \quad (3.2.8)$$

Let now $A(a, b, c)$ denote the area of a triangle with sides of length a, b and c . Since A is a Lipschitz function in an ε -neighborhood of (r, r, ℓ) (where both (\bar{r}, \bar{r}, ℓ) and (r_i, r_{i+1}, ℓ_i) lie by (3.2.5)), by (3.2.7), (3.2.8) and by $|\Pi_0| = |\Pi|$ we find

$$\left| n A(r, r, \ell) - n A(\bar{r}, \bar{r}, \ell) \right| = \left| \sum_{i=1}^n A(r_i, r_{i+1}, \ell_i) - n A(\bar{r}, \bar{r}, \ell) \right| \leq C \sqrt{\delta}.$$

Since $A(a, a, \ell) = (\ell/4) \sqrt{4a^2 - \ell^2}$ we immediately see that $|A(r, r, \ell) - A(a, a, \ell)| \geq c|a - r|$ whenever $|a - r| < \varepsilon$ and where $c = c(\ell) = c(n) > 0$. Thus, $|r - \bar{r}| \leq C\sqrt{\delta}$, and (3.2.7) and (3.2.8) give

$$\max_{1 \leq i \leq n} |r_i - r| + |\ell_i - \ell| \leq C\sqrt{\delta}. \quad (3.2.9)$$

If α_i denotes the interior angle between v_i and v_{i+1} (so that $|\alpha_i - 2\pi/n| = O(\varepsilon)$ by (3.2.5)), then

$$\alpha_i = f(r_i, r_{i+1}, \ell_i), \quad \text{where} \quad f(a, b, c) = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right).$$

Since f is a Lipschitz function in an ε -neighborhood of (r, r, ℓ) , we conclude from (3.2.9) that

$$\max_{1 \leq i \leq n} \left| \alpha_i - \frac{2\pi}{n} \right| = \max_{1 \leq i \leq n} \left| f(r_i, r_{i+1}, \ell_i) - f(r, r, \ell) \right| \leq C\sqrt{\delta}.$$

In particular, since $\theta_1 = 0$ (as $\rho_1 = \text{Id}$), we deduce from this last estimate that $|\theta_i| \leq C\sqrt{\delta}$ for $i = 1, \dots, n$. We plug this inequality and (3.2.9) in (3.2.6) to conclude the proof. \square

Coming to the torus \mathcal{T} , we shall use the following corollary of Proposition 3.2.1.

Corollary 3.2.2. *There exist positive constants η and c , independent from \mathcal{T} , with the following property. If Π is a convex hexagon in \mathcal{T} such that $\text{hd}(\partial\Pi, \partial H) \leq \eta$, then there exists a regular hexagon H_* in \mathcal{T} with $|\Pi| = |H_*|$ and for which the following holds:*

$$P(\Pi) - P(H)\sqrt{|\Pi|} \geq c \text{hd}(\partial\Pi, \partial H_*)^2. \quad (3.2.10)$$

Proof. We first notice that by Proposition 3.2.1 and by scaling, if $\hat{\Pi}$ is a convex hexagon in \mathbb{R}^2 , then there exists a regular hexagon \hat{H}_* with $|\hat{H}_*| = |\hat{\Pi}|$ and

$$P(\hat{\Pi})^2 - P(\hat{H})^2|\hat{\Pi}| \geq c \text{hd}(\partial\hat{\Pi}, \partial\hat{H}_*)^2. \quad (3.2.11)$$

Since Π is a convex hexagon in \mathcal{T} with $\text{hd}(\partial\Pi, \partial H) \leq \eta$, then there exists a convex hexagon $\hat{\Pi}$ in \mathbb{R}^2 isometric to Π with $\text{hd}(\partial\hat{\Pi}, \partial\hat{H}) \leq \eta$. In particular, for some constant C independent from \mathcal{T} , one has

$$P(\hat{\Pi}) - P(\hat{H})\sqrt{|\hat{\Pi}|} \leq C\eta, \quad P(\hat{\Pi}) + P(\hat{H})\sqrt{|\hat{\Pi}|} \leq C,$$

and thus (3.2.11) gives, up to further decrease the value of c ,

$$C\eta \geq P(\hat{\Pi}) - P(\hat{H})\sqrt{|\hat{\Pi}|} \geq c \text{hd}(\partial\hat{\Pi}, \partial\hat{H}_*)^2. \quad (3.2.12)$$

By (3.2.12) and $\text{hd}(\partial\hat{\Pi}, \partial\hat{H}) \leq \eta$ we have $\text{hd}(\partial\hat{H}, \partial\hat{H}_*) \leq C\sqrt{\eta}$. Now, since $\beta \geq 2$ and α is even one can find $\eta_* > 0$ (independent of α and β) such that $I_{\eta_*}(\hat{H}) = \{x \in \mathbb{R}^2 : \text{dist}(x, \hat{H}) \leq \eta_*\}$ is compactly contained into a rectangular box of height $3\ell\alpha/2$ and width $\sqrt{3}\ell\beta$. As a consequence, if \hat{J} is a polygon contained in $I_{\eta_*}(\hat{H})$, then $J = \hat{J}/\sim \subset \mathcal{T}$ is isometric to \hat{J} . Thus, if $C\sqrt{\eta} < \eta_*$, then $H_* = \hat{H}_*/\sim$ is a regular hexagon in \mathcal{T} with $|H_*| = |\Pi|$ and $\text{hd}(\partial\hat{\Pi}, \partial\hat{H}_*) = \text{hd}(\partial\Pi, \partial H_*)$, and (3.2.10) follows from (3.2.12). \square

3.3 Small deformations of the reference honeycomb

The main result of this section is Theorem 3.3.1, which provides us, on a restricted class of unit-area tilings, with a stronger stability estimate than the one in Theorem 3.1.1. Before stating this result we briefly recall the following terminology.

Regular and singular sets: Given a N -tiling \mathcal{E} of \mathcal{T} one sets

$$\partial\mathcal{E} = \bigcup_{h=1}^N \partial\mathcal{E}(h), \quad \partial^*\mathcal{E} = \bigcup_{h=1}^N \partial^*\mathcal{E}(h),$$

$$\Sigma(\mathcal{E}) = \partial\mathcal{E} \setminus \partial^*\mathcal{E}, \quad [\partial\mathcal{E}]_\mu = \{x \in \partial\mathcal{E} : \text{dist}(x, \Sigma(\mathcal{E})) > \mu\}, \quad \mu > 0,$$

where ∂^*E denotes the reduced boundary of a set of finite perimeter E in \mathcal{T} , and where the normalization convention $\partial E = \overline{\partial^*E} = \text{spt } \mu_E$ for sets of finite perimeter is always assumed to be in force, see Subsection 1.2.8 above. We call $\partial^*\mathcal{E}$ and $\Sigma(\mathcal{E})$ the *regular set* and the *singular set* of $\partial\mathcal{E}$ respectively. In this way, $\partial^*\mathcal{H}$ and $\Sigma(\mathcal{H})$ are, respectively, the union of the open edges and the union of the vertices of the hexagons $\mathcal{H}(h)$ for $h = 1, \dots, N$.

Tilings and maps of class $C^{k,\alpha}$: Given $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, one says that a tiling \mathcal{E} of \mathcal{T} is of class $C^{k,\alpha}$ if there exist a finite family $\{\gamma_i\}_{i \in I}$ of compact $C^{k,\alpha}$ -curves with boundary and a finite family $\{p_j\}_{j \in J}$ of points such that

$$\partial\mathcal{E} = \bigcup_{i \in I} \gamma_i, \quad \partial^*\mathcal{E} = \bigcup_{i \in I} \text{int}(\gamma_i), \quad \Sigma(\mathcal{E}) = \bigcup_{i \in I} \text{bd}(\gamma_i) = \bigcup_{j \in J} \{p_j\}, \quad (3.3.1)$$

where $\text{int}(\gamma_i)$ and $\text{bd}(\gamma_i)$ denote the interior and the boundary of γ_i respectively. Moreover, given a function $f : \partial\mathcal{E} \rightarrow \mathcal{T}$, one says that $f \in C^{k,\alpha}(\partial\mathcal{E}; \mathcal{T})$ if f is continuous on $\partial\mathcal{E}$ and

$$\|f\|_{C^{k,\alpha}(\partial\mathcal{E})} := \sup_{i \in I} \|f\|_{C^{k,\alpha}(\gamma_i)} < \infty.$$

Finally, given two $C^{k,\alpha}$ -tilings \mathcal{E} and \mathcal{F} of \mathcal{T} , one says that f is a $C^{k,\alpha}$ -diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$ if f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$ with $f(\Sigma(\mathcal{E})) = \Sigma(\mathcal{F})$, $f(\partial\mathcal{E}(h)) = \partial\mathcal{F}(h)$ for every $h = 1, \dots, N$, $f \in C^{k,\alpha}(\partial\mathcal{E}; \mathcal{T})$ and $f^{-1} \in C^{k,\alpha}(\partial\mathcal{F}; \mathcal{T})$.

Tangential component of a map and (ε, μ, L) -perturbations of \mathcal{H} : Given a tiling \mathcal{E} of \mathcal{T} of class C^1 , by taking (3.3.1) into account one can define $\nu_{\mathcal{E}} \in C^0(\partial^*\mathcal{E}; S^1)$ in such a way that $\nu_{\mathcal{E}}$ is a unit normal vector to γ_i for every i . Correspondingly, given a map $f : \partial\mathcal{E} \rightarrow \mathcal{T}$, we define $\tau_{\mathcal{E}}f : \partial^*\mathcal{E} \rightarrow \mathcal{T}$, the tangential component of f with respect to $\partial\mathcal{E}$, as

$$\tau_{\mathcal{E}}f(x) = f(x) - (f(x) \cdot \nu_{\mathcal{E}}(x)) \nu_{\mathcal{E}}(x), \quad x \in \partial^*\mathcal{E}.$$

Finally, one says that \mathcal{E} is an (ε, μ, L) -perturbation of \mathcal{H} if \mathcal{E} is of class $C^{1,1}$ and there exists an homeomorphism f between $\partial\mathcal{H}$ and $\partial\mathcal{E}$ with

$$\begin{aligned} \|f\|_{C^{1,1}(\partial\mathcal{H})} &\leq L, \\ \|f - \text{Id}\|_{C^1(\partial\mathcal{H})} &\leq \varepsilon, \\ \|\tau_{\mathcal{H}}(f - \text{Id})\|_{C^1(\partial^*\mathcal{H})} &\leq \frac{L}{\mu} \sup_{\Sigma(\mathcal{H})} |f - \text{Id}|, \\ \tau_{\mathcal{H}}(f - \text{Id}) &= 0, \quad \text{on } [\partial\mathcal{H}]_\mu. \end{aligned} \quad (3.3.2)$$

Note that the definition of $\partial\mathcal{E}$, $\partial^*\mathcal{E}$ as well as the definition of singular set and tilings of class $C^{k,\alpha}$ are the suitable translations in the framework of *tilings on the torus* of the definition given in Section 1.5 above for clusters and planar tilings.

Theorem 3.3.1. *For every $L > 0$ there exist positive constants μ_0 , ε_0 and c_0 (depending on L and $|\mathcal{T}|$), C depending on $|\mathcal{T}|$ only, and C' depending on L only, with the following property. If \mathcal{E} is a unit-area $(\varepsilon_0, \mu_0, L)$ -perturbation of \mathcal{H} , then there exists $v \in \mathbb{R}^2$ such that*

$$P(\mathcal{E}) - P(\mathcal{H}) \geq c_0 \text{hd}(\partial\mathcal{E}, v + \partial\mathcal{H})^2, \quad |v| \leq C \varepsilon_0. \quad (3.3.3)$$

Moreover, there exists a $C^{1,1}$ -diffeomorphism f_0 between $v + \partial\mathcal{H}$ and $\partial\mathcal{E}$ such that

$$P(\mathcal{E}) - P(\mathcal{H}) \geq c_0 \left(\|f_0 - \text{Id}\|_{C^0(v+\partial\mathcal{H})}^2 + \|f_0 - \text{Id}\|_{C^1(v+\partial\mathcal{H})}^4 \right), \quad (3.3.4)$$

and $\|f_0\|_{C^{1,1}(v+\partial\mathcal{H})} \leq C'$.

We premise two lemmas to the proof of Theorem 3.3.1. The first one, Lemma 3.3.2 below, provide a quantitative version of (a particular case of) Hales's hexagonal isoperimetric inequality, the key step in the proof of (3.1.2) in [Hal01].

Lemma 3.3.2. *There exist positive constants ε_1 and c_1 with the following property. If \mathcal{E} is a unit-area tiling of \mathcal{T} such that there exists an homeomorphism f between $\partial\mathcal{H}$ and $\partial\mathcal{E}$ with $\|f - \text{Id}\|_{C^0(\partial\mathcal{H})} \leq \varepsilon_1$, if $E = \mathcal{E}(h)$ for some $h \in \{1, \dots, N\}$ and Π is the convex envelope of $\Sigma(\mathcal{E}) \cap \partial E$ (so that Π is convex hexagon with set of vertices $\Sigma(\mathcal{E}) \cap \partial E$ provided ε_1 is small enough), then there exists a regular hexagon H_* with $|H_*| = |\Pi|$ such that*

$$P(E) \geq P(H) + \frac{P(H)}{2} (|\Pi| - |E|) + c_1 \left(|E\Delta\Pi|^2 + \text{hd}(\partial\Pi, \partial H_*)^2 \right). \quad (3.3.5)$$

Remark 3.3.3. The constants ε_1 and c_1 will just depend on the metric properties of the unit-area hexagon. In particular they do not depend on \mathcal{T} .

Proof of Lemma 3.3.2. Let $\text{arc}_t(a)$ denote the length of a circular arc that bounds an area $a \geq 0$ and whose chord length is $t > 0$, and let us set $\text{arc}(a) = \text{arc}_1(a)$. In this way, $\text{arc} : [0, \infty) \rightarrow [1, \infty)$ is an increasing function. Since the derivative of arc at a is the curvature of any circular arc bounding an area a above a unit length chord, and since this curvature is increasing as a ranges from 0 to $\pi/8$ (the value $a = \pi/8$ corresponds to the case of an half-disk with unit diameter), we conclude that arc is convex on $[0, \pi/8]$ (and, in fact, also concave on $[\pi/8, \infty)$). Moreover, a Taylor expansion gives that $\text{arc}''(0^+) > 0$: hence there exists $\eta > 0$ such that

$$\text{arc}(a) \geq 1 + \eta a^2, \quad \forall a \in [0, \eta]. \quad (3.3.6)$$

Let ℓ_i denote the length of the i th side of Π , and let a_i denote the total area enclosed between the i th side of Π and the i th side of E ; see Figure 3.3.1. (If σ_i is the i th side of Π , then the i th side of E is a small C^0 -deformation of σ_i with fixed end-points). Noticing that $\text{arc}_t(a) = t \text{arc}(a/t^2)$, by Dido's inequality we find that

$$P(E) \geq \sum_{i=1}^6 \text{arc}_{\ell_i}(a_i) = \sum_{i=1}^6 \ell_i \text{arc}\left(\frac{a_i}{\ell_i^2}\right).$$

By $\|f - \text{Id}\|_{C^0(\partial\mathcal{H})} \leq \varepsilon_1$ and provided $\varepsilon_1 \leq 1$, one has

$$\text{hd}(\partial\Pi, \partial\mathcal{H}(h)) \leq \varepsilon_1, \quad \max_{1 \leq i \leq 6} \left\{ a_i, \left| \ell_i - \frac{P(H)}{6} \right| \right\} \leq C \varepsilon_1, \quad (3.3.7)$$

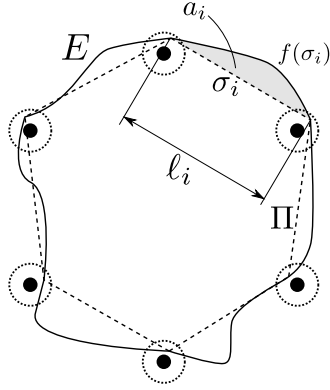


Figure 3.3.1: The convex hexagon Π spanned by $\Sigma(\mathcal{E}) \cap \partial E$. The vertices of Π are ε_1 -close to the vertices of the unit-area regular hexagon $\mathcal{H}(h)$ (as $E = \mathcal{E}(h)$ and $f(\partial\mathcal{H}(h)) = \partial\mathcal{E}(h)$) which are depicted as black dots. The boundaries of Π and E are depicted, respectively, by a dashed line and by a continuous line.

where a possible value for C in (3.3.7) is $2(\pi + \ell)$. By (3.3.7), by further decreasing ε_1 , we can assume that $a_i/\ell_i^2 \in [0, \pi/8]$ for every $i = 1, \dots, 6$. We thus apply Jensen inequality to find that

$$P(E) \geq \sum_{i=1}^6 \ell_i \operatorname{arc} \left(\frac{1}{\sum_{i=1}^6 \ell_i} \sum_{i=1}^6 \frac{a_i}{\ell_i^2} \right).$$

Since $P(H)/6 = (12)^{1/4}/3 < 1$, by (3.3.7) we may further assume that $\ell_i \leq 1$ for every $i = 1, \dots, 6$, and thus conclude by $P(\Pi) = \sum_{i=1}^6 \ell_i$, $|E\Delta\Pi| = \sum_{i=1}^6 a_i$, and the monotonicity of arc that

$$P(E) \geq P(\Pi) \operatorname{arc} \left(\frac{|E\Delta\Pi|}{P(\Pi)} \right). \quad (3.3.8)$$

(Inequality (3.3.8) is clearly related to the *chordal isoperimetric inequality* [Hal01, Proposition 6.1-A], see also [Mor09, 15.5].) By (3.3.6), (3.3.7) and (3.3.8),

$$P(E) \geq P(\Pi) + \eta \frac{|E\Delta\Pi|^2}{P(\Pi)^2} \geq P(\Pi) + c_1 |E\Delta\Pi|^2, \quad (3.3.9)$$

where $c_1 > 0$. Provided ε_1 is small enough, by (3.3.7) we can apply Corollary 3.2.2 to find a regular hexagon H_* with $|H_*| = |\Pi|$ and

$$P(\Pi) - P(H) \sqrt{|\Pi|} \geq c \operatorname{hd}(\partial\Pi, \partial H_*)^2.$$

Thus, up to further decrease the value of c_1 , (3.3.9) gives

$$P(E) \geq P(H) \sqrt{|\Pi|} + c_1 \left(\operatorname{hd}(\partial\Pi, \partial H_*)^2 + |E\Delta\Pi|^2 \right). \quad (3.3.10)$$

Finally, given $\tau > 0$ let $\lambda > 0$ be such that $\sqrt{1-s} \geq 1 - (s/2) - \tau s^2$ for $|s| < \lambda$: up to further decrease ε_1 , by $\|f - \operatorname{Id}\|_{C^0(\partial\mathcal{H})} \leq \varepsilon_1$ we entail $|\sigma| < \lambda$ for $\sigma = |E| - |\Pi|$, and thus deduce with the aid of (3.3.10) and $|E| = 1$ that

$$P(E) \geq P(H) - \frac{P(H)}{2} \sigma - P(H) \tau \sigma^2 + c_1 \left(\operatorname{hd}(\partial\Pi, \partial H_*)^2 + |E\Delta\Pi|^2 \right). \quad (3.3.11)$$

Since $|\sigma| = ||E| - |\Pi|| \leq |E\Delta\Pi|$, for τ small enough depending from c_1 , we prove (3.3.5). \square

Given \mathcal{E} an $(\varepsilon_0, \mu_0, L)$ -perturbation of \mathcal{H} , thanks to Lemma 3.3.2, we are able to perform the construction of a suitable translation $\mathcal{H}_0 = \mathcal{H} + v$ of \mathcal{H} having the singular set $\Sigma(\mathcal{H}_0)$ close to $\Sigma(\mathcal{E})$ in terms of the perimeter deficit. We thus try to show that the distance between $\partial\mathcal{E}$ and $\partial\mathcal{H}_0$ is estimated by the perimeter deficit but, in this situation, we cannot use just the information provided by the diffeomorphism $f : \partial\mathcal{H} \rightarrow \partial\mathcal{E}$. To achieve the proof of Theorem 3.3.1 we need to have some more information about the relation between the new tilings \mathcal{H}_0 and the tiling \mathcal{E} . In particular we are going to build a new diffeomorphism $f_0 : \partial\mathcal{H}_0 \rightarrow \partial\mathcal{E}$ having the tangential component small in terms of the distance between $\Sigma(\mathcal{E})$ and $\Sigma(\mathcal{H}_0)$. In order to do that we make use of the following lemma combined with the existence of the diffeomorphism f between $\partial\mathcal{E}$ and $\partial\mathcal{H}$ given by our definition of $(\varepsilon_0, \mu_0, L)$ -perturbation. In particular Lemma 3.3.4 establishes the existence of a diffeomorphism between a given segment σ_0 and a curve γ close enough to σ_0 . We set

$$[\sigma_0]_t = \{x \in \sigma_0 : \text{dist}(x, \text{bd}(\sigma_0)) > t\},$$

for $t > 0$ and we state the Lemma as follows.

Lemma 3.3.4. *For every $M, \lambda > 0, \alpha \in (0, 1]$ there exist two constants C_1 and $\bar{\mu}$ depending on M, λ and α only with the following property. Let σ_0 be a segment of length λ and γ be a $C^{1,\alpha}$ curve in the plane having $\text{bd}(\gamma) \neq \emptyset$ and*

$$\begin{cases} |\nu(x) - \nu(y)| & \leq M|x - y|^\alpha, \\ |\nu(x) \cdot (y - x)| & \leq M|y - x|^{1+\alpha}, \end{cases} \quad \text{for all } x, y \in \gamma \quad (3.3.12)$$

where $\nu(x)$ denotes the normal unit-vector to γ at x . Assume also that for some $\rho < \bar{\mu}^2$, the curve γ satisfies the following hypothesis

(a) $\text{hd}(\sigma_0, \gamma) \leq \rho$;

(b) setting $\text{bd}(\sigma_0) = \{p_0, q_0\}$ and $\text{bd}(\gamma) = \{p, q\}$ it holds

$$|\tau(p) - \tau_0| + |\tau(q) - \tau_0| \leq \rho,$$

where $\tau_0 = \frac{p_0 - q_0}{|p_0 - q_0|}$ is the tangent unit-vector to σ_0 and $\tau(x) = -\nu(x)^\perp$ denotes the tangent unit-vector to γ at x ;

(c) there exists a map $\psi_0 \in C^{1,1}([\sigma_0]_\rho)$ such that

$$[\gamma]_{3\rho} \subset (\text{Id} + \psi_0 \nu_0)([\sigma_0]_\rho) \subset \gamma,$$

$$\|\psi_0\|_{C^{1,1}([\sigma_0]_\rho)} \leq M, \quad \|\psi_0\|_{C^1([\sigma_0]_\rho)} \leq \rho;$$

Then, for every $\mu \in (\sqrt{\rho}, \bar{\mu})$, there exists a $C^{1,\alpha}$ -diffeomorphism f_0 between σ_0 and γ such that $f_0(\text{bd}(\sigma_0)) = \text{bd}(\gamma)$ and

$$\|f_0\|_{C^{1,\alpha}(\sigma_0)} \leq C_1, \quad \|f_0 - \text{Id}\|_{C^1(\sigma_0)} \leq \frac{C_1}{\mu} \rho^\alpha,$$

$$\|(f_0 - \text{Id}) \cdot \tau_0\|_{C^1(\sigma_0)} \leq \frac{C_1}{\mu} \sup_{\text{bd}(\sigma_0)} |f_0 - \text{Id}|.$$

Proof. The proof follows by applying [CLM15, Theorem 3.1] in the case $n = 2, k = 1$ and by setting $S_0 = \sigma_0, S = \gamma$. \square

Proof of Theorem 3.3.1. Step one: The reflection of \mathbb{R}^2 with respect to a generic line does not induce a map on \mathcal{T} . However, by (3.1.1), one has that if $R_\theta \hat{H}$ denotes the counterclockwise rotation of \hat{H} by an angle θ around the origin, then $R_\theta \hat{H}$ is compactly contained in a box of height $3\ell\alpha/2 \geq 3\ell$ and width $\sqrt{3}\ell\beta \geq 2\sqrt{3}\ell$ for every θ . As a consequence, given a unit-area regular hexagon K in \mathcal{T} , all the rotations of K are well-defined as unit-area regular hexagons in \mathcal{T} ; in particular, it always makes sense to define the reflection $g_\sigma(K)$ of K with respect to an edge σ of K . Taking this into account, we notice that there exist positive constants η and C (independent of \mathcal{T}) such that, if K and K' are unit-area regular hexagons in \mathcal{T} , and if σ and σ' are edges of K and K' respectively, then

$$\begin{cases} \text{hd}(\sigma, \sigma') \leq \eta, \\ |K \Delta K'| \geq 2 - \eta, \end{cases} \quad \Rightarrow \quad \text{hd}(\partial g_\sigma(K), \partial K') \leq C \text{hd}(\sigma, \sigma').$$

This geometric remark is going to be repeatedly used in the following arguments, where we shall denote by ε_1 and c_1 the constants of Lemma 3.3.2 and set $\delta = P(\mathcal{E}) - P(\mathcal{H})$. We notice that, by the area formula and since $\|f - \text{Id}\|_{C^1(\partial\mathcal{H})} \leq \varepsilon_0$, one has

$$\delta \leq C P(\mathcal{H}) \varepsilon_0^2, \quad (3.3.13)$$

where C is independent from \mathcal{T} and where $P(\mathcal{H}) = |\mathcal{T}| P(H)/2$.

Step two: We claim that, if ε_0 is small enough depending only from $|\mathcal{T}|$, and if Π_h denotes the convex envelope of $\partial\mathcal{E}(h) \cap \Sigma(\mathcal{E})$ (so that Π_h is a convex hexagon, not necessarily with unit-area), then for every $h = 1, \dots, N$ there exists a regular unit-area hexagon K_h such that

$$\text{hd}(\partial\Pi_h, \partial K_h) \leq C \sqrt{\delta}, \quad (3.3.14)$$

$$|K_h \Delta K_{h+1}| \geq 2 - C \sqrt{\delta}, \quad (3.3.15)$$

where here and in the rest of this step, C denotes a constant depending from $|\mathcal{T}|$ only. Indeed, since $\{\Pi_h\}_{h=1}^N$ is a partition of \mathcal{T} , one has $\sum_{h=1}^N |\Pi_h| = |\mathcal{T}| = \sum_{h=1}^N |\mathcal{E}(h)|$. By requiring $\varepsilon_0 \leq \varepsilon_1$ we can apply Lemma 3.3.2 to each $\mathcal{E}(h)$ in order to find regular hexagons H_h^* with $|H_h^*| = |\Pi_h|$ such that, by adding up (3.3.5) on h , one finds

$$2\delta = \sum_{h=1}^N (P(\mathcal{E}(h)) - P(H)) \geq c_1 \sum_{h=1}^N \left(|\mathcal{E}(h) \Delta \Pi_h|^2 + \text{hd}(\partial\Pi_h, \partial H_h^*)^2 \right). \quad (3.3.16)$$

By (3.3.16),

$$||\Pi_h| - 1| \leq |\mathcal{E}(h) \Delta \Pi_h| \leq \sqrt{\frac{2\delta}{c_1}}. \quad (3.3.17)$$

By (3.1.1), we may further decrease the value of η introduced in step one so to have that if J is a regular hexagon in \mathcal{T} with $||J| - 1| \leq \eta$, then it makes sense to scale J with respect to its barycenter in order to obtain a unit-area regular hexagon J' with $\text{hd}(\partial J, \partial J') \leq C ||J| - 1|$. In particular, by (3.3.13) and (3.3.17), up to decrease the value of ε_0 we can define unit-area hexagons K_h in \mathcal{T} with the property that

$$\text{hd}(\partial K_h, \partial H_h^*) \leq C ||H_h^*| - 1| = C ||\Pi_h| - 1| \leq C \sqrt{\delta}.$$

By combining this estimate with (3.3.16) we prove (3.3.14). By (3.3.14), $|K_j \Delta \Pi_j| \leq C \sqrt{\delta}$ for every j , and thus

$$|K_h \Delta K_{h+1}| \geq |\mathcal{E}(h) \Delta \mathcal{E}(h+1)| - \sum_{j=h}^{h+1} |\mathcal{E}(j) \Delta K_j| \geq 2 - C \sqrt{\delta} - \sum_{j=h}^{h+1} |\mathcal{E}(j) \Delta \Pi_j|.$$

In particular, (3.3.15) follows from (3.3.16).

Step three: We claim the existence of a tiling $\mathcal{H}_0 = v + \mathcal{H}$ of \mathcal{T} such that

$$\text{hd}(\Sigma(\mathcal{E}), \Sigma(\mathcal{H}_0)) \leq C\sqrt{\delta}, \quad |v| \leq C\varepsilon_0, \quad (3.3.18)$$

where here and in the rest of this step, C denotes a constant depending from $|\mathcal{T}|$ only. Let us recall from Figure 3.1.1 that the chambers of \mathcal{H} are ordered so that $\{\mathcal{H}(h)\}_{h=1}^\beta$ is the “bottom row” in the grid defined by \mathcal{H} and that $\mathcal{H}(1) = H$. Since \mathcal{E} is an $(\varepsilon_0, \mu_0, L)$ -perturbation of \mathcal{H} one has

$$\max \left\{ \text{hd}(\partial\mathcal{E}(h), \partial\mathcal{H}(h)), \text{hd}(\partial\Pi_h, \partial\mathcal{H}(h)) \right\} \leq \varepsilon_0, \quad \forall h = 1, \dots, N, \quad (3.3.19)$$

so that (3.3.14) implies $\text{hd}(\partial H, \partial K_1) \leq C\varepsilon_0$. In particular, there exists $|\theta|, |s|, |t| \leq C\varepsilon_0$ such that

$$K_1 = (t\sqrt{3}\ell, s\ell) + R_\theta H,$$

where, with a slight abuse of notation, $R_\theta H$ denotes the counterclockwise rotation of H by an angle θ around its left-bottom vertex (see step one). Of course, there is no reason to get a better estimate than $|s|, |t| \leq C\varepsilon_0$ here (indeed, \mathcal{E} itself could just be an ε_0 -size translation of \mathcal{H}). Nevertheless, if $\theta \neq 0$, then we cannot fit K_1 into an hexagonal honeycomb of \mathcal{T} : therefore one expects

$$|\theta| \leq C\sqrt{\delta}. \quad (3.3.20)$$

We prove (3.3.20): set $J_1 = K_1$, let τ_1 be the common edge between Π_1 and Π_2 , and let σ_1 and σ'_1 be the edges of K_1 and K_2 respectively such that, thanks to (3.3.14), $\text{hd}(\tau_1, \sigma_1) + \text{hd}(\tau_1, \sigma'_1) \leq C\sqrt{\delta}$. In this way $\text{hd}(\sigma_1, \sigma'_1) \leq C\sqrt{\delta}$, and by (3.3.15) we can apply step one to deduce

$$\text{hd}(\partial J_2, \partial K_2) \leq C\text{hd}(\sigma_1, \sigma'_1) \leq C\sqrt{\delta}, \quad |J_2 \Delta K_2| \leq C\sqrt{\delta}, \quad (3.3.21)$$

where J_2 is the reflection of J_1 with respect to σ_1 . Let now τ_2 be common side between Π_2 and Π_3 . By (3.3.14) and (3.3.21) we have $\text{hd}(\partial J_2, \partial\Pi_2) + \text{hd}(\partial K_3, \partial\Pi_3) \leq C\sqrt{\delta}$, thus there exist edges σ_2 and σ'_2 of J_2 and K_3 respectively such that $\text{hd}(\tau_2, \sigma) + \text{hd}(\tau_2, \sigma') \leq C\sqrt{\delta}$. By (3.3.15) and (3.3.21) one has $|J_2 \Delta K_3| \geq 2 - C\sqrt{\delta}$, so that by step one $\text{hd}(\partial J_3, \partial K_3) \leq C\sqrt{\delta}$ where J_3 is the reflection of J_2 with respect to σ_2 . If we repeat this argument β -times, then we find regular unit-area hexagons J_1, \dots, J_β such that $J_1 = K_1$, J_h is obtained by reflecting J_{h-1} with respect to its “vertical” right edge, and $\text{hd}(\partial J_h, \partial K_h) \leq C\sqrt{\delta}$ for $h = 1, \dots, \beta$. By construction, Π_β and Π_1 also share a common edge τ , and correspondingly J_β and K_1 have edges σ and σ' respectively with $\text{hd}(\tau, \sigma) + \text{hd}(\tau, \sigma') \leq C\sqrt{\delta}$. By reflecting J_β with respect to σ we thus find a regular unit area hexagon J_* with

$$\text{hd}(\partial J_*, \partial K_1) \leq C\sqrt{\delta}.$$

At the same time, since J_* has been obtained by iteratively reflecting $J_1 = K_1$ with respect to its “vertical” right edge, we find that

$$\text{hd}(\partial J_*, \partial J_1) \geq \frac{|\theta|}{C}.$$

Thus (3.3.20) holds. As a consequence, up to apply to K_1 a rotation of size $C\sqrt{\delta}$, one can assume that

$$K_1 = (t\sqrt{3}\ell, s\ell) + H, \quad \text{for some } |t|, |s| \leq C\varepsilon_0. \quad (3.3.22)$$

In particular, if we set $\mathcal{H}_0(h) = (t\sqrt{3}\ell, s\ell) + \mathcal{H}(h)$, then \mathcal{H}_0 defines a unit-area tiling of \mathcal{T} by regular hexagons. By arguing as in the proof of (3.3.20), one easily sees that

$$\text{hd}(\partial\Pi_h, \partial\mathcal{H}_0(h)) \leq C\sqrt{\delta}, \quad \forall h = 1, \dots, N. \quad (3.3.23)$$

In particular, the set of vertices of Π_h and $\mathcal{H}_0(h)$ lie at distance $C\sqrt{\delta}$. Since $\Sigma(\mathcal{E})$ is the set of all the vertices of the Π_h s, we complete the proof of (3.3.18).

Step four: We show that if μ_0 is small enough with respect to L , and ε_0 is small enough with respect to μ_0 and $|\mathcal{T}|$, then there exists a $C^{1,1}$ -diffeomorphism f_0 between $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$ such that

$$\|f_0\|_{C^{1,1}(\partial\mathcal{H}_0)} \leq C, \quad \|f_0 - \text{Id}\|_{C^1(\partial\mathcal{H}_0)} \leq C\mu_0, \quad (3.3.24)$$

$$\|(f_0 - \text{Id}) \cdot \tau_0\|_{C^1(\partial\mathcal{H}_0)} \leq C \sup_{\Sigma(\mathcal{H}_0)} |f_0 - \text{Id}|. \quad (3.3.25)$$

where C depends on L only. The map f_0 is built starting from Lemma 3.3.4 and is more useful than the map f appearing in (3.3.2) because the best estimate for $f - \text{Id}$ on $\Sigma(\mathcal{H})$ is of order ε_0 , while, thanks to (3.3.18), we have a much more precise information about $f_0 - \text{Id}$ on $\Sigma(\mathcal{H}_0)$, namely

$$\sup_{\Sigma(\mathcal{H}_0)} |f_0 - \text{Id}| \leq C\sqrt{\delta}. \quad (3.3.26)$$

(In (3.3.26), C depends on $|\mathcal{T}|$.) Let us also notice that we cannot just define f_0 by composing f with the translation bringing $\partial\mathcal{H}_0$ onto $\partial\mathcal{H}$, because this translation is $O(\varepsilon_0)$, and thus the resulting map f_0 would still have tangential displacement $O(\varepsilon_0)$. We thus need a more precise construction, directly relating $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$.

In order to apply Lemma 3.3.4, we fix an edge σ of \mathcal{H} , and set $\sigma_0 = v + \sigma$, so that σ_0 is an edge of \mathcal{H}_0 . We denote by τ_0 and $\nu_0 = \tau_0^\perp$ the constant tangent and normal unit-vector fields to σ_0 (and, obviously, to σ). We let $\gamma = f(\sigma)$ and set $\tau(x) = \nabla^\sigma f(f^{-1}(x))[\tau_0]$ and $\nu(x) = \tau(x)^\perp$, where $\nabla^\sigma f$ denotes the tangential gradient of f with respect to σ . We start by noticing that $\text{bd}(\gamma) \neq \emptyset$ and we argue as follows.

By applying Lemma 3.3.4 with $\lambda = \ell$ and $\alpha = 1$ we discover that for any given $M > 0$ for which γ satisfies (3.3.12) there exist positive constants C_1 and $\bar{\mu}$ (depending on M) such that if γ satisfies also hypothesis (a),(b) and (c) for some $\rho < \bar{\mu}^2$ then, for every $\mu \in (\sqrt{\rho}, \bar{\mu})$, there exists a $C^{1,1}$ -diffeomorphism f_0 between σ_0 and γ such that $f_0(\text{bd}(\sigma_0)) = \text{bd}(\gamma)$ and

$$\|f_0\|_{C^{1,1}(\sigma_0)} \leq C_1, \quad \|f_0 - \text{Id}\|_{C^1(\sigma_0)} \leq \frac{C_1}{\mu} \rho,$$

$$\|(f_0 - \text{Id}) \cdot \tau_0\|_{C^1(\sigma_0)} \leq \frac{C_1}{\mu} \sup_{\text{bd}(\sigma_0)} |f_0 - \text{Id}|.$$

Thus it is enough to show for μ_0, ε_0 small enough depending on L then properties (a),(b), (c) and condition (3.3.12) hold on γ with a suitable choice of $M = M(L)$ and ρ . Clearly γ satisfies (3.3.12) for some $M = M(L)$, since $\|f\|_{C^{1,1}(\sigma)} \leq L$ and $\|f - \text{Id}\|_{C^1(\sigma)} \leq \varepsilon_0$. We notice that property (a) holds provided $\rho \geq C\varepsilon_0$ for some C depending on $|\mathcal{T}|$ only: indeed, by $\|f - \text{Id}\|_{C^0(\sigma)} \leq \varepsilon_0$ one finds $\text{hd}(\sigma, \gamma) \leq \varepsilon_0$, while $|v| \leq C\varepsilon_0$ (recall (3.3.18)) gives $\text{hd}(\sigma, \sigma_0) \leq C\varepsilon_0$. Similarly, property (b) holds if $\rho \geq \varepsilon_0$, as $\tau(x) = \nabla^\sigma f(f^{-1}(x))[\tau_0]$ and $\|f - \text{Id}\|_{C^1(\sigma)} \leq \varepsilon_0$. Finally, concerning property (c), we notice that by exploiting the fact that \mathcal{E} is an $(\varepsilon_0, \mu_0, L)$ -perturbation of \mathcal{H} and setting $\psi = (f - \text{Id}) \cdot \nu_0$, one has $\psi \in C^{1,1}([\sigma]_{\mu_0})$ with

$$[\gamma]_{\mu_0+2\varepsilon_0} \subset (\text{Id} + \psi\nu_0)([\sigma]_{\mu_0}) \subset \gamma, \quad (3.3.27)$$

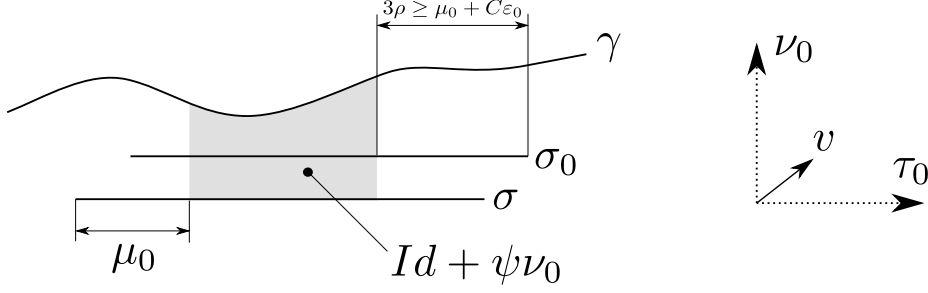


Figure 3.3.2: The function ψ_0 is defined by computing the values of ψ after a projection of σ_0 onto σ .

$$\|\psi\|_{C^{1,1}([\sigma]_{\mu_0})} \leq L, \quad \|\psi\|_{C^1([\sigma]_{\mu_0})} \leq \varepsilon_0, \quad (3.3.28)$$

where the first inclusion in (3.3.27) follows from $\|f - \text{Id}\|_{C^0(\sigma_0)} \leq \varepsilon_0$ and $\gamma = f(\sigma)$. By exploiting (3.3.27), (3.3.28), and the fact that $\sigma_0 = v + \sigma$ with $|v| \leq C\varepsilon_0$ by (3.3.18), one can find two constants $C_2 \leq C_3$ (both depending just on $|\mathcal{T}|$) and $\psi_0 \in C^{1,1}([\sigma]_{\mu_0 + C_2\varepsilon_0})$ such that properties (a), (b) and (c) hold with $\rho = \mu_0 + C_2\varepsilon_0$, and

$$[\gamma]_{\mu_0 + C_3\varepsilon_0} \subset (\text{Id} + \psi_0\nu_0)([\sigma]_{\mu_0 + C_2\varepsilon_0}) \subset \gamma, \quad (3.3.29)$$

$$\|\psi_0\|_{C^{1,1}([\sigma]_{\mu_0 + C_2\varepsilon_0})} \leq L, \quad \|\psi_0\|_{C^1([\sigma]_{\mu_0 + C_2\varepsilon_0})} \leq \varepsilon_0, \quad (3.3.30)$$

see Figure 3.3.2. Of course one can entail $3\rho > \mu_0 + C_3\varepsilon_0$ by requiring ε_0 small enough with respect to μ_0 : in this way, property (c) follows from (3.3.29) and (3.3.30). Summarizing, we have shown that if μ_0 is small enough depending on L (that is, depending on $M = M(L)$), and if ε_0 is small enough with respect to μ_0 and $|\mathcal{T}|$, then properties (a),(b), (c) hold with $\rho = \mu_0 + C_2\varepsilon_0$ and condition (3.3.12) holds with a suitable $M = M(L)$. Up to further decrease the values of μ_0 and ε_0 we may entail $\rho < \bar{\mu}^2$ and fix $\mu \in (\sqrt{\rho}, \bar{\mu})$ depending on L only. Thus, thanks to Lemma 3.3.4, we find a $C^{1,1}$ -diffeomorphism f_0 between σ_0 and γ such that $f_0(\text{bd}(\sigma_0)) = \text{bd}(\gamma)$ and

$$\begin{aligned} \|f_0\|_{C^{1,1}(\sigma_0)} &\leq C, & \|f_0 - \text{Id}\|_{C^1(\sigma_0)} &\leq C\mu_0, \\ \|(f_0 - \text{Id}) \cdot \tau_0\|_{C^{1,1}(\sigma_0)} &\leq C \sup_{\text{bd}(\sigma_0)} |f_0 - \text{Id}|, \end{aligned}$$

where C depends on L only. By repeating this construction on every edge σ_0 of $\partial\mathcal{H}_0$ we complete the proof of (3.3.24) and (3.3.25).

Step four: With a little abuse of notation, let us denote by $\{\sigma_i\}_{i=1}^{3N}$ the family of segments such that $\partial\mathcal{H}_0 = \bigcup_{i=1}^{3N} \sigma_i$. For every i let τ_i denote a constant tangent unit vector to σ_i . If we set $g = f_0 - \text{Id}$, then we have

$$P(\mathcal{E}) - P(\mathcal{H}) = \sum_{i=1}^{3N} \int_{\sigma_i} (|\nabla^{\sigma_i} g[\tau_i] + \tau_i| - 1) d\mathcal{H}^1,$$

where, by $\|g\|_{C^1(\partial\mathcal{H}_0)} \leq \mu_0$, $\sqrt{1+t} \geq 1 + t/2 - t^2/8 - C|t|^3$ ($t \geq -1$), and provided μ_0 is small enough,

$$\begin{aligned} |\nabla^{\sigma_i} g[\tau_i] + \tau_i| - 1 &= \sqrt{1 + 2\tau_i \cdot \nabla^{\sigma_i} g[\tau_i] + |\nabla^{\sigma_i} g[\tau_i]|^2} - 1 \\ &\geq \tau_i \cdot \nabla^{\sigma_i} g[\tau_i] + \frac{|\nabla^{\sigma_i} g[\tau_i]|^2}{2} - \frac{|2\tau_i \cdot \nabla^{\sigma_i} g[\tau_i]|^2}{8} - C\mu_0 |\nabla^{\sigma_i} g[\tau_i]|^2. \end{aligned}$$

Let $\Sigma(\mathcal{H}_0) = \{p_j\}_{j=1}^{2N}$, and for $p_j \in \text{bd}(\sigma_i)$ denote by v_j^i the tangent unit vector to σ_i at p_j pointing outside σ_i . In this way,

$$\sum_{i=1}^{3N} \int_{\sigma_i} \tau_i \cdot \nabla^{\sigma_i} g[\tau_i] d\mathcal{H}^1 = \sum_{j=1}^{2N} \sum_{\{i: p_j \in \text{bd}(\sigma_i)\}} v_j^i g(p_j) = 0,$$

since $\{i : p_j \in \text{bd}(\sigma_i)\} = \{i_1, i_2, i_3\}$ with $v_j^{i_2}$ and $v_j^{i_3}$ obtained from $v_j^{i_1}$ by counterclockwise rotations of $2\pi/3$ and $4\pi/3$ respectively. Hence, if we set $\nu_i = \tau_i^\perp$, then

$$P(\mathcal{E}) - P(\mathcal{H}) \geq \sum_{i=1}^{3N} \int_{\sigma_i} \frac{|\nu_i \cdot \nabla^{\sigma_i} g[\tau_i]|^2}{2} d\mathcal{H}^1 - C \mu_0 \int_{\sigma_i} |\nabla^{\sigma_i} g[\tau_i]|^2 d\mathcal{H}^1. \quad (3.3.31)$$

By (3.3.25) and (3.3.26) we find that

$$\sup_{1 \leq i \leq 3N} \|\tau_i \cdot \nabla^{\sigma_i} g[\tau_i]\|_{C^0(\sigma_i)} \leq C \sqrt{\delta},$$

where C depends on L and $|\mathcal{T}|$. By combining this last inequality with (3.3.31), and provided μ_0 is small enough with respect to L and $|\mathcal{T}|$, we find

$$C \sqrt{\delta} \geq \sum_{i=1}^{3N} \int_{\sigma_i} |\nabla^{\sigma_i} g[\tau_i]| \geq \sum_{i=1}^{3N} \|g - g(p_{j(i)})\|_{C^0(\sigma_i)}, \quad (3.3.32)$$

where for each $i = 1, \dots, 3N$ we have picked $p_{j(i)} \in \text{bd}(\sigma_i)$. By (3.3.26) we have $|g(p_{j(i)})| \leq C\sqrt{\delta}$, so that (3.3.32) implies

$$C \sqrt{\delta} \geq \sum_{i=1}^{3N} \|g\|_{C^0(\sigma_i)} = \|f_0 - \text{Id}\|_{C^0(\partial\mathcal{H}_0)}. \quad (3.3.33)$$

Since f_0 is a bijection between $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$, we find that $\|f_0 - \text{Id}\|_{C^0(\partial\mathcal{H}_0)} \geq \text{hd}(\partial\mathcal{H}_0, \partial\mathcal{E})$ and thus prove (3.3.3). We now notice that if $u : (a, b) \rightarrow \mathbb{R}$ is a Lipschitz function with $\|u\|_{C^0(a,b)} \leq 1$, then

$$\|u\|_{C^0(a,b)}^2 \leq 8 \max \left\{ \text{Lip}(u), \frac{1}{b-a} \right\} \|u\|_{L^1(a,b)}. \quad (3.3.34)$$

Indeed, let $x_0 \in [a, b]$ be such that $u(x_0) = \|u\|_{C^0(a,b)}$ and set $L = \text{Lip}(u)$, $r = |u(x_0)|/4L$. If $(x_0, x_0+r) \subset (a, b)$ or $(x_0-r, x_0) \subset (a, b)$, then by integrating $|u(y)| \geq |u(x_0)| - L|x_0 - y|$ in y over (x_0, x_0+r) or over (x_0-r, x_0) respectively, we find

$$\int_{(a,b)} |u| \geq r |u(x_0)| - L \frac{r^2}{2} \geq \frac{|u(x_0)|^2}{8L};$$

otherwise one has $b-a \leq 2r$ and thus $|u(y)| \geq |u(x_0)|/2$ for every $y \in (a, b)$. In order to complete the proof of (3.3.4) we just need to use (3.3.33) and to combine the first inequality in (3.3.32) with $\|f_0\|_{C^{1,1}(\partial\mathcal{H})} \leq C$ and with (3.3.34) (applied to the components of $\nabla^{\partial^*\mathcal{H}_0}(f_0 - \text{Id})$). \square

3.4 Proof of Theorem 3.1.1, Theorem 3.1.3 and Theorem 3.1.4

We start by introducing the following fundamental tool in the study of isoperimetric problems with multiple volume constraints. This kind of construction is originally found in [Alm76], and it is fully detailed in our setting in [Mag12, Sections 29.5-29.6], see also [CLM15, Theorem B.1]. Since the version of this lemma needed here does not seem to appear elsewhere, we give some details of the proof.

Lemma 3.4.1 (Volume-fixing variations). *If \mathcal{E}_0 is a N -tiling of \mathcal{T} , $\gamma \in (0, 1]$ and $L > 0$, then there exist positive constants r_0 , σ_0 , ε_0 , and C_0 (depending on \mathcal{E}_0 , L and γ only) with the following property: if $\eta \in \mathbb{R}^N$ with $\sum_{h=1}^N \eta_h = 0$, $\Phi \in \text{Lip}(\mathcal{T} \times S^1; (0, \infty))$, $\psi \in C^{1,\gamma}(\mathcal{T}; (0, \infty))$, $x \in \mathcal{T}$, and \mathcal{E} and \mathcal{F} are N -tilings of \mathcal{T} with*

$$\|\Phi\|_{C^{0,1}(\mathcal{T} \times S^1)} + \|\psi\|_{C^{1,\gamma}(\mathcal{T})} \leq L, \quad (3.4.1)$$

$$d(\mathcal{E}, \mathcal{E}_0) \leq \varepsilon_0, \quad (3.4.2)$$

$$\mathcal{F} \Delta \mathcal{E} \subset \subset B_{r_0}(x), \quad |\eta| < \sigma_0, \quad (3.4.3)$$

then there exists a N -cluster \mathcal{F}' such that

$$\mathcal{F}' \Delta \mathcal{F} \subset \subset \mathcal{T} \setminus \bar{B}_{r_0}(x), \quad (3.4.4)$$

$$\int_{\mathcal{F}'(h)} \psi = \eta_h + \int_{\mathcal{E}(h)} \psi, \quad (3.4.5)$$

$$|\Phi(\mathcal{F}') - \Phi(\mathcal{F})| \leq C_0 P(\mathcal{E}) \left(\sum_{h=1}^N \left| \int_{\mathcal{F}'(h)} \psi - \int_{\mathcal{E}(h)} \psi \right| + |\eta| \right), \quad (3.4.6)$$

$$|d(\mathcal{F}', \mathcal{E}) - d(\mathcal{F}, \mathcal{E})| \leq C_0 P(\mathcal{E}) \left(\sum_{h=1}^N \left| \int_{\mathcal{F}'(h)} \psi - \int_{\mathcal{E}(h)} \psi \right| + |\eta| \right). \quad (3.4.7)$$

Remark 3.4.2. In practice we are going to apply this lemma either with $\eta = 0$ and $\mathcal{F} \Delta \mathcal{E} \neq \emptyset$, or with $\eta \neq 0$ and $\mathcal{F} = \mathcal{E}$. In the first case, we are given a compactly supported variation \mathcal{F} of \mathcal{E} , and we want to modify \mathcal{F} outside of $B_{r_0}(x)$ into a new N -tiling \mathcal{F}' so that $\int_{\mathcal{F}'(h)} \psi = \int_{\mathcal{E}(h)} \psi$ for every $h = 1, \dots, N$. In the second case we want to modify \mathcal{E} so that $\int_{\mathcal{E}(h)} \psi$ is changed into $\eta_h + \int_{\mathcal{E}(h)} \psi$ for every $h = 1, \dots, N$. In both cases, we want to control the change in Φ -energy and the change in distance from \mathcal{E} needed to pass from \mathcal{F} to \mathcal{F}' . The name attached to the lemma is motivated by the fact that one usually takes $\psi \equiv 1$.

Proof of Lemma 3.4.1. The basic step consists in picking up a ball $B_{z,\varepsilon}$ and notice that if $T \in C_c^\infty(B_{z,\varepsilon}; \mathbb{R}^2)$ and $f_t(x) = x + tT(x)$ for $x \in \mathcal{T}$, then for every Borel set $E \subset \mathcal{T}$ the function $\Psi_E(t) = \int_{f_t(E)} \psi = \int_E \psi(f_t) Jf_t$ is of class $C^{1,\gamma}(-t_0, t_0)$ with

$$\|\Psi_E\|_{C^{1,\gamma}(-t_0, t_0)} \leq C, \quad \left| \int_{f_t(E)} \psi - \int_E \psi - t \int_E \text{div}(\psi T) \right| \leq C |t|^{1+\gamma}, \quad (3.4.8)$$

where t_0 and C denote positive constants depending only on γ , L , $|\mathcal{T}|$, and $\|T\|_{C^1(\mathcal{T})}$. Next, one considers two families of balls $\{B_{z_i,\varepsilon}\}_{i=1}^M$ and $\{B_{y_i,\varepsilon}\}_{i=1}^M$ with $z_i, y_i \in \partial^* \mathcal{E}_0(h(i)) \cap \partial^* \mathcal{E}_0(k(i))$ (for $1 \leq h(i) \neq k(i) \leq N$ to be properly chosen – see condition (3.4.13) below)

and with $|z_i - z_j| > 2\varepsilon$ and $|y_i - y_j| > 2\varepsilon$ for $1 \leq i < j \leq M$ and $|y_i - z_j| > 2\varepsilon$ for $1 \leq i \leq j \leq M$. For each i we can find $T_i \in C_c^\infty(B_{z_i, \varepsilon}; \mathbb{R}^2)$ such that

$$\int_{\mathcal{E}_0(h(i))} \operatorname{div}(\psi T_i) = 1 = - \int_{\mathcal{E}_0(k(i))} \operatorname{div}(\psi T_i), \quad (3.4.9)$$

$$\int_{\mathcal{E}_0(j)} \operatorname{div}(\psi T_i) = 0, \quad j \neq h(i), k(i). \quad (3.4.10)$$

Let us consider the smooth map $f : (-t_0, t_0)^M \times \mathcal{T} \rightarrow \mathcal{T}$ defined by $f(\mathbf{t}, x) = x + \sum_{i=1}^M t_i T_i(x)$, $\mathbf{t} = (t_1, \dots, t_M)$, so that for $t_0 > 0$ small enough $f(\mathbf{t}, \cdot)$ is a smooth diffeomorphism of \mathcal{T} with

$$\operatorname{spt}(f(\mathbf{t}, \cdot) - \operatorname{Id}) \subset \subset \bigcup_{i=1}^M B_{z_i, \varepsilon}. \quad (3.4.11)$$

If we let $\alpha = (\alpha_1, \dots, \alpha_N) \in C^{1, \gamma}((-t_0, t_0)^M; \mathbb{R}^N)$ be defined by

$$\alpha_h(\mathbf{t}) = \int_{f(\mathbf{t}, \mathcal{E}(h))} \psi - \int_{\mathcal{E}(h)} \psi, \quad h = 1, \dots, N,$$

then $\alpha((-t_0, t_0)^M) \subset V = \{\eta \in \mathbb{R}^N : \sum_{h=1}^N \eta_h = 0\}$, $\|\alpha\|_{C^{1, \gamma}((-t_0, t_0)^M)} \leq C$, and, by (3.4.1), (3.4.2), (3.4.8), (3.4.9) and (3.4.10), one finds

$$\left| \frac{\partial \alpha_{h(i)}}{\partial t_i}(\mathbf{t}) - 1 \right| + \left| \frac{\partial \alpha_{k(i)}}{\partial t_i}(\mathbf{t}) + 1 \right| + \max_{j \neq h(i), k(i)} \left| \frac{\partial \alpha_j}{\partial t_i}(\mathbf{t}) \right| \leq C \varepsilon_0, \quad (3.4.12)$$

where, from now on, C denotes a constant depending only on L , γ , $|\mathcal{T}|$, and \mathcal{E}_0 (through $\|T_i\|_{C^1(\mathcal{T})}$). Provided $h(i)$ and $k(i)$ are suitable defined (see [Mag12, Step one, Proof of Theorem 29.14]) one can entail from (3.4.12) that

$$\dim \nabla \alpha(\mathbf{0}) = N - 1. \quad (3.4.13)$$

By the implicit function theorem there exists $\sigma_1 > 0$ and an open neighborhood U of $\mathbf{0} \in \mathbb{R}^M$ such that $\alpha^{-1} \in C^{1, \gamma}(V_{\sigma_1}; U)$ with $V_{\sigma_1} = \{\eta \in V : |\eta| < \sigma_1\}$, and

$$|\alpha^{-1}(\eta)| \leq C |\eta|, \quad \forall \eta \in V_{\sigma_1}. \quad (3.4.14)$$

Similarly, we may construct functions g and β , analogous to f and α , starting from the family of balls $\{B_{y_i, \varepsilon}\}_{i=1}^M$. Now let \mathcal{F} be as in (3.4.3), and assume that

$$\sigma_0 + \|\psi\|_{C^0(\mathcal{T})} \pi r_0^2 < \sigma_1. \quad (3.4.15)$$

Up to further decrease the value of r_0 with respect to ε , we may also assume that $\overline{B}_{r_0}(x) \cap \overline{B}_\varepsilon(z_i) = \emptyset$ for every $i = 1, \dots, M$, or that $\overline{B}_{r_0}(z) \cap \overline{B}_\varepsilon(y_i) = \emptyset$ for every $i = 1, \dots, M$. Without loss of generality we may assume to be in the former case, and set

$$\mathcal{F}'(h) = (\mathcal{F}(h) \cap B_{r_0}(x)) \cup (f(\alpha^{-1}(w), \mathcal{E}(h)) \setminus B_{r_0}(x)), \quad 1 \leq h \leq N,$$

where w_h is defined by the identity

$$\int_{\mathcal{F}(h) \cap B_{r_0}(x)} \psi = \eta_h - w_h - \int_{\mathcal{E}(h) \cap B_{r_0}(x)} \psi, \quad 1 \leq h \leq N.$$

By construction one has (3.4.4). Moreover, by definition of w_h , by (3.4.11) and since $\overline{B_{r_0}}(x) \cap \overline{B_\varepsilon}(z_i) = \emptyset$ for every $i = 1, \dots, M$, one has

$$\begin{aligned} \int_{\mathcal{F}'(h)} \psi - \int_{\mathcal{E}(h)} \psi &= \int_{\mathcal{F}(h) \cap B_{r_0}(x)} \psi + \int_{f(\alpha^{-1}(w), \mathcal{E}(h)) \setminus B_{r_0}(x)} \psi - \int_{\mathcal{E}(h)} \psi \\ &= \eta_h - w_h + \int_{f(\alpha^{-1}(w), \mathcal{E}(h)) \setminus B_{r_0}(x)} \psi - \int_{\mathcal{E}(h) \setminus B_{r_0}(x)} \psi \\ &= \eta_h - w_h + \int_{f(\alpha^{-1}(w), \mathcal{E}(h))} \psi - \int_{\mathcal{E}(h)} \psi = \eta_h - w_h + \alpha_h(\alpha^{-1}(w)). \end{aligned}$$

By (3.4.3) and (3.4.15) one has $|w| < \sigma_1$, so that (3.4.5) is proved. We now notice that by [DPM14, Equation (2.9)]

$$\Phi(f(\mathbf{t}, E)) = \int_{f(\mathbf{t}, \partial^* E)} \Phi(y, \nu_{f_t(E)}(y)) d\mathcal{H}^1(y) = \int_{\partial^* E} \Phi(f_t(x), \text{cof} \nabla f_t(x) [\nu_E(x)]) d\mathcal{H}^1(x),$$

so that, by (3.4.1), $|\Phi(f(\mathbf{t}, E)) - \Phi(E)| \leq C |t| P(E)$. By (3.4.14) we immediately deduce (3.4.6). Finally (3.4.7) is obtained by exploiting [CLM15, Lemma B.2]. \square

We now translate the improved convergence Theorem for planar bubble clusters 1.9.3 in the case of tilings of \mathcal{T} . One says that a N -tiling \mathcal{E} of \mathcal{T} is (Λ, r_0) -*minimizer* if

$$P(\mathcal{E}) \leq P(\mathcal{F}) + \Lambda d(\mathcal{E}, \mathcal{F}),$$

whenever \mathcal{F} is a N -tiling of \mathcal{T} and $\mathcal{E} \Delta \mathcal{F} \subset\subset B_{r_0}(x)$ for some $x \in \mathcal{T}$. If \mathcal{E} is a (Λ, r_0) -minimizing tiling of \mathcal{T} , then (by a trivial adaptation of Theorem 1.8.1 above) \mathcal{E} is of class $C^{1,1}$. Moreover, the curves γ_i and the points p_j in (3.3.1) are such that each γ_i has distributional curvature bounded by Λ , and for every p_j there exists exactly three curves from $\{\gamma_i\}_{i \in I}$ which share p_j as a common boundary point, and meet at p_j by forming three 120 degrees angles.

We notice that, by (3.1.2), the reference honeycomb \mathcal{H} is a $(0, \infty)$ -minimizing unit-area tiling of \mathcal{T} . The following result is what we call an *improved convergence theorem*. The proof comes as a simple variant of Theorem 1.9.3 (or [CLM15, Theorem 1.5]) and therefore we omit the details.

Theorem 3.4.3. *Given $\Lambda \geq 0$, there exist positive constants L and $\mu_* > 0$ (depending on Λ and \mathcal{H}) with the following property. If $N = |\mathcal{T}|$, $\mu < \mu_*$ and $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ is a sequence of (Λ, r_0) -minimizing N -tilings of \mathcal{T} (for some $r_0 > 0$) with $d(\mathcal{E}_k, \mathcal{H}) \rightarrow 0$ as $k \rightarrow \infty$, then there exist $k(\mu) \in \mathbb{N}$ and, for every $k \geq k(\mu)$, a $C^{1,1}$ -diffeomorphism f_k with*

$$\sup_{k \geq k(\mu)} \|f_k\|_{C^{1,1}(\partial \mathcal{H})} \leq L, \quad \lim_{k \rightarrow \infty} \|f_k - \text{Id}\|_{C^1(\partial \mathcal{H})} = 0, \quad (3.4.16)$$

$$\tau_{\mathcal{H}}(f_k - \text{Id}) = 0 \quad \text{on } [\partial \mathcal{H}]_\mu, \quad \|\tau_{\mathcal{H}}(f_k - \text{Id})\|_{C^1(\partial^* \mathcal{H})} \leq \frac{L}{\mu} \sup_{\Sigma(\mathcal{H})} |f_k - \text{Id}|. \quad (3.4.17)$$

In particular, \mathcal{E}_k is a (ε_k, μ, L) -perturbation of \mathcal{H} whenever $k \geq k(\mu)$.

Let us now set

$$\kappa = \kappa(\mathcal{T}) = \inf \liminf_{k \rightarrow \infty} \frac{P(\mathcal{F}_k) - P(\mathcal{H})}{\alpha(\mathcal{F}_k)^2}, \quad (3.4.18)$$

where the infimum is taken among all sequences $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ of unit-area tilings of \mathcal{T} such that $\alpha(\mathcal{F}_k) > 0$ for every $k \in \mathbb{N}$ and $\alpha(\mathcal{F}_k) \rightarrow 0$ as $k \rightarrow \infty$. By a compactness argument, Theorem 3.1.1 is equivalent in saying that $\kappa > 0$.

Lemma 3.4.4. *If $\kappa = 0$, then there exists a sequence of (Λ, r_0) -minimizing unit-area tilings $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ such that $\alpha(\mathcal{E}_k) > 0$ for every $k \in \mathbb{N}$, $\alpha(\mathcal{E}_k) \rightarrow 0$ as $k \rightarrow \infty$, and*

$$P(\mathcal{E}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{E}_k)^2), \quad \text{as } k \rightarrow \infty. \quad (3.4.19)$$

Proof. By definition of κ , and since we are assuming $\kappa = 0$, there exist unit-area tilings $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ of \mathcal{T} such that $\alpha(\mathcal{F}_k) > 0$ for every $k \in \mathbb{N}$, and

$$\alpha(\mathcal{F}_k) \rightarrow 0, \quad P(\mathcal{F}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{F}_k)^2), \quad \text{as } k \rightarrow \infty. \quad (3.4.20)$$

For every $k \in \mathbb{N}$, let \mathcal{E}_k be a minimizer in the variational problem

$$\inf \left\{ P(\mathcal{E}) + d(\mathcal{E}, \mathcal{F}_k)^2 \mid \mathcal{E} \text{ unit-area tiling of } \mathcal{T} \text{ with } \alpha(\mathcal{E}) > 0 \right\}.$$

By comparing \mathcal{E}_k with \mathcal{F}_k and then subtracting $P(\mathcal{H})$ one has

$$P(\mathcal{E}_k) - P(\mathcal{H}) + d(\mathcal{E}_k, \mathcal{F}_k)^2 \leq P(\mathcal{F}_k) - P(\mathcal{H}) = o(\alpha(\mathcal{F}_k)^2). \quad (3.4.21)$$

Since $|\alpha(\mathcal{E}_k) - \alpha(\mathcal{F}_k)| \leq d(\mathcal{E}_k, \mathcal{F}_k)$ and $P(\mathcal{E}_k) \geq P(\mathcal{H})$, we conclude that

$$\lim_{k \rightarrow \infty} \frac{\alpha(\mathcal{E}_k)}{\alpha(\mathcal{F}_k)} = 1, \quad (3.4.22)$$

so that, in particular, $\alpha(\mathcal{E}_k) \rightarrow 0$ as $k \rightarrow \infty$. Dividing by $\alpha(\mathcal{E}_k)^2$ in (3.4.21) and using (3.4.22), we complete the proof of (3.4.19). We now show that each \mathcal{E}_k is (Λ, r_0) -minimizer in \mathcal{T} . Indeed, let $r_0, \varepsilon_0, \sigma_0$ and C_0 be the constants associated by Lemma 3.4.1 to $\mathcal{E}_0 = \mathcal{H}$, $\Phi = P$ and $\psi \equiv 1$. Since $\alpha(\mathcal{E}_k) \rightarrow 0$, up to translations we have $d(\mathcal{E}_k, \mathcal{H}) \leq \varepsilon_0$ for k large. We apply Lemma 3.4.1 with $\mathcal{E} = \mathcal{E}_k$, \mathcal{F} a N -tiling with $\mathcal{E}_k \Delta \mathcal{F} \subset \subset B_{r_0}(x)$ for some $x \in \mathcal{T}$, and $\eta = 0$, to find a unit-area tiling \mathcal{F}' such that

$$\begin{aligned} P(\mathcal{E}_k) &\leq P(\mathcal{E}_k) + d(\mathcal{E}_k, \mathcal{F}_k)^2 \leq P(\mathcal{F}') + d(\mathcal{F}', \mathcal{E}_k)^2 \\ &\leq P(\mathcal{F}) + C_0 P(\mathcal{E}_k) |\text{vol}(\mathcal{F}) - \text{vol}(\mathcal{E}_k)| + \left(d(\mathcal{F}, \mathcal{E}_k) + C_0 P(\mathcal{E}_k) |\text{vol}(\mathcal{F}) - \text{vol}(\mathcal{E}_k)| \right)^2. \end{aligned}$$

Hence $P(\mathcal{E}_k) \leq P(\mathcal{F}) + \Lambda d(\mathcal{E}_k, \mathcal{F})$ thanks to $|\text{vol}(\mathcal{F}) - \text{vol}(\mathcal{E}_k)| \leq d(\mathcal{F}, \mathcal{E}_k)$ and since, for k large enough, $P(\mathcal{E}_k) \leq 2P(\mathcal{H})$. \square

Proof of Theorem 3.1.1. We argue by contradiction. If the theorem is false, then $\kappa = 0$ and thus by Lemma 3.4.4 there exists a sequence $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ of (Λ, r_0) -minimizing unit-area tilings of \mathcal{T} such that $\alpha(\mathcal{E}_k) > 0$, $\alpha(\mathcal{E}_k) \rightarrow 0$ as $k \rightarrow \infty$ and

$$P(\mathcal{E}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{E}_k)^2), \quad \text{as } k \rightarrow \infty.$$

Up to translation we may assume that $\alpha(\mathcal{E}_k) = d(\mathcal{E}_k, \mathcal{H}) \rightarrow 0$ as $k \rightarrow \infty$. Let L and μ_* be the constants of Theorem 3.4.3 (which depends on Λ and \mathcal{H}) so that for every $\mu < \mu_*$ there exists $k(\mu) \in \mathbb{N}$ such that \mathcal{E}_k is a (ε_k, μ, L) -perturbation of \mathcal{H} for every $k \geq k(\mu)$, with $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Let ε_0 and μ_0 be determined as in Theorem 3.3.1 depending on L and $|\mathcal{T}|$. If we set $\mu = \min\{\mu_*, \mu_0\}$ and increase $k(\mu)$ so that $\varepsilon_k \leq \varepsilon_0$ for $k \geq k(\mu)$, then by Theorem 3.3.1, one finds $v_k \in \mathbb{R}^2$ with $|v_k| \leq C \varepsilon_k$ such that

$$P(\mathcal{E}_k) - P(\mathcal{H}) \geq c_0 \text{hd}(\partial \mathcal{E}_k, v_k + \partial \mathcal{H})^2 \geq c d(\mathcal{E}_k, v_k + \mathcal{H})^2 \geq c \alpha(\mathcal{E}_k)^2,$$

for some positive constant c . We have thus reached a contradiction, and proved the theorem. \square

Proof of Theorem 3.1.3. Let $\mathcal{E}_j = \mathcal{E}_{m^j}$ be minimizers in (3.1.4) for a sequence $\{m^j\}_{j \in \mathbb{N}}$ such that $\sum_{h=1}^N m_h^j = N$, $m_h^j > 0$ and $m_h^j \rightarrow 1$ as $j \rightarrow \infty$. By an explicit construction, for every j large enough we can construct a small deformation \mathcal{H}_j of \mathcal{H} such that $|\mathcal{H}_j(h)| = m_h^j$ and $P(\mathcal{H}_j) \leq P(\mathcal{H}) + C \max_{1 \leq h \leq N} |m_h^j - 1|$, with C independent from j . (Alternatively, one can apply Lemma 3.4.1 with $\mathcal{E}_0 = \mathcal{E} = \mathcal{F} = \mathcal{H}$, $\Phi = P$, $\psi \equiv 1$ and $\eta_h = m_h^j - 1$.) As a consequence, $\sup_{j \in \mathbb{N}} P(\mathcal{E}_j) < \infty$, and thus, up to extracting subsequences, $d(\mathcal{E}_j, \mathcal{E}_0) \rightarrow 0$ where \mathcal{E}_0 is a unit-area tiling of \mathcal{T} . In particular,

$$P(\mathcal{H}) \leq P(\mathcal{E}_0) \leq \liminf_{j \rightarrow \infty} P(\mathcal{E}_j) \leq \liminf_{j \rightarrow \infty} P(\mathcal{H}) + C \max_{1 \leq h \leq N} |m_h^j - 1| = P(\mathcal{H}).$$

By Hales's theorem, up to a relabeling of \mathcal{E}_0 , $\mathcal{E}_0 = v + \mathcal{H}$ for $v = (t\sqrt{3}\ell, s\ell)$ and $t, s \in [0, 1]$. By performing the same relabeling on each \mathcal{E}_j , we have $d(\mathcal{E}_j, v + \mathcal{H}) \rightarrow 0$. By exploiting Lemma 3.4.1 as in the proof of Lemma 3.4.4 one sees that each \mathcal{E}_j is a (Λ, r_0) -minimizing tiling in \mathcal{T} , and then by arguing as in the proof of Theorem 3.1.1 we find a constant L (depending on Λ and \mathcal{H}) such that $\mathcal{E}_j - v$ is an $(\varepsilon_j, \mu_0, L)$ -perturbation of \mathcal{H} for μ_0 as in Theorem 3.3.1 and for $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. By Theorem 3.3.1, for j large enough there exist $v_j \rightarrow 0$ and $C^{1,1}$ -diffeomorphism f_j between $v_j + \partial\mathcal{H}$ and $\partial\mathcal{E}_j - v$, with

$$C \max_{1 \leq h \leq N} |m_h^j - 1| \geq P(\mathcal{E}_j) - P(\mathcal{H}) \geq c \left(\|f_j - \text{Id}\|_{C^0(v_j + \partial\mathcal{H})}^2 + \|f_j - \text{Id}\|_{C^1(v_j + \partial\mathcal{H})}^4 \right).$$

Theorem 3.1.3 is then deduced by a contradiction argument. \square

Proof of Theorem 3.1.4. In the following we denote by \mathcal{E}_δ a minimizing in (3.1.6), and set

$$\delta = \delta(\Phi, \psi) = \|\Phi - 1\|_{C^0(\mathcal{T} \times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})},$$

so that $\delta < \delta_0$. We notice that for every $E \subset \mathcal{T}$ of finite perimeter one has

$$\left| \int_E \psi - |E| \right| \leq C |E| \|\psi - 1\|_{C^0(\mathcal{T})}, \quad (3.4.23)$$

$$|\Phi(E) - P(E)| \leq C \min\{P(E), \Phi(E)\} \|\Phi - 1\|_{C^0(\mathcal{T} \times S^1)}, \quad (3.4.24)$$

where in (3.4.24) we have also used the fact that $P(E) \leq 2\Phi(E)$ provided $\delta_0 \leq 1$.

Step one: We claim that, provided δ_0 is small enough, then

$$\Phi(\mathcal{E}_\delta) \leq 2P(\mathcal{H}), \quad (3.4.25)$$

$$P(\mathcal{E}_\delta) \leq P(\mathcal{H}) + C\delta. \quad (3.4.26)$$

Indeed, by considering an explicit small modification of \mathcal{H} (or by applying Lemma 3.4.1 with $\mathcal{E} = \mathcal{E}_0 = \mathcal{F} = \mathcal{H}$ and $\eta \neq 0$) we can construct a N -tiling \mathcal{H}' of \mathcal{T} such that $\int_{\mathcal{H}'(h)} \psi = N^{-1} \int_{\mathcal{T}} \psi$ for every $h = 1, \dots, N$ and $\Phi(\mathcal{H}') \leq \Phi(\mathcal{H}) + C\delta$. By $\Phi(\mathcal{E}_\delta) \leq \Phi(\mathcal{H}')$ and by (3.4.24)

$$\Phi(\mathcal{E}_\delta) \leq \Phi(\mathcal{H}) + C\delta \leq P(\mathcal{H}) + C\delta, \quad (3.4.27)$$

which implies (3.4.25). Again by (3.4.24), $P(\mathcal{E}_\delta) \leq \Phi(\mathcal{E}_\delta) + C\delta$, and (3.4.27) gives (3.4.26).

Step two: We now show that if $\delta_j = \delta(\Phi_j, \psi_j) \rightarrow 0$ and \mathcal{E}_j is a minimizer in (3.1.6) associated to Φ_j and ψ_j , then (and up to subsequences and to relabeling the chambers of \mathcal{E}_j) $d(\mathcal{E}_j, v + \mathcal{H}) \rightarrow 0$ for some $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0, 1]$. By (3.4.25) and since $\Phi_j(E) \geq P(E)/2$ for every $E \subset \mathcal{T}$ we find that $\sup_{j \in \mathbb{N}} P(\mathcal{E}_j) \leq 4P(\mathcal{H})$. By compactness, there exists a N -tiling \mathcal{E}_* of \mathcal{T} such that $d(\mathcal{E}_j, \mathcal{E}_*) \rightarrow 0$ (up to subsequences). By (3.4.23),

$\int_{\mathcal{E}_j(h)} \psi_j = N^{-1} \int_{\mathcal{T}} \psi_j$ implies $m_j(h) = |\mathcal{E}_j(h)| \rightarrow 1$ for every $h = 1, \dots, N$. In particular, \mathcal{E}_* is a unit-area tiling of \mathcal{T} , and thus by (3.1.2), by lower semicontinuity and by (3.4.26)

$$P(\mathcal{H}) \leq P(\mathcal{E}_*) \leq \liminf_{j \rightarrow \infty} P(\mathcal{E}_j) \leq P(\mathcal{H}). \quad (3.4.28)$$

By Hales's theorem, up a relabeling, $\mathcal{E}_* = v + \mathcal{H}$.

Step three: Let $\varepsilon_0, r_0, \sigma_0$ and C_0 be the constants associated to $\mathcal{E}_0 = \mathcal{H}, \Phi$ and ψ by Lemma 3.4.1. (Notice that the same constants will work on any translation of \mathcal{H} , and that these constants ultimately depend on L and γ only.) By step two we can assume that δ_0 is small enough to entail $d(\mathcal{E}_\delta, v_\delta + \mathcal{H}) \leq \varepsilon_0$ for some translation v_δ . We now claim that there exist positive constants $r_1, c_0 > 0$ such that

$$|\mathcal{E}_\delta(h) \cap B_r(x)| \geq c_0 r^2, \quad \forall x \in \partial \mathcal{E}_\delta(h), r < r_1, h = 1, \dots, N. \quad (3.4.29)$$

This is a classical argument, see for example [Mag12, Lemma 30.2], and we include some details just for the sake of completeness. Without loss of generality let us set $h = 1$ and fix $x \in \partial \mathcal{E}_\delta(1)$ and $r < r_1 \leq r_0$ such that $P(\mathcal{E}_\delta; \partial B_r(x)) = 0$. There exists $j \in \{1, \dots, N\}$ such that

$$\mathcal{H}^1(\partial^* \mathcal{E}_\delta(1) \cap \partial^* \mathcal{E}_\delta(j) \cap B_r(x)) \geq \mathcal{H}^1(\partial^* \mathcal{E}_\delta(1) \cap \partial^* \mathcal{E}_\delta(h) \cap B_r(x)), \quad \forall h \neq 1, j. \quad (3.4.30)$$

If we set $\mathcal{F}(1) = \mathcal{E}_\delta(1) \setminus B_r(x)$, $\mathcal{F}(j) = \mathcal{E}_\delta(j) \cup (\mathcal{E}_\delta(1) \cap B_r(x))$ and $\mathcal{F}(h) = \mathcal{E}_\delta(h)$ for $h \neq 1, j$, then by applying Lemma 3.4.1 with $\mathcal{E}_0 = v_\delta + \mathcal{H}$, $\mathcal{E} = \mathcal{E}_\delta$, and $\eta = 0$ and setting $u(r) = |\mathcal{E}_\delta(1) \cap B_r(x)|$, we find that, if $\varepsilon < r_0 - r$, then

$$\begin{aligned} \Phi(\mathcal{E}_\delta; B_{r+\varepsilon}(x)) &\leq \Phi(\mathcal{F}; B_{r+\varepsilon}(x)) + C_0 P(\mathcal{E}_\delta) \left| \int_{\mathcal{E}_\delta(1) \cap B_r(x)} \psi \right| \\ &\leq \Phi(\mathcal{E}_\delta; B_{r+\varepsilon}) + \hat{\Phi}(B_r(x); \mathcal{E}_\delta(1)) \\ &\quad - \int_{\partial^* \mathcal{E}_\delta(1) \cap \partial^* \mathcal{E}_\delta(j) \cap B_r(x)} \hat{\Phi}(y, \nu_{\mathcal{E}_\delta(1)}(y)) d\mathcal{H}^1 + C u(r), \end{aligned}$$

where we have set $\hat{\Phi}(x, \nu) = (\Phi(x, \nu) + \Phi(x, -\nu))/2$. In particular, by (3.4.30) and by $2 \geq \Phi \geq 1/2$, for every $h \neq 1$ one finds

$$\mathcal{H}^1(\partial^* \mathcal{E}_\delta(1) \cap \partial^* \mathcal{E}_\delta(h) \cap B_r(x)) \leq C(\mathcal{H}^1(\mathcal{E}_\delta(1) \cap \partial B_r(x)) + u(r)),$$

i.e.

$$P(\mathcal{E}_\delta(1); B_r(x)) \leq C(u'(r) + u(r)), \quad \text{for a.e. } r < r_1.$$

By adding $u'(r) = \mathcal{H}^1(\mathcal{E}_\delta(1) \cap \partial B_r(x))$ to both sides we find that

$$C(u'(r) + u(r)) \geq P(\mathcal{E}_\delta(1) \cap B_r(x)) \geq 2\sqrt{\pi u(r)}.$$

In particular if r_1 is small enough to give $C u(r) \leq C\sqrt{\pi r_1^2 u(r)} \leq \sqrt{\pi u(r)}$, then we find $\sqrt{u(r)} \leq C u'(r)$ for a.e. $r < r_1$. This proves (3.4.29).

Step four: We now conclude the proof. Again by step two and by Lemma 3.4.1, one can find a unit-area tiling \mathcal{E}'_δ of \mathcal{T} such that $P(\mathcal{E}'_\delta) \leq P(\mathcal{E}_\delta) + C\delta$ and $d(\mathcal{E}'_\delta, \mathcal{E}_\delta) \leq C\delta$. By Theorem 3.1.1 and up to permutations of the chambers of \mathcal{E}_δ , we find a translation v_δ such that

$$c d(\mathcal{E}'_\delta, v_\delta + \mathcal{H})^2 \leq P(\mathcal{E}'_\delta) - P(\mathcal{H}) \leq P(\mathcal{E}_\delta) - P(\mathcal{H}) + C\delta \leq C\delta,$$

where in the last inequality we have used (3.4.26). Since $d(\mathcal{E}'_\delta, v_\delta + \mathcal{H}) \geq d(\mathcal{E}_\delta, v_\delta + \mathcal{H}) - d(\mathcal{E}'_\delta, \mathcal{E}_\delta)$ we conclude

$$d(\mathcal{E}_\delta, v_\delta + \mathcal{H})^2 \leq C \delta.$$

Setting for the sake of brevity $v_\delta = 0$, we now pick $x \in \partial\mathcal{E}_\delta(1)$ such that $\text{dist}(x, \partial\mathcal{H}(1)) \geq \text{dist}(y, \partial\mathcal{H}(1))$ for every $y \in \partial\mathcal{E}_\delta(1)$. Let $r = \min\{r_1, \text{dist}(x, \partial\mathcal{H}(1))\}$, so that either $B_r(x) \subset \mathcal{T} \setminus \mathcal{H}(1)$ or $B_r(x) \subset \mathcal{H}(1)$. In particular, provided δ_0 is small enough with respect to c_0 , either

$$d(\mathcal{E}_\delta, \mathcal{H}) \geq |\mathcal{E}_\delta(1) \setminus \mathcal{H}(1)| \geq |\mathcal{E}_\delta(1) \cap B_r(x)| \geq c_0 r^2 \geq c_0 \text{dist}(x, \partial\mathcal{H}(1))^2,$$

or

$$\begin{aligned} d(\mathcal{E}_\delta, \mathcal{H}) &\geq |\mathcal{H}(1) \setminus \mathcal{E}_\delta(1)| \geq |B_r(x) \setminus \mathcal{E}_\delta(1)| = \left| \bigcup_{h=2}^N B_r(x) \cap \mathcal{E}_\delta(h) \right| \\ &\geq (N-1)c_0 r^2 \geq c_0 \text{dist}(x, \partial\mathcal{H}(1))^2; \end{aligned}$$

in both cases, $\partial\mathcal{E}_\delta(1) \subset I_\varepsilon(\partial\mathcal{H}(1))$ for $\varepsilon = C \sqrt{d(\mathcal{E}_\delta, \mathcal{H})}$. By the same argument (based on area density estimates for \mathcal{H} , which hold trivially) one finds that $\partial\mathcal{H}(1) \subset I_\varepsilon(\partial\mathcal{E}_\delta(1))$. \square

Chapter 4

Cheeger N -clusters

4.1 Introduction

For a given open, bounded set Ω and an integer $N \in \mathbb{N}$ we introduce the N -Cheeger constant of Ω as:

$$H_N(\Omega) = \inf \left\{ \sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \mid \mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^N \subseteq \Omega, \text{ is an } N\text{-cluster} \right\}. \quad (4.1.1)$$

As shown below in Theorem 4.3.1, the infimum in (4.1.1) is always attained and we refer to the minimizers as the *Cheeger N -clusters of Ω* .

We focus on the quantity H_N because it seems to represent the right object to study in order to provide some non trivial lower bound on the optimal partition functional

$$\Lambda_N^{(p)}(\Omega) = \inf \left\{ \sum_{i=1}^N \lambda_1^{(p)}(\mathcal{E}(i)) \right\}, \quad (4.1.2)$$

where $\lambda_1^{(p)}$ denotes the first Dirichlet eigenvalue of the p -Laplacian, defined as:

$$\lambda_1^{(p)}(E) := \inf \left\{ \int_E |\nabla u|^p dx \mid u \in W_0^{1,p}(E), \|u\|_{L^p} = 1 \right\}.$$

The infimum in (4.1.2) is taken over all the N -clusters \mathcal{E} whose chambers are *quasi-open sets of Ω* . The family of quasi-open sets of an open bounded set Ω is a suitable sub-class of the Borel's algebra of Ω where the first Dirichlet eigenvalue of the p -Laplacian $\lambda_1^{(p)}$ can be defined. The definition of quasi-open set is related to the concept of p -capacitary measure in \mathbb{R}^n that we do not need to recall in here (see [EG91] for more details about it). For our purposes it is enough to recall that:

the quasi-open sets are the upper levels of $W^{1,p}$ functions as well as the open sets are the upper levels of continuous functions. Each open set of an open bounded set Ω is also a quasi-open set of Ω .

The importance of the partition problem (4.1.2) relies in the fact that it provides a way to look at the asymptotic behavior in N of the N -th Dirichlet eigenvalue of the classical

Laplacian (the 2-Laplacian), as Caffarelli and Lin show in [CL07]. The N -th Dirichlet eigenvalue of the Laplacian of an open set Ω is recursively defined as

$$\lambda_N^{(2)}(\Omega) = \inf_{u \in X_{N-1}} \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \right\}$$

$$X_{N-1} = \left\{ u \in W_0^{1,2}(\Omega) \mid \langle u, u_i \rangle_2 = 0, \text{ for all } i = 1, \dots, N-1 \right\}$$

where u_1, \dots, u_{N-1} are the first $N-1$ eigenfunctions

$$\lambda_i^{(2)}(\Omega) = \frac{\int_{\Omega} |\nabla u_i|^2 dx}{\int_{\Omega} |u_i|^2 dx} \quad \text{for all } i = 1, \dots, N-1$$

and $\langle \cdot, \cdot \rangle_2$ denotes the standard scalar product of $L^2(\Omega)$

$$\langle u, v \rangle_2 = \int_{\Omega} uv dx \quad \text{for all } u, v \in L^2(\Omega)$$

(see [EG91, Section 6.5] for a detailed discussion about eigenvalues and eigenfunctions). In [CL07], Caffarelli and Lin prove that there exist two constants C_1 and C_2 depending only on the dimension such that

$$C_1 \frac{\Lambda_N^{(2)}(\Omega)}{N} \leq \lambda_N^{(2)}(\Omega) \leq C_2 \frac{\Lambda_N^{(2)}(\Omega)}{N}, \quad (4.1.3)$$

where $\lambda_N^{(2)}$ is the N -th Dirichlet eigenvalue. The detailed study of $\lambda_N^{(2)}(\Omega)$ for $N \geq 2$ seems to be an hard task (so far only the case $N = 1, 2$ are well known in details, see for instance [Hen06]) and that is why the asymptotic approach suggested by Caffarelli and Lin could be a good way to look at the spectral problem. We also refer the reader to [Buc12] where the existence of minimizers for $\lambda_N^{(2)}$ is proved.

Caffarelli and Lin's conjecture (appearing in [CL07]) about the asymptotic behavior of $\Lambda_N^{(2)}(\Omega)$ in the planar case states that

$$\Lambda_N^{(2)}(\Omega) = \frac{N^2}{|\Omega|} \lambda_1^{(2)}(H) + o(N^2),$$

where H denotes a unit-area regular hexagon. So far, no progress has been made in proving the conjecture, anyway numerical simulations (see [BBO09]) point out that the conjecture could be true. If the conjecture turns out to be true, relation (4.1.3) could be improved, in the planar case, as:

$$C_1 \frac{N \lambda_1^{(2)}(H)}{|\Omega|} + o(N) \leq \lambda_N^{(2)}(\Omega) \leq C_2 \frac{N \lambda_1^{(2)}(H)}{|\Omega|} + o(N). \quad (4.1.4)$$

In order to explain the connection between H_N and $\Lambda_N^{(p)}$ we recall some well-known fact about the classical Cheeger constant of a Borel set Ω :

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} \mid E \subseteq \Omega \right\}, \quad (4.1.5)$$

(note that $h(\Omega) = H_1(\Omega)$). Given an open set Ω , each set $E \subseteq \Omega$ such that $h(\Omega) = \frac{P(E)}{|E|}$ is called *Cheeger set for Ω* . It is possible to prove that each Cheeger set E for Ω is a (Λ, r_0) -perimeter-minimizing inside Ω and that $\partial^* E \cap \Omega$ is a constant mean curvature analytic

hypersurface relatively open inside ∂E . Furthermore, the mean curvature C of the set E in the open set Ω is equal to $C = \frac{1}{n-1}h(E)$. We refer the reader to [Par11] and [Leo15]: two exhaustive surveys on Cheeger sets and Cheeger constant.

The Cheeger constant was introduced by Jeff Cheeger in [Che70] and provides a lower bound on the first Dirichlet eigenvalue of the p -Laplacian of a domain Ω . By exploiting the coarea formula and Hölder's inequality it is possible to show that for every domain Ω and for every $p > 1$ it holds,

$$\lambda_1^{(p)}(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p. \quad (4.1.6)$$

The Cheeger constant is also called the *first Dirichlet eigenvalue of the 1-laplacian* since, thanks to (4.1.6) and to a comparison argument

$$\lim_{p \rightarrow 1} \lambda_1^{(p)}(\Omega) = h(\Omega). \quad (4.1.7)$$

See, for example, [KN08] for more details about the relation between the Cheeger constant and the first Dirichlet eigenvalue of the p -Laplacian or [BB05] and [But11] for more details about the spectral problems and shape optimization problems.

We note here that the constant H_N is the analogous of the Cheeger constant in the optimal partition problem for p -laplacian eigenvalues. We refer the reader to [Par09], where a generalized type of Cheeger constant for the 2-nd Dirichlet eigenvalue of the Laplacian is also studied. As we show in Proposition 4.6.2 below, we can always give a lower bound on $\Lambda_N^{(p)}$ by making use of (4.1.6) and Jensen's inequality:

$$\Lambda_N^{(p)}(\Omega) \geq \frac{1}{N^{p-1}} \left(\frac{H_N(\Omega)}{p} \right)^p. \quad (4.1.8)$$

By combining (4.1.8) with a comparison argument (see Theorem 4.6 below) we are also able to compute the limit as p goes to 1 and obtain

$$\lim_{p \rightarrow 1} \Lambda_N^{(p)}(\Omega) = H_N(\Omega). \quad (4.1.9)$$

Thus, the constant H_N seems to provide the suitable generalization of the Cheeger constant for the study of $\Lambda_N^{(p)}$.

In this chapter we mainly focus on the general structure and regularity of Cheeger N -clusters in order to lay the basis for future investigations on H_N . In the final section, once we have proved (4.1.9), we study the asymptotic behavior of H_N in the planar case. The statements involving regularity are quite technical and we reserve to them the whole Section 4.2 (Theorems 4.2.1, 4.2.3 and 4.2.7), we just point out here that if \mathcal{E} is a Cheeger N -cluster of Ω the following statement holds.

For every $i = 1, \dots, N$ the reduced boundary of each chambers $\partial^ \mathcal{E}(i) \cap \Omega$ is a $C^{1,\alpha}$ -hypersurfaces (for every $\alpha \in (0, 1)$) that is relatively open inside $\partial \mathcal{E}(i) \cap \Omega$. Furthermore it is possible to characterize the singular set of a Cheeger N -cluster \mathcal{E} as a suitable collection of points with density zero for the external chamber*

$$\mathcal{E}(0) = \Omega \setminus \bigcup_{i=1}^N \mathcal{E}(i).$$

Moreover if the dimension is $n = 2$ then the singular set is discrete and the chambers $\mathcal{E}(i) \subset \subset \Omega$ are indecomposable.

Note that, in this context, the external chambers should be intended as $\Omega \setminus (\cup_i \mathcal{E}(i))$ instead of $\mathbb{R}^n \setminus \cup_i \mathcal{E}(i)$ as usual (that is because the ambient space is Ω in place of \mathbb{R}^n). As we are pointing out below, also the definition of "singular set of a Cheeger N -cluster" must be given in a slightly different way (see (4.2.3)) from the standard one $\partial \mathcal{E} \setminus \partial^* \mathcal{E}$, since this last set turns out to be too small. Let us postpone this discussion below to Section 4.2 where precise statements are given, and let us, instead, briefly focus on the asymptotic properties of H_N (to which Subsection 4.6.1 is devoted).

We note that for H_N it is reasonable to expect a behavior of the type

$$H_N(\Omega) = C(\Omega)N^{\frac{3}{2}} + o(N^{\frac{3}{2}}), \quad (4.1.10)$$

for some constant $C(\Omega)$. In Theorem 4.6.3 (Property 3) we provide some asymptotic estimate for H_N showing that the exponent $\frac{3}{2}$ in (4.1.10) is the correct one and proving that for any given bounded open set $\Omega \subset \mathbb{R}^2$ it holds

$$\frac{h(B)\sqrt{\pi}}{\sqrt{|\Omega|}} \leq \liminf_{N \rightarrow +\infty} \frac{H_N(\Omega)}{N^{\frac{3}{2}}} \leq \limsup_{N \rightarrow +\infty} \frac{H_N(\Omega)}{N^{\frac{3}{2}}} \leq \frac{h(H)}{\sqrt{|\Omega|}}, \quad (4.1.11)$$

We here conjecture that

$$C(\Omega) = \frac{h(H)}{\sqrt{|\Omega|}},$$

which is nothing more than Caffarelli and Lin's conjecture for the case $p = 1$. Note that, thanks to (4.1.8) this would imply

$$\Lambda_N^{(2)}(\Omega) \geq \frac{N^2}{|\Omega|} \left(\frac{h(H)}{2} \right)^2 + o(N^2), \quad (4.1.12)$$

a "weak" version of Caffarelli and Lin's conjecture. It seems coherent and natural to expect this kind of behavior for $H_N(\Omega)$.

The chapter is organized as follows. In Section 4.2 we present and comment the three main statements describing the regularity property and the structure of Cheeger N -clusters. Sections 4.3, 4.4 and 4.5 are devoted to the proof of the Theorems introduced in Section 4.2. In the final section 4.6 we show the connection between H_N and $\Lambda_N^{(p)}$ and we establish the asymptotic trend of H_N for N large in the planar case.

4.2 Basic definitions and regularity theorems for Cheeger N -clusters

We present three statements that we are going to prove in Section 4.3 and in Subsections 4.4.2 and 4.5.2.

In Section 4.3 after we have shown existence of Cheeger N -clusters for any given bounded ambient space Ω with finite perimeter (Theorem 4.3.1) we provide the partial

regularity Theorem 4.2.1 in the spirit of Theorems 1.8.1 and 1.3.2. Set, for a generic Borel set F and for $i = 1, \dots, N$

$$\Sigma(\mathcal{E}(i); F) := [\partial\mathcal{E}(i) \setminus \partial^*\mathcal{E}(i)] \cap F, \quad (4.2.1)$$

$$\Sigma(\mathcal{E}(i)) := \Sigma(\mathcal{E}(i); \mathbb{R}^n). \quad (4.2.2)$$

Theorem 4.2.1. *Let $n \geq 1, N \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with finite perimeter and \mathcal{E} be a Cheeger N -cluster of Ω . Then for every $i = 1, \dots, N$ the following statements hold true:*

- (i) *For every $\alpha \in (0, 1)$ the set $\Omega \cap \partial^*\mathcal{E}(i)$ is a $C^{1,\alpha}$ -hypersurface that is relatively open in $\Omega \cap \partial\mathcal{E}(i)$ and it is \mathcal{H}^{n-1} equivalent to $\Omega \cap \partial^*\mathcal{E}(i)$;*
- (ii) *For every $i = 1, \dots, N$ the set $\partial\mathcal{E}(i) \cap \Omega$ can meet $\partial^*\Omega$ only in a tangential way, that is: $\partial^*\Omega \cap \partial\mathcal{E}(i) \subseteq \partial^*\mathcal{E}(i)$. Moreover for every $x \in \partial^*\Omega \cap \partial\mathcal{E}(i)$ it holds:*

$$\nu_{\mathcal{E}(i)}(x) = \nu_{\Omega}(x).$$

Here $\nu_{\mathcal{E}(i)}, \nu_{\Omega}$ denote, respectively, the measure theoretic outer unit normal to $\mathcal{E}(i)$ and to Ω ;

- (iii) $\Sigma(\mathcal{E}(i); \Omega)$ is empty if $n \leq 7$;
- (iv) $\Sigma(\mathcal{E}(i); \Omega)$ is discrete if $n = 8$;
- (v) if $n \geq 9$, then $\mathcal{H}^s(\Sigma(\mathcal{E}(i); \Omega)) = 0$ for every $s > n - 8$.

For proving (i), (iii), (iv) and (v) we simply show (in Theorem 4.3.3) that each chamber $\mathcal{E}(i)$ is a (Λ, r_0) -perimeter-minimizing in Ω (see Definition 1.3.1) and then we make use of the De Giorgi's regularity Theorem 1.3.2. We re-adapt an idea from [BM82] based on the fact that a solution of an obstacle problem having bounded distributional mean curvature is regular. Assertion (ii) follows as a consequence of [LP14, Proposition 2.5, Assertion (vii)] retrieved below (Proposition 4.3.4).

Remark 4.2.2. We need to ask that Ω is bounded otherwise no Cheeger N -clusters are attained. Indeed if Ω is unbounded, by intersecting Ω with N suitable disjoint balls of radius approaching $+\infty$ we easily obtain $H_N(\Omega) = 0$.

4.2.1 The role of the singular set $\Sigma(\mathcal{E})$

Note that Theorem 4.2.1 yields the inner regularity of *all* the chambers, differently from Theorem 1.8.1 which involves the reduced boundary of the cluster $\partial^*\mathcal{E}$ (which in general is smaller than the union of the reduced boundary of the chambers). This stronger regularity of the chambers somehow affect the behavior of the singular set. For example consider the case $n \leq 7$. In this case, according to Theorem 4.2.1, for a Cheeger N -cluster it must hold that

$$(\partial\mathcal{E} \setminus \partial^*\mathcal{E}) \cap \Omega = \emptyset,$$

and this would lead us to say that the singular set of a Cheeger N -cluster is empty which is clearly not the case. Indeed let us highlights that there is somehow an "hidden chamber" that plays a key role and influences the behavior of the global structure of these objects, namely the *external chamber*:

$$\mathcal{E}(0) = \Omega \setminus \left(\bigcup_{i=1}^N \mathcal{E}(i) \right).$$

Note that this definition of external chamber is slightly different from the ones adopted in the previous chapters. Since our ambient space in this context is the open bounded set Ω instead of \mathbb{R}^n we found convenient and coherent to keep this notation throughout this chapter.

Even if Theorem 4.2.1 provides a satisfactory description of $\Sigma(\mathcal{E}(i), \Omega)$, this does not exhaust the analysis of the singular set of \mathcal{E} . Indeed the chamber $\mathcal{E}(0)$ is not regular after all and there are points in $\partial\mathcal{E}(0)$ of cuspidal type. For a complete description of the singularity, the correct definition of *singular set of a Cheeger N -cluster \mathcal{E} in the Borel set F* must be given as

$$\Sigma(\mathcal{E}; F) := \Sigma(\mathcal{E}(0); F) \cup \bigcup_{i=1}^N \Sigma(\mathcal{E}(i); F), \quad (4.2.3)$$

where for $i \neq 0$ the set $\Sigma(\mathcal{E}(i))$ are the ones defined in (4.2.1) and (4.2.2), while for $i = 0$ we clearly set

$$\Sigma(\mathcal{E}(0); F) = [\partial\mathcal{E}(0) \setminus \partial^*\mathcal{E}(0)] \cap F.$$

With this definitions, $(\partial\mathcal{E} \setminus \partial^*\mathcal{E}) \cap \Omega \subseteq \Sigma(\mathcal{E}; \Omega)$. With a slight abuse of notation we denote by $\Sigma(\mathcal{E})$ the singular set of a Cheeger N -cluster, even if it is different from the one defined in (1.8.2). Since Theorem 4.2.1 do not provides information about $\Sigma(\mathcal{E}(0))$, we focus our attention on it in Subsection 4.4.2 where the following theorem is proved.

Theorem 4.2.3. *Let $1 \leq n \leq 7, N \geq 2$, Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded set with C^1 boundary and finite perimeter and \mathcal{E} be a Cheeger N -cluster of Ω . Then the following statements hold true.*

- (i) $\mathcal{E}(0)$ is not empty and $\mathcal{H}^{n-1}(\partial\mathcal{E}(0) \cap \partial\mathcal{E}(j)) > 0$ for all $i = 1, \dots, N$;
- (ii) $\Sigma(\mathcal{E}(0); \Omega) = \partial\mathcal{E}(0) \cap \mathcal{E}(0)^{(0)}$, $\Sigma(\mathcal{E}(0); \Omega)$ is closed and

$$\Sigma(\mathcal{E}(0); \Omega) = \Omega \cap \bigcup_{\substack{j,k=1, \\ k \neq j}}^N (\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \partial\mathcal{E}(0)) \quad (4.2.4)$$

Remark 4.2.4. Note that Assertion (ii) of Theorem 4.2.3, stated as above, would be meaningless if we do not ensure that $|\mathcal{E}(0)| > 0$ (that is Assertion (i), proved in Proposition 4.4.4). The assumption on Ω to be connected and with C^1 -boundary are the necessary ones to ensure the validity of this fact. Probably, the theorem remains true also by replacing C^1 boundary with Lipschitz boundary. Anyway we prefer to state and prove it by taking advantage of this stronger regularity on $\partial\Omega$ in order to avoid some technicality. Let us also point out that there are situations where Ω is not connected or $\partial\Omega$ is not Lipschitz and where $\mathcal{E}(0)$ turns out to be empty. For example, given a set Ω and one of its Cheeger N -cluster \mathcal{E} , we provide a counterexample by defining the new open set

$$\Omega_0 = \left(\bigcup_{j=1}^N \mathcal{E}(j) \right).$$

The N -cluster \mathcal{E} will be a Cheeger N -clusters of Ω_0 also and, by construction, $|\mathcal{E}(0)| = 0$ (see Figure 4.2.1). The reason is that Ω_0 has no regular boundary. As a further example one may also consider the case when Ω is the union of N disjoint balls. Anyway, it is reasonable to expect that, no matter what kind of ambient space Ω we choose, for N sufficiently large the chamber $\mathcal{E}(0)$ will be not empty.

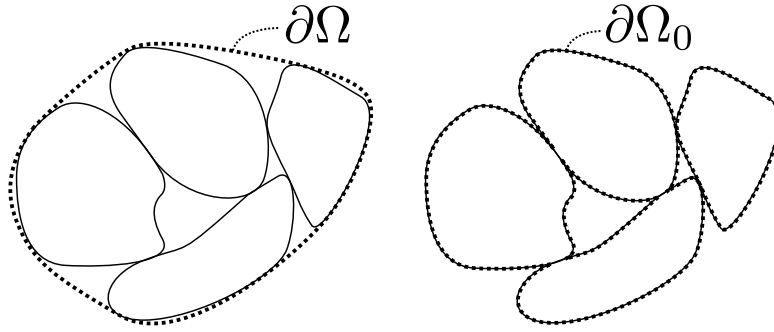


Figure 4.2.1: The set Ω_0 built as the union of the interior of the Cheeger N -cluster of an open set Ω . The external chamber of this Cheeger N -cluster of Ω_0 is empty because of the cusps at the boundary of the open set.

Remark 4.2.5. Note that we ask for the dimension n to be less than 7. That is because, to prove Theorem 4.2.3, we exploit the regularity given by 4.2.1 and we prefer to deal with the favorable case $n \leq 7$ where the singular set $\Sigma(\mathcal{E}(i); \Omega) = \emptyset$ for $i \neq 0$. Let us also point out that Assertion (ii) remains true also in dimension bigger than 7 up to replace Ω with $\Omega_0 = \Omega \setminus \cup_{i \neq 0} \Sigma(\mathcal{E}(i); \Omega)$. The interesting and not-trivial fact is that we actually do not know if assertion (i) remains true in dimension bigger than 7 since, in the proof of Proposition 4.4.4 (the crucial one in order to prove assertion (i)), we make a strong use of the fact $\Sigma(\mathcal{E}(i); \Omega) = \emptyset$. Roughly speaking in dimension bigger than 7 it could happen that the chambers, by taking advantage of the possible presence of singular points $x \in \Sigma(\mathcal{E}(i); \Omega)$, can be combined in a way that kill $\mathcal{E}(0)$ even under a strong regularity assumption on Ω .

Remark 4.2.6. Somehow assertion (ii) of Theorem 4.2.3 is saying that the only singular points of \mathcal{E} are the one where a cusp is attained. Now we can give a complete description of the singular set $\Sigma(\mathcal{E}; \Omega)$ of a Cheeger N -cluster of an open, bounded, connected set Ω with finite perimeter and C^1 boundary in dimension less than or equal to 7. By combining Assertion (iii) in Theorem 4.2.1 and assertion (i) in 4.2.3 we can write

$$\Sigma(\mathcal{E}; \Omega) = \Sigma(\mathcal{E}(0); \Omega) = (\partial\mathcal{E}(0) \cap \mathcal{E}(0)^{(0)}) \cap \Omega.$$

4.2.2 The planar case

Theorem 4.2.3 gives us a precise structure of $\Sigma(\mathcal{E}; \Omega)$. We do not focus here on the singular set $\Sigma(\mathcal{E}; \partial\Omega)$ anyway, by exploiting the C^1 -regularity assumption on $\partial\Omega$, it is possible to prove a result in the spirit of Theorem 4.2.3 also for the singular set $\Sigma(\mathcal{E}(0); \partial\Omega)$ (and thus characterize $\Sigma(\mathcal{E}; \partial\Omega)$). Let us point out that, at the present, the crucial information $\mathcal{H}^{n-1}(\Sigma(\mathcal{E}(0); \Omega)) = 0$ is missing. We are able to fill this gap when the ambient space dimension is $n = 2$, together with some remarkable facts stated in the following theorem (proved in Subsection 4.5.2).

Theorem 4.2.7. *Let $n = 2, N \geq 2$. Let $\Omega \subset \mathbb{R}^2$ be an open, connected, bounded set with C^1 boundary and finite perimeter and \mathcal{E} be a Cheeger N -cluster of Ω . Then the following statements hold true.*

- (i) *The singular set $\Sigma(\mathcal{E}(0); \Omega)$ is a finite union of points $\{x_j\}_{j=1}^k \subset \Omega$.*

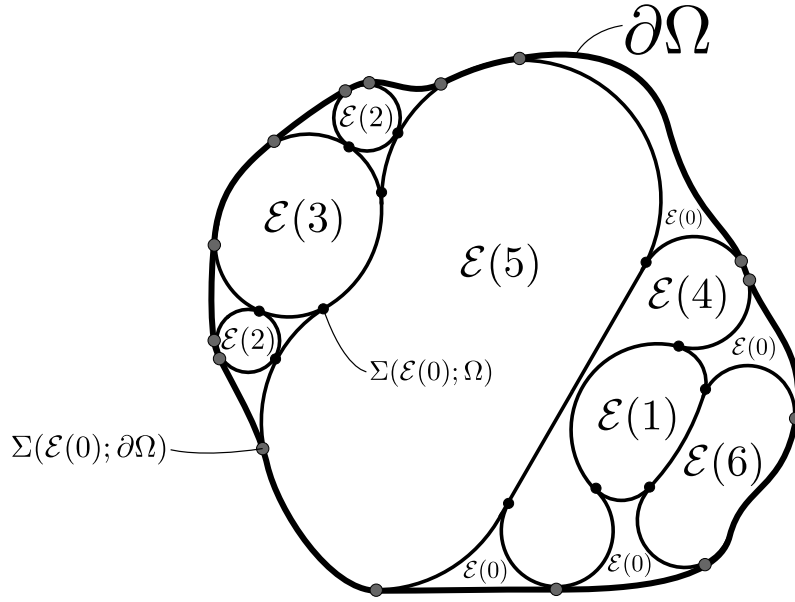


Figure 4.2.2: An example of a possible Cheeger 6-cluster in dimension $n = 2$ suggested by Theorems 4.2.1, 4.2.3 and 4.2.7

(ii) For every $j, k = 0, \dots, N$, $k \neq j$ the set

$$E_{j,k} := [\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega] \setminus \Sigma(\mathcal{E}(0); \Omega)$$

is relatively open in $\partial\mathcal{E}(j)$ ($\partial\mathcal{E}(k)$) and is the finite union of segments and circular arcs. Moreover the set $\mathcal{E}(j)$ has constant curvature $C_{j,k}$ inside each open set A such that $A \cap \partial\mathcal{E}(j) \subseteq E_{j,k}$. The constant $C_{j,k}$ is equal to:

$$C_{j,k} = \begin{cases} \frac{|\mathcal{E}(k)|h(\mathcal{E}(j)) - |\mathcal{E}(j)|h(\mathcal{E}(k))}{|\mathcal{E}(j)| + |\mathcal{E}(k)|}, & \text{if } k \neq 0, \\ h(\mathcal{E}(j)), & \text{if } k = 0. \end{cases} \quad (4.2.5)$$

As a consequence the set $\mathcal{E}(k)$ has constant curvature $C_{j,k} = -C_{k,j}$ inside each open set A such that $A \cap \partial\mathcal{E}(k) \subseteq E_{k,j}$ ($= E_{j,k}$);

(iii) Each chamber $\mathcal{E}(j) \subset\subset \Omega$ is indecomposable.

Remark 4.2.8. Theorems 4.2.1, 4.2.3 and 4.2.7 allow us to provide examples of planar Cheeger N -cluster. The one depicted in Figure 4.2.2 is a possible Cheeger 6-clusters. Let us highlight that we do not want to suggest that the object in the figure is exactly the Cheeger 6-cluster of the set Ω . We just want to point out the possible structure of such objects.

Remark 4.2.9. Let us notice that Assertion (i) of Theorem 4.2.7 could fail when we replace $\Sigma(\mathcal{E}(0); \Omega)$ with $\Sigma(\mathcal{E}(0); \partial\Omega)$. Indeed we can always modify Ω at the boundary in order to produce a set Ω_0 having the same Cheeger N -clusters of Ω and kissing the boundary of some $\partial\mathcal{E}(i)$ in a countable number of points (see Figure 4.2.3).

Remark 4.2.10. We speak of ‘‘curvature of chambers’’ $\mathcal{E}(j)$, $\mathcal{E}(k)$, instead of curvature of interfaces $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k)$ in order to point out that the sign of the constant $C_{j,k}$ depends on

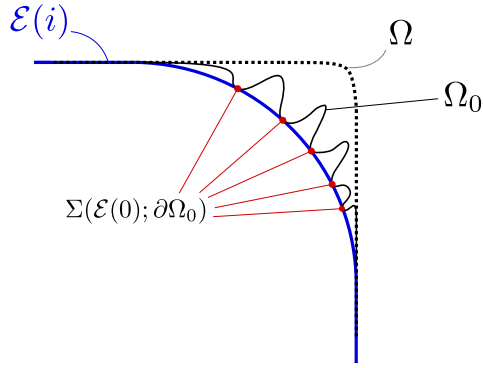


Figure 4.2.3: By gently pushing $\partial\Omega$ we can build as many contact points as we want. This proves that Assertion (i) in Theorem 4.5.4 does not hold in general for $\Sigma(\mathcal{E}(0); \partial\Omega)$.

whether we are looking at $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k)$ as a piece of the boundary of $\mathcal{E}(j)$ or as a piece of the boundary of $\mathcal{E}(k)$ (namely it depends on the direction of the unit-normal vector to $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k)$ that we choose).

Remark 4.2.11. Note that the set $E_{j,k}$ could be empty. For example, if $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega = \{x\}$ consists of a single point, thanks to our characterization (assertion (ii) Theorem 4.2.3) $x \in \Sigma(\mathcal{E}(0); \Omega)$. However, for some $k = 0, \dots, N$, $k \neq j$ it must clearly hold $\mathcal{H}^1(E_{j,k}) > 0$. The natural question is whether there exists a chamber $\mathcal{E}(j)$ such that $E_{j,k} = \emptyset$ for all $k \neq 0$. We provide a lemma (Lemma 4.5.2) that excludes this possibility whenever $\mathcal{E}(j) \subset\subset \Omega$ and this will be our starting point for proving assertion (iii) in Theorem 4.2.7.

Remark 4.2.12. Since $E_{j,k}$ is relatively open in $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k)$ we can find an open set A such that $A \cap \partial\mathcal{E}(k) \cap \partial\mathcal{E}(j) = E_{j,k}$ and conclude that $\partial\mathcal{E}(j)$ must have constant mean curvature in A (that is, on $E_{j,k}$). In the sequel we sometimes refer to the distributional mean curvature of $\mathcal{E}(i)$ on $E \subset \partial\mathcal{E}(i)$, a relatively open subset of $\partial\mathcal{E}(i)$, as the distributional mean curvature of $\mathcal{E}(i)$ inside the open set A such that $A \cap \partial\mathcal{E}(i) = E$.

Remark 4.2.13. Assertions (i) and (ii) in Theorem 4.2.7 tell us that a chamber $\mathcal{E}(j)$ has distributional curvature inside Ω equal to

$$H_{\mathcal{E}(j)}(x) = \sum_{\substack{k=0 \\ k \neq j}}^N C_{j,k} \mathbb{1}_{E_{j,k}}(x), \quad \text{for } \mathcal{H}^1\text{-almost every } x \in \partial\mathcal{E}(j) \cap \Omega .$$

Indeed, since the set $\Sigma(\mathcal{E}(0); \Omega)$ is finite, $\partial\mathcal{E}(j) \cap \Omega$ is \mathcal{H}^1 -equivalent to $\bigcup_{k \neq j} E_{j,k}$. In

particular, if $T \in C_c^\infty(\Omega; \mathbb{R}^2)$ then

$$\begin{aligned} \int_{\partial\mathcal{E}(j)\cap\Omega} \operatorname{div}_{\mathcal{E}(j)}(T) \, d\mathcal{H}^1(x) &= \sum_{\substack{k=0 \\ k \neq j}}^N \int_{\partial\mathcal{E}(j)\cap E_{j,k}\cap\Omega} \operatorname{div}_{\mathcal{E}(j)}(T) \, d\mathcal{H}^1(x) \\ &= \sum_{\substack{k=0 \\ k \neq j}}^N \int_{\partial\mathcal{E}(j)\cap E_{j,k}\cap\Omega} C_{j,k}(T \cdot \nu_{\mathcal{E}(j)})(x) \, d\mathcal{H}^1(x) \\ &= \int_{\partial\mathcal{E}(j)\cap\Omega} (T \cdot \nu_{\mathcal{E}(j)})(x) \sum_{\substack{k=0 \\ k \neq j}}^N C_{j,k} \mathbb{1}_{E_{j,k}}(x) \, d\mathcal{H}^1(x). \end{aligned}$$

We finally point out that, even if the indecomposability of the chambers is usually an hard task in the tessellation problems, in this case, thanks to a general fact for Cheeger sets (Proposition 4.5.2), we can easily achieve the proof of Assertion (iii) in Theorem 4.2.7. This will be particularly useful when focusing our attention on the asymptotic behavior of H_N .

4.3 Existence and regularity: proof of Theorem 4.2.1

We start by proving the existence and then, separately, we prove the regularity for Cheeger N -clusters. We present all the results needed to prove Theorem 4.2.1.

Theorem 4.3.1 (Existence of Cheeger N -clusters.). *Let Ω be a bounded set with finite perimeter and $0 < |\Omega|$. For every $N \in \mathbb{N}$ there exists a Cheeger N -cluster of Ω , i.e. an N -cluster $\mathcal{E} \subseteq \Omega$ such that:*

$$H_N(\Omega) = \sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|}.$$

Moreover each Cheeger N -cluster of Ω has the following properties:

$$|\mathcal{E}(i)| \geq \frac{n^n \omega_n}{2^n H_N(\Omega)^n} \quad \text{for all } i = 1, \dots, N, \quad (4.3.1)$$

$$h(\mathcal{E}(i)) = \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \quad \text{for all } i = 1, \dots, N. \quad (4.3.2)$$

Proof. Clearly $H_N(\Omega) < +\infty$ since we can always choose, for example, B_1, \dots, B_N disjoint balls such that $|B_i \cap \Omega| > 0$ and obtain

$$H_N(\Omega) \leq \sum_{i=1}^N \frac{P(B_i \cap \Omega)}{|B_i \cap \Omega|} < +\infty. \quad (4.3.3)$$

Moreover, thanks to the fact that Ω is bounded we deduce $H_N(\Omega) > 0$. Indeed for every N -cluster $\mathcal{E} \subseteq \Omega$, the isoperimetric inequality for sets of finite perimeter (1.4.1) implies

$$\sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \geq nN \left(\frac{\omega_n}{|\Omega|} \right)^{1/n}$$

hence

$$H_N(\Omega) \geq nN \left(\frac{\omega_n}{|\Omega|} \right)^{1/n} > 0.$$

Consider a minimizing sequence $\mathcal{E}^k = \{\mathcal{E}^k(i)\}_{i=1}^N$ of N -clusters such that

$$\lim_{k \rightarrow +\infty} \sum_{i=1}^N \frac{P(\mathcal{E}^k(i))}{|\mathcal{E}^k(i)|} = H_N(\Omega).$$

Note that

$$P(\mathcal{E}^k(i)) \leq |\Omega| \sum_{j=1}^N \frac{P(\mathcal{E}^k(j))}{|\mathcal{E}^k(j)|} \leq 2|\Omega|H_N(\Omega).$$

Moreover, by exploiting again the isoperimetric inequality for sets of finite perimeter (1.4.1), we provide the bound

$$n \left(\frac{\omega_n}{|\mathcal{E}^k(i)|} \right)^{\frac{1}{n}} \leq \frac{P(\mathcal{E}^k(i))}{|\mathcal{E}^k(i)|} \leq 2H_N(\Omega)$$

and thus

$$\sup_k \left\{ \max_i \left\{ P(\mathcal{E}^k(i)) \right\} \right\} \leq 2|\Omega|H_N(\Omega), \quad (4.3.4)$$

$$\inf_k \left\{ \min_i \left\{ |\mathcal{E}^k(i)| \right\} \right\} \geq \frac{n^n \omega_n}{2^n H_N(\Omega)^n}. \quad (4.3.5)$$

Thanks to the boundedness of Ω and to (4.3.4), we can apply the compactness theorem for sets of finite perimeter (Theorem 1.2.1 in Subsection 1.2.5) and deduce that, up to a subsequence, each sequence of chambers $\mathcal{E}^k(i)$ is converging in L^1 to some $\mathcal{E}(i) \subseteq \Omega$ as $k \rightarrow +\infty$. Equation (4.3.5) implies the lower bound (4.3.1) while the lower semicontinuity of distributional perimeter (Theorem 1.2.2 in Subsection 1.2.5) yields:

$$\begin{aligned} H_N(\Omega) &\leq \sum_{i=1}^N \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \leq \sum_{i=1}^N \liminf_{k \rightarrow \infty} \frac{P(\mathcal{E}^k(i))}{|\mathcal{E}^k(i)|} \\ &\leq \liminf_{k \rightarrow +\infty} \sum_{i=1}^N \frac{P(\mathcal{E}(i)^k)}{|\mathcal{E}^k(i)|} = H_N(\Omega). \end{aligned}$$

Property (4.3.2) immediately follows from minimality. \square

Remark 4.3.2. Thanks to property (4.3.2) H_N can be equivalently defined as

$$H_N(\Omega) = \left\{ \sum_{i=1}^N h(\mathcal{E}(i)) \mid \mathcal{E} \subseteq \Omega \text{ N-Cluster} \right\}. \quad (4.3.6)$$

We now show that every Cheeger N -cluster of a given open set is a (Λ, r_0) -perimeter-minimizing inside Ω that will implies immediately assertion (i), (iii), (iv), (v) in Theorem 4.2.1 by applying the regularity Theorem 1.3.2.

Note that, for proving regularity in the case of Cheeger N -clusters we have to deal with the possible non trivial components $\partial\mathcal{E}(i) \cap \partial\mathcal{E}(j)$. Roughly speaking, property (4.3.2), implies that both $\mathcal{E}(i)$ and $\mathcal{E}(j)$ must have mean curvature bounded from above. This leads us to say that the mean curvature of $\mathcal{E}(i)$ ($\mathcal{E}(j)$) on $\partial\mathcal{E}(i) \cap \mathcal{E}(j)$ must be bounded from below as well and so neither outer nor inner cusps can be attained. This approach is based on an idea from [BM82], where the authors prove a regularity result for the solutions of some obstacle problems.

Theorem 4.3.3. *Let Ω be an open bounded set and \mathcal{E} be a Cheeger N -cluster of Ω . Then there exists $\Lambda, r_0 > 0$ depending on \mathcal{E} with $\Lambda r_0 \leq 1$, such that each $\mathcal{E}(i)$ is a (Λ, r_0) -perimeter-minimizing in Ω . As a consequence, for every $i = 1, \dots, N$ the set $\Omega \cap \partial\mathcal{E}(i)$ has the regularity of Theorem 1.3.2.*

Before entering in the details of the proof of Theorem 4.3.3 let us remark that, by exploiting formula (1.2.29) contained in Subsection 1.2.9, it is possible to derive the inequality

$$P(F \setminus E; A) + P(E \setminus F; A) \leq P(F; A) + P(E; A) \quad (4.3.7)$$

holding for every couple of sets E, F of locally finite perimeter and for open set A .

In order to prove Theorem 4.3.3 we also recall the following definition. We say that a set of finite perimeter M has *distributional mean curvature less than $g \in L^1_{loc}(\Omega)$* in Ω if, there exists r_0 such that for every $B_r \subset\subset \Omega$ with $r < r_0$ and for every $L \subseteq M$ with $M \setminus L \subset\subset B_r$, it holds

$$P(M; B_r) \leq P(L; B_r) + \int_{M \setminus L} g(x) \, dx. \quad (4.3.8)$$

Proof of theorem 4.3.3. We start by fixing $i \in \{1, \dots, N\}$ and by defining

$$M_i = \bigcup_{\substack{j=1, \\ j \neq i}}^N \mathcal{E}(j).$$

We divide the proof in two steps.

Step one. We prove that each M_i has distributional mean curvature less than $H_N(\Omega)$ in Ω . Let $B_r \subset\subset \Omega$ be a ball and $L \subseteq M_i$ be a subset of finite perimeter of M_i with $M_i \setminus L \subset\subset B_r$. What we need to prove is

$$P(M_i; B_r) \leq P(L; B_r) + H_N(\Omega) |M_i \setminus L|. \quad (4.3.9)$$

Note that, up to choosing $r < r_0 = \frac{n}{4H_N(\Omega)}$ we can always assume $|\mathcal{E}(j) \cap L| > 0$ for every $j \neq i$. Indeed $M_i \setminus L \subset\subset B_r$ and, if by contradiction we assume $|\mathcal{E}(j) \cap L| = 0$ for some $j \neq i$, this would mean $\mathcal{E}(j) \subset B_r$ up to a set of measure 0 which implies (because of property (4.3.1) and thanks to the choice of r_0) :

$$\frac{n^n \omega_n}{2^n H_N(\Omega)^n} < |\mathcal{E}(j)| < \omega_n r^n < \frac{n^n \omega_n}{4^n H_N(\Omega)^n}$$

that is impossible.

By minimality it must hold:

$$\frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|} \leq \frac{P(\mathcal{E}(j) \cap L)}{|\mathcal{E}(j) \cap L|} \quad \text{for every } j \neq i,$$

that leads to:

$$\begin{aligned} \frac{P(\mathcal{E}(j); B_r) + P(\mathcal{E}(j); B_r^c)}{|\mathcal{E}(j)|} &\leq \frac{P(\mathcal{E}(j) \cap L; B_r) + P(\mathcal{E}(j); B_r^c)}{|\mathcal{E}(j)| - |\mathcal{E}(j) \setminus L|} \\ P(\mathcal{E}(j); B_r) &\leq P(\mathcal{E}(j) \cap L; B_r) + |\mathcal{E}(j) \setminus L| h(\mathcal{E}(j)). \end{aligned} \quad (4.3.10)$$

By exploiting (1.2.20) in Lemma 1.2.5 and (4.3.10) above we obtain

$$\begin{aligned}
P(M_i; B_r) &= P(\cup_{j \neq i} \mathcal{E}(j); B_r) \\
(1.2.20) \text{ in Lemma 1.2.5} &= \sum_{j \neq i} P(\mathcal{E}(j); B_r) - \sum_{k, j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r) \\
(4.3.10) &\leq \sum_{j \neq i} P(\mathcal{E}(j) \cap L; B_r) + |\mathcal{E}(j) \setminus L| h(\mathcal{E}(j)) \\
&\quad - \sum_{k, j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r) \\
&\leq \sum_{j \neq i} P(\mathcal{E}(j) \cap L; B_r) - \sum_{k, j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(j) \cap \partial^* \mathcal{E}(k) \cap B_r) \\
&\quad + H_N(\Omega) |M_i \setminus L|, \tag{4.3.11}
\end{aligned}$$

where in the last inequality we have used the formulation of H_N as in (4.3.6). By exploiting again Lemma 1.2.5 for $\{\mathcal{E}(j) \cap L\}_{j \neq i}$ we obtain

$$\begin{aligned}
P(M_i \cap L; B_r) &= \sum_{j \neq i} P(\mathcal{E}(j) \cap L; B_r) \\
&\quad - \sum_{k, j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^*(\mathcal{E}(j) \cap L) \cap \partial^*(\mathcal{E}(k) \cap L) \cap B_r). \tag{4.3.12}
\end{aligned}$$

After some quick computations, by exploiting formula (1.2.24) for the reduced boundary of the intersections and the fact that the chambers $\mathcal{E}(j)$ are disjoint (up to a set of zero Lebesgue measure), we discover that

$$\partial^*(\mathcal{E}(j) \cap L) \cap \partial^*(\mathcal{E}(k) \cap L) \cap B_r \approx L^{(1)} \cap \partial^* \mathcal{E}(k) \cap \partial^* \mathcal{E}(j) \cap B_r$$

which plugged into (4.3.12) leads to

$$\sum_{j \neq i} P(\mathcal{E}(j) \cap L; B_r) - \sum_{k, j \neq i, k \neq j} \mathcal{H}^{n-1}(\partial^* \mathcal{E}(k) \cap \partial^* \mathcal{E}(j) \cap B_r) \leq P(L; B_r), \tag{4.3.13}$$

where we have exploited also $[(M_i \cap L) \Delta L] \cap B_r = \emptyset$. By combining (4.3.13) with (4.3.11) we reach

$$P(M_i; B_r) \leq P(L; B_r) + H_N(\Omega) |M_i \setminus L|,$$

and we achieve the proof of Step one.

Step two. We now prove that $\mathcal{E}(i)$ is a (Λ, r_0) -perimeter-minimizing for a suitable choice of Λ and $r_0 < \frac{n}{4H_N(\Omega)}$ (according to Step one). Let $B_r \subset\subset \Omega$ and F be such that $F \Delta \mathcal{E}(i) \subset\subset B_r$. Define $E := F \setminus M_i$ and observe, by minimality of \mathcal{E} and by the relation $\mathcal{E}(i) \cap B_r^c = (F \setminus M_i) \cap B_r^c$, that:

$$\frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \leq \frac{P(E)}{|E|}.$$

Hence

$$\begin{aligned}
\frac{P(\mathcal{E}(i); B_r) + P(\mathcal{E}(i); B_r^c)}{|\mathcal{E}(i)|} &\leq \frac{P(F \setminus M_i; B_r) + P(F \setminus M_i; B_r^c)}{|F| - |F \cap M_i|}, \\
&\leq \frac{P(F \setminus M_i; B_r) + P(\mathcal{E}(i); B_r^c)}{|\mathcal{E}(i)| + (|F \cap B_r| - |\mathcal{E}(i) \cap B_r|) - |F \cap M_i|}.
\end{aligned}$$

If we expand the last inequality we get:

$$P(\mathcal{E}(i); B_r)|\mathcal{E}(i)| \leq P(F \setminus M_i; B_r)|\mathcal{E}(i)| + P(\mathcal{E}(i))(|F \cap M_i| + |\mathcal{E}(i) \cap B_r| - |F \cap B_r|),$$

which means (by observing that $F \cap M_i \subseteq F \setminus \mathcal{E}(i)$),

$$P(\mathcal{E}(i); B_r) \leq P(F \setminus M_i; B_r) + 2h(\mathcal{E}(i))|\mathcal{E}(i)\Delta F|. \quad (4.3.14)$$

By making use of (4.3.7) we obtain

$$P(F \setminus M_i; B_r) \leq P(F; B_r) + P(M_i; B_r) - P(M_i \setminus F; B_r) \quad (4.3.15)$$

Since $M_i \setminus F \subset M_i$ and $(M_i \setminus F)\Delta M_i \subset\subset B_r$ we can use step one (relation (4.3.9)) with $L = M_i \setminus F$ for conclude that

$$\begin{aligned} P(M_i; B_r) &\leq P(M_i \setminus F; B_r) + H_N(\Omega)|M_i \setminus (M_i \setminus F)| \\ &\leq P(M_i \setminus F; B_r) + H_N(\Omega)|M_i \cap F|. \end{aligned}$$

Hence

$$P(M_i; B_r) - P(M_i \setminus F; B_r) \leq H_N(\Omega)|F \setminus \mathcal{E}(i)|. \quad (4.3.16)$$

By plugging (4.3.16) in (4.3.15) we obtain

$$P(F \setminus M_i; B_r) \leq P(F; B_r) + H_N(\Omega)|\mathcal{E}(i)\Delta F| \quad (4.3.17)$$

and by using (4.3.17) in (4.3.14) we find

$$P(\mathcal{E}(i); B_r) \leq P(F; B_r) + 3H_N(\Omega)|\mathcal{E}(i)\Delta F|.$$

By choosing $\Lambda = 3H_N(\Omega)$ and $r_0 = \frac{1}{4H_N(\Omega)}$ we conclude that each $\mathcal{E}(i)$ is a (Λ, r_0) -perimeter-minimizing with $\Lambda r_0 < 1$ and we achieve the proof. \square

Proof of assertion (ii) can be viewed as a consequence of [LP14, Proposition 2.5, Assertion (vii)] recalled below for the sake of clarity.

Proposition 4.3.4. *Let A be an open and bounded set and let E be a Cheeger set A . Then*

$$\partial^* A \cap \partial E \subseteq \partial^* E.$$

Moreover for every $x \in \partial^* A \cap \partial E$ it holds that

$$\nu_E(x) = \nu_A(x),$$

where ν_E, ν_A denotes the measure theoretic outer unit normal to E and A respectively.

The proof follows by combining the fact that each Cheeger set E is a (Λ, r_0) -perimeter-minimizing in A with the fact that the blow-ups of ∂A at a point $x \in \partial^* A$ converge to an half plane.

Proof of Theorem 4.2.1. Assertion (i), (iii), (iv), (v) follow by combining Theorems 4.3.3 and 1.3.2. Assertion (ii) is obtained by noticing that each chambers $\mathcal{E}(i)$ is a Cheeger set for

$$A = \Omega \setminus \bigcup_{\substack{j=1, \\ j \neq i}}^N \mathcal{E}(j)$$

and then by applying Proposition 4.3.4. \square

4.4 The singular set $\Sigma(\mathcal{E})$ of Cheeger N -clusters in low dimension

The following results are all stated and proved for open bounded and connected sets $\Omega \subset \mathbb{R}^n$ having C^1 boundary and with the ambient space dimension less than 8. We ask Ω to be connected and with C^1 boundary because this is enough to avoid degenerate situations where $|\mathcal{E}(0)| = 0$ (see Remark 4.2.4 where a Cheeger N -cluster with $|\mathcal{E}(0)| = 0$ is provided).

We obtain the proof of Theorem 4.2.3 by combining different results, sated and proved separately in Subsection 4.4.2. We premise some technical lemmas.

4.4.1 Technical lemmas

Lemma 4.4.1. *If $n \leq 7$, Ω is an open, bounded, connected sets with C^1 boundary and finite perimeter and \mathcal{E} is a Cheeger N -cluster of Ω it holds*

$$\partial^* \mathcal{E}(i) = \partial \mathcal{E}(i) \quad \text{for all } i \neq 0.$$

Proof. We decompose $\partial \mathcal{E}(i)$ as

$$\partial \mathcal{E}(i) = (\partial \mathcal{E}(i) \cap \Omega) \cup (\partial \mathcal{E}(i) \cap \partial \Omega).$$

and we note that

$$\partial \mathcal{E}(i) \cap \Omega = \partial^* \mathcal{E}(i) \cap \Omega,$$

because of Assertion (iii) of 4.2.1. Moreover, since Ω has C^1 boundary $\partial^* \Omega = \partial \Omega$ and thus, thanks to Assertion (ii) we have also

$$\partial \mathcal{E}(i) \cap \partial \Omega = \partial \mathcal{E}(i) \cap \partial^* \Omega \subseteq \partial^* \mathcal{E}(i).$$

Hence

$$\partial^* \mathcal{E}(i) \subseteq \partial \mathcal{E}(i) = (\partial \mathcal{E}(i) \cap \Omega) \cup (\partial \mathcal{E}(i) \cap \partial \Omega) \subseteq \partial^* \mathcal{E}(i),$$

and we achieve the proof. \square

Remark 4.4.2. If $n \leq 7$, Ω is an open, bounded, connected set with finite perimeter and C^1 -boundary and \mathcal{E} is a Cheeger N -cluster of Ω , by considering $\mathcal{F}(k) = \mathcal{E}(k) \cup \partial \mathcal{E}(k)$ for $k \neq 0$, thanks to Lemma 4.4.1 we must have $|\mathcal{F}(k) \Delta \mathcal{E}(k)| \leq |\partial \mathcal{E}(k)| = 0$ and thus

$$P(\mathcal{F}(k)) = P(\mathcal{E}(k)).$$

For this reason in the sequel we are always assuming that each chamber $\mathcal{E}(k)$ for $k \neq 0$ is **a closed set with $\partial^* \mathcal{E}(k) = \partial \mathcal{E}(k)$.**

Lemma 4.4.3. *Let $n \leq 7$, $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in \mathbb{R}^n . If \mathcal{E} is a Cheeger N -cluster of Ω , then for every $x \in \mathbb{R}^n$ and every $k = 1, \dots, N$ there exists the n -dimensional density $\vartheta_n(x, \mathcal{E}(k))$ and it takes values:*

$$\vartheta_n(x, \mathcal{E}(k)) = \begin{cases} 0 & \text{if } x \notin \mathcal{E}(k); \\ \frac{1}{2} & \text{if } x \in \partial \mathcal{E}(k); \\ 1 & \text{if } x \in \mathring{\mathcal{E}}(k). \end{cases}$$

Proof. Each chamber $\mathcal{E}(k)$ for $k \neq 0$ is a closed set (see Remark 4.4.2) and thus

$$\begin{aligned}\mathcal{E}(k)^c &= \mathcal{E}(k)^{(0)}, \\ \mathring{\mathcal{E}}(k) &= \mathcal{E}(k)^{(1)}.\end{aligned}$$

Lemma 4.4.1 implies

$$\partial\mathcal{E}(k) = \partial^*\mathcal{E}(k) \subseteq \mathcal{E}(k)^{(\frac{1}{2})} \subseteq \partial\mathcal{E}(k).$$

□

4.4.2 Proof of theorem 4.2.3

We are now ready to prove two propositions that immediately imply Theorem 4.2.3. The following Proposition is needed in order to prove Assertion (i) in Theorem 4.2.3.

Proposition 4.4.4. *Let $n \leq 7$, $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in \mathbb{R}^n . Let \mathcal{E} be a Cheeger N -cluster of Ω . Then for every $i \in \{1, \dots, N\}$ there exists $x \in \partial\mathcal{E}(i)$ such that $|B_s(x) \cap \mathcal{E}(0)| > 0$ for all $s > 0$.*

Proof. Without loss of generality (and for the sake of clarity) we can assume $i = 1$. We note that the proof of the lemma is a consequence of the following claim.

$$\text{Claim. } \partial\mathcal{E}(1) \setminus \left[\partial\Omega \cup \bigcup_{k=2}^N \partial\mathcal{E}(k) \right] \neq \emptyset.$$

Indeed, if the claim is in force then there exists $x \in \partial\mathcal{E}(1) \cap \Omega$ and $x \notin \mathcal{E}(k)$ for all $k \neq 1$. Since the chambers are closed we can also find a small ball $B_s(x) \subset\subset \Omega$ such that $B_s(x) \cap \mathcal{E}(k) = \emptyset$ for all $k \neq 1$, implying (thanks to Lemma 4.4.3)

$$|\mathcal{E}(0) \cap B_s(x)| = |B_s| - \sum_{k=1}^n |\mathcal{E}(k) \cap B_s| = |B_s| - |\mathcal{E}(1) \cap B_s| > 0$$

(because $x \in \partial\mathcal{E}(1) = \mathcal{E}(1)^{(\frac{1}{2})}$) and achieving the proof.

Let us focus on the proof of the claim. Thanks to the connectedness of Ω it is easy to show that $\partial\mathcal{E}(1) \setminus \partial\Omega \neq \emptyset$. If also $\partial\mathcal{E}(1) \cap \partial\mathcal{E}(k) = \emptyset$ for $k \neq 1$ the claim trivially holds. Otherwise it must exist at least an index $j \in \{2, \dots, N\}$ such that $\partial\mathcal{E}(1) \cap \partial\mathcal{E}(j) \neq \emptyset$. Assume without loss of generality $j = 2$:

$$\partial\mathcal{E}(1) \cap \partial\mathcal{E}(2) \neq \emptyset.$$

Choose $x \in \partial\mathcal{E}(1) \cap \partial\mathcal{E}(2)$ and let us denote by M the connected component of $\partial\mathcal{E}(1)$ containing x . Note that $x \notin \partial\Omega$. Otherwise we would have $x \in \partial\mathcal{E}(1) \cap \partial\mathcal{E}(2) \cap \partial\Omega$ and because of the regularity of Ω and thanks to Lemma 4.4.3 this leads to a contradiction:

$$\begin{aligned}\frac{1}{2} &= \lim_{r \rightarrow 0^+} \frac{|\Omega \cap B_r(x)|}{|B_r(x)|} = \lim_{r \rightarrow 0^+} \sum_{h=0}^N \frac{|\mathcal{E}(h) \cap B_r(x)|}{|B_r(x)|} \\ &\geq \lim_{r \rightarrow 0^+} \frac{|\mathcal{E}(1) \cap B_r(x)|}{|B_r(x)|} + \frac{|\mathcal{E}(2) \cap B_r(x)|}{|B_r(x)|} = 1.\end{aligned}$$

Hence the following are in force:

$$M \setminus \partial\Omega \neq \emptyset, \quad M \cap \partial\mathcal{E}(2) \neq \emptyset. \tag{4.4.1}$$

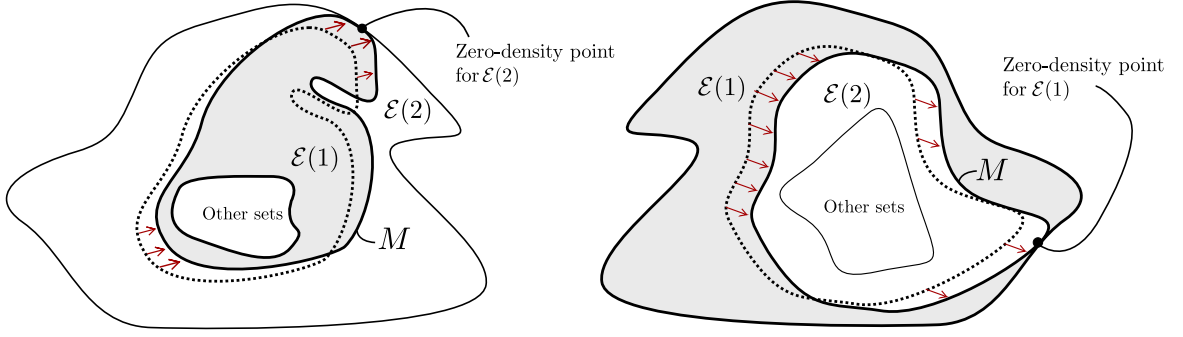


Figure 4.4.1: If (4.4.2) holds, then one of these two situations must be in force and we can contradict regularity by simply translate M until it kisses another part of the boundary yielding a not allowed point of density zero.

We now note that, if

$$M \setminus \left[\partial\Omega \cup \bigcup_{k=2}^N \partial\mathcal{E}(k) \right] = \emptyset \quad (4.4.2)$$

then, necessarily $M \subseteq \partial\mathcal{E}(2) \cap \Omega$. Indeed considered

$$y \in \overline{M \cap \partial\mathcal{E}(2)} \cap \overline{(M \setminus \partial\mathcal{E}(2))} = \text{bd}_M(M \cap \partial\mathcal{E}(2)),$$

since (4.4.2) is in force (and since $y \in \text{bd}_M(M \cap \partial\mathcal{E}(2))$) either there exists an index $k \neq 1, 2$ such that $y \in \partial\mathcal{E}(k)$ or $y \in \partial\Omega$. In both cases we reach a contradiction because y would be a point of density $\frac{1}{2}$ for three disjoint sets ($\mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(k)$ or $\mathcal{E}(1), \mathcal{E}(2), \Omega^c$). Thus the only possibility is that $\text{bd}_M(M \cap \partial\mathcal{E}(2)) = \emptyset$ and since (4.4.1) is in force, by applying the following (topological) fact (4.4.3) we conclude that $M = M \cap \partial\mathcal{E}(2) \subseteq \partial\mathcal{E}(2)$.

$$\text{If } M \subset \mathbb{R}^n \text{ is a closed connected set and } C \subseteq M \text{ is a non empty subset of } M, \text{ then } \text{bd}_M(C) := \overline{C} \cap \overline{(M \setminus C)} = \emptyset \text{ if and only if } M = C. \quad (4.4.3)$$

As before $M \cap \partial\mathcal{E}(2) \cap \partial\Omega = \emptyset$ otherwise we would have a point of density $\frac{1}{2}$ for three disjoint set ($\mathcal{E}(1), \mathcal{E}(2), \Omega^c$) and hence $M \subseteq \partial\mathcal{E}(2) \cap \Omega$. This means that M must be a closed $C^{1,\alpha}$ surface without boundary contained in Ω and disjoint from the other sets $\mathcal{E}(k)$ and from $\partial\Omega$, which means that one of the situation of Figure 4.4.1 has to be in force. We are thus able to move a little bit M , and whatever is bounded by M , inside Ω as in Figure 4.4.1 until it kisses $\partial\mathcal{E}(2)$ or $\partial\mathcal{E}(1)$ (we easily exclude that M bounds a hole of Ω with a slight variation of this previous argument). In this way we produce a zero-density point for $\mathcal{E}(1)$ or for $\mathcal{E}(2)$ without changing $\sum_j \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|}$ and this contradicts the regularity.

Hence (4.4.2) cannot holds and the claim is true. □

The next Proposition implies Assertion (ii) in Theorem 4.2.3.

Proposition 4.4.5. *Let $n \leq 7$, $N \geq 2$ and Ω be an open, bounded and connected set with*

C^1 boundary and finite perimeter in \mathbb{R}^n . Let \mathcal{E} be a Cheeger N -cluster of Ω . Then

$$\begin{aligned}\Sigma(\mathcal{E}; \Omega) &= \Sigma(\mathcal{E}(0); \Omega) \\ &= \{x \in \partial\mathcal{E}(0) \cap \Omega \mid \vartheta_n(x, \mathcal{E}(0)) = 0\} \\ &= \Omega \cap \bigcup_{\substack{i,j=1, \\ i \neq j}}^N \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0).\end{aligned}$$

Proof. Thanks to Proposition 4.4.4 the set $\mathcal{E}(0)$ is not empty. As pointed out in Remark 4.2.6, thanks to the regularity of each chambers, it is immediate that $\Sigma(\mathcal{E}; \Omega) = \Sigma(\mathcal{E}(0); \Omega)$. Let us denote (for the sake of brevity) by

$$\begin{aligned}\Sigma_0 &= \Sigma(\mathcal{E}(0); \Omega), \\ A &= \{x \in \partial\mathcal{E}(0) \cap \Omega \mid \vartheta_n(x, \mathcal{E}(0)) = 0\} \\ B &= \Omega \cap \bigcup_{\substack{i,j=1, \\ k \neq j}}^N \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0).\end{aligned}$$

We note that $B \subseteq A$ is immediate and also $A \subseteq \Sigma_0$ is immediate, since if $x \in A$ then $x \notin \mathcal{E}(0)^{\left(\frac{1}{2}\right)} \supseteq \partial^*\mathcal{E}(0)$. We are left to show that $\Sigma_0 \subseteq B$. In order to do this we define the following family of subsets of Ω .

$$E_i := \mathring{\mathcal{E}}(i) \quad \text{for all } 0 \leq i \leq N \quad (4.4.4)$$

$$F_{i,j} := \Omega \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \setminus \left(\bigcup_{\substack{k=0, \\ k \neq i,j}}^N \partial\mathcal{E}(k) \right), \quad \text{for all } 0 \leq i < j \leq N \quad (4.4.5)$$

$$G_{i,j} := \Omega \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0), \quad \text{for all } 1 \leq i < j \leq N. \quad (4.4.6)$$

It is easy to verify that the Borel sets defined in (4.4.4),(4.4.5),(4.4.6) form a partition of Ω . Now, for a given point $x \in \Sigma_0$, clearly $x \notin E_i$ for all $i = 0, \dots, N$. Thus either $x \in F_{i,j}$ for some $0 \leq i < j \leq N$ or $x \in G_{i,j}$ for some $1 \leq i < j \leq N$. If $x \in F_{i,j}$, by closedness there exists a small ball $B_s(x)$ such that $\partial\mathcal{E}(k) \cap B_s(x) = \emptyset$ for all $k \neq i, j$. This implies that either $i = 0$ or $j = 0$ (since we have chosen $x \in \Sigma_0 \subset \partial\mathcal{E}(0)$) and that $\partial\mathcal{E}(0) \cap B_s(x) = \partial\mathcal{E}(j) \cap B_s(x)$ leading to say that $\partial\mathcal{E}(0)$ must be regular in a small neighborhood of x and contradicting $x \in \Sigma_0$. Hence necessarily $x \in G_{i,j}$ for some $1 \leq i < j \leq N$ and we achieve the proof: $\Sigma_0 \subseteq B$. \square

The following corollary is an easy consequence of Proposition 4.4.5.

Corollary 4.4.6. *Let $n \leq 7$, $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in \mathbb{R}^n . If \mathcal{E} is a Cheeger N -cluster for Ω , then $\Sigma(\mathcal{E}(0); \Omega)$ is closed.*

Proof. Thanks to Proposition 4.4.5 we have that

$$\Sigma(\mathcal{E}; \Omega) = \Omega \cap \bigcup_{\substack{i,j=1, \\ i \neq j}}^N \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0). \quad (4.4.7)$$

Let $\{x_k\}_{k \in \mathbb{N}} \subseteq \Sigma(\mathcal{E}; \Omega)$ such that $x_k \rightarrow x$. Up to extract a subsequence we have that $\{x_k\}_{k \in \mathbb{N}} \subset \Omega \cap \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0)$ for some $1 \leq i < j \leq N$ (since (4.4.7) is in force). By closedness we obtain $x \in \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\mathcal{E}(0)$ and we need to prove that $x \in \Omega$. If $x \in \partial\Omega$ we have $x \in \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial\Omega$ which is a contradiction since x would be a point of density $\frac{1}{2}$ for three disjoint sets $\mathcal{E}(1), \mathcal{E}(2), \Omega^c$. Hence $x \in \Omega$ and thus $x \in \Sigma(\mathcal{E}(0); \Omega)$. \square

The proof of Theorem 4.2.3 is now obtained as an easy consequence of the previous results.

Proof of Theorem 4.2.3. Follows by Propositions 4.4.4, 4.4.5 and by Corollary 4.4.6. \square

4.5 The planar case

As in the previous sections, the proof of Theorem 4.2.7 is attained by combining different results that we state and prove in Subsection 4.5.2. We premise some technical lemmas.

4.5.1 Technical lemmas

Lemma 4.5.1. *Let $n \leq 7$, $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in \mathbb{R}^n . Let \mathcal{E} be a Cheeger N -cluster for Ω . If E is an indecomposable component of $\mathcal{E}(0)$ such that $E \subset\subset \Omega$, then there exist at least three different indexes $i, j, k \neq 0$ such that $\partial E \cap \mathcal{E}(i) \neq \emptyset$, $\partial E \cap \mathcal{E}(j) \neq \emptyset$, and $\partial E \cap \mathcal{E}(k) \neq \emptyset$. In particular, E shares boundary at least with three different chambers.*

Proof. Let E be a generic indecomposable component of $\mathcal{E}(0)$. Assume that E shares its boundary with exactly two other different chambers $j, k \geq 1$ and $\partial E \cap \partial\Omega = \emptyset$. Then either

$$a) \quad \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(j)) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(k)),$$

or

$$b) \quad \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(k)) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(j))$$

hold. Assume that $a)$ holds and define $\mathcal{E}_1(j) := \mathcal{E}(j) \cup E$, $\mathcal{E}_1(i) = \mathcal{E}(i)$ for $i \neq j$. Since

$$P(\mathcal{E}_1(j)) = P(\mathcal{E}(j)) - \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(j)) + \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(k))$$

we obtain:

$$\begin{aligned} H_N(\Omega) &\leq \sum_{i=1}^N \frac{P(\mathcal{E}_1(i))}{|\mathcal{E}_1(i)|} \\ &= \frac{P(\mathcal{E}_1(j))}{|\mathcal{E}_1(j)|} + \sum_{i \neq j} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \\ &= \frac{P(\mathcal{E}(j)) - \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(j)) + \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{E}(k))}{|\mathcal{E}(j)| + |E|} + \sum_{i \neq j} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \\ &\leq \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)| + |E|} + \sum_{i \neq j} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|}. \end{aligned}$$

If $|E| > 0$ we are led to $H_N(\Omega) < H_N(\Omega)$ which is a contradiction, so $|E| = 0$. Since E is open (because $\mathcal{E}(0)$ is open), then $E = \emptyset$. If E shares its boundary with exactly one chamber we argue in the same way. We have discovered that every decomposable component of $\mathcal{E}(0)$ that shares boundary with exactly one or two chambers is empty. That complete the proof. \square

Lemma 4.5.2. *Let E be a Cheeger set of an open bounded set $A \subset \mathbb{R}^2$. Assume that the following properties hold for E :*

- 1) $\#(\text{bd}_{\partial A}(\partial A \cap \partial E)) < +\infty$,
- 2) every $x \in \partial A \cap \partial E$ is a regular point for ∂A , namely $x \in \partial A \cap \partial E \subseteq \partial^* A$;

where

$$\text{bd}_{\partial A}(\partial A \cap \partial E) = [\partial A \cap \partial E] \cap \overline{[\partial A \setminus \partial E]}.$$

Then $\mathcal{H}^1(\partial E \cap \partial A) > 0$.

Remark 4.5.3. It seems that it is possible to generalize Lemma 4.5.2 to dimension $n \geq 2$ by making use of Alexandrov's Theorem [Ale62] for the characterization of the Constant Mean Curvature (CMC) embedded hypersurface in \mathbb{R}^n . In this (more technical) framework hypothesis 1) can be weakened. Anyway, since we do not have to deal (at least here) with $n \geq 2$ and since for our purposes Lemma 4.5.2 is all we need to complete the proof of Theorem 4.2.7 we decide to not add this generalization.

Proof of Lemma 4.5.2. Assume by contradiction that $\mathcal{H}^1(\partial E \cap \partial A) = 0$. In this case

$$\text{bd}_{\partial A}(\partial A \cap \partial E) = \partial E \cap \partial A.$$

Let F be an indecomposable component of E and note that, since E is a Cheeger set for A it must hold

$$\frac{P(F)}{|F|} = h(A). \quad (4.5.1)$$

Set

$$M = \text{bd}_{\partial A}(\partial A \cap \partial F)$$

and $\#(M) = k < +\infty$. The well-known regularity theory for Cheeger sets, combined with the fact that $k < +\infty$ tells us that

$$\partial F \cap A = \bigcup_{i=1}^k \alpha_i$$

where each α_i is a piece of the boundary of a suitable ball B_i (relatively open inside ∂B_i) of radius $\frac{1}{h(A)}$. The finiteness of M implies that for a suitably small r it holds $B_r(x) \cap M = \{x\}$ for all $x \in M$ and this means that for every $x \in M$ there exists two arcs α_i, α_j (with possibly $i = j$) such that $x \in \overline{\alpha_i} \cap \overline{\alpha_j}$.

Let B_i, B_j the balls from which such arcs come from: $\alpha_i \in \partial B_i, \alpha_j \in \partial B_j$. Hypothesis 2) implies that the outer unit normal to B_i and to B_j at x must coincide with $\nu_A(x)$ and hence the balls B_i and B_j must coincide as well. Since $k < +\infty$, by iterating this argument we conclude that there exists only one ball B of radius $\frac{1}{h(A)}$ such that $M \subset \partial B$ and $\partial F \cap A = \partial B \cap A$. In particular ∂F is equal to ∂B and by exploiting (4.5.1) we bump

into a contradiction

$$\begin{aligned}
\frac{P(F)}{|F|} &= h(A) \\
\frac{P(B)}{|B|} &= h(A) \\
\frac{\frac{2\pi}{h(A)}}{\frac{\pi}{h(A)^2}} &= h(A) \\
\frac{\frac{2}{h(A)}}{\frac{1}{h(A)^2}} &= h(A) \\
2h(A) &= h(A).
\end{aligned}$$

The contradiction comes from the fact that we have assumed $\mathcal{H}^1(\partial E \cap \partial A) = 0$, hence $\mathcal{H}^1(\partial E \cap \partial A) > 0$ and the proof is complete. \square

4.5.2 Proof of Theorem 4.2.7

Proposition 4.5.4. *Let $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in the plane. Let \mathcal{E} be a Cheeger N -cluster of Ω . Then $\Sigma(\mathcal{E}; \Omega) = \Sigma(\mathcal{E}(0); \Omega)$ is a finite union of points.*

Proof. We prove that $\Sigma(\mathcal{E}(0); \Omega)$ has no accumulation point. In this way we show that $\Sigma(\mathcal{E}(0); \Omega)$ is a closed (thanks to Theorem 4.2.3), bounded set of \mathbb{R}^2 (since Ω is bounded) without accumulation points which means that $\Sigma(\mathcal{E}(0); \Omega)$ must be a finite union of points.

Set $\Sigma_0 = \Sigma(\mathcal{E}(0); \Omega)$ for the sake of brevity. Let $\xi \in \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j)$ for some $1 \leq i < j \leq N$ that without loss of generality we assume to be $i = 1, j = 2$. We can assume (up to a translation) also that $\xi = (0, 0)$. Since $\partial\mathcal{E}(1), \partial\mathcal{E}(2)$ are regular up to a rotation we can find a small closed cube

$$Q_\varepsilon := [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \subset\subset \Omega$$

centered at $\xi = (0, 0)$ and two C^1 functions $f_1, f_2 : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$ such that $f_1(x) \leq f_2(x)$ for all $x \in [-\varepsilon, \varepsilon]$ and:

$$\begin{aligned}
\mathcal{E}(1) \cap Q_\varepsilon &= \{(x, y) \in Q_\varepsilon \mid -\varepsilon \leq y \leq f_1(x)\}, \\
\partial\mathcal{E}(1) \cap Q_\varepsilon &= \{(x, f_1(x)) \mid x \in [-\varepsilon, \varepsilon]\}, \\
\mathcal{E}(2) \cap Q_\varepsilon &= \{(x, y) \in Q_\varepsilon \mid f_2(x) \leq y \leq \varepsilon\}, \\
\partial\mathcal{E}(2) \cap Q_\varepsilon &= \{(x, f_2(x)) \mid x \in [-\varepsilon, \varepsilon]\} \\
\partial\mathcal{E}(2) \cap \partial\mathcal{E}(1) \cap Q_\varepsilon &= \{(x, y) \in Q_\varepsilon \mid y = f_1(x) = f_2(x) \leq \varepsilon\}, \\
\mathcal{E}(0) \cap Q_\varepsilon &= \{(x, y) \in Q_\varepsilon \mid -\varepsilon \leq f_1(x) < y < f_2(x) \leq \varepsilon\}, \\
\mathcal{E}(k) \cap Q_\varepsilon &= \emptyset \text{ for all } k \geq 3,
\end{aligned}$$

(see Figure 4.5.1). Since the blow-up of $\partial\mathcal{E}(1) \cap \partial\mathcal{E}(2)$ at $\xi = (0, 0)$ is a line, up to further decrease ε , we can also assume that $\mathcal{E}(1) \cap Q_\varepsilon$ and $\mathcal{E}(2) \cap Q_\varepsilon$ are indecomposable, which is equivalent to say:

$$|f_1(x)| < \varepsilon, \quad |f_2(x)| < \varepsilon \quad \forall x \in [-\varepsilon, \varepsilon].$$

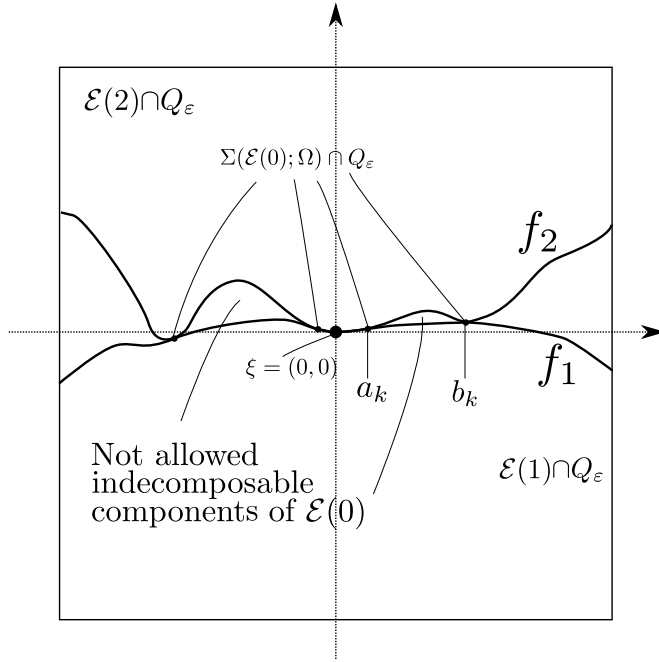


Figure 4.5.1: This kind of behavior contradicts the minimality property of \mathcal{E} , in particular it contradicts Lemma 4.5.1.

We consider the set

$$E_0 := \{x \in [-\varepsilon, \varepsilon] \mid f_1(x) < f_2(x)\}.$$

which is relatively open inside $[-\varepsilon, \varepsilon]$ (is the counter-image of the open set $(-2\varepsilon, 0)$ through the continuous function $f_1 - f_2$). Hence, E_0 must be the union of countably many disjoint (open) intervals:

$$E_0 = [-\varepsilon, a) \cup (b, \varepsilon] \cup \left(\bigcup_{k=2}^{+\infty} (a_k, b_k) \right)$$

for $\{a_k\}_{k=1}^{+\infty}, \{b_k\}_{k=1}^{+\infty} \subset [-\varepsilon, \varepsilon]$ (with a slight abuse of notation we are allowing also the possible cases $a = -\varepsilon, b = \varepsilon$ or even $a = a_1, b_\infty = b$ as in Figure 4.5.1). It is immediate that each

$$A_k := \{(x, y) \in Q_\varepsilon \mid a_k < x < b_k, f_1(x) < y < f_2(x)\}$$

is an indecomposable component of $\mathcal{E}(0)$. Observe that $E_k \subset\subset Q_\varepsilon \subset\subset \Omega$ is an indecomposable component of $\mathcal{E}(0)$ that share its boundary with exactly two chambers ($\mathcal{E}(1), \mathcal{E}(2)$) and hence contradicts Lemma 4.5.1. This means that the only possibility is

$$E_0 = [-\varepsilon, a) \cup (b, \varepsilon]$$

for some $a, b \in [-\varepsilon, \varepsilon]$. By possibly decreasing ε we can assume that $(-\varepsilon, f_1(-\varepsilon)), (\varepsilon, f_1(\varepsilon)) \notin \Sigma_0 \cap Q_\varepsilon$. The only possibilities remained are

- 1) $a = -\varepsilon$ and $b = \varepsilon$, thus $\Sigma_0 \cap Q_\varepsilon = \emptyset$;
- 2) $a \neq -\varepsilon$ and $b = \varepsilon$, thus $\Sigma_0 \cap Q_\varepsilon = \{(a, f_1(a))\} = \{(a, f_2(a))\}$;
- 3) $a = -\varepsilon$ and $b \neq \varepsilon$, thus $\Sigma_0 \cap Q_\varepsilon = \{(b, f_1(b))\} = \{(b, f_2(b))\}$;

4) $a \neq -\varepsilon$ and $b \neq \varepsilon$, thus $\Sigma_0 \cap Q_\varepsilon = \{(a, f_1(a)), (b, f_1(b))\} = \{(a, f_2(a)), (b, f_2(b))\}$.

In particular $\#(\Sigma_0 \cap Q_\varepsilon) \leq 2$ which means that Σ_0 has no accumulation points. \square

We now exploit the stationarity of Cheeger N -clusters in order to derive information on their structure.

Proposition 4.5.5. *Let $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in the plane. Let \mathcal{E} be a Cheeger N -cluster of Ω . For every $j, k = 0, \dots, N$, $k \neq j$ the set*

$$E_{j,k} := [\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega] \setminus \Sigma(\mathcal{E}(0); \Omega)$$

is relatively open in $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega$ and is the finite union of segments and circular arcs. Moreover the set $\mathcal{E}(j)$ has constant curvature $C_{j,k}$ on each open set A such that $A \cap \partial\mathcal{E}(j) \subseteq E_{j,k}$. The constant $C_{j,k}$ is equal to:

$$C_{j,k} = \begin{cases} \frac{|\mathcal{E}(k)|h(\mathcal{E}(j)) - |\mathcal{E}(j)|h(\mathcal{E}(k))}{|\mathcal{E}(j)| + |\mathcal{E}(k)|}, & \text{if } k \neq 0 \\ h(\mathcal{E}(j)), & \text{if } k = 0. \end{cases} \quad (4.5.2)$$

As a consequence the set $\mathcal{E}(k)$ has constant curvature $C_{k,j} = -C_{j,k}$ on each open set A such that $A \cap \partial\mathcal{E}(k) \subseteq E_{k,j} (= E_{j,k})$.

Proof. If $k = 0$ (or $j = 0$) we just notice that $\mathcal{E}(j)$ is a Cheeger set for

$$\Omega_0 = \Omega \setminus \bigcup_{\substack{i=1, \\ i \neq j}}^N \mathcal{E}(i)$$

so the free boundary $E_{j,0}$ is the finite union of segments and circular arcs and $\mathcal{E}(j)$ has constant mean curvature $C_{j,0} = h(\mathcal{E}(j))$ on each open set A such that $A \cap \partial^*\mathcal{E}(j) \subseteq E_{j,0}$.

Thus, we consider a couple $j, k \in \{1, \dots, N\}$ such that

$$[\partial\mathcal{E}(i) \cap \partial\mathcal{E}(k) \cap \Omega] \setminus \Sigma(\mathcal{E}(0); \Omega) \neq \emptyset$$

(otherwise there is nothing to prove and the proposition is trivial). The set $\Sigma(\mathcal{E}(0); \Omega)$ is closed and is the finite union of points (thanks to Lemma 4.5.4). Hence

$$E_{j,k} := [\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega] \setminus \Sigma(\mathcal{E}(0); \Omega)$$

is relatively open in $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap \Omega$. For every $x \in E_{j,k}$, by closedness, there exists a ball $B_r(x)$ such that

$$B_r(x) \cap \mathcal{E}(i) = \emptyset \quad \forall i = 1, \dots, N, \quad i \neq j, k.$$

Note that, up to further decrease the value of r it must hold as well

$$B_r(x) \cap \mathcal{E}(0) = \emptyset.$$

Indeed if this is not the case, we would have that $x \in \partial\mathcal{E}(0) \cap \mathcal{E}(0)^{(0)}$ and thus (thanks to Proposition 4.4.5) $x \in \Sigma(\mathcal{E}(0); \Omega)$ which is a contradiction since $x \in E_{j,k}$. Hence, because of the minimality of \mathcal{E} , the set $\partial\mathcal{E}(j) \cap \partial\mathcal{E}(k) \cap B_r(x)$ must solve an isoperimetric problem

with volume constraint inside $B_r(x)$ and by exploiting stationarity it is possible to prove that each solution to an isoperimetric problem with volume constraint must be an analytic constant mean curvature hypersurface ([Mag12, Theorems 17.20, 24.4]). Set $C_{j,k}$ and $C_{k,j}$ to be respectively the value of the mean curvature of $\mathcal{E}(j)$ and of $\mathcal{E}(k)$ in $B_r(x)$. Observe that, since $\mathcal{E}(k) \cap B_r = \mathcal{E}(j)^c \cap B_r$ it holds trivially that $C_{k,j} = -C_{j,k}$. Let us compute the (constant) value of $C_{j,k}$.

Consider a map $T \in C_c^\infty(B_r; \mathbb{R}^2)$, define for all $|t| < \varepsilon$ the diffeomorphism $f_t(y) = y + tT(y)$ and the cluster $\mathcal{E}_t := \{f_t(\mathcal{E}(i))\}_{i=1}^N$. Of course, for t suitably small, $\mathcal{E}_t \Delta \mathcal{E} \subset \subset B_r$. Note that $\{f_t \mid -\varepsilon < t < \varepsilon\}$ is a local variation in B_r and that T is the initial velocity (according to the definitions given in Subsection 1.2.11). By minimality it holds:

$$\frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|} + \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|} \leq \frac{P(\mathcal{E}_t(j))}{|\mathcal{E}_t(j)|} + \frac{P(\mathcal{E}_t(k))}{|\mathcal{E}_t(k)|}, \quad \forall |t| < \varepsilon.$$

Thus

$$0 \leq \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(j))}{|\mathcal{E}_t(j)|} + \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(k))}{|\mathcal{E}_t(k)|}. \quad (4.5.3)$$

With some easy computations

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(j))}{|\mathcal{E}_t(j)|} &= \frac{|\mathcal{E}(j)| \frac{d}{dt} \Big|_{t=0} P(\mathcal{E}_t(j)) - P(\mathcal{E}(j)) \frac{d}{dt} \Big|_{t=0} |\mathcal{E}_t(j)|}{|\mathcal{E}(j)|^2}, \\ \frac{d}{dt} \Big|_{t=0} P(\mathcal{E}_t(j)) &= C_{i,k} \int_{\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \cap B_r} (T(y) \cdot \nu_{\mathcal{E}(j)}(y)) \, d\mathcal{H}^1(y) \\ \frac{d}{dt} \Big|_{t=0} |\mathcal{E}_t(j)| &= \int_{\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \cap B_r} (T(y) \cdot \nu_{\mathcal{E}(j)}(y)) \, d\mathcal{H}^1(y). \end{aligned}$$

where we have used formulas (1.2.36) and (1.2.34) combined with the fact that the mean curvature exists and it is constantly equal to $C_{j,k}$ in B_r (and hence on $\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \cap B_r$). By denoting with

$$\begin{aligned} f_j &= \int_{\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \cap B_r} (T(y) \cdot \nu_{\mathcal{E}(j)}(y)) \, d\mathcal{H}^1(y) \\ f_k &= \int_{\partial \mathcal{E}(j) \cap \partial \mathcal{E}(k) \cap B_r} (T(y) \cdot \nu_{\mathcal{E}(k)}(y)) \, d\mathcal{H}^1(y), \end{aligned}$$

we can write:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(j))}{|\mathcal{E}_t(j)|} &= \frac{|\mathcal{E}(j)| f_j C_{j,k} - P(\mathcal{E}(j)) f_j}{|\mathcal{E}(j)|^2}, \\ \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(k))}{|\mathcal{E}_t(k)|} &= \frac{|\mathcal{E}(k)| f_k C_{k,j} - P(\mathcal{E}(k)) f_k}{|\mathcal{E}(k)|^2}, \end{aligned}$$

that plugged into (4.5.3), by observing that $f_j = -f_k$, lead to the relation:

$$\begin{aligned} 0 &\leq \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(j))}{|\mathcal{E}_t(j)|} + \frac{d}{dt} \Big|_{t=0} \frac{P(\mathcal{E}_t(k))}{|\mathcal{E}_t(k)|} \\ &= \frac{|\mathcal{E}(j)| f_j C_{j,k} - P(\mathcal{E}(j)) f_j}{|\mathcal{E}(j)|^2} + \frac{|\mathcal{E}(k)| f_k C_{k,j} - P(\mathcal{E}(k)) f_k}{|\mathcal{E}(k)|^2} \\ &= f_j \left[\frac{|\mathcal{E}(j)| C_{j,k} - P(\mathcal{E}(j))}{|\mathcal{E}(j)|^2} - \frac{|\mathcal{E}(k)| C_{k,j} - P(\mathcal{E}(k))}{|\mathcal{E}(k)|^2} \right]. \end{aligned}$$

By choosing a T_1 such that f_j is positive and then a T_2 such that f_j is negative we conclude that

$$0 = \frac{|\mathcal{E}(j)|C_{j,k} - P(\mathcal{E}(j))}{|\mathcal{E}(j)|^2} - \frac{|\mathcal{E}(k)|C_{k,j} - P(\mathcal{E}(k))}{|\mathcal{E}(k)|^2}.$$

Finally, by exploiting $C_{j,k} = -C_{k,j}$ we reach

$$\begin{aligned} 0 &= \frac{C_{j,k}}{|\mathcal{E}(j)|} - \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|^2} - \frac{C_{k,j}}{|\mathcal{E}(k)|} + \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|^2} \\ &= \frac{C_{j,k}}{|\mathcal{E}(j)|} - \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|^2} + \frac{C_{j,k}}{|\mathcal{E}(k)|} + \frac{P(\mathcal{E}(k))}{|\mathcal{E}(k)|^2}, \end{aligned}$$

that can be re-arranged as:

$$\begin{aligned} C_{i,k}(|\mathcal{E}(j)| + |\mathcal{E}(k)|) &= h(\mathcal{E}(j))|\mathcal{E}(k)| - h(\mathcal{E}(k))|\mathcal{E}(j)|, \\ C_{j,k} &= \frac{h(\mathcal{E}(j))|\mathcal{E}(k)| - h(\mathcal{E}(k))|\mathcal{E}(j)|}{|\mathcal{E}(j)| + |\mathcal{E}(k)|}. \end{aligned}$$

In particular, since $C_{j,k}$ do not depend on $x \in E_{j,k}$ and since the ambient space dimension is $n = 2$, $E_{j,k}$ must be a finite union of circular arcs or segments with curvature $|C_{j,k}|$. \square

Our last proposition of the section put together Lemma 4.5.2 Proposition 4.5.4 and Proposition 4.5.5 and tells us that the interior chambers of a Cheeger N -cluster are always indecomposable. We are making strong use of Proposition 4.5.4 which does not hold on $\partial\Omega$ (see Figure 4.2.3 and Remark 4.2.9) and that is why we cannot extend the proof of the Proposition 4.5.6 to all the chambers.

Proposition 4.5.6. *Let $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in the plane. Let \mathcal{E} be a Cheeger N -cluster for Ω . Then, every chamber $\mathcal{E}(i) \subset\subset \Omega$ for $i \neq 0$ is indecomposable.*

Proof. Assume, without loss of generality $i = 1$ and let E_1 and E_2 be two different components of $\mathcal{E}(1)$. By minimality it must hold

$$\frac{P(E_1)}{|E_1|} = \frac{P(E_2)}{|E_2|} = \frac{P(\mathcal{E}(1))}{|\mathcal{E}(1)|}. \quad (4.5.4)$$

The component E_2 is a Cheeger set for

$$A = \left(\bigcup_{j \neq 1} \Omega \setminus \mathcal{E}(j) \right) \cup E_1$$

and by Theorem 4.2.1, every $x \in \partial E_2 \cap \partial A$ is a regular point for ∂A . Moreover $\text{bd}_{\partial A}(\partial A \cap \partial E_2) \subseteq \Sigma(\mathcal{E}(0); \Omega)$ (since $\mathcal{E}(i) \subset\subset \Omega$) and thus, thanks to Proposition 4.5.4, we have

$$\#(\text{bd}_{\partial A}(\partial A \cap \partial E_2)) \leq \#(\Sigma(\mathcal{E}(0); \Omega)) < +\infty.$$

Therefore we can exploit Lemma 4.5.2 on E_2 and conclude that $\mathcal{H}^1(\partial E_2 \cap \partial A) > 0$. In particular we deduce that there exists an index $k \neq 0, 1$ such that $\mathcal{H}^1(\partial E_2 \cap \partial \mathcal{E}(k)) > 0$. Define the new cluster $\mathcal{F}(1) = E_1$, $\mathcal{F}(j) = \mathcal{E}(j)$ for $j \neq 1$ (see Figure 4.5.2). Thanks to (4.5.4) it holds:

$$H_N(\Omega) = \sum_{i=1}^N \frac{P(\mathcal{F}(i))}{|\mathcal{F}(i)|}. \quad (4.5.5)$$

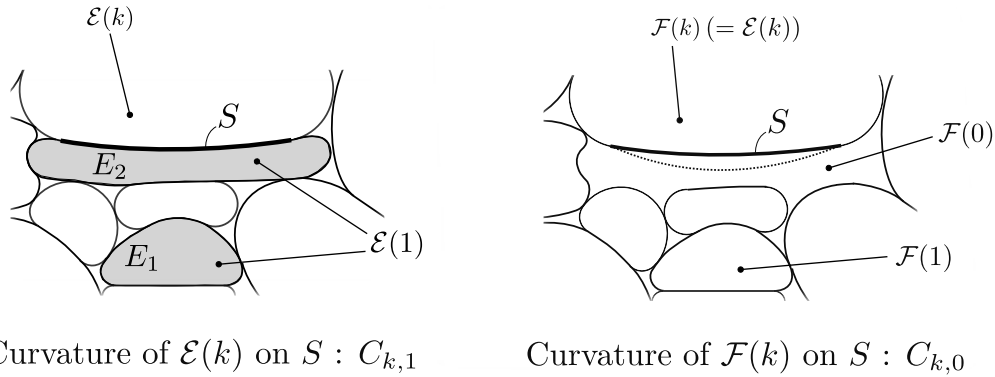


Figure 4.5.2:

Hence \mathcal{F} it is also a Cheeger N -cluster for Ω . Consider the piece of boundary

$$S = [\partial\mathcal{E}(k) \cap \partial E_2] \setminus \Sigma(\mathcal{E}(0); \Omega) \neq \emptyset$$

from the old cluster \mathcal{E} . Proposition 4.5.5 tells us that S must be a circular arc and that $\mathcal{E}(k)$ must has constant mean curvature on S equal to:

$$C_{k,1} = \frac{|\mathcal{E}(1)|h(\mathcal{E}(k)) - |\mathcal{E}(k)|h(\mathcal{E}(1))}{|\mathcal{E}(1)| + |\mathcal{E}(k)|}.$$

From the other side it holds $\mathcal{F}(k) = \mathcal{E}(k)$ and, since S is now a part of the free boundary of $\mathcal{F}(k)$ (we have removed the component E_2), we have that $\mathcal{F}(k) = \mathcal{E}(k)$ must has constant mean curvature on S also equal to:

$$C_{k,0} = h(\mathcal{F}(k)) = h(\mathcal{E}(k)).$$

Thus equality $C_{k,1} = C_{k,0}$ must be in force, implying $(h(\mathcal{E}(k)) + h(\mathcal{E}(1)))|\mathcal{E}(k)| = 0$ which is impossible. \square

Proof of Theorem 4.2.7. It follows from Propositions 4.5.4, 4.5.5 and 4.5.6. \square

4.6 The limit of $\Lambda_N^{(p)}$ as p goes to one

We conclude this Chapter by focusing on the asymptotic trend of H_N . We first briefly state the following Theorem involving the existence of the optimal partition for problem (4.1.2).

Theorem 4.6.1. *For every $1 < p \leq n$ there exists an optimal partition for Ω in quasi-open sets $\{\Omega_i\}_{i=1}^N$ such that*

$$\Lambda_N^{(p)}(\Omega) = \sum_{i=1}^N \lambda_1^{(p)}(\Omega_i).$$

Proof. The existence of an optimal partition for $\Lambda_N^{(p)}(\Omega)$ follow as a simple variation of the argument in [CL07], or as a consequence of more general results contained in [BBH98], [BV13] or [BDM93] and thus we omit the details. \square

In the following Proposition we compute the limit of $\Lambda_N^{(p)}$ as p goes to one.

Proposition 4.6.2. *If Ω is an open bounded set with C^1 boundary then*

$$\lim_{p \rightarrow 1} \Lambda_N^{(p)}(\Omega) = H_N(\Omega).$$

Proof. Let \mathcal{E} be a Cheeger N -cluster for Ω . Since $\partial\mathcal{E}(i)$ is C^1 , for every $i = 1, \dots, N$ there exists a sequence of open sets $\{\mathcal{E}_t(i)\}_{t>0}$ such that $\mathcal{E}_t(i) \subset \subset \mathcal{E}(i)$ for all $t > 0$ and

$$\mathcal{E}_t(i) \rightarrow \mathcal{E}(i) \text{ in } L^1, \quad P(\mathcal{E}_t(i)) \rightarrow P(\mathcal{E}(i)),$$

as $t \rightarrow 0$ (see [Sch15]). In this way, since $\mathcal{E}_t(i)$ are open sets (and thus quasi-open sets) strictly contained into Ω and with disjoint closure, by exploiting (4.1.7) we reach:

$$\begin{aligned} H_N(\Omega) &= \lim_{t \rightarrow 0} \sum_{i=1}^N \frac{P(\mathcal{E}_t(i))}{|\mathcal{E}_t(i)|} \geq \lim_{t \rightarrow 0} \sum_{i=1}^N h(\mathcal{E}_t(i)) \\ &\geq \lim_{t \rightarrow 0} \limsup_{p \rightarrow 1} \sum_{i=1}^N \lambda_N^{(p)}(\mathcal{E}_t(i)) \geq \limsup_{p \rightarrow 1} \Lambda_N^{(p)}(\Omega). \end{aligned}$$

On the other hand, thanks to (4.1.6) and to Jensen's inequality we get (4.1.8):

$$\begin{aligned} \sum_{j=1}^N \lambda_1^{(p)}(\mathcal{E}(i)) &\geq \sum_{j=1}^N \left(\frac{h(\mathcal{E}(i))}{p} \right)^p \geq \frac{1}{N^{p-1}} \left(\sum_{j=1}^N \frac{h(\mathcal{E}(i))}{p} \right)^p \\ &\geq \frac{1}{N^{p-1}} \left(\frac{H_N(\Omega)}{p} \right)^p \end{aligned}$$

which completes the proof. □

4.6.1 On the asymptotic behavior of H_N in dimension $n = 2$

Theorem 4.6.3. *Denote with B a ball of unit radius and with H a unit-area regular hexagon. Let $N \geq 2$ and Ω be an open, bounded and connected set with C^1 boundary and finite perimeter in the plane. Then the following assertions hold true:*

1) *If \mathcal{E} is a Cheeger $(N + 1)$ -cluster for Ω then:*

$$|\mathcal{E}(i)| \geq \frac{h(B)^2 \pi}{(H_{N+1}(\Omega) - H_N(\Omega))^2} \quad \forall i = 1, \dots, N + 1;$$

2) $H_N(\Omega) + \frac{h(B)\sqrt{\pi}}{\sqrt{|\Omega|}} \sqrt{N + 1} \leq H_{N+1}(\Omega)$, for all $N \in \mathbb{N}$;

3) for every $0 \leq \varepsilon < \frac{1}{2}$ there exists $N_0(\Omega, \varepsilon)$ such that:

$$\frac{\sqrt{\pi}h(B)}{\sqrt{|\Omega|}} N^{\frac{3}{2}} \leq H_N(\Omega) \leq \frac{h(H)}{\sqrt{|\Omega|}} N^{\frac{3}{2}} + N^{\frac{3}{2}-\varepsilon} \quad \text{for all } N \geq N_0(\Omega, \varepsilon).$$

Proof. Thanks to the planar Cheeger inequality (1.4.4)

$$h(E) \geq \sqrt{\pi} \frac{h(B)}{\sqrt{|E|}} \tag{4.6.1}$$

we observe that, given \mathcal{E} a Cheeger $(N + 1)$ -cluster of Ω , it holds:

$$\begin{aligned} H_{N+1}(\Omega) &= \sum_{i=1}^{N+1} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \geq \frac{P(\mathcal{E}(j))}{|\mathcal{E}(j)|} + \sum_{i=1, i \neq j}^{N+1} \frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|} \\ &\geq h(\mathcal{E}(j)) + H_N(\Omega) \geq \frac{\sqrt{\pi}h(B)}{\sqrt{|\mathcal{E}(j)|}} + H_N(\Omega) \end{aligned}$$

which, implies Property 1).

Property 2) follows from Property 1):

$$|\Omega| - \frac{(N + 1)h(B)^2\pi}{(H_{N+1}(\Omega) - H_N(\Omega))^2} \geq |\Omega| - \sum_{i=1}^{N+1} |\mathcal{E}(i)| \geq 0$$

and so

$$|\Omega| \geq \frac{(N + 1)h(B)^2\pi}{(H_{N+1}(\Omega) - H_N(\Omega))^2},$$

which implies

$$(H_{N+1}(\Omega) - H_N(\Omega))^2 \geq \frac{(N + 1)h(B)^2\pi}{|\Omega|}$$

and thus

$$H_{N+1}(\Omega) \geq H_N(\Omega) + \sqrt{(N + 1)} \frac{\sqrt{\pi}h(B)}{\sqrt{|\Omega|}}.$$

Let us prove Property 3). Let \mathcal{E} be a Cheeger N -cluster for Ω . We exploit again the Cheeger inequality (4.6.1) and we obtain the lower bound

$$\begin{aligned} H_N(\Omega) &= \sum_{i=1}^N h(\mathcal{E}(i)) \geq \sqrt{\pi}h(B) \sum_{i=1}^N \frac{1}{\sqrt{|\mathcal{E}(i)|}} \\ &\geq \sqrt{\pi}h(B)N^{\frac{3}{2}} \left(\frac{1}{\sum_{i=1}^N |\mathcal{E}(i)|} \right)^{\frac{1}{2}} \geq \sqrt{\pi} \frac{h(B)}{\sqrt{|\Omega|}} N^{\frac{3}{2}}. \end{aligned}$$

Here we have used the inequality

$$\sum_{i=1}^N \frac{1}{x_i^{\frac{1}{n}}} \geq N^{\frac{n+1}{n}} \left(\frac{1}{\sum_{i=1}^N x_i} \right)^{\frac{1}{n}}, \quad \forall N, n \geq 2, x_i > 0.$$

Let us focus on the upper bound. Let \mathcal{H}_δ be the standard hexagonal grid of the plane, made by hexagons of area δ (the one depicted in Figure 2.1.1 in Chapter 2). Define

$$\begin{aligned} I(\delta) &:= \{i \in \mathbb{N} \mid \mathcal{H}_\delta(i) \subset\subset \Omega\}. \\ k(\delta) &:= \#(I(\delta)). \end{aligned}$$

Up to a relabeling, let us assume that $I(\delta) = \{1, \dots, k(\delta)\}$. Note that since $\mathcal{H}_\delta(i) \subset\subset \Omega$ we get

$$H_{k(\delta)}(\Omega) \leq \sum_{i=1}^{k(\delta)} h(\sqrt{\delta}H) = \frac{k(\delta)}{\sqrt{\delta}} h(H).$$

From $\mathcal{H}_\delta(i) \subset \Omega$ for all $i = 1, \dots, k(\delta)$ it follows

$$k(\delta) \leq \frac{|\Omega|}{\delta}.$$

If we set $\delta(N) = \frac{|\Omega|}{N} - \frac{|\Omega|}{N^\alpha}$ for some $\alpha > 1$ to be chosen, we are led to

$$H_{k(N)}(\Omega) \leq \frac{N^{\frac{3}{2}}}{\sqrt{|\Omega|} (1 - N^{1-\alpha})^{\frac{3}{2}}} h(H). \quad (4.6.2)$$

where $k(N) = k(\delta(N))$. Note that, by setting

$$(\partial\Omega)_{r(N)} := \partial\Omega + B_{r(N)}$$

where $r(N) = \sqrt{\delta(N)} \text{diam}(H)$, it must hold

$$\left(\Omega \setminus \bigcup_{i=1}^{k(N)} \mathcal{H}(i) \right) \subseteq (\partial\Omega)_{r(N)}.$$

Since Ω has Lipschitz boundary, for N bigger than $N_0(\Omega)$, it also holds that

$$|(\partial\Omega)_{r(N)}| \leq 4r(N)P(\Omega)$$

and so:

$$|\Omega| - \delta(N)k(N) \leq |(\partial\Omega)_{r(N)}| \leq 4r(N)P(\Omega) = 4\sqrt{\delta(N)} \text{diam}(H)P(\Omega),$$

that imply

$$k(N) \geq \frac{N}{1 - N^{1-\alpha}} - 4\sqrt{N} \frac{P(\Omega) \text{diam}(H)}{\sqrt{|\Omega|} \sqrt{1 - N^{1-\alpha}}}.$$

For all N bigger than some fixed N_0 depending only on Ω . It is easy to show that, for all $\alpha < \frac{3}{2}$, up to further increase N_0 in dependence only on Ω and α , it holds

$$\frac{N}{1 - N^{1-\alpha}} - 4\sqrt{N} \text{diam}(H) \frac{P(\Omega)}{\sqrt{|\Omega|} \sqrt{1 - N^{1-\alpha}}} \geq N.$$

Hence by choosing $\alpha < \frac{3}{2}$ we obtain

$$k(N) \geq N \quad \forall N \geq N_0,$$

and, thanks to the monotonicity given by Property 2) and to (4.6.2), provided also $\alpha > 1 + \varepsilon$ we reach:

$$\begin{aligned} H_N(\Omega) &\leq H_{k(N)}(\Omega) \\ &\leq \frac{N^{\frac{3}{2}}}{\sqrt{|\Omega|} (1 - N^{1-\alpha})^{\frac{3}{2}}} h(H) \\ &\leq \frac{h(H)}{\sqrt{|\Omega|}} (N^{\frac{3}{2}} + N^{\frac{3}{2}-\varepsilon}) \quad \text{for all } N > N_0(\Omega, \varepsilon). \end{aligned}$$

□

Chapter 5

Final remarks and future research interests

We conclude our discussion about the isoperimetric properties of N -clusters with some remarks and ideas for possible future investigations.

As a first remark we highlight that it could be interesting to explore whether the topics contained in Chapter Two can be extended to a dimension higher than $n = 2$. A first apparent obstacle arises when one tries to generalize the arguments contained in the proofs of Theorems 2.3.2 and 2.4.2: so far there is no theorem in the spirit of Hales's Theorem 1.9.2 that holds in dimension $n > 2$. Another interesting direction could be to weaken the hypothesis involving indecomposability and boundedness of the chambers. For example we can try to implement the argument exposed in Chapter Two with the assumption that each chamber can split into a controlled number of pieces $p(N) \leq N^a$ for a that varies in $[0, 1)$. We can ask how to combine the result contained in Chapter Three, about the quantitative version of the hexagonal honeycomb theorem, with the energetic estimates of Chapter two. In particular:

*Is there a way, starting from an energetic estimate,
to obtain information about the shape of the chambers?* (5.0.1)

As far as Chapter Three is concerned, we could investigate how the constant $\kappa(N)$ in Theorem 3.1.1 depends on N . This could provide an answer to (5.0.1). For example it is reasonable to expect that we can combine an energetic estimate of the type of the one appearing in Theorem 2.4.2 with the quantitative version of the Hexagonal Honeycomb Theorem for conclude that each chamber of an indecomposable minimizing N -cluster of Ω converges in the L^1 sense to a regular hexagon. In order to do that we need to understand what is the dependence of the constant $\kappa(N)$ in Theorem 3.1.1 with respect to N .

Chapter Four instead left us with many open questions; for example:

*Is there a way to prove the Caffarelli and Lin's conjecture
in the case $p = 1$, by importing tools and instrument
from the Hales's proof of Hexagonal honeycom Theorem?* (5.0.2)

Is this easier than the Caffarelli and Lin's conjecture for $p = 2$? It could be interesting to understand whether the approach proposed in Chapter One can be adapted to the context

of Cheeger N -cluster in order to obtain a sort of periodicity (from an energetic point of view) in the asymptotic behavior of Cheeger N -clusters.

Let us conclude this work, that is the result of three years of efforts and study at the University of Pisa, by remarking that these are just a few questions, the more natural that occur when we deal with such problems, that will lead our research in the future, together with many (and we hope fruitful) others interesting queries.

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