A NOTE ON SOME POINCARÉ INEQUALITIES ON CONVEX SETS BY OPTIMAL TRANSPORT METHODS

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Abstract. We show that a class of Poincaré-Wirtinger inequalities on bounded convex sets can be obtained by means of the dynamical formulation of Optimal Transport. This is a consequence of a more general result valid for convex sets, possibly unbounded.

Contents

1. Introduction

1.1. **Overview.** Let $1 < p < \infty$ and $0 < r < \infty$. For an open set $\Omega \subset \mathbb{R}^N$, we introduce the Sobolev spaces

$$\mathrm{W}^{1,p}_r(\Omega) := \left\{ \phi \in L^r(\Omega) \,:\, \nabla \phi \in L^p(\Omega;\mathbb{R}^N) \right\},$$

and

$$\ddot{\mathrm{W}}^{1,p}_r(\Omega) := \left\{ \phi \in \mathrm{W}^{1,p}_r(\Omega) \,:\, \int_\Omega |\phi|^{r-1}\,\phi\,dx = 0 \right\}.$$

In the particular case $r = p$, we will omit to indicate it and simply write $\mathrm{W}^{1,p}(\Omega)$ and $\ddot{\mathrm{W}}^{1,p}(\Omega)$.

The aim of this note is to prove some functional inequalities for the space $\ddot{\mathrm{W}}^{1,p}_r(\Omega)$, by means of Optimal Transport techniques. The use of Optimal Transport to prove functional

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and geometric inequalities is nowadays classical. We are not concerned here with geometric inequalities, thus we only refer to Sections 2.5.3 and 7.4.2 of [22] for a brief discussion on the subject (in particular on the isoperimetric and the Brunn-Minkowski inequalities). As for functional inequalities obtained via Optimal Transport techniques, which is the main concern of this paper, after the fundamental paper [7] the literature on the subject is now quite rich. In addition to [7], we encourage the reader to look in details into the papers [3, 6, 13, 14] and [18], for example.

It is useful to observe that most of these papers use the geometric properties of the optimal transport map as a tool to obtain a clever change-of-variable. This is indeed the case for the transport-based proof of the isoperimetric, Sobolev and Gagliardo-Nirenberg inequalities. We could say that they are based on the "statical" version of Optimal Transport problems.

On the contrary, the proof that we propose here is based on the "dynamical" counterpart of Optimal Transport (the so-called *Benamou-Brenier formula*, see [5]) and on *displacement convexity* considerations, see [17]. In this respect, it can be more suitably compared to the transport-based proof of the Brunn-Minkowski inequality.

It is also useful to remark that while the above cited papers deal with functional inequalities which are invariant for the trasformation $\phi \mapsto |\phi|$, such as Sobolev and Gagliardo-Nirenberg ones, this is not the case here. Indeed, if a function ϕ belongs to our space $\ddot{\mathrm{W}}^{1,p}_r(\Omega)$, then $|\phi| \not\in \ddot{\mathrm{W}}^{1,p}_r(\Omega)$. Thus, in order to prove our main result (see Theorem 1.1 below), we can not reduce to the case of positive functions and then use an optimal transport to transform any positive function ϕ into an extremal of the relevant functional inequality, as in [7]. Roughly speaking, what we do is to perform an optimal transport between the positive and negative parts ϕ_+ and ϕ_- (suitably renormalized).

Our proof has some points in common with the one presented by Rajala in [21], which is valid in general metric measure spaces under *Ricci curvature conditions*. Indeed, it is well-known that Ricci curvature conditions are linked to the displacement convexity of suitable functionals (see for instance the work [12] by Lott and Villani, to which [21] is inspired). However, even if the result of [21, Theorem 1.1] holds in a much more general setting, we stress that the tools used in [21] are not the same as ours. Moreover, the result of [21] only concerns with Poincaré inequalities on balls in the case $q = 1$ (with our notation below).

1.2. Main result.

In order to neatly present the main result, we first need to recall some basic definitions and notations.

We indicate by $\mathcal{P}(\Omega)$ the set of all Borel probability measures over Ω. Then for $1 < m < \infty$, we define

$$\mathcal{P}_m(\Omega) = \left\{ \mu \in \mathcal{P}(\Omega) \, : \, \int_\Omega |x|^m \, d\mu < \infty \right\}, \tag{1.1}$$

i.e. the set of probability measure over Ω with finite moment of order m. For every $\mu, \nu \in \mathcal{P}_m(\Omega)$ their $m-$*Wasserstein distance* is defined through the optimal transport problem

$$W_m(\mu,\nu) = \left(\min_{\gamma\in\Pi(\mu,\nu)} \int_{\Omega\times\Omega} |x-y|^m\, d\gamma\right)^{\frac{1}{m}}.$$

Here $\Pi(\mu,\nu) \subset \mathcal{P}(\Omega\times\Omega)$ is the set of *transport plans*, i.e. the probability measures on the product space $\Omega\times\Omega$ such that

$$\gamma(A\times\Omega) = \mu(A) \qquad \gamma(\Omega\times B) = \nu(B), \qquad \text{for every } A, B \subset \Omega \text{ Borel sets.}$$

In what follows, we will note by $\mathcal{L}^N$ the $N-$dimensional Lebesgue measure. For a function $f \in L^1$, the writing

$$\mu = f\cdot \mathcal{L}^N,$$

will indicate the Radon measure which is absolutely continuous with respect to $\mathcal{L}^N$ and whose Radon-Nikodym derivative is given by f.

In this note we prove the following scaling invariant inequality, which is valid for general convex sets.

Theorem 1.1. *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open convex set. For every $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ such that*

$$\int_\Omega |x|^{\frac{p}{p-q}}\, |\phi|^{q-1}\, dx < \infty, \tag{1.2}$$

we define the two probability measures $\rho_0, \rho_1 \in \mathcal{P}_{p/(p-q)}(\Omega)$

$$\rho_0 = \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+\, dx}\cdot\mathcal{L}^N \qquad \text{and} \qquad \rho_1 = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_-\, dx}\cdot\mathcal{L}^N.$$

Then there holds

$$\left(\int_\Omega |\phi|^q\, dx\right)^{p-q+1} \le \frac{\left(W_{\frac{p}{p-q}}(\rho_0,\rho_1)\right)^p}{2^{p-1}} \int_\Omega |\nabla\phi|^p\, dx \left(\int_\Omega |\phi|^{q-1}\, dx\right)^{p-q}. \tag{1.3}$$

The proof of this result is postponed to Section 3. We point out that inequality (1.3) in turn implies a handful of Poincaré-type inequalities with explicit constants. The reader is invited to jump directly to Section 4 in order to discover them. In particular, as a corollary we can obtain a lower bound for the first non-trivial Neumann eigenvalue of the $p-$Laplacian, see Corollary 4.5. This can be seen as a weak version of the *Payne-Weinberger inequality* (see [4, 9, 19]): though the explicit constant we get is not optimal, we believe the method of proof to be of independent interest.

Remark 1.2. We point out that the hypothesis $\phi \in L^q(\Omega)$ *is not needed* in Theorem 1.1. Rather, inequality (1.3) permits to show that on a convex set, functions in $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ verifying (1.2) are automatically in $L^q(\Omega)$.

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2. Preliminaries

2.1. **An embedding result.**

We will need some basic inequalities for Sobolev spaces in bounded sets. The proofs are standard, but we give it for the reader's convenience. The values of the constants appearing in the inequalities below will have no bearing in what follows.

Lemma 2.1. *Let $1<p<\infty$ and let $\Omega\subset\mathbb{R}^N$ be an open connected and bounded set, with Lipschitz boundary. Then for every $\phi\in \mathrm{W}^{1,p}(\Omega)$ we have*

$$\int_\Omega |\phi|^p\,dx\le C\,\frac{|\Omega|}{|A_\phi|}\int_\Omega |\nabla \phi|^p\,dx,\qquad A_\phi:=\{x\in\Omega\,:\,|\phi(x)|=0\},\tag{2.1}$$

for some $C=C(N,p,\Omega)>0$.

Proof. The proof is an adaptation of that of [10, Theorem 3.16]. We first observe that if we indicate by $\overline{\phi}_\Omega$ the mean of ϕ over Ω, then

$$|A_\phi|\,|\overline{\phi}_\Omega|^p=\int_{A_\phi}|\overline{\phi}_\Omega|^p\,dx=\int_{A_\phi}|\phi-\overline{\phi}_\Omega|^p\,dx\le\int_\Omega|\phi-\overline{\phi}_\Omega|^p\,dx.$$

By using this information, with elementary manipulations we then get

$$\int_\Omega|\phi|^p\,dx\le 2^{p-1}\int_\Omega|\phi-\overline{\phi}_\Omega|^p\,dx+2^{p-1}\,\frac{|\Omega|}{|A_\phi|}\int_\Omega|\phi-\overline{\phi}_\Omega|^p\,dx.$$

We can conclude by applying Poincaré inequality for functions with vanishing mean, see for example [10, Theorem 3.14]. □

The next interpolation inequality for the Sobolev space $\mathrm{W}^{1,p}_r(\Omega)$ will be useful.

Lemma 2.2. *Let $1<p<\infty$ and $0<r<p$. Let $\Omega\subset\mathbb{R}^N$ be a open connected and bounded set with Lipschitz boundary. Then $\mathrm{W}^{1,p}_r(\Omega)\subset L^p(\Omega)$. More precisely, for every $\phi\in\mathrm{W}^{1,p}_r(\Omega)$ we have*

$$\int_\Omega|\phi|^p\,dx\le C\int_\Omega|\nabla\phi|^p\,dx+C\left(\int_\Omega|\phi|^r\,dx\right)^{\frac{p}{r}},$$

for some $C=C(N,p,\Omega)>0$.

Proof. Given $\phi\in\mathrm{W}^{1,p}_r(\Omega)$, for every $t>0$ and $M>0$ we define

$$\phi_t(x)=(|\phi(x)|-t)_+\qquad\text{and}\qquad\phi_{t,M}(x)=\min\{\phi_t(x),\,M\}.$$

The function $\phi_{t,M}$ belongs to $\mathrm{W}^{1,p}(\Omega)$, if we set $A_{t,M} = \{x \in \Omega : \phi_{t,M}(x) = 0\}$ then by Chebyshev's inequality

$$|\Omega \setminus A_{t,M}| := \big|\{x \in \Omega : \phi_{t,M}(x) \neq 0\}\big| \leq \frac{1}{t^r} \int_\Omega |\phi|^r \, dx. \tag{2.2}$$

From (2.1) we get

$$\int_\Omega |\phi_{t,M}|^p \, dx \leq C \frac{|\Omega|}{|A_{t,M}|} \int_\Omega |\nabla \phi_{t,M}|^p \, dx,$$

and observe that from (2.2)

$$\frac{|\Omega|}{|A_{t,M}|} = \frac{|\Omega|}{|\Omega| - |\Omega \setminus A_{t,M}|} \leq \frac{1}{2}, \qquad \text{if we choose } t = \left(\frac{2}{|\Omega|}\right)^{1/r} \|\phi\|_{L^r(\Omega)}.$$

We thus obtain

$$\int_\Omega |\phi_{t,M}|^p \, dx \leq \frac{C}{2} \int_\Omega |\nabla \phi|^p \, dx.$$

It is now possible to take the limit as M goes to ∞, thus getting by Fatou's Lemma

$$\int_\Omega |\phi_t|^p \, dx \leq \frac{C}{2} \int_\Omega |\nabla \phi|^p \, dx.$$

By recalling the choice of t and observing that $|\phi| \leq t + \phi_t$, we get the desired conclusion. □

2.2. **Some tools from Optimal Transport.** We recall a couple of standard results in Optimal Transport that will be needed for the proof of the main result. For more details, the reader is invited to refer to classical monographs such as [2] or [23], or to the more recent one [22].

Definition 2.3. The *$m-$Wasserstein space over Ω* is the set $\mathcal{P}_m(\Omega)$ defined in (1.1), equipped with the Wasserstein distance W_m. This metric space will be denoted by $\mathbb{W}_m(\Omega)$.

The first important tool we need is a characterization of geodesics in the Wasserstein space. This is essentially a refined version of the celebrated Benamou-Brenier formula, firstly introduced in [5]. The proof can be found in [22, Theorem 5.14 & Proposition 5.30].

Proposition 2.4 (Wasserstein geodesics)**.** *Let $1 < m < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Let $\rho_0, \rho_1 \in \mathbb{W}_m(\Omega)$, then there exist an absolutely continuous curve $(\mu_t)_{t\in[0,1]}$ in the Wasserstein space $\mathbb{W}_m(\Omega)$ and a vector field $\mathbf{v}_t \in L^m(\Omega; \mu_t)$ such that*

- *$\mu_0 = \rho_0$ and $\mu_1 = \rho_1$;*
- *the continuity equation*

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mathbf{v}_t \, \mu_t) &= 0, \quad \text{in } \Omega, \\ \langle \mathbf{v}_t, \nu_\Omega \rangle &= 0, \quad \text{on } \partial\Omega \end{cases}$$

holds in distributional sense, i.e. for every $\phi \in C^1([0,1] \times \overline{\Omega})$ there holds

$$\int_0^1 \int_\Omega \partial_t \phi \, d\mu_t \, dt + \int_0^1 \int_\Omega \langle \nabla \phi, \mathbf{v}_t \rangle \, d\mu_t \, dt = \int_\Omega \phi(1,\cdot) \, d\rho_1 - \int_\Omega \phi(0,\cdot) \, d\rho_0;$$

- *we have*

$$\left(\int_0^1 \|\mathbf{v}_t\|^m_{L^m(\Omega;\mu_t)}\,dt\right)^{\frac{1}{m}} = W_m(\rho_0,\rho_1).$$

The other expedient result from Optimal Transport we need is the following convexity property of L^q norms. For $m = 2$, this is a particular case of a result by McCann, see [17]. The proof can be found, for instance, in [22, Theorem 7.28].

Proposition 2.5 (Geodesic convexity of L^p norms)**.** *Let $1 < m < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Let $\rho_0 = f_0 \cdot \mathcal{L}^N$ and $\rho_1 = f_1 \cdot \mathcal{L}^N$ be two probability measures on Ω, such that $f_0, f_1 \in L^q(\Omega)$ for some $1 \le q \le \infty$. If $(\mu_t)_{t\in[0,1]} \subset \mathbb{W}_m(\Omega)$ is the curve of Theorem 2.4, then we have*

$$\mu_t = f_t \cdot \mathcal{L}^N \qquad \text{and} \qquad \|f_t\|_{L^q(\Omega)} \le \Big((1-t)\,\|f_0\|^q_{L^q(\Omega)} + t\,\|f_1\|^q_{L^q(\Omega)}\Big)^{\frac{1}{q}}, \quad t \in [0,1].$$

3. Proof of the main result

3.1. **An expedient estimate.** We first need the following preliminary result. The idea of the proof is similar to that of [11, Proposition 2.6] and [15, Lemma 3.5], though the final outcome is different. We also cite the short unpublished note [20] containing interesting uniform estimates on these topics.

Lemma 3.1. *Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. For every $\phi \in \mathrm{W}^{1,p}(\Omega)$ and every $f_0, f_1 \in L^{q'}(\Omega)$ such that*

$$\int_\Omega f_0\,dx = \int_\Omega f_1\,dx = 1, \qquad f_0, f_1 \ge 0,$$

we have

$$\int_\Omega \phi\,(f_1 - f_0)\,dx \le W_{\frac{p}{p-q}}(\rho_0,\rho_1)\,\|\nabla\phi\|_{L^p(\Omega)} \left(\frac{\|f_0\|^{q'}_{L^{q'}(\Omega)} + \|f_1\|^{q'}_{L^{q'}(\Omega)}}{2}\right)^{\frac{q-1}{p}}, \tag{3.1}$$

where

$$\rho_i = f_i \cdot \mathcal{L}^N, \qquad i = 0, 1,$$

Proof. Let us first suppose that $\phi \in C^1(\overline{\Omega})$. In this case we clearly have $C^1(\overline{\Omega}) \subset \mathrm{W}^{1,p}(\Omega)$.

For notational simplicity we set $r := p/(p-q)$. Then, by using Propositions 2.4 and 2.5 with $\rho_0 = f_0 \cdot \mathcal{L}^N$ and $\rho_1 = f_1 \cdot \mathcal{L}^N$ and observing that ϕ does not depend on t, with the previous notation we can infer

$$\begin{aligned}\int_\Omega \phi\,(f_1 - f_0)\,dx &= \int_0^1 \int_\Omega \langle \nabla \phi, \mathbf{v}_t\rangle\, f_t\,dx\,dt \\ &\le \left(\int_0^1 \int_\Omega |\nabla \phi|^{\frac{p}{q}}\, f_t\,dx\,dt\right)^{\frac{q}{p}} \left(\int_0^1 \int_\Omega |\mathbf{v}_t|^r\, f_t\,dx\,dt\right)^{\frac{1}{r}} \\ &\le \left(\int_0^1 \int_\Omega |\nabla \phi|^p\,dx\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt\right)^{\frac{q-1}{p}} W_r(\rho_0,\rho_1).\end{aligned}$$

Observe that the last term is finite, since $f_t \in L^{q'}(\Omega)$ and its $L^{q'}$ norm is integrable in time, thanks to Proposition 2.5.

Since ϕ does not depend on t, from the previous estimate we get in particular

$$\int_\Omega \phi\,(f_1 - f_0)\,dx \le W_r(\rho_0,\rho_1)\,\|\nabla\phi\|_{L^p(\Omega)} \left(\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt\right)^{\frac{q-1}{p}}.$$

We now observe that by Proposition 2.5

$$\begin{aligned}\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt &\le \int_0^1 \left[\|f_0\|^{q'}_{L^{q'}(\Omega)} + t\left(\|f_1\|^{q'}_{L^{q'}(\Omega)} - \|f_0\|^{q'}_{L^{q'}(\Omega)}\right)\right] dt \\ &= \frac{\|f_0\|^{q'}_{L^{q'}(\Omega)} + \|f_1\|^{q'}_{L^{q'}(\Omega)}}{2}.\end{aligned}$$

thus we obtain the desired estimate (3.1), for $\phi \in C^1(\overline{\Omega})$.

Finally, we get the general case by using that for a convex set $C^1(\overline{\Omega})$ is dense in $\mathrm{W}^{1,p}(\Omega)$, see [16, Theorem 1, Section 1.1.6]. □

3.2. **Proof of Theorem 1.1.** We divide the proof in two steps: we first prove the inequality for bounded convex sets and then consider the general case. For notational simplicity, we set again $r := p/(p-q)$.

<u>*Bounded convex sets.*</u> Let $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)\setminus\{0\}$, the hypothesis $\int_\Omega |\phi|^{q-2}\,\phi = 0$ implies

$$\int_\Omega |\phi|^{q-1}\,dx = 2\int_\Omega |\phi|^{q-2}\,\phi_+\,dx = 2\int_\Omega |\phi|^{q-2}\,\phi_-\,dx. \tag{3.2}$$

By Lemma 2.2, we have $\phi \in \mathrm{W}^{1,p}(\Omega)$ as well, thus we can now apply (3.1) with the choices

$$\rho_1 = f_1 \cdot \mathcal{L}^N := \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+\,dx}\cdot \mathcal{L}^N \quad \text{and} \quad \rho_0 = f_0 \cdot \mathcal{L}^N = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_-\,dx}\cdot \mathcal{L}^N.$$

For the left-hand side of (3.1), by using (3.2) we get

$$\int_\Omega \phi\,(f_1 - f_0)\,dx = 2\,\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\displaystyle\int_\Omega |\phi|^{q-1}\,dx}.$$

For the right-hand side of (3.1), we observe that again by (3.2) and using that

$$|\phi|^{q-2}\,\phi_+ = \phi_+^{q-1}, \qquad |\phi|^{q-2}\,\phi_- = \phi_-^{q-1},$$

we get

$$\begin{aligned}\|f_0\|_{L^{q'}(\Omega)}^{q'} + \|f_1\|_{L^{q'}(\Omega)}^{q'} &= \frac{\displaystyle\int_\Omega \left(|\phi|^{q-2}\,\phi_-\right)^{\frac{q}{q-1}}\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_-\,dx\right)^{\frac{q}{q-1}}} + \frac{\displaystyle\int_\Omega \left(|\phi|^{q-2}\,\phi_+\right)^{\frac{q}{q-1}}\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+\,dx\right)^{\frac{q}{q-1}}}\\ &= 2^{\frac{q}{q-1}}\,\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-1}\,dx\right)^{\frac{q}{q-1}}}.\end{aligned}$$

Then from (3.1) we finally obtain

$$\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\displaystyle\int_\Omega |\phi|^{q-1}\,dx} \le \frac{W_r(\rho_0,\rho_1)}{2^{\frac{p-1}{p}}}\,\left(\int_\Omega |\nabla\phi|^p\,dx\right)^{\frac{1}{p}}\,\frac{\left(\displaystyle\int_\Omega |\phi|^q\,dx\right)^{\frac{q-1}{p}}}{\left(\displaystyle\int_\Omega |\phi|^{q-1}\,dx\right)^{\frac{q}{p}}}.$$

After a simplification, this proves the desired inequality (1.3) when Ω is a bounded set.

General convex sets. We now take a generic open convex set Ω and $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)\setminus\{0\}$. We can suppose that the origin belongs to Ω, then for $k \in \mathbb{N}\setminus\{0\}$ we define

$$\Omega_k = \{x\in\Omega\,:\,|x|<k\} \qquad \text{and} \qquad \delta_k = \left(\frac{\displaystyle\int_{\Omega_k} |\phi_+|^{q-1}\,dx}{\displaystyle\int_{\Omega_k} |\phi_-|^{q-1}\,dx}\right)^{1/(q-1)}.$$

Notice that, at least for k large, δ_k is well-defined, since

$$\lim_{k\to\infty}\int_{\Omega_k} |\phi_-|^{q-1}\,dx = \int_\Omega |\phi_-|^{q-1}\,dx,$$

and the last quantity is strictly positive, since $\phi \not\equiv 0$.

The function $\phi_k = \phi_+ - \delta_k\,\phi_-$ belongs to $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega_k)$, by construction. Moreover, since $\phi\in\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$, we have

$$\lim_{k\to\infty}\delta_k = 1. \tag{3.3}$$

We also set

$$\rho_{1,k} := \frac{|\phi_k|^{q-2}\,(\phi_k)_+}{\displaystyle\int_{\Omega_k} |\phi_k|^{q-2}\,(\phi_k)_+\,dx} \cdot \mathcal{L}^N = \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_{\Omega_k} |\phi|^{q-2}\,\phi_+\,dx} \cdot \mathcal{L}^N$$

and

$$\rho_{0,k} := \frac{|\phi_k|^{q-2}\,(\phi_k)_-}{\displaystyle\int_{\Omega_k} |\phi_k|^{q-2}\,(\phi_k)_-\,dx} \cdot \mathcal{L}^N = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_{\Omega_k} |\phi|^{q-2}\,\phi_-\,dx} \cdot \mathcal{L}^N.$$

Since Ω_k is bounded, from the previous step we obtain

$$\left(\int_{\Omega_k} |\phi_k|^q\,dx\right)^{p-q+1} \le \frac{(W_r(\rho_{0,k},\rho_{1,k}))^p}{2^{p-1}} \int_{\Omega_k} |\nabla\phi_k|^p\,dx \left(\int_{\Omega_k} |\phi_k|^{q-1}\,dx\right)^{p-q}. \tag{3.4}$$

We now observe that

$$\lim_{k\to\infty} W_r(\rho_{0,k},\rho_{1,k}) = W_r(\rho_0,\rho_1).$$

Indeed, it is enough to remark that we have $\rho_{i,k} \to \rho_i$ in $\mathbb{W}_r(\Omega)$ for $i = 0,1$. This follows from the fact that the convergence in $\mathbb{W}_r$ is equivalent to the weak convergence plus the convergence of the moments of order r (see for instance [22, Theorem 5.11]). Both conditions are easily seen to hold true here.

Moreover, by construction we have

$$|\phi_k|^{q-1} \cdot 1_{\Omega_k} \le (\max\{1,\delta_k\})^{q-1}\,|\phi|^{q-1} \cdot 1_\Omega,$$

and

$$|\nabla\phi_k|^p \cdot 1_{\Omega_k} \le (\max\{1,\delta_k\})^p\,|\nabla\phi|^p \cdot 1_\Omega.$$

If we use (3.3), we can pass to the limit as k goes to ∞ in (3.4), by using the Dominated Convergence Theorem on the right-hand side and Fatou's Lemma on the left-hand side. This finally gives (1.3) for a generic function $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$.

4. Some consequences

In this section, we discuss some functional inequalities which are contained in nuce in Theorem 1.1.

4.1. **General convex sets.**

We start with the following inequality, valid for general convex sets. We observe again that it is not necessary to assume $\phi \in L^q(\Omega)$.

Corollary 4.1. *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open convex set. For every $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ such that*

$$\int_\Omega |x|^{\frac{p}{p-q}}\,|\phi|^{q-1}\,dx < \infty,$$

we have

$$\left(\int_\Omega |\phi|^q\,dx\right)^{p-q+1} \le 2\left(\inf_{x_0\in\Omega}\int_\Omega |x-x_0|^{\frac{p}{p-q}}\,|\phi|^{q-1}\,dx\right)^{p-q}\int_\Omega |\nabla\phi|^p\,dx. \tag{4.1}$$

Proof. Let ϕ be a function as in the statement. We use the notations of Theorem 1.1 and take $\gamma_{opt} \in \Pi(\rho_0, \rho_1)$ an optimal transport plan for $W_r(\rho_0, \rho_1)$ (where, as usual, $r = p/(p-q)$). By using the triangle inequality we get

$$\begin{aligned} W_r(\rho_0, \rho_1) &\leq \left(\int_{\Omega\times\Omega} |x-x_0|^r \, d\gamma_{opt}\right)^{1/r} + \left(\int_{\Omega\times\Omega} |y-x_0|^r \, d\gamma_{opt}\right)^{1/r} \\ &= \left(\int_{\Omega} |x-x_0|^r \, d\rho_0\right)^{1/r} + \left(\int_{\Omega} |y-x_0|^r \, d\rho_1\right)^{1/r}, \end{aligned}$$

for every $x_0 \in \Omega$. By using concavity of the map $\tau \mapsto \tau^{1/r}$, this in turn gives

$$\begin{aligned} W_r(\rho_0, \rho_1) &\leq 2^{\frac{q}{p}} \left(\int_{\Omega} |x-x_0|^r \, (d\rho_0 + d\rho_1)\right)^{1/r} \\ &= 2 \left(\int_{\Omega} |x-x_0|^r \, |\phi|^{q-1} \, dx\right)^{1/r} \left(\int_{\Omega} |\phi|^{q-1} \, dx\right)^{\frac{q-p}{p}}, \end{aligned} \tag{4.2}$$

where we used again (3.2), by assumption. By using (4.2) in (1.3) and using the arbitrariness of $x_0 \in \Omega$, we get the desired result. □

4.2. **Bounded convex sets.** In this case, Theorem 1.1 implies some known inequalities, with explicit constants depending on simple geometric quantities and p only.

Corollary 4.2 (Nash-type inequality)**.** *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. Then for every $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$*

$$\left(\int_{\Omega} |\phi|^q \, dx\right)^{p-q+1} \leq \frac{\mathrm{diam}(\Omega)^p}{2^{p-1}} \int_{\Omega} |\nabla\phi|^p \, dx \left(\int_{\Omega} |\phi|^{q-1} \, dx\right)^{p-q}. \tag{4.3}$$

Proof. In order to prove (4.3), it is sufficient to observe that for a bounded set we have

$$W_r(\rho_0, \rho_1) \leq \mathrm{diam}(\Omega).$$

If we spend this information in (1.3), we can then conclude. □

Corollary 4.3 (Poincaré-Wirtinger inequality)**.** *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. Then for every $\phi \in \mathrm{W}^{1,p}_{q-1}(\Omega)$, there holds*

$$\min_{t\in\mathbb{R}} \left(\int_{\Omega} |\phi - t|^q \, dx\right)^{\frac{p}{q}} \leq \frac{\mathrm{diam}(\Omega)^p}{2^{p-1}} \, |\Omega|^{\frac{p}{q}-1} \int_{\Omega} |\nabla\phi|^p \, dx\,. \tag{4.4}$$

Proof. Let $\phi \in \mathrm{W}^{1,p}_{q-1}(\Omega)$, by Lemma 2.2 we know in particular that $\phi \in L^q(\Omega)$. Then we can define the unique minimizer t_q of

$$t \mapsto \left(\int_{\Omega} |\phi - t|^q \, dx\right)^{\frac{p}{q}}.$$

By minimality, we have

$$\int_{\Omega} |\phi - t_q|^{q-2} \, (\phi - t_q) \, dx = 0.$$

Thus the function $\phi - t_q$ belongs to $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$. We just need to observe that since $\phi - t_q \in L^q(\Omega)$, then

$$\left(\int_\Omega |\phi - t_q|^{q-1}\, dx\right)^{p-q} \le |\Omega|^{\frac{p-q}{q}} \left(\int_\Omega |\phi - t_q|^q\, dx\right)^{\frac{p-q}{q}\,(q-1)}.$$

By using this in (4.3) for the function $\phi - t_q$, we get the conclusion. □

Remark 4.4. Observe that the constant in (4.4) degenerates to 0 as the measure $|\Omega|$ gets smaller and smaller. This behaviour is optimal, as one may easily verify. Indeed, by taking $n \in \mathbb{N} \setminus \{0\}$ and

$$\Omega_n = [0,1] \times \left[0, \frac{1}{n}\right] \times \cdots \times \left[0, \frac{1}{n}\right] \qquad \text{and} \qquad \phi(x) = x_1, \tag{4.5}$$

we have

$$\frac{\displaystyle\min_{t\in\mathbb{R}} \left(\int_{\Omega_n} |\phi - t|^q\, dx\right)^{\frac{p}{q}}}{\displaystyle\int_{\Omega_n} |\nabla \phi|^p\, dx} \le \frac{\displaystyle\left(\int_{\Omega_n} |\phi|^q\, dx\right)^{\frac{p}{q}}}{\displaystyle\int_{\Omega_n} |\nabla \phi|^p\, dx} \simeq \left(\frac{1}{n}\right)^{(N-1)\,\frac{p-q}{q}} = |\Omega_n|^{\frac{p-q}{q}}.$$

We conclude this list with an application to spectral problems. Let $1 < p < \infty$, for every $\Omega \subset \mathbb{R}^N$ open and bounded set we introduce its *first non-trivial Neumann eigenvalue of the $p-$Laplacian*, i.e.

$$\mu(\Omega; p) := \inf_{\phi \in \mathrm{W}^{1,p}(\Omega)\setminus\{0\}} \left\{ \frac{\displaystyle\int_\Omega |\nabla \phi|^p\, dx}{\displaystyle\int_\Omega |\phi|^p\, dx} \,:\, \int_\Omega |\phi|^{p-2}\, \phi\, dx = 0 \right\}.$$

The terminology is justified by the fact that for a connected set with Lipschitz boundary, the costant $\mu(\Omega; p)$ attained and coincides with the smallest number different from 0 such that the Neumann boundary value problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\, \nabla u) &= \mu\, |u|^{p-2}\, u, \qquad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_\Omega} &= 0, \qquad \text{on } \partial\Omega \end{cases}$$

admits non-trivial weak solutions. We then have the following result, which corresponds to the limit case $q = p$ of Theorem 1.1.

Corollary 4.5 (Payne-Weinberger type estimate)**.** *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. We have the lower bound*

$$\left(\frac{2^{\frac{p-1}{p}}}{\mathrm{diam}(\Omega)}\right)^p \le \mu(\Omega; p). \tag{4.6}$$

Proof. We take $\phi \in W^{1,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega |\phi|^{p-2}\,\phi\,dx = 0$. Then we have

$$\min_{t\in\mathbb{R}} \int_\Omega |\phi - t|^p\,dx = \int_\Omega |\phi|^p\,dx. \tag{4.7}$$

For $1 < q < p$, we take $t_q \in \mathbb{R}$ to be the unique minimizer of

$$t \mapsto \left(\int_\Omega |\phi - t|^q\,dx\right)^{\frac{p}{q}}.$$

By minimality of t_q and Minkowski inequality, we have

$$t_q\,|\Omega|^{\frac{1}{q}} - \left(\int_\Omega |\phi|^q\,dx\right)^{\frac{1}{q}} \le \left(\int_\Omega |\phi - t_q|^q\,dx\right)^{\frac{1}{q}} \le \left(\int_\Omega |\phi|^q\,dx\right)^{\frac{1}{q}}.$$

This shows that $\{t_q\}_{q<p}$ is bounded, thus if we take the limit as q goes to p, then t_q converges (up to a subsequence) to some $\bar{t}$. By passing to the limit in (4.4) we get

$$\int_\Omega |\phi - \bar{t}|^p\,dx \le \frac{\operatorname{diam}(\Omega)^p}{2^{p-1}} \int_\Omega |\nabla \phi|^p\,dx.$$

By keeping into account (4.7), we get the desired conclusion. □

Remark 4.6. As mentioned in the Introduction, the constant appearing in the left-hand side of (4.6) is not sharp. Indeed, the sharp lower bound is known to be

$$\left(\frac{\pi_p}{\operatorname{diam}(\Omega)}\right)^p < \mu(\Omega;p), \qquad \text{where} \quad \pi_p = 2\,\pi\,\frac{(p-1)^{\frac{1}{p}}}{p\,\sin\left(\dfrac{\pi}{p}\right)}, \tag{4.8}$$

as proved by Payne and Weinberger in [19] for $p = 2$ (see also [4]). The general case $p \neq 2$ has been proved in [8, 9]. We recall that (4.8) is sharp in the following sense: for every convex set Ω the inequality in (4.8) is strict and it becomes asymptotically an equality along the sequence (4.5).

In the limit case $p = 1$, a related result can be found in [1].

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