# A Bernstein-type result for the minimal surface equation

Alberto Farina

LAMFA, CNRS UMR 7352, Université de Picardie Jules Verne
33 rue Saint-Leu, 80039 Amiens, France
and
Institut Camille Jordan, CNRS UMR 5208, Université Claude Bernard Lyon I
43 boulevard du 11 novembre 1918, 69622 Villeurbane cedex, France
email: alberto.farina@u-picardie.fr

**Abstract**

We prove the following Bernstein-type theorem: if $u$ is an entire solution to the minimal surface equation, such that $N-1$ partial derivatives $\frac{\partial u}{\partial x_j}$ are bounded on one side (not necessarily the same), then $u$ is an affine function. Its proof relies *only* on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, our theorem also provides a new and self-contained proof of celebrated results of Moser and of Bombieri & Giusti.

**MSC:** 53A10, 58JO5, 35J15

## 1 Introduction and main results

In this short article we are concerned with a Bernstein-type theorem for solutions to the minimal surface equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \qquad \text{in} \qquad \mathbb{R}^N, \qquad N\geq 2. \tag{1.1}$$

The classical Bernstein Theorem ([2],[7]) asserts that the affine functions are the only solutions of (1.1) in $\mathbb{R}^2$. This result has been generalized to $\mathbb{R}^3$ by E. De Giorgi [5], to $\mathbb{R}^4$ by J.F. Almgren [1] and, up to dimension $N=7$, by J. Simons [9]. On the other hand, E. Bombieri, E. De Giorgi and E. Giusti [3] proved the existence of a non-affine solution of the minimal surface equation (1.1) for any $N\geq 8$. Nevertheless, J. Moser [8] was able to prove that, if $\nabla u$ is bounded on $\mathbb{R}^N$, then $u$ must be again an affine function, and this for every dimension $N\geq 2$. Later, E. Bombieri and E. Giusti [4] generalized Moser's result by assuming that *only* $N-1$ partial derivatives of $u$ are bounded on $\mathbb{R}^N$, $N\geq 2$. To prove their result, the Authors of [4] demonstrate a Harnack inequality for uniformly elliptic equations on minimal surfaces (oriented boundary of least area) and then they use it to show that, if $N-1$ partial derivatives of $u$ are bounded on $\mathbb{R}^N$, then $u$ has bounded gradient on $\mathbb{R}^N$, and they conclude by invoking the result of Moser. Our main theorem (see Theorem 1.1 below) provides a further extension of the above results. Its proof relies *only* on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, it also provides a new and self-contained proof of the celebrated results of Moser and of Bombieri & Giusti. We believe that this is another interesting feature of our work.

Our main result is stated in the following theorem.

**Theorem 1.1.** *Assume $N \geq 2$. Let $u$ be a solution of the minimal surface equation (1.1) such that $N-1$ partial derivatives $\frac{\partial u}{\partial x_j}$ are bounded on one side (not necessarily the same). Then $u$ is an affine function.*

## 2 Auxiliary results and proofs

To prove our results we briefly recall some standard notations and some well-known facts concerning the solutions of the minimal surface equation (1.1) (cfr. [4], [6]). For a given solution $u$ of equation (1.1), we denote by $S$ the minimal graph $x_{N+1} = u(x)$ over $\mathbb{R}^N$ (i.e., the complete smooth area minimizing hypersurface without boundary $S \subset \mathbb{R}^{N+1}$, given by the graph of $u$ over the entire $\mathbb{R}^N$). Then the (upward pointing) unit normal to $S$ at a point $(x, u(x))$ is $\nu = (\nu_1, ..., \nu_{N+1}) = \frac{(-\nabla u(x), 1)}{\sqrt{1+|\nabla u(x)|^2}}$ and we can define the tangential derivatives $\delta_k$ by

$$\delta_k := \frac{\partial}{\partial x_k} - \nu_k \sum_{h=1}^{N+1} \nu_h \frac{\partial}{\partial x_h} \qquad \forall\, k = 1, ..., N+1. \tag{2.1}$$

Moreover the functions $\nu_h$ satisfy the equation

$$\sum_{k=1}^{N+1} \delta_k \delta_k \nu_h + c^2 \nu_h = 0 \qquad \text{on} \quad S, \qquad \forall\, h = 1, ..., N+1 \tag{2.2}$$

where $c^2 := \sum_{j,k=1}^{N+1} (\delta_j \nu_k)^2$ denotes the sum of the squares of the principal curvatures of the hypersurface $S$ at the point $(x, u(x))$. Therefore, for any vector $a := (a_1, ..., a_{N+1}) \in \mathbb{R}^{N+1}$, the function $(\, a \cdot \nu) = \sum_{j=1}^{N+1} a_j v_j$ also solves

$$\sum_{k=1}^{N+1} \delta_k \delta_k (\, a \cdot \nu) + c^2 (\, a \cdot \nu) = 0 \qquad \text{on} \quad S. \tag{2.3}$$

**Lemma 2.1.** *Assume $N \geq 2$ and let $S$ be a minimal graph $x_{N+1} = u(x)$ over $\mathbb{R}^N$. If $v > 0$ and $w$ are smooth solutions of the equation (2.3) on $S$, then the smooth function $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation*

$$\sum_{k=1}^{N+1} \delta_k \left[ (v^2 + w^2) \delta_k \theta \right] = 0 \qquad \text{on} \quad S. \tag{2.4}$$

*Proof.* Consider the smooth complex-valued function $z := v + iw$. Since $v > 0$ everywhere, we have that $z = \rho e^{i\theta}$ on $S$ and

$$\sum_{k=1}^{N+1} \delta_k \delta_k z + c^2 z = 0 \qquad \text{on} \quad S, \tag{2.5}$$

where $\rho := \sqrt{v^2 + w^2} > 0$ everywhere on $S$. Hence, by definition of $\delta_k$ we get

$$0 = \sum_{k=1}^{N+1} \delta_k \delta_k (\rho e^{i\theta}) + c^2 \rho e^{i\theta} = \sum_{k=1}^{N+1} \delta_k \left( e^{i\theta} \delta_k \rho + i\rho e^{i\theta} \delta_k \theta \right) + c^2 \rho e^{i\theta} =$$

$$\sum_{k=1}^{N+1} e^{i\theta} \delta_k \delta_k \rho + i e^{i\theta} \delta_k \theta \delta_k \rho + i\rho e^{i\theta} \delta_k \delta_k \theta + i \left( e^{i\theta} \delta_k \rho + i \rho e^{i\theta} \delta_k \theta \right) \delta_k \theta + c^2 \rho e^{i\theta} =$$

$$\sum_{k=1}^{N+1} e^{i\theta} \delta_k \delta_k \rho - \rho e^{i\theta} \delta_k \theta \delta_k \theta + i e^{i\theta} (\rho \delta_k \delta_k \theta + 2 \delta_k \rho \delta_k \theta) + c^2 \rho e^{i\theta} \qquad \text{on} \quad S.$$

Hence

$$0 = \sum_{k=1}^{N+1} \delta_k \delta_k \rho - \rho \delta_k \theta \delta_k \theta + i (\rho \delta_k \delta_k \theta + 2 \delta_k \rho \delta_k \theta) + c^2 \rho \qquad \text{on} \quad S$$

and taking the imaginary part of the latter identity we obtain

$$0 = \sum_{k=1}^{N+1} \rho \delta_k \delta_k \theta + 2 \delta_k \rho \delta_k \theta = \frac{1}{\rho} \sum_{k=1}^{N+1} \delta_k \left[ \rho^2 \delta_k \theta \right] \qquad \text{on} \quad S$$

which immediately implies (2.4). □

Now we are in position to prove our main result.

*Proof of Theorem 1.1.* We divide the proof into three steps.

*Step 1: Every partial derivative of $u$ is bounded on one side.*

By assumption there exists an integer $n \in \{1, ..., N\}$ such that for every integer $j \in \{1, ..., N\} \setminus \{n\} := J$, the partial derivative $\frac{\partial u}{\partial x_j}$ is bounded on one side. We set $A := \{\alpha \in J : \frac{\partial u}{\partial x_\alpha}$ is bounded from below$\}$ and $B := \{\beta \in J : \frac{\partial u}{\partial x_\beta}$ is bounded from above$\}$. Hence

$$\forall\,\alpha\in A \qquad \exists\, c_\alpha>0 \quad : \quad \frac{\partial u}{\partial x_\alpha}+c_\alpha>1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.6}$$

$$\forall\,\beta\in B \qquad \exists\, c_\beta>0 \quad : \quad c_\beta-\frac{\partial u}{\partial x_\beta}>1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}. \tag{2.7}$$

Now we observe that

$$|\nabla u|^2=\left(\frac{\partial u}{\partial x_n}\right)^2+\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}\right)^2+\sum_{\beta\in B}\left(\frac{\partial u}{\partial x_\beta}\right)^2= \tag{2.8}$$

$$\left(\frac{\partial u}{\partial x_n}\right)^2+\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha-c_\alpha\right)^2+\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}-c_\beta\right)^2= \tag{2.9}$$

$$\left(\frac{\partial u}{\partial x_n}\right)^2+\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)^2+\sum_{\alpha\in A}c_\alpha^2-2\sum_{\alpha\in A}c_\alpha\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)+ \tag{2.10}$$

$$\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)^2+\sum_{\beta\in B}c_\beta^2-2\sum_{\beta\in B}c_\beta\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)\le \tag{2.11}$$

$$\left(\frac{\partial u}{\partial x_n}\right)^2+\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)^2+\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)^2+\sum_{j\in J}c_j^2\le \tag{2.12}$$

$$\left(\frac{\partial u}{\partial x_n}\right)^2+\left[\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)+\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)\right]^2+\sum_{j\in J}c_j^2 \tag{2.13}$$

where in the latter we have used (2.6) and (2.7).

Now we set $\xi:=\sum_{\alpha\in A}e_\alpha-\sum_{\beta\in B}e_\beta\in\mathbb{R}^N$, $k_1:=\sum_{j\in J}c_j^2>0$, $k_2:=\sum_{j\in J}c_j>0$, where $\{e_1,...,e_N\}$ denotes the canonical basis of $\mathbb{R}^N$ and we rewrite (2.13) as

$$\left(\frac{\partial u}{\partial x_n}\right)^2+(\nabla u\cdot\xi+k_2)^2+k_1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}} \tag{2.14}$$

and observe that

$$\nabla u\cdot\xi+k_2>1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}. \tag{2.15}$$

again by (2.6) and (2.7).

Combining (2.8)-(2.14) and (2.15) we find

$$1 + |\nabla u|^2 \leq \left(\frac{\partial u}{\partial x_n}\right)^2 + (2 + k_1)\,(\nabla u \cdot \xi + k_2)^2 \tag{2.16}$$

$$\leq (2 + k_1)\left[\left(\frac{\partial u}{\partial x_n}\right)^2 + (\nabla u \cdot \xi + k_2)^2\right] \tag{2.17}$$

Set $\chi := (-e_n, 0) \in \mathbb{R}^{N+1}$, $\tau := (-\xi, k_2) \in \mathbb{R}^{N+1}$ and consider the functions $w := \frac{\frac{\partial u}{\partial x_n}}{\sqrt{1+|\nabla u|^2}} = (\,\chi \cdot \nu)$ and $v := \frac{\nabla u \cdot \xi + k_2}{\sqrt{1+|\nabla u|^2}} = (\,\tau \cdot \nu) > 0$. Since $v > 0$ and $w$ are solutions of the equation (2.3), an application of Lemma 2.1 implies that $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation

$$\sum_{k=1}^{N+1} \delta_k\left[(v^2 + w^2)\delta_k\theta\right] = 0 \qquad \text{on} \quad S. \tag{2.18}$$

Thanks to (2.16)-(2.17) we see that the above equation (2.18) is uniformly elliptic on $S$. Indeed, from (2.16)-(2.17) we get

$$\frac{1 + |\nabla u|^2}{2 + k_1} \leq \left[\left(\frac{\partial u}{\partial x_n}\right)^2 + (\nabla u \cdot \xi + k_2)^2\right] \leq 2(N + k_2^2)\left[1 + |\nabla u|^2\right] \tag{2.19}$$

which implies

$$\frac{1}{2 + k_1} \leq v^2 + w^2 \leq 2(N + k_2^2) \qquad \text{on} \qquad S. \tag{2.20}$$

Thus $\theta$ must be constant, by an application of the Harnack inequality proved by Bombieri and Giusti (cfr. Theorem 5 of [4]), i.e., $w = \lambda v$ on $S$, for some $\lambda \in \mathbb{R}$. The latter immediately implies that $\frac{\partial u}{\partial x_n}$ has a sign. In particular, all the partial derivatives of $u$ are bounded on one side.

*Step 2: For every unit vector $\eta \in \mathbb{R}^N$ the directional derivative $\frac{\partial u}{\partial \eta}$ has a sign, that is, one and only one of the following assertions holds: (i) $\frac{\partial u}{\partial \eta}(x) = 0 \quad \forall x \in \mathbb{R}^N$, (ii) $\frac{\partial u}{\partial \eta}(x) > 0 \quad \forall x \in \mathbb{R}^N$, (iii) $\frac{\partial u}{\partial \eta}(x) < 0 \quad \forall x \in \mathbb{R}^N$.*

Let $\sigma$ be any unit vector of $\mathbb{R}^N$ and set $I := \{1, ..., N\}$, $A := \{\alpha \in I : \frac{\partial u}{\partial x_\alpha}$ is bounded from below$\}$ and $B := \{\beta \in I : \frac{\partial u}{\partial x_\beta}$ is bounded from above$\}$. Hence

$$\forall\, \alpha \in A \qquad \exists\, c_\alpha > 0 \quad : \quad \frac{\partial u}{\partial x_\alpha} + c_\alpha > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.21}$$

$$\forall\, \beta \in B \qquad \exists\, c_\beta > 0 \quad : \quad c_\beta - \frac{\partial u}{\partial x_\beta} > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}. \tag{2.22}$$

and proceeding as before we obtain

$$\left(\frac{\partial u}{\partial \sigma}\right)^2 \leq |\nabla u|^2 \leq \left[\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)+\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)\right]^2+\sum_{j\in I}c_j^2 = \tag{2.23}$$

$$= (\nabla u\cdot\xi+k_4)^2+k_3 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}} \tag{2.24}$$

and

$$\nabla u\cdot\xi+k_4>1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.25}$$

where $\xi := \sum_{\alpha\in A} e_\alpha - \sum_{\beta\in B} e_\beta \in \mathbb{R}^N$, $k_3 := \sum_{j=1}^N c_j^2 > 0$, $k_4 := \sum_{j=1}^N c_j > 0$.

We notice that $\xi$, $k_3$ and $k_4$ are independent of the unit vector $\sigma$ and let $\{\eta, \sigma_2, ..., \sigma_N\}$ be an orthonormal basis of $\mathbb{R}^N$. From (2.23)-(2.24) we get

$$1+|\nabla u|^2 = 1+\left(\frac{\partial u}{\partial \eta}\right)^2+\sum_{j=2}^N\left(\frac{\partial u}{\partial \sigma_j}\right)^2 \leq 1+\left(\frac{\partial u}{\partial \eta}\right)^2+(N-1)\left[(\nabla u\cdot\xi+k_4)^2+k_3\right] \tag{2.26}$$

and using (2.25) in the latter we immediately infer that

$$1+|\nabla u|^2 \leq (N+(N-1)k_3)\left[\left(\frac{\partial u}{\partial \eta}\right)^2+(\nabla u\cdot\xi+k_4)^2\right] \leq \tag{2.27}$$

$$3(N+(N-1)k_3)(N+k_4^2)\left[1+|\nabla u|^2\right]. \tag{2.28}$$

Setting $\chi := (-\eta, 0) \in \mathbb{R}^{N+1}$, $\tau := (-\xi, k_4) \in \mathbb{R}^{N+1}$, $w := \frac{\frac{\partial u}{\partial \eta}}{\sqrt{1+|\nabla u|^2}} = (\chi\cdot\nu)$ and $v := \frac{\nabla u\cdot\xi+k_4}{\sqrt{1+|\nabla u|^2}} = (\tau\cdot\nu) > 0$, and applying Lemma 2.1 as before, we see that the function $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation (2.4), which is again unifomly elliptic on $S$ in view of the above (2.27)-(2.28). It follows that $\theta$ is constant, which implies that the directional derivative $\frac{\partial u}{\partial \eta}$ has a sign.

*Step 3: End of the proof.*

Either $u$ is constant, and in this case we are done, or there exists $x_0 \in \mathbb{R}^N$ such that $\nabla u(x_0) \neq 0$. In the latter case there are $N-1$ unit vectors of $\mathbb{R}^N$, denoted by $\sigma_1, ..., \sigma_{N-1}$, which are orthogonal to $\nabla u(x_0)$, i.e., such that

$$0 = \nabla u(x_0)\cdot\sigma_j = \frac{\partial u}{\partial \sigma_j}(x_0) \qquad \forall j = 1, ..., N-1. \tag{2.29}$$

By the previous step, we must have

$$\frac{\partial u}{\partial \sigma_j}(x) \equiv 0 \qquad \text{on} \quad \mathbb{R}^N, \qquad \forall\, j = 1, ..., N-1, \tag{2.30}$$

thus $u(x) = h(\tau \cdot x)$, where $\tau = \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$ and $h = h(t)$ is a non constant solution of the ODE $-\left(\frac{h'}{\sqrt{1+|h'|^2}}\right)' = 0$ on $\mathbb{R}$. A direct integration of the latter gives $h(t) = at + b$, $a \neq 0$. Thus $u$ is an affine function. □

**Acknowledgements:** The author thanks A. Savas-Halilaj and E. Valdinoci for a careful reading of a first version of this article. The author is supported by the ERC grant EPSILON (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).

## References

[1] F.J.Jr. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem.* Ann. of Math. (2) 84 (1966), 277-292.

[2] S. Bernstein, *Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique*, Comm. Soc. Math. de Kharkov 2 (15) (1915-1917), 38-45. German translation: Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus, Math. Z. 26 (1) (1927), 551-558.

[3] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969), 243-268.

[4] E. Bombieri, E. Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces* Invent. Math. 15, 1, (1972), 24-46.

[5] E. De Giorgi, *Una estensione del teorema di Bernstein.* Ann. Scuola Norm. Sup. Pisa (3) 19 (1965), 79-85.

[6] E. Giusti, *Minimal surfaces and functions of bounded variation.* Monographs in Mathematics, 80. Birkhauser Verlag, Basel, 1984.

[7] E. Hopf, *On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature.* Proc. Amer. Math. Soc. 1, (1950), 80-85.

[8] J. Moser, *On Harnack's theorem for elliptic differential equations.* Comm. Pure Appl. Math. 14 (1961), 577-591.

[9] J. Simons, *Minimal varieties in riemannian manifolds.* Ann. of Math. (2) 88 (1968), 62-105.