AN ELEMENTARY PROOF OF THE RANK-ONE THEOREM FOR BV FUNCTIONS

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ABSTRACT. We provide a simple proof of a result, due to G. Alberti, concerning a rank-one property for the singular part of the derivative of vector-valued functions of bounded variation.

In this paper we provide a short, elementary proof of the following result by G. Alberti [1] concerning a rank-one property for the derivative of a function with bounded variation.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u : \Omega \to \mathbb{R}^m$ a function of bounded variation and let $D_s u$ be the singular part of Du with respect to the Lebesgue measure $\mathcal{L}^n$. Then $D_s u$ is a rank-one measure, i.e., the (matrix-valued) function $\frac{D_s u}{|D_s u|}(x)$ has rank one for $|D_s u|$-a.e. $x \in \Omega$.*

We recall that a function $u \in L^1(\Omega, \mathbb{R}^m)$ has *bounded variation* in Ω ($u \in BV(\Omega, \mathbb{R}^m)$) if the derivatives Du of u in the sense of distributions are represented by a (matrix-valued) measure with finite total variation. The measure Du can then be decomposed as the sum $Du = D_a u + D_s u$ of a measure $D_a u$, that is absolutely continuous with respect to $\mathcal{L}^n$, and a measure $D_s u$ that is singular with respect to $\mathcal{L}^n$. The Radon-Nikodym derivative $\frac{D_s u}{|D_s u|}$ of $D_s u$ with respect to its total variation $|D_s u|$ is a $|D_s u|$-measurable map from Ω to $\mathbb{R}^{m \times n}$. The Theorem states that this map takes values in the space of rank-one matrices. See [2] for more details on BV functions.

The Theorem above was conjectured by L. Ambrosio and E. De Giorgi in [3]. It was first proved by G. Alberti in [1] by introducing new tools and using sophisticated techniques in Geometric Measure Theory. A new proof has been announced by G. De Philippis and F. Rindler as one of the consequences of the forthcoming, profound PDE result [4].

On the contrary, our proof of the Theorem above is elementary: it stems from well-known geometric properties relating the derivative of a BV function and the *perimeter* of its subgraph. The main new tool is the following lemma, where we denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the canonical projection $\pi(x_1, \dots, x_{n+1}) := (x_1, \dots, x_n)$.

Lemma. *Let Σ_1, Σ_2 be C^1 hypersurfaces in $\mathbb{R}^{n+1}$ with unit normals $\nu_{\Sigma_1}, \nu_{\Sigma_2}$. Then, the set*

$$T := \left\{ p \in \Sigma_1 : \exists\, q \in \Sigma_2 \cap \pi^{-1}(\pi(p)) \text{ with } (\nu_{\Sigma_1}(p))_{n+1} = (\nu_{\Sigma_2}(q))_{n+1} = 0 \text{ and } \nu_{\Sigma_1}(p) \neq \pm \nu_{\Sigma_2}(q) \right\}$$

is $\mathcal{H}^n$-negligible.

We postpone the proof of the Lemma in order to directly address the proof of the main result.

Proof of the Theorem. Let $u = (u_1, \dots, u_m) \in BV(\Omega, \mathbb{R}^m)$. It is not restrictive to assume that Ω is bounded. For any $i = 1, \dots, m$ we write $D_s u_i = \sigma_i |D_s u_i|$ for a $|D_s u_i|$-measurable map $\sigma_i : \Omega \to \mathbb{S}^{n-1}$. We also let $E_i := \{(x,t) \in \Omega \times \mathbb{R} : t < u_i(x)\}$ be the subgraph of u_i; it is well known that E_i has finite perimeter in $\Omega \times \mathbb{R}$. Denoting by $\partial^* E_i$ the *reduced boundary* of E_i and by ν_i the measure theoretic inner normal to E_i, we have (see e.g. [5, Section 4.1.5])

$$|D_s u_i| = \pi_\#(\mathcal{H}^n \llcorner S_i) \quad \text{for } S_i := \left\{ p \in \partial^* E_i : (\nu_i(p))_{n+1} = 0 \right\},$$

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where $\pi_\#$ denotes push-forward of measures. The set S_i is n-rectifiable and we can assume that it is contained in the union $\cup_{h\in\mathbb{N}}\Sigma^i_h$ of C^1 hypersurfaces Σ^i_h in $\mathbb{R}^{n+1}$.

By [5, Section 4.1.5], the Lemma above and well-known properties of rectifiable sets, the following properties hold for $\mathcal{H}^n$-a.e. $p \in S_i$:

$$\nu_{\partial^* E_i}(p) = (\sigma_i(\pi(p)), 0) \tag{1}$$

$$\text{if } p \in \Sigma^i_h, \text{ then } \nu_i(p) = \pm\nu_{\Sigma^i_h}(p) \tag{2}$$

$$\text{if } p \in \Sigma^i_h \text{ and } q \in S_j \cap \Sigma^j_k \cap \pi^{-1}(\pi(p)), \text{ then } \nu_{\Sigma^i_h}(p) = \pm\nu_{\Sigma^j_k}(q). \tag{3}$$

Up to modifying S_i on a $\mathcal{H}^n$-negligible set and σ_i on a $|D_s u_i|$-negligible set, we can assume that (1),(2) and (3) hold everywhere on S_i and that $\sigma_i = 0$ on $\Omega \setminus \pi(S_i)$.

Since $D_s u = (\sigma_1|D_s u_1|, \dots, \sigma_m|D_s u_m|)$ and $|D_s u|$ is concentrated on $\pi(S_1) \cup \dots \cup \pi(S_m)$, it is enough to prove that the matrix-valued function $(\sigma_1, \dots, \sigma_m)$ has rank 1 on $\pi(S_1) \cup \dots \cup \pi(S_m)$. This will follow if we prove that the implication

$$i, j \in \{1, \dots, m\},\ i \neq j,\ x \in \pi(S_i) \implies \sigma_j(x) \in \{0, \sigma_i(x), -\sigma_i(x)\}$$

holds. If i, j, x are as above and $x \notin \pi(S_j)$, then $\sigma_j(x) = 0$. Otherwise, $x \in \pi(S_i) \cap \pi(S_j)$, i.e., there exist $p \in S_i$ and $h \in \mathbb{N}$ such that $\pi(p) = x$ and $\sigma_i(x) = \pm\nu_{\Sigma^i_h}(p)$ and there exist $q \in S_j$ and $k \in \mathbb{N}$ such that $\pi(q) = x$ and $\sigma_j(x) = \pm\nu_{\Sigma^j_k}(p)$. By (3) we obtain $\sigma_j(x) = \pm\sigma_i(x)$, as wished. □

Proof of the Lemma. Consider the sets $\Sigma := \Sigma_1 \times \Sigma_2 \subset \mathbb{R}^{2n+2}$ and

$$\Delta := \{\xi = (x, t, y, s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+2} : x = y\}.$$

Then Σ is a $2n$-dimensional manifold of class C^1 and Δ is a smooth $(n+2)$-dimensional manifold in $\mathbb{R}^{2n+2}$. Let us consider the set

$$R := \{\xi = (x, t, x, s) \in \Delta \cap \Sigma : (\nu_{\Sigma_1}(x, t))_{n+1} = (\nu_{\Sigma_2}(x, s))_{n+1} = 0 \text{ and } \nu_{\Sigma_1}(x, t) \neq \pm\nu_{\Sigma_2}(x, s)\}.$$

By construction, the intersection between Δ and Σ is transversal at every point of R, thus R is contained in a n-dimensional submanifold $\tilde{R} \subset \Delta \cap \Sigma$ of class C^1. Let ϕ be the projection $\phi(x, t, y, s) := (x, t)$; notice that $\phi(\tilde{R}) \subset \Sigma_1$ and $T = \phi(R)$. Moreover, for every $\xi \in R$ the differential $d\phi_\xi : T_\xi\tilde{R} \to T_{\phi(\xi)}\Sigma_1$ is not surjective, because the vector $(0, \dots, 0, 1)$ is in the kernel of $d\phi_\xi$. By the area formula we deduce that $\mathcal{H}^n(T) = \mathcal{H}^n(\phi(R)) = 0$, as desired. □

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References

[1] G. Alberti, *Rank one property for derivatives of functions with bounded variation.* Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), no. 2, 239–274.

[2] L. Ambrosio, N. Fusco & D. Pallara, *Functions of bounded variation and free discontinuity problems.* The Clarendon Press, Oxford University Press, New York, 2000.

[3] E. De Giorgi & L. Ambrosio, *New functionals in the calculus of variations* (Italian. English summary.) Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 82 (1988), no. 2, 199–210.

[4] G. De Philippis & F. Rindler, *On the structure of $\mathcal{A}$-free measures and applications.* In preparation.

[5] M. Giaquinta, G. Modica & J. Souček, *Cartesian currents in the calculus of variations. I.* Cartesian currents. Springer-Verlag, Berlin, 1998.

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