A LOWER SEMICONTINUITY RESULT FOR A FREE DISCONTINUITY FUNCTIONAL WITH A BOUNDARY TERM

STEFANO ALMI, GIANNI DAL MASO, AND RODICA TOADER

Abstract. We study the lower semicontinuity in $GSBV^p(\Omega;\mathbb{R}^m)$ of a free discontinuity functional $\mathcal{F}(u)$ that can be written as the sum of a crack term, depending only on the jump set S_u, and of a boundary term, depending on the trace of u on $\partial\Omega$. We give necessary and sufficient conditions on the integrands for the lower semicontinuity of $\mathcal{F}$. Moreover, we prove a relaxation result, which shows that the lower semicontinuous envelope of $\mathcal{F}$ can still be represented as the sum of two integrals on S_u and $\partial\Omega$, respectively.

Keywords: Free discontinuity problems, special function of bounded variation, lower semicontinuity, relaxation.

MSC 2010: 49J45, 74R10.

1. Introduction

In this paper we are interested in the study of free discontinuity functionals of the form

$$\mathcal{G}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} g(x,u)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 1$, with Lipschitz boundary, $u \in GSBV(\Omega;\mathbb{R}^m)$, the space of generalized special functions of bounded variation, S_u denotes the discontinuity set of u, ν_u is the approximate unit normal vector to S_u, and ∇u stands for the approximate gradient of u (we refer to [2, 5] and Section 2 for definitions and notation).

In the framework of fracture mechanics, see for instance [10, 15], the functional (1.1) represents the energy of an elastic body Ω, with a crack S_u, subject to a displacement u and to external volume and surface forces whose potentials are given by f and g, respectively. In particular, W is the density of the stored elastic energy, while ψ stands for the energy per unit surface needed to extend the crack.

As usual in elasticity, the equilibrium condition of such a body is expressed in terms of the minimum problem

$$\min\{\mathcal{G}(u) : u \in GSBV(\Omega;\mathbb{R}^m)\}\,. \tag{1.2}$$

To apply the direct method of the calculus of variations, we need to know the lower semicontinuity properties of $\mathcal{G}$.

Let us briefly discuss on the usual hypotheses on the volume terms of (1.1), see, e.g., [10, Section 3]. We suppose that the bulk energy density $W(x,\xi)$ is quasiconvex in ξ and satisfies a p-growth condition for some $p \in (1,+\infty)$. These assumptions on W imply that in (1.1) the approximate gradient ∇u is p-summable when $\mathcal{G}(u) < +\infty$, thus the domain of $\mathcal{G}$ is

Preprint SISSA 61/2015/MATE.

actually $GSBV^p(\Omega;\mathbb{R}^m)$, a proper subspace of $GSBV(\Omega;\mathbb{R}^m)$ (we refer to Section 2 for the definition and some properties). Moreover, they guarantee that the volume term

$$\mathcal{W}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x$$

is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ (we refer to Section 2 for this notion of weak convergence and to [17, Theorem 1.2] for the proof).

With mild hypotheses on f, such as continuity with respect to the second variable and a q-growth condition for some $q\in(1,+\infty)$, we may assume that the second volume term in (1.1) is lower semicontinuous with respect to the same notion of convergence.

Therefore, to prove the existence of a solution to (1.2), we are led to study the lower semicontinuity of the surface part of (1.1)

$$\int_{S_u}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial\Omega} g(x,u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{1.3}$$

In this paper, we will consider a slightly more general free discontinuity functional of the form

$$\mathcal{F}(u) := \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma} g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{1.4}$$

where Σ is a prescribed orientable Lipschitz manifold of dimension $n-1$ contained in $\overline{\Omega}$ with $\mathcal{H}^{n-1}(\Sigma)<+\infty$, $\mathcal{H}^{n-1}(\overline{\Sigma}\setminus\Sigma)=0$, and $\mathcal{H}^{n-1}((\overline{\Sigma\cap\Omega})\cap\partial\Omega)=0$, and u^+ and u^- are the traces of u on the positive and negative side of Σ. To give a precise definition of $\mathcal{F}$ when $\Sigma\cap\partial\Omega\neq\emptyset$, the function u is extended to 0 out of Ω, so that u^+ and u^- are well defined $\mathcal{H}^{n-1}$-a.e. on Σ. The functional in (1.3) corresponds to the case $\Sigma=\partial\Omega$.

In Section 3 we prove that $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ under the following assumptions: ψ is a continuous function on $\overline{\Omega}\times\mathbb{R}^n$ such that

$$\begin{gathered}\psi(x,\cdot)\text{ is a norm on }\mathbb{R}^n\text{ for every }x\in\overline{\Omega}\,,\\ c_1|\nu|\le\psi(x,\nu)\le c_2|\nu|\text{ for every }(x,\nu)\in\overline{\Omega}\times\mathbb{R}^n\end{gathered} \tag{1.5}$$

for some $0<c_1\le c_2$, and g is a Borel function on $\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ satisfying

$$(s,t)\mapsto g(x,s,t)\text{ is lower semicontinuous on }\mathbb{R}^m\times\mathbb{R}^m\text{ for every }x\in\Sigma, \tag{1.6}$$

and, for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,s',t,t'\in\mathbb{R}^m$,

$$g(x,s,t)\le g(x,s',t)+\psi(x,\nu_\Sigma(x))\quad\text{and}\quad g(x,s,t)\le g(x,s,t')+\psi(x,\nu_\Sigma(x))\,, \tag{1.7}$$

where $\nu_\Sigma(x)$ denotes the unit normal to Σ at x.

We notice that the hypotheses (1.5) on ψ are quite standard and guarantee that

$$\Psi(u) := \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\qquad\text{for }u\in GSBV^p(\Omega;\mathbb{R}^m) \tag{1.8}$$

is already lower semicontinuous. Indeed, this result has been obtained, for a more general integrand, first in [3, 4] in the case of functions defined on Caccioppoli's partitions, i.e., functions belonging to $BV(\Omega;T)$ for some finite subset T of $\mathbb{R}^m$, and then generalized in [2, Theorem 3.7] to the space $GSBV^p(\Omega;\mathbb{R}^m)$.

The novelty of this paper, in comparison with [2, 3, 4], is the presence of an integral over a fixed surface Σ which is not lower semicontinuous on its own because of the lack of regularity of the function u near Σ. Indeed, we only know that the traces u^+ and u^- of u on the two sides of Σ are measurable functions, but we do not have any continuity or compactness property of the trace operator at our disposal, due to the presence of the jump set. As a matter of fact, it could happen that, along a sequence u_k converging to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$, the jump set S_{u_k} approaches Σ as $k\to+\infty$. In this case, we have no information on the convergence of the traces of u_k. Condition (1.7) will allow us to control the behavior of $\mathcal{F}$ along such sequences.

The proof of the lower semicontinuity theorem is divided into three steps. By the blow-up technique introduced in [6, 13, 14] we first prove that

$$\mathcal{F}(u) \le \liminf_k \mathcal{F}(u_k) \tag{1.9}$$

whenever u_k converges to u pointwise and $u_k, u \in BV(\Omega; T)$ for some finite subset T of $\mathbb{R}^m$ (see Theorem 3.7). We notice that, since the surface Σ is supposed to be only Lipschitz regular, we have to slightly modify the usual choice of the blow-up sets. In Theorem 3.10 we extend (1.9) by approximation to functions belonging to $SBV^p(\Omega; \mathbb{R}^m)$. The third step is a truncation argument, which allows us to conclude in the general case $u \in GSBV^p(\Omega; \mathbb{R}^m)$. In Theorem 3.11 we show that condition (1.7) is also necessary for the lower semicontinuity of the functional $\mathcal{F}$ in $GSBV^p(\Omega; \mathbb{R}^m)$, provided that g is a Carathéodory function satisfying the following properties:

(1.10) there exists $a \in L^1(\Sigma)^+$ such that $g(x, s, t) \ge -a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$,

$$g(\cdot, s, t) \in L^1(\Sigma) \text{ for every } s, t \in \mathbb{R}^m . \tag{1.11}$$

We conclude Section 3 by proving that the minimum problem (1.2) admits a solution (Theorem 3.12).

Finally, in Section 4 we prove a relaxation result for a functional $\mathcal{F}$ of the form (1.4), i.e., we give an integral representation formula for $sc^-\mathcal{F}$, defined as the greatest sequentially weakly lower semicontinuous functional on $GSBV^p(\Omega; \mathbb{R}^m)$ which is less than or equal to $\mathcal{F}$. In (1.4) we still assume that ψ satisfies (1.5). As for g, instead of (1.6) and (1.7), we suppose that g is a Carathéodory function such that $g(x, \cdot, \cdot)$ is uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m$, (1.10) holds, and, for every $M > 0$, $g(x, s, t) \le a_M(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$ with $|s|, |t| \le M$, where $a_M \in L^1(\Sigma)$.

In Theorem 4.3 we show that

$$sc^-\mathcal{F}(u) = \int_{S_u \setminus \Sigma} \psi(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x, u^+, u^-)\, \mathrm{d}\mathcal{H}^{n-1} ,$$

where, for $(x, s, t) \in \Sigma \times \mathbb{R}^m \times \mathbb{R}^m$, we have set

$$g_{12}(x, s, t) := \min\Big\{ g_1(x, s, t), \inf_{\tau \in \mathbb{R}^m} g_1(x, s, \tau) + \psi(x, \nu_\Sigma(x)) \Big\} ,$$
$$g_1(x, s, t) := \min\Big\{ g(x, s, t), \inf_{\sigma \in \mathbb{R}^m} g(x, \sigma, t) + \psi(x, \nu_\Sigma(x)) \Big\} .$$

In this theorem the uniform continuity of $g(x, \cdot, \cdot)$ is replaced by the weaker assumption of continuity of $g_{12}(x, \cdot, \cdot)$.

Therefore, the relaxed functional $sc^-\mathcal{F}$ is again of the form (1.4) and the density g_{12} on Σ is a Carathéodory function which satisfies properties (1.6) and (1.7). The mechanical interpretation of this result is that, if the potential g of the surface force is too strong, it is energetically more convenient to create a new crack near the surface Σ.

We conclude the paper with a relaxation result for the functional $\mathcal{G}$ introduced in (1.1). More precisely, we characterize the functional $sc^-\mathcal{G}$, defined this time as the greatest lower semicontinuous functional in $L^q(\Omega; \mathbb{R}^m)$ which is less than or equal to $\mathcal{G}$. We assume that $W(x, \xi)$ is quasiconvex and has a p-growth with respect to ξ, and that $f(x, s)$ has a q-growth with respect to s. In Theorem 4.5 we prove that

$$sc^-\mathcal{G}(u) = \int_\Omega W(x, \nabla u)\, \mathrm{d}x + \int_\Omega f(x, u)\, \mathrm{d}x + \int_{S_u \setminus \Sigma} \psi(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x, u^+, u^-)\, \mathrm{d}\mathcal{H}^{n-1}$$

if $u \in GSBV^p(\Omega; \mathbb{R}^m) \cap L^q(\Omega; \mathbb{R}^m)$, and $sc^-\mathcal{G}(u) = +\infty$ if $u \in L^q(\Omega; \mathbb{R}^m) \setminus GSBV^p(\Omega; \mathbb{R}^m)$.

If $W(x,\xi)$ has linear growth with respect to ξ, a similar relaxation problem for the functional

$$\int_\Omega W(x,\nabla u)\,\mathrm{d}x+\int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,,\qquad u\in \mathrm{W}^{1,1}(\Omega\setminus\Sigma;\mathbb{R}^m)\,,$$

has been studied in [7] and leads to a functional defined on $BV(\Omega;\mathbb{R}^m)$.

2. Preliminaries and notation

Throughout the paper $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure, respectively. If $p\in[1,+\infty]$ and E is a measurable set, we use the notation $\|\cdot\|_p$ or $\|\cdot\|_{p,E}$ for the L^p-norm on E with respect to $\mathcal{L}^n$ or $\mathcal{H}^{n-1}$, according to the context. Moreover, we denote by $\mathcal{H}^{n-1}\lfloor E$ the measure $\mathcal{H}^{n-1}$ restricted to E, which is defined by $\mathcal{H}^{n-1}\lfloor E\,(F):=\mathcal{H}^{n-1}(F\cap E)$ for every measurable set F.

Definition 2.1. A subset $\Sigma\subseteq\mathbb{R}^n$ is said to be a Lipschitz manifold of dimension $n-1$ with Lipschitz constant L if for every $x\in\Sigma$ there exist a vector $\xi(x)\in\mathbb{S}^{n-1}$, an $(n-1)$-dimensional rectangle Δ_x contained in the hyperplane orthogonal to $\xi(x)$ and passing through the origin, an interval I_x, and a Lipschitz function $\varphi_x\colon\Delta_x\to\mathrm{I}_x$ with Lipschitz constant L such that

$$\{y+t\xi(x):\,y\in\Delta_x,\,t\in\mathrm{I}_x\}\cap\Sigma=\{y+\varphi_x(y)\xi(x):\,y\in\Delta_x\}\,.$$

If Σ is a Lipschitz manifold with Lipschitz constant L, for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ there exists a unit normal vector $\nu_\Sigma(x)$. The tangent space to Σ at x is then

$$T_x(\Sigma):=\{y\in\mathbb{R}^n:\,y\cdot\nu_\Sigma(x)=0\}\,. \tag{2.1}$$

Definition 2.2. An orientable Lipschitz manifold is a pair (Σ,ν_Σ), where Σ is a Lipschitz manifold of dimension $n-1$ and Lipschitz constant L and $\nu_\Sigma\colon\Sigma\to\mathbb{S}^{n-1}$ is a Borel vector field with the following properties:

- $\nu_\Sigma(x)$ is normal to Σ for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$;
- for every $x_0\in\Sigma$ there exist $\xi(x_0)$, Δ_{x_0}, and I_{x_0} as in Definition 2.1 such that $\nu_\Sigma(x)\cdot\xi(x_0)>0$ for $\mathcal{H}^{n-1}$-a.e. $x\in\{y+t\xi(x_0):\,y\in\Delta_{x_0},\,t\in\mathrm{I}_{x_0}\}\cap\Sigma$.

If U is an open set in $\mathbb{R}^n$ with Lipschitz boundary, $\nu_U(x)$ denotes the inner unit normal to U at x, which exists for $\mathcal{H}^{n-1}$-a.e. $x\in\partial U$. It is easy to see that $(\partial U,\nu_U)$ is an orientable Lipschitz manifold.

A set $\Gamma\subseteq\mathbb{R}^n$ is said to be countably $(\mathcal{H}^{n-1},n-1)$-rectifiable if there exists a sequence Γ_j of $(n-1)$-dimensional C^1-manifolds such that $\Gamma=\bigcup\Gamma_j$ up to an $\mathcal{H}^{n-1}$-negligible set. It is well known that every countably $(\mathcal{H}^{n-1},n-1)$-rectifiable set Γ admits an approximate unit normal vector $\nu_\Gamma(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Gamma$ (see, for instance, [11, Sections 3.2.14-16]). Moreover, every Lipschitz manifold Σ is countably $(\mathcal{H}^{n-1},n-1)$-rectifiable (see, e.g., [5, Proposition 2.76]) and its approximate unit normal coincides $\mathcal{H}^{n-1}$-a.e. with the vector ν_Σ considered above.

For every $x\in\mathbb{R}^n$, every $\xi\in\mathbb{S}^{n-1}$, and every $\rho>0$, on the hyperplane orthogonal to ξ and passing through the origin we consider an $(n-1)$-dimensional cube $\mathrm{Q}^{n-1}_{\rho,\xi}(x)$ of side length ρ and centered in the projection $x-(x\cdot\xi)\xi$ of x onto that hyperplane. Given $C>0$, we consider also the n-dimensional rectangle centered in x defined by

$$\mathrm{R}^C_{\rho,\xi}(x):=\{y+t\xi:\,y\in\mathrm{Q}^{n-1}_{\rho,\xi}(x),\,|t-x\cdot\xi|<C\rho\}\,. \tag{2.2}$$

Moreover, we denote by $\mathrm{B}_\rho(x)$ the n-dimensional open ball of radius ρ and center x.

Given a bounded open subset U of $\mathbb{R}^n$, $\mathcal{B}(U)$ denotes the set of Borel subsets of U and $\mathcal{M}_b(U)$ stands for the set of bounded Radon measures on U. For every $\mu,\lambda\in\mathcal{M}_b(U)$, we denote by $\mathrm{d}\mu/\mathrm{d}\lambda$ the Radon-Nikodym derivative of μ with respect to λ.

Let $u\colon U\to\mathbb{R}^m$ be a measurable function. We define the jump set S_u of u as the set of $x\in U$ such that u does not have an approximate limit at x (see [5, Section 4.5]).

The space $BV(U;\mathbb{R}^m)$ of functions of bounded variation is the set of $u \in L^1(U;\mathbb{R}^m)$ whose distributional gradient Du is a bounded Radon measure on U with values in the space $\mathbb{M}^{m\times n}$ of $m\times n$ matrices. Given $u \in BV(U;\mathbb{R}^m)$, we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to $\mathcal{L}^n$. The function u is approximatively differentiable $\mathcal{L}^n$-a.e. in U and its approximate gradient ∇u belongs to $L^1(U;\mathbb{M}^{m\times n})$ and coincides $\mathcal{L}^n$-a.e. in U with the density of $D^a u$ with respect to $\mathcal{L}^n$. Note that the jump set S_u agrees with the complement of the set of Lebesgue points of u, up to an $\mathcal{H}^{n-1}$-negligible set. For all these notions we refer to [5, Sections 3.6 and 3.9].

The space $SBV(U;\mathbb{R}^m)$ of special functions of bounded variation is defined as the set of all $u \in BV(U;\mathbb{R}^m)$ such that $D^s u$ is concentrated on the jump set S_u, i.e., $|D^s u|(U\setminus S_u) = 0$.

As usual, $SBV_{loc}(U;\mathbb{R}^m)$ denotes the space of functions which belong to $SBV(U';\mathbb{R}^m)$ for every $U' \subset\subset U$.

For $p \in (1,+\infty)$, the space $SBV^p(U;\mathbb{R}^m)$ is the set of functions $u \in SBV(U;\mathbb{R}^m)$ with approximate gradient $\nabla u \in L^p(U;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S_u) < +\infty$. We now give the definition of weak convergence in $SBV^p(U;\mathbb{R}^m)$.

Definition 2.3. Let $u_k, u \in SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$. The sequence u_k converges to u weakly in $SBV^p(U;\mathbb{R}^m)$ if $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. in U, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U;\mathbb{M}^{m\times n})$, and $\|u_k\|_\infty$ and $\mathcal{H}^{n-1}(S_{u_k})$ are uniformly bounded with respect to k.

The following compactness theorem is proved in [1].

Theorem 2.4. *Let $p \in (1,+\infty)$ and let u_k be a sequence in $SBV^p(U;\mathbb{R}^m)$ such that $\|u_k\|_\infty$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $SBV^p(U;\mathbb{R}^m)$.*

This result is in general not enough for some applications since it requires an a priori bound on the L^∞-norm. To overcome this difficulty, we have to work in the larger space $GSBV(U;\mathbb{R}^m)$ of generalized special functions of bounded variation, defined as the set of measurable functions $u\colon U \to \mathbb{R}^m$ such that $\varphi(u) \in SBV_{loc}(U;\mathbb{R}^m)$ for every $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ whose gradient has compact support. If $u \in GSBV(U;\mathbb{R}^m)$, then the approximate gradient ∇u exists $\mathcal{L}^n$-a.e. in U and the jump set S_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable. Its approximate unit normal vector is denoted by ν_u.

In the case $m = 1$, we have that $u \in GSBV(U;\mathbb{R})$ if and only if $T_h(u) \in SBV_{loc}(U;\mathbb{R})$ for every $h \in \mathbb{N}$, where T_h is the truncation function defined by $T_h(s) := \min\{\max\{s,-h\}, h\}$ for $s \in \mathbb{R}$ (see for instance [5, Section 4.5]).

For $p \in (1,+\infty)$, we define $GSBV^p(U;\mathbb{R}^m)$ as the set of functions $u \in GSBV(U;\mathbb{R}^m)$ such that $\nabla u \in L^p(U;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S_u) < +\infty$. In particular, if $u \in GSBV^p(U;\mathbb{R}^m)$, then the function $\varphi(u)$ belongs to $SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$ for every $\varphi \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^m$. We notice that $GSBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m) = SBV^p(U;\mathbb{R}^m) \cap L^\infty(U;\mathbb{R}^m)$.

We now recall some basic properties of $GSBV^p(U;\mathbb{R}^m)$, which can be found in [5, Section 4.5] and [10, Section 2].

Proposition 2.5. *$GSBV^p(U;\mathbb{R}^m)$ is a vector space. A function $u := (u^1,\dots,u^m)\colon U \to \mathbb{R}^m$ belongs to $GSBV^p(U;\mathbb{R}^m)$ if and only if each component u^i belongs to $GSBV^p(U;\mathbb{R})$.*

If U has a Lipschitz boundary, for every $u \in GSBV^p(U;\mathbb{R}^m)$ there exists a function $\tilde u\colon \partial U \to \mathbb{R}^m$ such that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$, $\tilde u(x)$ is the approximate limit of u at x, and we write

$$\tilde u(x) := \operatorname*{ap\,lim}_{\substack{y\to x\\ y\in U}} u(y) \tag{2.3}$$

(see, e.g., [11, Section 2.9.12]). The function $\tilde u$ is called the trace of u on ∂U.

Remark 2.6. If (Σ, ν_Σ) is an orientable Lipschitz manifold of dimension $n-1$, with $\Sigma \subseteq U$, for every $x \in \Sigma$ there exists an open neighborhood V of x contained in U such that $V \setminus \Sigma$ has two connected components V^+ and V^-, with Lipschitz boundaries and with $\nu_\Sigma(x)$ pointing towards V^+. For every function $u \in GSBV^p(U;\mathbb{R}^m)$ the traces on $\Sigma \cap V$ of the restriction of u to $V^\pm$ are denoted by $u^\pm$. This allows us to define the traces $u^\pm$ of u $\mathcal{H}^{n-1}$-a.e. on Σ.

We now recall the notion of weak convergence in $GSBV^p(U;\mathbb{R}^m)$.

Definition 2.7. Let $u_k, u \in GSBV^p(U;\mathbb{R}^m)$. The sequence u_k converges to u weakly in $GSBV^p(U;\mathbb{R}^m)$ if $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. in U, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U;\mathbb{M}^{m\times n})$, and $\mathcal{H}^{n-1}(S_{u_k})$ is uniformly bounded with respect to k.

The following compactness theorem has been proved in [2] (see also [5, Section 4.5]).

Theorem 2.8. *Let $p \in (1,+\infty)$ and let u_k be a sequence in $GSBV^p(U;\mathbb{R}^m)$ such that $\|u_k\|_1$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $GSBV^p(U;\mathbb{R}^m)$.*

We recall a lower semicontinuity result in $GSBV^p(U;\mathbb{R}^m)$, proved in [17, Theorem 1.2].

Theorem 2.9. *Let $W\colon U \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function such that*

$$W(x,\cdot) \text{ is quasiconvex for every } x \in U, \tag{2.4}$$

$$a_1|\xi|^p - b_1(x) \le W(x,\xi) \le a_2|\xi|^p + b_2(x) \quad \text{for every } (x,\xi) \in U \times \mathbb{M}^{m\times n} \tag{2.5}$$

for some $1 < p < +\infty$, $0 < a_1 \le a_2$, and $b_1, b_2 \in L^1(U)$.

Then the functional $\mathcal{W}\colon GSBV^p(U;\mathbb{R}^m) \to \mathbb{R}$ defined by

$$\mathcal{W}(u) := \int_U W(x,\nabla u)\,\mathrm{d}x \tag{2.6}$$

is lower semicontinuous with respect to the weak convergence in $GSBV^p(U;\mathbb{R}^m)$.

We say that $E \subseteq \mathbb{R}^n$ is a set of finite perimeter if the distributional gradient of its characteristic function $\mathbf{1}_E$ is a bounded Radon measure on $\mathbb{R}^n$. The essential boundary $\partial^* E$ of E is defined by

$$\partial^* E := \Big\{ x \in \mathbb{R}^n : \limsup_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap E)}{\rho^n} > 0 \quad \text{and} \quad \limsup_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \setminus E)}{\rho^n} > 0 \Big\}.$$

For $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$, there exists the measure theoretical inner unit normal vector $\nu_E(x)$ to E at x. We refer to [5, Sections 3.3 and 3.5] for further properties of sets of finite perimeter.

We conclude this preliminary section with a simple lemma which will be useful in the proof of Theorem 3.7.

Lemma 2.10. *Let U be an open set with Lipschitz boundary and let $E \subseteq U$ be a set of finite perimeter. Let us set*

$$\mathrm{tr}E := \{ x \in \partial U : \widetilde{\mathbf{1}_E}(x) = 1 \},$$

where $\widetilde{\mathbf{1}_E}$ is the trace on ∂U of the restriction of $\mathbf{1}_E$ to U. Then $\mathrm{tr}E = \partial U \cap \partial^* E$ up to an $\mathcal{H}^{n-1}$-negligible set.

Proof. We first notice that the trace of $\mathbf{1}_E$ on ∂U is either 1 or 0 for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$. Therefore, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U \setminus \mathrm{tr}E$ we have that $\widetilde{\mathbf{1}_E}(x) = 0$, hence, by definition of trace,

$$\lim_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap E)}{\rho^n} = \lim_{\rho \searrow 0} \frac{1}{\rho^n} \int_{\mathrm{B}_\rho(x) \cap U} \mathbf{1}_E(y)\,\mathrm{d}y = 0. \tag{2.7}$$

This implies that $\partial U \cap \partial^* E \subseteq \mathrm{tr}E$ up to an $\mathcal{H}^{n-1}$-negligible set.

Viceversa, let $x \in \mathrm{tr}E$ be such that the inner unit normal $\nu_U(x)$ to U at x exists. As in (2.7), by the properties of the trace we have that

$$\lim_{\rho\searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap (U \setminus E))}{\rho^n} = \lim_{\rho\searrow 0} \frac{1}{\rho^n} \int_{\mathrm{B}_\rho(x)\cap U} |\mathbf{1}_E(y) - 1| \,\mathrm{d}y = 0\,.$$

From the previous equality and the properties of $\nu_U(x)$ we deduce that

$$\lim_{\rho\searrow 0} \frac{\mathcal{L}^n(\mathrm{B}^+_\rho(x) \setminus E)}{\rho^n} = 0\,, \tag{2.8}$$

where we have set

$$\mathrm{B}^+_\rho(x) := \big\{ y \in \mathrm{B}_\rho(x) : \, (y - x) \cdot \nu_U(x) > 0 \big\}\,.$$

In view of (2.8) we obtain that

$$\limsup_{\rho\searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap E)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} \geq \frac{1}{2}\,. \tag{2.9}$$

Moreover, since $E \subseteq U$, by the properties of $\nu_U(x)$ we get

$$\limsup_{\rho\searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \setminus E)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} \geq \lim_{\rho\searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \setminus U)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} = \frac{1}{2}\,. \tag{2.10}$$

Inequalities (2.9) and (2.10) imply that $x \in \partial U \cap \partial^* E$, and the proof is thus complete. □

3. Lower semicontinuity

In this section we prove a lower semicontinuity result for a free discontinuity functional with a boundary term or, more in general, for a functional of the form (1.4). Let us introduce the setting of the problem. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, and let (Σ, ν_Σ) be an orientable Lipschitz manifold of dimension $n-1$ and Lipschitz constant L, with $\Sigma \subseteq \overline{\Omega}$ and

$$\mathcal{H}^{n-1}(\Sigma) < +\infty\,, \qquad \mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0\,, \qquad \mathcal{H}^{n-1}((\overline{\Sigma \cap \Omega}) \cap \partial\Omega) = 0\,. \tag{3.1}$$

We consider two functions $\psi \colon \overline{\Omega} \times \mathbb{R}^n \to [0, +\infty)$ and $g \colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ with the following properties:

(H1) ψ is continuous;

(H2) there exist $0 < c_1 \leq c_2$ such that

$$c_1 |\nu| \leq \psi(x, \nu) \leq c_2 |\nu|$$

for every $(x, \nu) \in \overline{\Omega} \times \mathbb{R}^n$;

(H3) $\psi(x, \cdot)$ is a norm on $\mathbb{R}^n$ for every $x \in \overline{\Omega}$;

(H4) g is a Borel function;

(H5) $g(\cdot, 0, 0) \in L^1(\Sigma)$;

(H6) $g(x, \cdot, \cdot)$ is lower semicontinuous for every $x \in \Sigma$;

(H7) for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and for every $s, t, s', t' \in \mathbb{R}^m$

$$g(x, s, t) \leq g(x, s', t) + \psi(x, \nu_\Sigma(x))\,, \tag{3.2}$$

$$g(x, s, t) \leq g(x, s, t') + \psi(x, \nu_\Sigma(x))\,. \tag{3.3}$$

Remark 3.1. If $\Sigma = \partial\Omega$ and $g \colon \partial\Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies

$$s \mapsto g(x, s) \text{ is lower semicontinuous for every } x \in \Sigma,$$

$$g(x, s) \leq g(x, t) + \psi(x, \nu_\Omega(x)) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma \text{ and every } s, t \in \mathbb{R}^m,$$

then $(x, s, t) \mapsto g(x, s + t)$ fulfills (H6) and (H7).

Remark 3.2. The inequalities (3.2) and (3.3) in (H7) are equivalent to

$$\operatorname{osc} g(x,\cdot,t) \leq \psi(x,\nu_\Sigma(x)) \qquad \text{and} \qquad \operatorname{osc} g(x,s,\cdot) \leq \psi(x,\nu_\Sigma(x))\,,$$

where for every function $\gamma\colon \mathbb{R}^m \to \mathbb{R}$

$$\operatorname{osc}\gamma := \sup_{s,t\in\mathbb{R}^m} |\gamma(s)-\gamma(t)| = \sup_{s\in\mathbb{R}^m} \gamma(s) - \inf_{s\in\mathbb{R}^m} \gamma(s)\,.$$

Remark 3.3. Let us set

$$\mathrm{N}^{\pm} := \{x \in \Sigma \cap \partial\Omega :\ \nu_\Sigma(x) = \pm\nu_\Omega(x)\}\,. \tag{3.4}$$

In view of our convention on the traces $u^\pm$ on $\Sigma\cap\partial\Omega$, it is not restrictive to assume that

$$\begin{aligned}&\text{if } x\in N^+, \text{ then } g(x,s,t) = g(x,s,0) \text{ for every } s,t\in\mathbb{R}^m,\\ &\text{if } x\in N^-, \text{ then } g(x,s,t) = g(x,0,t) \text{ for every } s,t\in\mathbb{R}^m.\end{aligned} \tag{3.5}$$

For $p\in(1,+\infty)$, we consider the functionals $\mathcal{F},\Psi\colon GSBV^p(\Omega;\mathbb{R}^m)\to\mathbb{R}$ defined by (1.4) and (1.8), respectively.

The finiteness of $\mathcal{F}(u)$ and $\Psi(u)$ is an easy consequence of (H1)-(H7). For simplicity of notation, from now on, if not necessary, we will not indicate the dependence of $u^\pm$ and ν_u on the space variable x.

As we have already noticed in the Introduction, the lower semicontinuity of Ψ with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ has been proved in [2, Theorem 3.7] for a more general integrand. Here we are interested in the connection between the free discontinuity term and the fixed surface integral in (1.4). In particular, in Theorem 3.4 we prove that conditions (H1)-(H7) are sufficient for the lower semicontinuity of the functional $\mathcal{F}$ in $GSBV^p(\Omega;\mathbb{R}^m)$. Viceversa, in Theorem 3.11 we show that (H7) is also a necessary condition if g is a Carathéodory function.

We now state the main result of this section.

Theorem 3.4. *Let $p\in(1,+\infty)$ and $\mathcal{F}$ be defined as in (1.4) with ψ and g satisfying (H1)-(H7). Then $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$.*

The strategy of the proof of Theorem 3.4 is the following: by the blow-up technique developed in [13, 14] we first prove the lower semicontinuity property for functions belonging to $BV(\Omega;T)$ for some finite set $T\subseteq\mathbb{R}^m$. Then we extend this result to $SBV^p(\Omega;\mathbb{R}^m)$ by approximation and, finally, to $GSBV^p(\Omega;\mathbb{R}^m)$ by a simple truncation argument.

The following lemma shows that, in order to prove Theorem 3.4, it is not restrictive to assume that g is a nonnegative Carathéodory function satisfying (H5) and (H7).

Lemma 3.5. *There exists a sequence $g_\lambda\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ of nonnegative Carathéodory functions satisfying (H5) and (H7) such that $g_\lambda(x,\cdot,\cdot)$ is Lipschitz continuous with Lipschitz constant λ, and, setting*

$$\mathcal{F}_\lambda(u) := \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_\lambda(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $u\in GSBV^p(\Omega;\mathbb{R}^m)$, the following property holds: if $u_k,u\in GSBV^p(\Omega;\mathbb{R}^m)$ satisfy

$$\mathcal{F}_\lambda(u) \leq \liminf_k \mathcal{F}_\lambda(u_k) \quad \textit{for every } \lambda\,,$$

then

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k)\,. \tag{3.6}$$

Proof. For every $(x,s,t)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ and every $\lambda\in\mathbb{N}$ let

$$g_\lambda(x,s,t):=\inf_{\sigma,\tau\in\mathbb{R}^m}\{g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|\}\,, \tag{3.7}$$

where c_2 is the constant in (H2). Let us prove that g_λ is a Carathéodory function. For every $s,t\in\mathbb{R}^m$ and every $c\in\mathbb{R}$, we have that

$$\begin{aligned}&\{x\in\Sigma:\ g_\lambda(x,s,t)<c\}\\&=\{x\in\Sigma:\ \exists\,\sigma,\tau\in\mathbb{R}^m \text{ such that } g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|<c\}\\&=\Pi_\Sigma\big(\{(x,\sigma,\tau)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m:\ g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|<c\}\big)\,,\end{aligned}$$

where $\Pi_\Sigma\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\Sigma$ denotes the projection onto Σ. Since g is Borel, applying the projection theorem (see, e.g., [9, Proposition 8.4.4]), we get that the set $\{x\in\Sigma:\ g_\lambda(x,s,t)<c\}$ is $\mathcal{H}^{n-1}$-measurable. Hence $g_\lambda(\cdot,s,t)$ is $\mathcal{H}^{n-1}$-measurable for every $s,t\in\mathbb{R}^m$. It is easy to see that for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ the function $g_\lambda(x,\cdot,\cdot)$ is Lipschitz continuous with Lipschitz constant λ, thus g_λ is a Carathéodory function.

By (H2) and (H7) for g we have that g_λ is nonnegative and satisfies (H7). The inequalities $0\le g_\lambda(x,0,0)\le 4c_2$ imply that $g_\lambda(\cdot,0,0)\in L^1(\Sigma)$. Since $g(x,\cdot,\cdot)$ is lower semicontinuous and g_λ is the Yosida approximation of $g(x,\cdot,\cdot)-g(x,0,0)+2c_2$, we have that $g_\lambda(x,s,t)\nearrow g(x,s,t)-g(x,0,0)+2c_2$ for every $(x,s,t)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ (see for instance [8, Section 1.3]).

Let u_k,u be as in the statement of the lemma. Then, by definition of g_λ and $\mathcal{F}_\lambda$,

$$\mathcal{F}_\lambda(u)\le\liminf_k\mathcal{F}_\lambda(u_k)\le\liminf_k\mathcal{F}(u_k)-\int_\Sigma g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}+2c_2\,\mathcal{H}^{n-1}(\Sigma)\,. \tag{3.8}$$

By the monotone convergence theorem, we get that

$$\lim_\lambda\mathcal{F}_\lambda(u)=\mathcal{F}(u)-\int_\Sigma g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}+2c_2\,\mathcal{H}^{n-1}(\Sigma)\,.$$

The previous equality, together with (3.8), implies (3.6). □

In the sequel, we will also need the following technical lemma, where $\mathrm{R}^C_{\rho,\xi}(x)$ is defined as in (2.2).

Lemma 3.6. *Let $g\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be a Carathéodory function satisfying properties (H5) and (H7). Then, for every $C>0$ and for every compact subset K of $\mathbb{R}^m\times\mathbb{R}^m$ we have that*

$$\lim_{\rho\searrow0}\frac{1}{\rho^{n-1}}\int_{\Sigma\cap\mathrm{R}^C_{\rho,\xi}(x)}\sup_{(s,t)\in K}|g(y,s,t)-g(x,s,t)|\,\mathrm{d}\mathcal{H}^{n-1}(y)=0 \tag{3.9}$$

for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $\xi\in\mathbb{S}^{n-1}$.

Proof. For every $(x,\delta)\in\Sigma\times(0,+\infty)$ we set

$$\omega(x,\delta):=\sup_{\substack{(s,t),(\sigma,\tau)\in K\\|(s,t)-(\sigma,\tau)|\le\delta}}|g(x,s,t)-g(x,\sigma,\tau)|\,. \tag{3.10}$$

Then $\omega(x,\delta)\to0$ as $\delta\searrow0$ for every $x\in\Sigma$ such that $g(x,\cdot,\cdot)$ is continuous on $\mathbb{R}^m\times\mathbb{R}^m$. Moreover, by properties (H2) and (H7), we have that $\omega(\cdot,\delta)\in L^1(\Sigma)$ for every $\delta>0$.

Fix a sequence $\delta_k\searrow0$. For every $k\in\mathbb{N}$, let $(s^k_1,t^k_1),\dots,(s^k_{l_k},t^k_{l_k})\in K$ satisfy

$$K\subseteq\bigcup_{i=1}^{l_k}\mathrm{B}_{\delta_k}(s^k_i,t^k_i)\,,$$

where, in this proof, $\mathrm{B}_r(s,t)$ denotes the open ball in $\mathbb{R}^m\times\mathbb{R}^m$ of radius r and center (s,t).

Fix $x\in\Sigma$ with the following properties: x is a Lebesgue point of $\omega(\cdot,\delta_k)$ and of $g(\cdot,s^k_i,t^k_i)$ for every k and every $i=1,\dots,l_k$, $\omega(x,\delta_k)\to0$ as $k\to+\infty$, and $\nu_\Sigma(x)$ is normal to Σ at x. Note that these properties are satisfied by $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$.

Finally, fix $k \in \mathbb{N}$. For every $(s,t) \in K$, let $j_s \in \{1,\dots,l_k\}$ be such that $|(s,t) - (s^k_{j_s}, t^k_{j_s})| < \delta_k$. Then, for $\mathcal{H}^{n-1}$-a.e. $y \in \Sigma$ we have that

$$
\begin{aligned}
|g(y,s,t) - g(x,s,t)| &\le |g(y,s,t) - g(y,s^k_{j_s},t^k_{j_s})| + |g(y,s^k_{j_s},t^k_{j_s}) - g(x,s^k_{j_s},t^k_{j_s})| \\
&\quad + |g(x,s^k_{j_s},t^k_{j_s}) - g(x,s,t)| \\
&\le \omega(y,\delta_k) + \sup_{i=1,\dots,l_k} |g(y,s^k_i,t^k_i) - g(x,s^k_i,t^k_i)| + \omega(x,\delta_k)\,.
\end{aligned} \tag{3.11}
$$

Inequality (3.11) implies that, for every $\xi \in \mathbb{S}^{n-1}$ and every $C > 0$,

$$
\begin{aligned}
&\frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}^C_{\rho,\xi}(x)} \sup_{(s,t)\in K} |g(y,s,t) - g(x,s,t)| \,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\le \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}^C_{\rho,\xi}(x)} (\omega(y,\delta_k) + \omega(x,\delta_k)) \,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\quad + \sum_{i=1}^{l_k} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}^C_{\rho,\xi}(x)} |g(y,s^k_i,t^k_i) - g(x,s^k_i,t^k_i)| \,\mathrm{d}\mathcal{H}^{n-1}(y)\,.
\end{aligned} \tag{3.12}
$$

Since, by assumption, $x \in \Sigma$ is a Lebesgue point of $\omega(\cdot,\delta_k)$ and of $g(\cdot,s^k_i,t^k_i)$, passing to the $\limsup$ as $\rho \searrow 0$ in (3.12) we obtain that for every $k \in \mathbb{N}$

$$
\begin{aligned}
&\limsup_{\rho\searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}^C_{\rho,\xi}(x)} \sup_{(s,t)\in K} |g(y,s,t) - g(x,s,t)| \,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\qquad \le 2\mathcal{H}^{n-1}(T_x(\Sigma) \cap \mathrm{R}^C_{1,\xi}(0))\,\omega(x,\delta_k)\,,
\end{aligned} \tag{3.13}
$$

where $T_x(\Sigma)$ is the tangent space defined in (2.1). Passing to the limit as $k \to +\infty$ in (3.13) we get (3.9). □

Let us introduce some notation which will be useful in the sequel. Let T be a finite subset of $\mathbb{R}^m$, U an open subset of Ω such that $\{x \in \Omega : d(x, \Sigma \cup \partial\Omega) < \eta\} \subseteq U$ for some $\eta > 0$, and let Ω' be a bounded smooth open subset of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega'$. For every $u \in BV(U;T) := \{v \in BV(U;\mathbb{R}^m) : v(x) \in T \text{ for } \mathcal{L}^n\text{-a.e. } x \in U\}$, its extension to 0 on $\Omega' \setminus \Omega$ is still denoted by u. We notice that $U' := (\Omega' \setminus \Omega) \cup U$ is open and that this extension belongs to $BV(U';T')$, where $T' := T \cup \{0\}$. For every $B \in \mathcal{B}(U')$ we set

$$
\mathcal{F}_U(u,B) := \int_{U \cap S_u \cap B \setminus \Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma \cap B} g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{3.14}
$$

where, in the second integral, $u^\pm$ denote the traces on the two faces of Σ of u, according to Remark 2.6.

Since ψ and g satisfy (H2), (H5), and (H7), we have that $\mathcal{F}_U(u,\cdot)$ is a measure defined on $\mathcal{B}(U')$. If, in addition, $\mathcal{H}^{n-1}(S_u) < +\infty$, in view of (3.1) $\mathcal{F}_U(u,\cdot)$ belongs to $\mathcal{M}_b(U')$ (this is always the case if $u \in BV(U;T)$ for some finite set $T \subseteq \mathbb{R}^m$). Finally, we notice that if g is nonnegative, then $\mathcal{F}_U(u,\cdot)$ is nonnegative.

We are now ready to state the lower semicontinuity result on $BV(U;T)$.

Theorem 3.7. *Let ψ and g be functions satisfying (H1)-(H7). Assume in addition that g is a nonnegative Carathéodory function. Let T be a finite subset of $\mathbb{R}^m$ and let U be an open subset of Ω such that $\{x \in \Omega : d(x,\Sigma \cup \partial\Omega) < \eta\} \subseteq U$ for some $\eta > 0$. Then*

$$
\mathcal{F}_U(u, U \cup \Sigma) \le \liminf_k \mathcal{F}_U(u_k, U \cup \Sigma)
$$

for every $u_k, u \in BV(U;T)$ such that u_k converges to u pointwise $\mathcal{L}^n$-a.e. in U.

In order to prove Theorem 3.7, we need the following blow-up lemma.

Lemma 3.8. *Let ψ, g, T, U, η, u_k, and u be as in Theorem 3.7 and let $U' := (\Omega' \setminus \Omega) \cup U$. For every $x \in \Sigma$, let $\xi(x) \in \mathbb{S}^{n-1}$ be as in Definition 2.2. Assume that $\mathcal{F}_U(u_k, U \cup \Sigma)$ is bounded and that $\mathcal{F}_U(u_k, \cdot) \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(U')$ for some $\mu \in \mathcal{M}_b(U')$. Then*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor\Sigma}(x) \geq g(x, u^+(x), u^-(x)) \tag{3.15}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$, and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor(S_u \setminus \Sigma)}(x) \geq \psi(x, \nu_u(x)) \tag{3.16}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in S_u \setminus \Sigma$.

Proof. Let us perform the blow-up on Σ. Let $L > 0$ be the Lipschitz constant of Σ and $\Lambda := L\sqrt{n}$. Let $x_0 \in \Sigma$ be such that $\nu_\Sigma(x_0)$ is normal to Σ at x_0 and (H7) holds. We introduce the simplified notation $\mathrm{R}_\rho(x_0) := \mathrm{R}^\Lambda_{\rho,\xi(x_0)}(x_0)$, $\mathrm{R}_\rho := \mathrm{R}^\Lambda_{\rho,\xi(x_0)}(0)$, and $\mathrm{R}^\pm_\rho(x_0) := \{y \in \mathrm{R}_\rho(x_0) : (y - x_0)\cdot \nu_\Sigma(x_0) \gtrless 0\}$, where $\mathrm{R}^C_{\rho,\xi}(x)$ is defined in (2.2). We assume in addition that x_0 satisfies the following conditions:

$$x_0 \notin (\overline{\Sigma \cap \Omega}) \cap \partial\Omega \tag{3.17}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}_\rho(x_0)} |\nu_\Sigma(x) - \nu_\Sigma(x_0)| \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0\,, \tag{3.18}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^n} \int_{\mathrm{R}^\pm_\rho(x_0)} |u(x) - u^\pm(x_0)| \,\mathrm{d}x = 0\,, \tag{3.19}$$

$$\text{there exists } \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))} = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor\Sigma}(x_0)\,, \tag{3.20}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}_\rho(x_0)} \sup_{s,t \in T} |g(x,s,t) - g(x_0,s,t)| \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0\,. \tag{3.21}$$

We notice that conditions (3.17)-(3.21) are satisfied for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \Sigma$ as a consequence of the properties of the traces of BV functions, of hypotheses (3.1), of Lemma 3.6, and of a generalized version of Besicovitch differentiation theorem (see [18] and [12, Sections 1.2.1-1.2.2]).

Since $\nu_\Sigma(x_0)$ is normal to Σ at x_0, we have that

$$\lim_{\rho \searrow 0} \frac{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))}{\rho^{n-1}} = \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1)\,, \tag{3.22}$$

where $T_{x_0}(\Sigma)$ is the tangent space defined in (2.1) and, according to the notation introduced above, $\mathrm{R}_1 = \mathrm{R}^\Lambda_{1,\xi(x_0)}(0)$. Let

$$\gamma(x_0) := \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\rho^{n-1}}\,. \tag{3.23}$$

From (3.20) and (3.22) we get that the limit in (3.23) exists and

$$\gamma(x_0) = \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1) \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))}\,. \tag{3.24}$$

Using the definition (3.23), we shall first express $\gamma(x_0)$ as limit of suitable rescalings of the functional $\mathcal{F}_U$. Then we shall estimate $\gamma(x_0)$ from below using g, and finally we shall deduce (3.15) thanks to (3.24).

By the weak*-convergence of $\mathcal{F}_U(u_k, \cdot)$ to μ, we have that

$$\mathcal{F}_U(u_k, \mathrm{R}_\rho(x_0)) \to \mu(\mathrm{R}_\rho(x_0)) \tag{3.25}$$

for every $\rho > 0$ out of an at most countable set. Thus, we can fix a sequence $\rho_j \searrow 0$ such that $\Omega \cap \mathrm{R}_{\rho_j}(x_0) \subseteq U$, (3.25) holds for every ρ_j, and

$$\lim_j \frac{\mu(\mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}} = \gamma(x_0)\,. \tag{3.26}$$

Since Σ is a Lipschitz manifold with Lipschitz constant L, for j sufficiently large the function φ_{x_0} of Definition 2.1 is well-defined and Lipschitz continuous on the $(n-1)$-dimensional cube $\mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$, with Lipschitz constant L. Let $\tilde{x} := x_0 - (x_0 \cdot \xi(x_0))\xi(x_0)$ be the center of $\mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$. Then, for every $y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$ we have that

$$|\varphi_{x_0}(y) - \varphi_{x_0}(\tilde{x})| \leq L|y - \tilde{x}| \leq \frac{\Lambda}{2}\rho_j\,. \tag{3.27}$$

In view of the definition of the rectangle $\mathrm{R}_{\rho_j}(x_0)$, inequality (3.27) implies that

$$\mathrm{R}_{\rho_j}(x_0) \cap \Sigma = \{y + \varphi_{x_0}(y)\xi(x_0) : \, y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)\}\,.$$

We define

$$A^{\rho_j}_{\pm} := \{y + t\xi(x_0) : \, y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0),\, |t - x_0 \cdot \xi(x_0)| < \Lambda\rho_j,\, t \gtrless \varphi_{x_0}(y)\}\,. \tag{3.28}$$

It is easy to see that $A^{\rho_j}_{+}$ and $A^{\rho_j}_{-}$ are connected, have Lipschitz boundaries, and that $\nu_\Sigma(x)$ points towards $A^{\rho_j}_{+}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathrm{R}_{\rho_j}(x_0) \cap \Sigma$. Moreover, thanks to (3.17), it is not restrictive to assume that if $x_0 \in \Sigma \cap \partial\Omega$, then $A^{\rho_j}_{+} = \mathrm{R}_{\rho_j}(x_0) \cap \Omega$ and $A^{\rho_j}_{-} = \mathrm{R}_{\rho_j}(x_0) \setminus \overline{\Omega}$, or viceversa, according to the orientation of $\nu_\Omega(x_0)$ with respect to $\nu_\Sigma(x_0)$. Conversely, if $x_0 \in \Sigma \setminus \partial\Omega$, we assume that $\mathrm{R}_{\rho_j}(x_0) \subseteq \Omega$.

It is now convenient to rescale $\mathcal{F}_U$ to the rectangle R_1 and, consequently, to define the corresponding rescaled sets and functions: let $\Omega_j := \{y \in \mathbb{R}^n : \, x_0 + \rho_j y \in \Omega\}$, $\Sigma_j := \{y \in \mathbb{R}^n : \, x_0 + \rho_j y \in \Sigma\}$,

$$A^{\pm}_j := \{y \in \mathbb{R}^n : \, x_0 + \rho_j y \in A^{\rho_j}_{\pm}\}\,, \tag{3.29}$$

and $u^j_k(y) := u_k(x_0 + \rho_j y)$ for $y \in \mathrm{R}_1$, noticing that $u^j_k(y) = 0$ for $y \in \mathrm{R}_1 \setminus \Omega_j$. By the change of variables $x = x_0 + \rho_j y$ with $y \in \mathrm{R}_1$ we have

$$\begin{aligned} \frac{\mathcal{F}_U(u_k, \mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}} &= \int_{\Omega_j \cap S_{u^j_k} \cap \mathrm{R}_1 \setminus \Sigma_j} \psi(x_0 + \rho_j y, \nu_{u^j_k}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\Sigma_j \cap \mathrm{R}_1} g(x_0 + \rho_j y, (u^j_k)^+(y), (u^j_k)^-(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) = \mathcal{F}^{\rho_j}(u^j_k, \mathrm{R}_1)\,, \end{aligned} \tag{3.30}$$

where

$$\mathcal{F}^{\rho_j}(v, B) := \int_{\Omega_j \cap S_v \cap B \setminus \Sigma_j} \psi(x_0 + \rho_j y, \nu_v(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\Sigma_j \cap B} g(x_0 + \rho_j y, v^+(y), v^-(y))\, \mathrm{d}\mathcal{H}^{n-1}(y)$$

for every $j \in \mathbb{N}$, every $v \in BV(\mathrm{R}_1; T')$, and every $B \in \mathcal{B}(\mathrm{R}_1)$.

Let us introduce $u^j(y) := u(x_0 + \rho_j y)$ and

$$u^{x_0}(y) := \begin{cases} u^+(x_0) & \text{if } y \in \mathrm{R}^+_1\,, \\ u^-(x_0) & \text{if } y \in \mathrm{R}^-_1\,, \end{cases} \tag{3.31}$$

where we have set $\mathrm{R}^{\pm}_1 := \{y \in \mathrm{R}_1 : \, y \cdot \nu_\Sigma(x_0) \gtrless 0\}$. By hypothesis, $u^j_k \to u^j$ in $L^1(\mathrm{R}_1; T')$ as $k \to +\infty$ and, by (3.19), $u^j \to u^{x_0}$ in $L^1(\mathrm{R}_1; T')$. Therefore, we can find a sequence $k_j \nearrow +\infty$ such that $u^j_{k_j} \to u^{x_0}$ in $L^1(\mathrm{R}_1; T')$ as $j \to +\infty$ and, by (3.25) and (3.30),

$$\left| \mathcal{F}^{\rho_j}(u^j_{k_j}, \mathrm{R}_1) - \frac{\mu(\mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}} \right| < \frac{1}{j}\,. \tag{3.32}$$

By (3.26) and (3.32) we get that

$$\gamma(x_0) = \lim_j \mathcal{F}^{\rho_j}(u^j_{k_j}, \mathrm{R}_1)\,. \tag{3.33}$$

Besides $\mathcal{F}^{\rho_j}(v,B)$, it is convenient to consider also the functional $\mathcal{F}^{\rho_j}_{x_0}(v,B)$ defined by "freezing" the value of the first argument of ψ and g at x_0:

$$\mathcal{F}^{\rho_j}_{x_0}(v,B) := \int_{\Omega_j \cap S_v \cap B \setminus \Sigma_j} \psi(x_0,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma_j \cap B} g(x_0,v^+,v^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $j\in\mathbb{N}$, every $v\in BV(\mathrm{R}_1;T')$, and every $B\in\mathcal{B}(\mathrm{R}_1)$.

Equalities (3.21) and (3.33), together with the uniform continuity of ψ on $\overline{\Omega}\times\mathbb{S}^{n-1}$, imply that

$$\gamma(x_0) = \lim_j \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j}, \mathrm{R}_1)\,. \tag{3.34}$$

The next step of the proof is to show that, in order to give an estimate of $\gamma(x_0)$ in terms of g, we can restrict ourselves to functions which are equal to $u^+(x_0)$ or $u^-(x_0)$ near $\partial\mathrm{R}_1$. To this end, let us define, for every $j\in\mathbb{N}$, the functions

$$u^{x_0}_j(y) := \begin{cases} u^+(x_0) & \text{if } y\in A^+_j\,,\\ u^-(x_0) & \text{if } y\in A^-_j\,,\end{cases}$$

where $A^\pm_j$ are introduced in (3.29). The difference between this definition and (3.31) is that in (3.31) the interface is flat and coincides with $T_{x_0}(\Sigma)\cap\mathrm{R}_1$, while here the interface is the rescaled version Σ_j of Σ. It is clear that $u^{x_0}_j\in BV(\mathrm{R}_1;T')$ and $u^{x_0}_j\to u^{x_0}$ in $L^1(\mathrm{R}_1;T')$ as $j\to+\infty$.

Given $\varepsilon>0$, we now modify the functions $u^j_{k_j}$ near $\partial\mathrm{R}_1$ in order to obtain new functions v_j in $BV(\mathrm{R}_1;T')$ such that $v_j\to u^{x_0}$ in $L^1(\mathrm{R}_1;T')$, $v_j=u^{x_0}_j$ in a neighborhood of $\partial\mathrm{R}_1$, and

$$\limsup_j \mathcal{F}^{\rho_j}_{x_0}(v_j,\mathrm{R}_1) \le \lim_j \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_1)+\varepsilon = \gamma(x_0)+\varepsilon\,. \tag{3.35}$$

This will be done following the lines of an interpolation argument proposed in [3, Lemma 4.4].

To this aim, we consider the distance function $d\colon T'\times T'\to\{0,1\}$ defined by $d(i,j):=1$ for $i,j\in T'$ with $i\neq j$ and $d(i,i):=0$. Let us fix $0<r_1<r_2<1$ and a function $\varphi\in C^\infty(\mathrm{R}_1)$ such that $0\le\varphi\le1$, $\varphi=1$ in $\mathrm{R}_1\setminus\overline{\mathrm{R}}_{r_2}$, and $\varphi=0$ in $\overline{\mathrm{R}}_{r_1}$. By Sard Lemma and Coarea Formula, for every j we can find $t_j\in(0,1)$ such that

$$\partial\{\varphi<t_j\}=\{\varphi=t_j\} \text{ is } C^\infty\,, \tag{3.36}$$

$$\mathcal{H}^{n-1}(\{\varphi=t_j\})<+\infty\,, \tag{3.37}$$

$$\mathcal{H}^{n-1}(S_{u^j_{k_j}}\cap\{\varphi=t_j\})=\mathcal{H}^{n-1}(\Sigma_j\cap\{\varphi=t_j\})=0\,, \tag{3.38}$$

$$\int_{\{\varphi=t_j\}\cap\mathrm{R}_{r_2}\setminus\mathrm{R}_{r_1}} d(u^{x_0}_j,u^j_{k_j})\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{\mathrm{R}_{r_2}\setminus\mathrm{R}_{r_1}} d(u^{x_0}_j,u^j_{k_j})\,|\nabla\varphi|\,\mathrm{d}x \le C\,\mathcal{L}^n(\{u^{x_0}_j\neq u^j_{k_j}\}\cap\mathrm{R}_{r_2}\setminus\mathrm{R}_{r_1})\,, \tag{3.39}$$

where $C:=\|\nabla\varphi\|_\infty$. For such a t_j we set

$$v^{r_1,r_2}_j(x) := \begin{cases} u^j_{k_j}(x) & \text{if } \varphi(x)<t_j\,,\\ u^{x_0}_j(x) & \text{if } \varphi(x)\ge t_j\,.\end{cases}$$

Then $v^{r_1,r_2}_j\in BV(\mathrm{R}_1;T')$, $v^{r_1,r_2}_j=u^{x_0}_j$ in $\mathrm{R}_1\setminus\mathrm{R}_{r_2}$, $v^{r_1,r_2}_j=u^j_{k_j}$ in R_{r_1}, and $v^{r_1,r_2}_j\to u^{x_0}$ in $L^1(\mathrm{R}_1;T')$ as $j\to+\infty$. By (3.37), $\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\cdot)$ is a nonnegative bounded Radon measure on R_1. Thus, to estimate $\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\mathrm{R}_1)$, we integrate separately on the sets

$\{\varphi < t_j\}$ and $\{\varphi > t_j\}$, and on the interface $\{\varphi = t_j\}$. Taking into account (H2) and (3.36)-(3.39), we get that

$$\begin{aligned}\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j, \mathrm{R}_1) &\le \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j}, \mathrm{R}_{r_2}) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j, \mathrm{R}_1\backslash\mathrm{R}_{r_1}) + c_2\int_{\{\varphi=t_j\}\cap\mathrm{R}_{r_2}\backslash\mathrm{R}_{r_1}} d(u^{x_0}_j, u^j_{k_j})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j}, \mathrm{R}_1) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j, \mathrm{R}_1\backslash\mathrm{R}_{r_1}) + c_2 C\,\mathcal{L}^n(\{u^{x_0}_j \neq u^j_{k_j}\}\cap\mathrm{R}_{r_2}\backslash\mathrm{R}_{r_1}).\end{aligned} \tag{3.40}$$

Since $u^j_{k_j}, u^{x_0}_j$ converge to u^{x_0} in $L^1(\mathrm{R}_1; T')$, passing to the lim sup as $j \to +\infty$ in (3.40) we deduce that

$$\limsup_j \mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j, \mathrm{R}_1) \le \limsup_j \big(\mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j}, \mathrm{R}_1) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j, \mathrm{R}_1\setminus\mathrm{R}_{r_1})\big). \tag{3.41}$$

Obviously, $u^{x_0}_j$ does not have jump points in $\mathrm{R}_1\setminus\Sigma_j$. Hence, recalling that (Σ, ν_Σ) is an orientable Lipschitz manifold, we have that

$$\begin{aligned}\mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j, \mathrm{R}_1\setminus\mathrm{R}_{r_1}) &= \int_{\Sigma_j\cap\mathrm{R}_1\backslash\mathrm{R}_{r_1}} g(x_0, (u^{x_0}_j)^+, (u^{x_0}_j)^-)\,\mathrm{d}\mathcal{H}^{n-1}\\ &= g(x_0, u^+(x_0), u^-(x_0))\,\mathcal{H}^{n-1}(\Sigma_j\cap\mathrm{R}_1\setminus\mathrm{R}_{r_1}).\end{aligned} \tag{3.42}$$

Since $\nu_\Sigma(x_0)$ is normal to Σ at x_0, $\mathcal{H}^{n-1}(\Sigma_j\cap\mathrm{R}_1\setminus\mathrm{R}_{r_1}) \to \mathcal{H}^{n-1}(T_{x_0}(\Sigma)\cap\mathrm{R}_1\setminus\mathrm{R}_{r_1})$ as $j\to+\infty$. Therefore, given $\varepsilon>0$, we can choose $0<r_1<r_2<1$ such that

$$g(x_0, u^+(x_0), u^-(x_0))\lim_j \mathcal{H}^{n-1}(\Sigma_j\cap\mathrm{R}_1\setminus\mathrm{R}_{r_1}) < \varepsilon,$$

and set $v_j := v^{r_1,r_2}_j$. By (3.41) and (3.42), we get (3.35).

We now study the behavior of v_j and $\mathcal{F}^{\rho_j}_{x_0}(v_j,\cdot)$ on the interface between the sets $\{v_j = u^{x_0}_j\}$ and $\{v_j \neq u^{x_0}_j\}$. To this aim, we define, for every j,

$$E^\pm_j := A^\pm_j \cap \{v_j \neq u^{x_0}_j\}.$$

Since $v_j, u^{x_0}_j \in BV(\mathrm{R}_1; T')$ and $v_j = u^{x_0}_j$ in a neighborhood of $\partial\mathrm{R}_1$, the sets $E^\pm_j$ have finite perimeter and $E^\pm_j \subset\subset \mathrm{R}_1$. We set also

$$\mathrm{tr}E^\pm_j := \{y\in\partial A^\pm_j : \widetilde{\mathbf{1}_{E^\pm_j}}(y) = 1\}.$$

By the definitions of $A^\pm_j$, of $E^\pm_j$, and of $\mathrm{tr}E^\pm_j$, for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma_j\setminus\mathrm{tr}E^\pm_j$ we have that

$$\begin{aligned}\lim_{r\searrow 0}\frac{1}{r^n}\int_{\mathrm{B}_r(x)\cap A^\pm_j}|v_j(y)-u^\pm(x_0)|\,\mathrm{d}y &= \lim_{r\searrow 0}\frac{1}{r^n}\int_{\mathrm{B}_r(x)\cap E^\pm_j}|v_j(y)-u^\pm(x_0)|\,\mathrm{d}y\\ &\le c\lim_{r\searrow 0}\frac{\mathcal{L}^n(\mathrm{B}_r(x)\cap E^\pm_j)}{r^n} = c\lim_{r\searrow 0}\frac{1}{r^n}\int_{\mathrm{B}_r(x)\cap A^\pm_j}\mathbf{1}_{E^\pm_j}(y)\,\mathrm{d}y = 0,\end{aligned} \tag{3.43}$$

where $c := 2\max\{|s| : s\in T\}$. Equality (3.43) implies that for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma_j\setminus\mathrm{tr}E^\pm_j$ the traces $v^\pm_j(x)$ on the two sides of Σ_j are equal to $u^\pm(x_0)$, respectively.

We now prove that

$$\int_{\partial^*E^\pm_j\backslash\mathrm{tr}E^\pm_j}\nu_{E^\pm_j}\,\mathrm{d}\mathcal{H}^{n-1} = \mp\int_{\mathrm{tr}E^\pm_j}\nu_{\Sigma_j}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.44}$$

By Lemma 2.10 and by the definition of v_j, we have that, up to an $\mathcal{H}^{n-1}$-negligible set,

$$\mathrm{tr}E^\pm_j = \Sigma_j\cap\partial^*E^\pm_j. \tag{3.45}$$

Since $E_j^{\pm}$ have finite perimeter, we get that

$$\int_{\partial^* E_j^{\pm}} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} = (D\mathbf{1}_{E_j^{\pm}})(\mathbb{R}^n) = 0\,, \tag{3.46}$$

where $\nu_{E_j^{\pm}}$ are the inner unit normals to $E_j^{\pm}$. By the definitions of $A_{\pm}^{\rho_j}$ and of $A_j^{\pm}$ given in (3.28)-(3.29), by Definition 2.2, and by the equality (3.45), for j large enough $\nu_{E_j^{\pm}} = \pm\nu_{\Sigma_j}$ $\mathcal{H}^{n-1}$-a.e. on $\mathrm{tr}E_j^{\pm}$. Hence, by (3.46) we have that

$$\begin{aligned} 0 &= \int_{\partial^* E_j^{\pm}} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* E_j^{\pm}\cap\Sigma_j} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E_j^{\pm}\setminus\Sigma_j} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \pm \int_{\mathrm{tr}E_j^{\pm}} \nu_{\Sigma_j} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E_j^{\pm}\setminus\mathrm{tr}E_j^{\pm}} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned}$$

which implies (3.44).

From (3.18) we obtain that

$$\lim_j \int_{\mathrm{tr}E_j^{\pm}} |\nu_{\Sigma_j}(y) - \nu_{\Sigma}(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \le \lim_j \int_{\Sigma_j\cap \mathrm{R}_1} |\nu_{\Sigma_j}(y) - \nu_{\Sigma}(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(y) = 0\,. \tag{3.47}$$

Therefore, thanks to the continuity of ψ, to hypothesis (H3), and to equalities (3.44) and (3.47), we get that

$$\lim_j \left| \psi\Big(x_0, \int_{\mathrm{tr}E_j^{\pm}} \nu_{\Sigma_j}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) - \psi(x_0, \nu_{\Sigma}(x_0))\, \mathcal{H}^{n-1}(\mathrm{tr}E_j^{\pm}) \right| = 0\,, \tag{3.48}$$

and, by Jensen inequality, for every j it holds

$$\begin{aligned} \psi\Big(x_0, \int_{\mathrm{tr}E_j^{\pm}} \nu_{\Sigma_j}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) &= \psi\Big(x_0, \int_{\partial^* E_j^{\pm}\setminus\mathrm{tr}E_j^{\pm}} \nu_{E_j^{\pm}}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) \\ &\le \int_{\partial^* E_j^{\pm}\setminus\mathrm{tr}E_j^{\pm}} \psi(x_0, \nu_{E_j^{\pm}}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) = \int_{A_j^{\pm}\cap\partial^* E_j^{\pm}} \psi(x_0, \nu_{E_j^{\pm}}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y)\,, \end{aligned} \tag{3.49}$$

where in the last step we have used the equality $\partial^* E_j^{\pm} \setminus \mathrm{tr}E_j^{\pm} = A_j^{\pm} \cap \partial^* E_j^{\pm}$.

We are now ready to estimate from below $\gamma(x_0)$ in terms of $g(x_0, u^+(x_0), u^-(x_0))$ and then to conclude the blow-up argument on Σ. Recalling inequality (3.43) and the inclusions $\partial^* E_j^{\pm} \setminus \Sigma_j \subseteq S_{v_j} \cap A_j^{\pm} \subseteq S_{v_j} \cap \mathrm{R}_1 \setminus \Sigma_j$, we can write (3.35) as

$$\begin{aligned} \gamma(x_0) + \varepsilon \ge \limsup_j \Big(&\int_{\Omega_j\cap A_j^+\cap\partial^* E_j^+} \psi(x_0, \nu_{E_j^+}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &+ \int_{\Omega_j\cap A_j^+\cap(S_{v_j}\setminus\partial^* E_j^+)\cap\mathrm{R}_1} \psi(x_0, \nu_{v_j}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\Omega_j\cap A_j^-\cap\partial^* E_j^-} \psi(x_0, \nu_{E_j^-}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &+ \int_{\Omega_j\cap A_j^-\cap(S_{v_j}\setminus\partial^* E_j^-)\cap\mathrm{R}_1} \psi(x_0, \nu_{v_j}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathrm{tr}E_j^+\cup\mathrm{tr}E_j^-} g(x_0, v_j^+(y), v_j^-(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &+ g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}((\Sigma_j \setminus (\mathrm{tr}E_j^+ \cup \mathrm{tr}E_j^-) \cap \mathrm{R}_1)\Big)\,. \end{aligned} \tag{3.50}$$

Taking into account (3.48)-(3.43) and splitting the set $\mathrm{tr}E_j^+ \cup \mathrm{tr}E_j^-$ into the union of the pairwise disjoint sets $\mathrm{tr}E_j^+ \setminus \mathrm{tr}E_j^-$, $\mathrm{tr}E_j^- \setminus \mathrm{tr}E_j^+$, and $\mathrm{tr}E_j^+ \cap \mathrm{tr}E_j^-$, from (3.50) we obtain

$$\begin{aligned}
&\gamma(x_0)+\varepsilon \geq \limsup_j \Big(\psi(x_0,\nu_\Sigma(x_0))(\mathcal{H}^{n-1}(\mathrm{tr}E_j^+)+\mathcal{H}^{n-1}(\mathrm{tr}E_j^-))\\
&+\int\limits_{\mathrm{tr}E_j^+\setminus\mathrm{tr}E_j^-} g(x_0,v_j^+(y),u^-(x_0))\mathrm{d}\mathcal{H}^{n-1}(y)+\int\limits_{\mathrm{tr}E_j^-\setminus\mathrm{tr}E_j^+} g(x_0,u^+(x_0),v_j^-(y))\mathrm{d}\mathcal{H}^{n-1}(y)\\
&+\int\limits_{\mathrm{tr}E_j^+\cap\mathrm{tr}E_j^-} g(x_0,v_j^+(y),v_j^-(y))\mathrm{d}\mathcal{H}^{n-1}(y)+g(x_0,u^+(x_0),u^-(x_0))\mathcal{H}^{n-1}((\Sigma_j\setminus(\mathrm{tr}E_j^+\cup\mathrm{tr}E_j^-)\cap\mathrm{R}_1)\Big)\\
&=\limsup_j \Big(\psi(x_0,\nu_\Sigma(x_0))\big(\mathcal{H}^{n-1}(\mathrm{tr}E_j^+\setminus\mathrm{tr}E_j^-)+\mathcal{H}^{n-1}(\mathrm{tr}E_j^-\setminus\mathrm{tr}E_j^+)+2\mathcal{H}^{n-1}(\mathrm{tr}E_j^+\cap\mathrm{tr}E_j^-)\big)\\
&+\int\limits_{\mathrm{tr}E_j^+\setminus\mathrm{tr}E_j^-} g(x_0,v_j^+(y),u^-(x_0))\mathrm{d}\mathcal{H}^{n-1}(y)+\int\limits_{\mathrm{tr}E_j^-\setminus\mathrm{tr}E_j^+} g(x_0,u^+(x_0),v_j^-(y))\mathrm{d}\mathcal{H}^{n-1}(y)\\
&+\int\limits_{\mathrm{tr}E_j^+\cap\mathrm{tr}E_j^-} g(x_0,v_j^+(y),v_j^-(y))\mathrm{d}\mathcal{H}^{n-1}(y)+g(x_0,u^+(x_0),u^-(x_0))\,\mathcal{H}^{n-1}((\Sigma_j\setminus(\mathrm{tr}E_j^+\cup\mathrm{tr}E_j^-))\cap\mathrm{R}_1)\Big).
\end{aligned}$$

Using (H7) in the previous inequality we get

$$\begin{aligned}
\gamma(x_0)+\varepsilon &\geq g(x_0,u^+(x_0),u^-(x_0))\,\limsup_j\,\mathcal{H}^{n-1}(\Sigma_j\cap\mathrm{R}_1)\\
&= g(x_0,u^+(x_0),u^-(x_0))\,\mathcal{H}^{n-1}(T_{x_0}(\Sigma)\cap\mathrm{R}_1)\,,
\end{aligned} \tag{3.51}$$

where in the last equality we have used the fact that $\nu_\Sigma(x_0)$ is normal to Σ at x_0. Passing to the limit in (3.51) as $\varepsilon \searrow 0$ we get

$$\gamma(x_0) \geq g(x_0,u^+(x_0),u^-(x_0))\,\mathcal{H}^{n-1}(T_{x_0}(\Sigma)\cap\mathrm{R}_1)$$

for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \Sigma$. In view of (3.24) we have (3.15).

We already know that the functional

$$\Psi_U(v) := \int_{U\cap S_v} \psi(x,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1}$$

is lower semicontinuous in $BV(U;T)$ with respect to the pointwise convergence (see [2, 4]). Now we show, using the blow-up technique, that (3.16) holds for $\mathcal{H}^{n-1}$-a.e. $x \in S_u \setminus \Sigma$. Indeed, let $x \in S_u \setminus \Sigma$ be such that

$$x \notin \overline{\Sigma}\,, \tag{3.52}$$

(3.53) there exists the approximate unit normal vector $\nu_u(x)$ to S_u at x,

$$\lim_{\rho\searrow 0} \frac{1}{\rho^{n-1}} \int_{S_u\cap\mathrm{B}_\rho(x)} |\nu_u(y)-\nu_u(x)|\,\mathrm{d}\mathcal{H}^{n-1}(y) = 0\,, \tag{3.54}$$

(3.55) there exists $\displaystyle\lim_{\rho\searrow 0} \frac{\mu(\mathrm{B}_\rho(x))}{\mathcal{H}^{n-1}(\mathrm{B}_\rho(x)\cap(S_u\setminus\Sigma))} = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor(S_u\setminus\Sigma)}(x)\,.$

We notice that properties (3.52)-(3.55) are satisfied by $\mathcal{H}^{n-1}$-a.e. $x \in S_u \setminus \Sigma$ as a consequence of hypotheses (3.1), of well-known properties of BV functions, and of the Besicovitch differentiation theorem.

Let $\rho_j \searrow 0$ be such that, for every $j \in \mathbb{N}$, $\mathrm{B}_{\rho_j}(x) \subseteq U \setminus \Sigma$ and $\mathcal{F}_U(u_k, \mathrm{B}_{\rho_j}(x)) \to \mu(\mathrm{B}_{\rho_j}(x))$ as $k \to +\infty$. Then, in view of the continuity of ψ, of the definition (3.14) of $\mathcal{F}_U$, and of conditions (3.52) and (3.54), we have

$$\begin{aligned}\lim_{\rho\searrow 0}\frac{\mu(\mathrm{B}_\rho(x))}{\mathcal{H}^{n-1}(\mathrm{B}_\rho(x)\cap(S_u\setminus\Sigma))}&=\lim_j\frac{\mu(\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}\\ &=\lim_j\lim_k\frac{\mathcal{F}_U(u_k,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}=\lim_j\lim_k\frac{\Psi_U(u_k,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}\\ &\geq\lim_j\frac{\Psi_U(u,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}=\psi(x,\nu_u(x))\,,\end{aligned}\tag{3.56}$$

Since (3.55) holds, the previous inequality implies (3.16). This concludes the proof of the lemma. □

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. Let ψ, g, T, U, η, u_k, u be as in the statement of the theorem, and let $U':=(\Omega'\setminus\Omega)\cup U$, as in Lemma 3.8.

Assume that

$$\liminf_k \mathcal{F}_U(u_k,U\cup\Sigma)<+\infty.\tag{3.57}$$

Up to a subsequence, we may suppose that the $\liminf$ in (3.57) is a limit and that there exists $M>0$ such that $\mathcal{F}_U(u_k,U\cup\Sigma)\leq M$. Then the sequence of nonnegative measures $\mathcal{F}_U(u_k,\cdot)$ is bounded in $\mathcal{M}_b(U')$. Therefore, there exists a nonnegative measure $\mu\in\mathcal{M}_b(U')$ such that, up to a subsequence, $\mathcal{F}_U(u_k,\cdot)\rightharpoonup\mu$ weakly* in $\mathcal{M}_b(U')$.

Applying Lemma 3.8 and recalling the definition (3.14) of $\mathcal{F}_U$, we get that

$$\begin{aligned}\mathcal{F}_U(u,U\cup\Sigma)&=\int_{U\cap S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\leq\mu(U\cup\Sigma)\leq\mu(U')\\ &\leq\liminf_k\mathcal{F}_U(u_k,U')=\liminf_k\mathcal{F}_U(u_k,U\cup\Sigma)\,,\end{aligned}$$

and the proof is thus concluded. □

Remark 3.9. We notice that, if we assume g to be symmetric on $\mathbb{R}^m\times\mathbb{R}^m$, that is, $g(x,s,t)=g(x,t,s)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$, then the orientability property given in Definition 2.2 is not needed to prove Theorem 3.7 and Lemma 3.8: indeed in this case it is enough to assume Σ to be a Lipschitz manifold of dimension $n-1$.

In the following theorem we prove the lower semicontinuity of the functional $\mathcal{F}$ with respect to the weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$, $p\in(1,+\infty)$.

Theorem 3.10. *Let $p\in(1,+\infty)$. Let ψ and g satisfy (H1)-(H7). Then the functional $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$.*

Proof. Through this proof, the superscript j, with $1\leq j\leq m$, stands for the j-th component of a vector in $\mathbb{R}^m$.

Thanks to Lemma 3.5 we restrict our attention to the case of a nonnegative Carathéodory function g.

We apply the approximation argument of [2, Theorem 3.3]. Let $u_k,u\in SBV^p(\Omega;\mathbb{R}^m)$ be such that u_k converges to u weakly in $SBV^p(\Omega;\mathbb{R}^m)$. By Definition 2.3, we have that

$$\sup_k\|u_k\|_\infty<+\infty\,,\qquad\sup_k\|\nabla u_k\|_p<+\infty\,,\qquad\sup_k\mathcal{H}^{n-1}(S_{u_k})<+\infty\,.\tag{3.58}$$

Hence, for the sake of simplicity, we may assume that u_k takes values in $(0,1)^m$ for every k. Moreover, thanks to (3.58) and to hypotheses (3.1), (H2), (H5), and (H7), we have

$$\liminf_k\mathcal{F}(u_k)<+\infty\,.\tag{3.59}$$

By the second inequality in (3.58) for every $l \in \mathbb{N}$, $l \geq 1$, we can find an open subset A_l of Ω such that

$$\bigcup_{k\in\mathbb{N}} S_{u_k} \cup S_u \subseteq A_l\,, \qquad \sup_k \int_{A_l} |\nabla u_k|\,\mathrm{d}x < 2^{-l}\,,$$

and $\{x \in \Omega :\, d(x, \Sigma \cup \partial\Omega) < \eta_l\} \subseteq A_l$ for some $\eta_l > 0$. We also set $B_{k,l} := A_l \setminus S_{u_k}$.

Let us fix $l \in \mathbb{N}$. By the Coarea Formula, for every $k \in \mathbb{N}$, every $i = 1, \dots, l$, and every $j = 1, \dots, m$, we can find $\xi^j_{i,k}$ such that

$$\xi^j_{i,k} \in \Big(\frac{i-1}{l}, \frac{2i-1}{2l}\Big]\,, \tag{3.60}$$

$$\{x \in \Omega :\, u^j_k(x) > \xi^j_{i,k}\} \text{ is of finite perimeter,} \tag{3.61}$$

$$\mathcal{L}^n(\{x \in \Omega :\, u^j_k(x) = \xi^j_{i,k}\}) = 0\,, \tag{3.62}$$

$$\begin{aligned} &\mathcal{H}^{n-1}(B_{k,l} \cap \partial^*\{x \in \Omega :\, u^j_k(x) > \xi^j_{i,k}\}) \\ &\qquad \leq 2l \int_{\frac{i-1}{l}}^{\frac{i}{l}} \mathcal{H}^{n-1}(B_{k,l} \cap \partial^*\{x \in \Omega :\, u^j_k(x) > t\})\,\mathrm{d}t \leq 2l|Du^l_k|(B_{k,l})\,. \end{aligned} \tag{3.63}$$

We set also $\xi^j_{0,k} := 0$ and $\xi^j_{l+1,k} := 1$.

We denote by $\mathcal{S}$ the family of functions $\sigma\colon \{1, \dots, m\} \to \{0, \dots, l\}$. For every $\sigma \in \mathcal{S}$ we define $\eta^j_\sigma := \sigma(j)/l$ and

$$\begin{aligned} &Q_{\sigma,k} := \{s \in \mathbb{R}^m :\, \xi^j_{\sigma(j),k} < s^j < \xi^j_{\sigma(j)+1,k} \text{ for } j = 1, \dots, m\}\,, \\ &E_{\sigma,k} := \{x \in \Omega :\, u_k(x) \in Q_{\sigma,k}\}\,. \end{aligned} \tag{3.64}$$

We notice that $\eta_\sigma \in \overline{Q}_{\sigma,k}$ and the sets $\{E_{\sigma,k}\}_{\sigma\in\mathcal{S}}$ are pairwise disjoint and of finite perimeter by (3.61).

For every k we define a piecewise constant function v_k by

$$v_k(x) := \begin{cases} \eta_\sigma & \text{if } x \in E_{\sigma,k} \text{ for some } \sigma \in \mathcal{S}, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.65}$$

If we set $T := \{\eta_\sigma\}_{\sigma\in\mathcal{S}}$, from (3.61) we infer $v_k \in BV(\Omega; T)$. Moreover, by construction of η_σ and of v_k, we have that $\|u_k - v_k\|_{\infty,\Omega} \leq 2m/l$ and $\|u^\pm_k - v^\pm_k\|_{\infty,\Sigma} \leq 2m/l$.

We now estimate $\mathcal{F}_{A_l}(v_k, A_l \cup \Sigma)$. Since $A_l \setminus B_{k,l} \subseteq S_{u_k}$, we get

$$\begin{aligned} &\int_{A_l\cap S_{v_k}\setminus(B_{k,l}\cup\Sigma)} \psi(x, \nu_{v_k})\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x, v^+_k, v^-_k)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad \leq \int_{S_{u_k}\setminus\Sigma} \psi(x, \nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x, u^+_k, u^-_k)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma \omega\Big(x, \frac{2\sqrt{2}m}{l}\Big)\,\mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned} \tag{3.66}$$

where ω is a modulus of continuity defined as in (3.10) with $K = [0,1]^m \times [0,1]^m$. We recall that $\omega(\cdot, \delta) \to 0$ in $L^1(\Sigma)$ as $\delta \to 0$. By (H2) and (3.63), on the set $B_{k,l}$ we have

$$\begin{aligned} \int_{S_{v_k}\cap B_{k,l}\setminus\Sigma} \psi(x, \nu_{v_k})\,\mathrm{d}\mathcal{H}^{n-1} &\leq c_2\,\mathcal{H}^{n-1}(B_{k,l} \cap S_{v_k}) \leq c_2\,\mathcal{H}^{n-1}\Big(B_{k,l} \cap \bigcup_{\sigma\in\mathcal{S}} \partial^* E_{\sigma,k}\Big) \\ &\leq c_2 \sum_{j=1}^m \sum_{i=1}^l \mathcal{H}^{n-1}(B_{k,l} \cap \partial^*\{x \in \Omega :\, u^j_k(x) > \xi^j_{i,k}\}) \\ &\leq 2c_2 l \sum_{j=1}^m |Du^j_k|(B_{k,l}) \leq 2c_2 ml|Du_k|(B_{k,l}) \leq Cl2^{1-l} \end{aligned} \tag{3.67}$$

for some $C > 0$ independent of l. Summing up (3.66) and (3.67) and recalling definition (3.14) of $\mathcal{F}_{A_l}$, we obtain

$$\begin{aligned} \mathcal{F}_{A_l}(v_k, A_l \cup \Sigma) &= \int_{A_l \cap S_{v_k} \setminus \Sigma} \psi(x, \nu_{v_k})\, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g(x, v_k^+, v_k^-)\, \mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_{S_{u_k} \setminus \Sigma} \psi(x, \nu_{u_k})\, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g(x, u_k^+, u_k^-)\, \mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\Sigma} \omega\Big(x, \frac{2\sqrt{2}m}{l}\Big)\, \mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l} \\ &= \mathcal{F}(u_k) + \int_{\Sigma} \omega\Big(x, \frac{2\sqrt{2}m}{l}\Big)\, \mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l}. \end{aligned} \tag{3.68}$$

Assumptions (3.1) and (H2), together with inequalities (3.59) and (3.68), imply that

$$\sup_k \mathcal{H}^{n-1}(S_{v_k} \cap A_l) < +\infty.$$

Hence v_k satisfies the hypotheses of the compactness Theorem 2.4 in $SBV(A_l; \mathbb{R}^m)$: there exists $w_l \in SBV(A_l; \mathbb{R}^m)$ such that, up to a subsequence, $v_k \to w_l$ pointwise $\mathcal{L}^n$-a.e. in A_l. Moreover, $w_l \in BV(A_l; T)$. Thus, we are in a position to apply Theorem 3.7 on A_l:

$$\begin{aligned} \mathcal{F}_{A_l}(w_l, A_l \cup \Sigma) &\le \liminf_k \mathcal{F}_{A_l}(v_k, A_l \cup \Sigma) \\ &\le \liminf_k \mathcal{F}(u_k) + \int_{\Sigma} \omega\Big(x, \frac{2\sqrt{2}m}{l}\Big)\, \mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l}. \end{aligned} \tag{3.69}$$

Since $u_k \to u$ and $v_k \to w_l$ pointwise $\mathcal{L}^n$-a.e. in A_l, we have that

$$\|w_l - u\|_{\infty, A_l} \le 2m/l \qquad \text{and} \qquad \|w_l^{\pm} - u^{\pm}\|_{\infty, \Sigma} \le 2m/l. \tag{3.70}$$

In addition, for every $\sigma \in \mathcal{S}$ there exists $E_{\sigma,l}$ of finite perimeter such that

$$w_l = \sum_{\sigma \in \mathcal{S}} \eta_\sigma \mathbf{1}_{E_{\sigma,l}}.$$

Up to a subsequence, we may assume that

$$\xi_{i,k}^j \to \xi_i^j \in \Big[\frac{i-1}{l}, \frac{2i-1}{2l}\Big]. \tag{3.71}$$

We define the cube

$$Q_\sigma := \{s \in \mathbb{R}^m : \xi_{\sigma(j)}^j < s^j < \xi_{\sigma(j)+1}^j\}.$$

By the pointwise convergence of u_k to u and by (3.60), (3.64), (3.65), and (3.71), it is easy to see that $\mathcal{L}^n(E_{\sigma,l} \setminus u^{-1}(\overline{Q}_\sigma)) = 0$. Thus, up to a negligible set, we have

$$E_{\sigma,l} \subseteq A_l \cap u^{-1}(\overline{Q}_\sigma). \tag{3.72}$$

We now pass to the limit as $l \to +\infty$. For every $\varepsilon > 0$, let $l_0 \in \mathbb{N}$ be such that $\operatorname{diam}(Q_\sigma) < \varepsilon/3$ for every $\sigma \in \mathcal{S}$ and every $l \ge l_0$. Then, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ such that $|u^+(x) - u^-(x)| > \varepsilon$ we have that the sets

$$\{y \in \Omega : |u(y) - u^+(x)| < \varepsilon/3\} \qquad \text{and} \qquad \{y \in \Omega : |u(y) - u^-(x)| < \varepsilon/3\}$$

have density $1/2$ at x. Therefore, from (3.72) we deduce that, up to an $\mathcal{H}^{n-1}$-negligible set,

$$S_\varepsilon := \{x \in S_u : |u^+(x) - u^-(x)| > \varepsilon\} \subseteq A_l \cap S_{w_l}. \tag{3.73}$$

In view of (3.73), we have that $\nu_u = \pm\nu_{w_l}$ $\mathcal{H}^{n-1}$-a.e. in S_ε for every $l \geq l_0$, and, by (3.70), $\|w_l^\pm - u^\pm\|_{\infty,\Sigma} \to 0$ as $l \to +\infty$. Thus, recalling (3.69) and applying Fatou Lemma, we get

$$
\begin{aligned}
&\int_{S_\varepsilon\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \leq \liminf_l \int_{S_\varepsilon\setminus\Sigma} \psi(x,\nu_{w_l})\,\mathrm{d}\mathcal{H}^{n-1} + \liminf_l \int_\Sigma g(x,w_l^+,w_l^-)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \leq \liminf_l \mathcal{F}_{A_l}(w_l, A_l\cup\Sigma) \leq \liminf_k \mathcal{F}(u_k)\,.
\end{aligned}
\tag{3.74}
$$

Since $S_\varepsilon \nearrow S_u$, we conclude the proof of the theorem by passing to the limit in (3.74) as $\varepsilon \searrow 0$. □

We now conclude with the proof of Theorem 3.4.

Proof of Theorem 3.4. Let us assume that g is a nonnegative Carathéodory function such that, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$, $g(x,\cdot,\cdot)$ is Lipschitz continuous with Lipschitz constant $\lambda > 0$. Let $u_k, u \in GSBV^p(\Omega;\mathbb{R}^m)$ be such that u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ and $\liminf_k \mathcal{F}(u_k) < +\infty$.

By Proposition 2.5, for every $h,k \in \mathbb{N}$ we have that $T_h(u_k) := (T_h(u_k^1),\dots,T_h(u_k^m)) \in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. By definition of T_h and by the weak convergence of u_k in $GSBV^p(\Omega;\mathbb{R}^m)$, for every h the sequences $\{T_h(u_k)\}_k$ and $\{\nabla(T_h(u_k))\}_k$ are bounded in $L^\infty(\Omega;\mathbb{R}^m)$ and in $L^p(\Omega;\mathbb{M}^{m\times n})$, respectively. Moreover, $S_{T_h(u_k)} \subseteq S_{u_k}$ for every $h,k \in \mathbb{N}$. Therefore, by the compactness Theorem 2.4, we deduce that $T_h(u_k)$ converges to $T_h(u)$ weakly in $SBV^p(\Omega;\mathbb{R}^m)$ as $k \to +\infty$.

Let $h \in \mathbb{N}$ be fixed. We now construct a new function $g_h\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ such that $0 \leq g_h \leq g$ and g_h satisfies (H4)-(H7). For every $x \in \Sigma$ and every $s,t \in \mathbb{R}^m$ we set

$$
g_h(x,s,t) := \begin{cases}
g(x,s,t) & \text{if } |s|,|t| < h\,,\\
\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) & \text{if } |s| \geq h, |t| < h\,,\\
\inf_{\tau\in\mathbb{R}^m} g(x,s,\tau) & \text{if } |s| < h, |t| \geq h\,,\\
\inf_{\sigma,\tau\in\mathbb{R}^m} g(x,\sigma,\tau) & \text{if } |s|,|t| \geq h\,.
\end{cases}
$$

It is clear that $0 \leq g_h \leq g$. Let us prove that g_h satisfies properties (H4)-(H7). By construction, g_h is a Borel function and $g_h(\cdot,0,0) = g(\cdot,0,0) \in L^1(\Sigma)$, hence (H4) and (H5) hold.

To prove (H6) we consider two sequences $s_j, t_j \in \mathbb{R}^m$ converging to s and t, respectively. By definition of g_h and by the continuity of $g(x,\cdot,\cdot)$, there is only one non-trivial alternative:

$$|s_j|, |s| \geq h \text{ and } |t_j|, |t| < h\,.$$

In this case, by the Lipschitz continuity of $g(x,\cdot,\cdot)$ we have that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $\tau,\tau' \in \mathbb{R}^m$

$$\Big|\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,\tau) - \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,\tau')\Big| \leq \lambda|\tau-\tau'|\,,$$

which implies that

$$\lim_j g_h(x,s_j,t_j) = \lim_j \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t_j) = \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) = g_h(x,s,t)\,.$$

This concludes the proof of (H6).

To prove (3.2), we fix $s,s',t \in \mathbb{R}^m$ and distinguish between the cases $|t| < h$ and $|t| \geq h$. If $|t| < h$, since g satisfies (H7) we have that, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$,

$$g_h(x,s,t) \leq g(x,s,t) \leq \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x)) \leq g_h(x,s',t) + \psi(x,\nu_\Sigma(x))\,. \tag{3.75}$$

Otherwise, if $|t| \geq h$,

$$\begin{aligned} g_h(x,s,t) &\leq \inf_{\tau\in\mathbb{R}^m} g(x,s,\tau) \leq \inf_{\sigma,\tau\in\mathbb{R}^m} g(x,\sigma,\tau) + \psi(x,\nu_\Sigma(x)) \\ &\leq g_h(x,s',t) + \psi(x,\nu_\Sigma(x)) \,. \end{aligned} \tag{3.76}$$

Thanks to (3.75) and (3.76), we get that g_h satisfies (3.2). Inequality (3.3) can be proved in the same way. Therefore, g_h fulfills property (H7).

Finally, it is easy to see that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$, every $s,t \in \mathbb{R}^m$, and every $h \in \mathbb{N}$

$$g_h(x,s,t) = g_h(x,T_h(s),T_h(t)) \,. \tag{3.77}$$

Let us define the functional $\mathcal{F}_h \colon GSBV^p(\Omega;\mathbb{R}^m) \to \mathbb{R}$ by

$$\mathcal{F}_h(v) := \int_{S_v\setminus\Sigma} \psi(x,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_h(x,v^+,v^-)\,\mathrm{d}\mathcal{H}^{n-1} \,.$$

Since ψ and g_h satisfy (H1)-(H7), we can apply Theorem 3.10 to $\mathcal{F}_h$. Hence, in view of the weak convergence of $T_h(u_k)$ to $T_h(u)$ in $SBV^p(\Omega;\mathbb{R}^m)$, we get that

$$\mathcal{F}_h(T_h(u)) \leq \liminf_k \mathcal{F}_h(T_h(u_k)) \,. \tag{3.78}$$

As a consequence of (3.77), of the inclusion $S_{T_h(u_k)} \subseteq S_{u_k}$, and of the inequality $g_h \leq g$, we have that $\mathcal{F}_h(T_h(u_k)) \leq \mathcal{F}(u_k)$ for every h,k. Thus, from (3.78) we deduce that

$$\int_{S_{T_h(u)}\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_h(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_k \mathcal{F}(u_k) \,. \tag{3.79}$$

Since $g_h(\cdot,u^+,u^-) \nearrow g(\cdot,u^+,u^-)$ pointwise $\mathcal{H}^{n-1}$-a.e. in Σ and $S_{T_h(u)} \nearrow S_u$, passing to the limit in (3.79) as $h \to +\infty$ we obtain

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k) \,,$$

which concludes the proof of the theorem in the particular case of a nonnegative Carathéodory function g with $g(x,\cdot,\cdot)$ Lipschitz continuous with Lipschitz constant λ. The general case follows by Lemma 3.5. □

We now show that condition (H7) is also necessary for the lower semicontinuity of the functional $\mathcal{F}$, provided that $g \colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying the following properties:

(H8) there exists $a \in L^1(\Sigma)^+$ such that $g(x,s,t) \geq -a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s,t \in \mathbb{R}^m$;
(H9) $g(\cdot,s,t) \in L^1(\Sigma)$ for every $s,t \in \mathbb{R}^m$.

Theorem 3.11. *Let ψ satisfy (H1)-(H3) and let g be a Carathéodory function such that* (3.5)*, (H8), and (H9) hold. Let $\mathcal{F} \colon GSBV^p(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ be the functional defined in* (1.4)*. Assume that $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$. Then ψ and g fulfill property (H7).*

Proof. Let $L > 0$ be the Lipschitz constant of Σ and $\Lambda := L\sqrt{n}$. Let us prove that g satisfies the inequality (3.2) on $\Sigma \cap \Omega$. Let $x_0 \in \Sigma \cap \Omega$ be such that $\nu_\Sigma(x_0)$ is normal to Σ at x_0, and let $\xi(x_0) \in \mathbb{S}^{n-1}$ be as in Definition 2.2. As in the proof of Lemma 3.8, we set $\mathrm{R}_\rho(x_0) := \mathrm{R}^\Lambda_{\rho,\xi(x_0)}(x_0)$, where $\mathrm{R}^C_{\rho,\xi}(x)$ is defined in (2.2). In particular, for ρ sufficiently small we may suppose that $\mathrm{R}_\rho(x_0) \subseteq \Omega$, that $\mathcal{H}^{n-1}(\Sigma \cap \partial\mathrm{R}_\rho(x_0)) = 0$, that the function φ_{x_0} of Definition 2.1 is well-defined and Lipschitz continuous on the $(n-1)$-dimensional cube $\mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0)$, and that

$$\mathrm{R}_\rho(x_0) \cap \Sigma = \{y + \varphi_{x_0}(y)\xi(x_0) : \, y \in \mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0)\} \,.$$

We assume in addition that x_0 satisfies the following conditions:

$$x_0 \text{ is a Lebesgue point for } g(\cdot,\sigma,\tau) \text{ for every } \sigma,\tau\in\mathbb{Q}^m\,, \tag{3.80}$$

$$g(x_0,\cdot,\cdot) \text{ is continuous on } \mathbb{R}^m\times\mathbb{R}^m\,, \tag{3.81}$$

$$\lim_{\rho\searrow 0}\frac{1}{\rho^{n-1}}\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}|\nu_\Sigma(x)-\nu_\Sigma(x_0)|\,\mathrm{d}\mathcal{H}^{n-1}(x)=0\,. \tag{3.82}$$

We notice that properties (3.80)-(3.82) are satisfied for $\mathcal{H}^{n-1}$-a.e. $x_0\in\Sigma\cap\Omega$.

We define the sets $A^\rho_\pm$ as in (3.28). For every $k\in\mathbb{N}$ we set

$$\Sigma_k:=(\mathrm{R}_\rho(x_0)\cap\Sigma)+\tfrac{1}{k}\xi(x_0)=\Big\{y+\big(\varphi_{x_0}(y)+\tfrac{1}{k}\big)\xi(x_0):\,y\in\mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0)\Big\}\,, \tag{3.83}$$

and

$$A^{\rho,k}_+:=\Big\{y+t\xi(x_0):\,y\in\mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0),\ \varphi_{x_0}(y)+\tfrac{1}{k}<t<x_0\cdot\xi(x_0)+\Lambda\rho\Big\}\,. \tag{3.84}$$

It is easy to see that for k large enough we have $\Sigma_k, A^{\rho,k}_+\subseteq A^\rho_+$ and

$$\nu_{\Sigma_k}(x)=\pm\nu_\Sigma\big(x-\tfrac{1}{k}\xi(x_0)\big) \tag{3.85}$$

for $\mathcal{H}^{n-1}$ a.e. $x\in\Sigma_k$.

Let us fix three distinct points $s,s',t\in\mathbb{Q}^m\setminus\{0\}$. We introduce the functions

$$u_k(x):=\begin{cases} s' & \text{if } x\in A^\rho_+\setminus A^{\rho,k}_+\,,\\ s & \text{if } x\in A^{\rho,k}_+\,,\\ t & \text{if } x\in A^\rho_-\,,\\ 0 & \text{if } x\in\Omega\setminus\mathrm{R}_\rho(x_0)\,,\end{cases}\qquad u(x):=\begin{cases} s & \text{if } x\in A^\rho_+\,,\\ t & \text{if } x\in A^\rho_-\,,\\ 0 & \text{if } x\in\Omega\setminus\mathrm{R}_\rho(x_0)\,.\end{cases} \tag{3.86}$$

It is clear that $u_k,u\in GSBV^p(\Omega;\mathbb{R}^m)$ and that u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ as $k\to+\infty$. Moreover,

$$S_{u_k}=\partial\mathrm{R}_\rho(x_0)\cup\Sigma_k\cup(\Sigma\cap\mathrm{R}_\rho(x_0))\qquad\text{and}\qquad S_u=\partial\mathrm{R}_\rho(x_0)\cup(\Sigma\cap\mathrm{R}_\rho(x_0))\,. \tag{3.87}$$

Thanks to the lower semicontinuity of the functional $\mathcal{F}$, to hypothesis (H3), and to (3.85)-(3.87), we have that

$$\begin{aligned}&\int_{\partial\mathrm{R}_\rho(x_0)}\psi(x,\nu_{R_\rho(x_0)})\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}g(x,s,t)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma\setminus\mathrm{R}_\rho(x_0)}g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad=\mathcal{F}(u)\le\liminf_k\mathcal{F}(u_k)=\liminf_k\int_{\Sigma_k}\psi\big(x,\nu_\Sigma\big(x-\tfrac{1}{k}\xi(x_0)\big)\big)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad+\int_{\partial\mathrm{R}_\rho(x_0)}\psi(x,\nu_{R_\rho(x_0)})\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}g(x,s',t)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma\setminus\mathrm{R}_\rho(x_0)}g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.88}$$

Therefore, by the change of coordinates $y=x-\frac{1}{k}\xi(x_0)$ and taking into account the uniform continuity of ψ on $\overline{\Omega}\times\mathbb{S}^{n-1}$, from (3.88) we get

$$\begin{aligned}&\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}g(x,s,t)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad\le\liminf_k\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}\psi\big(x+\tfrac{1}{k}\xi(x_0),\nu_\Sigma(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}g(x,s',t)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad=\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}\psi(x,\nu_\Sigma(x))\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\mathrm{R}_\rho(x_0)\cap\Sigma}g(x,s',t)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.89}$$

Dividing (3.89) by ρ^{n-1} and passing to the limit as $\rho \searrow 0$, thanks to properties (3.80)-(3.82) and to (H1), we obtain that

$$g(x_0, s, t) \leq g(x_0, s', t) + \psi(x_0, \nu_\Sigma(x_0)) \tag{3.90}$$

for every triple of distinct points $s, s', t \in \mathbb{Q}^m \setminus \{0\}$. By density and by (3.81), we conclude that g satisfies (3.2) for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma \cap \Omega$. To prove the same result for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma \cap \partial\Omega$, we use a similar argument and take into account (3.5). The proof of (3.3) is analogous. □

We conclude this section with an existence result whose proof follows directly from Theorems 2.9 and 3.4. Let $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy (2.4) and (2.5), and let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that

$$a_3|s|^q - b_3(x) \leq f(x,s) \leq a_4|s|^q + b_4(x) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^m \tag{3.91}$$

for some $1 < q < +\infty$, $0 < a_3 \leq a_4$, and $b_3, b_4 \in L^1(\Omega)$.

We define the functional $\mathcal{G}\colon L^q(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ by

$$\mathcal{G}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.92}$$

for every $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$, and $\mathcal{G}(u) := +\infty$ otherwise in $L^q(\Omega;\mathbb{R}^m)$. From a mechanical point of view, according to [15, 16], the functional $\mathcal{G}$ is the energy of an elastic body Ω, with a crack S_u, subject to a displacement u. Indeed, the first two volume integrals in (3.92) represent the stored elastic energy and the work done by the volume forces, while the two surface contributions stand for the energy spent in order to create a jump surface S_u and the work done by the surface forces acting on Σ, respectively.

In the following theorem we state an existence result for the minimum problem

$$\min\{\mathcal{G}(u) : u \in L^q(\Omega;\mathbb{R}^m)\}. \tag{3.93}$$

Theorem 3.12. *Let ψ and g satisfy (H1)-(H7). Let $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy* (2.4) *and* (2.5)*, and let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that* (3.91) *holds. Then the minimum problem* (3.93) *admits a solution.*

Proof. The proof is based on the direct method of the calculus of variations. Let $u_k \in L^q(\Omega;\mathbb{R}^m)$ be a minimizing sequence for (3.93). Then $u_k \in GSBV^p(\Omega;\mathbb{R}^m)$ and, by hypotheses (2.4), (2.5), (3.1), (H1)-(H7), and (3.91), we have that $\|u_k\|_q$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. By the compactness Theorem 2.8, there exists $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$ such that, up to a subsequence, u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$.

Applying Theorems 2.9 and 3.4, and the Fatou Lemma, we get that

$$\mathcal{G}(u) \leq \liminf_k \mathcal{G}(u_k),$$

thus u is a solution of (3.93). □

Remark 3.13. Theorem 3.12 is the starting point for the study of problems of quasi-static evolution of brittle fractures in nonlinear elasticity, with energies involving also a boundary term as it has been done in [10]. Indeed, we remark that, in the case $\Sigma = \partial\Omega$, the novelty of our result is that we allow the jump set to reach the boundary of Ω, while in [10, Section 3] an unbreakable part was introduced in a neighborhood of the Neumann part of $\partial\Omega$, in order to prevent this situation.

4. Relaxation result

In this section we give a relaxation result for functionals of the form (1.4) in $GSBV^p$.

Let us be more precise on the setting of the problem. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, let (Σ,ν_Σ) be an orientable Lipschitz manifold of dimension $n-1$ and Lipschitz constant L with $\Sigma\subseteq\overline{\Omega}$ and such that (3.1) holds. We consider a function $\psi\colon\overline{\Omega}\times\mathbb{R}^n\to[0,+\infty)$ satisfying properties (H1)-(H3), and a function $g\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ such that:

(A1) g is Borel measurable;
(A2) $g(x,\cdot,\cdot)$ is continuous on $\mathbb{R}^m\times\mathbb{R}^m$ for every $x\in\Sigma$;
(A3) there exists $a\in L^1(\Sigma)^+$ such that $g(x,s,t)\geq -a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$;
(A4) for every $M>0$ there exists $a_M\in L^1(\Sigma)$ such that $g(x,s,t)\leq a_M(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$ with $|s|,|t|\leq M$.

In the statement and in the proof of Theorem 4.3 we will use the following functions

$$g_1(x,s,t):=\min\Big\{g(x,s,t),\ \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t)+\psi(x,\nu_\Sigma(x))\Big\}\,,\tag{4.1}$$

$$g_2(x,s,t):=\min\Big\{g(x,s,t),\ \inf_{\tau\in\mathbb{R}^m} g(x,s,\tau)+\psi(x,\nu_\Sigma(x))\Big\}\,,\tag{4.2}$$

$$g_{12}(x,s,t):=\min\Big\{g_1(x,s,t),\ \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau)+\psi(x,\nu_\Sigma(x))\Big\}\,,\tag{4.3}$$

$$g_{21}(x,s,t):=\min\Big\{g_2(x,s,t),\ \inf_{\sigma\in\mathbb{R}^m} g_2(x,\sigma,t)+\psi(x,\nu_\Sigma(x))\Big\}\,.\tag{4.4}$$

We will prove in Lemma 4.2 that $g_{12}=g_{21}$. In Theorem 4.3 we need the additional hypothesis

(A5) $g_{12}(x,\cdot,\cdot)$ is continuous on $\mathbb{R}^m\times\mathbb{R}^m$ for every $x\in\Sigma$.

Note that (A5) is not a consequence of (A2). Indeed, there are easy examples where g_1 and g_{12} are not even lower semicontinuous. However, if $g(x,\cdot,\cdot)$ is uniformly continuous on $\mathbb{R}^m\times\mathbb{R}^m$ for every $x\in\Sigma$, then the functions $g_1(x,\cdot,\cdot)$, $g_2(x,\cdot,\cdot)$, $g_{12}(x,\cdot,\cdot)$, $g_{21}(x,\cdot,\cdot)$ are uniformly continuous on $\mathbb{R}^m\times\mathbb{R}^m$ for every $x\in\Sigma$.

Remark 4.1. If (3.5) holds, it is easy to see that for every $s,t\in\mathbb{R}^m$

$$\begin{array}{lll} g_1(x,s,t)=g_1(x,s,0) \quad\text{and}\quad g_2(x,s,t)=g(x,s,0) & \text{if } x\in\mathrm{N}^+\,,\\ g_1(x,s,t)=g(x,0,t) \quad\text{and}\quad g_2(x,s,t)=g_2(x,0,t) & \text{if } x\in\mathrm{N}^-\,,\\ g_{12}(x,s,t)=g_{21}(x,s,t)=g_1(x,s,0) & \text{if } x\in\mathrm{N}^+\,,\\ g_{12}(x,s,t)=g_{21}(x,s,t)=g_2(x,0,t) & \text{if } x\in\mathrm{N}^-\,,\end{array}$$

where $\mathrm{N}^\pm$ are as in (3.4).

In the following lemma we discuss some properties of the functions introduced in (4.1)-(4.4).

Lemma 4.2. *Assume (A1)-(A4). Then the functions* $g_1, g_2, g_{12}, g_{21}\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ *are Borel measurable and satisfy the inequalities*

$$g_1,\ g_2,\ g_{12},\ g_{21}\geq -a\,,\tag{4.5}$$

$$g_1,\ g_2,\ g_{12},\ g_{21}\leq g\,.\tag{4.6}$$

Moreover, for every $x\in\Sigma$ *they are upper semicontinuous with respect to* (s,t)*. Finally,* g_{12} *and* g_{21} *fulfill property (H7) and*

$$g_{12}(x,s,t)=g_{21}(x,s,t)=\sup_{\gamma\in\Gamma_g}\gamma(x,s,t)\,,\tag{4.7}$$

where Γ_g *is the set of all functions* $\gamma\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ *satisfying (H7) and such that* $\gamma\leq g$*.*

Proof. Since for every $x \in \Sigma$ the function $g(x,\cdot,\cdot)$ is upper semicontinuous on $\mathbb{R}^m \times \mathbb{R}^m$, we have that

$$g_1(x,s,t) := \min\Big\{g(x,s,t),\ \inf_{\sigma\in\mathbb{Q}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x))\Big\}.$$

Since g is also Borel measurable, this implies that g_1 is Borel measurable and, for every $x \in \Sigma$, $g_1(x,\cdot,\cdot)$ is upper semicontinuous. The same argument applies to g_2, g_{12}, g_{21}. The inequalities (4.5) and (4.6) follow immediately from (A3) and (4.1)-(4.4).

Let us prove that g_{12} fulfills property (H7). By definitions (4.1) and (4.2), it is easy to see that g_1 satisfies (3.2) and g_{12} satisfies (3.3). Therefore, for every $x \in \Sigma$ and every $s, s', t \in \mathbb{R}^m$, the following inequalities hold:

$$g_{12}(x,s,t) \le g_1(x,s,t) \le g_1(x,s',t) + \psi(x,\nu_\Sigma(x))\,, \tag{4.8}$$

$$g_{12}(x,s,t) \le \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x)) \le \inf_{\tau\in\mathbb{R}^m} g_1(x,s',\tau) + 2\psi(x,\nu_\Sigma(x))\,. \tag{4.9}$$

From (4.2), (4.8), and (4.9), we infer that g_{12} satisfies (3.2), which completes the proof of (H7). A similar argument can be used for g_{21}.

We now prove (4.7). To this end, we first check that

$$g_1(x,s,t) = \sup_{\gamma\in\Gamma_g^1} \gamma(x,s,t)\,, \tag{4.10}$$

where Γ_g^1 is the set of all functions $\gamma\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ satisfying (3.2) for every $x \in \Sigma$ and such that $\gamma \le g$. Let $G_1(x,s,t)$ be the right-hand side of (4.10). Since g_1 satisfies (3.2) and $g_1 \le g$, we have that $g_1 \le G_1$. Conversely, let $\gamma \in \Gamma_g^1$. Then

$$\gamma(x,s,t) \le \inf_{\sigma\in\mathbb{R}^m} \gamma(x,\sigma,t) + \psi(x,\nu_\Sigma(x)) \le \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x))$$

for every $x \in \Sigma$ and every $s,t \in \mathbb{R}^m$. Since $\gamma \le g$, the previous inequality implies that $\gamma \le g_1$. Taking the supremum for $\gamma \in \Gamma_g^1$, we deduce that $G_1 \le g_1$. Since the opposite inequality has already been proved, we have that (4.10) holds. With the same argument it is possible to show that

$$g_2(x,s,t) = \sup_{\gamma\in\Gamma_g^2} \gamma(x,s,t)\,, \tag{4.11}$$

where Γ_g^2 is the set of all functions $\gamma\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ satisfying (3.3) for every $x \in \Sigma$ and such that $\gamma \le g$.

Since g_{12} satisfies (H7) and $g_{12} \le g$, we have that

$$g_{12}(x,s,t) \le \sup_{\gamma\in\Gamma_g} \gamma(x,s,t)\,. \tag{4.12}$$

For the converse inequality, let us fix $\gamma \in \Gamma_g$ and let $G(x,s,t)$ be the right-hand side of (4.12). Then, in view of (4.10), we have that $\gamma \le g_1$ and

$$\gamma(x,s,t) \le \inf_{\tau\in\mathbb{R}^m} \gamma(x,s,\tau) + \psi(x,\nu_\Sigma(x)) \le \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x)) \tag{4.13}$$

for every $x \in \Sigma$ and every $s,t \in \mathbb{R}^m$. In view of (4.13) we get that $\gamma \le g_{12}$. Thus, $G \le g_{12}$, which, together with (4.12), gives $g_{12} = G$. In the same way, using (4.11), we can show that $g_{21} = G$, and this concludes the proof of the lemma. ☐

Given $p \in (1,+\infty)$, we define the functional $\mathcal{F}\colon GSBV^p(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ as in (1.4) and the functional $sc^-\mathcal{F}\colon GSBV^p(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ as the greatest sequentially lower semicontinuous functional on $GSBV^p(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{F}$. We are now ready to state the main theorem of this section.

Theorem 4.3. *Let ψ and g satisfy (H1)-(H3), (A1)-(A5), and* (3.5)*. Then we have*

$$sc^-\mathcal{F}(u) = \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \tag{4.14}$$

for every $u \in GSBV^p(\Omega;\mathbb{R}^m)$ *.*

For what follows, it is convenient to define the functionals $\mathcal{F}_{12}, \mathcal{F}_1 \colon GSBV^p(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ by

$$\mathcal{F}_{12}(u) := \int_{S_u \setminus \Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.15}$$

$$\mathcal{F}_{1}(u) := \int_{S_u \setminus \Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g_{1}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.16}$$

for every $u \in GSBV^p(\Omega;\mathbb{R}^m)$. The functional $\mathcal{F}_1$ is "intermediate" between $\mathcal{F}$ and $\mathcal{F}_{12}$ and will be used in the proof of Theorem 4.3.

In order to prove Theorem 4.3 we need the following approximation lemma.

Lemma 4.4. *Let* $r \in [1,+\infty)$*. Then for every* $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ *and every* $\varepsilon > 0$ *there exists* $v \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ *such that*

$$\|v-u\|_{r,\Omega} < \varepsilon\,, \qquad \|\nabla v - \nabla u\|_{p,\Omega} < \varepsilon\,, \tag{4.17}$$

$$\mathcal{H}^{n-1}(S_v) < \mathcal{H}^{n-1}(S_u) + 4\mathcal{H}^{n-1}(\Sigma) + \varepsilon\,, \tag{4.18}$$

$$\mathcal{F}(v) < \mathcal{F}_{12}(u) + \varepsilon\,. \tag{4.19}$$

Proof. Let us set $\Sigma' := \big(\Sigma \setminus (\overline{\Sigma \cap \Omega})\big) \cup (\Sigma \cap \Omega)$. In view of hypotheses (3.1), we have

$$\mathcal{H}^{n-1}(\Sigma \setminus \Sigma') = 0\,. \tag{4.20}$$

Moreover, Σ' is open in the relative topology of Σ.

By Definition 2.2 and by Lindelöff theorem, there exists a sequence of points $x_i \in \Sigma'$ and corresponding $(n-1)$-dimensional rectangles Δ_{x_i}, intervals I_{x_i}, vectors $\xi(x_i) \in \mathbb{S}^{n-1}$, and Lipschitz functions $\varphi_{x_i} \colon \Delta_{x_i} \to \mathrm{I}_{x_i}$ such that the following conditions hold, where, for simplicity of notation, we have set $V_i := \{y + t\xi(x_i) : y \in \Delta_{x_i},\, t \in \mathrm{I}_{x_i}\}$:

$$V_i \cap \Sigma = \{y + \varphi_{x_i}(y)\xi(x_i) : y \in \Delta_{x_i}\}\,, \tag{4.21}$$

$$\nu_\Sigma(x) \cdot \xi(x_i) \text{ has constant sign for } x \in V_i \cap \Sigma\,, \tag{4.22}$$

$$\Sigma' \subseteq \bigcup_{i \in \mathbb{N}} \{y + \varphi_{x_i}(y)\xi(x_i) : y \in \Delta_{x_i}\}\,, \tag{4.23}$$

$$V_i \cap \Sigma \subset\subset \Omega \quad \text{or} \quad V_i \cap \Sigma \subset\subset \Sigma' \cap \partial\Omega\,. \tag{4.24}$$

As in the proof of Lemma 3.8, we define

$$A_i^{\pm} := \{y + t\xi(x_i) : y \in \Delta_{x_i},\, t \in \mathrm{I}_{x_i},\, t \gtrless \varphi_{x_i}(y)\}\,. \tag{4.25}$$

Therefore, for every $i \in \mathbb{N}$, Σ splits the set V_i into two disjoint connected open subsets A_i^+ and A_i^-, with $\nu_\Sigma(x)$ pointing towards A_i^+ for $\mathcal{H}^{n-1}$-a.e. $x \in V_i \cap \Sigma$.

Let u be as in the statement of the lemma. We set

$$B_{g_1} := \big\{x \in \Sigma' : g_1(x,u^+(x),u^-(x)) > \inf_{\tau \in \mathbb{R}^m} g_1(x,u^+(x),\tau) + \psi(x,\nu_\Sigma(x))\big\}\,, \tag{4.26}$$

where g_1 is defined in (4.1). Clearly, B_{g_1} is an $\mathcal{H}^{n-1}$-measurable subset of Σ'. By Remark 4.1, we have that $B_{g_1} \cap \partial\Omega \subseteq \mathrm{N}^-$, where the set N^- is defined in (3.4). This implies that $\nu_\Sigma(x) = -\nu_\Omega(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in B_{g_1} \cap \partial\Omega$. Therefore, from (4.24) and (4.25) we deduce that

$$\mathcal{H}^{n-1}(V_i \cap B_{g_1} \cap \partial\Omega) > 0 \quad \Longrightarrow \quad \nu_\Sigma = -\nu_\Omega \;\; \mathcal{H}^{n-1}\text{-a.e. in } V_i \cap \Sigma \text{ and } A_i^- \subseteq \Omega\,. \tag{4.27}$$

Moreover, by (4.2), (4.20), and (4.26), for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ we have

$$g_{12}(x,u^+(x),u^-(x)) = \begin{cases} g_1(x,u^+(x),u^-(x)) & \text{if } x \in \Sigma \setminus B_{g_1}\,, \\ \inf_{\tau \in \mathbb{R}^m} g_1(x,u^+(x),\tau) + \psi(x,\nu_\Sigma(x)) & \text{if } x \in B_{g_1}\,. \end{cases} \tag{4.28}$$

Given $\varepsilon > 0$, our first aim is to construct a new function $w \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that

$$\|w-u\|_{r,\Omega} < \frac{\varepsilon}{2}\,, \qquad \|\nabla w - \nabla u\|_{p,\Omega} < \frac{\varepsilon}{2}\,, \tag{4.29}$$

$$\mathcal{H}^{n-1}(S_w) < \mathcal{H}^{n-1}(S_u) + 2\mathcal{H}^{n-1}(\Sigma) + \frac{\varepsilon}{2}\,, \tag{4.30}$$

$$\mathcal{F}_1(w) < \mathcal{F}_{12}(u) + \frac{\varepsilon}{2}\,, \tag{4.31}$$

where $\mathcal{F}_1$ is defined in (4.16). Roughly speaking, the idea of the proof of (4.29)-(4.31) is to construct a sort of copy of the "bad" set B_{g_1} inside Ω near Σ. This modified version of B_{g_1} will be part of the jump set of the new function w which will be constructed in such a way that $w = u$ far from B_{g_1}, $w^+ = u^+$ on Σ, and $g_1(x, w^+(x), w^-(x))$ is close to $\inf_{\tau\in\mathbb{R}^m} g_1(x, u^+(x), \tau)$ for $x \in B_{g_1}$.

We now start our construction. Let us fix an auxiliary parameter $\delta > 0$ which will be chosen at the end of the proof in order to get (4.29)-(4.31) and (4.17)-(4.19). Given an enumeration $\{q_j\}_{j\in\mathbb{N}}$ of $\mathbb{Q}^m$, for every j we define

$$B^j_{g_1} := \big\{x \in B_{g_1} : g_1(x, u^+(x), q_j) < \inf_{\tau\in\mathbb{R}^m} g_1(x, u^+(x), \tau) + \delta\big\} \setminus \bigcup_{l=1}^{j-1} B^l_{g_1}\,. \tag{4.32}$$

The sets $\{B^j_{g_1}\}_{j\in\mathbb{N}}$ are pairwise disjoint $\mathcal{H}^{n-1}$-measurable subsets of Σ' such that $B_{g_1} = \bigcup_j B^j_{g_1}$. By taking suitable intersections with the sets V_i and their complements, it is not restrictive to assume that for every j there exists $i_j \in \mathbb{N}$ such that $B^j_{g_1} \subseteq V_{i_j}$ and $B^j_{g_1} \cap V_l = \emptyset$ for $l \neq i_j$.

The next step is to approximate B_{g_1} with the union of a finite number of relatively open subsets of Σ'. Let us set $M := \|u\|_{\infty,\Omega}$. Since $\{B^j_{g_1}\}_{j\in\mathbb{N}}$ are pairwise disjoint and $\mathcal{H}^{n-1}(\Sigma) < +\infty$, we can find $N \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}\Big(\bigcup_{j>N} B^j_{g_1}\Big) < \delta\,, \qquad \int_{\bigcup_{j>N} B^j_{g_1}} a\,\mathrm{d}\mathcal{H}^{n-1} < \delta\,, \qquad \int_{\bigcup_{j>N} B^j_{g_1}} a_M\,\mathrm{d}\mathcal{H}^{n-1} < \delta\,. \tag{4.33}$$

where a, $a_M \in L^1(\Sigma)$ have been defined in (A3) and (A4), respectively.

For every $j \in \{1,\dots,N\}$, we choose a compact set $K_j \subseteq \Sigma'$ such that $K_j \subseteq B^j_{g_1}$ and

$$\mathcal{H}^{n-1}(B^j_{g_1}\setminus K_j) < \frac{\delta}{2^j}\,, \qquad \int_{B^j_{g_1}\setminus K_j} a\,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,, \qquad \int_{B^j_{g_1}\setminus K_j} a_M\,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,. \tag{4.34}$$

Let us set $\widetilde{M} := \max\{M, q_1, \dots, q_N\}$, where q_j are associated to each $B^j_{g_1}$ through definition (4.32). Since $\mathcal{H}^{n-1}\lfloor\Sigma$ is a bounded Radon measure, we can find a family $\{U_j\}_{j=1}^N$ of relatively open subsets of Σ such that the following conditions hold, where $\partial_\Sigma U_j$ denotes the boundary of U_j in the relative topology of Σ:

$$K_j \subseteq U_j \subset\subset V_{i_j} \cap \Sigma\,, \quad \overline{U}_l \cap \overline{U}_j \text{ for } l \neq j, \tag{4.35}$$

$$\mathcal{H}^{n-1}(U_j\setminus K_j) < \frac{\delta}{2^j}\,, \qquad \int_{U_j\setminus K_j} a\,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,, \qquad \int_{U_j\setminus K_j} a_{\widetilde{M}}\,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j} \tag{4.36}$$

$$\mathcal{H}^{n-2}(\partial_\Sigma U_j) < +\infty\,, \tag{4.37}$$

where $a_{\widetilde{M}} \in L^1(\Sigma)$ has been defined in (A4).

We now move each U_j inside $\Omega\setminus\Sigma$ by translation. Let $j\in\{1,\dots,N\}$ be fixed. Thanks to (4.24), (4.25), (4.27), and (4.35), we may choose $\eta_j>0$ such that

$$U_j-\zeta\xi(x_{i_j})\subseteq A^-_{i_j}\quad\text{and}\quad U_j-\zeta\xi(x_{i_j})\subset\subset V_{i_j}\cap\Omega\quad\text{for every }\zeta\in(0,\eta_j]. \tag{4.38}$$

Moreover, by the uniform continuity of ψ on $\overline{\Omega}\times\mathbb{S}^{n-1}$ and by (4.37), we may assume that:

$$\sup_{x\in U_j}|\psi(x-\eta_j\xi(x_{i_j}),\nu_\Sigma(x))-\psi(x,\nu_\Sigma(x))|\le\frac{\delta}{2^j}\,, \tag{4.39}$$

$$2^j\eta_j\in\Big(0,\delta\,\min\Big\{1,\frac{1}{|q_j|^r},\frac{1}{\mathcal{H}^{n-2}(\partial_\Sigma U_j)}\Big\}\Big)\,. \tag{4.40}$$

We denote by C_j the open "cylinders"

$$C_j:=\bigcup_{\zeta\in(0,\eta_j)}(U_j-\zeta\xi(x_{i_j}))\,. \tag{4.41}$$

By possibly changing η_j, by (4.38) we may assume that

$$C_j\subseteq A^-_{i_j}\cap\Omega\,, \tag{4.42}$$

$$\{U_j-\eta_j\xi(x_{i_j})\}_{j=1}^N\text{ are pairwise disjoint,} \tag{4.43}$$

$$\{C_j\}_{j=1}^N\text{ are pairwise disjoint,} \tag{4.44}$$

$$\|u\|_{r,C_j}<\frac{\delta}{2^j}\quad\text{and}\quad\|\nabla u\|_{p,C_j}<\frac{\delta}{2^j}\,. \tag{4.45}$$

Moreover, if

$$L_j:=\bigcup_{\zeta\in(0,\eta_j)}(\partial U_j-\zeta\xi(x_{i_j}))$$

is the lateral surface of the cylinder C_j, by (4.40) we have that

$$\mathcal{H}^{n-1}\Big(\bigcup_{j=1}^N L_j\Big)\le\sum_{j=1}^N\eta_j\,\mathcal{H}^{n-2}(\partial_\Sigma U_j)<\sum_{j=1}^N\frac{\delta}{2^j}<\delta\,. \tag{4.46}$$

Note that the trasversality condition $\nu_\Sigma(x)\cdot\xi(x_{i_j})>0$ for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$ implies that

$$\partial C_j=U_j\cup(U_j-\eta_j\xi(x_{i_j}))\cup L_j\,. \tag{4.47}$$

We are now ready to define the function $w\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ satisfying inequalities (4.29)-(4.31). For every $x\in\Omega$, we set

$$w(x):=\begin{cases}q_j & \text{if }x\in C_j\text{ for some }j\in\{1,\dots,N\},\\ u(x) & \text{if }x\in\Omega\setminus\bigcup_{j=1}^N C_j\,.\end{cases} \tag{4.48}$$

By definition, $\|w\|_{\infty,\Omega}=\widetilde{M}$, $\nabla w\in L^p(\Omega;\mathbb{M}^{m\times n})$, and

$$S_w\subseteq S_u\cup\Sigma\cup\bigcup_{j=1}^N\big(L_j\cup(U_j-\eta_j\xi(x_{i_j}))\big)\,, \tag{4.49}$$

thus, by (4.35) and (4.46), we get that

$$\begin{aligned}\mathcal{H}^{n-1}(S_w)&\le\mathcal{H}^{n-1}(S_u)+\mathcal{H}^{n-1}(\Sigma)+\mathcal{H}^{n-1}\Big(\bigcup_{j=1}^N U_j-\eta_j\xi(x_{i_j})\Big)+\mathcal{H}^{n-1}\Big(\bigcup_{j=1}^N L_j\Big)\\&<\mathcal{H}^{n-1}(S_u)+2\mathcal{H}^{n-1}(\Sigma)+\delta\,.\end{aligned} \tag{4.50}$$

Estimate (4.50) implies that $\mathcal{H}^{n-1}(S_w) < +\infty$, hence $w \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. Thanks to (4.40), (4.41), (4.45), and (4.48), we have that, for some $c_r > 0$ independent of δ,

$$\begin{aligned} \|w-u\|_{r,\Omega} &= \sum_{j=1}^N \|q_j - u\|_{r,C_j} \le \sum_{j=1}^N |q_j|\,(\mathcal{L}^n(C_j))^{1/r} + \|u\|_{r,C_j} \\ &< \sum_{j=1}^N |q_j|\,\eta_j^{1/r}\,(\mathcal{H}^{n-1}(U_j))^{1/r} + \delta \le \sum_{j=1}^N \Big(\frac{\delta}{2^j}\Big)^{1/r} (\mathcal{H}^{n-1}(\Sigma))^{1/r} + \delta \\ &< c_r \delta^{1/r} + \delta\,, \end{aligned} \tag{4.51}$$

and

$$\|\nabla w - \nabla u\|_{p,\Omega} = \sum_{j=1}^N \|\nabla u\|_{p,C_j} < \delta\,. \tag{4.52}$$

We now have to estimate $\mathcal{F}_1(w)$ in terms of $\overline{\mathcal{F}}(u)$. Let us start with the the jump term in $\mathcal{F}_1(w)$. Since $U_j \subseteq \Sigma$ for every $j = 1, \dots, N$, for $\mathcal{H}^{n-1}$-a.e. $x \in U_j - \eta_j \xi(x_{i_j})$ we have

$$\nu_{C_j}(x) = \nu_\Sigma(x + \eta_j \xi(x_{i_j}))\,,$$

which implies that

$$\nu_w(x) = \pm \nu_\Sigma(x + \eta_j\xi(x_{i_j})) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_w \cap (U_j - \eta_j \xi(x_{i_j}))\,. \tag{4.53}$$

Moreover, it is clear that

$$\begin{aligned} &\nu_w(x) = \pm\nu_{C_j}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_w \cap L_j, \\ &\nu_w(x) = \pm\nu_u(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_w \cap S_u. \end{aligned} \tag{4.54}$$

Therefore, thanks to (H3), (4.49), (4.53), and (4.54), we deduce that

$$\begin{aligned} \int_{S_w\setminus\Sigma} \psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^N L_j} \psi(x,\nu_{C_j})\,\mathrm{d}\mathcal{H}^{n-1} \\ + \sum_{j=1}^N \int_{U_j - \eta_j\xi(x_{i_j})} \psi(x, \nu_\Sigma(x + \eta_j \xi(x_{i_j})))\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{4.55}$$

Hypothesis (H2) on ψ and inequality (4.46) imply that

$$\int_{\bigcup_{j=1}^N L_j} \psi(x,\nu_{C_j})\,\mathrm{d}\mathcal{H}^{n-1} \le c_2 \mathcal{H}^{n-1}\Big(\bigcup_{j=1}^N L_j\Big) < c_2\delta\,. \tag{4.56}$$

By (4.35), (4.39), and by the change of variables $y = x + \eta_j \xi(x_{i_j})$ in the last term of (4.55), we obtain that

$$\begin{aligned} \sum_{j=1}^N \int_{U_j - \eta_j\xi(x_{i_j})} \psi(x, \nu_\Sigma(x+\eta_j\xi(x_{i_j})))\,\mathrm{d}\mathcal{H}^{n-1}(x) &= \sum_{j=1}^N \int_{U_j} \psi(y - \eta_j\xi(x_{i_j}), \nu_\Sigma(y))\,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &\le \sum_{j=1}^N \int_{U_j} \psi(y,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \delta\,\mathcal{H}^{n-1}(\Sigma)\,. \end{aligned} \tag{4.57}$$

By (4.35), we can split the sum in the right-hand side of (4.57) in the following way:

$$\sum_{j=1}^N \int_{U_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} = \sum_{j=1}^N \int_{K_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \sum_{j=1}^N \int_{U_j\setminus K_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.58}$$

In view of (H2) and of (4.36) and recalling that the sets K_j are pairwise disjoint and contained in B_{g_1}, (4.58) becomes

$$\sum_{j=1}^{N}\int_{U_j}\psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}\leq\int_{\bigcup_{j=1}^{N}K_j}\psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}+c_2\,\delta\leq\int_{B_{g_1}}\psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}+c_2\,\delta\,. \tag{4.59}$$

Therefore, collecting inequalities (4.55)-(4.57), and (4.59), we get that the jump term in $\mathcal{F}_1(w)$ can be controlled from above by

$$\begin{aligned}&\int_{S_w\setminus\Sigma}\psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1}\\&\qquad\leq\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{B_{g_1}}\psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,(2c_2+\mathcal{H}^{n-1}(\Sigma))\,.\end{aligned} \tag{4.60}$$

Finally, we give an estimate of the integral over Σ of $\mathcal{F}_1(w)$. We first split it into the contribution on $\bigcup_{j=1}^{N}K_j$ and on $\Sigma\setminus\bigcup_{j=1}^{N}K_j$:

$$\int_{\Sigma}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}=\int_{\bigcup_{j=1}^{N}K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma\setminus\bigcup_{j=1}^{N}K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.61}$$

We notice that by (4.25) and (4.42), for every $j=1,\dots,N$ and for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$ the unit normal $\nu_\Sigma(x)$ to Σ at x points outside C_j. Thus, by (4.48), we have that $w^-(x)=q_j$ for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$. Moreover, since $w=u$ in $\Omega\setminus\bigcup_{j=1}^{N}C_j$,

$$w^+=u^+\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\Sigma. \tag{4.62}$$

Therefore, recalling that the sets K_j are pairwise disjoint, we can write the first integral in the right-hand side of (4.61) as

$$\int_{\bigcup_{j=1}^{N}K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}=\sum_{j=1}^{N}\int_{K_j}g_1(x,u^+,q_j)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.63}$$

Taking into account definition (4.32) of the sets $B^j_{g_1}$, the inclusion $K_j\subseteq B^j_{g_1}$, and inequalities (4.5), (4.33), and (4.34), we can continue (4.63) in the following way:

$$\begin{aligned}\int_{\bigcup_{j=1}^{N}K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}&\leq\sum_{j=1}^{N}\int_{K_j}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\mathcal{H}^{n-1}(K_j)\\&\leq\int_{\bigcup_{j=1}^{N}B^j_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j=1}^{N}B^j_{g_1}\setminus K_j}a\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,\mathcal{H}^{n-1}(\Sigma)\\&\leq\int_{B_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j>N}B^j_{g_1}}a\,\mathrm{d}\mathcal{H}^{n-1}+\delta(\mathcal{H}^{n-1}(\Sigma)+1)\\&\leq\int_{B_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,(\mathcal{H}^{n-1}(\Sigma)+2)\,.\end{aligned} \tag{4.64}$$

We now consider the last term in (4.61). By (4.35), (4.37), and (4.48), we have that $w^-=u^-$ $\mathcal{H}^{n-1}$-a.e. on $\Sigma\setminus\bigcup_{j=1}^{N}U_j$. Thus, by (4.62), we obtain

$$\begin{aligned}\int_{\Sigma\setminus\bigcup_{j=1}^{N}K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}&=\int_{\Sigma\setminus\bigcup_{j=1}^{N}U_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j=1}^{N}U_j\setminus K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}\\&=\int_{\Sigma\setminus\bigcup_{j=1}^{N}U_j}g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j=1}^{N}U_j\setminus K_j}g_1(x,u^+,q_j)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned} \tag{4.65}$$

In view of (A4), (4.5), (4.6), and of (4.33)-(4.36), inequality (4.65) becomes

$$
\begin{aligned}
&\int_{\Sigma\setminus\bigcup_{j=1}^N K_j} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{\Sigma\setminus\bigcup_{j=1}^N U_j} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^N U_j\setminus K_j} a_{\widetilde{M}}\,\mathrm{d}\mathcal{H}^{n-1}\\
&< \int_{\Sigma\setminus\bigcup_{j=1}^N K_j} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^N U_j\setminus K_j} a\,\mathrm{d}\mathcal{H}^{n-1} + \delta\\
&< \int_{\Sigma\setminus\bigcup_{j=1}^N B^j_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^N B^j_{g_1}\setminus K_j} a_M\,\mathrm{d}\mathcal{H}^{n-1} + 2\delta\\
&< \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j>N} B^j_{g_1}} a_M\,\mathrm{d}\mathcal{H}^{n-1} + 3\delta < \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + 4\delta\,.
\end{aligned}
\tag{4.66}
$$

Therefore, (4.61), (4.64), and (4.66) imply that

$$
\begin{aligned}
\int_{\Sigma} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} &< \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad + \int_{B_{g_1}} \inf_{\tau\in\mathbb{R}^m} g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1} + \delta\,(\mathcal{H}^{n-1}(\Sigma)+4)\,.
\end{aligned}
\tag{4.67}
$$

Collecting inequalities (4.60) and (4.67) and using (4.28) in the last equality, we obtain that, for some $c>0$ independent of δ,

$$
\begin{aligned}
\mathcal{F}_1(w) &= \int_{S_w\setminus\Sigma} \psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}\\
&< \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{B_{g_1}} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad + \int_{B_{g_1}} \inf_{\tau\in\mathbb{R}^m} g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1} + c\delta = \mathcal{F}_{12}(u) + c\delta\,.
\end{aligned}
\tag{4.68}
$$

Choosing $0<\delta<\varepsilon/2$ such that $c\delta<\varepsilon/2$ and $c_r\delta^{1/r}+\delta<\varepsilon/2$ in estimates (4.50), (4.51), (4.52), and (4.68), we deduce (4.29)-(4.31).

If we repeat the above argument replacing u and B_{g_1} of (4.26) with the function w and the set

$$
\begin{aligned}
B_g :&= \big\{x\in\Sigma' : g(x,w^+(x),w^-(x)) > \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,w^-(x)) + \psi(x,\nu_\Sigma(x))\big\}\\
&= \big\{x\in\Sigma' : g(x,u^+(x),w^-(x)) > \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,w^-(x)) + \psi(x,\nu_\Sigma(x))\big\}\,,
\end{aligned}
$$

we are able to construct a new function $v\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ such that:

$$
\|v-w\|_{r,\Omega} < \frac{\varepsilon}{2}\,,\qquad \|\nabla v-\nabla w\|_{p,\Omega} < \frac{\varepsilon}{2}\,,
$$

$$
\mathcal{H}^{n-1}(S_v) < \mathcal{H}^{n-1}(S_w) + 2\mathcal{H}^{n-1}(\Sigma) + \frac{\varepsilon}{2}\,,
$$

$$
\mathcal{F}(v) < \mathcal{F}_1(w) + \frac{\varepsilon}{2}\,.
$$

The previous inequalities, together with (4.29)-(4.31), imply that v satisfies (4.17)-(4.19). This concludes the proof of the lemma. □

Proof of Theorem 4.3. By the hypotheses of the theorem and by Lemma 4.2, the functions ψ and g_{12} satisfy hypotheses (H1)-(H7). Hence, from Theorem 3.4 we deduce that the functional $\mathcal{F}_{12}$ defined in (4.15) is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$. Since $g_{12}\le g$, we have that $\mathcal{F}_{12}\le\mathcal{F}$. Thus, by definition of $sc^-\mathcal{F}$, we easily get that $\mathcal{F}_{12} \leq sc^{-}\mathcal{F}$ on $GSBV^p(\Omega;\mathbb{R}^m)$. Therefore, we only need to show the converse inequality, that is,

$$sc^{-}\mathcal{F}(u) \leq \mathcal{F}_{12}(u) \qquad \text{for every } u \in GSBV^p(\Omega;\mathbb{R}^m). \tag{4.69}$$

Let us first prove (4.69) for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. To this end, we need to construct a recovery sequence for u. Applying Lemma 4.4, we can find a sequence $v_k \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that v_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ and

$$\mathcal{F}(v_k) < \mathcal{F}_{12}(u) + \frac{1}{k} \quad \text{for every } k. \tag{4.70}$$

Passing to the $\liminf$ as $k \to +\infty$ in (4.70) we get

$$sc^{-}\mathcal{F}(u) \leq \liminf_k \, sc^{-}\mathcal{F}(v_k) \leq \liminf_k \, \mathcal{F}(v_k) \leq \mathcal{F}_{12}(u).$$

This concludes the proof of (4.69) for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$.

Let us now consider $u \in GSBV^p(\Omega;\mathbb{R}^m)$. Given a function $\varphi \in C_c^\infty(\mathbb{R}^m;\mathbb{R}^m)$ with $\varphi(s) = s$ if $|s| \leq 1$, we can approximate u in $GSBV^p(\Omega;\mathbb{R}^m)$ with the sequence $\varphi_k(u) \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, where we have set $\varphi_k(s) := k\varphi(s/k)$. Clearly, $\varphi_k(u)$ converges to u pointwise $\mathcal{L}^n$-a.e. in Ω and $\nabla\varphi_k(u) \to \nabla u$ in $L^p(\Omega;\mathbb{M}^{m\times n})$. Moreover, $S_{\varphi_k(u)} \subseteq S_u$ for every k. Hence, by Definition 2.7, $\varphi_k(u)$ converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ and

$$\limsup_k \int_{S_{\varphi_k(u)}\setminus\Sigma} \psi(x,\nu_{\varphi_k(u)})\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.71}$$

Recalling that $\varphi_k \in C_c^\infty(\mathbb{R}^m;\mathbb{R}^m)$, we have that $\varphi_k(u)^\pm = \varphi_k(u^\pm)$ $\mathcal{H}^{n-1}$-a.e. in Σ. Therefore, since g_{12} is a Carathéodory function, we get that $g_{12}(x,\varphi_k(u)^+(x),\varphi_k(u)^-(x)) \to g_{12}(x,u^+(x),u^-(x))$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$. Thanks to hypothesis (A4) and to inequalities (4.5) and (4.6) of Lemma 4.2, we can apply the dominated convergence theorem to deduce that

$$\lim_k \int_\Sigma g_{12}(x,\varphi_k(u)^+,\varphi_k(u)^-)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.72}$$

Collecting (4.71) and (4.72), we get that

$$sc^{-}\mathcal{F}(u) \leq \liminf_k \, sc^{-}\mathcal{F}(\varphi_k(u)) \leq \limsup_k \, \mathcal{F}_{12}(\varphi_k(u)) \leq \mathcal{F}_{12}(u),$$

which concludes the proof of (4.69) in the general case.

□

We conclude this section with a generalization of Theorem 4.3 which takes into account also the presence of volume terms. Let $q \in (1,+\infty)$, let $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy (2.4) and (2.5), and let $f\colon \Omega\times\mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that (3.91) holds. We consider the functional $\mathcal{G}\colon L^q(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ defined as in (3.92). With the same notation used before, $sc^{-}\mathcal{G}$ denotes the greatest sequentially lower semicontinuous functional on $L^q(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{G}$. Moreover, we define

$$\mathcal{G}_{12}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$. We extend $\mathcal{G}_{12}$ to $+\infty$ in $L^q(\Omega;\mathbb{R}^m) \setminus GSBV^p(\Omega;\mathbb{R}^m)$.

Theorem 4.5. *Let ψ and g satisfy (H1)-(H3), (A1)-(A5), and* (3.5). *Then the functionals $sc^{-}\mathcal{G}$ and $\mathcal{G}_{12}$ coincide on $L^q(\Omega;\mathbb{R}^m)$.*

Proof. By (4.6) of Lemma 4.2, $\mathcal{G}_{12} \leq \mathcal{G}$. Recalling that g_{12} satisfies properties (H4)-(H7), from Theorems 2.9 and 3.4 and from the hypotheses on f we deduce that $\mathcal{G}_{12}$ is sequentially lower semicontinuous in $L^q(\Omega;\mathbb{R}^m)$. Thus $\mathcal{G}_{12} \leq sc^{-}\mathcal{G}$. By Lemma 4.4 and by the hypotheses on the volume densities W and f, we get also the opposite inequality in $SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. The conclusion follows by the truncation argument used in the last part of the proof of Theorem 4.3. □

Acknowledgements. This material is based on work supported by the Italian Ministry of Education, University, and Research under the Project "Calculus of Variations" (PRIN 2010-11) and by the European Research Council under Grant No. 290888 "Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture". The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] L. Ambrosio, *A compactness theorem for a new class of functions of bounded variation*, Boll. Un. Mat. Ital. B (7), 3 (1989), pp. 857–881.
[2] ———, *Existence theory for a new class of variational problems*, Arch. Rational Mech. Anal., 111 (1990), pp. 291–322.
[3] L. Ambrosio and A. Braides, *Functionals defined on partitions in sets of finite perimeter. I. Integral representation and* Γ*-convergence*, J. Math. Pures Appl. (9), 69 (1990), pp. 285–305.
[4] ———, *Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization*, J. Math. Pures Appl. (9), 69 (1990), pp. 307–333.
[5] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[6] G. Bouchitté, I. Fonseca, and L. Mascarenhas, *A global method for relaxation*, Arch. Rational Mech. Anal., 145 (1998), pp. 51–98.
[7] G. Bouchitté, I. Fonseca, and L. Mascarenhas, *Relaxation of variational problems under trace constraints*, Nonlinear Anal., 49 (2002), pp. 221–246.
[8] G. Buttazzo, *Semicontinuity, relaxation and integral representation in the calculus of variations*, vol. 207 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.
[9] D. L. Cohn, *Measure theory*, Birkhäuser, Boston, Mass., 1980.
[10] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal., 176 (2005), pp. 165–225.
[11] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
[12] I. Fonseca and G. Leoni, *Modern methods in the calculus of variations:* L^p *spaces*, Springer Monographs in Mathematics, Springer, New York, 2007.
[13] I. Fonseca and S. Müller, *Quasi-convex integrands and lower semicontinuity in* L^1, SIAM J. Math. Anal., 23 (1992), pp. 1081–1098.
[14] ———, *Relaxation of quasiconvex functionals in* $\mathrm{BV}(\Omega,\mathbf{R}^p)$ *for integrands* $f(x,u,\nabla u)$, Arch. Rational Mech. Anal., 123 (1993), pp. 1–49.
[15] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.
[16] A. A. Griffith, *The phenomena of rupture and flow in solids*, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 221 (1921), pp. 163–198.
[17] J. Kristensen, *Lower semicontinuity in spaces of weakly differentiable functions*, Math. Ann., 313 (1999), pp. 653–710.
[18] A. P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc., 61 (1947), pp. 418–442.

(Stefano Almi) SISSA, Via Bonomea 265, 34136 Trieste, Italy
E-mail address, Stefano Almi: salmi@sissa.it

(Gianni Dal Maso) SISSA, Via Bonomea 265, 34136 Trieste, Italy
E-mail address, Gianni Dal Maso: dalmaso@sissa.it

(Rodica Toader) DIMI, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy
E-mail address, Rodica Toader: rodica.toader@uniud.it