

The continuity equation on metric measure spaces

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Abstract

Aim of this paper is to show that it makes sense to write the continuity equation on a metric measure space (X, d, \mathfrak{m}) and that absolutely continuous curves (μ_t) w.r.t. the distance W_2 can be completely characterized as solutions of the continuity equation itself, provided we impose the condition $\mu_t \leq C\mathfrak{m}$ for every t and some $C > 0$.

Keywords: Absolutely continuous curve, continuity equation, optimal transport, metric measure spaces.

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1 Introduction

A crucial intuition of Otto [13], inspired by the work of Benamou-Brenier [6], has been to realize that absolutely continuous curves of measures (μ_t) w.r.t. the quadratic transportation distance W_2 on \mathbb{R}^d can be interpreted as solutions of the continuity equation

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0, \tag{1.1}$$

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where the vector fields v_t should be considered as the ‘velocity’ of the moving mass μ_t and, for curves with square-integrable speed, satisfy

$$\int_0^1 \int |v_t|^2 d\mu_t dt < \infty. \quad (1.2)$$

This intuition has been made rigorous by the first author, Ambrosio and Savaré in [3], where it has been used to develop a solid first order calculus on the space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$, with particular focus on the study of gradient flows.

Heuristically speaking, the continuity equation describes the link existing between the ‘vertical derivative’ $\partial_t \mu_t$ (think to it as variation of the densities, for instance) and the ‘horizontal displacement’ v_t . In this sense it provides the crucial link between analysis made on the L^p spaces, where the distance is measured ‘vertically’, and the one based on optimal transportation, where distances are measured by ‘horizontal’ displacement. This is indeed the heart of the crucial substitution made by Otto in [13] who, to define the metric tensor g_μ on the space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ at a measure $\mu = \rho \mathcal{L}^d$ considers a ‘vertical’ variation $\delta \rho$ such that $\int \delta \rho d\mathcal{L}^d = 0$, then looks for solutions of

$$\delta \rho = -\nabla \cdot (\nabla \varphi \rho), \quad (1.3)$$

and finally defines

$$g_\mu(\delta \rho, \delta \rho) := \int |\nabla \varphi|^2 d\mu. \quad (1.4)$$

The substitution (1.3) is then another way of thinking at the continuity equation, while the definition (1.4) corresponds to the integrability requirement (1.2).

It is therefore not surprising that each time one wants to put in relation the geometry of optimal transport with that of L^p spaces some form of continuity equation must be studied. In the context of analysis on non-smooth structures, this has been implicitly done in [5, 10] to show that the gradient flow of the relative entropy on the space $(\mathcal{P}_2(X), W_2)$ produces the same evolution of the gradient flow of the energy (sometime called Cheeger energy or Dirichlet energy) in the space $L^2(X, \mathbf{m})$, where $(X, \mathbf{d}, \mathbf{m})$ is some given metric measure space.

The purpose of this paper is to make these arguments more explicit and to show that:

- i) It is possible to formulate the continuity equation on general metric measure spaces $(X, \mathbf{d}, \mathbf{m})$,
- ii) Solutions of the continuity equation completely characterize absolutely continuous curves $(\mu_t) \subset \mathcal{P}_2(X)$ with square-integrable speed w.r.t. W_2 and such that $\mu_t \leq C \mathbf{m}$ for every $t \in [0, 1]$ and some $C > 0$.

In fact, the techniques we use can directly produce similar results for the distances W_p , $p \in (1, \infty)$, and for curves whose speed is in L^1 rather than in some L^p , $p > 1$. Yet, we prefer not to discuss the full generality in order to concentrate on the main ideas.

Let us discuss how to formulate the continuity equation on a metric measure space where no a priori smooth structure is available. Notice that in the smooth setting (1.1) has to be understood in the sense of distributions. If we assume weak continuity of (μ_t) , this is equivalently formulated as the fact that for every $f \in C_c^\infty(\mathbb{R}^d)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous and the identity

$$\frac{d}{dt} \int f d\mu_t = \int df(v_t) d\mu_t,$$

holds for a.e. $t \in [0, 1]$. In other words, the vector fields v_t only act on differential of smooth functions and can therefore be thought of as linear functionals L_t from the space of differentials of smooth functions to \mathbb{R} . Recalling (1.2), the norm $\|L_t\|_{\mu_t}^*$ of L_t should be defined as

$$\frac{1}{2} (\|L_t\|_{\mu_t}^*)^2 = \sup_{f \in C_c^\infty(\mathbb{R}^d)} L_t(f) - \frac{1}{2} \int |df|^2 d\mu_t,$$

so that being (μ_t) 2-absolutely continuous is equivalent to require that $t \mapsto \|L_t\|_{\mu_t}^* \in L^2(0, 1)$.

Seeing the continuity equation in this way allows for a formulation of it in the abstract context of metric measure spaces $(X, \mathbf{d}, \mathbf{m})$. Indeed, recall that there is a well established notion of ‘space of functions having distributional differential in $L^2(X, \mathbf{m})$ ’, which we will denote by $S^2(X) = S^2(X, \mathbf{d}, \mathbf{m})$ and that for each function $f \in S^2(X)$ it is well defined the ‘modulus of the distributional differential $|Df| \in L^2(X, \mathbf{m})$ ’.

Then given a linear map $L : S^2(X) \rightarrow \mathbb{R}$ and μ such that $\mu \leq C\mathbf{m}$ for some $C > 0$ we can define the norm $\|L\|_\mu^*$ as

$$\frac{1}{2}(\|L\|_\mu^*)^2 := \sup_{f \in S^2(X)} L(f) - \frac{1}{2} \int |Df|^2 d\mu. \quad (1.5)$$

Hence given a curve $(\mu_t) \subset \mathcal{P}_2(X)$ such that $\mu_t \leq C\mathbf{m}$ for some $C > 0$ and every $t \in [0, 1]$ and a family $\{L_t\}_{t \in [0, 1]}$ of maps from $S^2(X)$ to \mathbb{R} such that $\int_0^1 (\|L_t\|_{\mu_t}^*)^2 dt < \infty$, we can say that the curve $(\mu_t) \subset \mathcal{P}_2(X)$ solves the continuity equation

$$\partial_t \mu_t = L_t,$$

provided:

- i) for every $f \in S^2(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous,
- ii) the identity

$$\frac{d}{dt} \int f d\mu_t = L_t(f), \quad (1.6)$$

holds for a.e. t .

Then we show that such formulation of the continuity equation fully characterizes absolutely continuous curves (μ_t) with square-integrable speed on the space $(\mathcal{P}_2(X), W_2)$, provided we restrict the attention to curves such that $\mu_t \leq C\mathbf{m}$ for some $C > 0$ and every $t \in [0, 1]$. See Theorem 3.5.

Concerning the proof of this result, we remark that the implication from absolute continuity of (μ_t) to the ‘PDE’ (1.6) is quite easy to establish and follows essentially from the definition of Sobolev functions. This is the easy implication even in the smooth context whose proof carries over quite smoothly to the abstract setting, the major technical difference being that we don’t know if in general the space $S^2(X)$ is separable or not, a fact which causes some complications in the way we can really write down the equation (1.6), see Definition 3.4.

The converse one is more difficult, as it amounts in proving that the differential identity (1.6) is strong enough to guarantee absolute continuity of the curve. The method used in the Euclidean context consists in regularizing the curve, apply the Cauchy-Lipschitz theory to the approximating sequence to find a flow of the approximating vector fields which can be used to transport μ_t to μ_s and finally in passing to the limit. By nature, this approach cannot be used in non-smooth situations. Instead, we use a crucial idea due to Kuwada which has already been applied to study the heat flow [5, 10]. It amounts in passing to the dual formulation of the optimal transport problem by noticing that

$$\frac{1}{2}W_2^2(\mu_1, \mu_0) = \sup \int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0, \quad (1.7)$$

the sup being taken among all Lipschitz and bounded $\varphi : X \rightarrow \mathbb{R}$, where $Q_t \varphi$ is the evolution of φ via the Hopf-Lax formula. A general result obtained in [5] has been that it holds

$$\frac{d}{dt} Q_t \varphi(x) + \frac{\text{lip}(Q_t \varphi)^2(x)}{2} \leq 0, \quad (1.8)$$

for every t except a countable number, where $\text{lip}(f)$ is the local Lipschitz constant f . Thus we can formally write

$$\begin{aligned} \int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0 &= \int_0^1 \frac{d}{dt} \int Q_t \varphi d\mu_t dt \\ \text{by (1.6)} \quad &= \int_0^1 \int \frac{d}{dt} Q_t \varphi d\mu_t dt + \int_0^1 L_t(Q_t \varphi) dt, \\ \text{by (1.5), (1.8)} \quad &\leq \int_0^1 -\frac{\text{lip}(Q_t \varphi)^2}{2} d\mu_t dt + \frac{1}{2} \int_0^1 (\|L_t\|_{\mu_t}^*)^2 dt + \frac{1}{2} \int_0^1 \int |DQ_t \varphi|^2 d\mu_t dt. \end{aligned}$$

Using the fact that $|Df| \leq \text{lip}(f)$ \mathfrak{m} -a.e. for every Lipschitz f we then conclude that

$$\int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0 \leq \frac{1}{2} \int_0^1 (\|L_t\|_{\mu_t}^*)^2 dt.$$

Here the right hand side does not depend on φ , hence by (1.7) we deduce

$$W_2^2(\mu_1, \mu_0) \leq \int_0^1 (\|L_t\|_{\mu_t}^*)^2 dt,$$

which bounds W_2 in terms of the L_t 's only. Replacing $0, 1$ with general $t, s \in [0, 1]$ we deduce the desired absolute continuity. As presented here, the computation is only formal, but a rigorous justification can be given, thus leading to the result. See the proof of Theorem 3.5.

It is worth pointing out that Kuwada's lemma works even if we don't know any uniqueness result for the initial value problem (1.6). That is we don't know if given μ_0 and a family of operators L_t from $S^2(X)$ to \mathbb{R} the solution of (1.6) is unique or not, because we "can't follow the flow of the L_t 's". Yet, it is possible to deduce anyway that any solution is absolutely continuous.

It is also worth to make some comments about the assumption $\mu_t \leq C\mathfrak{m}$. Notice that if we don't impose any condition on the μ_t 's, we could consider curves of the kind $t \mapsto \delta_{\gamma_t}$, where γ is a given Lipschitz curve. In the smooth setting we see that such curve solves

$$\partial_t \delta_{\gamma_t} + \nabla \cdot (\gamma_t' \delta_{\gamma_t}) = 0,$$

so that to write the continuity equation for such curve amounts to know the value of γ_t' at least for a.e. t . In the non-smooth setting to do this would mean to know who is the tangent space at γ_t for a.e. t along a Lipschitz curve γ , an information which without any assumption on X seems quite too strong. Instead, the process of considering only measures with bounded density has the effect of somehow 'averaging out the unsmoothness of the space' and allows for the possibility of building a working differential calculus, a point raised and heavily used in [9]. Here as application of the continuity equation to differential calculus we provide a Benamou-Brenier formula and describe the derivative of $\frac{1}{2}W_2^2(\cdot, \nu)$ along an absolutely continuous curve.

We then study situations where the operators L_t can be given somehow more explicitly. Recall that on the Euclidean setting the optimal (in the sense of energy-minimizer) vector fields v_t appearing in (1.1) always belong to the $L^2(\mu_t)$ -closure of the set of gradients of smooth functions and that in some case they are really gradient of functions, so that (1.1) can be written as

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0, \tag{1.9}$$

for some given smooth $\{\phi_t\}_{t \in [0, 1]}$, which means that for f smooth it holds

$$\frac{d}{dt} \int f d\mu_t = \int df(\nabla \phi_t) d\mu_t.$$

To interpret the equation (1.9) in the abstract framework we need to understand the duality relation between differentials and gradients of Sobolev functions on metric measure spaces. This has been done in [8], where for given $f, g \in S^2(X)$ the two functions $D^-f(\nabla g)$ and $D^+f(\nabla g)$ have been introduced. If the space is a Riemannian manifold or a Finsler one with norms strictly convex, then we have $D^-f(\nabla g) = D^+f(\nabla g)$ a.e. for every f, g , these being equal to the value of the differential of f applied to the gradient of g obtained by standard means. In the general case we do not have such single-valued duality, due to the fact that even in a flat normed situation the gradient of a function is not uniquely defined should the norm be not strictly convex. Thus the best we can do is to define $D^-f(\nabla g)$ and $D^+f(\nabla g)$ as being, in a sense, the minimal and maximal value of the differential of f applied to all the gradients of g .

Then we can interpret (1.9) in the non-smooth situation by requiring that for $f \in S^2(X)$ it holds

$$\int D^-f(\nabla \phi_t) d\mu_t \leq \frac{d}{dt} \int f d\mu_t \leq \int D^+f(\nabla \phi_t) d\mu_t, \quad a.e. t,$$

and it turns out that this way of writing the continuity equation, which requires two inequalities rather than an equality, is still sufficient to grant absolute continuity of the curve.

Notice that in the Euclidean setting, if the functions ϕ_t are smooth enough we can construct the flow associated to $\nabla\phi_t$ by solving

$$\begin{cases} \frac{d}{dt}T(x, t, s) = \nabla\phi_t(T(x, t, s)), \\ T(x, t, t) = x, \end{cases}$$

so that the curves $t \mapsto T(x, t, s)$ are gradient flows of the evolving function ϕ_t and a curve (μ_t) solves (1.6) if and only if $\mu_t = T(\cdot, t, 0)_\# \mu_0$ for every $t \in [0, 1]$. Interestingly enough, this point of view can be made rigorous even in the setting of metric measure spaces and a similar characterization of solutions of (1.6) can be given, see Theorem 4.6.

We conclude the paper by showing that the heat flows and the geodesics satisfy the same sort of continuity equation they satisfy in the smooth case, namely

$$\partial_t \mu_t + \nabla \cdot (\nabla(-\log(\rho_t))\mu_t) = 0,$$

for the heat flow, where $\mu_t = \rho_t \mathbf{m}$, and

$$\partial_t \mu_t + \nabla \cdot (\nabla\phi_t \mu_t) = 0,$$

with $\phi_t = -Q_{1-t}(-\varphi^c)$ for the geodesics, where φ is a Kantorovich potential inducing the geodesic. Here the aim is not to prove new results, as these two examples were already considered in the literature [4, 5, 9, 10], but rather to show that they are compatible with the theory we propose. We also discuss in which sense and under which circumstances an heat flow and a geodesic can be considered not just as absolutely continuous curves on $(\mathcal{P}_2(X), W_2)$, but rather as C^1 curves.

2 Preliminaries

2.1 Metric spaces and optimal transport

We quickly recall here those basic facts about analysis in metric spaces and optimal transport we are going to use in the following. Standard references are [3], [14] and [2].

Let (X, d) be a metric space. Given $f : X \rightarrow \mathbb{R}$ the local Lipschitz constant $\text{lip}(f) : X \rightarrow [0, \infty]$ is defined as

$$\text{lip}(f)(x) := \overline{\lim}_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(x, y)},$$

if x is not isolated and 0 otherwise. Recall that the Lipschitz constant of Lipschitz function is defined as:

$$\text{Lip}(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{d(x, y)}$$

In particular, if (X, d) is a geodesic space, we have $\text{Lip}(f) = \sup_x \text{lip}(f)(x)$.

A curve $\gamma : [0, 1] \rightarrow X$ is said absolutely continuous provided there exists $f \in L^1(0, 1)$ such that

$$d(\gamma_s, \gamma_t) \leq \int_t^s f(r) dr, \quad \forall t, s \in [0, 1], t < s. \quad (2.1)$$

For an absolutely continuous curve γ it can be proved that the limit

$$\lim_{h \rightarrow 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|},$$

exists for a.e. t and thus defines a function, called metric speed and denoted by $|\dot{\gamma}_t|$, which is in $L^1(0, 1)$ and is minimal, in the a.e. sense, among the class of L^1 -functions f for which (2.1) holds.

If there exists $f \in L^2(0, 1)$ for which (2.1) holds, we say that the curve is 2-absolutely continuous (2-a.c. in short). In the following we will often write $\int_0^1 |\dot{\gamma}_t|^2 dt$ for a curve γ which a priori is only continuous: in this case the value of the integral is taken by definition $+\infty$ if γ is not absolutely continuous.

The space of continuous curves on $[0, 1]$ with values in X will be denoted by $C([0, 1], X)$ and equipped with the sup distance. Notice that if (X, d) is complete and separable, then $C([0, 1], X)$ is complete and separable as well. For $t \in [0, 1]$ we denote by $e_t : C([0, 1], X) \rightarrow X$ the *evaluation map* defined by

$$e_t(\gamma) := \gamma_t, \quad \forall \gamma \in C([0, 1], X).$$

For $t, s \in [0, 1]$ the map restr_t^s from $C([0, 1], X)$ to itself is given by

$$(\text{restr}_t^s \gamma)_r := \gamma_{t+r(s-t)}, \quad \forall \gamma \in C([0, 1], X).$$

The set of Borel probability measures on X is denoted by $\mathcal{P}(X)$ and $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ is the space of probability measures with finite second moment. We equip $\mathcal{P}_2(X)$ with the quadratic transportation distance W_2 defined by

$$W_2^2(\mu, \nu) := \inf \int d^2(x, y) d\gamma(x, y), \quad (2.2)$$

the inf being taken among all $\gamma \in \mathcal{P}(X^2)$ such that

$$\begin{aligned} \pi_{\#}^1 \gamma &= \mu, \\ \pi_{\#}^2 \gamma &= \nu. \end{aligned}$$

Given $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ not identically $-\infty$ the c -transform $\varphi^c : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined by

$$\varphi^c(y) := \inf_{x \in X} \frac{d^2(x, y)}{2} - \varphi(x).$$

φ is said c -concave provided it is not identically $-\infty$ and $\varphi = \psi^c$ for some $\psi : X \rightarrow \mathbb{R} \cup \{-\infty\}$. Equivalently, φ is c -concave if it is not identically $-\infty$ and $\varphi^{cc} = \varphi$. Given a c -concave function φ , its c -superdifferential $\partial^c \varphi \subset X^2$ is defined as the set of (x, y) such that

$$\varphi(x) + \varphi^c(y) = \frac{d^2(x, y)}{2},$$

and for $x \in X$ the set $\partial^c \varphi(x)$ is the set of y 's such that $(x, y) \in \partial^c \varphi$. Notice that for general $(x, y) \in X^2$ we have $\varphi(x) + \varphi^c(y) \leq \frac{d^2(x, y)}{2}$, thus $y \in \partial^c \varphi(x)$ can be equivalently formulated as

$$\varphi(z) - \varphi(x) \leq \frac{d^2(z, y)}{2} - \frac{d^2(x, y)}{2}, \quad \forall z \in X.$$

It turns out that for $\mu, \nu \in \mathcal{P}_2(X)$ the distance $W_2(\mu, \nu)$ can be found as maximization of the dual problem of the optimal transport (2.2):

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup \int \varphi d\mu + \int \varphi^c d\nu, \quad (2.3)$$

the sup being taken among all c -concave functions φ . Notice that the integrals in the right hand side are well posed because for φ c -concave and $\mu, \nu \in \mathcal{P}_2(X)$ we always have $\max\{\varphi, 0\} \in L^1(\mu)$ and $\max\{\varphi^c, 0\} \in L^1(\nu)$. The sup is always achieved and any maximizing φ is called Kantorovich potential from μ to ν . For any Kantorovich potential we have in particular $\varphi \in L^1(\mu)$ and $\varphi^c \in L^1(\nu)$. Equivalently, the sup in (2.3) can be taken among all $\varphi : X \rightarrow \mathbb{R}$ Lipschitz and bounded.

We shall make frequently use of the following superposition principle, proved in [12] (see also the original argument in the Euclidean framework [3]):

Proposition 2.1. *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a 2-a.c. curve w.r.t. W_2 . Then there exists $\pi \in \mathcal{P}(C([0, 1], X))$ such that*

$$\begin{aligned} (e_t)_\# \pi &= \mu_t, & \forall t \in [0, 1], \\ \int_0^1 |\dot{\mu}_t|^2 dt &= \iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma), \end{aligned}$$

and in particular π is concentrated on the set of 2-a.c. curves on X . For any such π we also have

$$|\dot{\mu}_t|^2 = \int |\dot{\gamma}_t|^2 d\pi(\gamma), \quad \text{a.e. } t \in [0, 1].$$

Any plan π associated to the curve (μ_t) as in the above proposition will be called *lifting* of (μ_t) .

2.2 Metric measure spaces and Sobolev functions

Spaces of interest for this paper are metric measure spaces (X, d, \mathbf{m}) which will always be assumed to satisfy:

- (X, d) is complete and separable,
- the measure \mathbf{m} is a non-negative and non-zero Radon measure on X .

In this paper, for the abbreviation we will not distinguish X , (X, d) or (X, d, \mathbf{m}) when no ambiguities exist. For example, we write $S^2(X)$ instead of $S^2(X, d, \mathbf{m})$.

Given a curve $(\mu_t) \subset \mathcal{P}(X)$ we shall say that it has *bounded compression* provided there is $C > 0$ such that $\mu_t \leq C\mathbf{m}$ for every $t \in [0, 1]$. Similarly, given $\pi \in \mathcal{P}(C([0, 1], X))$ we shall say that it has bounded compression provided $t \mapsto \mu_t := (e_t)_\# \pi$ has bounded compression.

We shall now recall the definition of Sobolev functions ‘having distributional differential in $L^2(X, \mathbf{m})$ ’. The definition we adopt comes from [8] which in turn is a reformulation of the one proposed in [5]. For the proof that this approach produces the same concept as the one discussed in [11] and its references, see [5].

Definition 2.2 (Test plans). Let (X, d, \mathbf{m}) be a metric measure space and $\pi \in \mathcal{P}(C([0, 1], X))$. We say that π is a test plan provided it has bounded compression and

$$\iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) < +\infty.$$

Definition 2.3 (The Sobolev class $S^2(X)$). Let (X, d, \mathbf{m}) be a metric measure space. The Sobolev class $S^2(X) = S^2(X, d, \mathbf{m})$ is the space of all Borel functions $f : X \rightarrow \mathbb{R}$ such that there exists a function $G \in L^2(X, \mathbf{m})$, $G \geq 0$ such that:

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq \iint_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma),$$

for every test plan π . In this case, G is called a weak upper gradient of f .

It can be proved that for $f \in S^2(X)$ there exists a minimal, in the \mathbf{m} -a.e. sense, weak upper gradient: we shall denote it by $|Df|$.

Basic calculus rules for $|Df|$ are the following, all the expressions being intended \mathbf{m} -a.e.:

Locality For every $f, g \in S^2(X)$ we have

$$|Df| = 0, \quad \text{on } f^{-1}(N), \quad \forall N \subset \mathbb{R}, \text{ Borel with } \mathcal{L}^1(N) = 0, \quad (2.4)$$

$$|Df| = |Dg|, \quad \mathbf{m} - \text{a.e. on } \{f = g\}. \quad (2.5)$$

Weak gradients and local Lipschitz constants. For any $f : X \rightarrow \mathbb{R}$ locally Lipschitz it holds

$$|Df| \leq \text{lip}(f). \quad (2.6)$$

Vector space structure. $S^2(X)$ is a vector space and for every $f, g \in S^2(X)$, $\alpha, \beta \in \mathbb{R}$ we have

$$|D(\alpha f + \beta g)| \leq |\alpha| |Df| + |\beta| |Dg|. \quad (2.7)$$

Algebra structure. $L^\infty \cap S^2(X)$ is an algebra and for every $f, g \in L^\infty \cap S^2(X)$ we have

$$|D(fg)| \leq |f| |Dg| + |g| |Df|. \quad (2.8)$$

Similarly, if $f \in S^2(X)$ and g is Lipschitz and bounded, then $fg \in S^2(X)$ and the bound (2.8) holds.

Chain rule. Let $f \in S^2(X)$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz. Then $\varphi \circ f \in S^2(X)$ and

$$|D(\varphi \circ f)| = |\varphi'| \circ f |Df|, \quad (2.9)$$

where $|\varphi'| \circ f$ is defined arbitrarily at points where φ is not differentiable (observe that the identity (2.4) ensures that on $f^{-1}(\mathcal{N})$ both $|D(\varphi \circ f)|$ and $|Df|$ are 0 \mathbf{m} -a.e., \mathcal{N} being the negligible set of points of non-differentiability of φ).

We equip $S^2(X)$ with the seminorm $\|f\|_{S^2} := \| |Df| \|_{L^2(X)}$. In the following, we will sometimes need to work with spaces (X, d, \mathbf{m}) such that $S^2(X)$ is separable, we thus recall the following general criterion:

Proposition 2.4. *Let (X, d, \mathbf{m}) be a metric measure space where \mathbf{m} is locally bounded. If the Sobolev space $W^{1,2}(X, d, \mathbf{m})$ defined as $L^2 \cap S^2(X)$ equipped with the norm $\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + \|f\|_{S^2}^2$ is reflexive, then $S^2(X)$ is separable.*

In particular, let (X, d, \mathbf{m}) be a metric measure space satisfying one of the following properties below:

- i) (X, d) is doubling, i.e. there is $N \in \mathbb{N}$ such that for any $r > 0$ any ball of radius $2r$ can be covered by N balls of radius r . And \mathbf{m} is locally bounded.*
- ii) The seminorm $\|\cdot\|_{S^2}$ satisfies the parallelogram rule, and \mathbf{m} gives finite mass to bounded sets.*

We know $S^2(X)$ is separable.

Proof. In [1] it has been proved that if $W^{1,2}(X)$ is reflexive, then it is separable.

Thus to conclude it is sufficient to show that if $W^{1,2}(X)$ is separable and \mathbf{m} gives finite mass to bounded sets (this being trivially true in the case (i)), then $S^2(X)$ is separable as well. To this aim, let $f \in S^2(X)$, consider the truncated functions $f_n := \min\{n, \max\{-n, f\}\}$ and notice that thanks to (4.5) we have $\|f_n - f\|_{S^2} \rightarrow 0$ as $n \rightarrow \infty$. Thus we can reduce to consider the case of $f \in L^\infty \cap S^2(X)$. Let $B_n \subset X$ be bounded nondecreasing sequence of sets covering X , and for each $n \in \mathbb{N}$, $\chi_n : X \rightarrow [0, 1]$ a 1-Lipschitz function with bounded support and identically 1 on B_n . For $f \in L^\infty \cap S^2(X)$, by (2.8) we have $f\chi_n \in L^\infty \cap S^2(X)$ as well and furthermore $\text{supp}(\chi_n f)$ is bounded. Given that \mathbf{m} gives finite mass to bounded sets we deduce that $f\chi_n \in W^{1,2}(X)$, and the locality property (2.5) ensures that $\|\chi_n f - f\|_{S^2} \rightarrow 0$ as $n \rightarrow \infty$.

At last, still in [1], it has been shown that if (X, d) is doubling, then $W^{1,2}(X)$ is reflexive. On the other hand, if (ii) holds, then it is obvious that $W^{1,2}(X)$ is Hilbert, and hence reflexive. Therefore $S^2(X)$ is separable if (i) or (ii) holds. \square

2.3 Hopf-Lax formula and Hamilton-Jacobi equation

Here we recall the main properties of the Hopf-Lax formula and its link with the Hamilton-Jacobi equation in a metric setting. For a proof of these results see [5].

Definition 2.5 (Hopf-Lax formula). Given $f : X \rightarrow \mathbb{R}$ a function and $t > 0$ we define $Q_t f : X \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$Q_t f(x) := \inf_{y \in X} f(y) + \frac{d^2(x, y)}{2}.$$

We also put $Q_0 f := f$.

Proposition 2.6 (Basic properties of the Hopf-Lax formula). *Let $f : X \rightarrow \mathbb{R}$ be Lipschitz and bounded. Then the following hold.*

i) *For every $t \geq 0$ we have $\text{Lip}(Q_t f) \leq 2 \text{Lip}(f)$.*

ii) *For every $x \in X$ the map $[0, \infty) \ni t \mapsto Q_t f(x)$ is continuous, locally semiconcave on $(0, \infty)$ and the inequality*

$$\frac{d}{dt} Q_t f(x) + \frac{\text{lip}(Q_t f)^2(x)}{2} \leq 0,$$

holds for every $t \in (0, \infty)$ with at most a countable number of exceptions.

iii) *The map $(0, \infty) \times X \ni (t, x) \mapsto \text{lip}(Q_t f)(x)$ is upper-semicontinuous.*

3 The continuity equation $\partial_t \mu_t = L_t$

3.1 Some definitions and conventions

Let $\mu \in \mathcal{P}_2(X)$ be such that $\mu \leq C\mathbf{m}$ for some $C > 0$. We define the seminorm $\|\cdot\|_\mu$ on $S^2(X)$ as

$$\|f\|_\mu^2 := \int |Df|^2 d\mu.$$

Definition 3.1 (The cotangent space $\text{CoTan}_\mu(X)$). For $\mu \in \mathcal{P}_2(X)$ with $\mu \leq C\mathbf{m}$ for some $C > 0$ consider the quotient space $S^2(X)/\sim_\mu$, where $f \sim_\mu g$ if $\|f - g\|_\mu = 0$.

The cotangent space $\text{CoTan}_\mu(X)$ is then defined as the completion of $S^2(X)/\sim_\mu$ w.r.t. its natural norm. The norm on $\text{CoTan}_\mu(X)$ will still be denoted by $\|\cdot\|_\mu$.

Given a linear map $L : S^2(X) \rightarrow \mathbb{R}$ and μ as above, we denote by $\|L\|_\mu^*$ the quantity given by

$$\frac{1}{2}(\|L\|_\mu^*)^2 := \sup_{f \in S^2(X)} L(f) - \frac{1}{2}\|f\|_\mu^2.$$

Linear operators $L : S^2(X) \rightarrow \mathbb{R}$ such that $\|L\|_\mu^* < \infty$ are in 1-1 correspondence with elements of the dual of $\text{CoTan}_\mu(X)$. Abusing a bit the notation, we will often identify such operators L with the induced linear mapping on $\text{CoTan}_\mu(X)$.

3.2 A localization argument

In this section $(\mu_t) \subset \mathcal{P}_2(X)$ is a given W_2 -continuous curve with bounded compression and we consider a functional $L : S^2(X) \rightarrow L^1(0, 1)$ satisfying the inequality

$$\int_t^s L(f)(r) dr \leq \sqrt{\int_t^s G_r^2 dr} \sqrt{\int_t^s \|f\|_{\mu_r}^2 dr}$$

with $G \in L^2(0, 1)$, for every $f \in S^2(X)$ and $t, s \in [0, 1]$, $t < s$. The question we address is up to what extent we can deduce that for such L there are operators $L_t : S^2(X) \rightarrow \mathbb{R}$ such that $L(f)(t) = L_t(f)$ for \mathcal{L}^1 -a.e. $t \in [0, 1]$. We will see in a moment that this is always the case in an appropriate sense, but to deal with the case of $S^2(X)$ non separable we need to pay some attention to the set of Lebesgue points of $L(f)$.

Thus for given $g \in L^1(0, 1)$ we shall denote by $\text{Leb}(g) \subset (0, 1)$ the set of t 's such that the limit of

$$\int_{t-\varepsilon}^{t+\varepsilon} g_s ds,$$

as $\varepsilon \downarrow 0$ exists and is finite. Clearly the set $\text{Leb}(g)$ contains all the Lebesgue points of any representative of g and in particular we have $\mathcal{L}^1(\text{Leb}(g)) = 1$. We shall denote its value by \bar{g} , so that $\bar{g} : \text{Leb}(g) \rightarrow \mathbb{R}$ is a well chosen representative of g everywhere defined on $\text{Leb}(g)$.

It is obvious that for $t \in \text{Leb}(g)$, we have:

$$\overline{\lim}_{\varepsilon_1, \varepsilon_2 \downarrow 0} \int_{t-\varepsilon_1}^{t+\varepsilon_1} \int_{t-\varepsilon_2}^{t+\varepsilon_2} |g_s - g_r| \, ds \, dr = 0.$$

We then have the following result.

Lemma 3.2. *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a W_2 -continuous curve of bounded compression and $L : S^2(X) \rightarrow L^1(0, 1)$ be a linear map such that for some $G \in L^2(0, 1)$ the inequality*

$$\int_t^s L(f)(r) \, dr \leq \sqrt{\int_t^s G_r^2 \, dr} \sqrt{\int_t^s \|f\|_{\mu_r}^2 \, dr}, \quad \forall t, s \in [0, 1], \, t < s, \, \forall f \in S^2(X), \quad (3.1)$$

holds.

Then there exists a family $\{L_t\}_{t \in [0, 1]}$ of maps from $S^2(X)$ to \mathbb{R} such that for any $f \in S^2(X)$ we have

$$L(f)(t) = L_t(f), \quad \text{a.e. } t \in [0, 1], \quad (3.2)$$

$$|L_t(f)| \leq |G_t| \|f\|_{\mu_t}, \quad \text{a.e. } t \in [0, 1]. \quad (3.3)$$

Remark 3.3. As a direct consequence of (3.2), if $\{\tilde{L}_t\}_{t \in [0, 1]}$ is another family of maps satisfies the above, then for every $f \in S^2(X)$ we have $L_t(f) = \tilde{L}_t(f)$ for a.e. $t \in [0, 1]$. Thus L is unique as a map from $S^2(X)$ to $L^1(0, 1)$.

Proof. For $f \in S^2(X)$ consider the set $\text{Leb}(L(f))$ and for $t \in (0, 1)$ let $V_t \subset S^2(X)$ be the set of f 's in $S^2(X)$ such that $t \in \text{Leb}(L(f))$. The trivial inclusion

$$\text{Leb}(\alpha_1 g_1 + \alpha_2 g_2) \supset \text{Leb}(g_1) \cap \text{Leb}(g_2),$$

valid for any $g_1, g_2 \in L^1(0, 1)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ and the linearity of L grant that V_t is a vector space for every $t \in (0, 1)$.

The W_2 -continuity of (μ_t) grants in particular continuity w.r.t. convergence in duality with $C_b(X)$ and the further assumption that $\mu_t \leq C \mathbf{m}$ for any $t \in [0, 1]$ ensures continuity w.r.t. convergence in duality with $L^1(X, \mathbf{m})$. Thus the map $t \mapsto \int |Df|^2 \, d\mu_t$ is continuous for any $f \in S^2(X)$. Hence from inequality (3.1) we deduce that for any $f \in S^2(X)$ it holds

$$|\overline{L(f)}(t)| \leq \sqrt{G_t^2} \|f\|_{\mu_t}, \quad \forall t \in \text{Leb}(L(f)) \cap \text{Leb}(G^2),$$

which we can rewrite as: for any $t \in \text{Leb}(G^2)$ it holds

$$|\overline{L(f)}(t)| \leq \sqrt{G_t^2} \|f\|_{\mu_t}, \quad \forall f \in V_t.$$

In other words, for any $t \in \text{Leb}(G^2)$ the map $V_t \ni f \mapsto L_t(f) := \overline{L(f)}(t)$ is a well defined linear map from V_t to \mathbb{R} with norm bounded by $\sqrt{G_t^2}$.

By Hahn-Banach we can extend this map to a map from S^2 to \mathbb{R} with norm bounded by $G(t)$. Noticing that by construction we have $f \in V_t$ for a.e. $t \in [0, 1]$ for any $f \in S^2$, the family of maps \bar{L}_t fulfill the thesis. To conclude notice that trivially it holds $\sqrt{G_t^2} = |G_t|$ for \mathcal{L}^1 -a.e. t . \square

3.3 Main theorem

We start giving the definition of ‘distributional’ solutions of the continuity equation in our setting:

Definition 3.4 (Solutions of $\partial_t \mu_t = L_t$). Let (X, d, \mathbf{m}) be a metric measure space, $(\mu_t) \subset \mathcal{P}_2(X)$ a W_2 -continuous curve with bounded compression and $\{L_t\}_{t \in [0, 1]}$ a family of maps from $S^2(X)$ to \mathbb{R} .

We say that (μ_t) solves the continuity equation

$$\partial_t \mu_t = L_t, \quad (3.4)$$

provided:

i) for every $f \in S^2(X)$ the map $t \mapsto L_t(f)$ is measurable and the map $N : [0, 1] \rightarrow [0, \infty]$ defined by

$$\frac{1}{2}N_t^2 := \operatorname{ess\,sup}_{f \in S^2(X)} L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2, \quad (3.5)$$

belongs to $L^2(0, 1)$, i.e. for any f , $\frac{1}{2}N_t^2 \geq L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2$ a.e. t and for any other \bar{N}_t satisfying this property, we have: $N_t \leq \bar{N}_t$ for a.e. t .

ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \, d\mu_t$ is in absolutely continuous and the identity

$$\frac{d}{dt} \int f \, d\mu_t = L_t(f),$$

holds for a.e. t .

Our main result is that for curves of bounded compression, the continuity equation characterizes 2-absolute continuity.

Theorem 3.5. *Let $(\mu_t) \subset P(X)$ be a W_2 -continuous curve with bounded compression. Then the following are equivalent.*

- i) (μ_t) is 2-absolutely continuous w.r.t. W_2 .
- ii) There is a family of maps $\{L_t\}_{t \in [0,1]}$ from $S^2(X)$ to \mathbb{R} such that (μ_t) solves the continuity equation (3.4).

Finally, we have

$$N_t = |\dot{\mu}_t|, \quad \text{a.e. } t \in [0, 1].$$

Remark 3.6. It is obvious that if $\{\tilde{L}_t\}_{t \in [0,1]}$ is another family of maps such that (μ_t) solves $\partial_t \mu_t = \tilde{L}_t$, then for every $f \in L^1 \cap S^2(X)$ we have

$$L_t(f) = \tilde{L}_t(f), \quad \text{a.e. } t \in [0, 1],$$

thus $\{L_t\}_{t \in [0,1]}$ is unique as a map from $S^2(X)$ to $L^1(0, 1)$.

Proof. (i) \Rightarrow (ii) Let π be a lifting of (μ_t) and notice that π is a test plan. Hence for $f \in L^1 \cap S^2(X)$ we have

$$\begin{aligned} \left| \int f \, d\mu_s - \int f \, d\mu_t \right| &\leq \int |f(\gamma_s) - f(\gamma_t)| \, d\pi(\gamma) \leq \iint_t^s |Df|(\gamma_r) |\dot{\gamma}_r| \, dr \, d\pi(\gamma) \\ &\leq \sqrt{\int_t^s \int |Df|^2 \, d\mu_r} \sqrt{\int_t^s \int |\dot{\gamma}_r|^2 \, d\pi(\gamma)} \, dr. \end{aligned} \quad (3.6)$$

Taking into account that $\int |Df|^2 \, d\mu_r \leq C \int |Df|^2 \, dm$ for every $t \in [0, 1]$, this shows that $t \mapsto \int f \, d\mu_t$ is absolutely continuous.

Define $L : S^2(X) \rightarrow L^1(0, 1)$ by $L(f)(t) := \partial_t \int f \, d\mu_t$ and notice that the bound (3.6) gives

$$\left| \int_t^s L(f)(r) \, dr \right| \leq \sqrt{\int_t^s \int |Df|^2 \, d\mu_r} \sqrt{\int_t^s G_r^2 \, dr}, \quad \forall t, s \in [0, 1], \quad t < s,$$

for $G_t := |\dot{\mu}_t| = \sqrt{\int |\dot{\gamma}_r|^2 \, d\pi(\gamma)} \in L^2(0, 1)$. Hence we can apply Lemma 3.2 and deduce from (3.3) that for every $f \in S^2(X)$ we have

$$L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2 \leq \frac{1}{2}|\dot{\mu}_t|^2, \quad \text{a.e. } t \in [0, 1].$$

By the definition (3.5), this latter bound is equivalent to $N_t \leq |\dot{\mu}_t|$ for a.e. $t \in [0, 1]$.

(ii) \Rightarrow (i) To get the result it is sufficient to prove that

$$W_2^2(\mu_t, \mu_s) \leq |s - t| \int_t^s N_r^2 dr, \quad \forall t, s \in [0, 1], t < s.$$

We shall prove this bound for $t = 0$ and $s = 1$ only, the general case following by a simple rescaling argument. Recalling that

$$\frac{1}{2}W_2^2(\mu_0, \mu_1) = \sup_{\psi} \int \psi d\mu_0 + \int \psi^c d\mu_1 = \sup_{\varphi} \int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0,$$

the sup being taken among all Lipschitz and bounded ψ, φ , to get the claim it is sufficient to prove that

$$\int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0 \leq \frac{1}{2} \int_0^1 N_t^2 dt, \quad (3.7)$$

for any Lipschitz and bounded $\varphi : X \rightarrow \mathbb{R}$. Fix such φ and notice that

$$\int Q_1 \varphi d\mu_1 - \int \varphi d\mu_0 = \lim_{n \rightarrow \infty} \left\{ \sum_{i=0}^{n-1} \int (Q_{\frac{i+1}{n}} \varphi - Q_{\frac{i}{n}} \varphi) d\mu_{\frac{i+1}{n}} + \int Q_{\frac{i}{n}} \varphi d(\mu_{\frac{i+1}{n}} - \mu_{\frac{i}{n}}) \right\}. \quad (3.8)$$

Recalling point (ii) of Proposition 2.6 we have

$$\sum_{i=0}^{n-1} \int (Q_{\frac{i+1}{n}} \varphi - Q_{\frac{i}{n}} \varphi) d\mu_{\frac{i}{n}} \leq \sum_{i=0}^{n-1} \int \int_{\frac{i}{n}}^{\frac{i+1}{n}} -\frac{\text{lip}(Q_t \varphi)^2}{2} dt d\mu_{\frac{i}{n}} = \int_{X \times [0,1]} -\frac{\text{lip}(Q_t \varphi)^2(x)}{2} d\mu_n(x, t).$$

where $\mu_n := \sum_{i=0}^{n-1} \mu_{\frac{i}{n}} \times \mathcal{L}^1|_{[\frac{i}{n}, \frac{i+1}{n}]}$. The continuity of (μ_t) easily yields that (μ_n) converges to $\mu := d\mu_t(x) \otimes dt$ in duality with $C_b(X \times [0, 1])$. Furthermore, the assumption $\mu_t \leq C\mathbf{m}$ for every $t \in [0, 1]$ yields $\mu_n \leq C\mathbf{m} \times \mathcal{L}^1$ for every $n \in \mathbb{N}$ and thus by the Dunfort-Pettis theorem (see for instance Theorem 4.7.20 in [7]) we deduce that (μ_n) converges to $\mu \in \mathcal{P}(X \times [0, 1])$, $d\mu := d\mu_t \otimes dt$, in duality with $L^\infty(X \times [0, 1])$. Being $(t, x) \mapsto \frac{\text{lip}(Q_t \varphi)^2(x)}{2}$ bounded (point (i) of Proposition 2.6), we deduce that

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^{n-1} \int (Q_{\frac{i+1}{n}} \varphi - Q_{\frac{i}{n}} \varphi) d\mu_{\frac{i}{n}} \leq \iint_0^1 -\frac{\text{lip}(Q_t \varphi)^2(x)}{2} d\mu_t dt. \quad (3.9)$$

On the other hand we have

$$\begin{aligned} \sum_{i=0}^{n-1} \int Q_{\frac{i}{n}} \varphi d(\mu_{\frac{i+1}{n}} - \mu_{\frac{i}{n}}) &= \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} L_s(Q_{\frac{i}{n}} \varphi) ds \\ &\leq \sum_{i=0}^{n-1} \frac{1}{2} \int_{\frac{i}{n}}^{\frac{i+1}{n}} N_s^2 ds + \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int \frac{|DQ_{\frac{i}{n}} \varphi|^2}{2} d\mu_s ds \\ &\leq \frac{1}{2} \int_0^1 N_t^2 dt + \int_{X \times [0,1]} f_n(t, x) d\mu, \end{aligned}$$

where $f_n(t, x) := \frac{\text{lip}(Q_{\frac{i}{n}} \varphi)^2(x)}{2}$ for $t \in [\frac{i}{n}, \frac{i+1}{n})$ and $d\mu(t, x) := d\mu_t(x) \otimes dt$. Recall that by points (i), (iii) of Proposition 2.6 we have that the f_n 's are equibounded and satisfy $\overline{\lim}_n f_n(t, x) \leq f(t, x) := \frac{\text{lip}(Q_t \varphi)^2(x)}{2}$, thus Fatou's lemma gives

$$\overline{\lim}_{n \rightarrow \infty} \int_{X \times [0,1]} f_n(t, x) d\mu \leq \int_{X \times [0,1]} f(t, x) d\mu,$$

and therefore

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^{n-1} \int Q_{\frac{i+1}{n}} \varphi d(\mu_{\frac{i+1}{n}} - \mu_{\frac{i}{n}}) \leq \frac{1}{2} \int_0^1 N_t^2 dt + \iint_0^1 \frac{\text{lip}(Q_t \varphi)^2}{2} d\mu_t dt \quad (3.10)$$

The bounds (3.9) and (3.10) together with (3.8) give (3.7) and the thesis. \square

If we know that $S^2(X)$ is separable, the result is slightly stronger, as a better description of the operators $\{L_t\}$ is possible, as shown by the following statement.

Proposition 3.7. *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a 2-absolutely continuous curve w.r.t. W_2 of bounded compression. Assume furthermore that $S^2(X)$ is separable. Then there exists a \mathcal{L}^1 -negligible set $\mathcal{N} \subset [0, 1]$ and, for every $t \in [0, 1] \setminus \mathcal{N}$, a linear map $L_t : S^2(X) \rightarrow \mathbb{R}$ such that:*

i) *every $t \in [0, 1] \setminus \mathcal{N}$ is a Lebesgue point of $s \mapsto |\dot{\mu}_s|^2$, the metric speed $|\dot{\mu}_s|$ exists at $s = t$ and we have $|\dot{\mu}_t| = \|L_t\|_{\mu_t}^*$,*

ii) *for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous, differentiable at every $t \in [0, 1] \setminus \mathcal{L}$ and its derivative is given by*

$$\frac{d}{dt} \int f d\mu_t = L_t(f), \quad \forall t \in [0, 1] \setminus \mathcal{N}.$$

Proof. Let $\{f_n\}_{n \in \mathbb{N}} \subset S^2(X)$ be a countable dense set and $\mathcal{N} \subset [0, 1]$ the set of t 's such that either the metric speed $|\dot{\mu}_t|$ does not exist, or t is not a Lebesgue point of $s \mapsto |\dot{\mu}_s|^2$ or for some $n \in \mathbb{N}$ the map $s \mapsto \int f_n d\mu_s$ is not differentiable at t . Then by Theorem 3.5 we know that \mathcal{N} is negligible.

For $n \in \mathbb{N}$ and $t \in [0, 1] \setminus \mathcal{N}$, inequality (3.6) gives, after a division for $|s - t|$ and a limit $s \rightarrow t$, the bound

$$\left| \frac{d}{dt} \int f_n d\mu_t \right| \leq |\dot{\mu}_t| \sqrt{\int |Df_n|^2 d\mu_t} \leq C |\dot{\mu}_t| \sqrt{\int |Df_n|^2 m}.$$

This means that the map $S^2(X) \ni f_n \mapsto \frac{d}{dt} \int f_n d\mu_t$ can be uniquely extended to a linear operator L_t from $S^2(X)$ to \mathbb{R} which satisfies $\|L_t\|_{\mu_t}^* \leq |\dot{\mu}_t|$.

For $f \in L^1 \cap S^2(X)$ denote by $I_f : [0, 1] \rightarrow \mathbb{R}$ the function given by $I_f(t) := \int f d(\mu_t - \mu_0)$. Then the map $f \mapsto I_f$ is clearly linear and satisfies

$$|I_f(t)| \leq \int |f(\gamma_t) - f(\gamma_0)| d\pi(\gamma) \leq \iint_0^t |Df|(\gamma_s) |\dot{\gamma}_s| ds d\pi(\gamma) \leq \sqrt{C} \|f\|_{S^2} \sqrt{\iint_0^1 |\dot{\gamma}_t|^2 d\pi(\gamma)}.$$

Hence given that we have $I_{f_n}(t) = \int_0^t L_t(f_n) dt$ for every $n \in \mathbb{N}$ and that $\{f_n\}$ is dense in $L^1 \cap S^2(X)$ w.r.t. the (semi)distance of $S^2(X)$, from the bound $N_t \leq |\dot{\mu}_t|$ for \mathcal{L}^1 -a.e. t , we deduce that $I_f(t) = \int_0^t L_t(f) dt$ for every $f \in L^1 \cap S^2(X)$ and every $t \in [0, 1]$.

Along the same lines we have that

$$\begin{aligned} |I_f(s) - I_f(t)| &\leq \int |f(\gamma_s) - f(\gamma_t)| d\pi(\gamma) \leq \iint_t^s |Df|(\gamma_r) |\dot{\gamma}_r| dr d\pi(\gamma) \leq \\ &\leq \sqrt{C|s-t| \iint_t^s |\dot{\gamma}_s|^2 dr d\pi(\gamma)} \|f\|_{S^2} = \sqrt{C|s-t| \int_t^s |\dot{\mu}_r|^2 dr} \|f\|_{S^2} \end{aligned}$$

and therefore for every $t \in [0, 1]$ Lebesgue point of $s \mapsto |\dot{\mu}_s|^2$ and such that $|\dot{\mu}_t|$ exists we have

$$\overline{\lim}_{s \rightarrow t} \left| \frac{I_f(s) - I_f(t)}{s - t} \right| \leq \sqrt{C} \|f\|_{S^2} |\dot{\mu}_t|.$$

Taking into account that, by construction, we have $\lim_{s \rightarrow t} \frac{I_{f_n}(s) - I_{f_n}(t)}{s - t} = L_t(f_n)$ for every $t \in [0, 1] \setminus \mathcal{N}$ and the density of $\{f_n\}$, we deduce that $\lim_{s \rightarrow t} \frac{I_f(s) - I_f(t)}{s - t} = L_t(f)$ for every $f \in L^1 \cap S^2(X)$ and $t \in [0, 1] \setminus \mathcal{N}$.

It remains to prove that $\|L_t\|_{\mu_t}^* = |\dot{\mu}_t|$ for $t \in [0, 1] \setminus \mathcal{N}$. From Theorem 3.5 we know that $N_t = |\dot{\mu}_t|$ for \mathcal{L}^1 -a.e. t and to conclude use the separability of $S^2(X)$ to get

$$\frac{1}{2} (\|L_t\|_{\mu_t}^*)^2 = \sup_{n \in \mathbb{N}} L_t(f_n) - \frac{1}{2} \|f_n\|_{\mu_t}^2 = \operatorname{ess\,sup}_{f \in S^2(X)} L_t(f) - \frac{1}{2} \|f\|_{\mu_t}^2 = \frac{1}{2} N_t^2, \quad a.e. t,$$

so that up to enlarging \mathcal{N} we get the thesis. \square

3.4 Some consequences in terms of differential calculus

As discussed in [13], see also [3], the continuity equation plays a key role in developing a first order calculus on the space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$. In this section, we show that the continuity equation plays a similar role on metric measure spaces, where no smooth structure is a priori given. The only technical difference one needs to pay attention to is the fact that only curves with bounded compression should be taken into account.

We start with the Benamou-Brenier formula. Recall that on \mathbb{R}^d , and more generally Riemannian/Finslerian manifolds, we have the identity

$$W_2^2(\mu_0, \mu_1) = \inf \int_0^1 \int |v_t|^2 d\mu_t dt, \quad (3.11)$$

where the inf is taken among all 2-a.c. curves (μ_t) joining μ_0 to μ_1 and the v_t 's are such that the continuity equation

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0, \quad (3.12)$$

holds. We want to investigate the validity of this formula in the metric-measure context. To this aim, notice that formula (3.11) expresses the fact that the distance W_2 can be realized as inf of length of curves, where this length is measured in an appropriate way. Hence there is little hope to get an analogous of this formula on (X, d, \mathbf{m}) unless we require in advance that (X, d) is a length space. Furthermore, given that in the non-smooth case we are confined to work with curves with bounded compression, we need to enforce a length structure compatible with the measure \mathbf{m} , thus we are led to the following definition:

Definition 3.8 (Measured-length spaces). We say that (X, d, \mathbf{m}) is measured-length provided for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and satisfying $\mu_0, \mu_1 \leq C\mathbf{m}$ for some $C > 0$ the distance $W_2(\mu_0, \mu_1)$ can be realized as inf of length of absolutely continuous curves (μ_t) with bounded compression connecting μ_0 to μ_1 .

On measured-length spaces we then have a natural analog of formula (3.11), which is in fact a direct consequence of Theorem 3.5:

Proposition 3.9 (Benamou-Brenier formula on metric measure spaces). *Let (X, d, \mathbf{m}) be a measured-length space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and satisfying $\mu_0, \mu_1 \leq C\mathbf{m}$ for some $C > 0$.*

Then we have

$$W_2^2(\mu_0, \mu_1) = \inf \int_0^1 (\|L_t\|_{\mu_t}^*)^2 dt,$$

the inf being taken among all 2-absolutely continuous curves (μ_t) with bounded compression joining μ_0 to μ_1 and the operators (L_t) are those associated to the curve via Theorem 3.5.

Proof. By Theorem 3.5 we know that for a 2-absolutely continuous curve (μ_t) with bounded compression we have $|\dot{\mu}_t| = \|L_t\|_{\mu_t}^*$ for a.e. $t \in [0, 1]$, the operators $\{L_t\}$ being those associated to the curve via Theorem 3.5 itself. The conclusion then follows directly from the definition of measured-length space. \square

We now discuss the formula for the derivative of $t \mapsto \frac{1}{2}W_2^2(\mu_t, \nu)$, where (μ_t) is a 2-a.c. curve with bounded compression. Recall that on the Euclidean setting we have

$$\frac{d}{dt} \frac{1}{2}W_2^2(\mu_t, \nu) = \int \nabla \varphi_t \cdot v_t d\mu_t, \quad a.e. t,$$

where φ_t is a Kantorovich potential from μ_t to ν for every $t \in [0, 1]$ and the vector fields (v_t) are such that the continuity equation (3.12) holds. Due to our interpretation of the continuity equation in the metric measure setting, we are therefore lead to guess that in the metric-measure setting we have

$$\frac{d}{dt} \frac{1}{2}W_2^2(\mu_t, \nu) = L_t(\varphi_t), \quad a.e. t. \quad (3.13)$$

As we shall see in a moment (Proposition 3.11) this is actually the case in quite high generality, but before coming to the proof, we need to spend few words on how to interpret the right hand side of (3.13) because in general we don't have $\varphi_t \in \mathbb{S}^2(X)$ so that a priori φ_t is outside the domain of definition of L_t . This can in fact be easily fixed by considering φ_t as element of $\text{CoTan}_\mu(X)$, as defined in Section 3.1. This is the scope of the following lemma.

Lemma 3.10. *Let $(X, \mathbf{d}, \mathbf{m})$ be a m.m.s. such that \mathbf{m} gives finite mass to bounded sets and φ a c -concave function such that $\partial^c \varphi(x) \cap B \neq \emptyset$ for every $x \in X$ and some bounded set $B \subset X$. Define $\varphi_n := \min\{n, \varphi\}$.*

Then $\varphi_n \in \mathbb{S}^2(X)$ and (φ_n) is a Cauchy sequence w.r.t. the seminorm $\|\cdot\|_\mu \rightarrow 0$ for every $\mu \in \mathcal{P}_2(X)$ such that $\mu \leq C\mathbf{m}$ for some $C > 0$.

Proof. We first claim that $\sup_B \varphi^c < \infty$. Indeed, if not there is a sequence $(y_n) \subset B$ such that $\varphi^c(y_n) > n$ for every $n \in \mathbb{N}$. Hence for every $x \in X$ we would have

$$\varphi(x) \leq \inf_{n \in \mathbb{N}} \mathbf{d}^2(x, y_n) - \varphi^c(y_n) \leq \inf_{n \in \mathbb{N}} \frac{1}{2}(\mathbf{d}(x, B) + \text{diam}(B))^2 - n = -\infty,$$

contradicting the definition of c -concavity. Using the assumption we have

$$\varphi(x) = \inf_{y \in B} \frac{\mathbf{d}^2(x, y)}{2} - \varphi^c(y) \geq \frac{\mathbf{d}^2(x, B)}{2} - \sup_B \varphi^c. \quad (3.14)$$

This proves that φ is bounded from below and that it has bounded sublevels. Hence the truncated functions φ_n are constant outside a bounded set. Now let $x, x' \in X$ and $y \in \partial^c \varphi(x) \cap B$. Then we have

$$\varphi(x) - \varphi(x') \leq \frac{\mathbf{d}^2(x, y) - \mathbf{d}^2(x', y)}{2} \leq \mathbf{d}(x, x') \left(\frac{\mathbf{d}(x, B) + \mathbf{d}(x', B)}{2} + \text{diam}(B) \right).$$

Inverting the roles of x, x' we deduce that φ is Lipschitz on bounded sets and the pointwise estimate

$$\text{lip}(\varphi)(x) \leq \mathbf{d}(x, B) + \text{diam}(B), \quad \forall x \in X. \quad (3.15)$$

It follows that the φ_n 's are Lipschitz and, using the fact that \mathbf{m} gives finite mass to bounded sets, that $\varphi_n \in \mathbb{S}^2(X)$ for every $n \in \mathbb{N}$.

To conclude, notice that the bound (3.15) ensures that for any $\mu \in \mathcal{P}_2(X)$ we have $\text{lip}(\varphi) \in L^2(\mu)$. Thus for μ such that $\mu \leq C\mathbf{m}$ for some $C > 0$ we have $|D\varphi| \leq \text{lip}(\varphi)$ μ -a.e. and thus $|D\varphi| \in L^2(\mu)$ as well. Now observe that

$$\|\varphi_m - \varphi_n\|_\mu^2 = \int_{\{\varphi_m \neq \varphi_n\}} |D\varphi|^2 d\mu,$$

and that the right hand side goes to 0 as $n, m \rightarrow \infty$, because by (3.14) we know that $\cup_n \{\varphi = \varphi_n\} = X$. \square

Thanks to this lemma we can, and will, associate the Kantorovich potential φ as an element of $\text{CoTan}_\mu(X)$: it is the limit of the equivalence classes of the truncated functions φ_n .

Recall that for $\mu, \nu \in \mathcal{P}_2(X)$, there always exists a Kantorovich potential φ from μ to ν such that

$$\varphi(x) = \inf_{y \in \text{supp}(\nu)} \frac{\mathbf{d}^2(x, y)}{2} - \varphi^c(y), \quad \forall x \in X, \quad (3.16)$$

hence if ν has bounded support, a potential satisfying the assumption of Lemma 3.10 above can always be found.

We can now state and prove the following result about the derivative of $W_2^2(\cdot, \nu)$. It is worth noticing that formula (3.17) below holds even for spaces which are not length spaces.

Proposition 3.11 (Derivative of $W_2^2(\cdot, \nu)$). *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression, $\nu \in \mathcal{P}_2(X)$ with bounded support and notice that $t \mapsto \frac{1}{2}W_2^2(\mu_t, \nu)$ is absolutely continuous. Assume that $\mathbb{S}^2(X)$ is separable and that \mathbf{m} gives finite mass to bounded sets. Then the for a.e. $t \in [0, 1]$ the formula*

$$\frac{d}{dt} \frac{1}{2} W_2^2(\mu_t, \nu) = L_t(\varphi_t), \quad (3.17)$$

holds, where φ_t is any Kantorovich potential from μ_t to ν fulfilling the assumptions of Lemma 3.10.

Proof. Let $\mathcal{N} \subset [0, 1]$ be the \mathcal{L}^1 -negligible set given by Proposition 3.7. We shall prove formula (3.17) for every $t \in [0, 1] \setminus \mathcal{N}$ such that $\frac{1}{2}W_2^2(\mu_t, \nu)$ is differentiable at t . Fix such t , let φ_t be as in the assumptions and notice that

$$\begin{aligned} \frac{1}{2}W_2^2(\mu_t, \nu) &= \int \varphi_t d\mu_t + \int \varphi^c d\nu, \\ \frac{1}{2}W_2^2(\mu_s, \nu) &\geq \int \varphi_t d\mu_s + \int \varphi^c d\nu, \quad \forall s \in [0, 1], \end{aligned}$$

and thus

$$\frac{W_2^2(\mu_s, \nu) - W_2^2(\mu_t, \nu)}{2} \geq \int \varphi_t d(\mu_s - \mu_t).$$

Recall that $\max\{\varphi_t, 0\} \in L^1(\mu)$ for every $\mu \in \mathcal{P}_2(X)$ and that $\varphi_t \in L^1(\mu_t)$, so that the integral in the right hand side makes sense. Put $\varphi_{n,t} := \min\{n, \max\{-n, \varphi_t\}\}$ so that by Lemma 3.10 above we have $\varphi_{n,t} \in S^2(X)$ for every $n \in \mathbb{N}$ and $\|\varphi_{n,t} - \varphi_{m,t}\|_\mu \rightarrow 0$ as $n, m \rightarrow \infty$. For every $n \in \mathbb{N}$ we know that $\frac{d}{ds} \int \varphi_{n,t} d\mu_s|_{s=t} = L_t(\varphi_{n,t})$ and by Lemma 3.10 we know that $L_t(\varphi_{n,t}) \rightarrow L_t(\varphi_t)$ as $n \rightarrow \infty$. To conclude it is sufficient to notice that for any lifting π of (μ_t) we have the bound

$$\begin{aligned} \left| \int (\varphi_{n,t} - \varphi_{m,t}) d \frac{\mu_s - \mu_t}{s-t} \right| &\leq \frac{1}{|s-t|} \int (\varphi_{n,t} - \varphi_{m,t})(\gamma_s) - (\varphi_{n,t} - \varphi_{m,t})(\gamma_t) d\pi(\gamma) \\ &\leq \frac{1}{|s-t|} \iint_t^s |D(\varphi_{n,t} - \varphi_{m,t})|(\gamma_r) |\dot{\gamma}_r| dr d\pi(\gamma) \\ &\leq \sqrt{\int_t^s \|D(\varphi_{n,t} - \varphi_{m,t})\|_{\mu_r}^2 dr} \sqrt{\int_t^s |\dot{\gamma}_r|^2 dr} d\pi(\gamma), \end{aligned}$$

and that the dominated convergence theorem ensures that $\int_t^s \|D(\varphi_{n,t} - \varphi_{m,t})\|_{\mu_r}^2 dr \rightarrow 0$ as $n, m \rightarrow \infty$. \square

4 The continuity equation $\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0$

4.1 preliminaries: duality between differentials and gradients

On Euclidean spaces it is often the case that the continuity equation (3.12) can be written as

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0, \quad (4.1)$$

for some functions ϕ_t , so that the vector fields v_t can be represented as gradient of functions. In some sense, the ‘optimal’ velocity vector fields (i.e. those minimizing the $L^2(\mu_t)$ -norm) can always be thought of as gradients, as they always belong to the closure of the space of gradients of smooth functions w.r.t. the $L^2(\mu_t)$ -norm (i.e. they belong to the cotangent space $\text{CoTan}_{\mu_t}(\mathbb{R}^d)$), see [3]. Yet, the process of taking completion in general destroys the property of being the gradient of a smooth/Sobolev functions, so that technically speaking general absolutely continuous curves solve (3.12) and only in some cases one can write it as in (4.1).

It is then the scope of this part of the paper to investigate how one can give a meaning to (4.1) in the non-smooth setting and which sort of information on the curve we can obtain from such ‘PDE’. According to our interpretation of the continuity equation given in Theorem 3.5, the problem reduces to understand in what sense we can write $L_t(f) = \int Df(\nabla \phi_t) d\mu_t$, and thus ultimately to give a meaning to ‘the differential of a function applied to the gradient of another function’. This has been the scope of [8], we recall here the main definitions and properties.

Definition 4.1 (The objects $D^\pm f(\nabla g)$). Let (X, d, \mathbf{m}) be a m.m.s. and $f, g \in S^2(X)$.

The functions $D^\pm f(\nabla g) : X \rightarrow \mathbb{R}$ are \mathbf{m} -a.e. well defined by

$$\begin{aligned} D^+ f(\nabla g) &:= \lim_{\varepsilon \downarrow 0} \frac{|D(g + \varepsilon f)|^2 - |Dg|^2}{2\varepsilon}, \\ D^- f(\nabla g) &:= \lim_{\varepsilon \uparrow 0} \frac{|D(g + \varepsilon f)|^2 - |Dg|^2}{2\varepsilon}. \end{aligned}$$

It is immediate to check that for $\varepsilon_1 < \varepsilon_2$ we have

$$\frac{|D(g + \varepsilon_1 f)|^2 - |Dg|^2}{2\varepsilon_1} \leq \frac{|D(g + \varepsilon_2 f)|^2 - |Dg|^2}{2\varepsilon_2}, \quad \mathbf{m} - a.e.,$$

so that the limits above can be replaced by $\inf_{\varepsilon > 0}$ and $\sup_{\varepsilon < 0}$ respectively.

Heuristically, we should think to $D^+ f(\nabla g)$ (resp. $D^- f(\nabla g)$) as the maximal (resp. minimal) value of the differential of f applied to all possible gradients of g , see [8] for a discussion on this topic.

The basic algebraic calculus rules for $D^\pm f(\nabla g)$ are the following:

$$|D^\pm(f_1 - f_2)(\nabla g)| \leq |D(f_1 - f_2)| |Dg|, \quad (4.2)$$

$$D^- f(\nabla g) \leq D^+ f(\nabla g),$$

$$D^+(-f)(\nabla g) = D^+ f(\nabla(-g)) = -D^- f(\nabla g), \quad (4.3)$$

$$D^\pm g(\nabla g) = |Dg|^2. \quad (4.4)$$

We also have natural chain rules: given $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz we have

$$\begin{aligned} D^\pm(\varphi \circ f)(\nabla g) &= \varphi' \circ f D^{\pm \text{sign} \varphi' \circ f} f(\nabla g), \\ D^\pm f(\nabla \varphi \circ g) &= \varphi' \circ g D^{\pm \text{sign} \varphi' \circ g} f(\nabla g), \end{aligned} \quad (4.5)$$

where $\varphi' \circ f$ (resp. $\varphi' \circ g$) are defined arbitrarily at those x 's such that φ is not differentiable at $f(x)$ (resp. $g(x)$). In particular, $D^\pm f(\nabla(\alpha g)) = \alpha D^\pm f(\nabla g)$ for $\alpha > 0$.

Notice that as a consequence of the above we have that for given $g \in S^2(X)$ the map $S^2(X) \ni f \mapsto D^+ f(\nabla g)$ is \mathbf{m} -a.e. convex, i.e.

$$D^+((1 - \lambda)f_1 + \lambda f_2)(\nabla g) \leq (1 - \lambda)D^+ f_1(\nabla g) + \lambda D^+ f_2(\nabla g), \quad \mathbf{m} - a.e., \quad (4.6)$$

for every $f_1, f_2 \in S^2(X)$, and $\lambda \in [0, 1]$. Similarly, $f \mapsto D^- f(\nabla g)$ is \mathbf{m} -a.e. concave.

Furthermore, it is easy to see that for $g \in S^2(X)$ and π test plan we have

$$\overline{\lim}_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} d\pi(\gamma) \leq \frac{1}{2} \int |Dg|^2(\gamma_0) d\pi(\gamma) + \frac{1}{2} \overline{\lim}_{t \downarrow 0} \frac{1}{t} \iint_0^t |\dot{\gamma}_s|^2 ds \pi(\gamma). \quad (4.7)$$

We are therefore lead to the following definition:

Definition 4.2 (Plans representing gradients). Let (X, d, \mathbf{m}) be a m.m.s. $g \in S^2(X)$ and π a test plan. We say that π represents the gradient of g provided it is a test plan and we have

$$\underline{\lim}_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} d\pi(\gamma) \geq \frac{1}{2} \int |Dg|^2(\gamma_0) d\pi(\gamma) + \frac{1}{2} \underline{\lim}_{t \downarrow 0} \frac{1}{t} \iint_0^t |\dot{\gamma}_s|^2 ds \pi(\gamma). \quad (4.8)$$

It is worth noticing that plans representing gradients exist in high generality (see [8]).

Differentiation along plans representing gradients is tightly linked to the object $D^\pm f(\nabla g)$ defined above: this is the content of the following simple but crucial theorem proved in [8] as a generalization of a result originally appeared in [4].

Theorem 4.3 (Horizontal and vertical derivatives). *Let (X, d, \mathbf{m}) be a m.m.s., $f, g \in S^2(X)$ and π a plan representing the gradient of g .*

Then

$$\begin{aligned} \int D^- f(\nabla g) d(e_0)_\# \pi &\leq \underline{\lim}_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} d\pi(\gamma) \\ &\leq \overline{\lim}_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} d\pi(\gamma) \leq \int D^+ f(\nabla g) d(e_0)_\# \pi. \end{aligned} \quad (4.9)$$

Proof. Write inequality (4.7) for the function $g + \varepsilon f$ and subtract inequality (4.8) to get

$$\overline{\lim}_{t \downarrow 0} \varepsilon \int \frac{f(\gamma_t) - f(\gamma_0)}{t} d\pi \leq \frac{1}{2} \int |D(g + \varepsilon f)|^2 - |Dg|^2 d(e_0)_\# \pi$$

Divide by $\varepsilon > 0$ (resp. $\varepsilon < 0$), let $\varepsilon \downarrow 0$ (resp. $\varepsilon \uparrow 0$) and use the dominate convergence theorem to conclude. \square

4.2 The result

We are now ready to define what it is a solution of the continuity equation (4.1) in a metric measure context.

Definition 4.4 (Solutions of $\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0$). Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a W_2 -continuous curve with bounded compression and $\{\phi_t\}_{t \in [0,1]} \subset S^2(X)$ a given family. We say that (μ_t) solves the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0, \quad (4.10)$$

provided:

- i) for every $f \in S^2(X)$ the maps $(t, x) \mapsto D^\pm f(\nabla \phi_t)(x)$ are $L \times \mathfrak{m}$ measurable and the map $\tilde{N} : [0, 1] \rightarrow [0, \infty]$ given by

$$\frac{1}{2} \tilde{N}_t^2 := \operatorname{ess\,sup}_{f \in S^2(X)} \int D^+ f(\nabla \phi_t) d\mu_t - \frac{1}{2} \|f\|_{\mu_t}^2 \quad (4.11)$$

is in $L^2(0, 1)$ where $\operatorname{ess\,sup}$ has the same meaning as we mentioned before.

- ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous and satisfies

$$\int D^- f(\nabla \phi_t) d\mu_t \leq \frac{d}{dt} \int f d\mu_t \leq \int D^+ f(\nabla \phi_t) d\mu_t, \quad a.e. t. \quad (4.12)$$

We then have the following result, analogous to the implication $(ii) \Rightarrow (i)$ of Theorem 3.5. As already recalled, even in the smooth framework not all a.c. curves solve (4.1), so the other implication is in general false.

Proposition 4.5. *Let (X, d, \mathfrak{m}) be a m.m.s. and $(\mu_t) \subset \mathcal{P}_2(X)$ a continuous curve with bounded compression solving the continuity equation (4.10) for some given family $\{\phi_t\}_{t \in [0,1]} \subset S^2(X)$.*

Then (μ_t) is 2-a.c. and we have $|\dot{\mu}_t| \leq \tilde{N}_t$ for a.e. $t \in [0, 1]$.

If furthermore $S^2(X)$ is separable, then $\tilde{N}_t = |\dot{\mu}_t| = \|\phi_t\|_{\mu_t}$ for a.e. t .

Proof. We claim that for every $f \in S^2(X)$ it holds

$$\max \left\{ \left| \int D^+ f(\nabla \phi_t) d\mu_t \right|, \left| \int D^- f(\nabla \phi_t) d\mu_t \right| \right\} \leq \|f\|_{\mu_t} \tilde{N}_t, \quad a.e. t. \quad (4.13)$$

To this aim, fix a representative of \tilde{N} , a function $f \in S^2(X)$ and notice that for every $\lambda \geq 0$, by definition of \tilde{N} and the second of (4.5) we have

$$\lambda \int D^+ f(\nabla \phi_t) d\mu_t \leq \frac{\lambda^2}{2} \|f\|_{\mu_t}^2 + \frac{1}{2} \tilde{N}_t^2, \quad (4.14)$$

for \mathcal{L}^1 -a.e. t . Replacing f with $-f$ and recalling that

$$- \int D^+ f(\nabla \phi_t) d\mu_t = \int D^- (-f)(\nabla \phi_t) d\mu_t \leq \int D^+ (-f)(\nabla \phi_t) d\mu_t,$$

we deduce that (4.14) holds for every $\lambda \in \mathbb{R}$. In particular, there is a \mathcal{L}^1 -negligible set $\mathcal{N} \subset [0, 1]$ such that for every $t \in [0, 1] \setminus \mathcal{N}$ the inequality (4.14) holds for every $\lambda \in \mathbb{Q}$. Given that all the terms in (4.14) are continuous in λ , we deduce that (4.14) holds for every $t \in [0, 1] \setminus \mathcal{N}$ and every $\lambda \in \mathbb{R}$, which yields

$$\left| \int D^+ f(\nabla \phi_t) d\mu_t \right| \leq \|f\|_{\mu_t} \tilde{N}_t, \quad a.e. t.$$

Arguing analogously with $D^- f(\nabla \phi_t)$ in place of $D^+ f(\nabla \phi_t)$ we obtain (4.13).

Now define a linear operator $L : S^2(X) \rightarrow L^1(0, 1)$ as

$$L(f)(t) := \frac{d}{dt} \int f d\mu_t,$$

and observe that for every $t, s \in [0, 1]$, $t < s$, taking into account (4.12) and (4.13) we have

$$\begin{aligned} \int_t^s |L(f)(r)| \, dr &\leq \int_t^s \max \left\{ \left| \int D^+ f(\nabla \phi_r) \, d\mu_r \right|, \left| \int D^- f(\nabla \phi_r) \, d\mu_r \right| \right\} \, dr \\ &\leq \int_t^s \|f\|_{\mu_r} \tilde{N}_r \, dr \leq \sqrt{\int_t^s \tilde{N}_r^2 \, dr} \sqrt{\int_t^s \|f\|_{\mu_r}^2 \, dr}. \end{aligned}$$

Hence we can apply first Lemma 3.2 (with $G := \tilde{N}$) and then Theorem 3.5 to deduce that (μ_t) is 2-a.c. with $|\dot{\mu}_t| \leq \tilde{N}_t$ for a.e. $t \in [0, 1]$.

If $S^2(X)$ is separable, then arguing exactly as in the proof of Proposition 3.7 and using the convexity (resp. concavity) of $f \mapsto \int D^+ f(\nabla \phi_t) \, d\mu_t$ (resp. $f \mapsto \int D^- f(\nabla \phi_t) \, d\mu_t$) expressed in (4.6) we deduce the existence of a \mathcal{L}^1 -negligible set $\mathcal{N} \subset [0, 1]$ such that for $t \in [0, 1] \setminus \mathcal{N}$ the conclusions (i), (ii) of Proposition 3.7 hold and furthermore

$$\int D^- f(\nabla \phi_t) \, d\mu_t \leq L_t(f) \leq \int D^+ f(\nabla \phi_t) \, d\mu_t, \quad \forall f \in S^2(X).$$

Choosing $f := \phi_t$ and recalling (4.4) we obtain

$$\|\phi_t\|_{\mu_t}^2 = L_t(\phi_t) \leq \|\phi_t\|_{\mu_t} \|L_t\|_{\mu_t}^*, \quad \forall t \in [0, 1] \setminus \mathcal{N},$$

and hence $\|\phi_t\|_{\mu_t} \leq \|L_t\|_{\mu_t}^* = N_t = |\dot{\mu}_t|$ for a.e. t . On the other hand, letting $(f_n) \subset S^2(X)$ be a countable dense set, by (4.2) we know that $\frac{1}{2} \tilde{N}_t^2 = \sup_n D^+ f_n(\nabla \phi_t) - \frac{1}{2} \|f_n\|_{\mu_t}^2$ for a.e. t and thus $\tilde{N}_t \leq \|\phi_t\|_{\mu_t}$ for a.e. t . \square

The continuity equation (4.10) has very general relations with the concept of ‘plans representing gradients’, as shown by the following result:

Theorem 4.6. *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression, $(t, x) \mapsto \phi_t(x)$ a Borel map such that $\phi_t \in S^2(X)$ for a.e. $t \in [0, 1]$ and π a lifting of (μ_t) .*

The following are true.

- i) Assume that $(\text{restr}_t^1)_\# \pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t \in [0, 1]$. Then (μ_t) solves the continuity equation (4.10).*
- ii) Assume that $S^2(X)$ is separable and that (μ_t) solves the continuity equation (4.10). Then $(\text{restr}_t^1)_\# \pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t \in [0, 1]$.*

Proof.

(i) Let $\mathcal{A} \subset (0, 1)$ be the set of t 's such that $(\text{restr}_t^1)_\# \pi$ represents the gradient of $(1-t)\phi_t$, so that by assumption we know that $\mathcal{L}^1(\mathcal{A}) = 1$. Pick $f \in S^2(X)$ and recall that by Theorem 3.5 we know that

$$\frac{d}{dt} \int f \, d\mu_t = L_t(f), \quad \text{a.e. } t \in [0, 1]. \quad (4.15)$$

Fix $t \in \mathcal{A}$ such that (4.15) holds and notice that

$$\frac{d}{dt} \int f \, d\mu_t = \lim_{h \downarrow 0} \int \frac{f(\gamma_{t+h}) - f(\gamma_t)}{h} \, d\pi(\gamma) = \frac{1}{1-t} \lim_{h \downarrow 0} \int \frac{f(\gamma_h) - f(\gamma_0)}{h} \, d\pi_t(\gamma),$$

so that recalling (4.9) and (4.5) we conclude.

(ii) With exactly the same approximation procedure used in the proof of Proposition 3.7, we see that there exists a \mathcal{L}^1 -negligible set $\mathcal{N} \subset [0, 1]$ such that the thesis of Proposition 3.7 is fulfilled and furthermore for every $t \in [0, 1] \setminus \mathcal{N}$ we have

$$\int D^- f(\nabla \phi_t) \, d\mu_t \leq L_t(f) \leq \int D^+ f(\nabla \phi_t) \, d\mu_t, \quad \forall f \in S^2(X). \quad (4.16)$$

Fix $t \in [0, 1] \setminus \mathcal{N}$ and observe that by point (i) of Proposition 3.7 we have that

$$|\dot{\mu}_t|^2 = \lim_{h \downarrow 0} \frac{1}{h} \iint_t^{t+h} |\dot{\gamma}_s|^2 d\pi(\gamma) = \frac{1}{(1-t)^2} \lim_{h \downarrow 0} \frac{1}{h} \iint_0^h |\dot{\gamma}_s|^2 d\pi_t(\gamma). \quad (4.17)$$

Now pick $f := \phi_t$ in (4.16) and recall the identity $D^\pm f(\nabla f) = |Df|^2$ \mathbf{m} -a.e. valid for every $f \in S^2(X)$ to deduce

$$\int |D\phi_t|^2 d\mu_t = L_t(\phi_t) = \lim_{h \downarrow 0} \int \phi_t d \frac{\mu_{t+h} - \mu_t}{h} = \frac{1}{1-t} \lim_{h \downarrow 0} \int \frac{\phi_t(\gamma_h) - \phi_t(\gamma_0)}{h} d\pi_t(\gamma).$$

This last identity, (4.17) and the fact that $\|L_t\|_{\mu_t}^* = |\dot{\mu}_t|$ ensure that π_t represents the gradient of $(1-t)\phi_t$, as claimed. \square

5 Two important examples

We conclude the paper discussing two important examples of absolutely continuous curves on $\mathcal{P}_2(X)$: the heat flow and the geodesics. These examples already appeared in the literature [5,9,10], we report them here only to show that they are consistent with the concepts we introduced.

We start with the heat flow. Recall that the map $E : L^2(X, \mathbf{m}) \rightarrow [0, \infty]$ given by

$$E(f) := \begin{cases} \frac{1}{2} \int |Df|^2 d\mathbf{m}, & \text{if } f \in S^2(X), \\ +\infty, & \text{otherwise,} \end{cases}$$

is convex, lower semicontinuous and with dense domain. Being $L^2(X)$ an Hilbert space, we then know by the classical theory of gradient flows in Hilbert spaces (see e.g. [3] and references therein) that for any $\rho \in L^2(X)$ there exists a unique continuous curve $[0, \infty) \ni t \mapsto \rho_t \in L^2(X)$ with $\rho_0 = \rho$ which is locally absolutely continuous on $(0, \infty)$ and that satisfies

$$\frac{d}{dt} \rho_t \in -\partial^- E(\rho_t), \quad a.e. \ t > 0.$$

As in [5], we shall call any such gradient flow a *heat flow*. It is immediate to check that defining $D(\Delta) := \{\rho : \partial^- E(\rho) \neq \emptyset\}$ and for $\rho \in D(\Delta)$ the Laplacian $\Delta\rho$ as the opposite of the element of minimal norm in $\partial^- E(\rho)$, for any heat flow (ρ_t) we have $\rho_t \in D(\Delta)$ for any $t > 0$ and

$$\frac{d}{dt} \rho_t = \Delta\rho_t, \quad a.e. \ t > 0,$$

in accordance with the classical case. It is our aim now to check that, under reasonable assumptions, putting $\mu_t := \rho_t \mathbf{m}$, the curve (μ_t) solves

$$\partial_t \mu_t + \nabla \cdot (\nabla(-\log \rho_t) \mu_t) = 0.$$

To this aim, recall that for any heat flow (ρ_t) we have the weak maximum principle

$$\rho_0 \leq C \text{ (resp. } \rho_0 \geq c) \ \mathbf{m} - a.e. \quad \Rightarrow \quad \rho_t \leq C \text{ (resp. } \rho_t \geq c) \ \mathbf{m} - a.e. \text{ for any } t > 0, \quad (5.1)$$

where $c, C \in \mathbb{R}$ and the estimate

$$\int_0^\infty \|\Delta\rho_t\|_{L^2(X)}^2 dt \leq E(\rho_0). \quad (5.2)$$

Furthermore, if $\mathbf{m} \in \mathcal{P}(X)$ then $L^2(X, \mathbf{m}) \subset L^1(X, \mathbf{m})$ and the mass preservation property holds:

$$\int \rho_t d\mathbf{m} = \int \rho_0 d\mathbf{m}, \quad \forall t > 0. \quad (5.3)$$

See [5] for the simple proof of these facts.

We can now state our result concerning the heat flow as solution of the continuity equation.

Proposition 5.1 (Heat flow). *Let (X, d, \mathbf{m}) be a metric measure space with $\mathbf{m} \in \mathcal{P}_2(X)$ and ρ_0 a probability density such that $c \leq \rho_0 \leq C$ \mathbf{m} -a.e. for some $c, C > 0$ (in particular $\rho_0 \in L^2(X, \mathbf{m})$) and $E(\rho_0) < \infty$. Let (ρ_t) be the heat flow starting from ρ_0 .*

Then the curve $[0, 1] \ni t \mapsto \mu_t := \rho_t \mathbf{m}$ solves the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\nabla(-\log \rho_t) \mu_t) = 0.$$

Proof. By (5.3) and (5.1) we know that ρ_t is a probability density for every t and (5.1) again and the assumptions $\mathbf{m} \in \mathcal{P}_2(X)$ and $\rho_0 \leq C \mathbf{m}$ ensure that $\mu_t \in \mathcal{P}_2(X)$ for every $t \in [0, 1]$ and that (μ_t) has bounded compression. The W_2 -continuity of (μ_t) is a simple consequence of the L^2 -continuity of (ρ_t) and the bounds $\rho_t \leq C$, $\mathbf{m} \in \mathcal{P}_2(X)$. Also, recalling the chain rule (2.9) and the maximum principle (5.1) we have

$$\|\log \rho_t\|_{\mu_t}^2 = \int |D(\log \rho_t)|^2 d\mu_t = \int \frac{|D\rho_t|^2}{\rho_t} d\mathbf{m} \leq \frac{1}{c} \int |D\rho_t|^2 d\mathbf{m},$$

so that (5.2) ensures that $\int_0^1 \|\log \rho_t\|_{\mu_t}^2 < \infty$, which directly yields that point (i) of Definition 4.4 is fulfilled.

It remains to prove that for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous and fulfills

$$\int D^- f(\nabla(-\log \rho_t)) d\mu_t \leq \frac{d}{dt} \int f d\mu_t \leq \int D^+ f(\nabla(-\log \rho_t)) d\mu_t.$$

Taking into account the chain rule (4.5) the above can be written as

$$- \int D^+ f(\nabla \rho_t) d\mathbf{m} \leq \frac{d}{dt} \int f d\mu_t \leq - \int D^- f(\nabla \rho_t) d\mathbf{m}. \quad (5.4)$$

Pick $f \in L^2 \cap S^2(X)$ and notice that $t \mapsto \int f d\mu_t = \int f \rho_t d\mathbf{m}$ is continuous on $[0, 1]$ and locally absolutely continuous on $(0, 1]$. The inequality

$$\left| \frac{d}{dt} \int f \rho_t d\mathbf{m} \right| \leq \int |f| |\Delta \rho_t| d\mathbf{m} \leq \frac{1}{2} \|f\|_{L^2}^2 + \frac{1}{2} \|\Delta \rho_t\|_{L^2}^2,$$

the bound (5.2) and the assumption $E(\rho_0) < \infty$ ensure that the derivative of $t \mapsto \int f \rho_t d\mathbf{m}$ is in $L^1(0, 1)$, so that this function is absolutely continuous on $[0, 1]$.

The fact that $-\frac{d}{dt} \rho_t \in -\partial^- E(\rho_t)$ for a.e. t grants that for $\varepsilon \in \mathbb{R}$ we have

$$\frac{d}{dt} \int \varepsilon f \rho_t d\mathbf{m} = \int \varepsilon f \frac{d}{dt} \rho_t d\mathbf{m} \leq E(\rho_t - \varepsilon f) - E(f), \quad a.e. t. \quad (5.5)$$

Divide by $\varepsilon > 0$ and let $\varepsilon \downarrow 0$ to obtain

$$\frac{d}{dt} \int f \rho_t d\mathbf{m} \leq \lim_{\varepsilon \downarrow 0} \frac{E(\rho_t - \varepsilon f) - E(f)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \int \frac{|D(\rho_t - \varepsilon f)|^2 - |D\rho_t|^2}{2\varepsilon} d\mathbf{m} = - \int D^- f(\nabla \rho_t) d\mathbf{m},$$

which is the second inequality in (5.4). The first one is obtained starting from (5.5), dividing by $\varepsilon < 0$ and letting $\varepsilon \uparrow 0$.

The general case of $f \in L^1 \cap S^2(X)$ can now be obtained with a simple truncation argument, we omit the details. \square

We now turn to the study of geodesics. In the smooth Euclidean/Riemannian framework a geodesic (μ_t) solves

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

where $\phi_t := Q_{1-t}(-\varphi^c)$ and φ is a Kantorovich potential inducing (μ_t) .

We want to show that the same holds on metric measure spaces, at least for geodesics with bounded compressions. This will be achieved as a simple consequence of Theorem 4.6 and the following fact:

Theorem 5.2. *Let (X, d, \mathbf{m}) be a m.m.s., $(\mu_t) \subset \mathcal{P}_2(X)$ a geodesic with bounded compression and $\varphi \in S^2(X)$ a Kantorovich potential from μ_0 to μ_1 . Then:*

i) Any lifting π of (μ_t) represents the gradient of $-\varphi$.

ii) For any $t \in (0, 1]$ the function $(1-t)Q_{1-t}(-\varphi^c)$ is a Kantorovich potential from μ_t to μ_1 .

Point (i) of this theorem is a restatement of the metric Brenier theorem proved in [5], while point (ii) is a general fact about optimal transport in metric spaces whose proof can be found in [14] or [2].

In stating the continuity equation for geodesics we shall make use of the fact that for $\mu, \nu \in \mathcal{P}_2(X)$ with bounded support, there always exists a Kantorovich potential from μ to ν which is constant outside a bounded set: it is sufficient to pick any Kantorovich potential satisfying (3.16) and proceed with a truncation argument. This procedure ensures that if \mathbf{m} gives finite mass to bounded sets, then these Kantorovich potentials are in $S^2(X)$.

We then have the following result:

Proposition 5.3 (Geodesics). *Let (X, d, \mathbf{m}) be a m.m.s. with \mathbf{m} giving finite mass to bounded sets, $(\mu_t) \subset \mathcal{P}_2(X)$ a geodesic with bounded compression such that μ_0, μ_1 have bounded supports and φ a Kantorovich potential from μ_0 to μ_1 which is constant outside a bounded set.*

Then

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

where $\phi_t := -Q_{1-t}(-\varphi^c)$ for every $t \in [0, 1]$.

Proof. The assumption that φ is constant outside a bounded set easily yields that φ^c is Lipschitz and constant outside a bounded set and that for some $B \subset X$ bounded, ϕ_t is constant outside B for any $t \in [0, 1]$. Also, recalling point (i) of Proposition 2.6 we get that the ϕ_t 's are uniformly Lipschitz so that the assumption that \mathbf{m} gives finite mass to bounded sets yields that $\sup_{t \in [0, 1]} \|\phi_t\|_{S^2} < \infty$. In particular, the function \tilde{N} defined in (4.11) is bounded and hence in $L^2(0, 1)$, so that the statement makes sense.

Now let π be a lifting of (μ_t) and notice that π is a test plan so that $t \mapsto \int f \mu_t$ is absolutely continuous. For $t \in [0, 1)$ the plan $\pi_t := (\text{restr}_t^1)_\# \pi$ is a lifting of $s \mapsto \mu_{t+s(1-t)}$. Thus by Theorem 5.2 above we deduce that π_t represents the gradient of $(1-t)\phi_t$.

The conclusion follows by point (i) of Theorem 4.6. \square

In many circumstances, both heat flows and geodesics have regularity which go slightly beyond that of absolute continuity. Let us propose the following definition:

Definition 5.4 (Weakly C^1 curves). *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression. We say that (μ_t) is weakly C^1 provided for any $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f d\mu_t$ is C^1 .*

In presence of weak C^1 regularity, the description of the operators L_t in Theorem 3.5 can be simplified avoiding the use of the technical Lemma 3.2: it is sufficient to define $L_t : S^2(X) \rightarrow \mathbb{R}$ by

$$L_t(f) := \frac{d}{dt} \int f d\mu_t, \quad \forall f \in S^2(X).$$

Let us now discuss some cases where the heat flow and the geodesics are weakly C^1 . We recall that (X, d, \mathbf{m}) is said *infinitesimally strictly convex* provided

$$D^- f(\nabla g) = D^+ f(\nabla g), \quad \mathbf{m} - a.e. \forall f, g \in S^2(X).$$

We then have the following regularity result:

Proposition 5.5 (Weak C^1 regularity for the heat flow). *With the same assumptions of Proposition 5.1, assume furthermore that (X, d, \mathbf{m}) is infinitesimally strictly convex.*

Then (μ_t) is weakly C^1 .

Proof. We have already seen in the proof of Proposition 5.1 that for any $\rho \in D(\Delta) = D(\partial^- E)$ and $v \in -\partial^- E(\rho) \subset L^2(X, \mathbf{m})$ we have

$$\int D^- f(\nabla \rho) \, d\mathbf{m} \leq \int f v \, d\mathbf{m} \leq \int D^+ f(\nabla \rho) \, d\mathbf{m}.$$

Thus if (X, d, \mathbf{m}) is infinitesimally strictly convex, the set $\partial^- E(\rho)$ contains at most one element. The conclusion then follows from the weak-strong closure of $\partial^- E$. \square

We now turn to geodesics: we will discuss only the case of *infinitesimally Hilbertian* spaces, although weak C^1 regularity is valid on more general circumstances (see [9]). We recall that (X, d, \mathbf{m}) is infinitesimally Hilbertian provided

$$\|f + g\|_{\mathbb{S}^2}^2 + \|f - g\|_{\mathbb{S}^2}^2 = 2\|f\|_{\mathbb{S}^2}^2 + 2\|g\|_{\mathbb{S}^2}^2, \quad \forall f, g \in \mathbb{S}^2(X),$$

and that on infinitesimally Hilbertian spaces we have

$$D^- f(\nabla g) = D^+ f(\nabla g) = D^- g(\nabla f) = D^+ g(\nabla f), \quad \mathbf{m} - a.e. \, \forall f, g \in \mathbb{S}^2(X),$$

so that in particular infinitesimally Hilbertian spaces are infinitesimally strictly convex. The common value of the above expressions will be denoted by $\nabla f \cdot \nabla g$.

The proof of weak C^1 regularity is based on the following lemma, proved in [9]:

Lemma 5.6 ('Weak-strong' convergence). *Let (X, d, \mathbf{m}) be an infinitesimally Hilbert space. Also:*

i) *Let $(\mu_n) \subset \mathcal{P}_2(X)$ a sequence with uniformly bounded densities, such that letting ρ_n be the density of μ_n we have and $\rho_n \rightarrow \rho$ \mathbf{m} -a.e. for some probability density ρ . Put $\mu := \rho \mathbf{m}$.*

ii) *Let $(f_n) \subset \mathbb{S}^2(X)$ be such that:*

$$\sup_{n \in \mathbb{N}} \int |Df_n|^2 \, d\mathbf{m} < \infty,$$

and assume that $f_n \rightarrow f$ \mathbf{m} -a.e. for some Borel function f .

iii) *Let $(g_n) \subset \mathbb{S}^2(X)$ and $g \in \mathbb{S}^2(X)$ such that $g_n \rightarrow g$ \mathbf{m} -a.e. as $n \rightarrow +\infty$ and:*

$$\sup_{n \in \mathbb{N}} \int |Dg_n|^2 \, d\mathbf{m} < \infty, \quad \lim_{n \rightarrow \infty} \int |Dg_n|^2 \, d\mu_n = \int |Dg|^2 \, d\mu.$$

Then

$$\lim_{n \rightarrow \infty} \int \nabla f_n \cdot \nabla g_n \, d\mu_n = \int \nabla f \cdot \nabla g \, d\mu.$$

Proposition 5.7 (Weak C^1 regularity for geodesics). *With the same assumptions of Proposition 5.3 assume furthermore that (X, d, \mathbf{m}) is infinitesimally Hilbertian and that for the densities ρ_t of μ_t we have that $\rho_s \rightarrow \rho_t$ in $L^1(X, \mathbf{m})$ as $s \rightarrow t$.*

Then (μ_t) is a weakly C^1 curve.

Proof. By Proposition 5.3, its proof and taking into account the assumption of infinitesimal Hilbertianity we know that for every $t \in [0, 1)$ and $f \in L^1 \cap \mathbb{S}^2(X)$ we have

$$\lim_{h \downarrow 0} \frac{\int f \, d\mu_{t+h} - \int f \, d\mu_t}{h} = \int \nabla f \cdot \nabla \phi_t \, d\mu_t. \quad (5.6)$$

To conclude it is enough to show that the right hand side of the above expression is continuous in t . Pick $t \in [0, 1)$ and let $(t_n) \subset [0, 1]$ be a sequence converging to t . Up to pass to a subsequence, not relabeled, and using the assumption of strong convergence in $L^1(X, \mathbf{m})$ of ρ_{t_n} to ρ_t , we can assume that $\rho_{t_n} \rightarrow \rho_t$ \mathbf{m} -a.e. as $n \rightarrow \infty$. The proof of Proposition 5.3 grants that $\sup_n \|\phi_{t_n}\|_{\mathbb{S}^2} < \infty$ and that by point (ii) of Proposition 2.6 we know that $\phi_{t_n}(x) \rightarrow \phi_t(x)$ as $n \rightarrow \infty$ for every $x \in X$. Finally, it is obvious that $\lim_{n \rightarrow \infty} \int |Df|^2 \, d\mu_{t_n} = \int |Df|^2 \, d\mu_t$ (because weak convergence in duality with $C_b(X)$ plus uniform bound on the density grant weak convergence in duality with $L^1(X, \mathbf{m})$).

Thus by Lemma 5.6 we deduce the desired continuity of the right hand side of (5.6) for $t \in [0, 1)$. Continuity at $t = 1$ is obtained by considering the geodesic $t \mapsto \mu_{1-t}$. \square

It is worth recalling that the assumptions of Proposition 5.7 are fulfilled on $\text{RCD}(K, \infty)$ spaces when (μ_t) is a (in fact 'the') geodesic connecting two measures with bounded support and bounded density (see [9]).

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