Sobolev Spaces on Warped Products

Nicola Gigli* Bang-Xian Han †

December 10, 2015

Abstract

We study the structure of Sobolev spaces on the cartesian/warped products of a given metric measure space and an interval.

Keywords: warped product, Sobolev space, metric measure space.

Contents

1 Introduction

There is a well established definition of the space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ of real valued Sobolev functions defined on a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, see e.g. [14] for an overview of the topic and [1] for more recent developments. A function $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ comes with a function $|\mathrm{D}f|_X \in L^2(X, \mathfrak{m})$, called minimal weak upper gradient, playing the role of what the modulus of the distributional differential is in the smooth setting.

*SISSA, ngigli@sissa.it

†Universität Bonn, han@iam.uni-bonn.de

In this paper we are interested in the structure of the Sobolev spaces and the correspoding minimal weak upper gradients under some basic geometric constructions. The basic problem is the following. Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces and consider the space $X \times Y$ endowed with the product measure $\mathfrak{m}_c := \mathfrak{m}_X \times \mathfrak{m}_Y$ and the product distance d_c defined as

$$\mathrm{d}_c^2\big((x_1, y_1), (x_2, y_2)\big) := \mathrm{d}_X^2(x_1, x_2) + \mathrm{d}_Y^2(y_1, y_2), \qquad \forall x_1, x_2 \in X,\ y_1, y_2 \in Y.$$

Then one asks what is the relation between Sobolev functions on $X \times Y$ and those on X, Y. Guided by the Euclidean case, one might conjecture that $f \in W^{1,2}(X \times Y)$ if and only for $\mathfrak{m}_X$-a.e. x the function $y \mapsto f(x, y)$ is in $W^{1,2}(Y)$, for $\mathfrak{m}_Y$-a.e. y the function $x \mapsto f(x, y)$ is in $W^{1,2}(X)$ and the quantity

$$\sqrt{|\mathrm{D}f(\cdot, y)|_X^2(x) + |\mathrm{D}f(x, \cdot)|_Y^2(y)}$$

is in $L^2(X \times Y, \mathfrak{m}_c)$. Then one expects the above quantity to coincide with $|\mathrm{D}f|_{X \times Y}$.

Curiously, this kind of problem has not been studied until recently and, despite the innocent-looking statement, the full answer is not yet known.

The first result in this direction has been obtained in [6], where it has been proved that the conjecture is true under the very restrictive assumption that the spaces considered satisfy the, there introduced, $\mathrm{RCD}(K, \infty)$ condition for some $K \in \mathbb{R}$. Such restriction was necessary to use some regularization property of the heat flow.

The curvature condition has been dropped in the more recent paper [7]. There the authors prove that the above conjecture holds provided either both the base spaces are doubling and support a weak local 1-2 Poincaré inequality, or on both the spaces the integral of the local Lipschitz constant squared is a quadratic form on the space of Lipschitz functions.

Our contribution to the topic is the proof that the above conjecture is always true, provided one of the two spaces is $\mathbb{R}$ or a closed subinterval of $\mathbb{R}$. Our strategy is new and also allows to cover the case of warped product of a space and a closed interval, thus permitting to consider basic geometric constructions like that of cone and spherical subspension of a given space.

In fact, this line of research is motivated by the study of geometric properties of metric measure spaces, typically having Ricci curvature bounded from below in the appropriate weak sense, via the study of Sobolev functions on them (see in particular [11] and [12] for two examples where this project has been carried out).

In the last section of the paper we study the Sobolev-to-Lipschitz property (see Section 3.3 for the definition) of a warped product. Such notion, introduced in [11], is key to deduce precise metric information from the study of Sobolev functions. It is therefore interesting to ask whether warped products have this property. We will show that this is the case under very general assumptions on the warping functions and the base space.

2 Preliminaries

2.1 Metric measure spaces

Let (X, d) be a complete metric space. By a curve γ we shall typically denote a a continuous map $\gamma : [0,1] \mapsto X$, although sometimes curves defined on different intervals will be considered. The space of curves on $[0,1]$ with values in X is denoted by $C([0,1], X)$. The space $C([0,1], X)$ equipped with the uniform distance is a complete metric space.

We define the length of γ by

$$l[\gamma] := \sup_{\tau} \sum_{i=1}^{n} \mathrm{d}(\gamma(t_{i-1}), \gamma(t_i))$$

where $\tau := \{0 = t_0, t_1, ..., t_n = 1\}$ is a partition of $[0,1]$. The supreme here can be changed to 'lim' and the limit is taken with respect to the refinement ordering of partitions.

The space (X, d) is said to be a length space if for any $x, y \in X$ we have

$$\mathrm{d}(x, y) = \inf_{\gamma} l[\gamma]$$

where the infimum is taken among all $\gamma \in C([0,1], X)$ which connect x and y.

If the infimum is always a minimum, then the space is called geodesic space and we call the minimizers pre-geodesics. A geodesic from x to y is any pre-geodesic which is parameterized by constant speed. Equivalently, a geodesic from x to y is a curve γ such that:

$$\mathrm{d}(\gamma_s, \gamma_t) = |s - t| \mathrm{d}(\gamma_0, \gamma_1), \qquad \forall t, s \in [0,1], \quad \gamma_0 = x, \gamma_1 = y.$$

The space of all geodesics on X will be denoted by Geo(X). It is a closed subset of $C([0,1], X)$.

Given $p \in [1, +\infty]$ and a curve γ, we say that γ belongs to $AC^p([0,1], X)$ if

$$\mathrm{d}(\gamma_s, \gamma_t) \leq \int_s^t G(r) \, \mathrm{d}r, \quad \forall\, t, s \in [0,1],\ s < t$$

for some $G \in L^p([0,1])$. In particular, the case $p = 1$ corresponds to absolutely continuous curves, whose class is denoted by $AC([0,1], X)$. It is known (see for instance Theorem 1.1.2 of [3]) that for $\gamma \in AC([0,1], X)$, there exists an a.e. minimal function G satisfying this inequality, called the metric derivative which can be computed for a.e. $t \in [0,1]$ as

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|}.$$

It is known that (see for example [8], [9]) the length of a curve $\gamma \in AC([0,1], X)$ can be computed as

$$l[\gamma] := \int_0^1 |\dot{\gamma}_t| \, \mathrm{d}t.$$

In particular, on a length space X we have

$$\mathrm{d}(x,y) = \inf_{\gamma} \int_0^1 |\dot\gamma_t| \, \mathrm{d}t$$

where the infimum is taken among all $\gamma \in AC([0,1],X)$ which connect x and y.

Given $f : X \mapsto \mathbb{R}$, the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0,\infty]$ is defined as

$$\mathrm{lip}(f)(x) := \varlimsup_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}$$

if x is not isolated, 0 otherwise, while the (global) Lipschitz constant is defined as

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}.$$

If (X,d) is a length space, we have $\mathrm{Lip}(f) = \sup_x \mathrm{lip}(f)(x)$.

We are not only interested in metric structure, but also in the interaction between metric and measure. For the metric measure space $(X,\mathrm{d},\mathfrak{m})$, basic assumptions used in this paper are:

Assumption 2.1. The metric measure space $(X,\mathrm{d},\mathfrak{m})$ satisfies:

- (X,d) is a complete and separable length space,
- $\mathfrak{m}$ is a non-negative Borel measure with respect to d and finite on bounded sets,
- $\mathrm{supp}\,\mathfrak{m} = X$.

Moreover, for brevity we will not distinguish $X, (X,\mathrm{d})$ or $(X,\mathrm{d},\mathfrak{m})$ when no ambiguities exist. For example, we write $S^2(X)$ instead of $S^2(X,\mathrm{d},\mathfrak{m})$ (see the next section).

2.2 Optimal transport and Sobolev functions

The set of Borel probability measures on (X,d) will be denoted by $\mathcal{P}(X)$. We also use $\mathcal{P}_2(X) \subseteq P(X)$ to denote the set of measures with finite 2-moment, i.e. $\mu \in \mathcal{P}_2(X)$ if $\mu \in \mathcal{P}(X)$ and $\int \mathrm{d}^2(x,x_0)\,\mathrm{d}\mu(x) < +\infty$ for some (and thus every) $x_0 \in X$.

For $t \in [0,1]$, the evaluation map $e_t : C([0,1],X) \to X$ is given by

$$e_t(\gamma) := \gamma_t, \qquad \forall \gamma \in C([0,1],X).$$

Then we consider $(\mathcal{P}_2(X), \mathcal{W}_2)$, where we endow $\mathcal{P}_2(X)$ with the 2-Wasserstein distance $\mathcal{W}_2$ defined by:

$$\mathcal{W}_2^2(\mu,\nu) := \inf_{\pi} \int \mathrm{d}^2(x,y)\,\mathrm{d}\pi(x,y),$$

where the inf is taken among all plans π with marginal μ and ν, i.e. $(\Pi_1)_\sharp\pi = \mu$ and $(\Pi_2)_\sharp\pi = \nu$ where Π_i, $i = 1, 2$ are the projection maps onto the first and second coordinate respectively.

It is known that there exists an optimal transport plan π realizing the infimum in the Kantorovich problem. We denote the set of optimal transport plans between μ and ν by $\mathrm{Opt}(\mu, \nu)$.

Some other important properties of the distance $\mathcal{W}_2$ are the following.

Proposition 2.2 (Geodesics in the Wasserstein space, [2]). *Let (X, d) be a metric space and $\mu, \nu \in \mathcal{P}_2(X)$. Then the curve (μ_t) is a constant speed geodesic connecting μ and ν, i.e. it satisfies*

$$\mathcal{W}_2(\mu_s, \mu_t) = |s - t|\mathcal{W}_2(\mu_0, \mu_1), \qquad \forall\ s, t \in [0, 1] \tag{2.1}$$

and $\mu_0 = \mu, \mu_1 = \nu$, if and only if there exists a plan $\pi \in \mathcal{P}_2(\mathrm{Geo(X)}) \subseteq \mathcal{P}_2(\mathrm{C([0,1],X)})$ such that

$$\mu_t = (e_t)_\sharp\pi \quad \forall\ t \in [0, 1]; \quad (e_0)_\sharp\pi = \mu, \ (e_1)_\sharp\pi = \nu,$$

and $(e_0, e_1)_\sharp\pi \in \mathrm{Opt}(\mu_0, \mu_1)$. We denote the set of these measures in $\mathcal{P}_2(\mathrm{Geo(X)})$ by $\mathrm{OptGeo}(\mu_0, \mu_1)$.

In particular, if X is geodesic, the space $(\mathcal{P}_2(X), \mathcal{W}_2)$ is also a geodesic space.

Moreover, absolutely continuous curves in $(\mathcal{P}_2, \mathcal{W}_2)$ are characterized by the following theorem:

Theorem 2.3 (Superposition principle, [13]). *Let (X, d) be a complete and separable metric space, and $(\mu_t) \in AC^2([0, 1], \mathcal{P}_2(X))$. Then there exists a measure $\pi \in \mathcal{P}(C([0, 1], X))$ concentrated on $AC^2([0, 1], X)$ such that:*

$$\begin{aligned} (e_t)_\sharp\pi &= \mu_t, && \forall t \in [0, 1] \\ \int |\dot\gamma_t|^2 \, \mathrm{d}\pi(\gamma) &= |\dot\mu_t|^2, && a.e.\ t. \end{aligned}$$

Moreover, the minimum of the energy $\int_0^1 \int |\dot\gamma_t|^2 \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t$ among all the plans π' satisfying $(e_t)_\sharp\pi' = \mu_t$ for every $t \in [0, 1]$ is obtained by this plan π.

Definition 2.4 (Test plan). Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $\pi \in \mathcal{P}(C([0, 1], X))$. We say that π has bounded compression provided there exists $C > 0$ such that

$$(e_t)_\sharp\pi \le C\mathfrak{m}, \quad \forall t \in [0, 1].$$

Then we say that π is a test plan if it has bounded compression, is concentrated on $AC^2([0, 1], X)$ and

$$\int_0^1 \int |\dot\gamma_t|^2 \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t < +\infty.$$

The notion of Sobolev function is given by duality with that of test plan:

Definition 2.5 (Sobolev class)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. A Borel function $f : X \to \mathbb{R}$ belongs to the Sobolev class $S^2(X, \mathrm{d}, \mathfrak{m})$ (resp. $S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$) provided there exists a non-negative function $G \in L^2(X, \mathfrak{m})$ (resp. $L^2_{loc}(X, \mathfrak{m})$) such that

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \leq \int \int_0^1 G(\gamma_s)|\dot{\gamma}_s| \,\mathrm{d}s \,\mathrm{d}\pi(\gamma), \quad \forall \text{ test plan } \pi.$$

In this case, G is called a 2-weak upper gradient of f, or simply weak upper gradient.

It is known, see e.g. [5], that there exists a minimal function G in the $\mathfrak{m}$-a.e. sense among all the weak upper gradients of f. We denote such minimal function by $|\mathrm{D}f|$ or $|\mathrm{D}f|_X$ to emphasize which space we are considering and call it minimal weak upper gradient. Notice that if f is Lipschitz, then $|\mathrm{D}f| \leq \mathrm{lip}(f)$ $\mathfrak{m}$-a.e., because $\mathrm{lip}(f)$ is a weak upper gradient of f.

It is known that the locality holds for $|\mathrm{D}f|$, i.e. $|\mathrm{D}f| = |\mathrm{D}g|$ a.e. on the set $\{f = g\}$, moreover $S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$ is a vector space and the inequality

$$|\mathrm{D}(\alpha f + \beta g)| \leq |\alpha||\mathrm{D}f| + |\beta||\mathrm{D}g|, \qquad \mathfrak{m} - a.e., \tag{2.2}$$

holds for every $f, g \in S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$ and $\alpha, \beta \in \mathbb{R}$ and the space $S^2_{loc} \cap L^\infty_{loc}(X, \mathrm{d}, \mathfrak{m})$ is an algebra, with the inequality

$$|\mathrm{D}(fg)| \leq |f||\mathrm{D}g| + |g||\mathrm{D}f|, \qquad \mathfrak{m} - a.e., \tag{2.3}$$

being valid for any $f, g \in S^2_{loc} \cap L^\infty_{loc}(X, \mathrm{d}, \mathfrak{m})$.

Another basic - and easy to check - property of minimal weak upper gradients that we shall frequently use is their semicontinuity in the following sense: if $(f_n) \subset S^2(X, \mathrm{d}, \mathfrak{m})$ is a sequence $\mathfrak{m}$-a.e. converging to some f and such that $(|\mathrm{D}f_n|)$ is bounded in $L^2(X, \mathfrak{m})$, then $f \in S^2(X, \mathrm{d}, \mathfrak{m})$ and

$$|\mathrm{D}f| \leq G, \qquad \mathfrak{m} - a.e.,$$

for every L^2-weak limit G of some subsequence of $(|\mathrm{D}f_n|)$.

Then the Sobolev space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is defined as $W^{1,2}(X, \mathrm{d}, \mathfrak{m}) := S^2(X, \mathrm{d}, \mathfrak{m}) \cap L^2(X, \mathfrak{m})$ and is endowed with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

$W^{1,2}(X)$ is always a Banach space, but in general it is not an Hilbert space. Following [10], we say that $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}(X)$ is an Hilbert space.

In [4, 5] the following result has been proved.

Proposition 2.6 (Density in energy of Lipschitz functions)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $f \in W^{1,2}(X)$. Then there exists a sequence (f_n) of Lipschitz functions L^2-converging to f such that the sequence $(\mathrm{lip}(f_n))$ L^2-converges to $|\mathrm{D}f|$.*

2.3 Product spaces

In this subsection we recall the basic concepts and results about the Cartesian product and the warped product of two spaces. Both metric and metric measure structures are considered.

Given two metric measure spaces $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$, we define their (Cartesian) product as:

Definition 2.7 (Cartesian product)**.** We define the space $(Y \times X, \mathrm{d}_c, \mathfrak{m}_c)$ as the product space $Y \times X$ equipped with the distance $\mathrm{d}_c := \mathrm{d}_Y \times \mathrm{d}_X$ and the measure $\mathfrak{m}_c := \mathfrak{m}_Y \times \mathfrak{m}_X$. Here $\mathrm{d}_c = \mathrm{d}_Y \times \mathrm{d}_X$ means:

$$\mathrm{d}_c((y_1, x_1), (y_2, x_2)) = \sqrt{\mathrm{d}_Y^2(y_1, y_2) + \mathrm{d}_X^2(x_1, x_2)},$$

for any pairs $(y_1, x_1), (y_2, x_2) \in Y \times X$.

There is a natural and simple to prove (see e.g. [11]) link between Sobolev functions on the product depending on just one variable and Sobolev functions on the base spaces:

Proposition 2.8. *Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces, $g \in L^2_{loc}(X)$ and define $f \in L^2_{loc}(Y \times X)$ as $f(y, x) := g(x)$.*

Then $f \in S^2_{loc}(Y \times X)$ if and only if $g \in S^2_{loc}(X)$ and in this case the identity

$$|\mathrm{D}f|_{Y \times X}(y, x) = |\mathrm{D}g|_X(x),$$

holds for $\mathfrak{m}_c$-a.e. (y, x).

To define the warped product metric we need first to introduce the corresponding notion of length:

Definition 2.9 (Warped length of curves)**.** Let (X, d_X) and (Y, d_Y) be two complete and separable metric spaces and $w_\mathrm{d} : Y \to \mathbb{R}^+$ a continuous function. Let $\gamma = (\gamma^Y, \gamma^X)$ be a curve such that γ^X, γ^Y are absolutely continuous. Then the w_d-length of γ is defined as

$$l_w[\gamma] := \lim_\tau \sum_{i=1}^n \sqrt{\mathrm{d}_Y^2(\gamma^Y_{t_{i-1}}, \gamma^Y_{t_i}) + w_\mathrm{d}^2(\gamma^Y_{t_{i-1}})\mathrm{d}_X^2(\gamma^X_{t_{i-1}}, \gamma^X_{t_i})},$$

where $\tau := \{0 = t_0, t_1, ..., t_n = 1\}$ is a partition of $I = [0, 1]$ and the limit is taken with respect to the refinement ordering of partitions.

It is not hard to check that the limit exists and that the formula

$$l_w[\gamma] = \int_0^1 \sqrt{|\dot\gamma^Y_t|^2 + w_\mathrm{d}^2(\gamma^Y_t)|\dot\gamma^X_t|^2}\,\mathrm{d}t$$

holds.

Then we can define the metric d_w using this length structure:

Definition 2.10 (Warped product of metric spaces)**.** With the same assumptions of Definition 2.9, we define a pseudo-metric d_w on the space $Y \times X$ by

$$\mathrm{d}_w(p,q) := \inf\{l_w[\gamma] : \gamma \text{ is an absolutely continuous curve from } p \text{ to } q\},$$

for any $p, q \in Y \times X$.

d_w induces an equivalent relation on $Y \times X$: two points p, q are declared equivalent provided $\mathrm{d}_w(p,q) = 0$. The completion of the quotient of $Y \times X$ via this equivalence relation will be denoted by $Y \times_w X$. Then d_w induces a distance on $Y \times_w X$ which we shall continue to denote as d_w. Abusing a bit the notation, we shall also denote the typical element of $Y \times_w X$ as (y, x) with $y \in Y$ and $x \in X$ (there is no abuse in doing this if $w_{\mathrm{d}}(y) > 0$ and points in the completion not coming from points in $Y \times X$ will be negligible w.r.t. the warped product of measures and the same holds for the set of (y, x) such that $w_{\mathrm{d}}(y) = 0$, see below).

Notice that by definition $(Y \times_w X, \mathrm{d}_w)$ is a complete, separable and length space.

When considering the warped product of two metric measure spaces, we shall need to fix two warping functions: one for the distance and another for the measure.

Definition 2.11 (Warped products of metric measure spaces)**.** Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$, $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces and $w_{\mathrm{d}}, w_{\mathfrak{m}} : Y \to \mathbb{R}^+$ two functions. We say that $w_{\mathrm{d}}, w_{\mathfrak{m}}$ are warping functions provided they are continuous and such that $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$.

In this case, the measure $\mathfrak{m}_w$ is defined via the formula:

$$\int f(x)g(y)\,\mathrm{d}\mathfrak{m}_w(y,x) = \int \left(\int f(x)w_{\mathfrak{m}}(y)\,\mathrm{d}\mathfrak{m}_X(x)\right) g(y)\,\mathrm{d}\mathfrak{m}_Y(y), \tag{2.4}$$

for any Borel non-negative functions f and g.

The warped product of $(X, \mathrm{d}_X, \mathfrak{m}_X)$, $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ via the functions $w_{\mathrm{d}}, w_{\mathfrak{m}}$ is then defined as $(Y \times_w X, \mathrm{d}_w, \mathfrak{m}_w)$.

It is immediate to verify that the assumption $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$ grants that formula (2.4) truly defines a Borel measure on the space $(Y \times_w X, \mathrm{d}_w)$.

3 The results

3.1 Cartesian product

Throughtout this section $(X, \mathrm{d}, \mathfrak{m})$ is a fixed complete, separable and length space and $I \subset \mathbb{R}$ a closed, possibly unbounded, interval. We are interested in studying the Cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of I, endowed with its Euclidean structure, and $(X, \mathrm{d}, \mathfrak{m})$.

Given a function $f : X_c \to \mathbb{R}$ and $x \in X$ we denote by $f^{(x)} : I \to \mathbb{R}$ the function given by $f^{(x)}(t) := f(t, x)$. Similarly, for $t \in I$ we denote by $f^{(t)} : X \to \mathbb{R}$ the function given by $f^{(t)}(x) := f(t, x)$.

We start introducing the Beppo Levi space $\mathsf{BL}(X_c)$:

Definition 3.1 (The space $\mathsf{BL}(X_c)$)**.** The space $\mathsf{BL}(X_c) \subset L^2(X_c, \mathfrak{m}_c)$ is the space of functions $f \in L^2(X_c, \mathfrak{m}_c)$ such that

i) $f^{(x)} \in W^{1,2}(I)$ for $\mathfrak{m}$-a.e. x,

ii) $f^{(t)} \in W^{1,2}(X)$ for $\mathcal{L}^1$-a.e. t

iii) the function

$$|\mathrm{D}f|_c(t,x) := \sqrt{|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_I^2(t)},$$

belongs to $L^2(X_c, \mathfrak{m}_c)$.

On $\mathsf{BL}(X_c)$ we put the norm

$$\|f\|^2_{\mathsf{BL}(X_c)} := \|f\|^2_{L^2(X_c)} + \||\mathrm{D}f|_c\|^2_{L^2(X_c)}.$$

The space $\mathsf{BL}_{loc}(X_c)$ is the subset of $L^2_{loc}(X_c, \mathfrak{m}_c)$ of functions which are locally equal to some function in $\mathsf{BL}(X_c)$.

The main result of this section is the identification of the spaces $W^{1,2}(X_c)$ and $\mathsf{BL}(X_c)$ and of their corresponding weak gradients $|\mathrm{D}f|_{X_c}$ and $|\mathrm{D}f|_c$.

One inclusion has been proved in [6]:

Proposition 3.2 (Proposition 6.18 of [6])**.** *We have $W^{1,2}(X_c) \subset \mathsf{BL}(X_c)$ and*

$$\int_{X_c} |\mathrm{D}f|_c^2 \, \mathrm{d}\mathfrak{m}_c \leq \int_{X_c} |\mathrm{D}f|_{X_c}^2 \, \mathrm{d}\mathfrak{m}_c, \qquad \forall f \in W^{1,2}(X_c). \tag{3.1}$$

To prove the other one it is useful to introduce the following classes of functions:

Definition 3.3 (The classes $\mathcal{A}$ and $\tilde{\mathcal{A}}$)**.** We define the space of functions $\mathcal{A} \subset \mathsf{BL}_{loc}(X_c)$ as

$$\mathcal{A} := \Big\{ g_1(x) + h(t) g_2(x) \in \mathsf{BL}_{loc}(X_c) \ : \ g_1, g_2 \in W^{1,2}(X), \quad h : I \to \mathbb{R} \ \text{ is Lipschitz} \Big\},$$

and the space $\tilde{\mathcal{A}} \subset \mathsf{BL}_{loc}(X_c)$ as the set of functions $f \in \mathsf{BL}_{loc}(X_c)$ which are locally equal to some function in $\mathcal{A}$.

Notice that Proposition 2.8 and the calclulus rules (2.2), (2.3) ensure that

$$\tilde{\mathcal{A}} \subset S^2_{loc}(X_c). \tag{3.2}$$

We start with the following purely metric lemma:

Lemma 3.4. *Let $f : X_c \to \mathbb{R}$ be of the form $f(t,x) = g_1(x) + h(t) g_2(x)$ for Lipschitz functions g_1, g_2, h. Then*

$$\mathrm{lip}(f)^2(t,x) \leq \mathrm{lip}_X(f^{(t)})^2(x) + \mathrm{lip}_I(f^{(x)})^2(t)$$

for every $(t,x) \in X_c$.

Proof. Let $(t,x),(s,y)\in X_c$, and notice that

$$\begin{aligned}
&|f(s,y)-f(t,x)|\\
&=|g_1(y)+h(s)g_2(y)-g_1(x)-h(t)g_2(x)|\\
&\le |h(s)-h(t)||g_2(y)|+|g_1(y)-g_1(x)+h(t)(g_2(y)-g_1(x))|\\
&\le \frac{|h(s)-h(t)||g_2(y)|}{|s-t|}|s-t|+\frac{|g_1(y)-g_1(x)+h(t)(g_2(y)-g_1(x))|}{\mathrm{d}(x,y)}\mathrm{d}(x,y)\\
&\le \sqrt{\frac{|h(s)-h(t)|^2|g_2(y)|^2}{|s-t|^2}+\frac{|g_1(y)-g_1(x)+h(t)(g_2(y)-g_1(x))|^2}{\mathrm{d}^2(x,y)}}\,\mathrm{d}_c\big((s,y),(t,x)\big).
\end{aligned}$$

Dividing by $\mathrm{d}_c\big((s,y),(t,x)\big)$, letting $(s,y)\to(t,x)$ and using the continuity of g_2 we get the conclusion. □

The interest of functions in $\tilde{\mathcal{A}}$ is due to the next two results:

Proposition 3.5. *Let $f\in\tilde{\mathcal{A}}$. Then*

$$|\mathrm{D}f|_{X_c}=|\mathrm{D}f|_c \qquad \mathfrak{m}_c-a.e..$$

Proof. Notice that by (3.2) the statement makes sense. Moreover, due to the local nature of the statement we can assume that $f(t,x)=g_1(x)+h(t)g_2(x)\in\mathcal{A}$ with h having compact support. With this assumption we have that $f\in W^{1,2}(X_c)$ so that keeping in mind Proposition 3.2, to conclude it is sufficient to prove that

$$|\mathrm{D}f|^2_{\tilde{X}}(t,x)\le|\mathrm{D}f^{(x)}|^2_I(t)+|\mathrm{D}f^{(t)}|^2_X(x),\qquad \mathfrak{m}_c-a.e.\ (t,x). \tag{3.3}$$

To this aim, it is in turn sufficient to show that for any $[a,b)\subset I$ and any Borel set $E\subset X$ we have

$$\int_{\tilde{E}}|\mathrm{D}f|^2_{X_c}(t,x)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x)\le\int_{\tilde{E}}|\mathrm{D}f^{(x)}|^2_I(t)+|\mathrm{D}f^{(t)}|^2_X(x)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) \tag{3.4}$$

with $\tilde{E}:=[a,b)\times E$. Indeed if this holds, taking into account that open sets in X_c can always written as disjoint countable union of sets of the form $[a,b)\times E$, we deduce that (3.4) holds with $\tilde{E}$ generic open set in X_c, so that using the fact that the integrand are in $L^1(X_c,\mathfrak{m}_c)$ by exterior approximation we get that (3.4) holds for arbitrary Borel sets $\tilde{E}\subset X_c$ and thus (3.3) and the conclusion.

Thus fix $E\subset X$ Borel, let $\tilde{E}:=[a,b)\times E$ and up to a simple scaling argument assume also that $[a,b)=[0,1)$.

For $k,i\in\mathbb{N}$, $k>0$, we define $f_{k,i}\in W^{1,2}(X)$ as $f_{k,i}(x):=g_1(x)+h(\frac{i}{k})g_2(x)$ and $f_k\in\mathsf{BL}(X_c)$ as

$$f_k(t,x):=(kt-i)f_{k,i+1}(x)+(i+1-kt)f_{k,i}(x),\qquad \text{for } t\in\big[\tfrac{i}{k},\tfrac{i+1}{k}\big].$$

Notice that $f_k\to f$ in $L^2(X_c,\mathfrak{m}_x)$. By Proposition 2.6, for each (k,i) we can find a sequence of Lipschitz functions $f_{k,i,n}\in\mathrm{Lip}\,(X)$ converging to $f_{k,i}$ in $L^2(X,\mathfrak{m})$ such that $\lim_{n\to\infty}\mathrm{lip}(f_{k,i,n})=|\mathrm{D}f_{k,i}|_X$ in $L^2(X,\mathfrak{m})$.

Then we define $F_{k,n} \in \mathrm{Lip}(X_c)$ as

$$F_{k,n}(t,x) := (kt-i)f_{k,i+1,n}(x) + (i+1-kt)f_{k,i,n}(x), \qquad \text{for } t \in \left[\tfrac{i}{k}, \tfrac{i+1}{k}\right].$$

By construction we have $F_{k,n} \in \tilde{\mathcal{A}}$, so that Lemma 3.4 gives

$$|\mathrm{lip}(F_{k,n})|^2 \le |\mathrm{lip}_X(F_{k,n})|^2 + |\mathrm{lip}_I(F_{k,n})|^2, \qquad \mathcal{L}^1 \times \mathfrak{m} - a.e.,$$

moreover, since $\lim_{n\to\infty} F_{k,n} = f_k$ in $L^2(X_c, \mathfrak{m}_c)$ for every k, the lower semicontinuity of minimal weak upper gradients gives that

$$\int_{\tilde{E}} |\mathrm{D}f|^2_{\tilde{X}} \, \mathrm{d}\mathfrak{m}_c \le \varlimsup_{k\to\infty} \varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_X(F_{k,n})^2 \, \mathrm{d}\mathfrak{m}_c + \varlimsup_{k\to\infty} \varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_I(F_{k,n})^2 \, \mathrm{d}\mathfrak{m}_c. \quad (3.5)$$

Another direct consequence of the definition is that

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathrm{lip}_I(F^{(x)}_{k,n})(t) = |\mathrm{D}f^{(x)}|_I(t), \qquad \mathfrak{m}_c - a.e.\ (t,x),$$

which together with an application of the dominate convergence theorem grants that

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_I(F^{(x)}_{k,n})^2(t) \, \mathrm{d}\mathfrak{m}_c(t,x) = \int_{\tilde{E}} |\mathrm{D}f^{(x)}|^2_I(t) \, \mathrm{d}\mathfrak{m}_c(t,x). \quad (3.6)$$

On the other hand, the continuity of h grants that $\mathbb{R} \ni t \mapsto f^{(t)} \in W^{1,2}(X)$ is continuous so that also the map $I \ni t \mapsto \int_E |\mathrm{D}f^{(t)}|^2_X \, \mathrm{d}\mathfrak{m}$ is continuous. In particular, its integral on $[0,1]$ and coincides with the limit of the Riemann sums:

$$\begin{aligned}\int_{\tilde{E}} |\mathrm{D}f^{(t)}|^2_X(x) \, \mathrm{d}\mathfrak{m}_c(t,x) &= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \int_E |\mathrm{D}f_{k,i}|^2_X \, \mathrm{d}\mathfrak{m} \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{k} \sum_{i=0}^{k} \int_E \mathrm{lip}_X(f_{k,i,n})^2 \, \mathrm{d}\mathfrak{m}.\end{aligned} \quad (3.7)$$

From the very definition of $F_{k,n}$ we get that

$$\begin{aligned}\mathrm{lip}_X(F^{(t)}_{k,n})^2 &\le \big((kt-i)\mathrm{lip}_X(f_{k,i+1,n}) + (i+1-kt)\mathrm{lip}_X(f_{k,i,n})\big)^2 \\ &\le (kt-i)\mathrm{lip}_X(f_{k,i+1,n})^2 + (i+1-kt)\mathrm{lip}_X(f_{k,i,n})^2,\end{aligned}$$

on X for every $t \in [\frac{i}{k}, \frac{i+1}{k}]$, and thus

$$\begin{aligned}\int_{\tilde{E}} \mathrm{lip}_X(F^{(t)}_{k,n})^2(x) \, \mathrm{d}\mathfrak{m}_c(t,x) &\le \int_X \frac{1}{k} \sum_{i=0}^{k} \mathrm{lip}_X(f_{k,i,n})^2 - \frac{1}{2}\Big(\mathrm{lip}_X(f_{k,0,n})^2 + \mathrm{lip}_X(f_{k,k,n})^2\Big) \, \mathrm{d}\mathfrak{m} \\ &\le \int_X \frac{1}{k} \sum_{i=0}^{k} \mathrm{lip}_X(f_{k,i,n})^2 \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

This inequality together with (3.7) give

$$\varlimsup_{k\to\infty} \varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_X(F^{(t)}_{k,n})^2(x) \, \mathrm{d}\mathfrak{m}_c(t,x) \le \int_{\tilde{E}} |\mathrm{D}f^{(t)}|^2_X \, \mathrm{d}t \, \mathrm{d}\mathfrak{m},$$

which together with (3.6) and (3.5) gives (3.4) and the conclusion. □

Proposition 3.6 (Density in energy)**.** *For any function $f \in \mathsf{BL}(X_c)$ there exists a sequence $(f_n) \subset \mathsf{BL}(X_c) \cap \mathcal{A}_{loc}$ converging to f in $L^2(X_c, \mathfrak{m}_c)$ such that $|\mathrm{D} f_n|_c \to |\mathrm{D} f|_c$ in $L^2(X_c, \mathfrak{m}_c)$ as $n \to \infty$.*

Proof. We shall give the proof for the case $I = \mathbb{R}$, the argument for arbitrary I being similar.

With a standard cut-off, truncation and diagonalization argument we can, and will, assume that the given $f \in \mathsf{BL}(X_c)$ is bounded and with bounded support. Then for any $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ we define

$$g_{i,n}(x) := n \int_{\frac{i}{n}}^{\frac{(i+1)}{n}} f(x,s)\, ds,$$

and

$$h_{i,n}(t) := \chi_n\big(t - \frac{i}{n}\big),$$

where $\chi_n : \mathbb{R} \mapsto \mathbb{R}$ is given by:

$$\chi_n(t) := \begin{cases} 0, & \text{if} \quad t < -\frac{1}{n}, \\ nt + 1, & \text{if} \quad -\frac{1}{n} \le t < 0, \\ 1 - nt, & \text{if} \quad 0 \le t < \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < t. \end{cases} \tag{3.8}$$

Then we define the sequence (f_n) as:

$$f_n(t,x) := \sum_{i \in \mathbb{Z}} h_{i,n}(t) g_{i,n}(x),$$

the sum being well defined because $g_{i,n}$ is not zero only for a finite number of i's and it is immediate to check that $f_n \in \tilde{\mathcal{A}}$.

We claim that $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ as $n \to \infty$. Integrating the inequality

$$\begin{aligned} \big(f_n(t,x)\big)^2 &= \left(\sum_{i \in \mathbb{Z}} h_{i,n}(t) g_{i,n}(x) \right)^2 \\ &\le \sum_{i \in \mathbb{Z}} h_{i,n}(t) \big(g_{i,n}(x)\big)^2 \le \sum_{i \in \mathbb{Z}} h_{i,n}(t) n \int_{i/n}^{(i+1)/n} f^2(s,x)\, \mathrm{d}s, \end{aligned}$$

on x and t we obtain $\|f_n\|_{L^2(X_c)} \le \|f\|_{L^2(X_c)}$, for every $n \in \mathbb{N}$. This means that the linear operator T_n from $L^2(X_c, \mathfrak{m}_c)$ into itself assigning f_n to f is 1-Lipschitz for every $n \in \mathbb{N}$. Since obviously $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ if f is Lipschitz with bounded support, the uniform continuity of the T_n's grant that $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ for every $f \in L^2(\tilde{X}, \mathfrak{m}_c)$.

Now, taking into account the L^2-lower semicontinuity of the BL-norm, to conclude it is sufficient to show that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_{\tilde{X}} |\mathrm{D} f_n^{(t)}|_X^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x) &\le \int_{\mathbb{R} \times X} |\mathrm{D} f^{(t)}|_X^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x), \\ \int_{\tilde{X}} |\mathrm{D} f_n^{(x)}|_{\mathbb{R}}^2(t)\, \mathrm{d}\mathfrak{m}_c(t,x) &\le \int_{\mathbb{R} \times X} |\mathrm{D} f^{(x)}|_{\mathbb{R}}^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x). \end{aligned} \tag{3.9}$$

Start noticing that the definition of the functions $g_{i,n}$, the convexity of minimal weak upper gradients and their L^2-lower semicontinuity yields that $g_{i,n} \in W^{1,2}(X)$ for every i, n with

$$\int_X |\mathrm{D} g_{i,n}|_X^2 \, \mathrm{d}\mathfrak{m} \leq n \int_X \int_{i/n}^{(i+1)/n} |\mathrm{D} f^{(t)}|_{X'}^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}. \tag{3.10}$$

Then from the trivial identity

$$f_n^{(t)} = (1 + i - nt) g_{i,n} + (nt - i) g_{i+1,n},$$

valid for every n and a.e. $t \in [\frac{i}{n}, \frac{i+1}{n}]$ we know that $f_n^{(t)} \in W^{1,2}(X)$ and

$$\begin{aligned} |\mathrm{D} f_n^{(t)}|_X^2 &\leq \big((1 + i - nt)|\mathrm{D} g_{i,n}|_X + (nt - i)|\mathrm{D} g_{i+1,n}|_X\big)^2 \\ &\leq (1 + i - nt)|\mathrm{D} g_{i,n}|_X^2 + (nt - i)|\mathrm{D} g_{i+1,n}|_X^2, \end{aligned}$$

for every n and a.e. $t \in [\frac{i}{n}, \frac{i+1}{n}]$. This yields the bound

$$\begin{aligned} \int_{X_c} |\mathrm{D} f_n^{(t)}|_X^2(x) \, \mathrm{d}\mathfrak{m}_c(t, x) &\leq \frac{1}{n} \sum_{i \in \mathbb{Z}} \int_X |\mathrm{D} g_{i,n}|_X^2(x) \, \mathrm{d}\mathfrak{m}(x) \\ \text{by (3.10)} \qquad &\leq \sum_{i \in \mathbb{Z}} \int_X \int_{i/n}^{(i+1)/n} |\mathrm{D} f^{(t)}|_X^2(x) \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}(x) \\ &= \int_{X_c} |\mathrm{D} f^{(t)}|_X^2(x) \, \mathrm{d}\mathfrak{m}_c(t, x), \end{aligned} \tag{3.11}$$

which is the first in (3.9).

Similarly, for $\mathfrak{m}$-a.e. $x \in X$ the function $f_n^{(x)} : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}^1$-a.e. well defined and given by

$$f_n^{(x)}(t) = (1 + i - nt) g_{i,n}(x) + (nt - i) g_{i+1,n}(x), \qquad \mathcal{L}^1 - a.e.\ t \in \big[\tfrac{i}{n}, \tfrac{i+1}{n}\big].$$

Arguing as before we get that $f_n^{(x)} \in W^{1,2}(\mathbb{R})$ for $\mathfrak{m}$-a.e. x and

$$\begin{aligned} \int_{i/n}^{(i+1)/n} |\mathrm{D} f_n^{(x)}|_{\mathbb{R}}^2(t) \, \mathrm{d}t &= \int_{i/n}^{(i+1)/n} n^2 \big(g_{i+1,n}(x) - g_{i,n}(x)\big)^2 \, \mathrm{d}t \\ &= n \big(g_{i+1,n}(x) - g_{i,n}(x)\big)^2 \\ &= n^3 \left(\int_{(i+1)/n}^{(i+2)/n} f(t, x) \, \mathrm{d}t - \int_{i/n}^{(i+1)/n} f(t, x) \, \mathrm{d}t \right)^2 \\ &= n^3 \left(\int_{i/n}^{(i+1)/n} f^{(x)}(t + 1/n) - f^{(x)}(t) \, \mathrm{d}t \right)^2 \\ &\leq n^3 \left(\int_{i/n}^{(i+1)/n} \int_t^{t+1/n} |\mathrm{D} f^{(x)}|_{\mathbb{R}}(s) \, \mathrm{d}s \, \mathrm{d}t \right)^2 \\ &\leq n \int_{i/n}^{(i+1)/n} \int_t^{t+1/n} |\mathrm{D} f^{(x)}|_{\mathbb{R}}^2(s) \, \mathrm{d}s \, \mathrm{d}t, \end{aligned}$$

which yields
$$\int_{\tilde X} |\mathrm{D} f_n^{(x)}|^2_{\mathbb{R}}(t)\,\mathrm{d}\mathfrak{m}_c(t,x) \le \int_{\tilde X} |\mathrm{D} f^{(x)}|^2_{\mathbb{R}}(t)\,\mathrm{d}\mathfrak{m}(t,x),$$
which is the second in (3.9) and the conclusion. □

We now have all the tools to prove the main result of this section:

Theorem 3.7. *The sets $W^{1,2}(X_c)$ and $\mathsf{BL}(X_c)$ coincide and for every $f \in W^{1,2}(X_c) = \mathsf{BL}(X_c)$ the identity*

$$|\mathrm{D} f|_{X_c} = |\mathrm{D} f|_c \qquad \mathfrak{m}_c - a.e.,$$

holds.

Proof. Proposition 3.2 gives the inclusion $W^{1,2}(X_c) \subset \mathsf{BL}(X_c)$. Now pick $f \in \mathsf{BL}(X_c)$ and find a sequence $(f_n) \subset \mathsf{BL}(X_c) \cap \mathcal{A}_{loc}$ as in Proposition 3.6. By Proposition 3.5 we know that
$$|\mathrm{D} f_n|_{X_c} = |\mathrm{D} f_n|_c \qquad \mathfrak{m}_c - a.e.,\ \forall n \in \mathbb{N}.$$
By construction, the right hand side converges to $|\mathrm{D} f|_c$ in $L^2(X_c, \mathfrak{m}_c)$ as $n \to \infty$, and since $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$, by the lower semicontinuity of weak upper gradients we deduce that $f \in W^{1,2}(X_c)$ and

$$|\mathrm{D} f|_{X_c} \le |\mathrm{D} f|_c, \qquad \mathfrak{m}_c - a.e.,$$

which together with inequality (3.1) gives the thesis. □

3.2 Warped product

Throughtout this section $w_{\mathrm{d}}, w_{\mathfrak{m}} : I \to \mathbb{R}^+$ are given warping functions as in Definition 2.11. We are interested in studying Sobolev functions on the warped product space $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$, where $X_w := I \times_w X$.

Like in the Cartesian case, given $f : X_w \to \mathbb{R}$ and $t \in I$ we shall denote by $f^{(t)} : X \to \mathbb{R}$ the function given by $f^{(t)}(x) := f(t,x)$. Similarly $f^{(x)}(t) := f(t,x)$ for $x \in X$.

We then consider the Beppo-Levi space $\mathsf{BL}(X_w)$ defined as follows:

Definition 3.8 (The space $\mathsf{BL}(X_w)$)**.** As a set, $\mathsf{BL}(X_w)$ is the subset of $L^2(X_w, \mathfrak{m}_w)$ made of those functions f such that:

i) for $\mathfrak{m}$-a.e. $x \in X$ we have $f^{(x)} \in W^{1,2}(\mathbb{R}, w_{\mathfrak{m}}\mathcal{L}^1)$,

ii) for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $f^{(t)} \in W^{1,2}(X)$,

iii) the function

$$|\mathrm{D} f|_w(t,x) := \sqrt{w_d^{-2}(t)|\mathrm{D} f^{(t)}|^2_X(x) + |\mathrm{D} f^{(x)}|^2_{\mathbb{R}}(t)} \tag{3.12}$$

belongs to $L^2(X_w, \mathfrak{m}_w)$.

On $\mathsf{BL}(X_w)$ we put the norm

$$\|f\|_{\mathsf{BL}(X_w)} := \sqrt{\|f\|^2_{L^2(X_w)} + \||\mathrm{D}f|_w\|^2_{L^2(X_w)}}.$$

It will be useful to introduce the following auxiliary space:

Definition 3.9 (The space $\mathsf{BL}_0(X_w)$)**.** Let $V \subset \mathsf{BL}(X_w)$ be the space of functions f which are identically 0 on $\Omega \times X \subset X_w$ for some open set $\Omega \subset \mathbb{R}$ containing $\{w_\mathfrak{m} = 0\}$.

$\mathsf{BL}_0(X_w) \subset \mathsf{BL}(X_w)$ is defined as the closure of V in $\mathsf{BL}(X_w)$.

The goal of this section is to compare the spaces $\mathsf{BL}(X_w)$ and $W^{1,2}(X_w)$ and their respective notions of minimal weak upper gradients, namely $|\mathrm{D}f|_w$ and $|\mathrm{D}f|_{X_w}$. Under the sole continuity assumption of $w_d, w_\mathfrak{m}$ and the compatibility condition $\{w_d = 0\} \subset \{w_\mathfrak{m} = 0\}$ we can prove that

$$\mathsf{BL}_0(X_w) \subset W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$$

and that for any $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ the identity

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w$$

holds $\mathfrak{m}_w$-a.e., so that in particular the above inclusions are continuous. Without additional hypotheses it is unclear to us whether $W^{1,2}(X_w) = \mathsf{BL}(X_w)$ (on the other hand, it is easy to construct examples where $\mathsf{BL}_0(X_w)$ is strictly smaller than $\mathsf{BL}(X_w)$). Still, if we assume that

$$\text{the set } \{w_\mathfrak{m} = 0\} \subset I \text{ is discrete} \tag{3.13}$$

and that $w_\mathfrak{m}$ decays at least linearly near its zeros, i.e.

$$w_\mathfrak{m}(t) \leq C \inf_{s: w_\mathfrak{m}(s)=0} |t-s|, \qquad \forall t \in \mathbb{R}, \tag{3.14}$$

for some constant $C \in \mathbb{R}$, then we can prove - using basically arguments about capacities - that

$$\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w),$$

so that the three spaces considered are all equal. We remark that these two additional assumptions on $w_\mathfrak{m}$ are satisfied in all the geometric applications we have in mind, because typically one considers cone/spherical subspensions and in these cases $w_\mathfrak{m}$ has at most two zeros and decays polinomially near them.

We turn to the details. The following result is easily established:

Proposition 3.10. *Let $w_\mathrm{d}, w_\mathfrak{m}$ be warping functions. Then $W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$.*

Proof. Pick $f \in W^{1,2}(X_w)$ and use Proposition 2.6 to find a sequence (f_n) of Lipschitz functions on X_w such that $f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{D}f|_{X_w}$ in $L^2(X_w)$. Up to pass to a fast converging subsequence, not relabeled, we can further assume that for

$\mathfrak{m}$-a.e. $x \in X$, we have $f_n^{(x)} \to f^{(x)}$ in $L^2(I, w_{\mathfrak{m}}\mathcal{L}^1)$ and that for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in I$ we have $f_n^{(t)} \to f^{(t)}$ in $L^2(X, \mathfrak{m})$.

Observe that for every $(t, x) \in X_w$ we have

$$\begin{aligned} \mathrm{lip}(f_n)(t,x) &= \overline{\lim_{(s,y)\to(t,x)}} \frac{|f_n(s,y) - f_n(t,x)|}{\mathrm{d}_w((s,y),(t,x))} \\ &\geq \overline{\lim_{s\to t}} \frac{|f_n(s,x) - f_n(t,x)|}{\mathrm{d}_w((s,x),(t,x))} \\ &= \overline{\lim_{s\to t}} \frac{|f_n^{(x)}(s) - f_n^{(x)}(t)|}{|s-t|} = \mathrm{lip}_{\mathbb{R}}(f_n^{(x)})(t) \end{aligned}$$

and therefore by Fatou's lemma we deduce

$$\begin{aligned} \int_X \lim_{n\to\infty} \int_I \mathrm{lip}_I(f_n^{(x)})^2(t)\,\mathrm{d}(w_{\mathfrak{m}}\mathcal{L}^1)(t)\,\mathrm{d}\mathfrak{m}(x) &\leq \lim_{n\to\infty} \int_{X_w} \mathrm{lip}(f_n)^2(t,x)\,\mathrm{d}\mathfrak{m}_w(t,x) \\ &= \int_{X_w} |\mathrm{D}f|_{X_w}^2\,\mathrm{d}\mathfrak{m}_w < \infty. \end{aligned}$$

Since $f_n^{(x)} \to f^{(x)}$ in $L^2(I, w_{\mathfrak{m}}\mathcal{L}^1)$ for $\mathfrak{m}$-a.e. $x \in X$, this last inequality together with the lower semicontinuity of minimal weak upper gradients ensures that $f^{(x)} \in W^{1,2}(I, w_{\mathfrak{m}}\mathcal{L}^1)$ for $\mathfrak{m}$-a.e. $x \in X$ and

$$\int_{X_w} |\mathrm{D}f^{(x)}|_I^2(t)\,\mathrm{d}\mathfrak{m}_w(t,x) \leq \int_{X_w} |\mathrm{D}f|_{X_w}^2\,\mathrm{d}\mathfrak{m}_c. \tag{3.15}$$

An analogous argument starting from the bound

$$\begin{aligned} \mathrm{lip}(f_n)(t,x) &= \overline{\lim_{(s,y)\to(t,x)}} \frac{|f_n(s,y) - f_n(t,x)|}{\mathrm{d}_w((s,y),(t,x))} \\ &\geq \overline{\lim_{y\to x}} \frac{|f_n(t,y) - f_n(t,x)|}{\mathrm{d}_w((t,y),(t,x))} \\ &= \overline{\lim_{y\to x}} \frac{|f_n^{(t)}(y) - f_n^{(t)}(x)|}{w(t)\mathrm{d}(x,y)} = \frac{1}{w(t)}\mathrm{lip}_I(f_n^{(t)})(x) \end{aligned}$$

valid for every $t \in I$ such that $w_{\mathrm{d}}(t) > 0$, grants that $f^{(t)} \in W^{1,2}(X)$ for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in I$ (recall that $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$) and

$$\int_{X_w} |\mathrm{D}f^{(t)}|_X^2(x)\,\mathrm{d}\mathfrak{m}_w(t,x) \leq \int_{X_w} |\mathrm{D}f|_{X_w}^2\,\mathrm{d}\mathfrak{m}_w. \tag{3.16}$$

The bounds (3.15) and (3.16) ensure that $f \in \mathsf{BL}(X_w)$, so that the inclusion $W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ is proved. $\square$

In order to prove that for $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ the minimal weak upper gradient $|\mathrm{D}f|_{X_w}$ coincides with the 'warped' gradient $|\mathrm{D}f|_w$ defined in (3.12), we shall make use of the following simple comparison argument, which will then allow us to reduce the proof to the already known cartesian case.

Lemma 3.11. *Let X be a set, $\mathrm{d}_1, \mathrm{d}_2$ two distances on it and $\mathfrak{m}_1, \mathfrak{m}_2$ two measures. Assume that $(X, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2)$ are both metric measure spaces satisfying the Assumptions 2.1, that for some $C > 0$ we have $\mathfrak{m}_2 \le C\mathfrak{m}_1$ and that for some $L > 0$ we have $\mathrm{d}_1 \le L\mathrm{d}_2$.*

Then denoting by $S(X_1), S(X_2)$ the Sobolev classes relative to $(X, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2)$ respectively and by $|\mathrm{D}f|_1, |\mathrm{D}f|_2$ the associated minimal weak upper gradients, we have

$$S(X_1) \subset S(X_2)$$

and for every $f \in S(X_1)$ the inequality

$$|\mathrm{D}f|_2 \le L|\mathrm{D}f|_1,$$

holds $\mathfrak{m}_2$-a.e..

Proof. The assumptions ensure that the topology induced by d_2 is finer than the one induced by d_1, hence every d_1-Borel function is also d_2-Borel. Then observe that the assumption $\mathrm{d}_1 \le L\mathrm{d}_2$ ensures that d_2-absolutely continuous curves are also d_1-absoutely continuous, the d_1-metric speed being bounded by L-times the d_2-metric speed. Then considering also the assumption $\mathfrak{m}_2 \le C\mathfrak{m}_1$ we see that $(X, \mathrm{d}_2, \mathfrak{m}_2)$ -test plans are also $(X, \mathrm{d}_1, \mathfrak{m}_1)$-test plans, which, by definition, gives the inclusion $S(X_1) \subset S(X_2)$. The inequality $|\mathrm{D}f|_2 \le L|\mathrm{D}f|_1$ $\mathfrak{m}_2$-a.e. is then obtained by the $\mathfrak{m}_2$-a.e. minimality of $|\mathrm{D}f|_2$ and the opposite inequality valid for the metric speeds. □

We can then prove the following result:

Proposition 3.12. *Let $w_\mathrm{d}, w_\mathfrak{m}$ be warping functions and $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$. Then*

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w, \qquad \mathfrak{m}_w - a.e..$$

Proof. Fix $\epsilon > 0$ and $t_0 \in \mathbb{R}$ such that $w_\mathfrak{m}(t_0) > 0$ so that also $w_\mathrm{d}(t_0) > 0$. Use the continuity of w_d to find $\delta > 0$ so that

$$\Big|\frac{w_\mathrm{d}(t)}{w_\mathrm{d}(s)}\Big| \le 1 + \epsilon \qquad \forall t, s \in [t_0 - 2\delta, t_0 + 2\delta] \tag{3.17}$$

and let $\chi : \mathbb{R} \to [0, 1]$ be a Lipschitz function identically 1 on $[t_0 - \delta, t_0 + \delta]$ with support contained in $[t_0 - 2\delta, t_0 + 2\delta]$.

We introduce the continuous functions $\bar{w}_\mathrm{d}, \bar{w}_\mathfrak{m} : \mathbb{R} \to \mathbb{R}$ as

$$\bar{w}_\mathrm{d}(t) := \begin{cases} w_\mathrm{d}(t_0 - 2\delta), & \text{if } t < t_0 - 2\delta, \\ w_\mathrm{d}(t), & \text{if } t \in [t_0 - 2\delta, t_0 + 2\delta], \\ w_\mathrm{d}(t_0 + 2\delta), & \text{if } t > t_0 + 2\delta, \end{cases}$$

$$\bar{w}_\mathfrak{m}(t) := \begin{cases} w_\mathfrak{m}(t_0 - 2\delta), & \text{if } t < t_0 - 2\delta, \\ w_\mathfrak{m}(t), & \text{if } t \in [t_0 - 2\delta, t_0 + 2\delta], \\ w_\mathfrak{m}(t_0 + 2\delta), & \text{if } t > t_0 + 2\delta, \end{cases}$$

the corresponding product space $(X_{\bar{w}}, \mathrm{d}_{\bar{w}}, \mathfrak{m}_{\bar{w}})$ and consider the function $\bar{f} : X_w \to \mathbb{R}$ given by $\bar{f}(t, x) := \chi(t) f(t, x)$ which belongs to $W^{1,2}(X_w)$ and therefore, by what we just proved, to $\mathsf{BL}(X_w)$. The locality property of minimal weak upper gradients ensure that

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}\bar{f}|_{X_w} \quad \text{and} \quad |\mathrm{D}f|_w = |\mathrm{D}\bar{f}|_w \qquad \mathfrak{m}_w - a.e. \text{ on } [t_0-\delta, t_0+\delta]\times X.$$

Since $\bar{f}$ has support concentrated in the set of (t,x)'s with $t\in[t_0-2\delta, t_0+2\delta]$ and w_{d} is positive in such interval, we can think at $\bar{f}$ also as a real valued function $X_{\bar{w}}$. With this identification in mind it is clear that

$$|\mathrm{D}\bar{f}|_{X_w} = |\mathrm{D}\bar{f}|_{X_{\bar{w}}} \quad \text{and} \quad |\mathrm{D}\bar{f}|_w = |\mathrm{D}\bar{f}|_{\bar{w}} \qquad \mathfrak{m}_w - a.e. \text{ on } [t_0-2\delta, t_0+2\delta]\times X.$$

We now consider the cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of $(X, \mathrm{d}, \mathfrak{m})$ and $\mathbb{R}$. Notice that the sets $X_{\bar{w}}$ and X_c both coincide with $\mathbb{R}\times X$ and that by construction (recall also (3.17)) we have

$$c\mathfrak{m}_c \le \mathfrak{m}_{\bar{w}} \le C\mathfrak{m}_c \qquad \text{and} \qquad \frac{w_{\mathrm{d}}(t_0)}{1+\varepsilon}\mathrm{d}_c \le \mathrm{d}_{\bar{w}} \le w_{\mathrm{d}}(t_0)(1+\varepsilon)\mathrm{d}_c$$

for some $c, C>0$. Hence by Lemma 3.11 we deduce that $\mathfrak{m}_{\bar{w}}$-a.e. it holds

$$\frac{|\mathrm{D}\bar{f}|_{X_c}}{w_{\mathrm{d}}(t_0)(1+\varepsilon)} \le |\mathrm{D}\bar{f}|_{X_{\bar{w}}} \le \frac{1+\varepsilon}{w_{\mathrm{d}}(t_0)}|\mathrm{D}\bar{f}|_{X_c} \quad \text{and} \quad \frac{|\mathrm{D}\bar{f}|_c}{w_{\mathrm{d}}(t_0)(1+\varepsilon)} \le |\mathrm{D}\bar{f}|_{\bar{w}} \le \frac{1+\varepsilon}{w_{\mathrm{d}}(t_0)}|\mathrm{D}\bar{f}|_c.$$

Since by Theorem 3.7 we know that $|\mathrm{D}\bar{f}|_{X_c} = |\mathrm{D}\bar{f}|_c$ $\mathfrak{m}_c$-a.e., collecting what we proved we deduce that

$$\frac{|\mathrm{D}f|_{X_w}}{(1+\varepsilon)^2} \le |\mathrm{D}f|_w \le (1+\varepsilon)^2|\mathrm{D}f|_{X_w}$$

$\mathfrak{m}_w$-a.e. on $[t_0-\delta, t_0+\delta]\times X$. By the arbitrariness of t_0 such that $w_{\mathfrak{m}}(t_0)>0$ and the Lindelof property of $\{w_{\mathfrak{m}}>0\}\subset\mathbb{R}$ we deduce that the above inequality holds $\mathfrak{m}_w$-a.e.. The conclusion then follows letting $\varepsilon\downarrow 0$. □

We now turn to the general relation between $\mathsf{BL}_0(X_w)$ and $W^{1,2}(X_w)$:

Proposition 3.13. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions. Then $\mathsf{BL}_0(X_w)\subset W^{1,2}(X_w)$.*

Proof. Taking into account Proposition 3.12 it is sufficient to prove that $V\subset W^{1,2}(X_w)$. Notice that for arbitrary $f\in\mathsf{BL}(X_w)$, considering the functions $\chi_n(t) := 0\vee(n-|t|)\wedge 1$ and defining $f_n(t,x) := \chi_n(t)f(t,x)$, via a direct verification of the definitions we have $f_n\in\mathsf{BL}(X_w)$, while inequality (2.3) and the dominate convergence theorem grant that $f_n\to f$ in $\mathsf{BL}(X_w)$. Therefore, using again Proposition 3.12 whcih ensures that BL-convergence implies $W^{1,2}$-convergence, to conclude it is sufficient to show that any $f\in V$ with support contained in $(I\cap[-T,T])\times X\subset X_w$ for some $T>0$ belongs to $W^{1,2}(X_w)$.

Thus fix such $f\in V$, for $r>0$ denote by $\Omega_r\subset\mathbb{R}$ the r-neighborhood of $\{w_{\mathfrak{m}}=0\}$ and find $r\in(0,1)$ such that f is $\mathfrak{m}_w$-a.e. zero on $\Omega_{2r}\times X$. Then by continuity and compactness and recalling that $\{w_{\mathrm{d}}=0\}\subset\{w_{\mathfrak{m}}=0\}$ we deduce that there are constants $0<c\le C<\infty$ such that

$$c\le w_{\mathrm{d}}(t), w_{\mathfrak{m}}(t)\le C, \qquad \forall t\in I\cap[-T,T]\setminus\Omega_{r/2}.$$

We are now going to use a comparison argument similar to that used in the proof of Proposition 3.12. Find two continuous functions $w'_{\mathrm{d}}, w_{\mathfrak{m}}$ agreeing with $w_{\mathrm{d}}, w_{\mathfrak{m}}$ on $[-T,T]\setminus\Omega$ and such that $c \le w'_{\mathrm{d}}, w'_{\mathfrak{m}} \le C$ on the whole $\mathbb{R}$ and consider the warped product $(X_{w'}, \mathrm{d}_{w'}, \mathfrak{m}_{w'})$ and the cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of I and X. We then have the equalities of sets:

$$\mathsf{BL}(X_{w'}) = \mathsf{BL}(X_c) = W^{1,2}(X_c) = W^{1,2}(X_{w'}),$$

the first and last coming from Lemma 3.11 and the properties of $w'_{\mathrm{d}}, w'_{\mathfrak{m}}$ and the middle one being given by Theorem 3.7.

By the construction of $w'_{\mathrm{d}}, w'_{\mathfrak{m}}$ we see that $f \in \mathsf{BL}(X_{w'})$ and thus, by what we just proved, that $f \in W^{1,2}(X_{w'})$. Then Proposition 2.6 grants that there exists a sequence (f_n) of $\mathrm{d}_{w'}$-Lipschitz functions converging to f in $L^2(X_{w'})$ with

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}'(f_n)^2 \,\mathrm{d}\mathfrak{m}_{w'} < \infty$$

uniformly bounded in n, where by lip' we denote the local Lipschitz constant computed w.r.t. the distance $\mathrm{d}_{w'}$. Notice that up to replacing f_n with $(-C_n) \vee f_n \wedge C_n$ for a sufficiently large C_n, we can, and will, assume that f_n is bounded for every $n \in \mathbb{N}$.

Now find a Lipschitz function $\chi : I \to [0,1]$ identically 0 on $\Omega_r \cup (I \setminus [-T-1, T+1])$, identically 1 on $I \cap [-T,T] \setminus \Omega_{2r}$ and put $\tilde{f}_n(t,x) := \chi(t) f_n(t,x)$. By construction it is immediate to check that the $\tilde{f}_n$'s are still $\mathrm{d}_{w'}$-Lipschitz, converging to f in $L^2(\mathfrak{m}_{w'})$ and satisfying

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}'(\tilde{f}_n)^2 \,\mathrm{d}\mathfrak{m}_{w'} < \infty. \tag{3.18}$$

We now claim that the $\tilde{f}_n$'s are d_w-Lipschitz, converging to f in $L^2(X_w)$ and such that

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}(\tilde{f}_n)^2 \,\mathrm{d}\mathfrak{m}_{w} < \infty, \tag{3.19}$$

from which the conclusion follows by the lower semicontinuity of weak upper gradients and the bound $\mathrm{lip}(f_n) \le |\mathrm{D}f|_{X_w}$ valid $\mathfrak{m}_w$-a.e.. Since all the functions $\tilde{f}_n$ and f are concentrated on $([-T,T]\setminus\Omega_r)\times X$ and on this set the measures $\mathfrak{m}_w$ and $\mathfrak{m}_{w'}$ agree, we clearly have $L^2(X_w)$-convergence. Moreover, since w_{d} and $w_{\mathrm{d}'}$ agree on $([-T,T]\setminus\Omega_r)\times X$, the topologies on $([-T,T]\setminus\Omega_r)\times X$ induced by d_w and $\mathrm{d}_{w'}$ agree (with the product topology, given that these functions are positive) and a direct use of the definition yields

$$\lim_{(s,y)\to(t,x)} \frac{\mathrm{d}_w\big((s,y),(t,x)\big)}{\mathrm{d}_{w'}\big((s,y),(t,x)\big)} = 1, \qquad \forall (t,x) \in ([-T,T]\setminus\Omega_r)\times X.$$

In particular, we have $\mathrm{lip}(\tilde{f}_n) = \mathrm{lip}'(\tilde{f}_n)$ in $([-T,T]\setminus\Omega_r)\times X$, so that (3.19) follows from (3.18). Finally, recalling that a Borel function on $[0,1]$ whose local Lipschitz constant is uniformly bounded by some constant L is in fact L-Lipschitz (as shown by a direct covering argument) and using the fact that (X_w, d_w) is by definition a length space we see that for every $n \in \mathbb{N}$ it holds

$$\mathrm{Lip}(f_n) = \sup_{X_w} \mathrm{lip}(f_n) = \sup_{X_{w'}} \mathrm{lip}'(f_n) = \mathrm{Lip}'(f_n) < \infty,$$

where $\mathrm{Lip}'(f_n)$ denotes the $\mathrm{d}_{w'}$-Lipschitz constant. Hence f_n is d_w-Lipschitz for every $n \in \mathbb{N}$ and the proof is achieved. □

Finally, we prove that if the set of zeros of $w_{\mathfrak{m}}$ is discrete and $w_{\mathfrak{m}}$ decays at least linearly close to its zeros, then $\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w)$:

Proposition 3.14. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions and assume that $w_{\mathfrak{m}}$ has the properties* (3.13) *and* (3.14).

Then $\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w)$.

Proof. A standard truncation argument shows that $\mathsf{BL} \cap L^\infty(X_w)$ is dense in $\mathsf{BL}(X_w)$, so to conclude it is sufficient to show that for any $f \in \mathsf{BL} \cap L^\infty(X_w)$ we can find a sequence $(f_n) \subset V$ converging to it in $\mathsf{BL}(X_w)$.

Thus pick $f \in \mathsf{BL} \cap L^\infty(X_w)$, put $D(t) := \min_{s: w_{\mathfrak{m}}(s)=0} |t-s|$ and for $n, m \in \mathbb{N}$, $n > 1$ consider the cut-off functions

$$\begin{aligned} \sigma_m(x) &:= 0 \vee (m - \mathrm{d}(x, \bar{x})) \wedge 1, \\ \eta_n(t) &:= 0 \vee \Big(1 - \frac{|\log(D(t))|}{\log(n)}\Big) \wedge 1, \\ \tilde{\eta}_n(t) &:= 0 \vee \Big(n - |t|\Big) \wedge 1, \end{aligned}$$

where $\bar{x} \in X$ is a chosen, fixed point, and define $f_{n,m}(t,x) := \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x) f(t,x)$. Since $(t,x) \mapsto \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x)$ is Lipschitz and bounded for every n, m, a direct check of the definition of $\mathsf{BL}(X_w)$ shows that $f_{n,m} \in \mathsf{BL}(X_w)$ for every n, m and, since η_n is 0 on a neighborhood of $\{w_{\mathfrak{m}} = 0\}$, we also have $f_{n,m} \in V$ for every n, m.

Using the fact that the functions $(t,x) \mapsto \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x)$ are uniformly bounded by 1 and pointwise converge to 1 as $n, m \to \infty$ and the dominate convergece theorem we see that $f_{n,m} \to f$ in $L^2(X_w)$ as $n, m \to \infty$.

Next, recalling (2.3) and using that σ_m is 1-Lipschitz we see that

$$|\mathrm{D}(f^{(t)} - f^{(t)}_{n,m})|_X(x) \le \big|\eta_n(t)\tilde{\eta}_n(t)\sigma_m(x) - 1\big|\, |\mathrm{D}f^{(t)}|_X(x) + |f(t,x)| 1_{\{\mathrm{d}(\cdot,\bar{x}) \ge m-1\}}(x)$$

for $\mathfrak{m}_w$-a.e. (t,x), so that the dominate convergence theorem again gives that $\int |\mathrm{D}(f^{(t)} - f^{(t)}_{n,m})|^2_X(x)\, \mathrm{d}\mathfrak{m}_w(t,x) \to 0$ as $n, m \to \infty$.

Similarly, we have

$$\begin{aligned} |\mathrm{D}(f^{(x)} - f^{(x)}_{n,m})|_I(t) \le& \big|\eta_n(t)\tilde{\eta}_n(t)\sigma_m(x) - 1\big|\, |\mathrm{D}f^{(x)}|_I(t) + |f(t,x)| 1_{\{|\cdot| \ge n-1\}}(t) \\ &+ |f(t,x)| 1_{\{\mathrm{d}(\cdot,\bar{x}) \le m\}}(x) 1_{\{|\cdot| \le n\}}(t) |\partial_t \eta_n|(t) \end{aligned}$$

for $\mathfrak{m}_w$-a.e. (t,x) and again by dominate convergence we see that the first two terms in the right hand side go to 0 in $L^2(X_w)$ as $n, m \to \infty$. For the last term, we use the fact that f is bounded and our assumptions on $w_{\mathfrak{m}}$. Observe indeed that

$|\partial_t\eta_n|(t) \leq \frac{1_{D^{-1}([n^{-1},1])}(t)}{D(t)\log n}$ so that letting $x_1,\ldots,x_N$ be the finite number of zeros of $w_{\mathfrak{m}}$ in $[-n-1,n+1]$ we have

$$\begin{aligned}\int |f(t,x)|^2 1_{\{\mathrm{d}(\cdot,\bar{x})\leq m\}}(x)1_{\{|\cdot|\leq n\}}(t)|\partial_t\eta_n|^2(t)\,\mathrm{d}\mathfrak{m}_w \\ &\leq \frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\int_{[-n,n]\cap D^{-1}([n^{-1},1])}\frac{1}{D^2(t)}w_{\mathfrak{m}}(t)\,\mathrm{d}t\\ &\leq C\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\int_{[-n,n]\cap D^{-1}([n^{-1},1])}\frac{1}{D(t)}\,\mathrm{d}t\\ &\leq C\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\sum_{i=1}^N\int_{\{t:|t-x_i|\in[n^{-1},1]\}}\frac{1}{|t-x_i|}\,\mathrm{d}t\\ &= 2NC\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)}.\end{aligned}$$

Since the last term goes to 0 as $n\to\infty$ for every $m\in\mathbb{N}$, we just proved that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int |\mathrm{D}(f^{(x)}-f^{(x)}_{n,m})|_I^2(t)\,\mathrm{d}\mathfrak{m}_w(t,x)=0,$$

which is sufficient to conclude. □

3.3 Sobolev-to-Lipschitz property

We recall the following definition:

Definition 3.15 (Sobolev-to-Lipschitz property)**.** We say that a metric measure space $(X,\mathrm{d},\mathfrak{m})$ has Sobolev to Lipschitz property if for any function $f\in W^{1,2}(X)$ with $|\mathrm{D}f|_X\in L^\infty(X)$, we can find a function $\tilde{f}$ such that $f=\tilde{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(f)=\operatorname{ess\,sup}|\mathrm{D}f|_X$.

Aim of this section is to study the Sobolev-to-Lipschitz property on warped products.

Metric measure spaces with the Sobolev-to-Lipschitz property are, in some sense, those whose metric properties can be studied via Sobolev calculus and actually such notion has been introduced by the second author in [11] precisely with this scope. Only quite regular metric measure structures possess this property (for instance, doubling & Poincaré are not sufficient to ensure the Sobolev-to-Lipschitz property) and it is a non-trivial fact that $\mathrm{RCD}(K,\infty)$ spaces have such property (see [6] for the definition of $\mathrm{RCD}(K,\infty)$ spaces and the proof of the claim).

We observe that under the only assumption that $w_{\mathrm{d}},w_{\mathfrak{m}}$ are warping functions we cannot hope to prove that X_w has the Sobolev-to-Lipschitz property. Indeed, if $w_{\mathfrak{m}}$ is 0 on some subinterval of I which disconnects I, then the measure $\mathfrak{m}_w$ has disconnected support and therefore we can find non-constant functions on X_w which are locally constant on the support of $\mathfrak{m}_w$, which is easily seen to violate the Sobolev-to-Lipschitz condition.

We shall therefore only consider the case where $w_{\mathfrak{m}}$ is strictly positive in the interior of I, a condition which is satisfied in the standard geometric constructions like that of cone/spherical subspension.

It is unclear to us whether, even with this condition on $w_{\mathfrak{m}}$, the Sobolev-to-Lipschitz property passes to warped products or not. What we are able to do, instead, is to identify two quite general properties which imply the Sobolev-to-Lipschitz, each of which passes to warped products. This will also imply that whenever these two properties hold for a certain base space, one can consider multiple warped products and still get that the final space has the Sobolev-to-Lipschitz property.

We begin introducing the two auxiliary concepts we just alluded to. The first is a variant of the length property which takes into account the reference measure:

Definition 3.16 (Measured-length space)**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is measured-length if there exists a Borel set $A \subset X$ whose complement is $\mathfrak{m}$-negligible with the following property. For every $x_0, x_1 \in A$ there exists $\varepsilon > 0$ such that for every $\varepsilon_0, \varepsilon_1 \in (0, \varepsilon]$ there is a test plan $\pi^{\varepsilon_0,\varepsilon_1} \in \mathcal{P}(X)$ with:

a) the map $(0, \varepsilon]^2 \ni (\varepsilon_0, \varepsilon_1) \mapsto \pi^{\varepsilon_0,\varepsilon_1}$ is weakly Borel in the sense that for any $\varphi \in C_b(C([0, 1], X))$ the map

$$(0, \varepsilon]^2 \ni (\varepsilon_0, \varepsilon_1) \qquad \mapsto \qquad \int \varphi \, \mathrm{d}\pi^{\varepsilon_0,\varepsilon_1},$$

is Borel.

b) We have

$$(e_0)_\sharp \pi^{\varepsilon_0,\varepsilon_1} = \frac{1_{B_{\varepsilon_0}(x_0)}}{\mathfrak{m}(B_{\varepsilon_0}(x_0))} \, \mathfrak{m}, \qquad \text{and} \qquad (e_1)_\sharp \pi^{\varepsilon_0,\varepsilon_1} = \frac{1_{B_{\varepsilon_1}(x_1)}}{\mathfrak{m}(B_{\varepsilon_1}(x_1))} \, \mathfrak{m}$$

for every $\varepsilon_0, \varepsilon_1 \in (0, \varepsilon]$,

c) We have

$$\overline{\lim_{\varepsilon_0,\varepsilon_1 \downarrow 0}} \iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\pi(\gamma) \le \mathrm{d}^2(x_0, x_1). \tag{3.20}$$

The second definition is a simple modification of the usual doubling notion:

Definition 3.17 (a.e. locally doubling spaces)**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is a.e. doubling provided there exists a Borel set B whose complement is $\mathfrak{m}$-negligible such that for every $x \in B$ there are an open set Ω containing x and constants $C, R > 0$ such that

$$\mathfrak{m}(B_{2r}(y)) \le C\mathfrak{m}(B_r(y)), \qquad \forall r \in (0, R), \ y \in \Omega.$$

It is easy to check that a a.e. locally doubling and measured-length space has the Sobolev-to-Lipschitz property:

Proposition 3.18. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an a.e. locally doubling and measured-length space. Then it has the Sobolev-to-Lipschitz property.*

Proof. It is well known that on doubling spaces, for any given function in L^1_{loc}, a.e. point is a Lebesgue point. Since the property of being a Lebesgue point is local in nature, we immediately have that even on a a.e. locally doubling space a.e. point is a Lebesgue point of a given L^1_{loc} function.

With that said, let $A \subset X$ be the set given in Definition 3.16, pick $f \in W^{1,2}(X)$ with $L := \text{ess sup } |\mathrm{D}f| < \infty$ and let $B \subset X$ the set of its Lebesgue points of f.

Pick $x, y \in A \cap B$, let ε be given by Definition 3.16, consider $\varepsilon' \in (0, \varepsilon]$ and the test plan $\pi^{\varepsilon',\varepsilon'}$ given by Definition 3.16. Then we have

$$\begin{aligned}\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma) &\leq \iint_0^1 |\mathrm{D}f|(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma) \\ &\leq L \iint_0^1 |\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma) \leq L\sqrt{\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma)}\end{aligned}$$

Letting $\varepsilon' \downarrow 0$, using the fact that x, y are Lebesgue points and the bound (3.20) we obtain

$$\begin{aligned}|f(y) - f(x)| &= \lim_{\varepsilon' \downarrow 0} \left| \int f \,\mathrm{d}(e_0)_\sharp \pi^{\varepsilon',\varepsilon'} - \int f \,\mathrm{d}(e_1)_\sharp \pi^{\varepsilon',\varepsilon'} \right| \\ &= \lim_{\varepsilon' \downarrow 0} \left| \int f(\gamma_1) - f(\gamma_0) \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma) \right| \\ &\leq \varliminf_{\varepsilon' \downarrow 0} \int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma) \\ &\leq L \varliminf_{\varepsilon' \downarrow 0} \sqrt{\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi^{\varepsilon',\varepsilon'}(\gamma)} \leq L\mathrm{d}(x, y).\end{aligned}$$

This proves that the restriction of f to $A \cap B$ is Lipschitz with Lipschitz constant bounded by L. Since $A \cap B$ has full measure the proof is achieved. □

We shall now verify that both the properties of being a.e. locally doubling and measured-length pass to warped products. We start with the a.e. locally doubling, which is easier.

Proposition 3.19. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an a.e. locally doubling space and $w_\mathrm{d}, w_\mathfrak{m}$ a couple of warping functions. Then the warped product $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ is a.e. locally doubling as well.*

Proof. Let $B \subset X$ be the set given in Definition 3.17, put $\hat{B} := \{w_\mathfrak{m} > 0\} \times B \subset X_w$ and notice that $\hat{B}$ has negligible complement. Then recall that $\{w_\mathfrak{m} > 0\} \subset \{w_\mathrm{d} > 0\}$, notice that it is trivial that the cartesian product of a doubling space and an interval is doubling and conclude using the continuity of $w_\mathrm{d}, w_\mathfrak{m}$. □

To study the behavior of the measured-length property on warped products, we need to recall some facts about warped product distances. The content of the following lemma is well known, but we provide the simple proof for completeness.

Lemma 3.20. *Let (X, d) be a complete and separable length space, $I \subset \mathbb{R}$ a closed, possibly unbounded interval, $\mathrm{w_d} : I \to \mathbb{R}^+$ a continuous function and consider the warped product metric space (X_w, d_w).*

Then there exists a function $D : I^2 \times X \to \mathbb{R}^+$ such that

$$\mathrm{d}_w\big((t_0, x_0), (t_1, x_1)\big) = D(t_0, t_1, \mathrm{d}(x_0, x_1)), \qquad \forall t_0, t_1 \in I,\ x_0, x_1 \in X, \tag{3.21}$$

and for every sequence of curves $\gamma_n = (\gamma_n^I, \gamma_n^X)$ joining (t_0, x_0) to (t_1, x_1) whose d_w-length converge to $\mathrm{d}_w\big((t_0, x_0), (t_1, x_1)\big)$ we have that the d-length of the curves γ_n^X converge to $\mathrm{d}(x_0, x_1)$.

Finally, for $(t_0, x_0) \in X_w$ with $\mathrm{w_d}(t_0) > 0$ and given $\varepsilon > 0$ and $t' \in I$ we have that there exists $x' \in X$ with $\mathrm{d}_w\big((t_0, x_0), (t', x')\big) \in B_\varepsilon((t_0, x_0))$ if and only if $|t' - t| < \varepsilon$ and in this case the set of such x''s is a ball centered at x_0 whose d-radius $r(t_0, t', \varepsilon)$ satisfies

$$\varlimsup_{\varepsilon \downarrow 0} \sup_{t' \in [t_0 - \varepsilon, t_0 + \varepsilon]} r(t_0, t', \varepsilon) = 0. \tag{3.22}$$

Proof. Fix $t_0, t_1 \in I$, $x_0, x_1 \in X$ and let $\Gamma_w \subset C([0,1], X_w)$ be the set of absolutely continuous curves joining (t_0, x_0) to (t_1, x_1) and $\Gamma \subset C([0,1], I)$ the set of absolutely continuous curves joining t_0 to t_1. Also, define $L_w : \Gamma_w \to \mathbb{R}^+$ and $L : \Gamma \to \mathbb{R}^+$ as

$$E_w((\gamma^I, \gamma^X)) := \int_0^1 \sqrt{|\dot\gamma_s|^2 + w_{\mathrm{d}}^2(\gamma_s^I)|\dot\gamma_s^X|^2}\,\mathrm{d}s,$$
$$E(\gamma^I) := \int_0^1 \sqrt{|\dot\gamma_s^I|^2 + w_{\mathrm{d}}^2(\gamma_s^I)\mathrm{d}^2(x_0, x_1)}\,\mathrm{d}s.$$

We claim that

$$\inf_{\gamma \in \Gamma_w} E_w(\gamma) = \inf_{\gamma^I \in \Gamma} E(\gamma^I). \tag{3.23}$$

Indeed, to get inequality $\le$ find a sequence of curves γ_n^X joining x_0 to x_1 parametrized with constant speed and whose length converges to $\mathrm{d}(x_0, x_1)$. Then for every $\gamma^I \in \Gamma$ consider the curves $\gamma_n := (\gamma^I, \gamma_n^X) \in \Gamma_w$ and notice that $\lim_n E_w(\gamma_n) = E(\gamma^I)$.

To prove $\ge$, pick $\gamma = (\gamma^I, \gamma^X) \in \Gamma_w$ and up to a small perturbation which does alter E_w much, assume that the curve γ^X has always positive speed. Then let $\tilde\gamma = (\tilde\gamma^I, \tilde\gamma^X) \in \Gamma_w$ be the reparametrization of γ chosen so that $\tilde\gamma^X$ has constant speed, call it ℓ. Then we have $\ell \ge \mathrm{d}(x_0, x_1)$ and thus

$$E_w(\gamma) = E_w(\tilde\gamma) = \int_0^1 \sqrt{|\dot{\tilde\gamma}_s^I|^2 + w_{\mathrm{d}}^2(\gamma_s^I)\ell^2}\,\mathrm{d}s \ge E(\tilde\gamma^I),$$

concluding the proof of (3.23).

The identity (3.23) gives the existence of the function D claimed in the statement, because the left-hand side of (3.23) equals $\mathrm{d}_w\big((t_0, x_0), (t_1, x_1)\big)$ while the right-hand side depends on t_0, t_1 and $\mathrm{d}(x_0, x_1)$ only.

The same arguments just used also yield the claim about the convergence of the d-length of the curves γ_n^X.

Concerning the last statement, notice that $\mathrm{d}_w((t_0, x_0), (t', x')) \ge |t_0 - t'|$ for every $t_0, t' \in I$ and $x_0, x' \in X$ and that the equality holds if $x_0 = x'$. This addresses the claim about the existence of x'. The fact that the set of such x''s is a ball follows directly from (3.21) and the claim (3.22) is obvious. □

We turn to the proof that the warped product of an interval and a measured-length space is still measured-length. Unfortunately, the argument is a bit tedious.

Proposition 3.21. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a measured-length space, $I \subset \mathbb{R}$ a closed, possibly unbounded, interval and $w_\mathrm{d}, w_\mathfrak{m} : I \to \mathbb{R}$ a couple of warping functions. Assume that $w_\mathfrak{m}$ is strictly positive in the interior of I.*

Then the warped product $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ is a measured-length space as well.

Proof.
Step 1: set up of the construction. Let $A \subset X$ be the set given in the definition of measured-length space, $\mathring{I}$ the interior of I and put $\hat{A} := \mathring{I} \times A \subset X_w$. Notice that $\hat{A}$ has full $\mathfrak{m}_w$-measure and fix $(t_0, x_0), (t_1, x_1) \in \hat{A}$.

We assume for the moment that there is a d_w-geodesic $\gamma = (\gamma^I, \gamma^X)$ connecting (t_0, x_0) to (t_1, x_1) such that the image of γ^I, which we shall call K, is contained in $\mathring{I}$. Without loss of generality, we shall assume that γ has constant speed, so that

$$\int_0^1 |\dot\gamma_s|^2 \, \mathrm{d}s = \mathrm{d}_w^2\big((t_0, x_0), (t_1, x_1)\big).$$

Since K is compact, for some $\delta > 0$ its δ-neighborhood K_δ is still contained in $\mathring{I}$ and thus the quantities $\inf_{K_{\delta/2}} w_\mathrm{d}$ and $\inf_{K_{\delta/2}} w_\mathfrak{m}$ are strictly positive.

All our construction will take place in $K_{\delta/2} \times X \subset X_w$ and since the role of the reference measure $\mathfrak{m}_w$ is only for the L^∞-bound in (c) of Definition 3.16, without loss of generality, we shall assume for simplicity that $w_\mathfrak{m} \equiv 1$ so that that the warped product measure will be in fact the cartesian product one $\mathfrak{m}_c$.

Let ε be the number given in Definition 3.16 related to the space X and the points $x_0, x_1 \in A$, put $\hat\varepsilon := \min\{\frac{\delta}{4}, \varepsilon \inf_{K_{\delta/2}} w_\mathrm{d}\}$ and fix $\varepsilon_0, \varepsilon_1 \in (0, \hat\varepsilon]$. Most of the objects we shall define from now on will depend on $\varepsilon_0, \varepsilon_1$, but to keep the notation simple we shall often avoid explicitly referring to them.

Consider the two measures

$$\mu_0 = \frac{1_{B_{\varepsilon_0}(x_0)}}{\mathfrak{m}_c(B_{\varepsilon_0}(x_0))} \mathfrak{m}_c, \qquad\qquad \mu_1 = \frac{1_{B_{\varepsilon_1}(x_1)}}{\mathfrak{m}_c(B_{\varepsilon_1}(x_1))} \mathfrak{m}_c,$$

put $\nu_i := \pi^I_\sharp \mu_i \in \mathcal{P}(I)$ and let $\{\mu_{i,t}\}_{t \in I}$ be the disintegration of μ_i w.r.t. π^I, $i = 0, 1$. We shall think to the measures $\mu_{i,t}$ as measures on $(X, \mathrm{d}, \mathfrak{m})$ so that the construction and Lemma 3.20 ensure that

$$\mu_{i,t} = \frac{1_{B_{f_i(t)}(x_i)}}{\mathfrak{m}(B_{f_i(t)}(x_i))} \mathfrak{m}, \qquad \forall t \in \mathrm{spt}(\nu_i) = [t_i - \varepsilon_i, t_i + \varepsilon_i], \quad i = 0, 1,$$

for some functions f_0, f_1 which, due to the choice of $\hat\varepsilon, \varepsilon_0, \varepsilon_1$ and the fact that we are considering those balls in the non-rescaled space $(X, \mathrm{d}, \mathfrak{m})$, satisfy $f_0(t), f_1(t) \le \varepsilon$ for every t in the respective domain of definition. Notice that inequality (3.22) gives

$$\varlimsup_{\varepsilon_i \downarrow 0} \sup_{t \in [t_i - \varepsilon_i, t_i + \varepsilon_i]} f_i(t) = 0, \qquad \text{for } i = 0, 1. \tag{3.24}$$

Observe that $\nu_0, \nu_1 \ll \mathcal{L}^1$, let $T : I \to I$ be the optimal transport map from ν_0 to ν_1 and for $t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ consider the plan $\pi^{f_0(t), f_1(T(t))} \in \mathcal{P}(C([0,1], X))$

joining $\mu_{0,t}$ to $\mu_{1,T(t)}$ whose existence is ensured by Definition 3.16 and the fact that $f_0, f_1 \leq \varepsilon$. Then we also know that for some Borel function $t \mapsto C(t) \geq 1$ we have

$$(e_s)_\sharp \pi^{f_0(t),f_1(T(t))} \leq C(t)\mathfrak{m}, \qquad \forall s \in [0,1] \tag{3.25}$$

and point (c) in Definition 3.16 and (3.24) ensure that

$$\varlimsup_{\varepsilon_0,\varepsilon_1 \downarrow 0} \sup_{t \in [t_0-\varepsilon_0, t_0+\varepsilon_0]} \iint_0^1 |\dot\gamma_s|^2 \, \mathrm{d}s \, \mathrm{d}\pi^{f_0(t),f_1(T(t))}(\gamma) \leq \mathrm{d}^2(x_0,x_1). \tag{3.26}$$

For any continuous and monotone map $\mathfrak{s} : [0,1] \to [0,1]$ with $\mathfrak{s}(0) = 0$ and $\mathfrak{s}(1) = 1$, we shall denote by $\bar{\mathfrak{s}}$ the 'reparametrization' map from $C([0,1], X_w)$ to itself sending a curve γ to $\gamma \circ \mathfrak{s}$. Then by standard arguments we see that for any $t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ we can find $\mathfrak{s}$ such that the plan $\sigma^t := \bar{\mathfrak{s}}_\sharp \pi^{f_0(t),f_1(T(t))}$ is such that $s \mapsto \int |\dot\gamma_s|^2 \, \mathrm{d}\sigma^t(\gamma)$ is proportional to $s \mapsto |\dot\gamma_s^X|^2$. Denoting by $L^2(t)$ the proportionality constant we therefore have

$$\int |\dot\gamma_s|^2 \, \mathrm{d}\sigma^t(\gamma) = L^2(t)|\dot\gamma_s^X|^2, \qquad a.e.\ s \in [0,1],\ \forall t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0], \tag{3.27}$$

and on the other hand we have

$$\int_0^1 \sqrt{\int |\dot\gamma_s|^2 \, \mathrm{d}\sigma^t(\gamma)} \, \mathrm{d}s = \int_0^1 \sqrt{\int |\dot\gamma_s|^2 \, \mathrm{d}\pi^{f_0(t),f_1(T(t))}(\gamma)} \, \mathrm{d}s \tag{3.28}$$

Now let $F_i : [t_i - \varepsilon_i, t_i + \varepsilon_i] \to [0,1]$ be the cumulative distribution function of ν_i given by $F_i(t) := \nu_i((-\infty, t])$, $i = 0,1$, and consider the functions $u_i : [0,1] \to \mathbb{R}^+$ defined as $u_i := \rho_i \circ F_i^{-1}$, where ρ_i is the density of ν_i, $i = 0,1$.

Introduce also $\bar u : [0,1] \to \mathbb{R}^+$ as $\bar u := \frac{1}{C \circ F_0^{-1}}$ (recall that $t \mapsto C(t) \geq 1$ was defined in (3.25)), put $u := \min\{u_0, u_1, \bar u\}$ and define $v : [0,1] \to [0, \max\{\varepsilon_0, \varepsilon_1\}]$ as

$$v(t) := c \int_0^t \frac{1}{u(s)} \, \mathrm{d}s, \qquad \text{with} \qquad c := \frac{\max\{\varepsilon_0, \varepsilon_1\}}{\int_0^1 \frac{1}{u(s)} \, \mathrm{d}s}. \tag{3.29}$$

Then v is invertible and its inverse $v^{-1} : [0, \max\{\varepsilon_0, \varepsilon_1\}] \to [0,1]$ is absolutely continuous and satisfies

$$(v^{-1})' = c u \circ v^{-1}. \tag{3.30}$$

Define the functions $\eta_0, \eta_1 : I \to \mathbb{R}^+$ as

$$\eta_i(t) := \begin{cases} (v^{-1})'(t - t_i), & \text{if } t \in [t_i, t_i + \max\{\varepsilon_0, \varepsilon_1\}], \\ 0, & \text{otherwise} \end{cases}$$

and notice that by constructions these are probability densities.

Step 2: definition of the interpolation. The interpolation will be built in 3 separate steps.

Let T_0 be the optimal transport map from ν_0 to $\eta_0 \mathcal{L}^1$, define $\hat T_0 : I \times X \to C([0,1], X_w)$ as

$$\hat T_0(t,x)_s := \big((1-s)t + sT_0(t), x\big)$$

and put

$$\pi_0 := (\hat{T}_0)_\sharp \mu_0.$$

Notice that we trivially have $(e_0)_\sharp \pi_0 = \mu_0$. Similarly, considering the optimal map T_1 from ν_1 to $\eta_1 \mathcal{L}^1$ and the induced map $\hat{T}_1 : I \times X \to C([0,1], X_w)$ given by $\hat{T}_1(t,x)_s := ((1-s)T_1(t) + st, x)$, we put

$$\pi_1 := (\hat{T}_1)_\sharp \mu_1,$$

and, much like before, we have $(e_1)_\sharp \pi_1 = \mu_1$. We shall now build a plan interpolating from $(e_1)_\sharp \pi_0$ to $(e_0)_\sharp \pi_1$. Recalling that the curve γ^I was previously introduced, we define the map

$$\begin{aligned} \mathcal{G} : I \times C([0,1], X) &\to C([0,1], X_w), \\ (t, \gamma) &\mapsto s \to \mathcal{G}(t,\gamma)_s := (T_0(t) + \gamma^I_s - t_0, \gamma_s). \end{aligned}$$

Then we consider the plan $\sigma \in \mathcal{P}(I \times C([0,1], X))$ given by

$$\mathrm{d}\sigma(t,\gamma) := \mathrm{d}\nu_0(t) \times \mathrm{d}\sigma^t(\gamma),$$

and put

$$\pi_{mid} := \mathcal{G}_\sharp \sigma \quad \in P(C([0,1], X_w)).$$

We claim that

$$(e_0)_\sharp \pi_{mid} = (e_1)_\sharp \pi_0 \qquad \text{and} \qquad (e_1)_\sharp \pi_{mid} = (e_0)_\sharp \pi_1. \tag{3.31}$$

To see the first, fix a bounded Borel function $\varphi : X_w \to \mathbb{R}$ and notice that

$$\begin{aligned} \int \varphi(t,x)\,\mathrm{d}(e_0)_\sharp \pi_{mid}(t,x) &= \iint \varphi(\mathcal{G}(t,\gamma)_0)\,\mathrm{d}\sigma^t(\gamma)\,\mathrm{d}\nu_0(t) \\ &= \iint \varphi(T_0(t), \gamma_0)\,\mathrm{d}\sigma^t(\gamma)\,\mathrm{d}\nu_0(t) \\ &= \iint \varphi(T_0(t), x)\,\mathrm{d}(e_0)_\sharp \sigma^t(x)\,\mathrm{d}\nu_0(t) \\ &= \iint \varphi(T_0(t), x)\,\mathrm{d}\mu_{0,t}(x)\,\mathrm{d}\nu_0(t) \\ &= \iint \varphi(T_0(t), x)\,\mathrm{d}\mu_0(t,x) \\ &= \iint \varphi(\hat{T}_0(t,x)_1)\,\mathrm{d}\mu_0(t,x) = \int \varphi(t,x)\,\mathrm{d}(e_1)_\sharp \pi_0(t,x) \end{aligned} \tag{3.32}$$

The second in (3.31) follows by an analgous computation taking into account that $t \mapsto T_1^{-1}(T_0(t) + t_1 - t_0)$ is the optimal map $t \mapsto T(t)$ from ν_0 to ν_1 (because in 1-d 'optimal=monotone').

The compatibility conditions (3.31) and a gluing argument ensure the existence of a plan $\pi_{0mid1} \in \mathcal{P}(C([0,3], X_w))$ such that

$$(\mathrm{Res}_{[0,1]})_\sharp \pi_{0mid1} = \pi_0, \qquad (\mathrm{Res}_{[1,2]})_\sharp \pi_{0mid1} = \pi_{mid}, \qquad (\mathrm{Res}_{[2,3]})_\sharp \pi_{0mid1} = \pi_1,$$

where for $[a,b] \subset [0,3]$ the map $\mathrm{Res}_{[a,b]} : C([0,3], X_w) \to C([a,b], X_w)$ sends a curve to its restriction on $[a,b]$.

Finally, putting $\varepsilon_{01} := \max\{\varepsilon_0, \varepsilon_1\}$ we define the scaling map $\mathrm{Scal} : C([0,3], X_w) \to C([0,1], X_w)$ as

$$\mathrm{Scal}(\gamma)_s := \begin{cases} \gamma_{s\varepsilon_{01}^{-1}}, & \text{for } s \in [0, \varepsilon_{01}], \\ \gamma_{1+\frac{s-\varepsilon_{01}}{1-2\varepsilon_{01}}}, & \text{for } s \in [\varepsilon_{01}, 1-\varepsilon_{01}], \\ \gamma_{2+(s+\varepsilon_{01}-1)\varepsilon_{01}^{-1}}, & \text{for } s \in [1-\varepsilon_{01}, 1], \end{cases}$$

and put

$$\pi := \mathrm{Scal}_\sharp \pi_{0mid1}.$$

Step 3: estimate of the density. To prove that π has bounded compression it is sufficient to show that for every $s \in [0,1]$ it holds

$$(e_s)_\sharp \pi_i \le \frac{\max\{c,1\}}{\mathfrak{m}_c(B_{\varepsilon_i}(x_i))}\mathfrak{m}, \quad \text{for} \quad i = 0,1 \qquad \text{and} \qquad (e_s)_\sharp \pi_{mid} \le c\mathfrak{m}, \tag{3.33}$$

where c is the constant defined in (3.29).

We start with the bound for π_0. Recall that T_0 is the optimal transport map from ν_0 to $\eta_0\mathcal{L}^1$, thus letting $T_{0,s}(t) := (1-s)t + sT_0(t)$ and denoting by ρ_s the density of $(T_{0,s})_\sharp\nu_0$, the general theory of optimal transport ensures that $s \mapsto \rho_s(T_{0,s}(t))$ is convex for ν_0-a.e. t. In particular we have

$$\rho_s(T_{0,s}(t)) \le \max\{\rho_0(t), \eta_0(T_0(t))\}, \qquad \nu_0 - a.e.\ t. \tag{3.34}$$

Let $G_0 : I \to [0,1]$ be the cumulative distribution function of $\eta_0\mathcal{L}^1$, i.e. $G_0(t) := \int_{-\infty}^t \eta_0 \, \mathrm{d}\mathcal{L}^1$ and notice that by definition of η_0 and property (3.30) it satisfies

$$\eta_0(t) = c\, u(G_0(t)), \tag{3.35}$$

so that by the definition of u we have

$$\eta_0(t) = c\, u(G_0(t)) \le c\, u_0(G_0(t)) = c\, \rho_0(F_0^{-1}(G_0(t))), \qquad \eta_0\mathcal{L}^1 - a.e.\ t.$$

Since the optimal map T_0 is the inverse of $F_0^{-1} \circ G_0$, from (3.34) we get that

$$\rho_s(T_{0,s}(t)) \le \max\{c,1\}\, \rho_0(t), \qquad \rho_0\mathcal{L}^1 - a.e.\ t,$$

for every $s \in [0,1]$. Now notice that with computations similar to those in (3.32) we see that

$$\mathrm{d}(e_s)_\sharp \pi(t,x) = \rho_s(t)\mathrm{d}t \times \mathrm{d}\mu_{0,T_{0,s}^{-1}(t)}(x),$$

in particular $(e_s)_\sharp\pi \ll \mathfrak{m}_c$ and for its density we have

$$\begin{aligned} \frac{\mathrm{d}(e_s)_\sharp \pi_0}{\mathrm{d}\mathfrak{m}_c}((T_{0,s}(t), x) &= \rho_s(T_{0,s}(t))\frac{\mathrm{d}\mu_{0,t}}{\mathrm{d}\mathfrak{m}}(x) \\ &\le \max\{c,1\}\rho_0(t)\frac{\mathrm{d}\mu_{0,t}}{\mathrm{d}\mathfrak{m}}(x) = \max\{c,1\}\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathfrak{m}_c}(t,x) \le \frac{\max\{c,1\}}{\mathfrak{m}_c(B_{\varepsilon_0}(x_0))}, \end{aligned}$$

for every $s\in[0,1]$ so that in this case the claim (3.33) is proved. The bound for π_1 is obtained in the same way, so we turn to the one on π_{mid}.

To this aim, start noticing that from the identity

$$\begin{aligned}\int\varphi(t,s)\,\mathrm{d}(e_s)_\sharp\pi_{mid}&=\int\left(\int\varphi(T_0(t)+\gamma^I_s-t_0,\gamma_s)\,\mathrm{d}\sigma^t(\gamma)\right)\,\mathrm{d}\nu_0(t)\\&=\int\left(\int\varphi(T_0(t)+\gamma^I_s-t_0,\gamma_s)\,\mathrm{d}\pi^{f_0(t),f_1(T(t))}(\gamma)\right)\,\mathrm{d}\nu_0(t)\\&=\int\left(\int\varphi(t+\gamma^I_s-t_0,x)\,\mathrm{d}(e_s)_\sharp\pi^{f_0(T_0^{-1}(t)),f_1(T(T_0^{-1}(t)))}(x)\right)\eta_0(t)\,\mathrm{d}t\end{aligned}$$

valid for any bounded Borel function φ, we see that $(e_s)_\sharp\pi_{mid}\ll\mathfrak{m}_c$ and

$$\frac{\mathrm{d}(e_s)_\sharp\pi_{mid}}{\mathrm{d}\mathfrak{m}_c}(t-t_0+\gamma^I_s,x)=\eta_0(t)\frac{\mathrm{d}(e_s)_\sharp\pi^{f_0(T_0^{-1}(t)),f_1(T(T_0^{-1}(t)))}}{\mathrm{d}\mathfrak{m}}(x),\qquad\eta_0\mathcal{L}^1\times\mathfrak{m}-a.e.\ (t,x).$$

Recalling the bound (3.25) we therefore obtain

$$\frac{\mathrm{d}(e_s)_\sharp\pi_{mid}}{\mathrm{d}\mathfrak{m}_c}(t-t_0+\gamma^I_s,x)\le\eta_0(t)C(T_0^{-1}(t)),\qquad\eta_0\mathcal{L}^1\times\mathfrak{m}-a.e.\ (t,x).$$

Using again (3.35) and the definition of u we get

$$\eta_0(t)\le\frac{c}{C(F_0^{-1}(G_0(t)))},$$

and since $T_0^{-1}=F_0^{-1}\circ G_0$, our claim is proved.

Step 4: estimate of the kinetic energy. Notice that by the very definition of π we have

$$\begin{aligned}\iint_0^1|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)&=\frac{1}{\varepsilon_{01}}\iint_0^1|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_0(\gamma)+\frac{1}{\varepsilon_{01}}\iint_0^1|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_1(\gamma)\\&\quad+\frac{1}{1-2\varepsilon_{01}}\iint_0^1|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_{mid}(\gamma)\end{aligned}\tag{3.36}$$

We start estimating the energy of π_0. Notice that since the d_w-metric speed of the curve $s\mapsto\hat T_0(t,x)_s$ is constantly equal to $|T_0(t)-t|$, we have

$$\iint|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_0(\gamma)=\int|T_0(t)-t|^2\,\mathrm{d}\mu_0(t,x)=\int|T_0(t)-t|^2\,\mathrm{d}\nu_0(t).$$

Now observe that since $\mathrm{spt}(\nu_0)\subset[t_0-\varepsilon_0,t_0+\varepsilon_0]$ and $\mathrm{spt}(\eta_0)\subset[t_0,t_0+\varepsilon_{01}]$, in trasporting ν_0 to $\eta_0\mathcal{L}^1$ no point is moved for more than $\varepsilon_0+\varepsilon_{01}$, therefore we have

$$\iint|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_0(\gamma)\le|\varepsilon_0+\varepsilon_{01}|^2\le4\varepsilon_{01}^2.\tag{3.37}$$

An analogous argument yields the bound

$$\iint|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_1(\gamma)\le4\varepsilon_{01}^2.\tag{3.38}$$

We pass to the energy of π_{mid}. Lemma 3.20 ensures that $\int_0^1 |\dot\gamma^X_s|\,\mathrm{d}s = \mathrm{d}(x_0,x_1)$, thus recalling the definition of $L(t)$ given in (3.27) and the identity (3.28) we obtain

$$\begin{aligned} L(t)\mathrm{d}(x_0,x_1) = L(t)\int_0^1 |\dot\gamma^X_s|\,\mathrm{d}s &= \int_0^1 \sqrt{\int |\dot\gamma_s|^2\,\mathrm{d}\sigma^t(\gamma)}\,\mathrm{d}s \\ &= \int_0^1 \sqrt{\int |\dot\gamma_s|^2\,\mathrm{d}\pi^{f_0(t),f_1(T(t))}(\gamma)}\,\mathrm{d}s \le \sqrt{\int_0^1\int |\dot\gamma_s|^2\,\mathrm{d}\pi^{f_0(t),f_1(T(t))}\,\mathrm{d}s} \end{aligned}$$

and thus from the estimate (3.26) we deduce

$$\overline{\lim_{\varepsilon_0,\varepsilon_1\downarrow 0}}\ \sup_{t\in[t_0-\varepsilon_0,t_0+\varepsilon_0]} L(t)\le 1. \tag{3.39}$$

Now notice that by the very definition of d_w, the d_w-squared speed of $s\mapsto(\gamma^I_s+T_0(t)-t_0,\gamma_s)$ is equal to $|\dot\gamma^I_s|^2+w_{\mathrm{d}}^2(\gamma^I_s+T_0(t)-t_0)|\dot\gamma_s|^2$ and thus we have

$$\begin{aligned} &\iint_0^1 |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_{mid}(\gamma) \\ &\quad= \iiint_0^1 |\dot\gamma^I_s|^2 + w_{\mathrm{d}}^2(\gamma^I_s+T_0(t)-t_0)|\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\sigma^t(\gamma)\,\mathrm{d}\nu_0(t) \\ &\quad= \int_0^1 |\dot\gamma^I_s|^2\,\mathrm{d}s + \iint_0^1 w_{\mathrm{d}}^2(\gamma^I_s+T_0(t)-t_0)\left(\int |\dot\gamma_s|^2\,\mathrm{d}\sigma^t(\gamma)\right)\mathrm{d}s\,\mathrm{d}\nu_0(t) \\ &\quad= \int_0^1 |\dot\gamma^I_s|^2\,\mathrm{d}s + \iint_0^1 w_{\mathrm{d}}^2(\gamma^I_s+T_0(t)-t_0)|\dot\gamma^X_s|^2L^2(t)\,\mathrm{d}s\,\mathrm{d}\nu_0(t). \end{aligned}$$

Now use the continuity of w_{d}, the weak convergence of ν_0 to δ_{t_0} as $\varepsilon_0\downarrow 0$ and (3.39) to obtain

$$\begin{aligned} \overline{\lim_{\varepsilon_0,\varepsilon_1\downarrow 0}}\iint_0^1 |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\pi_{mid}(\gamma) &\le \int_0^1 |\dot\gamma^I_s|^2 + w_{\mathrm{d}}^2(\gamma^I_s)|\dot\gamma^X_s|^2\,\mathrm{d}s \\ &= \int_0^1 |\dot\gamma_s|^2\,\mathrm{d}s = \mathrm{d}_w^2\big((t_0,x_0),(t_1,x_1)\big), \end{aligned}$$

which together with (3.37), (3.38) and (3.36) gives the conclusion.

Step 5: conclusion. We assumed initially the existence of a geodesic $\gamma=(\gamma^I,\gamma^X)$ with γ^I having image in the interior of I. To remove such assumption, it is sufficient to observe that there always exists a sequence of curves $\gamma_n=(\gamma^I_n,\gamma^X_n)$ with the same boundary data whose length converges to $\mathrm{d}_w((t_0,x_0),(t_1,x_1))$ and such that γ^I_n has image in the interior of I for every n. Then we can repeat the above arguments with γ_n in place of γ and conclude by diagonalization. □

Summing up what we proved so far we obtain:

Theorem 3.22. *Let $(X,\mathrm{d},\mathfrak{m})$ be an a.e. locally doubling and measured-length space, $I\subset\mathbb{R}$ a closed, possibly unbounded, interval and $w_{\mathrm{d}},w_{\mathfrak{m}}:I\to\mathbb{R}^+$ a couple of warping functions. Assume that $w_{\mathfrak{m}}$ is strictly positive in the interior of I.*

Then the warped product space $(X_w,\mathrm{d}_w,\mathfrak{m}_w)$ is a.e. doubling and measured-length. In particular, it has the Sobolev-to-Lipschitz property.

Proof. Direct consequence of Propositions 3.19, 3.21 and 3.18 □

References

[1] L. Ambrosio, M. Colombo, and S. Di Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope.* Priprint, arXiv:1212.3779, 2012.

[2] L. Ambrosio and N. Gigli, *A user's guide to optimal transport.* Modelling and Optimisation of Flows on Networks, Lecture Notes in Mathematics, Vol. 2062, Springer, 2011.

[3] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.

[4] ——, *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, Revista Matemática Iberoamericana, 29 (2013), pp. 969–996.

[5] ——, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.

[6] ——, *Metric measure spaces with riemannian Ricci curvature bounded from below*, Duke Math. J., 163 (2014), pp. 1405–1490.

[7] L. Ambrosio, A. Pinamonti, and G. Speight, *Tensorization of Cheeger energies, the space $H^{1,1}$ and the area formula for graphs.* Priprint, arXiv:1407.3489, 2014.

[8] L. Ambrosio and P. Tilli, *Topics on analysis in metric space*, Oxford University Press, 2004.

[9] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, 2001.

[10] N. Gigli, *On the differential structure of metric measure spaces and applications.* Accepted at Mem. Amer. Math. Soc., arXiv:1205.6622, 2013.

[11] ——, *The splitting theorem in non-smooth context.* Preprint, arXiv:1302.5555, 2013.

[12] C. Ketterer, *Cones over metric measure spaces and the maximal diameter theorem*, Journal de Mathématiques Pures et Appliquées, 103 (2015), pp. 1228–1275.

[13] S. Lisini, *Characterization of absolutely continuous curves in Wasserstein spaces*, Calc. Var. Partial Differential Equations, 28 (2007), pp. 85–120.

[14] K.-T. Sturm, *On the geometry of metric measure spaces. II*, Acta Math., 196 (2006), pp. 133–177.