Symmetry of components, Liouville-type theorems and classification results for some nonlinear elliptic systems

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Abstract

We prove the symmetry of components and some Liouville-type theorems for, possibly sign changing, entire distributional solutions to a family of nonlinear elliptic systems encompassing models arising in Bose-Einstein condensation and in nonlinear optics. For these models we also provide precise classification results for non-negative solutions. The sharpness of our results is also discussed.

1 Introduction

The aim of this paper is to classify entire solutions (u, v) of the following nonlinear elliptic systems arising in Bose-Einstein condensation and in nonlinear optics (see for instance [7], [14], [19] and the references therein) :

$$\begin{cases} -\Delta u + \alpha |u|^{m-1} u = -\lambda |u|^{\theta-1} u + \beta |u|^s |v|^{s+\gamma-1} v & \text{in } \mathbb{R}^N \\ -\Delta v + \alpha |v|^{m-1} v = -\lambda |v|^{\theta-1} v + \beta |v|^s |u|^{s+\gamma-1} u & \text{in } \mathbb{R}^N \end{cases} \quad (1.1)$$

where α, β and λ are *continuous* functions defined on $\mathbb{R}^N$, $N \geq 1$.

The present paper is motivated by the recent and interesting work [21], as well as by the stimulating discussions with the Authors of [21].

More precisely, we shall prove that any entire distributional solution (u, v) of system (1.1), has the symmetry property $u = v$ (*symmetry of components*).

We use this result to establish some new Liouville-type theorems as well as some classification results.

Our method is different (and complementary) from the one used in [21]. It exploits the attractive character of the interaction between the two states u and v. It applies to any distributional entire solution, possibly sign-changing and without any other restriction. Also, it applies to systems with nonlinearities that are not necessarily positive (or cooperative) nor necessarily homogeneous. Section 2 is devoted to the main results, while in section 3 we consider their extension to more general models involving nonlinearities which are not necessarily of polynomial type. We also discuss the sharpness of our results.

2 Model problems and main results

Throughout the section α, β and λ will be *continuous* functions defined on $\mathbb{R}^N$, $N \geq 1$.

Theorem 2.1. (Symmetry of components) *Let $N \geq 1$ and assume $m > 0$, $\theta > 1$, $s \geq 0$, $\gamma \geq 1$, $\alpha = \alpha(x) \geq 0$, $\beta = \beta(x) \geq 0$, $\lambda = \lambda(x) \geq \lambda_0 > 0$.*

Let (u, v) be a distributional solution of system (1.1) such that $u, v \in L^p_{loc}(\mathbb{R}^N)$, with $p = \max\{m, \theta, 2s + \gamma\}$.

Then $u = v$.

Proof. From (1.1) we have

$$\Delta(u - v) = \alpha(|u|^{m-1}u - |v|^{m-1}v) +$$

$$+ \lambda(|u|^{\theta-1}u - |v|^{\theta-1}v) + \beta|u|^s|v|^s(|u|^{\gamma-1}u - |v|^{\gamma-1}v) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \quad (2.1)$$

Set $\psi = u - v$. The assumptions on u and v imply that $\psi \in L^p_{loc}(\mathbb{R}^N)$ and $\Delta\psi$ belongs to $L^1_{loc}(\mathbb{R}^N)$. Thus we can apply Kato's inequality [12, 2] to get

$$\Delta(\psi^+) \geq \Delta\psi 1_{\{u-v>0\}} \geq \lambda_0(|u|^{\theta-1}u - |v|^{\theta-1}v)1_{\{u-v>0\}} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \quad (2.2)$$

where 1_E denotes the characteristic function of the measurable subset $E \subset \mathbb{R}^N$.

Reminding the well known inequality

$$|t|^{q-1}t - |s|^{q-1}s \geq c_q(t - s)^q, \quad for \ \ t > s \qquad (q \geq 1), \qquad (2.3)$$

from (2.2) we obtain

$$\Delta(\psi^+) \geq \lambda_0 c_q(\psi^+)^\theta \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \qquad (2.4)$$

Since $\lambda_0 c_q > 0$ and $\theta > 1$ we immediately get $\psi^+ = 0$ (cfr. Lemma 2 of [2]). Hence $u \leq v$ a.e. on $\mathbb{R}^N$. Finally, exchanging the role of u and v we obtain the desired conclusion $u = v$. $\qquad \square$

Then we are in position to prove the following

Theorem 2.2. (of Liouville-type) *Assume $N \geq 1$ and let (u, v) be a distributional solution of system* (1.1), *where $m > 0$, $\theta > 1$, $s \geq 0$, $\gamma \geq 1$, $\alpha = \alpha(x) \geq 0$, $\beta = \beta(x) \geq 0$, $\lambda = \lambda(x) \geq \lambda_0 > 0$ and $u, v \in L^p_{loc}(\mathbb{R}^N)$, with $p = \max\{m, \theta, 2s + \gamma\}$.*

Assume further then $\theta = 2s + \gamma$.

i) If $\inf_{\mathbb{R}^N}(\lambda - \beta) > 0$, then $u = v = 0$.

ii) If $\lambda \geq \beta$, $\alpha \geq \alpha_0 > 0$ and $m > 1$, then $u = v = 0$.

iii) If $m = \theta$ and $\inf_{\mathbb{R}^N}(\alpha + \lambda - \beta) > 0$, then $u = v = 0$.

iv) If $\lambda = \beta$, $\alpha = 0$, then $u = v$ and u is a harmonic function. In particular, if either u or v is bounded on one side, then $u = v = const$.

Proof. By the previous theorem we have $u = v$. Hence

$$\Delta u = \alpha |u|^{m-1} u + (\lambda - \beta)|u|^{\theta-1} u \qquad in \quad \mathcal{D}'(\mathbb{R}^N) \tag{2.5}$$

and by Kato's inequality (once again) we see that

$$\Delta u^+ \geq \alpha (u^+)^m + (\lambda - \beta)(u^+)^\theta \qquad in \quad \mathcal{D}'(\mathbb{R}^N). \tag{2.6}$$

If i) (or ii) or iii)) is in force, there are $\epsilon > 0$ and $\eta > 1$ such that

$$\Delta u^+ \geq \epsilon (u^+)^\eta \qquad in \quad \mathcal{D}'(\mathbb{R}^N). \tag{2.7}$$

Thus $u^+ = 0$ and then $u \leq 0$ on $\mathbb{R}^N$. On the other hand, also $-u$ is a solution of (2.5), hence $u^- = 0$ and the desired conclusion follows.

If iv) holds true, u and v are harmonic functions. Hence $u = v = const$. by the classical Liouville Theorem.

$\square$

Some remarks are in order :

Remark 1.

1) The assumption : $\lambda = \lambda(x) \geq \lambda_0 > 0$ is necessary, as it easily seen by choosing $m = \alpha = 1$ and $\lambda = \beta = 0$ in (1.1). *In this case system* (1.1) *reduces to*

$$\begin{cases} -\Delta u + u = 0 & in \ \mathbb{R}^N \\ -\Delta v + v = 0 & in \ \mathbb{R}^N \end{cases} \tag{2.8}$$

which admits positive, non-trivial and non-symmetric solutions. For instance $u(x) = e^{x_1}$ and $v(x) = e^{-x_1} + 2e^{x_1}$, where $x = (x_1, ..., x_N) \in \mathbb{R}^N$.

2) The assumptions : " $\lambda - \beta \geq 0$ " in the above Thereom 2.2 are essentially necessary. Indeed, when this condition is not satisfied, there are non constant solutions (cfr. for instance the following Theorems 2.3, 2.4 and the existence results in [15]).

Remark 2. *Theorem 2.1 recovers and significantly improves some previous results demonstrated in [21] and [16]. Indeed, by choosing $\alpha = 0$, $\theta = 3$, $s = 1$, $\gamma = 1$ in (1.1), we recover the cubic system (1.6) in [21]*

$$\begin{cases} -\Delta u = uv^2 - \lambda u^3 & \text{in } \mathbb{R}^N \\ -\Delta v = vu^2 - \lambda v^3 & \text{in } \mathbb{R}^N \\ \quad u \geq 0,\, v \geq 0 & \text{on } \mathbb{R}^N \end{cases} \tag{2.9}$$

which appears in Bose-Einstein condensation, and by choosing $\alpha = 1$, $\theta = 2r+1$, $s = r$, $\gamma = 1$ in (1.1), we recover system (1.8) in [21] and system (6) in [16]

$$\begin{cases} -\Delta u + u^m = -\lambda u^{2r+1} + \beta u^r v^{r+1} & \text{in } \mathbb{R}^N \\ -\Delta v + v^m = -\lambda v^{2r+1} + \beta v^r u^{r+1} & \text{in } \mathbb{R}^N \\ \quad u \geq 0,\, v \geq 0 & \text{on } \mathbb{R}^N \end{cases} \tag{2.10}$$

arising in nonlinear optics.

For instance, we recover and extend Theorem 1.6 of [21] and Theorem 2 of [16] (cfr. also Remark 1.8 of [21]), since (u, v) is merely a distributional solution, possibly sign-changing and no further assumption is made on the solution (u, v). In particular, (u, v) need not to be neither a ground state nor a classical positive decaying solution. Furthermore, as far as system (2.9) is concerned, we do not have any restriction about the parameter $\lambda > 0$.

Moreover, if we restrict our attention to non-negative solutions, we can further extend the above mentioned results to obtain precise classification results for models naturally arising in physical applications. More precisely we have :

Theorem 2.3. (of classification I) *Let $N \geq 1$ and assume $s \geq 0$, $\gamma \geq 1$, $\beta \geq 0$, $\lambda > 0$. Let (u, v) be a non-negative distributional solution of*

$$\begin{cases} -\Delta u = \beta u^s v^{s+\gamma} - \lambda u^{2s+\gamma} & \text{in } \mathbb{R}^N \\ -\Delta v = \beta v^s u^{s+\gamma} - \lambda v^{2s+\gamma} & \text{in } \mathbb{R}^N \end{cases} \tag{2.11}$$

such that $u, v \in L_{loc}^{2s+\gamma}(\mathbb{R}^N)$.

Then $u = v$.

Furthermore,

i) if $\lambda > \beta$, then $u = v = 0$.

ii) If $\lambda = \beta$, then $u = v = const$.

iii) If $\lambda < \beta$ and

$$1 < 2s + \gamma \leq \begin{cases} +\infty & \text{if } N \leq 2, \\ \\ \frac{N}{N-2} & \text{if } N \geq 3, \end{cases} \tag{2.12}$$

then $u = v = 0$.

iv) If $\lambda < \beta$, $N \geq 3$, $u, v \in H^1_{loc}(\mathbb{R}^N) \cap L^{2s+\gamma}_{loc}(\mathbb{R}^N)$ *and*

$$\frac{N}{N-2} < 2s + \gamma < \frac{N+2}{N-2} \tag{2.13}$$

then $u = v = 0$.

v) If $\lambda < \beta$, $N \geq 3$, $u, v \in H^1_{loc}(\mathbb{R}^N) \cap L^{2s+\gamma}_{loc}(\mathbb{R}^N)$ *and*

$$2s + \gamma = \frac{N+2}{N-2} \tag{2.14}$$

then either $u = v = 0$ *or*

$$u = v = c(N, \lambda, \beta) \left[\frac{\eta}{|x - x_0|^2 + \eta^2} \right]^{\frac{N-2}{2}} \tag{2.15}$$

for some $\eta > 0$, $x_0 \in \mathbb{R}^N$ *and* $c(N, \lambda, \beta) > 0$.

Remark 3.

1) Note that system (2.9) *is obtained by setting* $\beta = 1$, $s = \gamma = 1$ *in* (2.11).

2) Theorem 2.3 *provides a complete classification in dimension* $N \leq 2$ *(no restriction is made on the parameters* s, γ, β *and* λ*). It also provides a complete classification for the cubic system* (2.9) *in dimension* $N \leq 4$ *(cfr. also Remark* 2*).*

3) The assumption $u, v \in H^1_{loc}(\mathbb{R}^N)$ *is necessary in (iv) and (v). Indeed, for instance, for all* $s \geq 0$ *and* $\gamma \geq 1$ *such that* $2s + \gamma > \frac{N}{N-2}$*, there is a singular radial positive solution* (u, v) *with* $u = v = c(N, s, \gamma, \lambda, \beta)|x|^{-\frac{2}{2s+\gamma-1}}$ *and where* $c(N, s, \gamma, \lambda, \beta) > 0$ *is an explicit constant.*

Proof of Theorem 2.3. By Theorem 2.1 we have $u = v$. Items *i)* and *ii)* follow directly from *i)* and *iv)* of Theorem 2.2. To proceed, we observe that system (2.11) reduces to the equation

$$-\Delta u = (\beta - \lambda)u^{2s+\gamma} \qquad \text{in} \quad \mathcal{D}^{'}(\mathbb{R}^N). \tag{2.16}$$

By a standard density argument, we can use test functions of class C^2_c in the above equation (2.16). Thus, the desired result follows, for instance, from Theorem 2.1 of [17].

When *iv)* (or *v)*) is in force, it is well-known that u is a classical solution (i.e. of class C^2) of the equation (2.16). Hence, the claims follow immediately from the celebrated results of Gidas and Spruck [10, 11] and of Caffarelli, Gidas and Spruck [3]. □

Now we turn our attention to the system (2.10) and we prove the following

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Theorem 2.4. (of classification II) *Let $N \geq 1$ and assume $m > 0$, $r > 0$, $\beta \geq 0$, $\lambda > 0$. Let (u, v) be a non-negative distributional solution of*

$$\begin{cases} -\Delta u + u^m = -\lambda u^{2r+1} + \beta u^r v^{r+1} & in \ \mathbb{R}^N \\ -\Delta v + v^m = -\lambda v^{2r+1} + \beta v^r u^{r+1} & in \ \mathbb{R}^N \end{cases} \tag{2.17}$$

such that $u, v \in L^p_{loc}(\mathbb{R}^N)$, with $p = \max\{m, 2r+1\}$.

Then $u = v$.

Furthermore,

i) if $\lambda > \beta$, then $u = v = 0$.

ii) If $\lambda = \beta$ and $m > 1$, then $u = v = 0$.

iii) If $\lambda < \beta$ and $2r + 1 < m$,
then (u, v) is a smooth, bounded and classical solution of (2.17), it satisfies the following universal and sharp L^∞-bound

$$\|u\|_\infty = \|v\|_\infty \leq (\beta - \lambda)^{\frac{1}{m-(2r+1)}}. \tag{2.18}$$

Moreover, if either $N \leq 2$ or, $N \geq 3$ and $2r + 1 \leq \frac{N+2}{N-2}$, then either $u = v = 0$ or $u = v = (\beta - \lambda)^{\frac{1}{m-(2r+1)}}$.

iv) If $\lambda < \beta$ and $2r + 1 = m$, we have :

1) if $0 < \beta - \lambda < 1$, then $u = v = 0$.

2) if $\beta - \lambda = 1$, then $u = v = const.$

3) if $\beta - \lambda > 1$ and

$$2r + 1 \leq \begin{cases} +\infty & if \ \ N \leq 2, \\ \\ \frac{N}{N-2} & if \ \ N \geq 3, \end{cases} \tag{2.19}$$

then $u = v = 0$.

4) if $\beta - \lambda > 1$, $N \geq 3$, $u, v \in H^1_{loc}(\mathbb{R}^N) \cap L^{2r+1}_{loc}(\mathbb{R}^N)$ and

$$2r + 1 < \begin{cases} +\infty & if \ \ N \leq 2, \\ \\ \frac{N+2}{N-2} & if \ \ N \geq 3, \end{cases} \tag{2.20}$$

then $u = v = 0$.

5) if $\beta - \lambda > 1$, $N \geq 3$, $u, v \in H^1_{loc}(\mathbb{R}^N) \cap L^{2r+1}_{loc}(\mathbb{R}^N)$ and

$$2r + 1 = \frac{N+2}{N-2} \tag{2.21}$$

then either $u = v = 0$ or

$$u = v = c(N, \lambda, \beta) \left[\frac{\eta}{|x - x_0|^2 + \eta^2} \right]^{\frac{N-2}{2}} \qquad (2.22)$$

for some $\eta > 0$, $x_0 \in \mathbb{R}^N$ and $c(N, \lambda, \beta) > 0$.

v) If $\lambda < \beta$ and $2r + 1 > m$ we have :

1) when $m \geq 1$, $u, v \in C^0(\mathbb{R}^N)$ and either u or v tends to zero uniformly, as $|x| \to +\infty$, then either $u = v = 0$ or
$u = v > 0$ everywhere on $\mathbb{R}^N$ and u is necessarily radially symmetric and strictly radially decreasing, i.e., $u(x) = v(x) = w(|x - x_0|)$, for some $x_0 \in \mathbb{R}^N$ and some positive function w such that $w'(0) = 0$ and $w'(r) < 0$ for $r > 0$. Moreover, the profile w is unique.

2) when $m < 1$, $u, v \in C^0(\mathbb{R}^N)$ and either u or v tends to zero uniformly, as $|x| \to +\infty$, then either $u = v = 0$ or, $u = v$ is compactly supported, u has open support (i.e. the set $\{ x \in \mathbb{R}^N : u(x) > 0 \}$) on a finite number of open balls in $\mathbb{R}^N$, on each of which it is radially symmetric about the center of the ball and its profile is unique.

Furthermore, when $N \geq 2$, system (2.17) admits a non-constant solution (u, v) such that u or v tends to zero uniformly, as $|x| \to +\infty$, if and only if

$$2r + 1 < \begin{cases} +\infty & if \quad N = 2, \\[2mm] \frac{N+2}{N-2} & if \quad N \geq 3. \end{cases} \qquad (2.23)$$

Remark 4. *The situation is more complicated when $m < 1$. Indeed,*

1) in view of conclusion 2) of item v), system (2.17) admits non-negative, non-constant, compactly supported classical solutions. This also shows the importance to consider non-negative solutions.

2) For every $x_0 \in \mathbb{R}$ set

$$w(t) := \begin{cases} 0 & if \quad t \leq 0, \\[3mm] [(\frac{2}{1-m})(\frac{2}{1-m} - 1)]^{-\frac{1}{1-m}} \, t^{\frac{2}{1-m}} & if \quad t \geq 0, \end{cases} \qquad (2.24)$$

and $u_{x_0}(x) := w(x_1 - x_0, ..., x_N)$. The couple (u_{x_0}, u_{x_0}) is a non-negative, non-constant, classical solution of (2.17) with $\lambda = \beta$ and $0 < m < 1$. Combining this example with the example of Remark 1, we see that the conclusion of item ii) is sharp.

Proof of Theorem 2.4. By Theorem 2.1 we have $u = v$. Items *i)* and *ii)* follow directly from *i)* and *ii)* of Theorem 2.2. Since $u = v$, system (2.17) reduces to the equation

$$-\Delta u + u^m = (\beta - \lambda)u^{2r+1} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \tag{2.25}$$

with $\beta - \lambda > 0$.

To prove $iii)$, for every $\varepsilon > 0$ we set $u_\varepsilon := u - (\beta - \lambda)^{\frac{1}{m-(2r+1)}} - \varepsilon$ and apply Kato's inequality to (2.25) to get

$$\Delta u_\varepsilon^+ \geq \Delta u 1_{\{u_\varepsilon > 0\}} = \tag{2.26}$$

$$u^{2r+1}[u^{m-(2r+1)} - (\beta - \lambda)]1_{\{u_\varepsilon > 0\}} \geq c[u_\varepsilon^+]^{2r+1} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \tag{2.27}$$

where c is a positive constant depending on ε, m, r, β and λ. From the latter we infer $u_\varepsilon^+ \leq 0$ on $\mathbb{R}^N$ and thus $u \leq (\beta - \lambda)^{\frac{1}{m-(2r+1)}}$ by letting $\varepsilon \to 0$. This gives the bound (2.18), whose sharpness follows by noticing that $u = v = (\beta - \lambda)^{\frac{1}{m-(2r+1)}}$ is a solution of (2.17). The smoothness of (u, v) immediately follows from standard elliptic regularity, since $(u, v) \in L^\infty$. In view of (2.18) and of (2.25) we see that u is a smooth positive superharmonic function. Thus u must be constant when $N \leq 2$ and the only possibilities are $u = 0$ or $u = (\beta - \lambda)^{\frac{1}{m-(2r+1)}}$ (since u solves (2.25)). To treat the case $N \geq 3$ we need to use, in an essential way, the fact that we proved that u is smooth and satisfies the bound (2.18). Indeed, in view of those properties of u, we can invoke Theorem 2.4 of [5], when $2r + 1 < \frac{N+2}{N-2}$, and Theorem 3 of [1], when $2r + 1 = \frac{N+2}{N-2}$, to obtain the desired conclusion.

When $iv)$ is in force, system (2.17) boils down to

$$-\Delta u = (\beta - \lambda - 1)u^{2r+1} \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \tag{2.28}$$

and the claims follows as in the proof of item $iii)$ of Theorem 2.3.

Proof of $v)$: 1) by the strong maximum principle, either $u = v = 0$ or $u = v > 0$ on $\mathbb{R}^N$. In the latter case u is radially symmetric and strictly radially decreasing by the well-known results of [8, 24]. Uniqueness of the profile w follows from [20, 23].

The claims of part 2) follows by applying the results of [4, 24] and [20, 23].

The results for $N \geq 2$ follow from [4, 23] and [9] (cfr. also the references therein).

$\square$

3 More general results

The method used to prove the above results also applies to more general systems and with nonlinearities which are not necessarily of polynomial type. To this end we need to recall the well-known *Keller-Osserman condition* [13, 18].

A non-decreasing function $f \in C^0([0, +\infty), [0, +\infty))$ is said to satisfy the *Keller-Osserman condition* if

$$\begin{cases} f(0) = 0, \\ f(t) > 0, \quad \text{if} \quad t > 0, \\ \int^{+\infty} \left[\int_0^s f(t)dt \right]^{-\frac{1}{2}} ds < +\infty. \end{cases} \tag{3.1}$$

A typical example of function satisfying the above condition (3.1) is $f(t) = t^q$, $q > 1$. Also $f(t) = t \log^\delta(t+1)$, $\delta > 2$, satisfies (3.1), while $f(t) = t$ does not fullfill (3.1).

Theorem 3.1. *Assume $N \geq 1$ and let (u, v) be a distributional solution of*

$$\begin{cases} -\Delta u = h(x, u, v) & in \ \mathbb{R}^N \\ -\Delta v = h(x, v, u) & in \ \mathbb{R}^N \end{cases} \tag{3.2}$$

where $h : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$h(x, v, u) - h(x, u, v) \geq f(u - v) \qquad \forall \, u \geq v, \quad \forall x \in \mathbb{R}^N \tag{3.3}$$

and f is a convex function fulfilling the Keller-Osserman condition.
If $u, v \in L^1_{loc}(\mathbb{R}^N)$ and $h(\cdot, u, v), h(\cdot, v, u) \in L^1_{loc}(\mathbb{R}^N)$, then $u = v$.

Proof. Set $\psi = u - v$. The assumptions on u and v imply that both ψ and $\Delta \psi$ belong to $L^1_{loc}(\mathbb{R}^N)$. Hence Kato's inequality yields

$$\Delta(\psi^+) \geq (h(x, v, u) - h(x, u, v)) 1_{\{u-v>0\}}$$

$$\geq f(u - v) 1_{\{u-v>0\}} = f(\psi^+) \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \tag{3.4}$$

Thus we can apply Theorem 4.7 of [6] (where we have set $f(t) = 0$ if $t \leq 0$) to get that $\psi^+ = 0$. To conclude we proceed as in the proof of Theorem 2.1. $\square$

The above theorem is not true if the Keller-Osserman conditon is not satisfied, as witness the example given by system (2.8), which is of the form (3.2) with $h(x, u, v) = -u$ and satisfies (3.3) with $f(t) = t$.

Nevertheless not all is lost, since we have the following:

Theorem 3.2. *Assume $N \geq 1$ and let (u, v) be a distributional solution of*

$$\begin{cases} -\Delta u = h(x, u, v) & in \ \mathbb{R}^N \\ -\Delta v = h(x, v, u) & in \ \mathbb{R}^N \end{cases} \tag{3.5}$$

where $h : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$h(x, v, u) - h(x, u, v) \geq \nu(u - v) \qquad \forall \, u \geq v, \quad \forall x \in \mathbb{R}^N, \tag{3.6}$$

for some constant $\nu > 0$.
If $u, v \in L^1_{loc}(\mathbb{R}^N)$ and $h(\cdot, u, v), h(\cdot, v, u) \in L^1_{loc}(\mathbb{R}^N)$, then $u = v$, whenever u and v have at most polynomial growth at infinity.

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Proof. Set $\psi = u - v$. As in the proof of Theorem 3.1 we obtain

$$\Delta(\psi^+) \geq (h(x,v,u) - h(x,u,v))1_{\{u-v>0\}}$$

$$\geq \nu(u-v)1_{\{u-v>0\}} = \nu\psi^+ \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N). \qquad (3.7)$$

We consider a $\mathcal{C}^\infty$ function $\varphi : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \varphi(t) = 1 & t \in [0,1], \\ \varphi(t) = 0 & t \in [2, +\infty), \\ 0 \leq \varphi(t) \leq 1 & t \in (1,2), \end{cases}$$

and we set, for every $R > 0$ and every $x \in \mathbb{R}^N$, $\varphi_R(x) := \varphi(|x|/R)$.

Using the cut-off functions φ_R as test functions in (3.7), and recalling that ψ^+ has at most polynomial growth at infinity, we have for any $R > 1$

$$\int_{B_R} \psi^+ \leq \frac{C}{\nu R^2} \int_{B_{2R}} \psi^+ \leq C' R^{N+k-2}$$

for some $k \geq 0$, $C' > 0$ independent of R.

Iterating the latter a finite number of times, we immediately obtain that

$$\forall R > 1 \qquad \int_{B_R} \psi^+ \leq C'' R^{-m}$$

for some $m > 0$, $C'' > 0$ independent of R. This leads to $\int_{\mathbb{R}^N} \psi^+ = 0$, which in turn yields $u \leq v$ a.e. on $\mathbb{R}^N$. Finally, exchanging the role of u and v we obtain the desired conclusion $u = v$. $\qquad\square$

We conclude this section by noticing that, classification results similar to those of Theorem 2.3 and/or Theorem 2.4, can also be established for solutions to the system (3.2), under suitable assumptions on the function h. Nevertheless, we do not want to stress on this point.

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