From volume cone to metric cone in the nonsmooth setting

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Abstract

We prove that 'volume cone implies metric cone' in the setting of RCD spaces, thus generalising to this class of spaces a well known result of Cheeger-Colding valid in Ricci-limit spaces.

Contents

1 Introduction

In the study of measured-Gromov-Hausdorff limits of Riemannian manifolds with Ricci curvature uniformly bounded from below, Ricci-limit spaces in short, as developed by Cheeger and Colding ([9], [10], [11], [12]), two almost rigidity results play a key role: the almost splitting theorem and the almost volume cone implies almost metric cone. By nature, both these results imply corresponding rigidity results for Ricci-limit spaces and in fact also the converse

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implication holds provided one is willing to give up the precise quantification given by the almost rigidity versions.

In the seminal papers [23] and [27], [28], Lott-Villani and Sturm proposed a synthetic definition of lower Ricci curvature bounds for metric-measure spaces based on optimal transport: according to their approach, spaces with Ricci curvature bounded from below by K and dimension bounded from above by N are called $\mathsf{CD}(K,N)$ spaces. Later on, mostly for technical reasons related to the local-to-global property, Bacher-Sturm introduced in [7] a variant of the $\mathsf{CD}(K,N)$ condition, called reduced curvature dimension condition and denoted $\mathsf{CD}^*(K,N)$.

Key features of both the CD and CD^* notions are the compatibility with the Riemannian case and the stability w.r.t. measured-Gromov-Hausdorff convergence. In particular, they include Ricci-limit spaces and it is natural to wonder whether the aforementioned geometric rigidity result hold for these structures. However, this is not the case, as both CD and CD^* structures include Finsler geometries (see the last theorem in [29] and [24]) and it is therefore natural to look for stricter conditions which, while retaining the crucial stability properties of Lott-Sturm-Villani spaces, rule out Finsler structures.

A first step in this direction has been made by Ambrosio, Savaré and the second author in [2], where the notion of $\mathsf{RCD}(K,\infty)$ spaces (the 'R' stands for Riemannian) have been introduced via means related to the study of the heat flow. Partly motivated by this approach the second author in [17] proposed a strengthening of the $\mathsf{CD}/\mathsf{CD}^*$ conditions based solely on properties of Sobolev functions: the added requirement is that the Sobolev space $W^{1,2}$ is an Hilbert space, a condition called infinitesimal Hilbertianity, and the resulting classes of spaces are denoted $\mathsf{RCD}/\mathsf{RCD}^*$.

It turns out that the a priori purely analytic notion of infinitesimal Hilbertianity grants geometric properties, the reason being that it allows to make computations mimicking the calculus in Riemannian (as opposed to Finslerian) manifolds. The first example in this direction has been the Abresch-Gromoll inequality proved by the second author and Mosconi in [20]. Other relevant geometric properties, both on their own and for the purposes of the current paper, are the splitting theorem, proved in [14] by the second author, and the maximal diameter theorem, proved by Ketterer in [22].

On the more analytic side of the story, in [3] (see also [19] for some earlier results) it has been carried out a throughout study of the Bochner inequality - both in infinite and in finite dimensions - on non-smooth structures showing in particular its stability under mGH convergence and, in the infinite dimensional case, its equivalence with the $\mathsf{RCD}(K,\infty)$ condition. This circle of ideas has been closed by Erbar-Kuwada-Sturm in [13] and independently by Ambrosio-Mondino-Savaré in [5]: in these papers the equivalence between the finite dimensional Bochner inequality and the $\mathsf{RCD}^*(K,N)$ condition has been proved.

The focus of this paper it to prove the non-smooth version of the 'volume cone implies metric cone', our result being:

Theorem 1.1. *Let* $K\in\mathbb{R}$, $N\in(0,\infty)$, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *a* $\mathsf{RCD}^*(0,N)$ *space with* $\mathrm{supp}(\mathfrak{m})=\mathrm{X}$, $\mathsf{O}\in\mathrm{X}$ *and* $\mathsf{R}>\mathsf{r}>0$ *such that*

$$\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))=\Big(\frac{\mathsf{R}}{\mathsf{r}}\Big)^N\mathfrak{m}(B_{\mathsf{r}}(\mathsf{O})).$$

Then exactly one of the following holds:

1) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains only one point. In this case (X,d) is isometric to $[0,\mathrm{diam}(\mathrm{X})]$ ($[0,\infty)$ if X is unbounded) with an isometry which sends O in 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}}(\mathsf{O})$ to the measure $c\,x^{N-1}\mathrm{d}x$ for $c := N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$.*

2) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains two points. In this case (X,d) is a 1-dimensional Riemannian manifold, possibly with boundary, and there is a bijective local isometry (in the sense of distance-preserving maps) from $B_{\mathsf{R}}(\mathsf{O})$ to $(-\mathsf{R},\mathsf{R})$ sending O to 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ to the measure $c\,|x|^{N-1}\mathrm{d}x$ for $c := \frac{1}{2}N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$. Moreover, such local isometry is an isometry when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$.*

3) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains more than two points. In this case $N \geq 2$ and there exists a $\mathsf{RCD}^*(N-2,N-1)$ space $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ with $\mathrm{diam}(\mathrm{Z}) \leq \pi$ such that the ball $B_{\mathsf{R}}(\mathsf{O})$ is locally isometric to the ball $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$ of the cone Y built over Z. Moreover, such local isometry is an isometry when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$.*

Some remarks are in order. First of all, much like the case of Ricci-limit spaces, this result has consequences for what concerns the study of tangent cones, in particular in connection with so-called non-collapsing limit spaces. We shall analyze some of these consequences in subsequent papers.

Also, much like the non-smooth splitting, this result gives some new information also about the smooth world: the space in now known to be locally a cone over a $\mathsf{RCD}^*(N-2,N-1)$ space, rather than simply over a length space. We remark that such information about the structure of the sphere will come after we proved the cone structure of the original space via a direct use of Ketterer's results [22] about lower Ricci bounds for cones.

Actually the the $\mathsf{RCD}^*(N-2,N-1)$ space $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ will be given by an appropriate rescaling of the sphere $S_{\mathsf{R}/2}(\mathsf{O})$ once this is endowed with the "natural" induced distance and measure, see Section 3.5 for more details.

As a side remark, let us also point out that the classical smooth version of the equality cases in Bishop-Gromov inequality (i.e. the smooth version of Theorem 1.1) is usually stated saying that $B_{\mathsf{R}}(\mathsf{O}) \subset \mathrm{X}$ and $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}}) \subset \mathrm{Y}$ are *isometric*. Hence Theorem 1.1 seems a priori weaker of the corresponding smooth version. This is actually not the case: indeed it is classical in Differential Geometry to say that two Riemannian manifolds are isometric if the pull back the metric tensors coincide. It is easy seen that this however implies only locally isometry as metric spaces. Optimality of our result (in the non smooth as well as in the smooth setting) can be checked by looking at the ball of radius 1 on the flat torus $\mathbb{T}^N$, $N \in \mathbb{N}$.

We focus on the case $K = 0$ for simplicity, but in fact our techniques can be easily adapted to general K's as well as to the case of 'volume annulus'. We shall briefly mention this in the last section.

Let us now discuss the proof of our result. The structure and techniques used closely resemble that of the splitting theorem (compare with [14] and [16]) with all the metric informations being read at the level of Sobolev functions and only at the very end translated back into metric properties. We shall take advantage of this analogy in order to skip some lengthy detail whenever a closely related lemma has been already proved for the case of the splitting. With this said, let us briefly outline the content of each section of the paper:

3.1) We consider the function $\mathrm{b}(x) := \frac{1}{2}\mathsf{d}^2(x,\mathsf{O})$ and show, thanks to the volume rigidity, that its Laplacian is equal to N on $B_{\mathsf{R}}(\mathsf{O})$. On the other hand, b is a Kantorovich potential and this grants, via the results in [21], that for any $\mu \ll \mathfrak{m}$, $\mu \in \mathscr{P}_2(\mathrm{X})$ it induces a unique

W_2-geodesic starting from μ. Coupling these informations together we shall see that the gradient flow Fl_t of b is well-defined and satisfies $(\mathsf{Fl}_t)_*(\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}) = e^{Nt}\mathfrak{m}|_{B_{e^{-t}\mathsf{R}}(\mathsf{O})}$.

3.2) We use the fact that for b the Bochner inequality holds as equality on $B_{\mathsf{R}}(\mathsf{O})$ to deduce that its gradient flow preserves, up to an exponential factor, the Dirichlet energy. Here the biggest difference w.r.t. the splitting is in the fact that the splitting is a result about the structure of the whole space, while here we only deal with a portion of it. This means that the 'regular' part of our space is confined in $B_{\mathsf{R}}(\mathsf{O})$ and appropriate cut-off arguments have to be used in order to justify the computations needed.

3.3) We use the fact that maps between RCD spaces which preserve the Dirichlet energy must be isometries ([14]) to deduce that the restriction of Fl_t to $B_{\mathsf{R}}(\mathsf{O})$ is, up to modification on a negligible set, a homothety.

3.4) At this stage we introduce the sphere $S_{\mathsf{R}/2}(\mathsf{O})$ which will be - up to scaling - the 'base' space for the cone construction. With tools coming from optimal transport we shall see that if it contains more than 2 points, then every couple of its points can be connected by a Lipschitz path lying entirely on $S_{\mathsf{R}/2}(\mathsf{O})$. On the other hand, if $\#S_{\mathsf{R}/2}(\mathsf{O}) \leq 2$ the conclusion can be derived easily, so that from now on we shall assume that $\#S_{\mathsf{R}/2}(\mathsf{O}) > 2$.

3.5) Under this assumption on $S_{\mathsf{R}/2}(\mathsf{O})$ and thanks to what previously proved, there is a natural geodesic distance d' on $S_{\mathsf{R}/2}(\mathsf{O})$ induced by d by considering paths lying entirely on $S_{\mathsf{R}/2}(\mathsf{O})$, a natural radial projection map $\mathsf{Pr} : B_{\mathsf{R}}(\mathsf{O})\setminus\{\mathsf{O}\} \to S_{\mathsf{R}/2}(\mathsf{O})$ and a canonical measure $\mathfrak{m}' := \mathsf{Pr}_*\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ on $S_{\mathsf{R}/2}(\mathsf{O})$. Showing that $B_{\mathsf{R}}(\mathsf{O})$ is isometric to the cone built over (a rescaling of) $S_{\mathsf{R}/2}(\mathsf{O})$ amounts to put in relation d' and d. It is trivial that $\mathsf{d} \leq \mathsf{d}'$ on $S_{\mathsf{R}/2}(\mathsf{O})$.

3.6) It much more complicated to prove a sort of converse of such inequality, namely that given a curve γ d-absolutely continuous with values in $B_{\mathsf{R}}(\mathsf{O})\setminus\{\mathsf{O}\}$ the curve $\tilde\gamma_t := \mathsf{Pr}(\gamma_t)$ is d'-absolutely continuous and satisfying $|\dot{\tilde\gamma}_t| \leq \frac{\mathsf{R}}{2\mathsf{d}(\gamma_t,\mathsf{O})}|\dot\gamma_t|$. Read in the smooth world, this would mean that the differential of Pr at $x \in B_{\mathsf{R}} \setminus \{\mathsf{O}\}$ has norm bounded above by $\frac{\mathsf{R}}{2\mathsf{d}(x,\mathsf{O})}$.

This step of the proof is technically the most challenging and also the one where the arguments differ the most from those in the proof of the splitting. Notice indeed that in the splitting the level set of the Busemann function is (proved to be) isometrically embedded in the original space and this fact can be used to quickly gain crucial informations about its relation with the ambient space. Here, instead, we deal with a sphere and thus it is not really its metric structure which is inherited by the ambient space, but rather its differential one and this shift from 'metric' to 'differential' calculus requires appropriate tools to be handled because concepts like 'vector fields' and 'Hessian' must come into play.

In practice, what we do is to recover Pr as the limit as $t \to \infty$ of the gradient flow $\hat{\mathsf{Fl}}_t$ of the function $\hat{\mathrm{b}} := \psi \circ \mathrm{b}$ with $\psi(z) := \frac{1}{2}\big(\sqrt{2z} - \frac{\mathsf{R}}{2}\big)^2$. Since we know from Bochner's formula who is the Hessian of b on $B_{\mathsf{R}}(\mathsf{O})$ (namely the identity), we know who is the Hessian of $\hat{\mathrm{b}}$ and henceforth the differential of $\hat{\mathsf{Fl}}_t$. Then by direct computation we obtain the desired result. All this analysis heavily depends on the vocabulary proposed by the second author in [15].

3.7) The standard definition of metric-measure cone forces the base space to be the sphere of radius 1 around the vertex. Thus in our case in order to prove that $B_{\mathsf{R}}(\mathsf{O})$ is locally isometric to a cone we need first to introduce the base space Z as the rescaling of $S_{\mathsf{R}/2}(\mathsf{O})$ by a factor $\frac{2}{\mathsf{R}}$. This is the scope of this short section where also the cone Y over Z is defined.

3.8) There is a natural map $\mathsf{S} : B_{\mathsf{R}}(\mathsf{O}) \to B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$ given by $\mathsf{S}(x) := (\mathsf{Pr}(x), \mathsf{d}(x, \mathsf{O}))$ and the results of Sections 3.5, 3.6 grant that it preserves the measure and the weak upper gradients of functions not depending on the radial component. From these facts, the infinitesimal Hilbertianity of X and the structure of Sobolev functions on Y (studied in [18]) we deduce that $|\mathrm{D}(f \circ \mathsf{S})| = |\mathrm{D}f| \circ \mathsf{S}$ for any Sobolev function on Y which is zero on a neighbourhood of the vertex and with support in $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$.

3.9) Although we don't know yet that Y is a RCD space, from its very construction we still know (from [18]) that a map from X to Y which (locally) preserves the Dirichlet energy must be (locally) an isometry. Hence we deduce from the previous step that S must be locally an isometry on $B_{\mathsf{R}}(\mathsf{O}) \setminus \{\mathsf{O}\}$ and from this the conclusion easily follows.

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2 Preliminaries

We assume the reader familiar with calculus on metric measure spaces and the RCD condition, here we recall only few basic things as reference for what comes next, mainly in order to fix the notation. Other results will be recalled in the body of the paper, whenever needed.

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric space equipped with a non-negative Borel measure. The local Lipschitz constant $\mathrm{lip}(f) : \mathrm{X} \to [0, \infty]$ of a function $f : \mathrm{X} \to \mathbb{R}$ is defined as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)},$$

if x is not isolated, 0 otherwise.

A Borel probability measure $\boldsymbol{\pi}$ on $C([0,1], \mathrm{X})$ is said of bounded compression provided for some $C > 0$ it holds

$$(\mathrm{e}_t)_* \boldsymbol{\pi} \leq C \mathfrak{m}, \qquad \forall t \in [0,1],$$

where $\mathrm{e}_t : C([0,1], \mathrm{X}) \to \mathrm{X}$ is the evaluation map sending a curve γ to γ_t. By kinetic energy of a curve γ we intend $\frac{1}{2} \int_0^1 |\dot\gamma_t|^2 \, \mathrm{d}t$, the integral being intended $+\infty$ if the curve is not absolutely continuous. Then the kinetic energy of a probability measure $\boldsymbol{\pi}$ on $C([0,1], \mathrm{X})$ is $\frac{1}{2} \iint_0^1 |\dot\gamma_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma)$.

If $\boldsymbol{\pi}$ has finite kinetic energy and is of bounded compression, then it is called *test plan*.

For $t_0, t_1 \in [0,1]$ the map $\mathrm{Restr}_{t_0}^{t_1} : C([0,1], \mathrm{X}) \to C([0,1], \mathrm{X})$ is defined as

$$\mathrm{Restr}_{t_0}^{t_1}(\gamma)_t := \gamma_{(1-t)t_0 + t t_1}.$$

Notice that if $\boldsymbol{\pi}$ is a test plan, $t_0 \neq t_1 \in [0,1]$ and $\Gamma \subset C([0,1], \mathrm{X})$ is such that $\boldsymbol{\pi}(\Gamma) > 0$, then the plan $\boldsymbol{\pi}(\Gamma)^{-1} (\mathrm{Restr}_{t_0}^{t_1})_* (\boldsymbol{\pi}_{|\Gamma})$ is also a test plan.

The Sobolev class $S^2(\mathrm{X})$ is the collection of all Borel functions $f : \mathrm{X} \to \mathbb{R}$ for which there is $G \in L^2(\mathrm{X})$, $G \geq 0$, such that

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot\gamma_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma),$$

for every test plan $\boldsymbol{\pi}$. Any such G is called weak upper gradient for f and it turns out that for $f \in S^2(\mathrm{X})$ there exists a minimal G in the $\mathfrak{m}$-a.e. sense, which is called minimal weak upper gradient and denoted by $|\mathrm{D}f|$. Among other properties, the minimal weak upper gradient is local in the sense that for every $f, g \in S^2(\mathrm{X})$ we have

$$|\mathrm{D}f| = |\mathrm{D}g|, \qquad \mathfrak{m} - a.e. \ on \ \{f = g\}, \tag{2.1}$$

see [1] and [17] for more details and for the proof of the following equivalent characterization:

Proposition 2.1. *The following are equivalent:*

i) $f \in S^2(\mathrm{X})$ *and* G *is a weak upper gradient.*

ii) for every test plan $\boldsymbol{\pi}$ *the following holds. For* $\boldsymbol{\pi}$*-a.e.* γ *the function* $t \mapsto f(\gamma_t)$ *is equal a.e. on* $[0,1]$ *and on* $\{0,1\}$ *to an absolutely continuous function* f_γ *such that*

$$|\partial_t f_\gamma|(t) \leq G(\gamma_t)|\dot\gamma_t|, \qquad a.e. \ t \in [0,1].$$

The Sobolev space $W^{1,2}(\mathrm{X})$ is defined as $L^2 \cap S^2(\mathrm{X})$ and is equipped with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||\mathrm{D}f|\|^2_{L^2}.$$

The locality property (2.1) allows to introduce the space $S^2_{loc}(\mathrm{X})$ (resp. $W^{1,2}_{loc}(\mathrm{X})$) as the space of those functions locally (= on bounded sets) equal to some function in $S^2(\mathrm{X})$ (resp. $W^{1,2}(\mathrm{X})$). These functions come with a natural weak upper gradient which belongs to $L^2_{loc}(\mathrm{X})$. Similarly, for $\Omega \subset \mathrm{X}$ open, the space $S^2(\Omega)$ (resp. $W^{1,2}(\Omega)$) is defined as the space of those functions locally in Ω (= on bounded subsets of Ω with positive distance from $\partial\Omega$) equal to some function in $S^2(\mathrm{X})$ (resp. $W^{1,2}(\mathrm{X})$) such that $|\mathrm{D}f| \in L^2(\Omega)$ (resp. $f, |\mathrm{D}f| \in L^2(\Omega)$). Notice that Lipschitz functions f always belong to $W^{1,2}_{loc}(\mathrm{X})$ and that

$$|\mathrm{D}f| \leq \mathrm{lip}(f), \quad \mathfrak{m} - a.e.. \tag{2.2}$$

A test plan $\boldsymbol{\pi}$ is said to *represent* the gradient of $f \in S^2(\mathrm{X})$ provided

$$\varliminf_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi}(\gamma) \geq \frac{1}{2} \int |\mathrm{D}f|^2(\gamma_0) \, \mathrm{d}\boldsymbol{\pi}(\gamma) + \frac{1}{2} \varlimsup_{t \downarrow 0} \iint_0^t |\dot\gamma_s|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma).$$

Notice that the opposite inequality always holds. If f is only in $S^2(\Omega)$ for some open set Ω, we add the requirement that $(\mathrm{e}_t)_*\boldsymbol{\pi}$ is concentrated on Ω for every $t \in [0,1]$ sufficiently small.

Spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ such that $W^{1,2}(\mathrm{X})$ is an Hilbert space are called *infinitesimally Hilbertian* (see [17], to which we also refer for the differential calculus recalled below).

It turns out that

$$\text{on inf. Hilb. spaces the map } S^2(\mathrm{X}) \ni f \mapsto |\mathrm{D}f|^2 \in L^1(\mathrm{X}) \text{ is a quadratic form.} \tag{2.3}$$

By polarization, it induces a bilinear and symmetric map

$$S^2(\mathrm{X}) \ni f, g \qquad \mapsto \qquad \langle \nabla f, \nabla g \rangle \in L^1(\mathrm{X}),$$

which satisfies $\langle \nabla f, \nabla f \rangle = |\mathrm{D}f|^2$, the 'Cauchy-Schwarz' inequality $|\langle \nabla f, \nabla g \rangle| \le |\mathrm{D}f||\mathrm{D}g|$ and the chain and Leibniz rules

$$\begin{aligned} \langle \nabla(\varphi \circ f), \nabla g \rangle &= \varphi' \circ f \langle \nabla f, \nabla g \rangle, \\ \langle \nabla f, \nabla(g_1 g_2) \rangle &= g_1 \langle \nabla f, \nabla g_2 \rangle + g_2 \langle \nabla f, \nabla g_1 \rangle, \end{aligned} \tag{2.4}$$

for any $f, g \in S^2(\mathrm{X})$, $g_1, g_2 \in S^2 \cap L^\infty(\mathrm{X})$, $\varphi \in C^1(\mathbb{R})$ with bounded derivative.

A simple yet crucial result is the following *first order differentiation formula*: if $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, $f, g \in S^2(\Omega)$ for some open set Ω and $\boldsymbol{\pi}$ represents the gradient of f and is such that $(\mathrm{e}_t)_* \boldsymbol{\pi}$ is concentrated on Ω for every $t \in [0,1]$ sufficiently small, then

$$\lim_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \mathrm{d}\boldsymbol{\pi}(\gamma) = \int \langle \nabla f, \nabla g \rangle (\gamma_0) \, \mathrm{d}\boldsymbol{\pi}(\gamma). \tag{2.5}$$

The space $D(\Delta) \subset W^{1,2}(\mathrm{X})$ is the space of functions f for which there is a function in $L^2(\mathrm{X})$, called the Laplacian of f and denoted by Δf, such that

$$\int g \Delta f \, \mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \in W^{1,2}(\mathrm{X}).$$

The Laplacian is local in the sense that for every $f, g \in D(\Delta)$ we have

$$\Delta f = \Delta g, \qquad \mathfrak{m} - a.e. \text{ on the interior of } \{f = g\},$$

and satisfies the natural chain and Leibniz rules.

Choosing to test with Lipschitz functions with prescribed support yields the notion of measure valued Laplacian: for $f \in W^{1,2}(\mathrm{X})$ and $\Omega \subset \mathrm{X}$ open, we say that f has a measure valued Laplacian in Ω provided there is a measure, denoted by $\boldsymbol{\Delta} f_{|\Omega}$, such that

$$\int g \, \mathrm{d}\boldsymbol{\Delta} f_{|\Omega} = -\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}\mathfrak{m} \qquad \forall g \text{ bounded and Lipschitz with compact support in } \Omega.$$

If $\Omega = \mathrm{X}$ we shall simply write $\boldsymbol{\Delta} f$ in place of $\boldsymbol{\Delta} f_{|\mathrm{X}}$. The 'measure-valued' and 'L^2-valued' notions of Laplacian are tightly linked, as for a given $f \in W^{1,2}(\mathrm{X})$ we have $f \in D(\Delta)$ if and only if f has a measure-valued Laplacian absolutely continuous w.r.t. $\mathfrak{m}$ and with density in L^2. In this case Δf coincides with such density.

The *heat flow* on an infinitesimally Hilbertian space is the L^2-gradient flow of the Dirichlet energy $\mathcal{E} : L^2(\mathrm{X}) \to [0, \infty]$ defined as

$$\mathcal{E}(f) := \begin{cases} \dfrac{1}{2} \displaystyle\int |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}, & \text{if } f \in W^{1,2}(\mathrm{X}), \\ +\infty, & \text{otherwise.} \end{cases}$$

It will be denoted by $\mathsf{h}_t : L^2(\mathrm{X}) \to L^2(\mathrm{X})$ so that for $f \in L^2(\mathrm{X})$ the curve $t \mapsto \mathsf{h}_t f \in L^2(\mathrm{X})$ is the unique gradient flow trajectory on $L^2(\mathrm{X})$ of $\mathcal{E}$ starting from f. The fact that $\mathcal{E}$ is a quadratic form ensures that the heat flow is linear.

The functional $\mathcal{U}_N : \mathscr{P}(\mathrm{X}) \to \mathbb{R}$ is defined as

$$\mathcal{U}_N(\mu) := -\int \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}, \qquad \text{for} \quad \mu = \rho\mathfrak{m} + \mu^s, \quad \mu^s \perp \mathfrak{m}.$$

$(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to be a $\mathsf{CD}^*(0, N)$ space provided $\mathcal{U}_N$ is geodesically convex on the space $(\mathscr{P}_2(\mathrm{supp}(\mathfrak{m})), W_2)$ (see [7] and the original papers [23], [28]). If $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is $\mathsf{CD}^*(0, N)$, then $(\mathrm{supp}(\mathfrak{m}), \mathsf{d})$ is proper and geodesic ([28]).

A space which is both $\mathsf{CD}^*(0, N)$ and infinitesimally Hilbertian will be called $\mathsf{RCD}^*(0, N)$ (see [17] and [2]). We recall that a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ has the *Sobolev-to-Lipschitz* property (see [14]) provided any $f \in W^{1,2}(\mathrm{X})$ with $|\mathrm{D}f| \leq 1$ $\mathfrak{m}$-a.e., admits a 1-Lipschitz representative. It is an important fact about $\mathsf{RCD}^*(0, N)$ spaces that they have the Sobolev-to-Lipschitz property (see [2]).

On a $\mathsf{RCD}^*(0, N)$ space we consider the following space of *test functions* (see [26] and [15]):

$$\mathrm{Test}(\mathrm{X}) := \Big\{ f \in D(\Delta) \ : \ f, |\mathrm{D}f| \in L^\infty(\mathrm{X}), \ \Delta f \in W^{1,2}(\mathrm{X}) \Big\}.$$

In particular, test functions have a Lipschitz representative and we shall always consider such representative when working with them. It is a remarkable property of test functions f the fact that $|\mathrm{D}f|^2 \in W^{1,2}(\mathrm{X})$ (see [26]) and this fact grants that $\mathrm{Test}(\mathrm{X})$ is an algebra.

By simple truncation and mollification via the heat flow we see that $\mathrm{Test}(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$ and it easy to check that it is stable by application of the heat flow. A slightly more refined argument grants that

$$\begin{gathered}\text{if } f \in W^{1,2}(\mathrm{X}) \text{ (resp. } L^2(\mathrm{X})) \text{ has support on a given open set } \Omega, \text{ then there exists}\\ \text{a sequence of test functions with support in } \Omega \text{ converging to } f \text{ in } W^{1,2}(\mathrm{X}) \text{ (resp. } L^2(\mathrm{X})).\end{gathered} \tag{2.6}$$

To see this, taking into account that $\mathrm{Test}(\mathrm{X})$ is an algebra it is sufficient to show that for $K \subset \Omega$ with K compact and Ω open there exists a test function identically 1 on K and with support contained in Ω. Such function can be built as in [4] by first taking a Lipschitz function χ with $\mathrm{supp}(\chi) \subset \Omega$ and $\chi \equiv 1$ on K, then using the fact that $\mathsf{h}_t\chi \to \chi$ uniformly as $t \downarrow 0$ (see [2]) and finally considering $\varphi \circ (\mathsf{h}_t\chi) \in \mathrm{Test}(\mathrm{X})$ for $t \ll 1$ and an appropriate choice of $\varphi \in C^\infty(\mathbb{R})$.

Finally, on $\mathsf{RCD}^*(0, N)$ spaces the *Bochner inequality* holds ([13]) in the sense that for $f \in \mathrm{Test}(\mathrm{X})$ the function $|\mathrm{D}f|^2$ has a measure valued Laplacian and

$$\mathbf{\Delta}\frac{|\mathrm{D}f|^2}{2} \geq \Big(\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle\Big)\mathfrak{m}. \tag{2.7}$$

3 Main

Throughout all the paper we shall make the following assumption:

Assumption 3.1. $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a* $\mathsf{RCD}^*(0, N)$ *space with* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$, $\mathsf{O} \in \mathrm{X}$ *is a given point and* $\mathsf{R} > \mathsf{r} > 0$ *are radii such that*

$$\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O})) = \Big(\frac{\mathsf{R}}{\mathsf{r}}\Big)^N \mathfrak{m}(B_{\mathsf{r}}(\mathsf{O})).$$

The proof of Theorem 1.1 will be based on the study of the *gradient flow* of the "Busemann" function $\mathrm{b} : \mathrm{X} \to \mathbb{R}^+$ given by

$$\mathrm{b}(x) := \frac{\mathsf{d}^2(x, \mathsf{O})}{2}, \qquad \forall x \in \mathrm{X},$$

where here and in the following by $\mathbb{R}^+$ we denote the closed half-line $[0, \infty)$.

3.1 Gradient flow of b: effect on the measure

We start recalling the following basic fact about geodesics on $\mathsf{RCD}^*(0, N)$ spaces, which directly follows from the existence of optimal maps established in [21] when one of the two measures considered is δ_{O}:

Proposition 3.2. *There exists a Borel* $\mathfrak{m}$*-negligible set* $\mathcal{N} \subset \mathrm{X}$ *and a Borel map* $\mathsf{G} : [0,1] \times \mathrm{X} \to \mathrm{X}$ *such that for every* $x \in \mathrm{X} \setminus \mathcal{N}$ *the curve* $[0,1] \ni t \mapsto \mathsf{G}_t(x)$ *is the unique constant speed geodesic from* O *to* x*. Moreover, for every* $t \in (0,1]$ *the map* $\mathsf{G}_t : \mathrm{X} \setminus \mathcal{N} \to \mathrm{X}$ *is injective, and the measure* $(\mathsf{G}_t)_*\mathfrak{m}$ *is absolutely continuous w.r.t.* $\mathfrak{m}$ *and its density* ρ_t *satisfies*

$$\rho_t \circ \mathsf{G}_t \leq t^{-N}, \qquad \mathfrak{m} - a.e.. \tag{3.1}$$

Coupling this proposition with Assumption 3.1 we get the following rigidity result:

Proposition 3.3. *For every* $t \in (0, 1]$ *we have*

$$\mathfrak{m}(B_{t\mathsf{R}}(\mathsf{O})) = t^N \mathfrak{m}(B_{\mathsf{R}}(\mathsf{O})), \tag{3.2}$$

and

$$(\mathsf{G}_t)_*(\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}) = \frac{1}{t^N} \mathfrak{m}|_{B_{t\mathsf{R}}(\mathsf{O})}. \tag{3.3}$$

proof We start with (3.2). Set $v(s) := \mathfrak{m}(B_s(\mathsf{O}))$ and notice that the Bishop-Gromov inequality (see Theorem 2.3 in [28]) ensures that v is continuous, locally semiconcave on $(0, \mathsf{R}]$ and that the map $\frac{v'(s)}{s^{N-1}}$ is decreasing. It follows that the map $g(s) := v(s^{\frac{1}{N}})$ is locally semiconcave on $(0, \mathsf{R}^N]$, continuous on $[0, \mathsf{R}^N]$ and from

$$g'(s) = v'(s^{\frac{1}{N}}) \frac{1}{N} s^{\frac{1}{N}-1} = \frac{1}{N} \frac{v'(s^{\frac{1}{N}})}{(s^{\frac{1}{N}})^{N-1}}, \qquad a.e.\ s \in [0, \mathsf{R}^N],$$

we see that g has decreasing derivative, i.e. that g is concave and in particular

$$g(t\mathsf{R}^N) \geq tg(\mathsf{R}^N) + (1-t)g(0), \qquad \forall t \in [0,1].$$

Since $g(0) = 0$, we see that Assumption 3.1 gives that the inequality is an equality for $t := \frac{\mathsf{r}^N}{\mathsf{R}^N}$. Then the concavity of g forces the equality for every $t \in [0, 1]$, which is (3.2).

To conclude, let $\mu_0 := \mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))^{-1} \mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ and put $\mu_t := (\mathsf{G}_{1-t})_*\mu_0$ so that $t \mapsto \mu_t$ is the only W_2-geodesic connecting μ_0 to $\mu_1 = \delta_{\mathsf{O}}$. Put also $\nu_t := \mathfrak{m}(B_{(1-t)\mathsf{R}}(\mathsf{O}))^{-1} \mathfrak{m}|_{B_{(1-t)\mathsf{R}}(\mathsf{O})}$, notice that μ_t is concentrated on $B_{(1-t)\mathsf{R}}(\mathsf{O})$ and thus for every $t \in [0, 1)$

$$\begin{aligned}
-(1-t)\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))^{\frac{1}{N}} &= (1-t)\mathcal{U}_N(\mu_0) && \text{by computation} \\
&\geq \mathcal{U}_N(\mu_t) && \text{from } \mathsf{CD}^*(0,N) \text{ and } \mathcal{U}_N(\delta_{\mathsf{O}}) = 0 \\
&\geq \mathcal{U}_N(\nu_t) && \text{by Jensen's inequality} \\
&= -\mathfrak{m}(B_{(1-t)\mathsf{R}}(\mathsf{O}))^{\frac{1}{N}} && \text{by computation} \\
&= -(1-t)\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))^{\frac{1}{N}} && \text{by (3.2).}
\end{aligned}$$

Therefore we have the equality in Jensen's inequality, which forces $\mu_t = \nu_t$ for every $t \in [0,1)$, which is the claim. □

We now introduce the reparametrized flow $\mathsf{Fl} : [0,\infty) \times (\mathrm{X} \setminus \mathcal{N}) \to \mathrm{X}$ defined by

$$\mathsf{Fl}_s(x) := \mathsf{G}_{e^{-s}}(x), \qquad \forall s \geq 0,\ x \in \mathrm{X} \setminus \mathcal{N}.$$

Notice that in the smooth setting, the flow Fl would be the gradient flow of b. In our context, some basic properties of Fl follow from those of G:

Corollary 3.4. *The following holds:*

i) For every $x \in \mathrm{X} \setminus \mathcal{N}$ the curve $[0,\infty) \ni t \mapsto \gamma_t := \mathsf{Fl}_t(x)$ is locally Lipschitz and satisfies

$$\mathrm{b}(\gamma_t) = \mathrm{b}(\gamma_s) + \frac{1}{2}\int_t^s |\dot\gamma_r|^2 + \mathrm{lip}(\mathrm{b})^2(\gamma_r)\,\mathrm{d}r, \qquad \forall 0 \leq t \leq s,$$

and in particular the metric speed of $t \mapsto \mathsf{Fl}_t(x)$ is equal to $\mathrm{lip}(\mathrm{b})(\mathsf{Fl}_t(x)) \leq \mathrm{lip}(\mathrm{b})(x)$.

ii) For every $t \geq 0$, Fl_t is an essentially invertible map from $B_{\mathsf{R}}(\mathsf{O})$ to $B_{e^{-t}\mathsf{R}}(\mathsf{O})$, i.e. there exists a map $\mathsf{Fl}_t^{-1} : B_{e^{-t}\mathsf{R}}(\mathsf{O}) \to B_{\mathsf{R}}(\mathsf{O})$ such that $\mathsf{Fl}_t \circ \mathsf{Fl}_t^{-1} = \mathrm{Id}$ $\mathfrak{m}$-a.e. on $B_{e^{-t}\mathsf{R}}(\mathsf{O})$ and $\mathsf{Fl}_t^{-1} \circ \mathsf{Fl}_t = \mathrm{Id}$ $\mathfrak{m}$-a.e. on $B_{\mathsf{R}}(\mathsf{O})$.

iii) For every $t \geq 0$ we have

$$(\mathsf{Fl}_t)_*\mathfrak{m} \leq e^{Nt}\mathfrak{m} \tag{3.4}$$

and

$$\frac{\mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m}}{\mathrm{d}\mathfrak{m}} \circ \mathsf{Fl}_t = e^{Nt}, \qquad \mathfrak{m} - a.e.\ on\ B_{\mathsf{R}}(\mathsf{O}). \tag{3.5}$$

iv) For every $t, s \geq 0$ we have

$$\begin{aligned} \mathsf{Fl}_t(\mathsf{Fl}_s(x)) &= \mathsf{Fl}_{t+s}(x), \\ \mathsf{d}(\mathsf{Fl}_s(x), \mathsf{Fl}_t(x)) &= \mathsf{d}(x,\mathsf{O})|e^{-s} - e^{-t}|, \end{aligned} \tag{3.6}$$

for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.

proof The triangle inequality shows that $\mathrm{lip}(\mathrm{b})(x) \leq \mathsf{d}(x,\mathsf{O})$. Now for $x \in X$ let γ be the geodesic connecting it to O and observe that

$$\mathrm{lip}(\mathrm{b})(x) \geq \overline{\lim_{t\downarrow 0}} \frac{\mathrm{b}(x) - \mathrm{b}(\gamma_t)}{\mathsf{d}(x,\gamma_t)} = \mathsf{d}(x,O)\overline{\lim_{t\downarrow 0}} \frac{1-(1-t)^2}{2t} = \mathsf{d}(x,\mathsf{O}),$$

showing that $\mathrm{lip}(\mathrm{b}) \equiv \mathsf{d}(\cdot,\mathsf{O})$. Point (i) then follows by direct computation. Concerning point (ii), notice that the essential injectivity of Fl_t follows from the one of $\mathsf{G}_{e^{-t}}$ and the essential surjectivity is a consequence of (3.3). The bound (3.4) follows from (3.1) and (3.5) is a restatement of (3.3). Finally, for property (iv) recall that $t \mapsto \mathsf{G}_t(x)$ is a constant speed geodesic from O to x for $\mathfrak{m}$-a.e. x and take into account the reparametrization. □

Few basic, yet interesting, properties of b and Fl_t can now be established. Notice that the result of Corollary 3.6 below is also a direct consequence of Cheeger's seminal paper [8] (because $\mathsf{RCD}^*(K,N)$ spaces are doubling and support a weak 1-2 Poincaré inequality [28], [25]) but the additional structure we have at disposal allows for a shorter independent proof.

Corollary 3.5. *Let $T : (\mathrm{X} \setminus \mathcal{N}) \to C([0,1],\mathrm{X})$ be the map sending x to the curve $[0,1] \ni t \mapsto \mathsf{Fl}_t(x)$ and $\mu \in \mathscr{P}(\mathrm{X})$ with bounded support and such that $\mu \le C\mathfrak{m}$ for some C. Then $\boldsymbol{\pi} := T_*\mu$ represents the gradient of $-\mathrm{b}$.*

proof The bound (3.4) ensures that $\boldsymbol{\pi}$ has bounded compression, while from point (i) of Corollary 3.4 we know that the metric speed of $T(x)$ is bounded by $\mathrm{lip}(\mathrm{b})(x) = \mathsf{d}(x, \mathsf{O})$, so that the fact that μ has bounded support ensures that $\boldsymbol{\pi}$ has finite kinetic energy.

Thus $\boldsymbol{\pi}$ is a test plan. Moreover, from (i) of Corollary 3.4 we have

$$\begin{aligned}\int \mathrm{b}(\gamma_0) - \mathrm{b}(\gamma_t)\,\mathrm{d}\boldsymbol{\pi}(\gamma) &= \frac{1}{2}\iint_0^t \mathrm{lip}^2(\mathrm{b})(\gamma_s) + |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\ge \frac{1}{2}\iint_0^t |\mathrm{Db}|^2(\gamma_s) + |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma).\end{aligned}$$

Dividing by t, letting $t \downarrow 0$, noticing that the measures $(\mathrm{e}_s)_*\boldsymbol{\pi}$ have uniformly bounded support and that $|\mathrm{Db}| \le \mathrm{lip}(\mathrm{b})$ is bounded on bounded sets, to conclude it is sufficient to prove that $(\mathrm{e}_s)_*\boldsymbol{\pi} \rightharpoonup \mu$ as $s \downarrow 0$ in duality with $L^1(\mathrm{X})$. This follows from W_2-convergence - which grants weak convergence - and the fact that these measures have uniformly bounded densities. □

Corollary 3.6. $|\mathrm{Db}|^2 = 2\mathrm{b}$ $\mathfrak{m}$-*a.e.*.

proof Let $T : (\mathrm{X} \setminus \mathcal{N}) \to C([0,1],\mathrm{X})$ be defined as in Corollary 3.5 and $\mu \in \mathscr{P}(\mathrm{X})$ be with bounded support such that $\mu \le C\mathfrak{m}$ for some C. Then Corollary 3.5 grants that $\boldsymbol{\pi}$ is a test plan and therefore, keeping in mind point (i) of Corollary 3.4, we have

$$\begin{aligned}\frac{1}{2}\iint_0^t |\dot\gamma_s|^2 + \mathrm{lip}(\mathrm{b})^2(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma) &= \int \mathrm{b}(\gamma_0) - \mathrm{b}(\gamma_t)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\le \iint_0^t |\mathrm{Db}|(\gamma_s)|\dot\gamma_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\le \frac{1}{2}\iint_0^t |\dot\gamma_s|^2 + |\mathrm{Db}|^2(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

which gives

$$\iint_0^t \mathrm{lip}(\mathrm{b})^2(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \iint_0^t |\mathrm{Db}|^2(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma).$$

Dividing by t, letting $t \downarrow 0$ and using the fact that $(\mathrm{e}_t)_*\boldsymbol{\pi} \rightharpoonup \mu$ as $t \downarrow 0$ in duality with $L^1(\mathrm{X})$ (like in Corollary 3.5) we deduce that $\int \mathrm{lip}(\mathrm{b})^2\,\mathrm{d}\mu \le \int |\mathrm{Db}|^2\,\mathrm{d}\mu$ which by the arbitrariness of μ and inequality (2.2) is sufficient to conclude, since $\mathrm{lip}(\mathrm{b})^2(x) = 2\mathrm{b}(x)$. □

Finally we show as equality in the Bishop-Gromov inequality combined with the Laplacian comparison Theorem proved in [17] allows to show that $\boldsymbol{\Delta}\mathrm{b} = N\mathfrak{m}$ on $B_{\mathsf{R}}(\mathsf{O})$:

Proposition 3.7. *We have* $\boldsymbol{\Delta}\mathrm{b}|_{B_{\mathsf{R}}(\mathsf{O})} = N\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$.

proof Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth, non-increasing and with support in $[0,1)$ and let us consider

$$h_\varphi(r) := \frac{1}{r^N}\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathfrak{m}.$$

Differentiating with respect to r, taking into account that $|\mathrm{Db}|^2 = 2\mathrm{b}$ and that $\mathbf{\Delta}\mathrm{b} \leq N\mathfrak{m}$, see [17], we obtain

$$\begin{aligned}
r^{N+1}h'_\varphi(r) &= -N\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathfrak{m} - 4\int \varphi'\Big(\frac{2\mathrm{b}}{r^2}\Big)\frac{b}{r^2}\,\mathrm{d}\mathfrak{m} \\
&= -N\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathfrak{m} - 2\int \varphi'\Big(\frac{2\mathrm{b}}{r^2}\Big)\frac{|\mathrm{Db}|^2}{r^2}\,\mathrm{d}\mathfrak{m} \\
&= -N\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathfrak{m} + \int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathbf{\Delta}\mathrm{b} \leq 0.
\end{aligned} \tag{3.7}$$

So that h_φ is also non-increasing. On the other hand by (3.3) and the layer cake formula, we see that for $r \leq \mathsf{R}$ and φ decreasing we have

$$\begin{aligned}
h_\varphi(r) = \frac{1}{r^N}\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathfrak{m} &= \frac{1}{r^N}\int_0^{\sup\varphi} \mathfrak{m}\Big(\Big\{\varphi\Big(\frac{2\mathrm{b}}{r^2}\Big) > c\Big\}\Big)\,\mathrm{d}c \\
&= \frac{1}{r^N}\int_0^r \mathfrak{m}\Big(\Big\{\varphi\Big(\frac{2\mathrm{b}}{r^2}\Big) > \varphi\Big(\frac{s^2}{r^2}\Big)\Big\}\Big)|\varphi'|\Big(\frac{s^2}{r^2}\Big)\frac{2s}{r^2}\,\mathrm{d}s \\
\text{(because } \varphi \text{ is non-increasing)} \qquad &= \frac{1}{r^N}\int_0^r \mathfrak{m}(B_s(\mathsf{O}))|\varphi'|\Big(\frac{s^2}{r^2}\Big)\frac{2s}{r^2}\,\mathrm{d}s \\
\text{(by (3.2))} \qquad &= \frac{\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{\mathsf{R}^N}\int_0^r \Big(\frac{s}{r}\Big)^N|\varphi'|\Big(\frac{s^2}{r^2}\Big)\frac{2s}{r^2}\,\mathrm{d}s \\
&= \frac{\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{\mathsf{R}^N}\int_0^1 t^N|\varphi'|(t^2)2t\,\mathrm{d}t.
\end{aligned}$$

Hence h_φ is constant on $[0, R]$. Combining this with (3.7) we immediately deduce that for every $r \leq \mathsf{R}$ we have

$$\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big)\,\mathrm{d}\mathbf{\Delta}\mathrm{b} = N\int \varphi\Big(\frac{2\mathrm{b}}{r^2}\Big).$$

By letting $\varphi \to \chi_{[0,1)}$ we deduce that $\mathbf{\Delta}\mathrm{b}(B_{\mathsf{R}}(\mathsf{O})) = N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$ and thus recalling that $\mathbf{\Delta}\mathrm{b} \leq N\mathfrak{m}$ we conclude that $\mathbf{\Delta}\mathrm{b}|_{B_{\mathsf{R}}(\mathsf{O})} = N\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$, as desired. □

In the following corollary as well as in the foregoing discussion we shall denote by $\mathsf{d}_{\mathsf{O}} : \mathrm{X} \to \mathbb{R}^+$ the map sending x to $\mathsf{d}(x, \mathsf{O})$.

Corollary 3.8 (Continuous disintegration of $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ along d_{O})**.** *We have*

$$\mathrm{d}(\mathsf{d}_{\mathsf{O}})_*(\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})})(r) = c\,\chi_{[0,\mathsf{R}]}(r)\,r^{N-1}\mathrm{d}r, \tag{3.8}$$

with $c := N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$ *and there exists a weakly continuous family of measures* $[0, \mathsf{R}] \ni r \mapsto \mathfrak{m}_r \in \mathscr{P}(\mathrm{X})$ *such that*

$$\int \varphi\,\mathrm{d}\mathfrak{m} = c\int_0^{\mathsf{R}}\int \varphi\,\mathrm{d}\mathfrak{m}_r\,r^{N-1}\,\mathrm{d}r, \qquad \forall \varphi \in C_c(B_{\mathsf{R}}(\mathsf{O})). \tag{3.9}$$

Moreover, for every $t \geq 0$ *the measures* $\mathfrak{m}_r$ *satisfy*

$$(\mathsf{Fl}_t)_*\mathfrak{m}_r = \mathfrak{m}_{e^{-t}r}, \qquad a.e.\ r \in [0, \mathsf{R}]. \tag{3.10}$$

proof The identity (3.8) follows from (3.3), which shows that $\mathfrak{m}(B_r(\mathsf{O})) = (\frac{r}{\mathsf{R}})^N \mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$ for every $r \in [0, \mathsf{R}]$.

Now let $\{\mathfrak{m}_r\}_{r\in[0,\mathsf{R}]} \subset \mathscr{P}(\mathrm{X})$ be a disintegration of $\mathfrak{m}_{|B_{\mathsf{R}}(\mathsf{O})}$ along d_{O}, hence such that (3.9) holds, and recall that a priori existence and uniqueness of the $\mathfrak{m}_r$'s is only given for a.e. $r \in [0, \mathsf{R}]$. Fix $t \geq 0$, $\varphi \in C_c(B_{e^{-t}\mathsf{R}}(\mathsf{O}))$, and notice that

$$\int \varphi \, \mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m} = \int \varphi \circ \mathsf{Fl}_t \, \mathrm{d}\mathfrak{m} = N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O})) \int_0^{\mathsf{R}} \int \varphi \circ \mathsf{Fl}_t \, \mathrm{d}\mathfrak{m}_r \, r^{N-1} \, \mathrm{d}r.$$

On the other hand, since $\mathsf{Fl}_t(B_{\mathsf{R}}(\mathsf{O})) \subset B_{e^{-t}\mathsf{R}}(\mathsf{O})$, by (3.5) we also have

$$\int \varphi \, \mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m} = e^{Nt} \int \varphi \, \mathrm{d}\mathfrak{m} = e^{Nt} N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O})) \int_0^{\mathsf{R}} \int \varphi \, \mathrm{d}\mathfrak{m}_s \, s^{N-1} \, \mathrm{d}s.$$

Thus the change of variable $s = e^{-t}r$ in the last integral shows that

$$\int_0^{\mathsf{R}} \int \varphi \circ \mathsf{Fl}_t \, \mathrm{d}\mathfrak{m}_r \, r^{N-1} \, \mathrm{d}r = \int_0^{\mathsf{R}} \int \varphi \, \mathrm{d}\mathfrak{m}_{e^{-t}r} \, r^{N-1} \, \mathrm{d}r,$$

which by the arbitrariness of $\varphi \in C_c(B_{e^{-t}\mathsf{R}}(\mathsf{O}))$ gives the claim (3.10).

It remains to prove that the $\mathfrak{m}_r$'s can be chosen to weakly depend on $r \in [0, \mathsf{R}]$ and to this aim, due to the existence of a countable set of Lipschitz functions dense in $C_c(B_{\mathsf{R}}(\mathsf{O}))$, it is sufficient to show that for a given $\varphi : \mathrm{X} \to \mathbb{R}$ Lipschitz with support in $B_{\mathsf{R}}(\mathsf{O})$, the map $r \mapsto I_\varphi(r) := \int \varphi \, \mathrm{d}\mathfrak{m}_r$ admits a continuous representative.

Thus fix such φ, put $J_\varphi(s) := I_\varphi(\mathsf{R}e^{-s})$ for $s \in \mathbb{R}^+$ and notice that (3.10) grants that for every $h \geq 0$ the identity

$$\begin{aligned} \big|J_\varphi(s+h) - J_\varphi(s)\big| &= \Big| \int \varphi \, \mathrm{d}\mathfrak{m}_{\mathsf{R}e^{-h}e^{-s}} - \int \varphi \, \mathrm{d}\mathfrak{m}_{\mathsf{R}e^{-s}} \Big| \\ &= \Big| \int \varphi \circ \mathsf{Fl}_h - \varphi \, \mathrm{d}\mathfrak{m}_{\mathsf{R}e^{-s}} \Big| \leq \mathsf{R}e^{-s} \operatorname{Lip}(\varphi)(1 - e^{-h}) \end{aligned}$$

holds for a.e. s, the inequality being a consequence of the second identity in (3.6) and the fact that $\mathfrak{m}_{\mathsf{R}e^{-s}}$ is concentrated on $B_{\mathsf{R}e^{-s}}(\mathsf{O})$. This is sufficient to show that the distributional derivative of $s \mapsto J_\varphi(s)$ is bounded by $\mathsf{R}\operatorname{Lip}(\varphi)e^{-s}$. Being $s \mapsto \mathsf{R}\operatorname{Lip}(\varphi)e^{-s}$ in $L^1(\mathbb{R}^+)$, we just proved that J_φ has an absolutely continuous representative admitting a limit when $s \to +\infty$. By construction, this is the same as to say that I_φ has a continuous representative on $[0, \mathsf{R}]$.

□

For the purpose of the foregoing analysis it will be convenient to replace the function b with a smoother one. We therefore fix once and for all $\bar{\mathsf{R}} < \mathsf{R}$. Later on $\bar{\mathsf{R}}$ will be sent to R but for the moment it is convenient to think it as fixed also in order to avoid mentioning the dependence on it of the various objects we are going to build.

Let $\varphi \in C^\infty(\mathbb{R})$ be a function with support contained in $(-\infty, \frac{\mathsf{R}^2}{2})$ which is the identity on $(-\infty, \frac{\bar{\mathsf{R}}^2}{2})$ and define $\bar{\mathrm{b}} : \mathrm{X} \to \mathbb{R}$ as

$$\bar{\mathrm{b}} := \varphi \circ \mathrm{b}.$$

Notice in particular that $\bar{\mathrm{b}}$ is Lipschitz, with support in $B_{\mathsf{R}}(\mathsf{O})$ and equal to b on $B_{\bar{\mathsf{R}}}(\mathsf{O})$. We then introduce the flow $\bar{\mathsf{Fl}}$ as follows. First define the reparametrization function rep :

$(\mathbb{R}^+)^2 \to \mathbb{R}^+$ by requiring that

$$\partial_t \mathrm{rep}_t(r) = \varphi'\Big(\frac{r^2}{2}e^{-2\mathrm{rep}_t(r)}\Big), \qquad\qquad \mathrm{rep}_0(r) = 0, \tag{3.11}$$

for every $r \geq 0$, then we define $\bar{\mathsf{Fl}} : [0,\infty) \times (\mathrm{X} \setminus \mathcal{N}) \to \mathrm{X}$ as

$$\bar{\mathsf{Fl}}_t(x) := \mathsf{Fl}_{\mathrm{rep}_t(\mathsf{d}(x,\mathsf{O}))}(x).$$

The advantage of dealing with $\bar{\mathrm{b}}$ and $\bar{\mathsf{Fl}}_t(x)$ in place of b and Fl_t is that the former are 'smooth' on the whole space while the latter only on $B_{\mathsf{R}}(\mathsf{O})$, see for instance Corollary 3.10 below.

The following proposition collects the basic properties of $\bar{\mathsf{Fl}}_t(x)$.

Proposition 3.9. *Let* $\bar{\mathrm{b}}$ *and* $\bar{\mathsf{Fl}}_t(x)$ *as above, then:*

a) for every $x \in \mathrm{X} \setminus \mathcal{N}$ *the curve* $t \mapsto \gamma_t := \bar{\mathsf{Fl}}_t(x)$ *satisfies*

$$\bar{\mathrm{b}}(\gamma_t) = \bar{\mathrm{b}}(\gamma_s) + \frac{1}{2}\int_t^s |\dot\gamma_r|^2 + \mathrm{lip}(\bar{\mathrm{b}})^2(\gamma_r)\,\mathrm{d}r, \qquad \forall 0 \leq t \leq s.$$

In particular, the speed of $t \mapsto \bar{\mathsf{Fl}}_t(x)$ *is equal to* $\mathrm{lip}(\bar{\mathrm{b}})(\bar{\mathsf{Fl}}_t(x))$ *for a.e.* t*, thus granting that* $t \mapsto \bar{\mathsf{Fl}}_t(x)$ *is* $\mathrm{Lip}(\bar{\mathrm{b}})$*-Lipschitz for every* $x \in \mathrm{X} \setminus \mathcal{N}$*.*

b) $\bar{\mathsf{Fl}}_t$ *is the identity on* $\mathrm{X} \setminus (B_{\mathsf{R}}(\mathsf{O}) \cup \mathcal{N})$ *and sends* $B_{\mathsf{R}}(\mathsf{O}) \setminus \mathcal{N}$ *into* $B_{\mathsf{R}}(\mathsf{O})$ *for every* $t \geq 0$*.*

c) $\bar{\mathsf{Fl}}_t$ *coincides with* Fl_t *in* $B_{\bar{\mathsf{R}}}(\mathsf{O}) \setminus \mathcal{N}$*.*

d) $\bar{\mathsf{Fl}}_t : \mathrm{X} \to \mathrm{X}$ *is essentially invertible for every* $t \geq 0$*.*

e) $(\bar{\mathsf{Fl}}_t)_*\mathfrak{m} \ll \mathfrak{m}$ *for every* $t \geq 0$*, more precisely*

$$c(t)\mathfrak{m} \leq (\bar{\mathsf{Fl}}_t)_*\mathfrak{m} \leq C(t)\mathfrak{m}, \tag{3.12}$$

for some continuous functions $c, C : \mathbb{R}^+ \to (0,\infty)$*.*

f) The maps $\bar{\mathsf{Fl}}_t$ *form a semigroup, i.e.* $\bar{\mathsf{Fl}}_0 = \mathrm{Id}$ $\mathfrak{m}$*-a.e. and*

$$\bar{\mathsf{Fl}}_t \circ \bar{\mathsf{Fl}}_s = \bar{\mathsf{Fl}}_{t+s}, \quad \mathfrak{m} - a.e. \tag{3.13}$$

for every $t, s \geq 0$*.*

In particular, from the essential invertibility of $\bar{\mathsf{Fl}}_t : \mathrm{X} \to \mathrm{X}$ we see that for $t > 0$ the map $\bar{\mathsf{Fl}}_{-t} := (\bar{\mathsf{Fl}}_t)^{-1}$ is well defined $\mathfrak{m}$-a.e. and with this definition property (3.12) holds for every $t \in \mathbb{R}$, for appropriate continuous extensions of $c(\cdot), C(\cdot)$, and (3.13) holds for every $t, s \in \mathbb{R}$.

proof [Proof of Proposition 3.9] Points (b), (c), (f) follows from the definition and (a) can be obtained by the same argument of 3.4. The injectivity part of point (d) is a consequence of the same property for Fl_t, while the surjectivity follows once we prove (e).

We are thus left to prove (e). Start observing that for $r \geq \mathsf{R}$ we have $\mathrm{rep}_t(r) = 0$ for any $t \geq 0$ and thus $\bar{\mathsf{Fl}}_t|_{X\setminus B_{\mathsf{R}}(\mathsf{O})} = \mathrm{Id}$ so that

$$(\bar{\mathsf{Fl}}_t)_*\mathfrak{m}|_{X\setminus B_{\mathsf{R}}(\mathsf{O})} = \mathfrak{m}|_{X\setminus B_{\mathsf{R}}(\mathsf{O})}, \qquad \forall t \geq 0. \tag{3.14}$$

To control the behaviour inside $B_{\mathsf{R}}(\mathsf{O})$ notice that by the very definition of $\bar{\mathsf{Fl}}_t$ and from (3.10) we have that for any $t \geq 0$ it holds

$$(\bar{\mathsf{Fl}}_t)_*(\mathfrak{m}_r) = \mathfrak{m}_{e^{-\mathrm{rep}_t(r)}r}, \qquad a.e.\ r \in [0, \mathsf{R}].$$

Put $f_t(r) := e^{-\mathrm{rep}_t(r)}r$. We claim that there are continuous functions $c, C : \mathbb{R}^+ \to (0, \infty)$ such that

$$c(t) \leq \partial_r f_t(r) \leq C(t), \qquad \forall r \in [0, \mathsf{R}],\ t \geq 0. \tag{3.15}$$

The bound from above is obvious by smoothness, for the one from below observe that

$$\partial_r f_t(r) = e^{-\mathrm{rep}_t(r)}(1 - r\partial_r \mathrm{rep}_t(r)), \tag{3.16}$$

put $g_t(r) := r\partial_r \mathrm{rep}_t(r)$ and differentiate (3.11) in r to deduce that $g_t(r)$ solves

$$\partial_t g_t(r) = \varphi''\Big(\tfrac{r^2}{2}e^{-2\mathrm{rep}_t(r)}\Big)r^2 e^{-2\mathrm{rep}_t(r)}\big(1 - g_t(r)\big) \tag{3.17}$$

with the initial condition $g_0(r) = 0$ for every $r \in [0, \mathsf{R}]$. Noticing that the function identically 1 solution of (3.17), taking into account the initial condition by comparison we deduce that

$$g_t(r) \leq 1 - \tilde{c}(t), \qquad \forall r \in [0, \mathsf{R}],\ t \geq 0,$$

for some continuous function $\tilde{c} : \mathbb{R}^+ \to (0, \infty)$. Plugging this bound in (3.16) we deduce the first inequality in (3.15).

Then for every $t \geq 0$ consider the function $f_t^{-1} : [0, \mathsf{R}] \to [0, \mathsf{R}]$, notice that $f_t^{-1}(0) = 0$, $f_t^{-1}(\mathsf{R}) = \mathsf{R}$ and that (3.15) grants that

$$\frac{1}{C(t)} \leq \partial_s(f_t^{-1})(s) \leq \frac{1}{c(t)}, \qquad \forall s \in [0, \mathsf{R}],\ t \geq 0 \tag{3.18}$$

and thus it also holds

$$\frac{s}{C(t)} \leq f_t^{-1}(s) \leq \frac{s}{c(t)} \tag{3.19}$$

Let now $\varphi \geq 0$ be a Borel function identically zero on $X \setminus B_{\mathsf{R}}(\mathsf{O})$. According to Corollary 3.8 we have

$$\begin{aligned}
\int \varphi \,\mathrm{d}(\bar{\mathsf{Fl}}_t)_*\mathfrak{m} = \int \varphi \circ \bar{\mathsf{Fl}}_t \,\mathrm{d}\mathfrak{m} &= c\int_0^{\mathsf{R}} \int \varphi \circ \bar{\mathsf{Fl}}_t \,\mathrm{d}\mathfrak{m}_r r^{N-1}\mathrm{d}r \\
&= c\int_0^{\mathsf{R}} \int \varphi \,\mathrm{d}(\bar{\mathsf{Fl}}_t)_*(\mathfrak{m}_r)\, r^{N-1}\mathrm{d}r = c\int_0^{\mathsf{R}} \int \varphi \,\mathrm{d}\mathfrak{m}_{f_t(r)}\, r^{N-1}\mathrm{d}r \\
&= c\int_0^{\mathsf{R}} \int \varphi \,\mathrm{d}\mathfrak{m}_s\, (f_t^{-1}(s))^{N-1}\partial_s f_t^{-1}(s)\mathrm{d}s,
\end{aligned}$$

thus from the bounds (3.18) and (3.19) and using again Corollary 3.8 we obtain

$$\frac{c}{C(t)^N}\int \varphi \,\mathrm{d}\mathfrak{m} \leq \int \varphi \,\mathrm{d}(\bar{\mathsf{Fl}}_t)_*\mathfrak{m} \leq \frac{c}{c(t)^N}\int \varphi \,\mathrm{d}\mathfrak{m}, \qquad \forall t \geq 0,$$

which together with (3.14) gives the claim (3.12). □

Proposition 3.7 easily implies the following useful corollary:

Corollary 3.10. $\bar{\mathrm{b}} \in \mathrm{Test}(\mathrm{X})$ *and* $\Delta\bar{\mathrm{b}} \in L^\infty(\mathrm{X})$*.*

proof Clearly $\mathrm{supp}(\bar{\mathrm{b}}) \subset B_{\mathsf{R}}(\mathsf{O})$, hence the identity $|\mathrm{Db}|^2 = 2\mathrm{b}$ $\mathfrak{m}$-a.e. given by Corollary 3.6, Proposition 3.7 and the chain rule for the distributional Laplacian (see Proposition 4.11 in [17]) yield

$$\boldsymbol{\Delta}\bar{\mathrm{b}} = \varphi'' \circ \mathrm{b}|\mathrm{Db}|^2\mathfrak{m} + \varphi' \circ \mathrm{b}\boldsymbol{\Delta}\mathrm{b} = \big(2\mathrm{b}\varphi'' \circ \mathrm{b} + N\varphi' \circ \mathrm{b}\big)\mathfrak{m},$$

Given that $2\mathrm{b}\varphi'' \circ \mathrm{b} + N\varphi' \circ \mathrm{b} \in L^\infty \cap W^{1,2}(\mathrm{X})$, this identity shows that $\bar{\mathrm{b}} \in D(\Delta)$ with $\Delta\bar{\mathrm{b}} \in L^\infty \cap W^{1,2}(\mathrm{X})$, yielding the desired conclusion. □

We conclude the section with the following useful lemma, which highly depends on the first-order differentiation formula (2.5).

Lemma 3.11 (Basic properties of right composition with $\bar{\mathsf{Fl}}_t$)**.** *Let* $f \in L^p(\mathrm{X})$*,* $p < \infty$*. Then* $t \mapsto f \circ \bar{\mathsf{Fl}}_t \in L^p(\mathrm{X})$ *is continuous on* $\mathbb{R}$*.*

If $f \in W^{1,2}(\mathrm{X})$*, then* $t \mapsto f \circ \bar{\mathsf{Fl}}_t \in L^2(\mathrm{X})$ *is* C^1 *and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \bar{\mathsf{Fl}}_t = -\langle \nabla f, \nabla \bar{\mathrm{b}} \rangle \circ \bar{\mathsf{Fl}}_t. \tag{3.20}$$

proof Property (3.12) grants that the linear operators from $L^p(\mathrm{X})$ into itself given by the right composition with $\bar{\mathsf{Fl}}_t$ are, locally in t, uniformly continuous and thus it is sufficient to check that $t \mapsto f \circ \bar{\mathsf{Fl}}_t \in L^p(\mathrm{X})$ is continuous for a dense set of f's. We then consider Lipschitz functions and notice that the uniform Lipschitz bound granted by point (a) of Proposition 3.9 gives

$$\sqrt[p]{\int |f \circ \bar{\mathsf{Fl}}_{t_1} - f \circ \bar{\mathsf{Fl}}_{t_0}|^p \,\mathrm{d}\mathfrak{m}} \leq \mathrm{Lip}(f) \sqrt[p]{\int \mathsf{d}^p(\bar{\mathsf{Fl}}_{t_1}(x), \bar{\mathsf{Fl}}_{t_0}(x)) \,\mathrm{d}\mathfrak{m}(x)} \leq \mathrm{Lip}(f)\,\mathrm{Lip}(\bar{\mathrm{b}})|t_1 - t_0|,$$

thus yielding the first claim.

Now let $\bar{\mathfrak{m}} \in \mathscr{P}(\mathrm{X})$ be such that $\mathfrak{m} \ll \bar{\mathfrak{m}} \leq C\mathfrak{m}$ for some $C > 0$, let $T : \mathrm{X} \to C([0,1], \mathrm{X})$ be the map sending x to the curve $t \mapsto \bar{\mathsf{Fl}}_t(x)$ and define $\boldsymbol{\pi} := T_*\bar{\mathfrak{m}}$. The same arguments used for Corollary 3.5 grant that $\boldsymbol{\pi}$ is a test plan. Thus Proposition 2.1 grants that for any $t_0, t_1 \in [0,1]$, $t_0 < t_1$, for $\mathfrak{m}$-a.e. x we have

$$|f(\bar{\mathsf{Fl}}_{t_1}(x)) - f(\bar{\mathsf{Fl}}_{t_0}(x))| \leq \mathrm{Lip}(\bar{\mathrm{b}}) \int_{t_0}^{t_1} |\mathrm{D}f|(\bar{\mathsf{Fl}}_t(x)) \,\mathrm{d}t.$$

Squaring and integrating we get

$$\begin{aligned}\int |f \circ \bar{\mathsf{Fl}}_{t_1} - f \circ \bar{\mathsf{Fl}}_{t_0}|^2 \,\mathrm{d}\mathfrak{m} &\leq |t_1 - t_0| \,\mathrm{Lip}^2(\bar{\mathrm{b}}) \iint_{t_0}^{t_1} |\mathrm{D}f|^2(\bar{\mathsf{Fl}}_t(x)) \,\mathrm{d}t \,\mathrm{d}\mathfrak{m} \\ &= |t_1 - t_0| \,\mathrm{Lip}^2(\bar{\mathrm{b}}) \int_{t_0}^{t_1} \int |\mathrm{D}f|^2 \,\mathrm{d}(\bar{\mathsf{Fl}}_t)_*\mathfrak{m} \,\mathrm{d}t \\ &\leq C|t_1 - t_0| \,\mathrm{Lip}^2(\bar{\mathrm{b}}) \int |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m},\end{aligned}$$

for some constant C, where in the last step we used the bound (3.12). This shows that $t \mapsto f \circ \bar{\mathsf{Fl}}_t \in L^2(\mathrm{X})$ is Lipschitz on $[0,1]$ and it is then clear that Lipschitz continuity holds on the whole $\mathbb{R}$.

We now claim that

$$\frac{f\circ \bar{\mathsf{Fl}}_t - f}{t} \quad\to\quad -\langle \nabla f, \nabla \bar{\rm b}\rangle \qquad\qquad \text{weakly in } L^2({\rm X}). \tag{3.21}$$

Since the incremental ratios are, by what we just proved, uniformly bounded in $L^2({\rm X})$, thanks to a simple density and linearity argument to get the claim it is sufficient to prove that for any bounded probability density ρ it holds

$$\lim_{t\to 0}\int \frac{f\circ \bar{\mathsf{Fl}}_t - f}{t}\rho\,{\rm d}\mathfrak m = -\int \langle \nabla f, \nabla \bar{\rm b}\rangle \rho\,{\rm d}\mathfrak m \tag{3.22}$$

Putting $\tilde{\boldsymbol\pi} := T_*(\rho\mathfrak m)$, where T is defined as in Corollary 3.5, and arguing as in the proof of the same corollary we see that $\tilde{\boldsymbol\pi}$ represents the gradient of $-\bar{\rm b}$ and thus (3.22) follows from (2.5). Hence (3.21) is proved.

Since the $\bar{\mathsf{Fl}}_t$'s form a group, we then have proved (3.20) for any $t\in\mathbb R$, provided we intend the left hand side as the weak limit in $L^2({\rm X})$ of the incremental ratios.

Now notice that since $t\mapsto f\circ\bar{\mathsf{Fl}}_t \in L^2({\rm X})$ is Lipschitz and Hilbert spaces have the Radon-Nikodym property, the curve is a.e. differentiable. Together with the weak convergence just established, this is sufficient to get (3.21) for a.e. t. To conclude it is now enough to notice that the first part of the statement ensures that the right hand side of (3.20) depends L^2-continuously on t. □

3.2 Gradient flow of b: effect on the Dirichlet energy

In the previous section we established the link between Fl_t and the reference measure. To get informations about the link between the flow and the distance we shall, as in [14], first look at what happens to the Dirichlet energy along the flow. The metric information will later be recovered via the Sobolev-to-Lipschitz property.

Much like in the smooth case, the starting point for any metric-related information on the flow is Bochner inequality, which here is used to obtain the following Euler equation for $\bar{\rm b}$:

Proposition 3.12 (Euler equation for $\bar{\rm b}$). *Let $g\in{\rm Test}({\rm X})$. Then*

$$\boldsymbol\Delta\langle\nabla g,\nabla\bar{\rm b}\rangle|_{B_{\bar{\mathsf R}}(\mathsf O)} = \big(\langle\nabla\Delta g,\nabla\bar{\rm b}\rangle + 2\Delta g\big)\mathfrak m|_{B_{\bar{\mathsf R}}(\mathsf O)}. \tag{3.23}$$

In particular, for $f\in{\rm Test}({\rm X})$ with ${\rm supp}(f)\subset B_{\bar{\mathsf R}}(\mathsf O)$ we have

$$\int \Delta f\langle\nabla g,\nabla\bar{\rm b}\rangle\,{\rm d}\mathfrak m = \int f\big(\langle\nabla\Delta g,\nabla\bar{\rm b}\rangle + 2\Delta g\big)\,{\rm d}\mathfrak m. \tag{3.24}$$

proof Let $\varepsilon\in\mathbb R$ and write the Bochner inequality (2.7) for $\bar{\rm b}+\varepsilon g\in{\rm Test}({\rm X})$ to get

$$\boldsymbol\Delta\frac{|{\rm D}(\bar{\rm b}+\varepsilon g)|^2}{2} \ge \Big(\frac{|\Delta(\bar{\rm b}+\varepsilon g)|^2}{N} + \langle\nabla(\bar{\rm b}+\varepsilon g),\nabla\Delta(\bar{\rm b}+\varepsilon g)\rangle\Big)\mathfrak m.$$

Expand the formula and use the fact that $|{\rm D}\bar{\rm b}|^2 = 2\bar{\rm b}$ on $B_{\bar{\mathsf R}}(\mathsf O)$ (by Corollary 3.6 and $\bar{\rm b}={\rm b}$ on $B_{\bar{\mathsf R}}(\mathsf O)$) and $\Delta\bar{\rm b} = N$ $\mathfrak m$-a.e. on $B_{\bar{\mathsf R}}(\mathsf O)$ (Proposition 3.7 and Corollary 3.10) to get

$$\boldsymbol\Delta\Big(\varepsilon\langle\nabla g,\nabla\bar{\rm b}\rangle + \frac{\varepsilon^2}{2}|{\rm D}g|^2\Big)|_{B_{\bar{\mathsf R}}(\mathsf O)} \ge \Big(\varepsilon\big(\langle\nabla\Delta g,\nabla\bar{\rm b}\rangle + 2\Delta g\big) + \varepsilon^2\big(\frac{|\Delta g|^2}{N} + \langle\nabla g,\nabla\Delta f\rangle\big)\Big)\mathfrak m|_{B_{\bar{\mathsf R}}(\mathsf O)}.$$

Divide by $\varepsilon>0$ (resp. $\varepsilon<0$) and let $\varepsilon\downarrow 0$ (resp. $\varepsilon\uparrow 0$) to obtain (3.23) and then, trivially, also (3.24). □

Our control of $\bar{\mathrm{b}}$ is good on the ball $B_{\bar{\mathsf{R}}}(\mathsf{O})$, but in our applications we shall need an analogous of formula (3.24) with the function f replaced by the 'mollified' function $\mathsf{h}_s f$ for $s > 0$ small. To get a control of the error terms that appear we need first to build appropriate cut-off functions (the construction is the same used in Lemma 6.7 in [4] with the finite dimensionality allowing for more precise quantitative estimates):

Lemma 3.13 ('Smooth' cut-offs). *For every $r > 0$ there exists a constant $C(r) > 0$ such that the following holds. Given $K \subset \Omega$ with K compact and Ω open such that $\inf_{x\in K, y\in\Omega^c} \mathsf{d}(x,y) \geq r$, there exists a test function χ with values in $[0,1]$, which is 1 on K, with support in Ω and such that*

$$\mathrm{Lip}(\chi) + \|\Delta\chi\|_{L^\infty} \leq C(r).$$

proof We shall use the moment estimate proved in [13] (see Theorem 3)

$$W_2^2(\mathsf{H}_t\delta_x, \delta_x) \leq Nt, \qquad \forall x \in \mathrm{X},$$

where $(\mathsf{H_t})$ is the heat flow at the level of probability measures, and the Bakry-Émery estimate proved in the same reference (see Theorem 4):

$$|\mathrm{D}\mathsf{h}_t f|^2 + \frac{2t}{N}|\Delta \mathsf{h}_t f|^2 \leq \mathsf{h}_t(|\mathrm{D}f|^2), \qquad \mathfrak{m} - a.e.,$$

valid for any $f \in L^2(\mathrm{X})$.

Pick $f(x) := \max\{0, 1 - \frac{1}{r}\mathsf{d}(x,K)\}$ and notice that for every $x \in \mathrm{X}$ and $t > 0$ we have

$$\begin{aligned}|f(x) - \mathsf{h}_t f(x)| = \Big|\int f(x) - f(y)\,\mathrm{d}\mathsf{H}_t\delta_x(y)\Big| &\leq \mathrm{Lip}(f)\int \mathsf{d}(x,y)\,\mathrm{d}\mathsf{H}_t\delta_x(y)\\ &\leq \mathrm{Lip}(f) W_2(\delta_x, \mathsf{H}_t\delta_x) \leq t\frac{N}{r}.\end{aligned}$$

Thus the function $\mathsf{h}_{\frac{r}{3N}} f$ takes values in $[0, \frac{1}{3}]$ on Ω^c and in $[\frac{2}{3}, 1]$ on K.

Now let $\psi : \mathbb{R} \to [0,1]$ be C^∞, identically 0 on $(-\infty, \frac{1}{2}]$ and identically 1 on $[\frac{2}{3}, \infty)$ and define $\chi := \psi \circ \mathsf{h}_{\frac{r}{3N}} f$. It is then clear that χ is 1 on K and with support in Ω. The fact that $\chi \in \mathrm{Test}(\mathrm{X})$ follows from the chain rules for minimal weak upper gradients and Laplacians (see [17]) and the Bakry-Émery estimate grants that

$$\||\mathrm{D}\mathsf{h}_{\frac{r}{3N}} f|\|_{L^\infty} + \||\Delta \mathsf{h}_{\frac{r}{3N}} f|\|_{L^\infty} \leq c(r),$$

for some constant $c(r)$. Thus using the Sobolev-to-Lipschitz property to replace $\||\mathrm{D}\mathsf{h}_{\frac{r}{3N}} f|\|_{L^\infty}$ by $\mathrm{Lip}(\mathsf{h}_{\frac{r}{3N}} f)$ and the chain rules for the Lipschitz constant and the Laplacian we conclude. □

The cut-offs just built allow the following estimates:

Lemma 3.14 (Tail estimates). *For every $r > 0$ there is a constant $C'(r) > 0$ such that the following holds. Given $K \subset \Omega$ with K compact and Ω open such that $\inf_{x\in K, y\in\Omega^c} \mathsf{d}(x,y) \geq r$ and $f \in L^2(\mathfrak{m})$ with $\mathrm{supp}(f) \subset K$, for every $t > 0$ the quantities*

$$\begin{gathered}\int_{\Omega^c} |\mathsf{h}_t f|^2\,\mathrm{d}\mathfrak{m}, \quad \int_0^t\int_{\Omega^c} |\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s, \quad \int_0^t\int_{\Omega^c} |\mathrm{D}\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s,\\ \int_0^t\int_{\Omega^c} |\Delta\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s, \quad \int_0^t\int_{\Omega^c} |\mathrm{D}\Delta\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\end{gathered} \tag{3.25}$$

are all bounded above by $t^2 C'(r)\|f\|^2_{L^2}$.

proof In the course of the proof the value of the constant $C'(r)$ may change in its various occurrences. Let K_ε be the ε-neighbourhood of K and use repeatedly Lemma 3.13 above to find test functions $\chi_i : \mathrm{X} \to [0,1]$, $i = 0, \dots, 3$, identically 1 on $\bar K_{\frac{(i+1)r}{5}}$ and with support in $K_{\frac{(i+2)r}{5}}$. Put $\eta_i := 1 - \chi_i$. Also, notice that the quantities we have to bound depend continuously on $f \in L^2(\mathrm{X})$, thus taking into account the density property (2.6) we can, and will, assume that $f \in \mathrm{Test}(\mathrm{X})$ with support in $K_{\frac{r}{6}}$.

Start noticing that

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int \eta_1^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} = \int \eta_1^2 \mathsf{h}_s f \Delta \mathsf{h}_s f \,\mathrm{d}\mathfrak{m} = \int \eta_1^2 \big(\Delta \frac{|\mathsf{h}_s f|^2}{2} - |\mathrm{D}\mathsf{h}_s f|^2\big) \,\mathrm{d}\mathfrak{m} \le \int \eta_1^2 \Delta \frac{|\mathsf{h}_s f|^2}{2} \,\mathrm{d}\mathfrak{m},$$

and use the fact that $\eta_1^2 = 1 - 2\chi_1 + \chi_1^2$, that the integral of a Laplacian is 0 and that $\mathrm{supp}(\chi_1) \subset \{\eta_0 = 1\}$ to deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int \eta_1^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} &\le \frac{1}{2}\int (-2\Delta\chi_1 + \Delta\chi_1^2)|\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \\ &\le \| -2\Delta\chi_1 + \Delta\chi_1^2\|_{L^\infty} \frac{1}{2}\int \eta_0^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le C'(r)\frac{1}{2}\int \eta_0^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m},\end{aligned} \tag{3.26}$$

having used the identity $\Delta\chi_1^2 = 2\chi_1\Delta\chi_1 + 2|\mathrm{D}\chi_1|^2$ to get the uniform bound on $\Delta\chi_1^2$. Analogously, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int \eta_0^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le C'(r)\frac{1}{2}\int_{(K_{\frac{r}{5}})^c} |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le C'(r)\|\mathsf{h}_s f\|_{L^2}^2 \le C'(r)\|f\|_{L^2}^2$$

and since $\int \eta_0^2 |f|^2 \,\mathrm{d}\mathfrak{m} = 0$ (because η_0 and f have disjoint supports), integrating in s we obtain

$$\int \eta_0^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le sC'(r)\|f\|_{L^2}^2, \qquad \forall s \ge 0, \tag{3.27}$$

which plugged in (3.26), integrating in s and using the fact that $\int \eta_1^2 |f|^2 \,\mathrm{d}\mathfrak{m} = 0$ yields the desired control

$$\int_{\Omega^c} |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le \int \eta_1^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \le s^2\, C'(r)\|f\|_{L^2}^2, \qquad \forall s \ge 0.$$

The bound for the second quantity in (3.25) follows directly.

Now notice that integrating in s the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\int \eta_1^2 |\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} = 2\int \eta_1^2\, \mathsf{h}_s f \Delta \mathsf{h}_s f \,\mathrm{d}\mathfrak{m} = -2\int 2\eta_1 \langle \nabla\eta_1, \nabla \mathsf{h}_s f\rangle \mathsf{h}_s f + \eta_1^2 |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m}$$

and using the fact that $\int \eta_1^2 |f|^2 \,\mathrm{d}\mathfrak{m} = 0$ we deduce

$$\begin{aligned}\iint_0^t \eta_1^2 |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}s \,\mathrm{d}\mathfrak{m} &\le \iint_0^t 2\eta_1 \langle \nabla\eta_1, \nabla \mathsf{h}_s f\rangle \mathsf{h}_s f \,\mathrm{d}r \,\mathrm{d}\mathfrak{m} \\ &\le 2\sqrt{\iint_0^t \eta_1^2 |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m}}\sqrt{\iint_0^t |\mathrm{D}\eta_1|^2 |\mathsf{h}_s f|^2 \,\mathrm{d}s \,\mathrm{d}\mathfrak{m}}.\end{aligned}$$

Taking into account that $\mathrm{supp}(\eta_1) \subset \{\eta_0 = 1\}$ we conclude that

$$\iint_0^t \eta_1^2 |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}s \,\mathrm{d}\mathfrak{m} \le 4\,\mathrm{Lip}(\eta_1)^2 \iint_0^t \eta_0^2 |\mathsf{h}_s f|^2 \,\mathrm{d}s \,\mathrm{d}\mathfrak{m} \overset{(3.27)}{\le} t^2\, C'(r)\|f\|_{L^2}^2.$$

Analogous computations starting from the derivatives of $\int \eta_2^2|\mathsf{D}\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}$ and $\int \eta_3^2|\Delta\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}$ yield the bounds

$$\iint_0^t \eta_2^2|\Delta\mathsf{h}_s f|^2\,\mathrm{d}s\,\mathrm{d}\mathfrak{m} \le 4\,\mathrm{Lip}(\eta_2)^2 \iint_0^t \eta_1^2|\mathrm{D}\mathsf{h}_s f|^2\,\mathrm{d}s\,\mathrm{d}\mathfrak{m},$$
$$\iint_0^t \eta_3^2|\mathrm{D}\Delta\mathsf{h}_s f|^2\,\mathrm{d}s\,\mathrm{d}\mathfrak{m} \le 4\,\mathrm{Lip}(\eta_3)^2 \iint_0^t \eta_2^2|\Delta\mathsf{h}_s f|^2\,\mathrm{d}s\,\mathrm{d}\mathfrak{m},$$

and the conclusion follows recalling that $\Omega^c \subset \{\eta_i = 1\}$ for $i = 1, 2, 3$. □

These estimates and the Euler equation for $\bar{\mathrm{b}}$ previously obtained yield the following:

Corollary 3.15. *For every $r > 0$ there is a constant $C''(r)$ such that the following holds. For $f \in L^2(\mathrm{X})$ with $\mathrm{supp}(f) \subset B_{\bar{\mathsf{R}}-r}(\mathsf{O})$ and every $s > 0$ we have*

$$\int \langle \nabla \mathsf{h}_{2s} f, \nabla \bar{\mathrm{b}}\rangle f\,\mathrm{d}\mathfrak{m} = \int -\frac{N}{2}|\mathsf{h}_s f|^2 + 2s|\mathrm{D}\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m} + \mathrm{Rem}(f, s), \tag{3.28}$$

where the remainder term $\mathrm{Rem}(f, s)$ can be estimated as

$$|\mathrm{Rem}(f, s)| \le s^2\, C''(r)\|f\|^2_{L^2}. \tag{3.29}$$

proof It is readily verified that all the integrals appearing in (3.28) vary continuously as f varies in $L^2(\mathrm{X})$, hence recalling the density property (2.6) we can, and will, assume that $f \in \mathrm{Test}(\mathrm{X})$ with support in $B_{\bar{\mathsf{R}}-r}(\mathsf{O})$.

It is also easy to see that the function $t \mapsto \int \langle \nabla \mathsf{h}_{s+t} f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_{s-t} f\,\mathrm{d}\mathfrak{m}$ is C^1 on $[0, s]$, thus

$$\int \langle \nabla \mathsf{h}_{2s} f, \nabla \bar{\mathrm{b}}\rangle f\,\mathrm{d}\mathfrak{m} = \int \langle \nabla \mathsf{h}_s f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_s f\,\mathrm{d}\mathfrak{m} + \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \int \langle \nabla \mathsf{h}_{s+t} f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_{s-t} f\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t. \tag{3.30}$$

For the first addend on the right, notice that

$$\int \langle \nabla \mathsf{h}_s f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_s f\,\mathrm{d}\mathfrak{m} = \int \langle \nabla \tfrac{|\mathsf{h}_s f|^2}{2}, \nabla \bar{\mathrm{b}}\rangle\,\mathrm{d}\mathfrak{m} = -\frac{N}{2}\int |\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m} + \int \frac{N - \Delta\bar{\mathrm{b}}}{2}|\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}, \tag{3.31}$$

and that since $\Delta\bar{\mathrm{b}} = N$ on $B_{\bar{\mathsf{R}}}(\mathsf{O})$, by Lemma 3.14 we have

$$\left|\int \frac{N - \Delta\bar{\mathrm{b}}}{2}|\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m}\right| \le (N + \|\Delta\bar{\mathrm{b}}\|_{L^\infty}) \int_{X\setminus B_{\bar{\mathsf{R}}}(\mathsf{O})} |\mathsf{h}_s f|^2\,\mathrm{d}\mathfrak{m} \le s^2 C'(r)(N + \|\Delta\bar{\mathrm{b}}\|_{L^\infty})\|f\|^2_{L^2}.$$

For the second added in the right of (3.30), it is readily verified that the derivative can be computed passing the limit inside the integral, thus obtaining

$$\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \int \langle \nabla \mathsf{h}_{s+t} f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_{s-t} f\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \iint_0^s \langle \nabla\Delta \mathsf{h}_{s+t} f, \nabla \bar{\mathrm{b}}\rangle \mathsf{h}_{s-t} f - \langle \nabla \mathsf{h}_{s+t} f, \nabla \bar{\mathrm{b}}\rangle \Delta f_{s-t}\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}. \tag{3.32}$$

Now let χ be the cut-off given by Lemma 3.13 relative to the compact set $\bar{B}_{\bar{\mathsf{R}}-\frac{r}{2}}(\mathsf{O})$ and the open set $B_{\bar{\mathsf{R}}}(\mathsf{O})$ and notice that

$$\begin{aligned}
&\iint_0^s \langle \nabla\Delta \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle f_{s-t} - \langle \nabla \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle \Delta \mathsf{h}_{s-t} f \,\mathrm{d}t\,\mathrm{d}\mathfrak{m} \\
&\qquad = \underbrace{\iint_0^s \langle \nabla\Delta \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle \chi \mathsf{h}_{s-t} f - \langle \nabla \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle \Delta(\chi \mathsf{h}_{s-t} f) \,\mathrm{d}t\,\mathrm{d}\mathfrak{m}}_{A} \\
&\qquad\quad + \underbrace{\iint_0^s \langle \nabla\Delta \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle (1-\chi) \mathsf{h}_{s-t} f - \langle \nabla \mathsf{h}_{s+t} f, \nabla\bar{\mathrm{b}}\rangle \Delta\big((1-\chi) \mathsf{h}_{s-t} f\big) \,\mathrm{d}t\,\mathrm{d}\mathfrak{m}}_{B},
\end{aligned} \tag{3.33}$$

thus using the fact that χf_{s-t} is a test function with support in $B_{\bar{\mathsf{R}}}(\mathsf{O})$, by Proposition 3.12 we get

$$A = 2\iint_0^s -\chi \mathsf{h}_{s-t} f \Delta \mathsf{h}_{s+t} f \,\mathrm{d}t\,\mathrm{d}\mathfrak{m} = 2s\int |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} + 2\iint_0^s (1-\chi)\mathsf{h}_{s-t} f \Delta \mathsf{h}_{s+t} f \,\mathrm{d}t\,\mathrm{d}\mathfrak{m}.$$

Since $1-\chi$ is identically 0 on $B_{\bar{\mathsf{R}}-r/2}(\mathsf{O})$, by Lemma 3.14 we obtain

$$\begin{aligned}
\left| A - 2s\int |\mathrm{D}\mathsf{h}_s f|^2 \,\mathrm{d}\mathfrak{m} \right| &\le \sqrt{\int_0^{2s}\int_{B^c_{\bar{\mathsf{R}}-r/2}(\mathsf{O})} |\mathsf{h}_t f|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t}\sqrt{\int_0^{2s}\int_{B^c_{\bar{\mathsf{R}}-r/2}(\mathsf{O})} |\Delta \mathsf{h}_t f|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t} \\
&\le 4s^2 C'(r/2)\|f\|^2_{L^2}.
\end{aligned} \tag{3.34}$$

For the same reason, letting $S := \max\{1, \mathrm{Lip}(\bar{\mathrm{b}}), \mathrm{Lip}(\chi), \|\Delta\chi\|_{L^\infty}\}$ we have

$$\begin{aligned}
|B| &\le S^2 \int_0^s \int_{B^c_{\bar{\mathsf{R}}-r/2}(\mathsf{O})} |\mathsf{h}_{s-t} f||\nabla\Delta \mathsf{h}_{s+t} f| + |\nabla \mathsf{h}_{s+t} f|(|\mathsf{h}_{s-t} f| + 2|\nabla \mathsf{h}_{s-t} f| + |\Delta \mathsf{h}_{s-t} f|) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t \\
&\le 10 s^2\, S^2 C(r/2)\|f\|^2_{L^2}.
\end{aligned} \tag{3.35}$$

The conclusion comes collecting the informations in (3.30), (3.31), (3.32), (3.33), (3.34), (3.35). □

We are now ready to prove the main result of this section.

Theorem 3.16. *Let* $f \in L^2(\mathfrak{m})$ *and* $T > 0$ *be such that* $\mathrm{supp}(f) \subset B_{e^{-T}\bar{\mathsf{R}}}(\mathsf{O})$. *Then*

$$\mathcal{E}(f \circ \bar{\mathsf{Fl}}_t) = e^{(N-2)t}\mathcal{E}(f), \qquad \forall t \in [0, T].$$

proof Let $f_t := f \circ \bar{\mathsf{Fl}}_t$ and notice that since $\mathrm{supp}(f_t) \subset B_{\bar{\mathsf{R}}}(\mathsf{O})$ for every $t \in [0, T]$, from (3.5) we have $\int |f_t|^2 \,\mathrm{d}\mathfrak{m} = \int |f|^2 \,\mathrm{d}(\bar{\mathsf{Fl}}_t)_*\mathfrak{m} = e^{Nt}\int |f|^2 \,\mathrm{d}\mathfrak{m}$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |f_t|^2 \,\mathrm{d}\mathfrak{m} = \frac{N}{2}\int |f_t|^2 \,\mathrm{d}\mathfrak{m}, \qquad \forall t \in [0, T]. \tag{3.36}$$

Now pick $s > 0$ and notice that since by Lemma 3.11 we have that $t \mapsto f_t \in L^2(\mathrm{X})$ is Lipschitz, we also have that $t \mapsto \mathsf{h}_s f_t \in L^2(\mathrm{X})$ is Lipschitz. Then for a.e. $t \in [0,T]$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |\mathsf{h}_s f_t|^2 \,\mathrm{d}\mathfrak{m} &= \lim_{h\to 0}\int \mathsf{h}_s f_t \frac{\mathsf{h}_s(f_t\circ \bar{\mathsf{Fl}}_h) - \mathsf{h}_s f_t}{h}\,\mathrm{d}\mathfrak{m} \\ &= \lim_{h\to 0}\int \mathsf{h}_{2s} f_t \frac{f_t\circ \bar{\mathsf{Fl}}_h - f_t}{h}\,\mathrm{d}\mathfrak{m} \\ &= \lim_{h\to 0}\int \frac{e^{Nh}\mathsf{h}_{2s} f_t\circ \bar{\mathsf{Fl}}_h^{-1} - \mathsf{h}_{2s} f_t}{h} f_t\,\mathrm{d}\mathfrak{m} \\ &= \int (N\mathsf{h}_{2s} f_t + \langle \nabla \mathsf{h}_{2s} f_t, \nabla \bar{\mathrm{b}}\rangle) f_t\,\mathrm{d}\mathfrak{m} \\ &= \int N|\mathsf{h}_s f_t|^2 + \langle \nabla \mathsf{h}_{2s} f_t, \nabla \bar{\mathrm{b}}\rangle f_t\,\mathrm{d}\mathfrak{m},\end{aligned}$$

having used Lemma 3.11 in the penultimate step. Thus from Corollary 3.15 we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int |\mathsf{h}_s f_t|^2\,\mathrm{d}\mathfrak{m} = \int \frac{N}{2}|\mathsf{h}_s f_t|^2 + 2s|\mathrm{D}\mathsf{h}_s f_t|^2\,\mathrm{d}\mathfrak{m} + \mathrm{Rem}(f_t, s), \tag{3.37}$$

with $\mathrm{Rem}(f_t,s)$ satisfying the bound (3.29).

Consider the quantity

$$G(t,s) := \int \frac{|f_t|^2 - |\mathsf{h}_s f_t|^2}{4s}\,\mathrm{d}\mathfrak{m},$$

and notice that (3.36) and (3.37) give that for any $s > 0$ the map $t \mapsto G(t,s)$ is Lipschitz with

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t,s) = NG(t,s) - 2\mathcal{E}(\mathsf{h}_s f_t) + \frac{\mathrm{Rem}(f_t,s)}{s}. \tag{3.38}$$

Now assume for the moment that $f \in W^{1,2}(\mathrm{X})$, let $c := \sup_{s\in(0,1)} \frac{\mathrm{Rem}(f_t,s)}{s} < \infty$, use the trivial bound $G(0,s) \le \mathcal{E}(f_0)$ to deduce from the last identity that

$$G(t,s) \le \mathcal{E}(f_0) + cT + N\int_0^t G(r,s)\,\mathrm{d}r, \qquad \forall t \in [0,T],\ s \in (0,1).$$

By the Gronwall inequality in the integral form we get the uniform bound

$$G(t,s) \le \big(\mathcal{E}(f_0) + cT\big)e^{Nt}, \qquad \forall t \in [0,T],\ \forall s \in (0,1).$$

Noticing that $G(t,s) \uparrow \mathcal{E}(f_t)$ as $s \downarrow 0$, this last bound ensures that $\mathcal{E}(f_t)$ is uniformly bounded in $t \in [0,T]$. We can therefore pass to the limit as $s \downarrow 0$ in (3.38) noticing that the right hand side is uniformly bounded and pointwise converges to $(N-2)\mathcal{E}(f_t)$ to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f_t) = (N-2)\mathcal{E}(f_t), \qquad a.e.\ t \in [0,T],$$

and the conclusion follows.

To remove the assumption that $f \in W^{1,2}(\mathrm{X})$ it is sufficient to show that if $f_T \in W^{1,2}(\mathrm{X})$ then $f \in W^{1,2}(\mathrm{X})$ as well: this follows from the very same arguments just used replacing $\bar{\mathsf{Fl}}_t$ with its inverse $\bar{\mathsf{Fl}}_t^{-1} = \bar{\mathsf{Fl}}_{-t}$. □

The statement of the previous theorem can be easily 'localized' thanks to the chain and Leibniz rules (2.4). Notice that in the following Corollary we will also come back to the original flow (Fl_t) in place of the regularized one $(\bar{\mathsf{Fl}}_t)$.

Corollary 3.17. *Let $f \in L^2(\mathfrak{m})$ and $T > 0$ be with* $\mathrm{supp}(f) \subset B_{e^{-T}\mathsf{R}}(\mathsf{O})$*. Then $f \in W^{1,2}(\mathrm{X})$ if and only if $f \circ \mathsf{Fl}_T \in W^{1,2}(\mathrm{X})$ and in this case*

$$|\mathrm{D}(f \circ \mathsf{Fl}_T)| = e^{-T}|\mathrm{D}f| \circ \mathsf{Fl}_T, \qquad \mathfrak{m} - a.e.$$

proof Fix f, T as in the assumptions and notice that by compactness there exists $\bar{\mathsf{R}} < \mathsf{R}$ such that $\mathrm{supp}(f) \subset B_{e^{-T}\bar{\mathsf{R}}}(\mathsf{O})$. Choosing such $\bar{\mathsf{R}}$ and building a corresponding function $\bar{\mathrm{b}}$ and its gradient flow $\bar{\mathsf{Fl}}$, we see that $f \circ \mathsf{Fl}_T = f \circ \bar{\mathsf{Fl}}_t$ $\mathfrak{m}$-a.e. and thus by Theorem 3.16 above we deduce that $f \in W^{1,2}(\mathrm{X})$ if and only if $f \circ \mathsf{Fl}_T \in W^{1,2}(\mathrm{X})$.

Now assume that $f \in W^{1,2}(\mathrm{X})$ and notice that by the locality property (2.1) we also have $|\mathrm{D}(f \circ \mathsf{Fl}_T)| = |\mathrm{D}(f \circ \bar{\mathsf{Fl}}_T)|$ $\mathfrak{m}$-a.e. so that our conclusion becomes

$$|\mathrm{D}(f \circ \bar{\mathsf{Fl}}_T)| = e^{-T}|\mathrm{D}f| \circ \bar{\mathsf{Fl}}_T, \qquad \mathfrak{m} - a.e.$$

and with an approximation argument based on the density of $L^\infty \cap W^{1,2}(\mathrm{X})$ in $W^{1,2}(\mathrm{X})$ (easy to establish from the definitions), we can, and will, assume that $f \in L^\infty \cap W^{1,2}(\mathrm{X})$.

Observe that for any two functions $f_1, f_2 \in W^{1,2}(\mathrm{X})$ with $\mathrm{supp}(f_i) \subset B_{e^{-T}\bar{\mathsf{R}}}(\mathsf{O})$, Theorem 3.16 gives, by polarization, that

$$\int \langle \nabla(f_1 \circ \bar{\mathsf{Fl}}_T), \nabla(f_2 \circ \bar{\mathsf{Fl}}_T)\rangle \,\mathrm{d}\mathfrak{m} = e^{(N-2)T} \int \langle \nabla f_1, \nabla f_2\rangle \,\mathrm{d}\mathfrak{m},$$

then pick an arbitrary Lipschitz function g with support in $B_{e^{-T}\bar{\mathsf{R}}}(\mathsf{O})$, notice that f, g, f^2, fg are all in $W^{1,2}(\mathrm{X})$ with support in $B_{e^{-T}\bar{\mathsf{R}}}(\mathsf{O})$, put for brevity $f_T := f \circ \bar{\mathsf{Fl}}_T$, $g_T := g \circ \bar{\mathsf{Fl}}_T$ and use this last identity to get

$$\begin{aligned} \int |\mathrm{D}f_T|^2 g_T \,\mathrm{d}\mathfrak{m} &= \int \langle \nabla(f_T g_T), \nabla f_T\rangle - \langle \nabla \tfrac{f_T^2}{2}, \nabla g_T\rangle \mathrm{d}\mathfrak{m} \\ &= e^{(N-2)T} \int \langle \nabla(fg), \nabla f\rangle - \langle \nabla \tfrac{f^2}{2}, \nabla g\rangle \mathrm{d}\mathfrak{m} = e^{(N-2)T} \int |\mathrm{D}f|^2 g \,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Since by (3.5) we have $\int |\mathrm{D}f_T|^2 g_T \,\mathrm{d}\mathfrak{m} = e^{NT} \int |\mathrm{D}f_T|^2 \circ \bar{\mathsf{Fl}}_T^{-1} g \,\mathrm{d}\mathfrak{m}$, we conclude that

$$\int |\mathrm{D}f_T|^2 \circ \bar{\mathsf{Fl}}_T^{-1} g \,\mathrm{d}\mathfrak{m} = e^{-2T} \int |\mathrm{D}f|^2 g \,\mathrm{d}\mathfrak{m},$$

which by the arbitrariness of g is the conclusion. □

3.3 Gradient flow of b: precise representative and first metric informations

We shall now use the Sobolev-to-Lipschitz property of X to obtain information about the behaviour of the metric under the flow (Fl_t):

Theorem 3.18. *The map Fl, seen as a map from $\mathbb{R}^+ \times B_{\mathsf{R}}(\mathsf{O})$ to $B_{\mathsf{R}}(\mathsf{O})$ admits a continuous representative w.r.t. the measure $(\mathcal{L}^1 \times \mathfrak{m})|_{\mathbb{R}^+ \times B_{\mathsf{R}}(\mathsf{O})}$. Still denoting such representative by Fl, we have:*

i) for every $t, s \in \mathbb{R}^+$ and $x \in B_{\mathsf{R}}(\mathsf{O})$ it holds

$$\begin{aligned} \mathsf{Fl}_t(\mathsf{Fl}_s(x)) &= \mathsf{Fl}_{t+s}(x), \\ \mathsf{d}(\mathsf{Fl}_s(x), \mathsf{Fl}_t(x)) &= |e^{-s} - e^{-t}|\mathsf{d}(x, \mathsf{O}). \end{aligned} \tag{3.39}$$

ii) for every $t \in \mathbb{R}^+$, Fl_t is an invertible locally Lipschitz map from $B_{\mathsf{R}}(\mathsf{O})$ to $B_{e^{-t}\mathsf{R}}(\mathsf{O})$ whose inverse is also locally Lipschitz. Moreover, given a curve γ with values in $B_{\mathsf{R}}(\mathsf{O})$, putting $\tilde{\gamma} := \mathsf{Fl}_t \circ \gamma$ we have

$$|\dot{\tilde{\gamma}}_s| = e^{-t}|\dot{\gamma}_s| \qquad \textit{for a.e. } s \in [0, 1], \tag{3.40}$$

meaning that one of the curves is absolutely continuous if and only if the other is and in this case their metric speeds are related by the stated identity.

proof Fix $t \in \mathbb{R}^+$ and notice that by construction the image of $B_{\mathsf{R}}(\mathsf{O}) \setminus \mathcal{N}$ under Fl_t is contained in $B_{e^{-t}\mathsf{R}}(\mathsf{O})$. Fix $x_0 \in B_{e^{-t}\mathsf{R}}(\mathsf{O})$, let $r > 0$ be such that $B_{3r}(x_0) \subset B_{e^{-t}\mathsf{R}}(\mathsf{O})$ and let $\mathcal{D}$ be a countable set of 1-Lipschitz functions, all with support in $B_{3r}(x_0)$ dense in the space of 1-Lipschitz functions with support in $B_{3r}(x_0)$ w.r.t. uniform convergence. It is then clear that

$$\mathsf{d}(y_0, y_1) = \sup_{f \in \mathcal{D}} |f(y_1) - f(y_0)|, \qquad \forall y_0, y_1 \in B_r(x_0).$$

Pick $f \in \mathcal{D}$ and apply Corollary 3.17 to deduce that $f \circ \mathsf{Fl}_t \in W^{1,2}(\mathrm{X})$ with

$$|\mathrm{D}(f \circ \mathsf{Fl}_t)| = e^{-t}|\mathrm{D}f| \circ \mathsf{Fl}_t \le e^{-t}, \qquad \mathfrak{m} - a.e..$$

Since X has the Sobolev-to-Lipschitz property, $f \circ \mathsf{Fl}_t$ has a e^{-t}-Lipschitz representative and since $\mathcal{D}$ is countable, we deduce that there exists a $\mathfrak{m}$-negligible Borel set $\mathcal{N}'$ such that the restriction of $f \circ \mathsf{Fl}_t$ to $\mathrm{X} \setminus (\mathcal{N} \cup \mathcal{N}')$ is e^{-t}-Lipschitz for every $f \in \mathcal{D}$.

Therefore for $x_0, x_1 \in \mathsf{Fl}_t^{-1}(B_r(x_0)) \setminus (\mathcal{N} \cup \mathcal{N}')$ we have

$$\mathsf{d}\big(\mathsf{Fl}_t(x_0), \mathsf{Fl}_t(x_1)\big) = \sup_{f \in \mathcal{D}} \big|f(\mathsf{Fl}_t(x_0)) - f(\mathsf{Fl}_t(x_1))\big| \le e^{-t}\mathsf{d}(x_0, x_1),$$

showing that Fl_t has a e^{-t}-Lipschitz representative on the preimage of $B_r(x_0)$. Then the arbitrariness of x_0, the Lindelof property of $B_{e^{-t}\mathsf{R}}(\mathsf{O})$ and the essential surjectivity of Fl_t ensure that $\mathsf{Fl}_t : B_{\mathsf{R}}(\mathsf{O}) \to B_{e^{-t}\mathsf{R}}(\mathsf{O})$ has a representative which is locally e^{-t}-Lipschitz and from now on we shall identify Fl_t with such representative.

It then follows from the arbitrariness of $t \in \mathbb{R}^+$ and the uniform continuity in t granted by the second identity in (3.6) that Fl, seen as a map from $\mathbb{R}^+ \times B_{\mathsf{R}}(\mathsf{O})$ to $B_{\mathsf{R}}(\mathsf{O})$, admits a continuous representative w.r.t. the measure $(\mathcal{L}^1 \times \mathfrak{m})|_{\mathbb{R}^+ \times B_{\mathsf{R}}(\mathsf{O})}$, as claimed.

The identities (3.39) then follow directly from (3.6) recalling that $(\mathsf{Fl}_t)_*\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})} \ll \mathfrak{m}$.

Inequality $\le$ in (3.40) is a direct consequence of the fact that Fl_t is locally e^{-t}-Lipschitz. To conclude it is therefore sufficient to prove that $\mathsf{Fl}_t^{-1} : B_{e^{-t}\mathsf{R}}(\mathsf{O}) \to B_{\mathsf{R}}(\mathsf{O})$, a priori well defined only $\mathfrak{m}$-a.e. (recall (ii) of Corollary 3.4) has a representative which is locally e^t-Lipschitz, as then it is clear that such representative is the inverse of the continuous representative of Fl_t. Such property of Fl_t^{-1} can be proved by the very same means used to prove the local e^{-t}-Lipschitzianity of Fl_t. Just notice that if f is a 1-Lipschitz function with support in $B_{\mathsf{R}}(\mathsf{O})$, then $f \circ \mathsf{Fl}_t^{-1}$ has support in $B_{e^{-t}\mathsf{R}}(\mathsf{O})$ and, by Corollary 3.17, it belongs to $W^{1,2}(\mathrm{X})$ with $|\mathrm{D}(f \circ \mathsf{Fl}_t^{-1})| = |\mathrm{D}f| \circ \mathsf{Fl}_t^{-1} \le e^t$ $\mathfrak{m}$-a.e.. The conclusion then follows arguing as above. $\square$

From now on, when considering the maps Fl_t on $B_{\mathsf{R}}(\mathsf{O})$ we shall always refer to their continuous versions.

Furthermore, for $t < 0$ and $x \in \mathrm{X}$ such that $x \in B_{e^t\mathsf{R}}(\mathsf{O})$, we put

$$\mathsf{Fl}_t(x) := \mathsf{Fl}_{-t}^{-1}(x).$$

3.4 Basic properties of the sphere $S_{\mathsf{R}/2}(\mathsf{O})$

We consider the sphere

$$S_{\mathsf{R}/2}(\mathsf{O}) := \{x \in \mathrm{X} \ : \ \mathsf{d}(x, \mathsf{O}) = \tfrac{\mathsf{R}}{2}\}$$

and the projection map $\mathsf{Pr} : B_{\mathsf{R}}(\mathsf{O}) \setminus \{\mathsf{O}\} \to S_{\mathsf{R}/2}(\mathsf{O})$ given by

$$\mathsf{Pr}(x) := \mathsf{Fl}_{\log(\frac{2\mathsf{d}(x,\mathsf{O})}{\mathsf{R}})}(x).$$

Notice that by Theorem 3.18 the map Pr is well defined and locally Lipschitz.

Proposition 3.19. *$S_{\mathsf{R}/2}(\mathsf{O})$ is either one point, or two points or a Lipschitz-path connected subset of* X, *i.e. a subset such that for every $x, y \in S_{\mathsf{R}/2}(\mathsf{O})$ there is a Lipschitz curve with values in $S_{\mathsf{R}/2}(\mathsf{O})$ connecting x to y.*

proof It will be convenient to work with the sphere $S_{\mathsf{R}/4}(\mathsf{O})$: notice that by Theorem 3.18 it has the same cardinality of $S_{\mathsf{R}/2}(\mathsf{O})$ and it is Lipschitz path connected if and only if $S_{\mathsf{R}/2}(\mathsf{O})$ is.

Thus we shall assume that $S_{\mathsf{R}/4}(\mathsf{O})$ contains at least 3 points x_1, x_2, x_3 and notice that to conclude it is sufficient to show that there are Lipschitz curves contained in $S_{\mathsf{R}/4}(\mathsf{O})$ connecting these. We argue by contradiction and assume for a moment that there are no Lipschitz curves whose image is contained in $S_{\mathsf{R}/4}(\mathsf{O})$ joining x_1 to x_2 and similarly no such curve joining x_1 to x_3.

Let $x \in B_{\mathsf{R}/4}(x_1)$ and notice that any given geodesic from x_1 to x does not pass through O and that the triangle inequality ensures that any given geodesic connecting x to x_2 must stay in $B_{\mathsf{R}}(\mathsf{O})$. Concatenate these two geodesics and assume that the resulting curve does not pass through O. Then composing it with the locally Lipschitz map $\mathsf{Fl}_{\log 2} \circ \mathsf{Pr}$ we would obtain a Lipschitz curve from x_1 to x_2 lying entirely on $S_{\mathsf{R}/4}(\mathsf{O})$, which contradicts our assumption. Thus the concatenation passes through O, which forces the geodesic from x to x_2 to pass through O. Hence $\mathsf{d}(x, x_2) = \mathsf{d}(x, \mathsf{O}) + \frac{\mathsf{R}}{4}$ and arguing symmetrically we deduce that

$$\mathsf{d}(x, x_2) = \mathsf{d}(x, x_3), \qquad \forall x \in B_{\mathsf{R}/4}(x_1). \tag{3.41}$$

Now consider the probability measures

$$\mu_0 := \frac{1}{\mathfrak{m}(B_{\mathsf{R}/4}(x_1))}\mathfrak{m}|_{B_{\mathsf{R}/4}(x_1)} \qquad\qquad \mu_1 := \frac{1}{2}\big(\delta_{x_2} + \delta_{x_3}\big)$$

and notice that the identity (3.41) gives that every admissible transport plan between them is optimal. In particular, this is the case for the plan $\mu_0 \times \mu_1$. But being it not induced by a map, we found a contradiction with the fact that optimal plans on $\mathsf{RCD}^*(0, N)$ spaces must be induced by maps (see [21]).

Hence there is a Lipschitz curve with image contained in $S_{\mathsf{R}/4}(\mathsf{O})$ connecting x_1 to either x_2 or x_3, say x_2. Repeating the argument swapping the roles of x_1 and x_3 we conclude. □

Corollary 3.20 (Conclusion in the easy cases)**.** *The following holds.*

i) Assume that $S_{\mathsf{R}/2}(\mathsf{O})$ consists of one point. Then (X,d) is isometric to $[0,\mathrm{diam}(\mathrm{X})]$ ($[0,\infty)$ if X is unbounded) with an isometry which sends O in 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}}(\mathsf{O})$ to the measure $c\,x^{N-1}\mathrm{d}x$ for $c := N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$.

ii) Assume that $S_{\mathsf{R}/2}(\mathsf{O})$ consists of two points. Then (X,d) is a 1-dimensional Riemannian manifold, possibly with boundary, and there is a bijective local isometry (in the sense of distance-preserving maps) from $B_{\mathsf{R}}(\mathsf{O})$ to $(-\mathsf{R},\mathsf{R})$ sending O to 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ to the measure $c\,|x|^{N-1}\mathrm{d}x$ for $c := \frac{1}{2}N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$. Moreover, such local isometry is an isometry when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$.

proof
(i) By Theorem 3.18 we know that Fl_t is a bijection of $S_r(\mathsf{O})$ into $S_{e^{-t}r}(\mathsf{O})$ for every $r < \mathsf{R}$, thus the assumption grants that for every $r < \mathsf{R}$ there is exactly one point at distance r from O. This shows that $B_{\mathsf{R}}(\mathsf{O})$ is isometric to $[0,\mathsf{R})$. The claim about the measure follows from Corollary 3.8. To conclude that the isometry can be extended to the whole X it is sufficient to show that the map $x \mapsto \mathsf{d}(x,\mathsf{O})$ is injective. This follows from the very same argument by contradiction based on existence and uniqueness of optimal maps used in Proposition 3.19 above, using the fact that the restriction of $\mathsf{d}(\cdot,\mathsf{O})$ to $B_{\mathsf{R}}(\mathsf{O})$ is injective.
(ii) Let x_L, x_R be the two points in $S_{\mathsf{R}/2}(\mathsf{O})$ and let $L := \mathsf{Pr}^{-1}(x_L)$, $R := \mathsf{Pr}^{-1}(x_R)$. Then $\{L,R\}$ is a partition of $B_{\mathsf{R}}(\mathsf{O})\setminus\{\mathsf{O}\}$ and the fact that X is geodesic ensures that both L and R are isometric to $(0,\mathsf{R})$. Now notice that a curve joining $x_0 \in B_{\mathsf{R}/2}(\mathsf{O})\cap L$ to $x_1 \in B_{\mathsf{R}/2}(\mathsf{O})\cap R$ either passes through O or through a point at distance R from O. Given that the length of a curve of the second kind is at least R, the metric claims all follow. The claim about the measure follows instead from Corollary 3.8 as before.

Finally let us prove that (X,d) is a 1-dimensional manifold, possibly with boundary. For let us define

$$L' = \big\{x \in X \setminus \{\mathsf{O}\} \text{ such that } \gamma \cap B_{\mathsf{R}}(\mathsf{O}) \subset L \text{ for every geodesic } \gamma \text{ between } x \text{ and } \mathsf{O}\big\}$$

and

$$R' = \big\{x \in X \setminus \{\mathsf{O}\} \text{ such that } \gamma \cap B_{\mathsf{R}}(\mathsf{O}) \subset R \text{ for every geodesic } \gamma \text{ between } x \text{ and } \mathsf{O}\big\}.$$

We claim that $\mathsf{d}(\cdot,O) : L' \to \mathbb{R}^+$ is injective. To prove this start observing that for $x' \in L' \setminus L = L' \setminus B_{\mathsf{R}}(\mathsf{O})$, any geodesic γ from x' to O must satisfy $\gamma \supset L$. It follows that $\mathsf{d}(x,x') = \mathsf{d}(x',\mathsf{O}) - \mathsf{d}(x,\mathsf{O})$ for any $x' \in L'$ and $x \in L$, thus if there where $x_1, x_2 \in L' \setminus L$ with $\mathsf{d}(x_1,\mathsf{O}) = \mathsf{d}(x_2,\mathsf{O})$ we would have, much like in Proposition 3.19, that any transport plan between the measures

$$\mu_0 := \mathfrak{m}(L)^{-1}\mathfrak{m}|_L, \qquad \mu_1 := \frac{1}{2}(\delta_{x_1} + \delta_{x_2}),$$

would be optimal and thus induced by a map (see [21]), contradicting the fact that $\mu_0 \times \mu_1$ is not induced by a map.

Thus $\mathsf{d}(\cdot,O) : L' \to \mathbb{R}^+$ is indeed injective and it is then clear that it is also an isometry. The same holds for R', hence the conclusion follows by elementary topology if we show that $\mathrm{X} = \overline{L'} \cup \overline{R'}$. This is the same as to say that the open set $\Omega := \mathrm{X} \setminus (\overline{L'} \cup \overline{R'})$ has measure 0. By definition of L and R, we know that for every $x \in \Omega$ there is a geodesic from x to O passing through x_L and another passing through x_R. Hence $\mathsf{d}(x, x_L) = \mathsf{d}(x, x_R)$ for any $x \in \Omega$ and if $\mathfrak{m}(\Omega) > 0$ we can consider the measures

$$\mu_0 := \mathfrak{m}(\Omega)^{-1}\mathfrak{m}_{|_\Omega}, \qquad \mu_1 := \frac{1}{2}(\delta_{x_L} + \delta_{x_R}),$$

and obtain a contradiction as before noticing that the plan $\mu_0 \times \mu_1$ would be optimal and not induced by a map. □

From now on, we shall always assume that $S_{\mathsf{R}}(\mathsf{O})$ contains at least 3 points.

3.5 The sphere equipped with the induced distance and measure

Definition 3.21. *We put* $\mathrm{X}' := S_{\mathsf{R}/2}(\mathsf{O})$*. For* $x', y' \in \mathrm{X}'$ *we define* $\mathsf{d}'(x', y')$ *as*

$$\mathsf{d}'(x', y')^2 := \inf \int_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t,$$

where the infimum is taken among all Lipschitz curves $\gamma : [0,1] \to \mathrm{X}' \subset \mathrm{X}$ *and the metric speed is computed w.r.t. the distance* d*.*

The measure $\mathfrak{m}'$ *on* X' *is defined as*

$$\mathfrak{m}' := \mathfrak{m}_{\mathsf{R}/2},$$

where $\mathfrak{m}_{\mathsf{R}/2}$ *is obtained disintegrating* $\mathfrak{m}$ *along* $\mathsf{d}(\cdot, \mathsf{O})$ *(recall Corollary 3.8).*

Notice that the definition directly ensures that on X' we have $\mathsf{d}' \geq \mathsf{d}$, because d is a geodesic distance and d' is obtained by minimizing the energy

Proposition 3.19 and the fact that we assumed $S_{\mathsf{R}/2}(\mathsf{O})$ to contain at least 3 points grant that d' is finite and it is then easy to see that it is indeed a distance on X' inducing the same topology coming from the inclusion $\mathrm{X}' \subset \mathrm{X}$. In particular, $\mathfrak{m}'$ is a Borel measure on $(\mathrm{X}', \mathsf{d}')$. It is also clear from (3.10) that

$$\mathsf{Pr}_*(\mathfrak{m}_{|_{B_{\mathsf{R}}(\mathsf{O})}}) = \mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))\mathfrak{m}'. \tag{3.42}$$

Moreover, since for any d-Lipschitz curve γ with values in X' we have - as it is easy to check - that $\mathsf{d}(\gamma_t, \gamma_s) \leq \mathsf{d}'(\gamma_t, \gamma_s) \leq \int_t^s |\dot{\gamma}_r| \,\mathrm{d}r$ for any $t < s$, $t, s \in [0,1]$, we deduce that

a curve γ with values in X' is absolutely continuous w.r.t. d' if and only if it is so w.r.t. d and in this case the metric speeds computed w.r.t. the two distances are the same. (3.43)

At this stage of the paper we begin considering Sobolev functions on different spaces. Although a priori there can be no confusion, for better clarity we shall denote the minimal weak upper gradient of the Sobolev function f defined, say, on the space X by $|\mathrm{D}f|_{\mathrm{X}}$ rather than by $|\mathrm{D}f|$.

Another notation that we introduce is $\mathrm{ms}_t(\gamma)$ for the metric speed $|\dot{\gamma}_t|$ of the absolutely continuous curve γ at time t.

With that said, the following link between Sobolev functions on X and on X' is easily established:

Proposition 3.22. *Let* $[a,b] \subset (0,\mathsf{R})$*,* $h \in \mathrm{Lip}(\mathbb{R})$ *with support in* $(0,\mathsf{R})$ *and identically 1 on* $[a,b]$ *and* $f \in L^2(\mathrm{X})$ *of the form* $f(x) = g(\mathsf{Pr}(x))h(\mathsf{d}(x,\mathsf{O}))$ *for some* $g \in L^2(\mathfrak{m}')$*.*

Assume that $f \in W^{1,2}(\mathrm{X})$*. Then* $g \in W^{1,2}(\mathrm{X}')$ *and*

$$|\mathrm{D}f|_{\mathrm{X}}(x) \geq \frac{\mathsf{R}}{2\mathsf{d}(x,\mathsf{O})}|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(x)), \qquad \text{for } \mathfrak{m}\text{-a.e. } x \text{ such that } \mathsf{d}(x,\mathsf{O}) \in [a,b]. \tag{3.44}$$

proof Fix $f \in W^{1,2}(\mathrm{X})$, let $\boldsymbol{\pi}'$ be a test plan on X' and pick $[a',b'] \subset [a,b]$ with $a' < b'$. Consider the map $P : \mathrm{X}' \times [a',b'] \to \mathrm{X}$ given by $P(x,d) := \mathsf{Fl}_{\log(\frac{\mathsf{R}}{2d})}(x)$, the induced map $\hat{P} : C([0,1],\mathrm{X}') \times [a',b'] \to C([0,1],\mathrm{X})$ defined as $\hat{P}(\gamma,d)_t := P(\gamma_t,d)$ and consider the plan

$$\boldsymbol{\pi} := \hat{P}_*(\boldsymbol{\pi}' \times (|b'-a'|^{-1}\mathcal{L}^1|_{[a',b']})) \in \mathscr{P}(C([0,1],\mathrm{X})).$$

Since $\mathsf{d}' \geq \mathsf{d}$ on X', we see from the fact that Fl_t is locally Lipschitz ((*ii*) in Theorem 3.18) and the compactness of $\mathrm{X}' \times [a',b']$ that P is Lipschitz, thus since $\boldsymbol{\pi}'$ has finite kinetic energy, we conclude that $\boldsymbol{\pi}$ has also finite kinetic energy. Moreover, from (3.8), the definition of $\mathfrak{m}'$ and the fact that $\boldsymbol{\pi}'$ has bounded compression, we deduce that $\boldsymbol{\pi}$ has also bounded compression. In summary: $\boldsymbol{\pi}$ is a test plan on X.

Notice that by construction, for $\boldsymbol{\pi}$-a.e. γ we have $f(\gamma_t) = g(\mathsf{Pr}(\gamma_t))$ and that from (3.43) and (3.40) we see that $\mathrm{ms}_t(\hat{P}(\gamma,d)) = \frac{2d}{\mathsf{R}}|\dot{\gamma}_t|$ for a.e. t. Then we have:

$$\begin{aligned}
\int |g(\gamma_1) - g(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}'(\gamma) &= \int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\leq \iint_0^1 |\mathrm{D}f|_{\mathrm{X}}(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&= \frac{1}{b'-a'}\iint_0^1\int_{a'}^{b'} |\mathrm{D}f|_{\mathrm{X}}(P(\gamma,d)_t)\mathrm{ms}_t(P(\gamma,d))\,\mathrm{d}d\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'(\gamma)\\
&= \iint_0^1 \left(\frac{1}{b'-a'}\int_{a'}^{b'} \frac{2d}{\mathsf{R}}|\mathrm{D}f|_{\mathrm{X}}(P(\gamma,d)_t)\,\mathrm{d}d\right)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'(\gamma),
\end{aligned}$$

which by the arbitrariness of $\boldsymbol{\pi}'$ shows that $g \in W^{1,2}(\mathrm{X}')$ and

$$|\mathrm{D}g|_{\mathrm{X}'}(x') \leq \frac{1}{b'-a'}\int_{a'}^{b'} \frac{2d}{\mathsf{R}}|\mathrm{D}f|(\mathsf{Fl}_{\log(\frac{\mathsf{R}}{2d})}(x'))\,\mathrm{d}d, \qquad \mathfrak{m}' - a.e.\ x'.$$

Then the arbitrariness of a',b' yields (3.44). □

In fact, also the inequality opposite of (3.44) holds, the proof being based on the following proposition:

Proposition 3.23. *Let* $[a,b] \subset (0,\mathsf{R})$ *and* $\boldsymbol{\pi}$ *be a test plan on* X *such that* $\mathsf{d}(\gamma_t,\mathsf{O}) \in [a,b]$ *for every* $t \in [0,1]$ *and* $\boldsymbol{\pi}$*-a.e.* γ*. Then for* $\boldsymbol{\pi}$*-a.e.* γ *the curve* $\tilde{\gamma} := \mathsf{Pr} \circ \gamma$ *is absolutely continuous and satisfies*

$$|\dot{\tilde{\gamma}}_t| \leq \frac{\mathsf{R}}{2\mathsf{d}(\gamma_t,\mathsf{O})}|\dot{\gamma}_t|, \qquad a.e.\ t.$$

The proof of this proposition is technically quite involved, as it heavily relies on the first and second order differential calculus recently developed in [15]. We postpone it to the next section, see Proposition 3.33, where all the necessary ingredients will be recalled and discussed. Here we show how to use this proposition to get the equality in (3.45) and then other basic informations about the structure of $(\mathrm{X}',\mathsf{d}',\mathfrak{m}')$:

Theorem 3.24. *Let $[a,b]\subset(0,\mathsf{R})$, $h\in\mathrm{Lip}(\mathbb{R})$ with support in $(0,\mathsf{R})$ and identically 1 on $[a,b]$ and $f\in L^2(\mathrm{X})$ of the form $f(x)=g(\mathsf{Pr}(x))h(\mathsf{d}(x,\mathsf{O}))$ for some $g\in L^2(\mathfrak{m}')$.*

Then $f\in W^{1,2}(\mathrm{X})$ if and only if $g\in W^{1,2}(\mathrm{X}')$ and in this case we have

$$|\mathrm{D}f|_{\mathrm{X}}(x)=\frac{\mathsf{R}}{2\mathsf{d}(x,\mathsf{O})}|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(x)),\qquad \text{for } \mathfrak{m}\text{-a.e. } x \text{ such that } \mathsf{d}(x,\mathsf{O})\in[a,b]. \tag{3.45}$$

proof The 'only if' and the inequality $\geq$ are the content of Proposition 3.22, so we turn to the 'if' and the inequality $\leq$.

Let $[a',b']\subset(0,\mathsf{R})$ be such that $\mathrm{supp}(h)\subset(a',b')$ and notice that by construction we have $\mathrm{supp}(f)\subset B_{b'}(\mathsf{O})\setminus B_{a'}(\mathsf{O})$, thus arguing as in the proof of Theorem 4.19 in [2] to conclude it is sufficient to check the weak upper gradient property for test plans $\boldsymbol{\pi}$ such that $\mathsf{d}(\gamma_t,O)\in[a',b']$ for every $t\in[0,1]$ and $\gamma\in\mathrm{supp}(\boldsymbol{\pi})$.

Fix such $\boldsymbol{\pi}$, let $G:\mathrm{X}\to\mathbb{R}$ be given by

$$G(x):=\frac{\mathsf{R}}{2\mathsf{d}(x,\mathsf{O})}|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(x))h(\mathsf{d}(x,\mathsf{O}))+g(\mathsf{Pr}(x))|h'|(\mathsf{d}(x,\mathsf{O})),$$

and notice that G is in $L^2(\mathfrak{m})$ and equal to $\frac{\mathsf{R}}{2\mathsf{d}(x,\mathsf{O})}|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(x))$ for x such that $\mathsf{d}(x,\mathsf{O})\in[a,b]$.

Therefore, taking into account Proposition 2.1, to conclude it is sufficient to prove that for $\boldsymbol{\pi}$-a.e. γ the function $t\mapsto f(\gamma_t)$ is equal a.e. on $[0,1]$ and in $\{0,1\}$ to an absolutely continuous map f_γ such that

$$|\partial_t f_\gamma|(t)\leq G(\gamma_t)|\dot\gamma_t|,\qquad a.e.\ t\in[0,1]. \tag{3.46}$$

Notice that $\mathsf{Pr}:B_{b'}(\mathsf{O})\setminus B_{a'}(\mathsf{O})\to S_{\mathsf{R}/2}(\mathsf{O})$ is Lipschitz, thus recalling (3.42) we deduce that the plan $\boldsymbol{\pi}':=\mathsf{Pr}_*\boldsymbol{\pi}$ is a test plan on X' (here we are abusing a bit the notation as we are interpreting Pr as the map from $C([0,1],\mathrm{X})$ to $C([0,1],\mathrm{X}')$ sending γ to $\mathsf{Pr}\circ\gamma$).

Since $g\in W^{1,2}(\mathrm{X}')$, by Proposition 2.1 we deduce that for $\boldsymbol{\pi}$-a.e. γ we have that the map $t\mapsto g(\mathsf{Pr}(\gamma_t))$ is equal a.e. on $[0,1]$ and in $\{0,1\}$ to an absolutely continuous map $g_{\mathsf{Pr}\circ\gamma}$ such that $|g'_{\mathsf{Pr}\circ\gamma}|(t)\leq|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(\gamma_t))\mathrm{ms}_t(\mathsf{Pr}\circ\gamma)$.

Here we use the key Proposition 3.23 to obtain that for $\boldsymbol{\pi}$-a.e. γ it holds

$$|g'_{\mathsf{Pr}\circ\gamma}|(t)\leq\frac{\mathsf{R}}{2\mathsf{d}(\gamma_t,\mathsf{O})}|\mathrm{D}g|_{\mathrm{X}'}(\mathsf{Pr}(\gamma_t))|\dot\gamma_t|.$$

Since for any absolutely continuous curve γ the map $t\mapsto h(\mathsf{d}(\gamma_t,\mathsf{O})$ is absolutely continuous with $|\partial_t h(\mathsf{d}(\gamma_t,\mathsf{O})|\leq|h'|(\mathsf{d}(\gamma_t,\mathsf{O})|\dot\gamma_t|$, we deduce that for $\boldsymbol{\pi}$-a.e. γ the map $t\mapsto f(\gamma_t)=g(\mathsf{Pr}(\gamma_t))h(\mathsf{d}(\gamma_t,\mathsf{O}))$ is equal a.e. on $[0,1]$ and in $\{0,1\}$ to the absolutely continuous map $t\mapsto f_\gamma(t):=g_{\mathsf{Pr}\circ\gamma}(t)h(\mathsf{d}(\gamma_t,\mathsf{O}))$ and that (3.46) holds. By the arbitrariness of $\boldsymbol{\pi}$, this is sufficient to conclude. □

In [18] the notion of 'measured-length space' has been introduced as key tool, in conjunction with some doubling property, to establish the Sobolev-to-Lipschitz property of a space and its warped products with an interval. We recall the definition, which consists in a variant of the well-known length property which takes into account the reference measure:

Definition 3.25 (Measured-length space)**.** *We say that a metric measure space $(\mathrm{Z},\mathrm{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ is measured-length if there exists a Borel set $A\subset\mathrm{Z}$ whose complement is $\mathfrak{m}_{\mathrm{Z}}$-negligible with the following property. For every $x_0,x_1\in A$ there exists $\varepsilon>0$ such that for every $\varepsilon_0,\varepsilon_1\in(0,\varepsilon]$ there is a test plan $\pi^{\varepsilon_0,\varepsilon_1}$ with:*

a) the map $(0,\varepsilon]^2 \ni (\varepsilon_0,\varepsilon_1) \mapsto \pi^{\varepsilon_0,\varepsilon_1}$ *is weakly Borel in the sense that for any* $\varphi \in C_b(C([0,1],\mathrm{Z}))$ *the map*

$$(0,\varepsilon]^2 \ni (\varepsilon_0,\varepsilon_1) \qquad \mapsto \qquad \int \varphi \,\mathrm{d}\pi^{\varepsilon_0,\varepsilon_1},$$

is Borel.

b) We have

$$(e_0)_*\pi^{\varepsilon_0,\varepsilon_1} = \frac{1_{B_{\varepsilon_0}(x_0)}}{\mathfrak{m}_{\mathrm{Z}}(B_{\varepsilon_0}(x_0))}\,\mathfrak{m}_{\mathrm{Z}}, \qquad \text{and} \qquad (e_1)_*\pi^{\varepsilon_0,\varepsilon_1} = \frac{1_{B_{\varepsilon_1}(x_1)}}{\mathfrak{m}_{\mathrm{Z}}(B_{\varepsilon_1}(x_1))}\,\mathfrak{m}_{\mathrm{Z}}$$

for every $\varepsilon_0,\varepsilon_1 \in (0,\varepsilon]$*,*

c) We have

$$\varlimsup_{\varepsilon_0,\varepsilon_1\downarrow 0} \iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t\,\mathrm{d}\pi(\gamma) \le \mathrm{d}^2_{\mathrm{Z}}(x_0,x_1).$$

Notice that if $(\mathrm{Z},\mathrm{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ is a measured-length metric measure space, then in particular $(\mathrm{supp}(\mathfrak{m}_Z),\mathsf{d}_Z)$ is a length space, but the converse is far from being true. Heuristically speaking, a measured-length metric measure space is a space admitting, for any couple of points x,y, a family of almost-geodesics starting in a neighbourhood of x and arriving in a neighbourhood of y which 'do not overlap too much', this fact being encoded in the requirement that the $\pi^{\varepsilon_0,\varepsilon_1}$'s above are test plans and in particular such that $(\mathrm{e}_t)_*\pi^{\varepsilon_0,\varepsilon_1} \le C_{\varepsilon_0,\varepsilon_1}\mathfrak{m}_Z$ for some $C_{\varepsilon_0,\varepsilon_1} > 0$ and any $t \in [0,1]$.

We then have the following result:

Proposition 3.26. $(\mathrm{X}',\mathsf{d}',\mathfrak{m}')$ *is infinitesimally Hilbertian, doubling and a measured-length space.*

proof
Infinitesimal Hilbertianity. Direct consequence of Theorem 3.24 and the infinitesimal Hilbertianity of X (recall also property (2.3)).
Doubling. We shall denote by B^{X}, $B^{\mathrm{X}'}$ balls in X, X′ respectively. Start noticing that being $(\mathrm{X}',\mathsf{d}')$ compact it is sufficient to prove that for some $c > 0$ we have

$$\mathfrak{m}'(B^{\mathrm{X}'}_{2r}(x')) \le c\,\mathfrak{m}'(B^{\mathrm{X}'}_r(x')), \qquad \forall x' \in \mathrm{X}',\ r < \mathsf{R}/8. \tag{3.47}$$

Then for $x' \in \mathrm{X}' \subset \mathrm{X}$ and $r \in (0,\mathsf{R}/8)$ define $A(x',r) \subset \mathrm{X}$ as

$$A(x',r) := \big\{x \in \mathrm{X}\ :\ \mathsf{d}(x,\mathsf{O}) \in [\mathsf{R}/2-r,\mathsf{R}/2+r],\ \mathsf{d}'(\mathsf{Pr}(x),x') \le r\big\} \tag{3.48}$$

and notice that from Corollary 3.8 we see that

$$\mathfrak{m}(A(x',r)) = N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))\mathfrak{m}'(B^{\mathrm{X}'}_r(x'))\int_{\mathsf{R}/2-r}^{\mathsf{R}/2+r} s^{N-1}\,\mathrm{d}s$$

and therefore for some constants $c_1,c_2 > 0$ we have

$$c_1 r\,\mathfrak{m}'(B^{\mathrm{X}'}_r(x')) \le \mathfrak{m}(A(x',r)) \le c_2 r\,\mathfrak{m}'(B^{\mathrm{X}'}_r(x')), \qquad \forall x' \in \mathrm{X}',\ r < \mathsf{R}/8, \tag{3.49}$$

while the construction ensures that

$$A(x',r)\subset B^{\mathrm{X}}_{2r}(x')\qquad \forall x'\in \mathrm{X}',\ r<\mathsf{R}/8. \tag{3.50}$$

Recall that $\mathsf{Pr}:B_{3\mathsf{R}/4}(\mathsf{O})\setminus B_{\mathsf{R}/4}(\mathsf{O})\to S_{\mathsf{R}/2}(\mathsf{O})$ is Lipschitz and let L be a bound on its Lipschitz constant. Observe also that the triangle inequality ensures that a geodesic with endpoints in $B_{2r}(x')$ for some $x'\in\mathrm{X}'$ and $r<\mathsf{R}/8$ never leaves $B_{3\mathsf{R}/4}(\mathsf{O})\setminus B_{\mathsf{R}/4}(\mathsf{O})$, which is sufficient to deduce that

$$\mathsf{d}'(\mathsf{Pr}(x_1),\mathsf{Pr}(x_2))\leq L\mathsf{d}(x_1,x_2),\qquad \forall x_1,x_2\in B_{2r}(x'),\ x'\in\mathrm{X}',\ r<\mathsf{R}/8.$$

This fact together with (3.50) grants that

$$B^{\mathrm{X}}_{r/L}(x')\subset A(x',r),\qquad \forall x'\in\mathrm{X}',\ r<\mathsf{R}/8. \tag{3.51}$$

Then the claim (3.47) follows from (3.49), (3.50) and (3.51) taking into account that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is doubling.

Measured-length property. Fix $x^0,x^1\in\mathrm{X}'$, put $\varepsilon:=\min\{\frac{1}{9},\frac{\mathsf{R}^2}{16(1+\mathsf{d}'(x^0,x^1))^2}\}$ and pick $\varepsilon_0,\varepsilon_1\in(0,\varepsilon)$. Put $\varepsilon_{01}:=\max\{\varepsilon_0,\varepsilon_1\}$, let γ be a geodesic in X' connecting x^0 to x^1, let n be the integer part of $1+\frac{1}{\sqrt{\varepsilon_{01}}}$ and for $i=0,\dots,n$ put $x_{\varepsilon_{01},i}:=\gamma_{\frac{i}{n}}$. Notice that

$$\sum_{i=0}^{n-1}\mathsf{d}(x_{\varepsilon_{01},i},x_{\varepsilon_{01},i+1})\leq \mathsf{d}'(x^0,x^1),\qquad \forall n\in\mathbb{N}.$$

For $x\in\mathrm{X}'$ and $r>0$ consider the sets $A(x,r)$ as defined in (3.48), then put $\varepsilon_i:=\varepsilon_0+\frac{i}{n}(\varepsilon_1-\varepsilon_0)$ and define the measures

$$\mu_i^{\varepsilon_0,\varepsilon_1}:=\frac{1}{\mathfrak{m}(A(x_{\varepsilon_{01},i},\varepsilon_i))}\mathfrak{m}|_{A(x_{\varepsilon_{01},i},\varepsilon_i)}\in\mathscr{P}(\mathrm{X}).$$

From Corollary 3.8 it follows that

$$\mathsf{Pr}_*\mu_i^{\varepsilon_0,\varepsilon_1}=\frac{1}{\mathfrak{m}'(B_{\varepsilon_i}(x_{\varepsilon_{01},i}))}\mathfrak{m}'|_{B_{\varepsilon_i}(x_{\varepsilon_{01},i})}, \tag{3.52}$$

the balls considered in the right hand side being in the space $(\mathrm{X}',\mathsf{d}')$.

For $i=0,\dots,n-1$ let $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$ be the only optimal geodesic plan from $\mu_i^{\varepsilon_0,\varepsilon_1}$ to $\mu_{i+1}^{\varepsilon_0,\varepsilon_1}$ (recall [21]). Taking into account that the distance between a point in $A(x_{\varepsilon_{01},i},\varepsilon_i)$ and a point in $A(x_{\varepsilon_{01},i+1},\varepsilon_{i+1})$ is bounded above by $4\varepsilon_{01}+\frac{\mathsf{d}'(x^0,x^1)}{n}$ we have

$$\iint_0^1|\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}(\gamma)=W_2^2(\mu_i^{\varepsilon_0,\varepsilon_1},\mu_{i+1}^{\varepsilon_0,\varepsilon_1})\leq\big(4\varepsilon_{01}+\tfrac{\mathsf{d}'(x^0,x^1)}{n}\big)^2 \tag{3.53}$$

and from the construction and the choice of $\varepsilon,\varepsilon_0,\varepsilon_1$ it is also easy to see that

$$\mathsf{d}(\gamma_t,\mathsf{O})\in\big[\tfrac{\mathsf{R}}{2}-\sqrt{\varepsilon_{01}}(1+\mathsf{d}'(x^0,x^1)),\tfrac{\mathsf{R}}{2}+\sqrt{\varepsilon_{01}}(1+\mathsf{d}'(x^0,x^1))\big]\subset\big[\tfrac{\mathsf{R}}{4},\tfrac{3\mathsf{R}}{4}\big], \tag{3.54}$$

for every $t\in[0,1]$ and $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$-a.e. γ.

With a gluing argument we can then build a plan $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}\in\mathscr{P}(C([0,1],\mathrm{X}))$ (which is in fact unique, being the $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$'s induced by maps) such that

$$\Big(\mathrm{Restr}_{\frac{i}{n}}^{\frac{i+1}{n}}\Big)_*\boldsymbol{\pi}_n^{\varepsilon_0,\varepsilon_1}=\boldsymbol{\pi}_{n,i}^{\varepsilon_0,\varepsilon_1},\qquad\forall i=0,\dots,n-1,$$

and property (3.53), taking into account the rescaling factor, gives

$$\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}(\gamma) = n\sum_{i=0}^n \iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}(\gamma)$$
$$\leq (4n\varepsilon_{01} + \mathsf{d}'(x^0,x^1))^2 \leq (8\sqrt{\varepsilon_{01}} + \mathsf{d}'(x^0,x^1))^2,$$

while the construction ensures that (3.54) holds also $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$-a.e. γ for every $t \in [0,1]$.

We now put

$$\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1} := \mathsf{Pr}_*\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1} \in \mathscr{P}(C([0,1],\mathrm{X}'))$$

and notice that:

- Since $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$ is a test plan on X for which (3.54) holds, by Corollary 3.8, property (3.42) and the fact that Pr is Lipschitz from $B_{3\mathsf{R}/4}(\mathsf{O}) \setminus B_{\mathsf{R}/4}(\mathsf{O})$ to X′ we deduce that $\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}$ is a test plan on X′.
- By (3.52) it follows that

$$(\mathrm{e}_0)_*\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1} = \frac{1}{\mathfrak{m}(B_{\varepsilon_0}(x^0))}\mathfrak{m}|_{B_{\varepsilon_0}(x^0)}, \qquad (\mathrm{e}_0)_*\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1} = \frac{1}{\mathfrak{m}(B_{\varepsilon_1}(x^1))}\mathfrak{m}|_{B_{\varepsilon_1}(x^1)}.$$

- From Proposition 3.23 we have that

$$\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}(\gamma) \leq \frac{\mathsf{R}^2}{4}\iint_0^1 \frac{|\dot\gamma_t|^2}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}(\gamma)$$

  and therefore using (3.53) and (3.54) we obtain

$$\varlimsup_{\varepsilon_0,\varepsilon_1\downarrow 0}\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}(\gamma) \leq \varlimsup_{\varepsilon_0,\varepsilon_1\downarrow 0}\frac{\mathsf{R}^2(8\sqrt{\varepsilon_{01}} + \mathsf{d}'(x^0,x^1))^2}{4(\frac{\mathsf{R}}{2} + \sqrt{\varepsilon_{01}}(1+\mathsf{d}'(x^0,x^1)))^2} = \mathsf{d}'(x^0,x^1)^2.$$

Since the Borel dependency of $\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}$ from $\varepsilon_0,\varepsilon_1$ is clear from the construction, the proof is achieved. □

3.6 Estimate on the speed of the projection

This section is devoted to the proof of Proposition 3.23, which, as already mentioned, relies on some definitions and results contained in [15]. In order to keep the presentation at reasonable length, we shall assume the reader familiar with the language developed in [15] and in particular with the concept of L^2-normed L^∞-module and related objects. We shall use the next subsection to recall the basic results we shall need, also in order to fix the notation. The subsequent one will then be devoted to the proof of Proposition 3.23.

3.6.1 Tools for differential calculus on metric measure spaces

(co)tangent vectors and speed of test plans. The tangent and cotangent modules of the metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ are denoted as $L^2(T\mathrm{X})$ and $L^2(T^*\mathrm{X})$ respectively. The pointwise norm on both spaces will be denoted by $|\cdot|$.

The differential of a function $f \in W^{1,2}(\mathrm{X})$ is denoted by $\mathrm{d}f$ and is an element of $L^2(T^*\mathrm{X})$. Both the module $L^2(T^*\mathrm{X})$ and the linear continuous map $\mathrm{d} : W^{1,2}(\mathrm{X}) \to L^2(T^*\mathrm{X})$ are characterized, up to unique isomorphism, by the properties:

$$\begin{aligned} &|\mathrm{d}f| = |\mathrm{D}f|, \quad \mathfrak{m} - a.e. \qquad \forall f \in W^{1,2}(\mathrm{X}) \\ &\{\mathrm{d}f : f \in W^{1,2}(\mathrm{X})\} \text{ generates, in the sense of modules, the whole } L^2(T^*\mathrm{X}). \end{aligned} \tag{3.55}$$

Seen as unbounded operator from $L^2(\mathrm{X})$ to $L^2(T^*\mathrm{X})$, the differential is a linear and closed operator. See Section 2.2 of [15] for details. In case X is infinitesimally Hilbertian, the gradient $\nabla f \in L^2(T\mathrm{X})$ of $f \in W^{1,2}(\mathrm{X})$ is the element associated to the differential $\mathrm{d}f$ via the Riesz isomorphism for modules.

Given another metric measure space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$, a map $\mathrm{p} : \mathrm{Y} \to \mathrm{X}$ is said of bounded compression provided $\mathrm{p}_*\mathfrak{m}_\mathrm{Y} \leq C\mathfrak{m}$ for some $C > 0$. If $\mathcal{M}$ is a module on X it is then possible to introduce the pullback module $p^*\mathcal{M}$, which is a module on Y, and the pullback operator $\mathrm{p}^* : \mathcal{M} \to p^*\mathcal{M}$. These are characterised, up to unique isomorphism, by the fact that $\mathrm{p}^* : \mathcal{M} \to p^*\mathcal{M}$ is linear and satisfying

$$\begin{aligned} &|\mathrm{p}^*v| = |v| \circ \mathrm{p}, \quad \mathfrak{m}_\mathrm{Y} - a.e. \qquad \forall v \in \mathcal{M}, \\ &\{p^*v : v \in \mathcal{M}\} \text{ generates, in the sense of modules, the whole } p^*\mathcal{M}. \end{aligned}$$

If $\mathcal{M}^*$ is the dual of $\mathcal{M}$, there is a unique duality relation $p^*\mathcal{M} \times p^*\mathcal{M}^* \to L^1(\mathrm{Y})$ which is $L^\infty(\mathrm{Y})$-bilinear, continuous and such that

$$\mathrm{p}^*\omega(\mathrm{p}^*v) = \omega(v) \circ \mathrm{p}, \quad \mathfrak{m}_\mathrm{Y} - a.e. \qquad \forall v \in \mathcal{M},\ \omega \in \mathcal{M}^*. \tag{3.56}$$

When $\mathcal{M}$ is the cotangent module $L^2(T^*\mathrm{X})$ (resp. the tangent module $L^2(T\mathrm{X})$) we shall denote the pullback by $L^2(T^*\mathrm{X}, \mathrm{p}, \mathfrak{m}_\mathrm{Y})$ (resp. $L^2(T\mathrm{X}, \mathrm{p}, \mathfrak{m}_\mathrm{Y})$). See Section 1.6 of [15] for details.

Remark 3.27. Let us briefly comment the above. The prototype example of L^2-normed L^∞-module is the space of L^2-sections of a normed vector bundle on a smooth manifold. On metric measure spaces we use use this fact to think at such modules as to the space of L^2-sections of some bundle which is not really given. The fact that this is a viable approach comes from the existence and uniqueness of the couple 'cotangent module - differential' in the sense described above. In this regard, the uniqueness part of the statement is trivial, while existence requires an explicit construction which we briefly outline. One starts considering the set 'Pre-Cotangent Module' Pcm defined as

$$\mathrm{Pcm} := \Big\{(A_i, f_i)_{i\in\mathbb{N}} : (A_i) \text{ is a Borel partition of X}, \ (f_i) \subset W^{1,2}(\mathrm{X}), \ \sum_i \int_{A_i} |\mathrm{D}f_i|^2 \,\mathrm{d}\mathfrak{m} < \infty\Big\}$$

and defines an equivalence relation on it by declaring $(A_i, f_i) \sim (B_j, g_j)$ provided

$$|\mathrm{D}(f_i - g_j)| = 0 \quad \mathfrak{m} - a.e. \text{ on } A_i \cap B_j \qquad \forall i, j \in \mathbb{N}.$$

The equivalence class $[A_i, f_i]$ of $(A_i, f_i) \in \mathrm{Pcm}$ should be thought of as the L^2 1-form which on A_i is equal to the differential of f_i. It is then easy to see that the quotient set $\mathrm{Pcm}/\sim$ carries a natural structure of normed vector space, the norm being given by $\|[A_i, f_i]\| := \sqrt{\sum_i \int_{A_i} |\mathrm{D}f_i|^2 \,\mathrm{d}\mathfrak{m}}$, and that its completion can canonically be given the structure of L^2-normed L^∞-module. Such module is the cotangent module and the differential of a function $f \in W^{1,2}(\mathrm{X})$ is defined by $\mathrm{d}f := [X, f]$.

For what concerns the pullback module, it is worth recalling first how things work in the smooth case. Let M, N be smooth manifolds, E a bundle on N with fiber E_y for $y \in N$ and $p : M \to N$ a map. Then the pullback p^*E of E via p is the bundle on M whose fiber at x is $E_{p(x)}$. The structure and regularity of p^*E, i.e. how the various fibers are 'glued' together, depend on those of p and E, so that, e.g., if both E and p are smooth then so is p^*E in a natural way. On the other hand, if E has only the structure of measurable bundle and p is only measurable, then p^*E will also be a measurable bundle. Given that the concept of L^2-normed module offers a replacement to the notion of (L^2 sections of a) measurable bundle, it is not surprising that there exists a notion of pullback of a module under mild regularity assumptions on the map p. In particular, notice that for the construction to work it is not necessary that p is Lipschitz and that this has nothing to do with the pullback of 1-forms described in the next section.

Finally, much like the case of cotangent module, the claimed uniqueness of the pullback module and pullback map is easy to prove. For existence, one starts introducing the 'Pre-pullback module' Ppb as

$$\mathrm{Ppb} := \Big\{(A_i, v_i)_{i\in\mathbb{N}} : (A_i) \text{ is a Borel partition of Y},\ (v_i) \subset \mathcal{M},\ \sum_i \int_{A_i} |v_i|^2 \circ p \, \mathrm{d}\mathfrak{m}_{\mathrm{Y}} < \infty\Big\}$$

and then defines an equivalence relation on it by declaring $(A_i, v_i) \sim (B_j, w_j)$ provided

$$|v_i - w_j| \circ p = 0 \quad \mathfrak{m}_{\mathrm{Y}} - a.e. \text{ on } A_i \cap B_j \qquad \forall i, j \in \mathbb{N}.$$

The equivalence class $[A_i, v_i]$ of $(A_i, v_i) \in \mathrm{Ppb}$ should be thought of as the element of $p^*\mathcal{M}$ which is equal to p^*v_i on A_i. Then the construction continues along the same lines used for the cotangent module. ■

Given a test plan $\boldsymbol{\pi}$, we shall consider such pullback construction for $\mathrm{Y} = C([0,1], \mathrm{X})$ equipped with the sup distance and $\boldsymbol{\pi}$ as reference measure. The maps of bounded compression of interest for us are the evaluation maps e_t.

If $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, as in our case, it turns out that for a.e. $t \in [0,1]$ there exists a unique element $\boldsymbol{\pi}'_t \in L^2(T\mathrm{X}, \mathrm{e}_t, \boldsymbol{\pi})$, called speed of $\boldsymbol{\pi}$ at time t, having the property that

$$\lim_{h\to 0} \frac{f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t}{h} = (\mathrm{e}_t^* \mathrm{d}f)(\boldsymbol{\pi}'_t), \qquad \forall f \in W^{1,2}(\mathrm{X}), \tag{3.57}$$

the limit being intended in the strong topology of $L^1(\boldsymbol{\pi})$. See Theorem 2.3.18 in [15]. The same theorem also provides a direct and tight link between the pointwise norm of the vector fields $\boldsymbol{\pi}'_t$ and the metric speed of curves, as for a.e. $t \in [0,1]$ it holds

$$|\boldsymbol{\pi}'_t|(\gamma) = |\dot\gamma_t|, \qquad \boldsymbol{\pi} - a.e.\ \gamma. \tag{3.58}$$

Maps of bounded deformation and their differential. Maps between metric measure spaces which are both Lipschitz and of bounded compression will be called of bounded deformation.

The right composition with a map of bounded deformation $F : \mathrm{X} \to \mathrm{Y}$ provides a linear and continuous map from $W^{1,2}(\mathrm{Y})$ to $W^{1,2}(\mathrm{X})$ and for any $\varphi \in W^{1,2}(\mathrm{Y})$ the bound

$$|\mathrm{d}(\varphi \circ F)| \le \mathrm{Lip}(F) |\mathrm{d}\varphi| \circ F, \quad \mathfrak{m}_{\mathrm{X}} - a.e., \tag{3.59}$$

holds.

If F is invertible with inverse also bounded deformation, then the differential $\mathrm{d}F$ is a well defined linear continuous map from $L^2(T\mathrm{X})$ to $L^2(T\mathrm{Y})$: for $v \in L^2(T\mathrm{X})$ the vector field $\mathrm{d}F(v) \in L^2(T\mathrm{Y})$ is charaterized by

$$\Big(\mathrm{d}\varphi\big(\mathrm{d}F(v)\big)\Big) \circ F = \mathrm{d}(\varphi \circ F)(v), \quad \mathfrak{m}_\mathrm{X} - a.e. \qquad \forall \varphi \in W^{1,2}(\mathrm{Y}), \tag{3.60}$$

and the bound (3.59) yields

$$|\mathrm{d}F(v)| \circ F \leq \mathrm{Lip}(F)|v|, \quad \mathfrak{m}_\mathrm{X} - a.e., \tag{3.61}$$

and in particular the differential $\mathrm{d}F$ is local in the sense that

$$\mathrm{d}F(v) = \mathrm{d}F(w), \quad \mathfrak{m}_\mathrm{Y} - a.e. \text{ on } F(\{v = w\}). \tag{3.62}$$

See Section 2.4 of [15] for details.

The left composition with F provides a Lipschitz map from $C([0,1],\mathrm{X})$ to $C([0,1],\mathrm{Y})$ which, abusing a bit the notation, we shall continue to denote by F. Hence for a given test plan $\boldsymbol{\pi}$ on X we can consider the measure $F_*\boldsymbol{\pi}$ on $C([0,1],\mathrm{Y})$ and it is trivial to see that the fact that F is of bounded deformation ensures that $F_*\boldsymbol{\pi}$ is a test plan on Y.

To clarify the notation in the foregoing discussion, we shall put $\bar{\boldsymbol{\pi}} := F_*\boldsymbol{\pi}$ and denote by $\bar{\mathrm{e}}_t$ the evaluation maps from $C([0,1],\mathrm{Y})$ to Y. We shall assume that F is of bounded deformation, invertible and with inverse of bounded deformation.

Notice that that for every $t \in [0,1]$, the differential $\mathrm{d}F : L^2(T\mathrm{X}) \to L^2(T\mathrm{Y})$ naturally induces a map, which we shall still denote $\mathrm{d}F$, from $L^2(T\mathrm{X}, \mathrm{e}_t, \boldsymbol{\pi})$ to $L^2(T\mathrm{X}, \bar{\mathrm{e}}_t, \bar{\boldsymbol{\pi}})$: it is the unique linear and continuous map such that

$$\begin{aligned} &\mathrm{d}F(\mathrm{e}_t^* v) = \bar{\mathrm{e}}_t^*(\mathrm{d}F(v)), && \forall v \in L^2(T\mathrm{X}),\\ &\mathrm{d}F(gV) = g \circ F^{-1}\,\mathrm{d}F(V), && \forall V \in L^2(T\mathrm{X}, \mathrm{e}_t, \boldsymbol{\pi}),\ g \in L^\infty(\boldsymbol{\pi}). \end{aligned} \tag{3.63}$$

Recalling that in the classical smooth setting we have the chain rule

$$(F \circ \gamma)'_t = \mathrm{d}F(\gamma'_t),$$

we are now going to show that the language just discussed allows to state and prove an analogous of this chain rule in the context of metric measure spaces. As we shall see, the proof is being just based on keeping track of the various definitions.

Proposition 3.28 (Chain rule for speeds). *Let $F : \mathrm{X} \to \mathrm{Y}$ be of bounded deformation, invertible and with inverse of bounded deformation. Then, recalling the definition of speed of a test plan given by* (3.57)*, for every test plan $\boldsymbol{\pi}$ on* X *we have*

$$(F_*\boldsymbol{\pi})'_t = (\mathrm{d}F)(\boldsymbol{\pi}'_t), \qquad a.e.\ t \in [0,1]. \tag{3.64}$$

proof Put $\bar{\boldsymbol{\pi}} := F_*\boldsymbol{\pi}$ as before and fix $f \in W^{1,2}(\mathrm{X})$. We claim that

$$(\bar{\mathrm{e}}_t^* \mathrm{d}f)\big((\mathrm{d}F)(V)\big) = \Big(\big(\mathrm{e}_t^* \mathrm{d}(f \circ F)\big)(V)\Big) \circ F^{-1}, \qquad \forall V \in L^2(T\mathrm{X}, \mathrm{e}_t, \boldsymbol{\pi}). \tag{3.65}$$

For V of the form $\mathrm{e}_t^* v$ for some $v \in L^2(T\mathrm{X})$ such identity comes from the chain of equalities

$$
\begin{aligned}
(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(\mathrm{e}_t^*v)\big) &= (\bar{\mathrm{e}}_t^*\mathrm{d}f)\big(\bar{\mathrm{e}}_t^*(\mathrm{d}F(v))\big) && \text{by the first in (3.63)}\\
&= \big(\mathrm{d}f(\mathrm{d}F(v))\big)\circ\bar{\mathrm{e}}_t && \text{by (3.56)}\\
&= \mathrm{d}(f\circ F)(v)\circ F^{-1}\circ\bar{\mathrm{e}}_t && \text{by (3.60)}\\
&= \big(\mathrm{d}(f\circ F)(v)\big)\circ\mathrm{e}_t\circ F^{-1} && \text{because } F^{-1}(\gamma_t)=(F^{-1}(\gamma))_t\\
&= \Big(\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(\mathrm{e}_t^*v)\Big)\circ F^{-1} && \text{by (3.56).}
\end{aligned}
$$

Therefore noticing that both sides of (3.65) are, as functions of V, linear and continuous from $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$ to $L^1(\bar{\boldsymbol{\pi}})$, and since the vector space generated by elements of the form $g\mathrm{e}_t^*v$ for $v\in L^2(T\mathrm{X})$ and $g\in L^\infty(\boldsymbol{\pi})$ is dense in $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$, the claim (3.65) follows if we show that

$$
\begin{aligned}
(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(gV)\big) &= g\circ F^{-1}\,(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(V)\big),\\
\Big(\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(gV)\Big)\circ F^{-1} &= g\circ F^{-1}\Big(\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(V)\Big)\circ F^{-1}.
\end{aligned}
$$

The second of these is obvious from the $L^\infty(\boldsymbol{\pi})$-linearity of $\mathrm{e}_t^*\mathrm{d}(f\circ F)$ as a map from $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$ to $L^1(\boldsymbol{\pi})$. For the first we use the second in (3.63) and the $L^\infty(\bar{\boldsymbol{\pi}})$-linearity of $\bar{\mathrm{e}}_t^*\mathrm{d}f$ as a map from $L^2(T\mathrm{Y},\bar{\mathrm{e}}_t,\bar{\boldsymbol{\pi}})$ to $L^1(\bar{\boldsymbol{\pi}})$:

$$
(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(gV)\big) = (\bar{\mathrm{e}}_t^*\mathrm{d}f)\big(g\circ F^{-1}(\mathrm{d}F)(V)\big) = g\circ F^{-1}\,(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(V)\big).
$$

Thus (3.65) is proved and writing it for $\boldsymbol{\pi}_t'$ in place of V we obtain

$$
(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(\boldsymbol{\pi}_t')\big) = \Big(\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(\boldsymbol{\pi}_t')\Big)\circ F^{-1},\qquad a.e.\ t\in[0,1].
$$

To conclude, recall that $\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(\boldsymbol{\pi}_t')$ is the strong limit in $L^1(\boldsymbol{\pi})$ as $h\to0$ of the maps

$$
\gamma\quad\mapsto\quad\frac{f(F(\gamma_{t+h}))-f(F(\gamma_t))}{h}
$$

and therefore the change of variable $\gamma=F^{-1}(\bar\gamma)$ and the fact that $\boldsymbol{\pi}=F_*^{-1}\bar{\boldsymbol{\pi}}$ show that $\Big(\big(\mathrm{e}_t^*\mathrm{d}(f\circ F)\big)(\boldsymbol{\pi}_t')\Big)\circ F^{-1}$ is the strong limit in $L^1(\bar{\boldsymbol{\pi}})$ as $h\to0$ of the maps

$$
\bar\gamma\quad\mapsto\quad\frac{f(\bar\gamma_{t+h})-f(\bar\gamma_t)}{h},
$$

but this latter limit is, by the very definition of $\bar{\boldsymbol{\pi}}_t'$, equal to $\bar{\mathrm{e}}_t^*\mathrm{d}f(\bar{\boldsymbol{\pi}}_t')$. We therefore proved that

$$
(\bar{\mathrm{e}}_t^*\mathrm{d}f)\big((\mathrm{d}F)(\boldsymbol{\pi}_t')\big) = (\bar{\mathrm{e}}_t^*\mathrm{d}f)(\bar{\boldsymbol{\pi}})_t',\qquad a.e.\ t\in[0,1],
$$

which, by the arbitrariness of $f\in W^{1,2}(\mathrm{X})$, is the thesis. □

Remark 3.29. In all this discussion the assumption that F was invertible with inverse of bounded deformation is not really necessary, being the differential $\mathrm{d}F$ of a map of bounded deformation $F:\mathrm{X}\to\mathrm{Y}$ always well defined as map from $L^2(T\mathrm{X})$ to the pullback $L^2(T\mathrm{Y},F,\mathfrak{m}_{\mathrm{X}})$ of the tangent module $L^2(T\mathrm{Y})$ of Y via F (see Section 2.4 in [15]).

We made this further assumption because it simplifies the exposition and will be present in our applications. ■

Bits of second order calculus Here we come back to our assumption that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{RCD}^*(0, N)$ space.

Test functions and the language of L^∞-modules allow to introduce the second order Sobolev space $W^{2,2}(\mathrm{X})$ as follows. First of all, we recall that being $L^2(T^*\mathrm{X})$ an Hilbert module, it is possible to consider the Hilbert tensor product of $L^2(T^*\mathrm{X})$ with itself, which we shall denote by $L^2((T^*)^{\otimes 2}\mathrm{X})$ (see Section 1.5 in [15] for the definition). We remark that in the smooth case the pointwise norm in $L^2((T^*)^{\otimes 2}\mathrm{X})$ is the Hilbert-Schmidt one.

Then we say that a function $f \in W^{1,2}(\mathrm{X})$ belongs to $W^{2,2}(\mathrm{X})$ provided there is an element of $L^2((T^*)^{\otimes 2}\mathrm{X})$, called the Hessian of f and denoted by $\mathrm{Hess}(f)$, such that for any $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$ it holds

$$2\int h\mathrm{Hess}(f)(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m} = \int -\langle \nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) - \langle \nabla f, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) - h\langle \nabla f, \nabla(\langle \nabla g_1, \nabla g_2\rangle)\rangle\,\mathrm{d}\mathfrak{m}. \tag{3.66}$$

The density of $\mathrm{Test}(\mathrm{X})$ in $W^{1,2}(\mathrm{X})$ grants that the above uniquely characterises $\mathrm{Hess}(f)$. It is then possible to see that $W^{2,2}(\mathrm{X})$ equipped with the norm

$$\|f\|^2_{W^{2,2}} := \int |f|^2 + |\nabla f|^2 + |\mathrm{Hess}(f)|^2\,\mathrm{d}\mathfrak{m},$$

is a separable Hilbert space.

An important fact is that $D(\Delta) \subset W^{2,2}(\mathrm{X})$ and that the inequality

$$\int |\mathrm{Hess}(f)|^2\,\mathrm{d}\mathfrak{m} \le \int |\Delta f|^2\,\mathrm{d}\mathfrak{m},$$

holds. In particular $W^{2,2}(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$.

The natural chain and Leibniz rules for the Hessian are in place. In particular, if $f \in W^{2,2}(\mathrm{X})$ is bounded and Lipschitz and $\varphi \in C^2_c(\mathbb{R})$ then $\varphi \circ f \in W^{2,2}(\mathrm{X})$ and

$$\mathrm{Hess}(\varphi \circ f) = \varphi'' \circ f \mathrm{d}f \otimes \mathrm{d}f + \varphi' \circ f \mathrm{Hess}(f),$$

and if $f_1, f_2 \in W^{2,2}(\mathrm{X})$ are bounded and Lipschitz, then $f_1 f_2 \in W^{2,2}(\mathrm{X})$ and

$$\mathrm{Hess}(f_1 f_2) = f_2 \mathrm{Hess}(f_1) + f_1 \mathrm{Hess}(f_2) + \mathrm{d}f_1 \otimes \mathrm{d}f_2 + \mathrm{d}f_2 \otimes \mathrm{d}f_1.$$

Moreover, for $f_1, f_2 \in \mathrm{Test}(\mathrm{X})$ we have that $\langle \nabla f_1, \nabla f_2\rangle \in W^{1,2}(\mathrm{X})$ with

$$\mathrm{d}\langle \nabla f_1, \nabla f_2\rangle = \mathrm{Hess}(f_1)(\nabla f_2) + \mathrm{Hess}(f_2)(\nabla f_1). \tag{3.67}$$

See Section 3.3 of [15] for more details and more general versions of these calculus rules.

Given an open set Ω we introduce the space $W^{2,2}(\Omega)$ as the subspace of $W^{1,2}_{loc}(\mathrm{X})$ made of functions f for which there is $\mathrm{Hess}(f) \in L^2((T^*)^{\otimes 2}\mathrm{X})$ such that (3.66) holds for any $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$ with support in Ω. In this case, evidently, $\mathrm{Hess}(f)$ is uniquely characterized only $\mathfrak{m}$-a.e. on Ω.

We conclude this preliminary section showing that

$$\mathrm{b} \in W^{2,2}(B_{\mathsf{R}}(\mathsf{O})) \qquad \text{with} \qquad \mathrm{Hess}(\mathrm{b}) = \mathrm{Id} \quad \mathfrak{m}-a.e.\ on\ B_{\mathsf{R}}(\mathsf{O}), \tag{3.68}$$

where $\mathrm{Id} \in L^\infty((T^*)^{\otimes 2}\mathrm{X})$ is defined by $\mathrm{Id}(v,w) = \langle v, w\rangle$ $\mathfrak{m}$-a.e. for any $v, w \in L^2(T\mathrm{X})$. For let us fix $\bar{\mathsf{R}} < \mathsf{R}$ and recall the that the function $\bar{\mathrm{b}} = \varphi \circ \mathrm{b}$ introduced in Section 3.1 belongs to $\mathrm{Test}(\mathrm{X})$ and that $\mathbf{\Delta}\bar{\mathrm{b}} = N\mathfrak{m}$ on $B_{\bar{\mathsf{R}}}(\mathsf{O})$. Let $g, h \in \mathrm{Test}(\mathrm{X})$ with support in $B_{\bar{\mathsf{R}}}(\mathsf{O})$, using the Euler equation (3.24) and few integration by parts we obtain:

$$
\begin{aligned}
&\int -\operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int -\Delta(hg)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + \operatorname{div}(g\nabla h)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int -hg\langle \nabla\Delta g, \nabla\bar{\mathrm{b}}\rangle - 2hg\Delta g + \langle \nabla h, \nabla g\rangle\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + g\Delta h\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int h\Delta g\langle \nabla g, \nabla\bar{\mathrm{b}}\rangle + g\Delta g\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + (N-2)hg\Delta g + \langle \nabla h, \nabla g\rangle\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + \Delta h\langle \nabla\bar{\mathrm{b}}, \nabla \tfrac{g^2}{2}\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + g\Delta g\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + (N-2)hg\Delta g + \Delta h\langle \nabla\bar{\mathrm{b}}, \nabla \tfrac{g^2}{2}\rangle \,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

Similarly, using (3.24) in the third equality we get

$$
\begin{aligned}
&\int - h\langle \nabla\bar{\mathrm{b}}, \nabla \tfrac{|\mathrm{D}g|^2}{2}\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \tfrac{|\mathrm{D}g|^2}{2}\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + Nh\tfrac{|\mathrm{D}g|^2}{2} \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \Delta\tfrac{g^2}{2}\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle - \tfrac{|\mathrm{D}g|^2}{2}\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle - g\Delta g\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + Nh\tfrac{|\mathrm{D}g|^2}{2} \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \tfrac{g^2}{2}\langle \nabla\Delta h, \nabla\bar{\mathrm{b}}\rangle + g^2\Delta h - \tfrac{|\mathrm{D}g|^2}{2}\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle - g\Delta g\langle \nabla h, \nabla \mathrm{b}\rangle + Nh\tfrac{|\mathrm{D}g|^2}{2} \,\mathrm{d}\mathfrak{m}\\
&\quad= \int -\Delta h\langle \nabla\tfrac{g^2}{2}, \nabla\bar{\mathrm{b}}\rangle - N\tfrac{g^2}{2}\Delta h + g^2\Delta h - \tfrac{|\mathrm{D}g|^2}{2}\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle - g\Delta g\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + Nh\tfrac{|\mathrm{D}g|^2}{2} \,\mathrm{d}\mathfrak{m}\\
&\quad= \int -\Delta h\langle \nabla\tfrac{g^2}{2}, \nabla\bar{\mathrm{b}}\rangle + (-\tfrac{N}{2}+1)g^2\Delta h + h\langle \nabla\bar{\mathrm{b}}, \nabla\tfrac{|\mathrm{D}g|^2}{2}\rangle - g\Delta g\langle \nabla h, \nabla\bar{\mathrm{b}}\rangle + Nh|\mathrm{D}g|^2 \,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

Adding up we get

$$
\begin{aligned}
&\int -\operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle - h\langle \nabla\bar{\mathrm{b}}, \nabla\tfrac{|\mathrm{D}g|^2}{2}\rangle \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + h\langle \nabla\bar{\mathrm{b}}, \nabla\tfrac{|\mathrm{D}g|^2}{2}\rangle + (N-2)hg\Delta g + (-\tfrac{N}{2}+1)g^2\Delta h + Nh|\mathrm{D}g|^2 \,\mathrm{d}\mathfrak{m}\\
&\quad= \int \operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle + h\langle \nabla\bar{\mathrm{b}}, \nabla\tfrac{|\mathrm{D}g|^2}{2}\rangle + 2h|\mathrm{D}g|^2 \,\mathrm{d}\mathfrak{m},
\end{aligned}
$$

or equivalently

$$
\int -\operatorname{div}(h\nabla g)\langle \nabla\bar{\mathrm{b}}, \nabla g\rangle - h\langle \nabla\bar{\mathrm{b}}, \nabla\tfrac{|\mathrm{D}g|^2}{2}\rangle \,\mathrm{d}\mathfrak{m} = \int h|\mathrm{D}g|^2 \,\mathrm{d}\mathfrak{m},
$$

which by a polarization argument gives $\mathrm{Hess}(\bar{\mathrm{b}}) = \mathrm{Id}$ in $B_{\bar{\mathsf{R}}}(\mathsf{O})$ from which (3.68) follows.

3.6.2 Proof of the Proposition 3.23

Fix $\mathsf{r}', \mathsf{R}' \in (0, \mathsf{R})$ so that $\mathsf{r}' < \frac{\mathsf{R}}{2} < \mathsf{R}'$. Later on r', R' will be sent to 0 and R respectively, but for the moment it is convenient to keep them fixed and to avoid mentioning the dependence on them of the various objects we are going to build.

Pick a function $\psi \in C^\infty(\mathbb{R})$ with support in $(0, \frac{\mathsf{R}^2}{2})$ so that

$$\psi(z) = \frac{1}{2}\big(\sqrt{2z} - \frac{\mathsf{R}}{2}\big)^2 = z - \frac{\mathsf{R}}{\sqrt{2}}\sqrt{z} + \frac{\mathsf{R}^2}{8} \qquad \text{for } z \in [\tfrac{\mathsf{r}'^2}{2}, \tfrac{\mathsf{R}'^2}{2}]$$

and define the reparametrization function $\mathrm{rep} : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ by requiring that

$$\partial_t \mathrm{rep}_t(r) = \psi'\Big(\frac{r^2}{2} e^{-2\mathrm{rep}_t(r)}\Big), \qquad \mathrm{rep}_0(r) = 0,$$

for every $r \geq 0$. Then define the function $\hat{\mathrm{b}} : \mathrm{X} \to \mathbb{R}$ and the flow $\hat{\mathsf{Fl}} : \mathbb{R}^+ \times \mathrm{X} \to \mathrm{X}$ as

$$\begin{aligned} \hat{\mathrm{b}} &:= \psi \circ \mathrm{b}, \\ \hat{\mathsf{Fl}}_s(x) &:= \mathsf{Fl}_{\mathrm{rep}_s(\mathsf{d}(x,\mathsf{O}))}(x), \qquad \forall s \in \mathbb{R}^+,\ x \in \mathrm{X}. \end{aligned}$$

As the reader might have noticed, the construction of $\hat{\mathrm{b}}, \hat{\mathsf{Fl}}_s$ follows closely that of $\bar{\mathrm{b}}, \bar{\mathsf{Fl}}_s$: the difference is in the choice of the function, ψ for $\hat{\mathrm{b}}$ and φ for $\bar{\mathrm{b}}$ (recall (3.11)), used to define the reparametrization functions.

Then with the same means used to study the basic properties of $\bar{\mathrm{b}}$ and $\bar{\mathsf{Fl}}_s$ (see Proposition 3.9) and minor algebraic manipulations we can deduce the following facts about $\hat{\mathrm{b}}$ and $\hat{\mathsf{Fl}}$:

a) From the chain rules for the gradient, Laplacian and Hessian we see that $\hat{\mathrm{b}} \in \mathrm{Test}(\mathrm{X})$ and

$$\begin{aligned} \nabla \hat{\mathrm{b}} &= \psi' \circ \mathrm{b}\, \nabla \mathrm{b}, \\ \Delta \hat{\mathrm{b}} &= \psi' \circ \mathrm{b}\, \Delta \mathrm{b} + \psi'' \circ \mathrm{b} |\nabla \mathrm{b}|^2, \\ \mathrm{Hess}(\hat{\mathrm{b}}) &= \psi' \circ \mathrm{b}\, \mathrm{Hess}(\mathrm{b}) + \psi'' \circ \mathrm{b} \nabla \mathrm{b} \otimes \nabla \mathrm{b}. \end{aligned}$$

In particular, recalling that $\boldsymbol{\Delta} \mathrm{b} = N\mathfrak{m}$ and $\mathrm{Hess}(\mathrm{b}) = \mathrm{Id}$ on $B_{\mathsf{R}}(\mathsf{O})$ we see that $\hat{\mathrm{b}}$ has bounded gradient, Laplacian and Hessian. Moreover, on $B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O})$ it holds

$$\begin{aligned} \hat{\mathrm{b}} &= \frac{1}{2}\big(\mathsf{d}(\cdot, \mathsf{O}) - \frac{\mathsf{R}}{2}\big)^2 = \mathrm{b} - \frac{\mathsf{R}}{\sqrt{2}}\sqrt{\mathrm{b}} + \frac{\mathsf{R}^2}{8}, \\ \nabla \hat{\mathrm{b}} &= \nabla \mathrm{b}\Big(1 - \frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}}\Big), \\ \mathrm{Hess}(\hat{\mathrm{b}}) &= \mathrm{Id}\Big(1 - \frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}}\Big) + \frac{\mathsf{R}}{4\sqrt{2}} \frac{1}{\mathrm{b}\sqrt{\mathrm{b}}} \nabla \mathrm{b} \otimes \nabla \mathrm{b}. \end{aligned} \tag{3.69}$$

b) For any $x \in \mathrm{X}$ the curve $[0,1] \ni s \mapsto \hat{\mathsf{Fl}}_s(x) \in \mathrm{X}$ is Lipschitz and satisfies

$$\mathrm{b}(\gamma_0) = \mathrm{b}(\gamma_t) + \frac{1}{2}\int_0^t |\dot{\gamma}_s|^2 + \mathrm{lip}^2(\hat{\mathrm{b}})(\gamma_s)\,\mathrm{d}s, \qquad \forall t \in [0,1].$$

In particular, using the fact that $\mathrm{lip}^2(\hat{\mathrm{b}}) = 2\hat{\mathrm{b}}$, which follows from the anlogous property of b, we see that $\partial_t \hat{\mathrm{b}}(\hat{\mathsf{Fl}}_t(x)) = 2\hat{\mathrm{b}}(\hat{\mathsf{Fl}}_t(x))$. Therefore for $x \in B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O})$ the quantity $|\mathsf{d}(\hat{\mathsf{Fl}}_t(x), \mathsf{O}) - \frac{\mathsf{R}}{2}|$ decreases exponentially as $t \to \infty$ and in particular

$$\hat{\mathsf{Fl}}_s \to \mathsf{Pr} \qquad \text{uniformly on } B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O}) \text{ as } s \to \infty. \tag{3.70}$$

Finally, the fact that $|\dot{\gamma}_t| = \mathrm{lip}(\hat{\mathrm{b}})(\gamma_t) \leq \mathrm{Lip}(\hat{\mathrm{b}})$ shows that $t \mapsto \hat{\mathsf{Fl}}_t(x)$ is $\mathrm{Lip}(\hat{\mathrm{b}})$-Lipschitz.

c) For any $s \geq 0$, the map $\hat{\mathsf{Fl}}_s$ is invertible map from X into itself, it is the identity on a neighbourhood of $\{\mathsf{O}\} \cup (\mathrm{X} \setminus B_{\mathsf{R}}(\mathsf{O}))$ and sends $B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O})$ into itself.

d) for any $t, s \in \mathbb{R}^+$ we have

$$\hat{\mathsf{Fl}}_s \circ \hat{\mathsf{Fl}}_t = \hat{\mathsf{Fl}}_{s+t}$$

and in particular putting $\hat{\mathsf{Fl}}_{-s} := \hat{\mathsf{Fl}}_s^{-1}$ we obtain a one parameter group of maps of X into itself.

e) The same arguments used to obtain the bound (3.12) grant that

$$c(s)\mathfrak{m} \leq (\hat{\mathsf{Fl}}_s)_*\mathfrak{m} \leq C(s)\mathfrak{m}, \qquad \forall s \in \mathbb{R},$$

for some positive continuous functions $c, C : \mathbb{R} \to (0, \infty)$.

f) Since by a compactness argument we have that Fl_s restricted to compact subsets of $B_{\mathsf{R}}(\mathsf{O})$ is Lipschitz, we deduce that $\hat{\mathsf{Fl}}_s : \mathrm{X} \to \mathrm{X}$ is Lipschitz. Moreover the chain of inequalities

$$\begin{aligned}
\mathsf{d}\big(\hat{\mathsf{Fl}}_s(x), \hat{\mathsf{Fl}}_s(y)\big) &= \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(x), \mathsf{Fl}_{\mathrm{rep}_s^y}(y)\big)\\
&\leq \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(x), \mathsf{Fl}_{\mathrm{rep}_s^x}(y)\big) + \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(y), \mathsf{Fl}_{\mathrm{rep}_s^y}(y)\big)\\
&\leq \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(x), \mathsf{Fl}_{\mathrm{rep}_s^x}(y)\big) + \mathsf{d}(y, \mathsf{O})|e^{-\mathrm{rep}_s^x} - e^{-\mathrm{rep}_s^y}|\\
&\leq \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(x), \mathsf{Fl}_{\mathrm{rep}_s^x}(y)\big) + \mathsf{d}(y, \mathsf{O})|\mathrm{rep}_s^x - \mathrm{rep}_s^y|
\end{aligned}$$

together with the fact that $\mathrm{rep}_s^y = y$ for any $s \in \mathbb{R}^+$ if $y \notin B_{\mathsf{R}}(\mathsf{O})$, the bound

$$\begin{aligned}
|\mathrm{rep}_s^x - \mathrm{rep}_s^y| &\leq \int_0^s |\psi'(\mathrm{b}(\mathsf{Fl}_r(x))) - \psi'(\mathrm{b}(\mathsf{Fl}_r(y)))|\,\mathrm{d}r\\
&\leq \mathrm{Lip}(\psi' \circ \mathrm{b}) \int_0^s \mathsf{d}(\mathsf{Fl}_r(x), \mathsf{Fl}_r(y))\,\mathrm{d}r,
\end{aligned}$$

and the fact that $\mathrm{lip}(\mathsf{Fl}_r)(x) = e^{-r} \leq 1$ for any $x \in B_{\mathsf{R}}(\mathsf{O})$, grant that

$$\mathsf{d}\big(\hat{\mathsf{Fl}}_s(x), \hat{\mathsf{Fl}}_s(y)\big) \leq \mathsf{d}(x, y)\Big(1 + s\mathsf{R}\,\mathrm{Lip}(\psi' \circ \mathrm{b}) + o(\mathsf{d}(x, y))\Big), \tag{3.71}$$

for every $x, y \in \mathrm{X}$ and $s \in \mathbb{R}^+$. To get a similar control for negative s, we start from

$$\mathsf{d}\big(\hat{\mathsf{Fl}}_s(x), \hat{\mathsf{Fl}}_s(y)\big) \geq \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(x), \mathsf{Fl}_{\mathrm{rep}_s^x}(y)\big) - \mathsf{d}\big(\mathsf{Fl}_{\mathrm{rep}_s^x}(y), \mathsf{Fl}_{\mathrm{rep}_s^y}(y)\big)$$

and use the bound $\mathrm{rep}_s^x \leq s \sup \psi'$, so that arguing as before we get

$$\mathsf{d}\big(\hat{\mathsf{Fl}}_s(x), \hat{\mathsf{Fl}}_s(y)\big) \geq \mathsf{d}(x, y)\Big(1 - s\big(\sup \psi' + \mathsf{R}\,\mathrm{Lip}(\psi' \circ \mathrm{b})\big) + o(\mathsf{d}(x, y))\Big), \tag{3.72}$$

for every $x, y \in \mathrm{X}$ and $s \in \mathbb{R}^+$. Putting $C := 2(\sup \psi' + \mathsf{R}\,\mathrm{Lip}(\psi' \circ \mathrm{b}))$, the bounds (3.71) and (3.72) give

$$\mathrm{lip}(\hat{\mathsf{Fl}}_s)(x) \leq 1 + |s|C, \qquad \forall x \in \mathrm{X},\ s \in [-\tfrac{1}{C}, \tfrac{1}{C}].$$

and since X is a geodesic space this further implies that $\mathrm{Lip}(\hat{\mathsf{Fl}}_s) \leq 1 + |s|C$ for every $s \in [-\frac{1}{C}, \frac{1}{C}]$. Recalling (3.59) and (3.61) we therefore deduce that

$$|\mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)| \leq (1 + |s|C)|\mathrm{d}f| \circ \hat{\mathsf{Fl}}_s, \qquad \text{and} \qquad |\mathrm{d}\hat{\mathsf{Fl}}_s(v)| \circ \hat{\mathsf{Fl}}_s \leq (1 + |s|C)|v| \tag{3.73}$$

$\mathfrak{m}$-a.e. for every $f \in W^{1,2}(\mathrm{X})$, $v \in L^2(T\mathrm{X})$ and $s \in [-\frac{1}{C}, \frac{1}{C}]$.

The following lemma will be useful.

Lemma 3.30. *Let $\varphi \in W^{1,2}(\mathrm{X})$. Then the map $\mathbb{R} \ni s \mapsto \varphi \circ \hat{\mathsf{Fl}}_s \in L^2(\mathrm{X})$ is C^1 and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi \circ \hat{\mathsf{Fl}}_s = -\langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \circ \hat{\mathsf{Fl}}_s. \tag{3.74}$$

If φ is further assumed to be in $\mathrm{Test}(\mathrm{X})$*, then the map $\mathbb{R} \ni s \mapsto \mathrm{d}(\varphi \circ \hat{\mathsf{Fl}}_s) \in L^2(T\mathrm{X})$ is also C^1 and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{d}(\varphi \circ \hat{\mathsf{Fl}}_s)\big) = -\mathrm{d}\big(\langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \circ \hat{\mathsf{Fl}}_s\big). \tag{3.75}$$

proof The first claim is proved exactly as Lemma 3.11, thus we pass to the second.

Start noticing that since $\varphi, \hat{\mathrm{b}} \in \mathrm{Test}(\mathrm{X})$, we have $\langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \in W^{1,2}(\mathrm{X})$ and thus, since $\hat{\mathsf{Fl}}_s$ is of bounded deformation, that $\langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \circ \hat{\mathsf{Fl}}_s \in W^{1,2}(\mathrm{X})$ as well.

We now claim that

$$\mathbb{R} \ni s \quad \mapsto \quad \mathrm{d}(\langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \circ \hat{\mathsf{Fl}}_s) \in L^2(T\mathrm{X}) \qquad \text{is continuous.} \tag{3.76}$$

Putting for a moment $f := \langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle$, using the group property of $(\hat{\mathsf{Fl}}_s)$ and (3.73) we get

$$\int |\mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)|^2 \,\mathrm{d}\mathfrak{m} \le (1 + |s - s_0|C)^2 \int |\mathrm{d}(f \circ \hat{\mathsf{Fl}}_{s_0})|^2 \circ \hat{\mathsf{Fl}}_{s-s_0} \,\mathrm{d}\mathfrak{m}, \qquad \forall s_0 \in \mathbb{R}.$$

Applying the first claim in Lemma 3.11 with the flow $\hat{\mathsf{Fl}}_s$ in place of $\bar{\mathsf{Fl}}_s$ (the proof is the same) to the L^1 function $|\mathrm{d}(f \circ \hat{\mathsf{Fl}}_{s_0})|^2$ we therefore deduce that

$$\varlimsup_{s \to s_0} \int |\mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)|^2 \,\mathrm{d}\mathfrak{m} \le \int |\mathrm{d}(f \circ \hat{\mathsf{Fl}}_{s_0})|^2 \,\mathrm{d}\mathfrak{m}.$$

Since $L^2(T^*\mathrm{X})$ is an Hilbert space, to get the claimed strong continuity of $s \mapsto \mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)$ it is now sufficient to prove the weak continuity. To this aim we start recalling that the density of $D(\Delta)$ in $W^{1,2}(\mathrm{X})$ and the weak* density of bounded Lipschitz functions in $L^\infty(\mathrm{X})$ grant that linear combinations of vector fields of the form $g\nabla f$ with $f \in D(\Delta)$ and $g \in L^\infty \cap \mathrm{Lip}(\mathrm{X})$ are dense in $L^2(T\mathrm{X})$. In particular, vector fields in $L^2(T\mathrm{X})$ with divergence in $L^2(\mathrm{X})$ are dense in $L^2(T\mathrm{X})$ and thus the weak continuity of $s \mapsto \mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)$ will follow if we show that for any such vector field v the map $s \mapsto \int \mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)(v) \,\mathrm{d}\mathfrak{m}$ is continuous. This is a consequence of the identity

$$\int \mathrm{d}(f \circ \hat{\mathsf{Fl}}_s)(v) \,\mathrm{d}\mathfrak{m} = -\int f \circ \hat{\mathsf{Fl}}_s \,\mathrm{div}(v) \,\mathrm{d}\mathfrak{m},$$

and the first claim in Lemma 3.11 (again applied to the flow $\hat{\mathsf{Fl}}_s$ in place of $\bar{\mathsf{Fl}}_s$) for the L^2 function f which grant the continuity in s of the right hand side. This settles property (3.76).

To conclude, notice that the first part of the statement ensures that

$$\varphi \circ \hat{\mathsf{Fl}}_{s_1} - \varphi \circ \hat{\mathsf{Fl}}_{s_0} = -\int_{s_0}^{s_1} \langle \nabla\varphi, \nabla\hat{\mathrm{b}}\rangle \circ \hat{\mathsf{Fl}}_s \,\mathrm{d}s, \qquad \forall s_0 < s_1,$$

the integral being the Bochner one. Take the differential on both sides and use the fact that it is, as a map from $W^{1,2}(\mathrm{X})$ to $L^2(T\mathrm{X})$, linear and continuous to bring it inside the integral. Then divide by $s_1 - s_0$, let $s_1 \to s_0$ and use the continuity property (3.76) to get the thesis.

□

Proposition 3.31. *Let $v \in L^2(T\mathrm{X})$ and put $v_s := \mathrm{d}\hat{\mathsf{Fl}}_s(v)$. Then the map $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{\mathsf{Fl}}_s \in L^1(\mathrm{X})$ is C^1 on $\mathbb{R}$ and its derivative is given by the formula*

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}|v_s|^2 \circ \hat{\mathsf{Fl}}_s = \mathrm{Hess}(\hat{\mathrm{b}})(v_s, v_s) \circ \hat{\mathsf{Fl}}_s, \tag{3.77}$$

the incremental ratios being converging both in $L^1(\mathrm{X})$ and $\mathfrak{m}$-a.e..

If v is also bounded, then the curve $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{\mathsf{Fl}}_s$ is C^1 also when seen with values in $L^2(\mathrm{X})$ and in this case the incremental ratios in (3.77) *also converge in $L^2(\mathrm{X})$ to the right hand side.*

proof

Step 1: v is the gradient of a test function and $s = 0$. Assume for the moment that $v = \nabla\varphi$ for some $\varphi \in \mathrm{Test}(\mathrm{X})$. Notice that for any $s \in \mathbb{R}$ we have

$$\frac{1}{2}|v_s|^2 \circ \hat{\mathsf{Fl}}_s \geq \mathrm{d}\varphi(v_s) \circ \hat{\mathsf{Fl}}_s - \frac{1}{2}|\mathrm{d}\varphi|^2 \circ \hat{\mathsf{Fl}}_s \qquad \mathfrak{m} - a.e., \tag{3.78}$$

with equality $\mathfrak{m}$-a.e. for $s = 0$. Recalling that (3.60) gives $\mathrm{d}\varphi(v_s) \circ \hat{\mathsf{Fl}}_s = \mathrm{d}(\varphi \circ \hat{\mathsf{Fl}}_s)(v)$, that $|\mathrm{d}\varphi|^2 \in W^{1,2}(\mathrm{X})$ and using Lemma 3.30 above, we see that the right hand side of this last inequality is C^1 when seen as a curve depending on s with values in $L^2(\mathrm{X})$. Formulas (3.74), (3.75) grant that its derivative at $s = 0$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{d}\varphi(v_s) \circ \hat{\mathsf{Fl}}_s - \frac{1}{2}|\mathrm{d}\varphi|^2 \circ \hat{\mathsf{Fl}}_s\Big)_{|s=0} &= \mathrm{d}\big(\langle\nabla\varphi, \nabla\hat{\mathrm{b}}\rangle\big)(v) - \langle\nabla\tfrac{|\mathrm{d}\varphi^2|}{2}, \nabla\hat{\mathrm{b}}\rangle \\ &= \mathrm{Hess}(\hat{\mathrm{b}})(v, v) \circ \hat{\mathsf{Fl}}_{s_0},\end{aligned}$$

having used the fact that $v = \nabla\varphi$ and (3.67). From (3.78) we then have

$$\overline{\lim_{s\uparrow 0}}\, \frac{|v_s|^2 \circ \hat{\mathsf{Fl}}_s - |v|^2}{2s} \leq \mathrm{Hess}(\hat{\mathrm{b}})(v, v) \leq \underline{\lim}_{s\downarrow 0}\, \frac{|v_s|^2 \circ \hat{\mathsf{Fl}}_s - |v|^2}{2s}, \tag{3.79}$$

where the $\underline{\lim}$ and $\overline{\lim}$ are intended in the $\mathfrak{m}$-essential sense (recall that the $\mathfrak{m}$-essential supremum of a family of Borel functions f_i, $i \in I$, is defined up to $\mathfrak{m}$-a.e. equality as the only function f such that $f \geq f_i$ $\mathfrak{m}$-a.e. for every $i \in I$ such that for any other $\tilde{f}$ with the same property it holds $f \leq \tilde{f}$ $\mathfrak{m}$-a.e. - then the definition of $\mathfrak{m}$-essential $\overline{\lim}\,/\,\underline{\lim}$ can be given following the classical ones of $\overline{\lim}\,/\,\underline{\lim}$ using $\mathfrak{m}$-essential inf / sup in place of the standard inf / sup).

Step 2: v is locally the gradient of a test function and $s = 0$. From the locality property of the differential (3.62) we deduce that (3.79) holds for v of the form $\sum_i \chi_{A_i}\nabla\varphi_i \in L^2(T\mathrm{X})$ for a given Borel partition $(A_i)_{i\in\mathbb{N}}$ of X and functions $\varphi_i \in \mathrm{Test}(\mathrm{X})$.

Step 3: generic $v \in L^2(T\mathrm{X})$ and $s = 0$. Let $Q_s : L^2(T\mathrm{X}) \to L^1(\mathrm{X})$ be the quadratic form defined by

$$Q_s(v) := \frac{|v_s|^2 \circ \hat{\mathsf{Fl}}_s - |v|^2}{s},$$

and notice that inequality (3.73) grants that

$$|Q_s(v)| \leq C'|v|^2, \quad \mathfrak{m} - a.e. \qquad \forall v \in L^2(T\mathrm{X}),\ s \in [-\tfrac{1}{C}, \tfrac{1}{C}], \tag{3.80}$$

for some $C'>0$. We claim that the Q_s's are locally uniformly continuous in $s\in[-\frac{1}{C},\frac{1}{C}]$ and to this aim we introduce the auxiliary quadratic forms

$$\tilde{Q}_s(v):=Q_s(v)+C'|v|^2,$$

and notice that (3.80) yields

$$0\le\tilde{Q}_s(v)\le 2C'|v|^2,\quad \mathfrak{m}-a.e.\qquad \forall v\in L^2(T\mathrm{X}),\ s\in[-\tfrac{1}{C},\tfrac{1}{C}].$$

Letting $\tilde{B}_s$ be the bilinear form associated to $\tilde{Q}_s$, the positivity of the latter and the Cauchy-Schwarz inequality - which is easily seen to be valid even in this context - yield

$$|\tilde{B}_s(v,w)|^2\le\tilde{Q}_s(v)\tilde{Q}_s(w),\quad \mathfrak{m}-a.e.\qquad \forall v,w\in L^2(T\mathrm{X}),\ s\in[-\tfrac{1}{C},\tfrac{1}{C}].$$

Therefore we have

$$\begin{aligned}|Q_s(v+w)-Q_s(v)|&\le|\tilde{Q}_s(v+w)-\tilde{Q}_s(v)|+C'\big||v+w|^2-|v|^2\big|\\&=|\tilde{Q}_s(w)+2\tilde{B}_s(v,w)|+C'\big||w|^2+2\langle v,w\rangle\big|\\&\le 2C|w|^2+2\sqrt{\tilde{Q}_s(v)\tilde{Q}_s(w)}+C'|w|^2+2C'|v||w|\\&\le 6C|v||w|+3C'|w|^2,\end{aligned}$$

$\mathfrak{m}$-a.e. for every $v,w\in L^2(T\mathrm{X})$ and $s\in[-\frac{1}{C},\frac{1}{C}]$. This estimate is the claimed local uniform continuity of the Q_s's.

Since the boundedness of $\mathrm{Hess}(\hat{\mathrm{b}})$ grants that $v\mapsto\mathrm{Hess}(\hat{\mathrm{b}})(v,v)$ is continuous from $L^2(T\mathrm{X})$ to $L^1(\mathrm{X})$ and in Step 2 we established (3.79) for a set of vector fields dense in $L^2(T\mathrm{X})$, we conclude that (3.79) holds for every $v\in L^2(T\mathrm{X})$.

Step 4: conclusion. Since the $\hat{\mathsf{Fl}}_s$'s form a group, by what we just proved we know that for any $v\in L^2(T\mathrm{X})$ and any $s_0\in\mathbb{R}$ we have

$$\varlimsup_{s\uparrow s_0}\frac{|v_s|^2\circ\hat{\mathsf{Fl}}_s-|v_{s_0}|^2\circ\hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)}\le\mathrm{Hess}(\hat{\mathrm{b}})(v_{s_0},v_{s_0})\circ\hat{\mathsf{Fl}}_{s_0}\le\varliminf_{s\downarrow s_0}\frac{|v_s|^2\circ\hat{\mathsf{Fl}}_s-|v_{s_0}|^2\circ\hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)},\quad(3.81)$$

$\mathfrak{m}$-a.e.. Now assume for a moment that $v\in L^2\cap L^\infty(T\mathrm{X})$, so that using again the group property and keeping in mind the bound (3.73) we have that $\mathbb{R}\ni s\mapsto\frac{1}{2}|v_s|^2\circ\hat{\mathsf{Fl}}_s\in L^2(\mathrm{X})$ is Lipschitz and hence - since Hilbert spaces have the Radon-Nikodym property - it is a.e. differentiable. Letting s_0 be a point of differentiability, from (3.81) we deduce that

$$\lim_{s\to s_0}\frac{|v_s|^2\circ\hat{\mathsf{Fl}}_s-|v_{s_0}|^2\circ\hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)}=\mathrm{Hess}(\hat{\mathrm{b}})(v_{s_0},v_{s_0})\circ\hat{\mathsf{Fl}}_{s_0},\quad(3.82)$$

the limit being intended both in $L^2(\mathrm{X})$ and $\mathfrak{m}$-a.e.. Since the right hand side of this last expression is, as a function of s_0 with values in $L^2(\mathrm{X})$, continuous, we deduce that $s\mapsto\frac{1}{2}|v_s|^2\circ\hat{\mathsf{Fl}}_s\in L^2(\mathrm{X})$ is C^1 and that (3.82) holds for any $s_0\in\mathbb{R}$.

Finally, let $v\in L^2(T\mathrm{X})$ be arbitrary and notice that the bound (3.73) grants that for any $s_0\in\mathbb{R}$ the incremental ratios $\frac{|v_s|^2\circ\hat{\mathsf{Fl}}_s-|v_{s_0}|^2\circ\hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)}$ are, for $s\in[s_0-1,s_0+1]$, dominated in $L^1(\mathrm{X})$.

To conclude, put $v_n := \chi_{\{|v|\leq n\}} v \in L^2 \cap L^\infty(T\mathrm{X})$, $n \in \mathbb{N}$, and $v_{n,s} := \mathrm{d}\hat{\mathsf{Fl}}_s(v_n)$ and notice that for every $s \in \mathbb{R}$ we have

$$\frac{|v_s|^2 \circ \hat{\mathsf{Fl}}_s - |v_{s_0}|^2 \circ \hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)} = \frac{|v_{n,s}|^2 \circ \hat{\mathsf{Fl}}_s - |v_{n,s_0}|^2 \circ \hat{\mathsf{Fl}}_{s_0}}{2(s-s_0)}, \qquad \mathfrak{m} - a.e.\ on\ \{|v| \leq n\},$$
$$\mathrm{Hess}(\hat{\mathrm{b}})(v_{s_0}, v_{s_0}) \circ \hat{\mathsf{Fl}}_{s_0} = \mathrm{Hess}(\hat{\mathrm{b}})(v_{n,s_0}, v_{n,s_0}) \circ \hat{\mathsf{Fl}}_{s_0}, \qquad \mathfrak{m} - a.e.\ on\ \{|v| \leq n\}.$$

By what we already proved, we deduce that the limit in (3.82) holds in the $\mathfrak{m}$-a.e. sense on $\{|v| \leq n\}$. Since $n \in \mathbb{N}$ is arbitrary, we conclude that the same limit is true $\mathfrak{m}$-a.e. and given that the incremental ratio on the left is dominated in $L^1(\mathrm{X})$, we obtain that (3.77) holds, the derivative being intended in $L^1(\mathrm{X})$.

The stated C^1 regularity is then a consequence of the continuity in $L^1(\mathrm{X})$ of $s \mapsto \mathrm{Hess}(\hat{\mathrm{b}})(v_s, v_s) \circ \hat{\mathsf{Fl}}_s$ which in turn is a consequence of the $L^2(T\mathrm{X})$ continuity of $s \mapsto v_s$ and the boundedness of $\mathrm{Hess}(\hat{\mathrm{b}})$. □

Corollary 3.32. *Let $v \in L^2(T\mathrm{X})$ be concentrated on $B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O})$ and put $v_s := \mathrm{d}\hat{\mathsf{Fl}}_s(v)$. Then for every $s_1 > s_0 \geq 0$ we have*

$$\frac{|v_{s_1}|^2}{\mathsf{d}^2(\cdot, \mathsf{O})} \circ \hat{\mathsf{Fl}}_{s_1} \leq \frac{|v_{s_0}|^2}{\mathsf{d}^2(\cdot, \mathsf{O})} \circ \hat{\mathsf{Fl}}_{s_0}, \qquad \mathfrak{m} - a.e.. \tag{3.83}$$

proof Up to replacing v with $v_n := \chi_{\{|v|\leq n\}} v$, using the fact that $|\mathrm{d}\hat{\mathsf{Fl}}_s(v_n)| \circ \hat{\mathsf{Fl}}_s = |\mathrm{d}\hat{\mathsf{Fl}}_s(v)| \circ \hat{\mathsf{Fl}}_s$ on $\{|v| \leq n\}$ and letting $n \to \infty$ we can assume that v is bounded.

Put for brevity $A := B_{\mathsf{R}'}(\mathsf{O}) \setminus B_{\mathsf{r}'}(\mathsf{O})$ and notice that on the complement of A both sides of (3.83) are 0 $\mathfrak{m}$-a.e., so that we are reduced to prove that

$$\frac{|v_{s_1}|^2}{2\mathrm{b}} \circ \hat{\mathsf{Fl}}_{s_1} \chi_A \leq \frac{|v_{s_0}|^2}{2\mathrm{b}} \circ \hat{\mathsf{Fl}}_{s_0} \chi_A, \qquad \mathfrak{m} - a.e..$$

Let $\tilde{\mathrm{b}} \in W^{1,2}(\mathrm{X})$ be a function bounded from below by a positive constant and agreeing with b on A. Then $\frac{1}{\tilde{\mathrm{b}}} \in W^{1,2}_{loc}(\mathrm{X})$ and Lemma 3.30 grants that the derivative of $s \mapsto \frac{1}{\tilde{\mathrm{b}}} \circ \hat{\mathsf{Fl}}_s$ in $L^2(\mathrm{X})$ is $\frac{1}{\tilde{\mathrm{b}}^2} \langle \nabla \tilde{\mathrm{b}}, \nabla \hat{\mathrm{b}} \rangle \circ \hat{\mathsf{Fl}}_s$. Since $\hat{\mathsf{Fl}}_s$ maps A into itself for every $s \geq 0$ (point (c) in the list of properties of $\hat{\mathsf{Fl}}_s$) and $\mathrm{b} = \tilde{\mathrm{b}}$ on A, we deduce that $s \mapsto \frac{1}{\mathrm{b}} \circ \hat{\mathsf{Fl}}_s \chi_A \in L^2(\mathrm{X})$ is C^1 with derivative equal to $\frac{1}{\mathrm{b}^2} \langle \nabla \mathrm{b}, \nabla \hat{\mathrm{b}} \rangle \circ \hat{\mathsf{Fl}}_s \chi_A$. Using then the second part of Proposition 3.31 we conclude that $s \mapsto \frac{|v_s|^2}{2\mathrm{b}} \circ \hat{\mathsf{Fl}}_s \chi_A \in L^1(\mathrm{X})$ is C^1 with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{|v_s|^2}{2\mathrm{b}} \circ \hat{\mathsf{Fl}}_s \chi_A\Big)_{|s=0} = \frac{\chi_A}{2\mathrm{b}^2}\Big(- 2\mathrm{b}\,\mathrm{Hess}(\hat{\mathrm{b}})(v,v) + |v|^2 \langle \nabla \mathrm{b}, \nabla \hat{\mathrm{b}} \rangle \Big). \tag{3.84}$$

Using the expressions for $\nabla \hat{\mathrm{b}}$ and $\mathrm{Hess}(\hat{\mathrm{b}})$ given in (3.69) we have that $\mathfrak{m}$-a.e. on A it holds

$$\begin{aligned} -2\mathrm{b}\,\mathrm{Hess}(\hat{\mathrm{b}})(v,v) &+ |v|^2 \langle \nabla \mathrm{b}, \nabla \hat{\mathrm{b}} \rangle \\ &= -2\mathrm{b}|v|^2\Big(1 - \frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}}\Big) - \frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}} \langle \nabla \mathrm{b}, v \rangle^2 + |v|^2 |\nabla \mathrm{b}|^2 \Big(1 - \frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}}\Big) \\ &= -\frac{\mathsf{R}}{2\sqrt{2\mathrm{b}}} \langle \nabla \mathrm{b}, v \rangle^2 \leq 0, \end{aligned}$$

having used the fact that $|\nabla \mathrm{b}|^2 = 2\mathrm{b}$ (Corollary 3.6). This computation together with (3.84) gives the conclusion. □

We are now ready to prove our key Proposition 3.23, which for convenience we restate:

Proposition 3.33. *Let* $[a,b] \subset (0,\mathsf{R})$ *and let* $\boldsymbol{\pi}$ *be a test plan such that* $\mathsf{d}(\gamma_t,\mathsf{O}) \in [a,b]$ *for every* $t \in [0,1]$ *and* $\boldsymbol{\pi}$*-a.e.* γ. *Then*

$$\mathrm{ms}_t(\mathsf{Pr}\circ\gamma) \le \frac{\mathsf{R}}{2\mathsf{d}(\gamma_t,\mathsf{O})}\mathrm{ms}_t(\gamma), \qquad a.e.\ t\in[0,1],\ \boldsymbol{\pi}-a.e.\ \gamma.$$

proof Build a function $\hat{\mathrm{b}}$ and the corresponding flow $\hat{\mathsf{Fl}}$ as in the beginning of the current subsection for $[\mathsf{r}',\mathsf{R}'] = [a,b]$.

With a slight abuse of notation we shall denote by $\hat{\mathsf{Fl}}_s$ the map from $C([0,1],\mathrm{X})$ into itself sending γ to $\hat{\mathsf{Fl}}_s\circ\gamma$. Then we put $\boldsymbol{\pi}_s := (\hat{\mathsf{Fl}}_s)_*\boldsymbol{\pi}$.

Recalling that for every $t\in[0,1]$ the differential of $\hat{\mathsf{Fl}}_s$ induces a map, still denoted by $\mathrm{d}\hat{\mathsf{Fl}}_s$ from $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$ to $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}_s)$ (recall the discussion in Section 3.6.1), we claim that for any $s_1\ge s_0\ge 0$ and any $V\in L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$ it holds

$$\frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(V)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} \le \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_0}(V)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_0}, \qquad \boldsymbol{\pi}-a.e.. \tag{3.85}$$

Indeed, for V of the form e_t^*v for some $v\in L^2(T\mathrm{X})$ the claim follows directly from Corollary 3.32 above via the computation:

$$\begin{aligned}\frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(\mathrm{e}_t^*v)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} &= \frac{|\mathrm{e}_t^*(\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(v))|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} = \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(v)|^2}{\mathsf{d}^2_{\mathsf{O}}}\circ\mathrm{e}_t\circ\hat{\mathsf{Fl}}_{s_1} = \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(v)|^2}{\mathsf{d}^2_{\mathsf{O}}}\circ\hat{\mathsf{Fl}}_{s_1}\circ\mathrm{e}_t\\ \text{(by (3.83))}\qquad &\le \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_0}(v)|^2}{\mathsf{d}^2_{\mathsf{O}}}\circ\hat{\mathsf{Fl}}_{s_0}\circ\mathrm{e}_t = \cdots = \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_0}(\mathrm{e}_t^*v)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_0}, \qquad \boldsymbol{\pi}-a.e..\end{aligned} \tag{3.86}$$

Then the locality property of $\mathrm{d}\hat{\mathsf{Fl}}_s : L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}) \to L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}_s)$ expressed by the second in (3.63) ensures that for V of the form $\sum_i \chi_{A_i}\mathrm{e}_t^*v_i$ for some Borel partition $(A_i)_{i\in\mathbb{N}}$ of $C([0,1],\mathrm{X})$ and $(v_i)\subset L^2(T\mathrm{X})$ it holds

$$\begin{aligned}\frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(\sum_i\chi_{A_i}\mathrm{e}_t^*v_i)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} &= \frac{|\sum_i\chi_{A_i}\circ\hat{\mathsf{Fl}}_{s_1}^{-1}\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(\mathrm{e}_t^*v_i)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} = \sum_i\chi_{A_i}\frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_1}(\mathrm{e}_t^*v_i)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1}\\ \text{(by (3.86))}\qquad &\le \sum_i\chi_{A_i}\frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_0}(\mathrm{e}_t^*v_i)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_0} = \cdots = \frac{|\mathrm{d}\hat{\mathsf{Fl}}_{s_0}(\sum_i\chi_{A_i}\mathrm{e}_t^*v_i)|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_0}\end{aligned}$$

$\boldsymbol{\pi}$-a.e.. Then the claim (3.85) follows by the density of the set of V's of the form just considered in $L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi})$ and the continuity of $\mathrm{d}\hat{\mathsf{Fl}}_s : L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}) \to L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}_s)$.

Denoting by $(\boldsymbol{\pi}_s)'_t \in L^2(T\mathrm{X},\mathrm{e}_t,\boldsymbol{\pi}_s)$ the speed at time t of the test plan $\boldsymbol{\pi}_s$, applying (3.85) to $\boldsymbol{\pi}'_t$ and recalling (3.64) we obtain that for $s_1\ge s_0\ge 0$ and a.e. $t\in[0,1]$ it holds

$$\frac{|(\boldsymbol{\pi}_{s_1})'_t|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_1} \le \frac{|(\boldsymbol{\pi}_{s_0})'_t|^2}{\mathsf{d}^2_{\mathsf{O}}\circ\mathrm{e}_t}\circ\hat{\mathsf{Fl}}_{s_0}, \qquad \boldsymbol{\pi}-a.e..$$

Integrating in t and recalling the link between pointwise norm and metric speed given in (3.58) we obtain that

$$\iint_0^1 \frac{|\dot\gamma_t|^2}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_{s_1}(\gamma) \le \iint_0^1 \frac{|\dot\gamma_t|^2}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_{s_0}(\gamma). \tag{3.87}$$

Recall that $A = B_{\bar{\mathsf{R}}}(\mathsf{O}) \setminus B_r(\mathsf{O})$ and let us consider the closure $\bar{A}$ of A and the functional $E : C([0,1],\bar{A}) \to [0,\infty]$ given by

$$E(\gamma) := \int_0^1 \frac{|\dot{\gamma}_t|^2}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t \quad \text{if } \gamma \in AC_2([0,1],\bar{A}), \qquad\qquad E(\gamma) = +\infty, \quad \text{otherwise.}$$

It is readily verified that E is lower semicontinuous. Indeed, let (γ_n) be with $\sup_n E(\gamma_n) < \infty$ and uniformly converging to γ. Then $\sup_n \int |\dot{\gamma}_{n,t}|^2\,\mathrm{d}t < \infty$ and the uniform convergence ensures that $\underline{\lim}_n E(\gamma_n) = \underline{\lim}_n \int \frac{|\dot{\gamma}_{n,t}|^2}{\mathsf{d}(\gamma_t,\mathsf{O})^2}\,\mathrm{d}t$. Since γ takes values in A, we have that $\frac{1}{\mathsf{d}(\gamma_t,\mathsf{O})}$ is bounded from above and thus the functions $|\dot{\gamma}_{n,t}|$ are uniformly bounded in $L^2([0,1],\frac{1}{\mathsf{d}(\gamma_t,\mathsf{O})^2}\mathrm{d}t)$. Up to pass to a subsequence, not relabeled, we can assume that $|\dot{\gamma}_{n,t}| \rightharpoonup G$ in $L^2([0,1],\frac{1}{\mathsf{d}(\gamma_t,\mathsf{O})^2}\mathrm{d}t)$ and passing to the limit in $\mathsf{d}(\gamma_{n,t_1},\gamma_{n,t_0}) \le \int_{t_0}^{t_1} |\dot{\gamma}_{n,t}|\,\mathrm{d}t$ deduce that $|\dot{\gamma}_t| \le G(t)$ for a.e. $t \in [0,1]$. Then the lower semicontinuity of E follows from the lower semicontinuity of the $L^2([0,1],\frac{1}{\mathsf{d}(\gamma_t,\mathsf{O})^2}\mathrm{d}t)$-norm w.r.t. weak convergence.

Since E is lower semicontinuous and non-negative, we deduce that the functional

$$\mathscr{P}(C([0,1],\bar{A})) \ni \boldsymbol{\sigma} \qquad \mapsto \qquad \int E(\gamma)\,\mathrm{d}\boldsymbol{\sigma}(\gamma) \in [0,\infty],$$

is lower semicontinuous w.r.t. weak convergence in duality with continuous and bounded functions on $C([0,1],\bar{A})$.

Now consider the functions $\hat{\mathsf{Fl}}_s$ as functions from $\bar{A}$ into itself and recall that they uniformly converge to Pr as $s \to +\infty$ (by (3.70)). It follows that $(\boldsymbol{\pi}_s)$ weakly converges to $\mathsf{Pr}_*\boldsymbol{\pi}$ as $s \to \infty$ and thus that

$$\int E(\gamma)\,\mathrm{d}\mathsf{Pr}_*\boldsymbol{\pi} \le \varliminf_{s\to+\infty} \int E(\gamma)\,\mathrm{d}\boldsymbol{\pi}_s.$$

Then the monotonicity property (3.87) and the definition of E give

$$\iint_0^1 \frac{\mathrm{ms}_t^2(\mathsf{Pr}\circ\gamma)}{|\frac{\mathsf{R}}{2}|^2}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} \le \iint_0^1 \frac{\mathrm{ms}_t^2(\gamma)}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}. \tag{3.88}$$

To pass from this integral formulation to the conclusion, we start claiming that for every $[t_0,t_1] \subset [0,1]$ and $\Gamma \subset C([0,1],\bar{A})$ Borel it holds

$$\iint_{[t_0,t_1]\times\Gamma} \frac{\mathrm{ms}_t^2(\mathsf{Pr}\circ\gamma)}{|\frac{\mathsf{R}}{2}|^2}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} \le \iint_{[t_0,t_1]\times\Gamma} \frac{\mathrm{ms}_t^2(\gamma)}{\mathsf{d}^2(\gamma_t,\mathsf{O})}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}. \tag{3.89}$$

Indeed, if $\boldsymbol{\pi}(\Gamma) = 0$ or $t_0 = t_1$ there is nothing to prove, otherwise simply write (3.88) for the plan $\frac{1}{\boldsymbol{\pi}(\Gamma)}(\mathrm{Restr}_{t_0}^{t_1})_*(\boldsymbol{\pi}|_\Gamma)$, which is still a test plan, in place of $\boldsymbol{\pi}$ to get the claim.

The class of sets of the form $[t_0,t_1] \times \Gamma$ just considered is a π-system which generates the Borel σ-algebra of $[0,1] \times C([0,1],\bar{A})$, hence by the π-λ theorem we deduce that we can replace $[t_0,t_1] \times \Gamma$ in (3.89) by an arbitrary Borel set $E \subset [0,1] \times C([0,1],\bar{A})$. Choosing as E the set where the integrand on the left is bigger than the one on the right, we conclude.

□

3.7 The rescaled sphere Z and the cone Y built over it

Let us define $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ as $(\mathrm{X}', 2\mathsf{d}'/\mathsf{R}, \mathfrak{m}')$ and note that $f : \mathrm{Z} = \mathrm{X}' \to \mathbb{R}$ belongs to $W^{1,2}(\mathrm{X}')$ if and only if it belongs to $W^{1,2}(\mathrm{Z})$ and in this case

$$|\mathrm{D}f|_{\mathrm{X}'} = \frac{2}{\mathsf{R}}|\mathrm{D}f|_{\mathrm{Z}} \qquad \mathfrak{m}' \text{ a.e. (equivalently } \mathfrak{m}_{\mathrm{Z}} \text{ a.e.).} \tag{3.90}$$

On the set $\mathrm{Z} \times [0, \infty)$ we put the semidistance d_{Y} defined by

$$\mathsf{d}^2_{\mathrm{Y}}\big((z_0, s_0), (z_1, s_1)\big) := \inf \int_0^1 \big|\frac{\mathrm{d}}{\mathrm{d}r}s(r)\big|^2 + s^2(r)|\dot\gamma_r|^2 \,\mathrm{d}r,$$

the inf being taken among all Lipschitz curves $[0, 1] \ni r \mapsto s(r) \in \mathbb{R}^+$ and $[0, 1] \ni r \mapsto \gamma_r \in \mathrm{Z}$ such that $(\gamma_i, s(i)) = (z_i, s_i)$ for $i = 0, 1$.

We denote by Y the quotient of $\mathrm{Z}\times[0, \infty)$ w.r.t. the equivalence relation given by $(z_0, s_0) \sim (z_1, s_1)$ provided $\mathsf{d}_{\mathrm{Y}}\big((z_0, s_0), (z_1, s_1)\big) = 0$. In particular, $(z_0, 0) = (z_1, 0)$ for any $z_0, z_1 \in \mathrm{Z}$ and their equivalence class will be denoted by O_{Y}.

The semidistance d_{Y} passes to the quotient and induces a distance on Y which we shall continue to denote as d_{Y}. It is easy to check that $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$ is a locally compact metric space whose topology is the same as the quotient topology. The typical element of Y will be denoted by (z, r) with $z \in \mathrm{Z} = \mathrm{X}'$ and $r \in [0, \infty)$.

We also endow Y with the measure $\mathfrak{m}_{\mathrm{Y}}$ defined by

$$\int_{\mathrm{Y}} f(z, s)\,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} := \frac{N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{\mathsf{R}^N}\int_0^{+\infty} s^{N-1}\int_{\mathrm{Z}} f(z, s)\,\mathrm{d}\mathfrak{m}_{\mathrm{Z}}(z)\,\mathrm{d}s,$$

for every non-negative Borel function f.

We shall now recall some results proved in [18] about the structure of Sobolev functions on Y. It is convenient to introduce the following notation: for $f : \mathrm{Y} \to \mathbb{R}$ given and $z \in \mathrm{Z}$ we shall denote by $f^{(z)}$ the function on $\mathbb{R}^+$ given by $r \mapsto f(z, r)$, similarly, for $r \in [0, \infty)$ the function $f^{(r)}$ on Z is defined as $z \mapsto f(z, r)$. We remark that to study the structure of Sobolev spaces on product metric measure spaces is not a trivial task, the first results in this direction have been obtained in [2], see also [6].

Theorem 3.34. *Let $f \in W^{1,2}(\mathrm{Y})$. Then:*

i) for $\mathfrak{m}_{\mathrm{Z}}$-a.e. $z \in \mathrm{Z}$ we have $f^{(z)} \in W^{1,2}([0, \infty), \mathsf{d}_{\mathrm{Eucl}}, r^{N-1}\mathrm{d}r)$

ii) for $\mathcal{L}^1$-a.e. $r \in [0, \infty)$ we have $f^{(r)} \in W^{1,2}(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$

iii) the identity

$$|\mathrm{D}f|^2_{\mathrm{Y}}(r, x) = |\mathrm{D}f^{(z)}|^2_{\mathbb{R}}(r) + \frac{1}{r^2}|\mathrm{D}f^{(r)}|^2_{\mathrm{Z}}(z) \tag{3.91}$$

holds for $\mathfrak{m}_{\mathrm{Y}}$-a.e. (z, r).

Conversely, if $f \in L^2(\mathrm{Y})$ is 0 on a neighbourhood of O_{Y}, (i) and (ii) hold and the right hand side of (3.91) is in $L^2(\mathrm{Y})$, then $f \in W^{1,2}(\mathrm{Y})$.

Furthermore, Y is infinitesimally Hilbertian and has the Sobolev-to-Lipschitz property.

proof The relation between $W^{1,2}(\mathrm{Y})$ and $(i), (ii)$ is one of the main results of [18]. Infinitesimal Hilbertianity then follows directly from the one of X' (Proposition 3.26). The Sobolev-to-Lipschitz property follows from the fact that $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ and thus $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ is doubling and measured-length, as proved in Proposition 3.26, and the results in the last section of [18].
□

3.8 From annuli in X to annuli in Y and viceversa

We shall now adapt the strategy used in [14] to prove that the natural map from $B_{\mathsf{R}}(\mathsf{O}_{\rm Y})$ to $B_{\mathsf{R}}(\mathsf{O})$ preserves the Sobolev norm of functions defined in appropriate annuli. As the proofs closely follows those in [14], we shall mostly only sketch them, highlighting the key points and providing precise references for their completion.

We introduce, for $0 < r < R < \mathsf{R}$, the annulus $\mathrm{Ann}^{\rm Y}_{r,R} := \{y \in {\rm Y} : \mathsf{d}_{\rm Y}(y, O_{\rm Y}) \in (r, R)\} \subset {\rm Y}$, and similarly the annulus $\mathrm{Ann}^{\rm X}_{r,R} := \{x \in {\rm X} : \mathsf{d}(x, \mathsf{O}) \in (r, R)\} \subset {\rm X}$.

We then introduce the map $\mathsf{T} : B_{\mathsf{R}}(O_{\rm Y}) \to B_{\mathsf{R}}(\mathsf{O})$ as

$$\mathsf{T}(z, r) := \mathsf{Fl}_{\log(\frac{\mathsf{R}}{2r})}(z), \quad \text{if } r > 0, \qquad\qquad \mathsf{T}(O_{\rm Y}) := \mathsf{O},$$

and the map $\mathsf{S} : B_{\mathsf{R}}(\mathsf{O}) \to B_{\mathsf{R}}(O_{\rm Y})$ as

$$\mathsf{S}(x) := (\mathsf{Pr}(x), \mathsf{d}(x, O)), \qquad \forall x \in B_{\mathsf{R}}(\mathsf{O}) \setminus \{\mathsf{O}\}, \qquad\qquad \mathsf{S}(\mathsf{O}) := O_{\rm Y}.$$

It is clear that T, S are one the inverse of the other, while the very definition of $\mathfrak{m}_{\rm Y}$ and Corollary 3.8 grant that

$$\mathsf{T}_* \mathfrak{m}_{\rm Y}|_{B_{\mathsf{R}}(\mathsf{O})} = \mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})} \qquad \text{and} \qquad \mathsf{S}_* \mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})} = \mathfrak{m}_{\rm Y}|_{B_{\mathsf{R}}(\mathsf{O})}. \tag{3.92}$$

Moreover, noticing that point *(ii)* of Theorem 3.18 gives that

$$\mathsf{d}(x_1, x_2) \le \frac{2d}{\mathsf{R}} \mathsf{d}'(\mathsf{Pr}(x_1), \mathsf{Pr}(x_2)) = d\, \mathsf{d}_{\rm Z}(\mathsf{Pr}(x_1), \mathsf{Pr}(x_2))$$

for any $x_1, x_2 \in {\rm X}$ with $\mathsf{d}(x_1, \mathsf{O}) = \mathsf{d}(x_2, \mathsf{O}) = d \in (0, \mathsf{R})$, we have the estimate

$$\begin{aligned} \mathsf{d}(x, y) &\le \mathsf{d}\Big(x, \mathsf{Fl}_{\log\big(\frac{\mathsf{d}(y,\mathsf{O})}{\mathsf{d}(x,\mathsf{O})}\big)}(y)\Big) + \mathsf{d}\Big(\mathsf{Fl}_{\log\big(\frac{\mathsf{d}(y,\mathsf{O})}{\mathsf{d}(x,\mathsf{O})}\big)}(y), y\Big) \\ &\le \mathsf{d}(x, \mathsf{O}) \mathsf{d}_{\rm Z}(\mathsf{Pr}(x), \mathsf{Pr}(y)) + |\mathsf{d}(x, \mathsf{O}) - \mathsf{d}(y, \mathsf{O})|. \end{aligned}$$

Thus the very definition of $\mathsf{d}_{\rm Y}$ grants that

$$\text{for every } \varepsilon \in (0, \mathsf{R}),\ \mathsf{T} \text{ is Lipschitz from } \mathrm{Ann}^{\rm Y}_{\varepsilon,\mathsf{R}} \text{ to } \mathrm{Ann}^{\rm X}_{\varepsilon,\mathsf{R}}. \tag{3.93}$$

Similarly, the fact that $\mathsf{d}(\cdot, \mathsf{O}) : {\rm X} \to \mathbb{R}$ is Lipschitz and that for every $\varepsilon, \varepsilon' \in (0, \mathsf{R}/2)$ the map $\mathsf{Pr} : \mathrm{Ann}^{\rm X}_{\varepsilon,\mathsf{R}-\varepsilon'} \to {\rm X}'$ is also Lipschitz (by local Lipschitzianity and a compactness argument) grant, together with the definition of $\mathsf{d}_{\rm Y}$, that

$$\text{for every } \varepsilon, \varepsilon' \in (0, \mathsf{R}/2),\ \mathsf{S} \text{ is Lipschitz from } \mathrm{Ann}^{\rm X}_{\varepsilon,\mathsf{R}-\varepsilon'} \text{ to } \mathrm{Ann}^{\rm Y}_{\varepsilon,\mathsf{R}-\varepsilon'}\ . \tag{3.94}$$

Finally, we define the following classes of functions:

$$\begin{aligned} \mathcal{G} &:= \Big\{g : {\rm Y} \to \mathbb{R}\ :\ g(x', r) = \tilde g(x') \text{ for some } \tilde g \in W^{1,2} \cap L^\infty({\rm Z})\Big\}, \\ \mathcal{H} &:= \Big\{h : {\rm Y} \to \mathbb{R}\ :\ h(x', r) = \tilde h(r) \text{ for some } \tilde h \in \mathrm{Lip}([0, \mathsf{R}]) \text{ with } \mathrm{supp}(h) \subset\ (0, \mathsf{R})\Big\}, \\ \mathcal{A} &:= \Big\{\sum_{i=1}^n g_i h_i\ :\ i \in \mathbb{N},\ \ g_i \in \mathcal{G},\ h_i \in \mathcal{H}\ \ \forall i = 1, \dots, n\Big\}. \end{aligned}$$

In the foregoing discussion, given a metric measure space $({\rm Z}, \mathsf{d}_{\rm Z}, \mathfrak{m}_{\rm Z})$ and an open set $\Omega \subset {\rm X}$, we shall denote by $W^{1,2}_0(\Omega) \subset W^{1,2}({\rm Z})$ the $W^{1,2}({\rm X})$-completion of the space of functions in $W^{1,2}({\rm X})$ with support in Ω. Using Theorem 3.34 we then see that every function in $\mathcal{A}$ belongs to $W^{1,2}_0(\mathrm{Ann}^{\rm Y}_{\varepsilon,\mathsf{R}})$ for any $\varepsilon \in (0, \mathsf{R})$ sufficiently small. In particular, the minimal weak upper gradient of such functions is well defined $\mathfrak{m}_{\rm Y}$-a.e..

Proposition 3.35. *For every* $\varepsilon \in (0, \mathsf{R})$*,* $\mathcal{A} \cap W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}})$ *is a dense subset of* $W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}})$*.*

proof It follows from the very same arguments used to prove the analogous statement in Proposition 6.6 in [14] keeping in mind Theorem 3.34 (see also Proposition 4.16 in [16] and [18]). □

Proposition 3.36. *For every* $\varepsilon, \varepsilon' \in (0, \mathsf{R}/2)$*, the map* $f \mapsto f \circ \mathsf{S}$ *is a homeomorphism from* $W_0^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ *to* $W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'})$*.*

proof Direct consequence of the measure preservation property (3.92), the Lipschitz properties (3.93), (3.94) and the fact that the Sobolev norm changes in a bi-Lipschitz way under a bi-Lipschitz change of the metric (see also Proposition 6.7 in [14], Proposition 4.16 in [16] and the arguments used in [18]). □

Proposition 3.37. *For every* $f \in \mathcal{A}$ *we have* $f \circ \mathsf{S} \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon})$ *for every* $\varepsilon > 0$ *sufficiently small. Moreover*

$$|\mathrm{D}f|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(f \circ \mathsf{S})|_{\mathrm{X}}, \qquad \mathfrak{m} - a.e. \text{ on } B_{\mathsf{R}}(\mathsf{O}).$$

proof The first claim follows from the fact, already noticed, that and $f \in \mathcal{A}$ belongs to $W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon})$ for $\varepsilon > 0$ sufficiently small and Proposition 3.36 above.

Now let $g \in \mathcal{G}$, $\varepsilon \in (0, \mathsf{R}/2)$ be arbitrary and pick $h \in \mathcal{H}$ identically 1 on $\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon}$. Theorem 3.24 grant that $(gh) \circ \mathsf{S} \in W^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}})$ and using also Theorem 3.34 and (3.90) we see that $|\mathrm{D}g|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(g \circ \mathsf{S})|_{\mathrm{X}}$ $\mathfrak{m}$-a.e. on $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon}$. As $\varepsilon > 0$ was chosen arbitrarily, we deduce that $|\mathrm{D}g|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(g \circ \mathsf{S})|_{\mathrm{X}}$ $\mathfrak{m}$-a.e. on $B_{\mathsf{R}}(\mathsf{O})$.

Similarly, given that a function $h \in \mathcal{H}$ is Lipschitz with support in $\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ for some $\varepsilon, \varepsilon' \in (0, \mathsf{R}/2)$, it is clear that $h \in W^{1,2}(\mathrm{Y})$ and, recalling (3.94), that $h \circ \mathsf{S} \in W^{1,2}(B_{\mathsf{R}}(\mathsf{O}))$. The fact that $|\mathrm{D}h|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(h \circ \mathsf{S})|_{\mathrm{X}}$ $\mathfrak{m}$-a.e. on $B_{\mathsf{R}}(\mathsf{O})$ then follows from the very same arguments used in Proposition 6.3 in [14] (see also Proposition 4.14 in [16] and [18]).

The conclusion for general $f \in \mathcal{A}$ then comes using the very same arguments of the proof of Proposition 6.5 in [14] (see also Proposition 4.15 in [16]), keeping in mind the infinitesimal Hilbertianity of X, Y, the characterisation of Sobolev functions on Y given by Theorem 3.34 and the first order differentiation formula (3.20). □

Theorem 3.38. *For any* $\varepsilon, \varepsilon' \in (0, \mathsf{R}/2)$ *we have* $f \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ *if and only if* $f \circ \mathsf{S} \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ *and in this case*

$$|\mathrm{D}f|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(f \circ \mathsf{S})|_{\mathrm{X}}, \qquad \mathfrak{m} - a.e. \text{ on } \mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}.$$

proof The fact that $f \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ if and only if $f \circ \mathsf{S} \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ has already been proved in Proposition 3.36. Now let $f \in W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ and use Proposition 3.35 to find a sequence $(f_n) \subset \mathcal{A}$ converging to it in $W_0^{1,2}(\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'})$. By Proposition 3.36 again we deduce that $f_n \circ \mathsf{S} \to f \circ \mathsf{S}$ in $W^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ and since by Proposition 3.37 we know that

$$|\mathrm{D}f_n|_{\mathrm{Y}} \circ \mathsf{S} = |\mathrm{D}(f_n \circ \mathsf{S})|_{\mathrm{X}}, \qquad \mathfrak{m} - a.e. \text{ on } \mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$$

for every $n \in \mathbb{N}$, passing to the limit (recall (3.92) for the left hand side) we conclude. □

3.9 Back to the metric properties and conclusion

We can now state and prove our main result.

Theorem 3.39 (Main result)**.** *Let $N \in [1,\infty)$, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ a $\mathsf{RCD}^*(0,N)$ space with* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$, $\mathsf{O} \in \mathrm{X}$ *and* $\mathsf{R} > \mathsf{r} > 0$ *such that*

$$\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O})) = \Big(\frac{\mathsf{R}}{\mathsf{r}}\Big)^N \mathfrak{m}(B_{\mathsf{r}}(\mathsf{O})).$$

Then exactly one of the following holds:

1) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains only one point. In this case (X,d) is isometric to $[0,\mathrm{diam}(\mathrm{X})]$ ($[0,\infty)$ if X is unbounded) with an isometry which sends O in 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}}(\mathsf{O})$ to the measure $c\,x^{N-1}\mathrm{d}x$ for $c := N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$.*

2) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains two points. In this case (X,d) is a 1-dimensional Riemannian manifold, possibly with boundary, and there is a bijective local isometry (in the sense of distance-preserving maps) from $B_{\mathsf{R}}(\mathsf{O})$ to $(-\mathsf{R},\mathsf{R})$ sending O to 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ to the measure $c\,|x|^{N-1}\mathrm{d}x$ for $c := \frac{1}{2}N\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))$. Moreover, such local isometry is an isometry when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$.*

3) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains more than two points. In this case:*
   - *$N \geq 2$ and the metric measure space $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ is a $\mathsf{RCD}^*(N-2,N-1)$ space.*
   - *The map $\mathsf{S} : B_{\mathsf{R}}(\mathsf{O}) \to \mathrm{Y}$ is a measure preserving local isometry which, when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$, is an isometry.*

proof Cases (1) and (2) have already been handled in Corollary 3.20, thus we assume that $S_{\mathsf{R}/2}(\mathsf{O})$ contains more than two points and notice that we already proved that T,S are measure preserving.

We now claim that for any $\varepsilon,\varepsilon' \in (0,\mathsf{R})$, the maps T,S are locally isometries from $\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ to $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ and viceversa.

To this aim, pick $y \in \mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ and let $r > 0$ be such that $B_{5r}(y) \subset \mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\bar{R}-\varepsilon'}$. Pick $y_1,y_2 \in B_r(y)$ and consider the function $f := \min\{\mathsf{d}_{\mathrm{Y}}(y_1,\cdot), 4r - \mathsf{d}_{\mathrm{Y}}(y_1,\cdot)\}$, which is supported in $B_{5r}(y)$.

By Theorem 3.38 we deduce that $\tilde{f} := f \circ \mathsf{S}$ is in $W^{1,2}(\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'})$ with $|\mathrm{D}\tilde{f}|_{\mathrm{X}} = |\mathrm{D}f|_{\mathrm{Y}} \circ \mathsf{S} \leq 1$ $\mathfrak{m}$-a.e. on $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$, the inequality being a consequence of the fact that f is 1-Lipschitz. Notice that being the support of $\tilde{f}$ contained in $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$, we can extend it to the whole X setting it to be 0 outside $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ and the new function, which we will continue to denote $\tilde{f}$, will still be in $W^{1,2}(\mathrm{X})$ with $|\mathrm{D}\tilde{f}| \leq 1$ $\mathfrak{m}$-a.e.. By the Sobolev-to-Lipschitz property of X - recall the discussion in the Section 2 -, we then deduce that $\tilde{f}$ has a 1-Lipschitz representative, but being $\tilde{f}$ continuous such representative must be equal to $\tilde{f}$ itself. In particular we have

$$\mathsf{d}_{\mathrm{Y}}(y_1,y_2) = |f(y_1) - f(y_2)| = |\tilde{f}(\mathsf{T}(y_1)) - \tilde{f}(\mathsf{T}(y_2))| \leq \mathsf{d}(\mathsf{T}(y_1),\mathsf{T}(y_2)).$$

Recalling that $y_1,y_2 \in B_r(y)$ were chosen arbitrarily and reversing the roles of $\mathrm{Ann}^{\mathrm{Y}}_{\varepsilon,\mathsf{R}-\varepsilon'}$, $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon'}$ in the argument (the Sobolev-to-Lipschitz property of Y being ensured by Theorem 3.34), we conclude.

Now let $x_0, x_1 \in \bar{B}_{\mathsf{R}/2}(\mathsf{O}) \setminus \{\mathsf{O}\}$ and, recalling that $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$ is a geodesic space, notice that their distance can be realized as limit of the lengths of a sequence of curves with range in $B_{\mathsf{R}}(\mathsf{O}) \setminus \{\mathsf{O}\}$. Any such curve has range in $\mathrm{Ann}^{\mathrm{X}}_{\varepsilon,\mathsf{R}-\varepsilon}$ for some $\varepsilon > 0$ and therefore its length is equal to the length in Y of its composition with S. This shows that the restriction of S to $\bar{B}_{\mathsf{R}/2}(\mathsf{O}) \setminus \{\mathsf{O}\}$ is 1-Lipschitz. In particular, it can be extended to a 1-Lipschitz map sending $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$ to $\bar{B}_{\mathsf{R}/2}(\mathsf{O}_{\mathrm{Y}})$ and it is obvious that such extension sends O in O_{Y} and thus agrees with $\mathsf{S}(\mathsf{O})$ as previously defined. Reversing the roles of X, Y we see that $\mathsf{S} : \bar{B}_{\mathsf{R}/2}(\mathsf{O}) \to \bar{B}_{\mathsf{R}/2}(\mathsf{O}_{\mathrm{Y}})$ is an isometry.

The fact that $N \geq 2$ now follows by considerations about the Hausdorff dimension. Indeed, the fact that Z is geodesic forces $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$ to contain a bi-Lipschitz copy of the square $[0,1]^2$. Thus the Hausdorff dimension of $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$ is at least 2 and since it is locally isometric to $B_{\mathsf{R}}(\mathsf{O})$, the same holds for X. The claim then follows recalling that N bounds the Hausdorff dimension of X from above (see Corollary 2.5 in [28]).

It remains to prove that $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ is a $\mathsf{RCD}^*(N-2, N-1)$ space. Thanks to the result of Ketterer [22, Theorem 5.28], to this aim it is sufficient to prove that Y, which by construction is a cone, is also a $\mathsf{RCD}^*(0, N)$ space.

This can be seen as follows. First of all, we recall that $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$ is isometric $\bar{B}_{\mathsf{R}/2}(\mathsf{O}_Y)$, then we observe that since $\bar{B}_{\mathsf{R}/4}(\mathsf{O}_{\mathrm{Y}})$ is a totally geodesic subset of Y, we must have that $\bar{B}_{\mathsf{R}/4}(\mathsf{O})$ is a totally geodesic subset of X (since geodesics with endpoints in $\bar{B}_{\mathsf{R}/4}(\mathsf{O})$ cannot leave $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$).

Now we use the Global-to-Local result established in [2] and [4] to deduce that $\bar{B}_{\mathsf{R}/4}(\mathsf{O})$ is $\mathsf{RCD}^*(0, N)$ and again the isomorphism of metric measure structures to see that $\bar{B}_{\mathsf{R}/4}(\mathsf{O}_{\mathrm{Y}})$ is $\mathsf{RCD}^*(0, N)$ as well.

Finally define the spaces $(\mathrm{Y}_r, \mathsf{d}_{\mathrm{Y}_r}, \mathfrak{m}_{\mathrm{Y}_r})$ as $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}/r, \mathfrak{m}_{\mathrm{Y}}/r^N)$ and note that Y_r is isomorphic to Y, the isomorphism being given by $(z, s) \mapsto (z, rs)$, so that trivially Y is the pointed-measured-Gromov-Hausdorff limit of the net $\{(\mathrm{Y}_r, \mathsf{d}_{\mathrm{Y}_r}, \mathfrak{m}_{\mathrm{Y}_r}, \mathsf{O}_{\mathrm{Y}_r})\}_{r>0}$ as $r \downarrow 0$. It is then clear that also the balls $B_{\mathsf{R}/(4r)}(\mathsf{O}_{\mathrm{Y}_r}) \subset \mathrm{Y}_r$ converge to Y and since the former are, by what said previously, $\mathsf{RCD}^*(0, N)$ spaces, the same is true for Y, as desired. □

4 Variants

There is nothing special about the choice $K = 0$ in the discussion we did. Here we, very briefly, discuss general lower bounds on the Ricci and the case of 'volume annulus implies metric annulus'.

For $K \in \mathbb{R}$, $N > 1$ define $s_{K,N} : \mathbb{R}^+ \to \mathbb{R}$ as

$$s_{K,N}(r) := \begin{cases} \sqrt{\frac{N-1}{K}} \sin(r\sqrt{\frac{K}{N-1}}), & \text{if } K > 0, \\ r, & \text{if } K = 0, \\ \sqrt{\frac{N-1}{|K|}} \sinh(r\sqrt{\frac{|K|}{N-1}}), & \text{if } K < 0, \end{cases}$$

and $v_{K,N} : \mathbb{R}^+ \to \mathbb{R}$ as

$$v_{K,N}(r) := \int_0^r \left| s_{K,N}(t) \right|^{N-1} \mathrm{d}t.$$

The following result is the analogous of Theorem 1.1 in the case X as a $\mathsf{RCD}^*(K, N)$ space, see [22] for the definition of (K, N)-cone.

Theorem 4.1. *Let $K \in \mathbb{R}$, $N \in [0,\infty)$, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ a $\mathsf{RCD}^*(K,N)$ space, $\mathsf{O} \in \mathrm{X}$ and $\mathsf{R} > \mathsf{r} > 0$ such that*

$$\frac{\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{v_{K,N}(\mathsf{R})} = \frac{\mathfrak{m}(B_{\mathsf{r}}(\mathsf{O}))}{v_{K,N}(\mathsf{r})}.$$

Then exactly one of the following holds:

1) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains only one point. In this case X is isometric to $[0,\mathrm{Diam}(\mathrm{X})]$ ($[0,\infty)$ if X is unbounded) with an isometry sending O to 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ to the measure $c\, s_{K,N}(x)\mathrm{d}x$ for $c := \frac{\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{v_{K,N}(\mathsf{R})}$.*

2) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains two points. In this case (X,d) is a 1-dimensional Riemannian manifold, possibly with boundary, and there is a bijective local isometry (in the sense of distance-preserving maps) from $B_{\mathsf{R}}(\mathsf{O})$ to $(-\mathsf{R},\mathsf{R})$ sending O to 0 and the measure $\mathfrak{m}|_{B_{\mathsf{R}}(\mathsf{O})}$ to the measure $c\, s_{K,N}(x)\mathrm{d}x$ for $c := \frac{\mathfrak{m}(B_{\mathsf{R}}(\mathsf{O}))}{2v_{K,N}(\mathsf{R})}$. Moreover, such local isometry is an isometry when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$.*

3) *$S_{\mathsf{R}/2}(\mathsf{O})$ contains more than two points. In this case there exists a $\mathsf{RCD}^*(N-2,N-1)$ space Z such that if Y is the (K,N) cone built over Z with "origin" O_Y, then $B_{\mathsf{R}}(\mathsf{O})$ is locally isometric to $B_{\mathsf{R}}(\mathsf{O}_{\mathrm{Y}})$, the corresponding metric ball in Y. Moreover the local isometry is a measure-preserving bijection, which, when restricted to $\bar{B}_{\mathsf{R}/2}(\mathsf{O})$, is an isometry.*

Exactly as in the case of Theorem 3.39 the space Z is given by a suitable rescaling of the sphere $S_{\mathsf{R}/2}(\mathsf{O})$ seen with its induced distance. The *proof* of this result follows along the very same lines used to obtain Theorem 3.39, the major difference being that here the 'Busemann' function is

$$\mathrm{b}(x) := \cos\Big(\mathsf{d}(x,\mathsf{O})\sqrt{\frac{K}{N-1}}\Big) \qquad \text{if } K > 0,$$
$$\mathrm{b}(x) := \cosh\Big(\mathsf{d}(x,\mathsf{O})\sqrt{\frac{|K|}{N-1}}\Big) \qquad \text{if } K < 0.$$

Then the computations can be carried over with the same modifications one would do in the smooth case. Eventually in order to apply Ketterer result in [22] to show that Z is a $\mathsf{RCD}^*(N-2,N-1)$ space one has to perform a blow-up analysis as the one done at the end of the proof of Theorem 3.39

In fact, the very same arguments can also be used to obtain the non-smooth version of the "volume annulus implies metric annulus" theorem. The main assumption in this case is that for our $\mathsf{RCD}^*(K,N)$ pointed space there are $0 < \mathsf{r}_1 < \mathsf{r}_2 < \mathsf{r}_3$ such that

$$\frac{\mathfrak{m}(B_{\mathsf{r}_3}(\mathsf{O})) - \mathfrak{m}(B_{\mathsf{r}_2}(\mathsf{O}))}{\int_{\mathsf{r}_2}^{\mathsf{r}_3} s_{K,N}^{N-1}(r)\,\mathrm{d}r} = \frac{\mathfrak{m}(B_{\mathsf{r}_2}(\mathsf{O})) - \mathfrak{m}(B_{\mathsf{r}_1}(\mathsf{O}))}{\int_{\mathsf{r}_1}^{\mathsf{r}_2} s_{K,N}^{N-1}(r)\,\mathrm{d}r}.$$

Then the same conclusions of the above theorem hold, with point (3) being replaced by:

3') $S_{\mathsf{R}/2}(\mathsf{O})$ contains more than two points. In this case, $N \geq 1$ and the annulus $\mathrm{Ann}^{\mathrm{X}}_{\mathsf{r}_1,\mathsf{r}_3}$ is locally isometric to the corresponding one in the (K,N)-cone built over a suitable rescaling of $S_{\mathsf{R}/2}(\mathsf{O})$ and the local isometry is a measure-preserving bijection.

The proof is the same. Note however that there is no ready-to-use result that would grant that the "sphere" of the cone is, with the induced distance and measure, a $\mathsf{RCD}^*(N-1,N)$ space. The problem is that we only have informations about the annulus $\mathrm{Ann}^{\mathrm{X}}_{\mathsf{r}_1,\mathsf{r}_3}$, which, being not convex, is certainly not a RCD^* space. A look to the proof of the aforementioned Ketterer's result, which is needed in Theorems 3.39 and 4.1, seems to suggest that the statement can be modified to be usable in our context, thus showing that even in this case the sphere is a $\mathsf{RCD}^*(N-2,N-1)$ space, but this is outside the scope of this paper.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.

[2] ——, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J., 163 (2014), pp. 1405–1490.

[3] ——, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, The Annals of Probability, 43 (2015), pp. 339–404. arXiv:1209.5786.

[4] L. Ambrosio, A. Mondino, and G. Savaré, *On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of $RCD^*(K,N)$ metric measure spaces*, The Journal of Geometric Analysis, 26 (2014), pp. 1–33.

[5] ——, *Nonlinear diffusion equations and curvature conditions in metric measure spaces.* Preprint, arXiv:1509.07273, 2015.

[6] L. Ambrosio, A. Pinamonti, and G. Speight, *Tensorization of Cheeger energies, the space $H^{1,1}$ and the area formula for graphs*, Adv. Math., 281 (2015), pp. 1145–1177.

[7] K. Bacher and K.-T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal., 259 (2010), pp. 28–56.

[8] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.

[9] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2), 144 (1996), pp. 189–237.

[10] ——, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom., 46 (1997), pp. 406–480.

[11] ——, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom., 54 (2000), pp. 13–35.

[12] ——, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom., 54 (2000), pp. 37–74.

[13] M. Erbar, K. Kuwada, and K.-T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Inventiones mathematicae, 201 (2014), pp. 1–79. arXiv:1303.4382.

[14] N. Gigli, *The splitting theorem in non-smooth context*. Preprint, arXiv:1302.5555, 2013.

[15] ——, *Nonsmooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below*. Accepted at Mem. Amer. Math. Soc., arXiv:1407.0809, 2014.

[16] ——, *An Overview of the Proof of the Splitting Theorem in Spaces with Non-Negative Ricci Curvature*, Analysis and Geometry in Metric Spaces, 2 (2014), pp. 169–213.

[17] ——, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc., 236 (2015), pp. vi+91. arXiv:1205.6622.

[18] N. Gigli and B. Han, *Sobolev spaces on warped products*. Preprint, arXiv:1512.03177.

[19] N. Gigli, K. Kuwada, and S.-i. Ohta, *Heat flow on Alexandrov spaces*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 307–331.

[20] N. Gigli and S. Mosconi, *The Abresch-Gromoll inequality in a non-smooth setting*, Discrete Contin. Dyn. Syst., 34 (2014), pp. 1481–1509.

[21] N. Gigli, T. Rajala, and K.-T. Sturm, *Optimal Maps and Exponentiation on Finite-Dimensional Spaces with Ricci Curvature Bounded from Below*, J. Geom. Anal., 26 (2016), pp. 2914–2929.

[22] C. Ketterer, *Cones over metric measure spaces and the maximal diameter theorem*, J. Math. Pures Appl. (9), 103 (2015), pp. 1228–1275.

[23] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2), 169 (2009), pp. 903–991.

[24] S.-i. Ohta, *Finsler interpolation inequalities*, Calc. Var. Partial Differential Equations, 36 (2009), pp. 211–249.

[25] T. Rajala, *Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm*, J. Funct. Anal., 263 (2012), pp. 896–924.

[26] G. Savaré, *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in* $\mathrm{RCD}(K,\infty)$ *metric measure spaces*, Discrete Contin. Dyn. Syst., 34 (2014), pp. 1641–1661.

[27] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math., 196 (2006), pp. 65–131.

[28] ——, *On the geometry of metric measure spaces. II*, Acta Math., 196 (2006), pp. 133–177.

[29] C. Villani, *Optimal transport. Old and new*, vol. 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2009.