Discrete double-porosity models for spin systems

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Abstract

We consider spin systems between a finite number N of "species" or "phases" partitioning a cubic lattice $\mathbb{Z}^d$. We suppose that interactions between points of the same phase are coercive, while between point of different phases (or, possibly, between points of an additional "weak phase") are of lower order. Following a discrete-to-continuum approach we characterize the limit as a continuum energy defined on N-tuples of sets (corresponding to the N strong phases) composed of a surface part, taking into account homogenization at the interface of each strong phase, and a bulk part which describes the combined effect of lower-order terms, weak interactions between phases, and possible oscillations in the weak phase.

Key words. spin systems, lattice energies, double porosity, gamma-convergence, homogenization, discrete-to-continuum

AMS subject classifications. 49M25, 39A12, 39A70, 35Q82

1 Introduction

In this paper we consider lattice spin energies mixing strong ferromagnetic interactions and weak (possibly, antiferromagnetic) pair interactions. The geometry that we have in mind is a periodic system of interactions, as that whose periodicity cell is represented in Fig. 1. In that picture, the strong interactions between nodes of the lattice (circles) are represented by solid lines and weak ones by dashed lines. In this particular case, we have two three-periodic systems of 'strong sites'; i.e., sites connected by strong interactions, and isolated 'weak sites' (pictured as white circles). Note that we may also have one or more infinite systems of connected weak interactions as in Fig. 2. In a discrete environment the topological requirements governing the interactions between the strong and weak phases characteristic of continuum high-contrast models are substituted by assumptions on long-range interactions. In particular, contrary to the continuum case, for discrete systems with second-neighbour (or longer-range) interactions

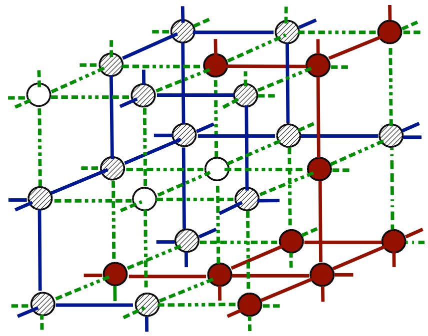

Figure 1: picture of a double-porosity system

we may have a limit multi-phase system even in dimension one (see the examples in the final section).

This paper is part of a general study of spin systems by means of variational techniques through the computation of continuum approximate energies, for which homogenization results have been proved in the ferromagnetic case (i.e, when all interactions are strong) by Caffarelli and de la Llave [17] and Braides and Piatnitski [14], and a general discrete-to-continuum theory of representation and optimization has been elaborated (see the survey article [10]). In particular, a discrete-to-continuum compactness result and an integral representation of the limit by means of surface energies defined on sets of finite perimeter has been proved by Alicandro and Gelli [3]. In that result, the coercivity of energies is obtained by assuming that nearest neighbours are always connected through a chain of strong interactions. Double-porosity systems can be interpreted as energies for which this condition does not hold, but is satisfied separately on (finitely many) infinite connected components.

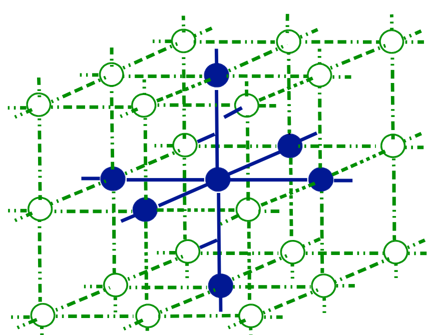

Figure 2: a double-porosity system with an infinite connected weak component

Double-porosity problems had been previously considered on the continuum for integral energies (see e.g. [4, 6, 11, 18, 19, 20]) and for interfacial energies (see the works by Solci [21, 22]). A study of discrete double-porosity models in the case of elastic energies has been recently carried on in [12]. With respect to that paper we remark that the case of spin systems allows a very easy proof of an extension lemma from connected discrete sets, and at the same time permits to highlight the possibility to include a weak phase with antiferromagnetic interactions, optimized by microscopic oscillations.

We are going to consider energies defined on functions parameterized on the cubic lattice $\mathbb{Z}^d$ of the following form

$$F_\varepsilon(u) = \sum_{(\alpha,\beta)\in\varepsilon\mathcal{N}_1\cap(\Omega\times\Omega)} \varepsilon^{d-1} a^\varepsilon_{\alpha\beta}(u_\alpha - u_\beta)^2 + \sum_{(\alpha,\beta)\in\varepsilon\mathcal{N}_0\cap(\Omega\times\Omega)} \varepsilon^d a^\varepsilon_{\alpha\beta}(u_\alpha - u_\beta)^2 + \sum_{\alpha\in\Omega\cap\varepsilon\mathbb{Z}^d} \varepsilon^d g(u_\alpha), \tag{1}$$

where Ω is a regular open subset of $\mathbb{R}^d$, and $u_\alpha \in \{-1,+1\}$ denote the values of a spin function. For explicatory purposes, in this formula and the rest of the Introduction, we use a simplified notation with respect to the rest of the paper, defining $u = \{u_\alpha\}$ on the nodes of $\Omega \cap \varepsilon\mathbb{Z}^d$ (instead of, equivalently, on the nodes of $\frac{1}{\varepsilon}\Omega \cap \mathbb{Z}^d$). We denote by $\mathcal{N}_1$ the set of pairs of nodes in $\mathbb{Z}^d \times \mathbb{Z}^d$ between which we have strong interactions, and by $\mathcal{N}_0$ the set of pairs in $\mathbb{Z}^d \times \mathbb{Z}^d$ between which we have weak interactions; the difference between these two types of interactions in the energy is the scaling factor: ε^{d-1} for strong interactions and ε^d for weak interaction. We suppose that all coefficients are obtained by scaling fixed coefficients on $\mathbb{Z}^d$; i.e.,

$$a^\varepsilon_{\alpha\beta} = a_{\alpha/\varepsilon\,\beta/\varepsilon} \qquad \text{if } \alpha, \beta \in \varepsilon\mathbb{Z}^d, \tag{2}$$

and a_{jk} are periodic of some integer period T. Moreover, we assume that the coefficients of the strong interactions are strictly positive; i.e., $a_{jk} > 0$ if $(j,k) \in \mathcal{N}_1$. The 'forcing' term containing g and depending only on the point values u_α is of lower-order with respect of strong interactions, but of the same order of the weak interactions.

We suppose that there there are N infinite connected components of the graph of points linked by strong interactions, which we denote by $C_1, \dots, C_N$. Note that weak interactions in $\mathcal{N}_0$ are due either to the existence of "weak sites" or to weak bonds between different "strong components", and, if we have more than one strong graph, the interactions in $\mathcal{N}_0$ are present also in the absence of a weak component.

If we consider only the strong interactions restricted to each strong connected component C_j, we obtain energies

$$F^j_\varepsilon(u) = \sum_{(\alpha,\beta)\in\varepsilon\mathcal{N}^j_1\cap(\Omega\times\Omega)} \varepsilon^{d-1} a^\varepsilon_{\alpha\beta}(u_\alpha - u_\beta)^2 \tag{3}$$

where $\mathcal{N}^j_1$ is the restriction to $C_j \times C_j$ of the set $\mathcal{N}_1$. This is a discrete analog of an energy on a *perforated domain*, the perforation being $\mathbb{Z}^d \setminus C_j$.

We prove an extension lemma that allows to define for each $j \in \{1, \dots, N\}$ a *discrete-to-continuum convergence* of (the restriction to C_j of) a sequence of function u^ε to a function $u^j \in BV(\Omega; \{\pm 1\})$, which is compact under a equi-boundedness assumptions for the energies $F^j_\varepsilon(u^\varepsilon)$. Thanks to this lemma, such energies behave as ferromagnetic energies with positive coefficients on the whole $\mathbb{Z}^d$, which can be *homogenized* thanks to [14]; i.e., their Γ-limit with respect to the convergence $u^\varepsilon \to u^j$ exists, and is of the form

$$F^j(u^j) = \int_{S(u^j)\cap\Omega} f^j_{\rm hom}(\nu_{u^j})\, d\mathcal{H}^{d-1}, \tag{4}$$

where $S(u^j)$ is the set of jump points of u^j, that can also be interpreted as the interface between $\{u^j = 1\}$ and $\{u^j = -1\}$.

Taking separately into account the restrictions of u^ε to all of the components C_j, we define a *vector-valued* limit function $u = (u^1, \dots, u^N)$ and a convergence $u^\varepsilon \to u$, and consider the

Γ-limit with respect of the whole energy with respect to that convergence. The combination of the weak interactions and the forcing term give rise to a term of the form

$$\int_\Omega \varphi(u)\, dx$$

depending on the values of all components of u. In the case that $\bigcup_{j=1}^N C_j$ is the whole $\mathbb{Z}^d$,the function $\varphi(z^1,\dots,z^N)$ is simply computed as the average of the T-periodic function

$$i \mapsto \sum_{k\in\mathbb{Z}^d} a_{ik}(u_i-u_k)^2 + g(u_i)$$

where u takes the value z^j on C_j. Note that with this conditions only (weak) interactions between different C_j are taken into account. Note moreover that the restriction of the last term g to εC_j is continuously converging to

$$K_j \int_\Omega g(u^j)\, dx,$$

where $K_j = T^{-d}\#\{i\in C^j; i\in\{0,\dots,T\}^d\}$ is the percentage of sites in C_j. In general, φ is obtained by optimizing the combined effect of weak pair-interactions and g on the free sites in the complement of all C_j.

Such different interactions can be summed up to describe the Γ-limit of F_ε that finally takes the form

$$F_{\rm hom}(u) = \int_{S(u)\cap\Omega} f_{\rm hom}(\nu_u) d\mathcal{H}^{d-1} + \int_\Omega \varphi(u)\, dx, \tag{5}$$

where $f_{\rm hom}(\nu) = \sum_{j=1}^N f^j_{\rm hom}(\nu)$.

We note that the presence of two terms of different dimensions in the limit highlights the combination of bulk homogenization effects due to periodic oscillations besides the optimization of the interfacial structure. The effect of those oscillations on the variational motions of such systems (in the sense of [2, 9]) is addressed in [16].

2 Notation

The numbers d, m, T and N are positive integers. We introduce a T periodic *label function* $J:\mathbb{Z}^d\to\{0,1\dots,N\}$, and the corresponding sets of sites

$$A_j = \{k\in\mathbb{Z}^d : J(k)=j\}, \qquad j=0,\dots,N.$$

Sites interact through possibly long (but finite)-range interactions, whose range is defined through a system $P^j = \{P^j_k\}$ of finite subsets $P^j_k\subset\mathbb{Z}^d$, for $j=0,\dots,N$ and $k\in A_j$. We suppose that

• (T*-periodicity*) $P^j_{k+m} = P^j_k$ for all $m\in T\mathbb{Z}^d$;

• (*symmetry*) if $k\in A_j$ for $j=1,\dots N$ (*hard components*) and $i\in P^j_k$ then $k+i\in A_j$ and $-i\in P^j_{k+i}$, and that $0\in P^j_k$.

We say that two points $k,k'\in A_j$ are P^j*-connected* in A_j if there exists a path $\{k_n\}_{n=0,\dots,K}$ such that $k_n\in A_j$, $k_0=k$, $k_K=k'$ and $k_n-k_{n-1}\in P^j_{k_{n-1}}$.

We suppose that

• (*connectedness*) there exists a unique infinite P^j-connected component of each A_j for $j=1,\dots,N$, which we denote by C_j.

Clearly, the connectedness assumption is not a modelling restriction upon introducing more labelling parameters, if the number of infinite connected components is finite. Note that we do not make any assumption on A_0 and P^0. In particular, if $k \in A_j$ for $j = 0, \dots N$ and $i \in P^0_k$ then $k + i$ may belong to any $A_{j'}$ with $j' \neq j$.

We consider the following sets of bonds between sites in $\mathbb{Z}^d$: for $j = 1, \dots, N$

$$N_j = \{(k, k') : k, k' \in A_j, k' - k \in P^j_k \setminus \{0\}\};$$

for $j = 0$

$$N_0 = \{(k, k') : k' - k \in P^0_k \setminus \{0\}, J(k)J(k') = 0 \text{ or } J(k) \neq J(k')\}.$$

Note that the set N_0 takes into account interactions not only among points of the set A_0, but also among pair of points in different A_j. A more refined notation could be introduced by defining range of interactions P^{ij} and the corresponding sets N_{ij}, in which case the sets N_j would correspond to N_{jj} for $j = 1, \dots, N$ and N_0 the union of the remaining sets. However, for simplicity of presentation we limit our notation to a single index.

We consider interaction energy densities associated to positive numbers $a_{kk'}$ for $k, k' \in \mathbb{Z}^d$, and forcing term g. We suppose that for all $k, k' \in \mathbb{Z}^d$

(i) (*coerciveness on the hard phase*) there exists $c > 0$ such that $a_{kk'} \geq c > 0$ if $k \in C_j$ and $k' - k \in P^j_k$ for $j \geq 1$;

(ii) (T-*periodicity*) $a_{k+m\,k'+m} = a_{kk'}$ for all $m \in T\mathbb{Z}^d$;

(iii) (*symmetry*) $a_{k'k} = a_{kk'}$;

(iv) (T-*periodicity of the forcing term*) $g(k + m, 1) = g(k, 1)$ and $g(k + m, -1) = g(k, -1)$ for all $m \in T\mathbb{Z}^d$.

Note that we do not suppose that $a_{kk'}$ be positive for weak interactions. They can as well be negative, thus favouring oscillations in the weak phase.

Given Ω a bounded regular open subset of $\mathbb{R}^d$, for $u : \frac{1}{\varepsilon}\Omega \cap \mathbb{Z}^d \to \{+1, -1\}$ we define the energies

$$\begin{aligned} F_\varepsilon(u) = F_\varepsilon\Big(u, \frac{1}{\varepsilon}\Omega\Big) \quad &= \quad \sum_{j=1}^{N} \sum_{(k,k') \in \mathcal{N}^\varepsilon_j(\Omega)} \varepsilon^{d-1} a_{kk'}(u_k - u_{k'})^2 \\ &\quad + \sum_{(k,k') \in \mathcal{N}^\varepsilon_0(\Omega)} \varepsilon^d a_{kk'}(u_k - u_{k'})^2 + \sum_{k \in Z^\varepsilon(\Omega)} \varepsilon^d g(k, u_k), \end{aligned} \tag{6}$$

where

$$\mathcal{N}^\varepsilon_j(\Omega) = N_j \cap \frac{1}{\varepsilon}(\Omega \times \Omega), j = 0, \dots, N, \qquad Z^\varepsilon(\Omega) = \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega. \tag{7}$$

The first sum in the energy takes into account all interactions between points in A_j (*hard phases*), which are supposed to scale differently than those between points in A_0 (*soft phase*) or between points in different phases. The latter are contained in the second sum. The third sum is a zero-order term taking into account all types of phases with the same scaling .

Note that the first sum may take into account also points in $A_j \setminus C_j$, which form “islands” of the hard phase P^j-disconnected from the corresponding infinite component. Furthermore, in this energy we may have sites that do not interact at all with hard phases.

Remark 2.1 (choice of the parameter space)**.** The energy is defined on discrete functions parameterized on $\frac{1}{\varepsilon}\Omega \cap \mathbb{Z}^d$. The choice of this notation, rather than interpreting u as defined on $\Omega \cap \varepsilon\mathbb{Z}^d$ allows a much easier notation for the coefficients, that in this way are ε-independent, rather than obtained by scaling as in (2).

3 Homogenization of perforated discrete domains

In this section we separately consider the interactions in each infinite connected component of the hard phases introduced above. To that end we fix one of the indices j, with $j > 0$, dropping it in the notation of this section (in particular we use the symbol C in place of C_j, etc.), and define the energies

$$\mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon\Big(u, \frac{1}{\varepsilon}\Omega\Big) = \sum_{(k,k')\in N^\varepsilon_C(\Omega)} \varepsilon^{d-1} a_{kk'}(u_k - u_{k'})^2\,, \tag{8}$$

where

$$N^\varepsilon_C(\Omega) = \Big\{(k,k') \in (C\times C)\cap \frac{1}{\varepsilon}(\Omega\times\Omega) : k'-k \in P_k, k\neq k'\Big\}, \tag{9}$$

We also introduce the notation $C^\varepsilon(\Omega) = C \cap \frac{1}{\varepsilon}\Omega$.

Definition 3.1. The *piecewise-constant interpolation* of a function $u : \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$, $k \mapsto u_k$ is defined as

$$u(x) = u_{\lfloor x/\varepsilon\rfloor},$$

where $\lfloor y\rfloor = (\lfloor y_1\rfloor, \ldots, \lfloor y_d\rfloor)$ and $\lfloor s\rfloor$ stands for the integer part of s. The *convergence* of a sequence (u^ε) of discrete functions is understood as the $L^1_{\rm loc}(\Omega)$ convergence of these piecewise-constant interpolations. Note that, since we consider local convergence in Ω, the value of $u(x)$ close to the boundary in not involved in the convergence process.

We prove an extension and compactness lemma with respect to the convergence of piecewise-constant interpolations.

Lemma 3.2 (extension and compactness)**.** *Let C be a T-periodic subset of $\mathbb{Z}^d$ P-connected in the notation of the previous section, let $u^\varepsilon : \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega \to \{+1,-1\}$ be a sequence such that*

$$\sup_\varepsilon\ \varepsilon^{d-1}\#\Big\{(k,k') \in N^\varepsilon_C(\Omega) : u^\varepsilon_k \neq u^\varepsilon_{k'}\Big\} < +\infty. \tag{10}$$

Then there exists a sequence $\widetilde{u}^\varepsilon : \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$ such that $\widetilde{u}^\varepsilon_k = u^\varepsilon_k$ if $k \in C^\varepsilon(\Omega)$ and $\mathrm{dist}(k, \partial\frac{1}{\varepsilon}\Omega) > c = c(P)$, with $\widetilde{u}^\varepsilon$ converging to some $u \in BV_{\rm loc}(\Omega; \{+1,-1\})$ up to subsequences.

Proof. For a fixed $M \in \mathbb{N}$ and $j \in \mathbb{Z}^d$ we consider the discrete cubes of side length M

$$Q_M(j) := jM + \{0, M-1\}^d.$$

For each j we also define the cube

$$Q'_{3M}(j) = \bigcup_{\|i-j\|_\infty\leq 1} Q_M(i),$$

which is a discrete cube centered in $Q_M(j)$ and with side length $3M$.

For all ε we consider the family

$$\mathcal{Q}^\varepsilon_M := \Big\{Q_M(j) : j \in \mathbb{Z}^d,\ Q'_{3M}(j) \subset \frac{1}{\varepsilon}\Omega\Big\}.$$

We suppose that M is large enough so that if $k, k' \in Q_M(j)\cap C$ then there exists a P-path connecting k and k' contained in $Q'_{3M}(j)$. The existence of such M follows easily from the connectedness hypotheses. Indeed, we may take M as the length of the longest shortest P-path connecting two points in C with distance not greater than $2\sqrt{d}$ (in particular belonging to neighbouring periodicity cubes), and construct such P-path by concatenating a family of those shortest paths.

We define the set of indices

$$\mathcal{S}_\varepsilon = \Big\{ j \in \mathbb{Z}^d : Q_M(j) \in \mathcal{Q}^\varepsilon_M \text{ and } u^\varepsilon \text{ is not constant on } C \cap Q_M(j) \Big\}.$$

By our choice of M if $j \in \mathcal{S}_\varepsilon$ then there exist $k, k' \in Q'_{3M}(j) \cap C$ with $k' - k \in P$ such that $u^\varepsilon_k \neq u^\varepsilon_{k'}$. Let

$$K := \sup_\varepsilon \; \varepsilon^{d-1} \# \Big\{ (k,k') \in N^\varepsilon_C(\Omega) : u^\varepsilon_k \neq u^\varepsilon_{k'} \Big\}.$$

Then we deduce that

$$\#\mathcal{S}_\varepsilon \leq 3^d K \frac{1}{\varepsilon^{d-1}} \tag{11}$$

(the factor 3^d comes from the fact that $k, k' \in Q'_{3M}(j)$ for 3^d possible j).

We define $\widetilde{u}^\varepsilon$ as follows

$$\widetilde{u}^\varepsilon = \begin{cases} \text{the constant value of } u^\varepsilon \text{ on } Q_M(j) \cap C & \text{on } Q_M(j), \text{ if } Q_M(j) \in \mathcal{Q}^\varepsilon_M \text{ and } j \notin \mathcal{S}_\varepsilon \\ u^\varepsilon & \text{elsewhere.} \end{cases}$$

This will be the required extension. However we will prove the convergence of $\widetilde{u}^\varepsilon$ as a consequence of the convergence of the functions v^ε defined as

$$v^\varepsilon = \begin{cases} \widetilde{u}^\varepsilon & \text{on } Q_M(j), \text{ if } Q_M(j) \in \mathcal{Q}^\varepsilon_M \text{ and } j \notin \mathcal{S}_\varepsilon \\ 1 & \text{elsewhere.} \end{cases}$$

By (11) we have that for fixed $\Omega' \subset\subset \Omega$

$$\|v^\varepsilon - \widetilde{u}^\varepsilon\|_{L^1(\Omega')} = O(\varepsilon)$$

(recall that we identify the function with their scaled interpolations in $L^1(\Omega)$).

If the value of v^ε differs on two neighbouring $Q_M(j)$ and $Q_M(j')$ with $\|j - j'\|_1 = 1$ then, upon taking a suitable larger M, we may also suppose that there exist $k, k' \in (Q'_{3M}(j) \cup Q'_{3M}(j)) \cap C$ with $k - k' \in P$ and $u^\varepsilon_k \neq u^\varepsilon_{k'}$. Arguing as for (11), we deduce that the number of such j is $O(\varepsilon^{1-d})$, so that

$$\mathcal{H}^{d-1}(\partial\{v^\varepsilon = 1\} \cap \Omega') = O(1),$$

which implies the compactness of the family (v^ε) in $BV_{\rm loc}(\Omega)$. □

Theorem 3.3 (homogenization on discrete perforated domains)**.** *The energies $\mathcal{F}_\varepsilon$ defined in (8) Γ-converge with respect to the $L^1_{\rm loc}(\Omega)$ topology to the energy*

$$\mathcal{F}_{\rm hom}(u) = \int_{\Omega \cap \partial^* E} f_{\rm hom}(\nu) \, d\mathcal{H}^{d-1}, \tag{12}$$

defined on $u = \chi_E$, $u \in BV(\Omega, \{+1, -1\})$ where the energy density $f_{\rm hom}$ satisfies

$$\begin{aligned} f_{\rm hom}(\xi) \;&=\; \lim_{T \to +\infty} \frac{1}{T^{d-1}} \inf \Big\{ \sum_{(k,k') \in \widetilde{N}_C(Q^\nu_T)} a_{kk'}(u_k - u_{k'})^2 : u_k = 1 \text{ if } k \notin Q^\nu_T \text{ and } \langle k, \nu \rangle > 0 \\ &\qquad u_k = -1 \text{ if } k \notin Q^\nu_T \text{ and } \langle k, \nu \rangle \leq 0 \Big\}, \end{aligned} \tag{13}$$

where Q^ν is a cube centered in 0 and with one side orthogonal to ν, $Q^\nu_T = T\,Q^\nu$, and $\widetilde{N}_C(Q^\nu_T)$ denote all pairs in $(k,k') \in N^1_C(\mathbb{R}^d)$ such that either $k \in Q^\nu_T$ or $k' \in Q^\nu_T$.

Proof. In [14] this theorem is proved under the additional assumption that the energies $\mathcal{F}_\varepsilon$ be equi-coercive with respect to the weak BV-convergence. This assumption can be substituted by Lemma 3.2. Indeed, if u^ε is a sequence converging to u in $L^1_{\rm loc}(\Omega)$ and with equibounded energies then by Lemma 3.2 we may find a sequence $\widetilde{u}^\varepsilon$ coinciding with u^ε on $C^\varepsilon(\Omega')$ for every fixed $\Omega' \subset\subset \Omega$ and ε sufficiently small, and converging to some $\widetilde{u}$ in $BV(\Omega;\{\pm 1\})$. Since $\widetilde{u}^\varepsilon = u_\varepsilon$ on $C^\varepsilon(\Omega')$ we have that $\widetilde{u} = u$ and $\mathcal{F}_\varepsilon\Big(\widetilde{u}^\varepsilon, \frac{1}{\varepsilon}\Omega'\Big) = \mathcal{F}_\varepsilon\Big(u^\varepsilon, \frac{1}{\varepsilon}\Omega'\Big)$. Then we can give a lower estimate on each Ω' fixed using the proof of [14], and hence on Ω by internal approximation. Note that neither the proof of the existence of the limit in (13) therein, nor the construction of the recovery sequences depend on the coerciveness assumption, so that the proof is complete. $\square$

4 Definition of the interaction term

The homogenization result in Theorem 3.3 will describe the contribution of the hard phases to the limiting behavior of energies F_ε. We now characterize their interactions with the soft phase.

For all M positive integer and $z_1, \dots, z_N \in \{+1, -1\}$ we define the minimum problem

$$\varphi_M(z_1,\dots,z_N) = \frac{1}{M^d}\min\Big\{\sum_{(k,k')\in N_0(Q_M)} a_{kk'}(v_k - v_{k'})^2 + \sum_{k\in Z(Q_M)} g(k, v_k) : v \in \mathcal{V}_M\Big\}, \quad (14)$$

where

$$Q_M = \Big[-\frac{M}{2}, \frac{M}{2}\Big)^d, \qquad N_0(Q_M) = N_0 \cap (Q_M \times Q_M), \qquad Z(Q_M) = \mathbb{Z}^d \cap Q_M, \qquad (15)$$

and the minimum is taken over the set $\mathcal{V}_M = \mathcal{V}_M(z_1,\dots,z_N)$ of all v that are constant on each connected component of $A_j \cap Q_M$ and $v = z_j$ on C_j for $j = 1, \dots N$.

Proposition 4.1. *There exists the limit φ of φ_M as $M \to +\infty$.*

Proof. We first show that

$$\varphi_{KM} \geq \varphi_M \text{ for all } K \in \mathbb{N}. \qquad (16)$$

To that end, let $\overline{v}$ be a minimizer for $\varphi_{KM}(z_1,\dots,z_N)$. Then we have

$$\begin{aligned}
&K^d M^d \varphi_{KM}(z_1,\dots,z_N) \\
&\quad = \sum_{(k,k')\in N_0(Q_{KM})} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 + \sum_{k\in Z(Q_{KM})} g(k,\overline{v}_k) \\
&\quad = \sum_{l\in\mathbb{Z}^d\cap Q_K}\Bigg(\sum_{(k,k')\in N_0(Q_M+lM)} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 + \sum_{k\in Z(Q_M+lM)} g(k,\overline{v}_k)\Bigg) \\
&\quad\quad + \sum_{(k,k')\in N_0(Q_{KM})\setminus\bigcup_l N_0(Q_M+lM)} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 \\
&\quad \geq \sum_{l\in\mathbb{Z}^d\cap Q_K}\Bigg(\sum_{(k,k')\in N_0(Q_M+lM)} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 + \sum_{k\in Z(Q_M+lM)} g(k,\overline{v}_k)\Bigg).
\end{aligned}$$

Let $\bar{l} \in \mathbb{Z}^d \cap Q_K$ minimize the expression in parenthesis. Then we deduce

$$K^d M^d \varphi_{KM}(z_1,\dots,z_N) \quad \geq \quad K^d\Bigg(\sum_{(k,k')\in N_0(Q_M+\bar{l}M)} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 + \sum_{k\in Z(Q_M+\bar{l}M)} g(k,\overline{v}_k)\Bigg),$$

from which (16) follows by taking $v_k = \overline{v}_{k-\bar{l}M}$ in the computation of $\varphi_M(z_1, \ldots, z_N)$.

We remark that for $L \geq L'$ we have

$$L^d \varphi_L \geq (L')^d \varphi_{L'} - \max|g| \; (L^d - (L')^d). \tag{17}$$

Hence, fixing n, L and M, $L \geq M2^n$, and taking $L' = \lfloor \frac{L}{M2^n} \rfloor M2^n$ in (17), we have, using (16) with $K = \lfloor \frac{L}{M2^n} \rfloor 2^n$

$$\begin{aligned} \varphi_L &\geq \frac{1}{L^d}\left(\left\lfloor \frac{L}{M2^n}\right\rfloor M2^n\right)^d \varphi_{\lfloor \frac{L}{M2^n} \rfloor M2^n} - \max|g|\left(1 - \left(\left\lfloor \frac{L}{M2^n}\right\rfloor \frac{M2^n}{L}\right)^d\right) \\ &\geq \left(\left\lfloor \frac{L}{M2^n}\right\rfloor \frac{M2^n}{L}\right)^d \varphi_M - \max|g|\left(1 - \left(\left\lfloor \frac{L}{M2^n}\right\rfloor \frac{M2^n}{L}\right)^d\right). \end{aligned}$$

Letting $L \to +\infty$ we then obtain

$$\liminf_{L \to +\infty} \varphi_L \geq \varphi_M$$

and the thesis by taking the upper limit in M. □

Let R be defined by

$$R = \max\{|k - k'| : \; k, k' \in A_j \setminus C_j \; P^j\text{-connected}, \; j = 1, \ldots, N\} \tag{18}$$

and for all M positive integer we set

$$D_M = \bigcup_{j=1}^{N} \bigcup \Big\{B : \; B \text{ a } P^j\text{-connected components of } A_j \setminus C_j \text{ not intersecting } Q_{M-R}\Big\}. \tag{19}$$

For all $z_1, \ldots, z_N \in \{+1, -1\}$ we define

$$\begin{aligned} \widetilde{\varphi}_M(z_1, \ldots, z_N) = \frac{1}{M^d} \min\Big\{ &\sum_{(k,k') \in N_0(Q_M)} a_{kk'}(v_k - v_{k'})^2 + \sum_{k \in Z(Q_M)} g(k, v_k) : \\ &v \in \mathcal{V}_M, v_k = 1 \text{ if } k \in D_M \Big\}. \end{aligned}$$

Proposition 4.2. *There is a positive constant c independent of M such that*

$$\widetilde{\varphi}_M \geq \varphi_M \geq \widetilde{\varphi}_M - \frac{c}{M}. \tag{20}$$

Proof. The first inequality is trivial. To prove the second, let $\overline{v}$ be a minimizer for $\varphi_M(z_1, \ldots, z_N)$ and define v by

$$v_k = \begin{cases} 1 & \text{if } k \in D_M \\ \overline{v}_k & \text{otherwise.} \end{cases}$$

Using v as a test function for $\widetilde{\varphi}_M(z_1, \ldots, z_N)$, we obtain

$$\begin{aligned} M^d \widetilde{\varphi}_M(z_1, \ldots, z_N) \leq\; & \sum_{(k,k') \in N_0(Q_M), \; k,k' \notin D_M} a_{kk'}(v_k - v_{k'})^2 + \sum_{k \in Z(Q_M) \setminus D_M} g(k, v_k) \\ & +2 \sum_{(k,k') \in N_0(Q_M), \; k \in D_M} a_{kk'}(v_k - v_{k'})^2 + \sum_{k \in Z(Q_M) \cap D_M} g(k, v_k) \\ \leq\; & \sum_{(k,k') \in N_0(Q_M), \; k,k' \notin D_M} a_{kk'}(\overline{v}_k - \overline{v}_{k'})^2 + \sum_{k \in Z(Q_M) \setminus D_M} g(k, \overline{v}_k) \\ & + \sum_{(k,k') \in N_0(Q_M), \; k \in D_M} a_{kk'} + \sum_{k \in Z(Q_M) \cap D_M} g(k, 1) \\ \leq\; & M^d \varphi_M(z_1, \ldots, z_N) + \#D_M \; \#P_0 \; \max a_{ij} + \#D_M \; 2\max|g|. \end{aligned}$$

Since $\#D_M \le 2^d M^{d-1} R$, the thesis follows with $c = 2^d R(\#P_0 \ \max a_{ij} + \ 2\max|g|)$. $\square$

5 Statement of the convergence result

We now have all the ingredients to characterize the asymptotic behavior of F_ε defined in (6).

Definition 5.1 (multi-phase discrete-to-continuum convergence)**.** *We define the* convergence

$$u^\varepsilon \to (u^1, \dots, u^N) \tag{21}$$

as the $L^1_{\rm loc}(\Omega; \mathbb{R}^m)$ *convergence* $\widetilde{u}^\varepsilon_j \to u^j$ *of the extensions of the restrictions of* u^ε *to* C_j *as in Lemma* 3.2*, which is a compact convergence as ensured by that lemma.*

The total contribution of the hard phases will be given separately by the contribution on the infinite connected components and the finite ones. The first one is obtained by computing independently the limit relative to the energy restricted to each component

$$\mathcal{F}^j_\varepsilon(u) = \sum_{(k,k')\in N^\varepsilon_j(\Omega)} \varepsilon^{d-1} a_{kk'} (v_k - v_{k'})^2 , \tag{22}$$

where

$$N^\varepsilon_j(\Omega) = N^\varepsilon_{C_j}(\Omega) = \Big\{(k,k') \in (C_j \times C_j) \cap \frac{1}{\varepsilon}(\Omega\times\Omega) : k - k' \in P^j_k, k \ne k'\Big\}, \tag{23}$$

which is characterized by Theorem 3.3 as

$$\mathcal{F}^j_{\rm hom}(u) = \int_{\Omega\cap\partial^*\{u=1\}} f^j_{\rm hom}(\nu) {\rm d}\mathcal{H}^{d-1}. \tag{24}$$

In the previous section we have introduced the energy density φ, which describes the interactions between the hard phases. Taking all contribution into account, we may state the following convergence result.

Theorem 5.2 (double-porosity homogenization)**.** *Let* Ω *be a Lipschitz bounded open set, and let* F_ε *be defined by* (6) *with the notation of Section* 2*. Then there exists the* Γ*-limit of* F_ε *with respect to the convergence* (21) *and it equals*

$$F_{\rm hom}(u^1, \dots, u^N) = \sum_{j=1}^N \int_{\Omega\cap\partial^*\{u^j=1\}} f^j_{\rm hom}(\nu) {\rm d}\mathcal{H}^{d-1} + \int_\Omega \varphi(u^1, \dots, u^N)\, dx \tag{25}$$

on functions $u = (u^1, \dots, u^N) \in (BV(\Omega; \{1, -1\}))^N$*, where* φ *is defined in Proposition* 4.1*,* $f^j_{\rm hom}$ *are defined by* (24)*.*

Note that there is no contribution of the finite connected components of A_j.

The proof of this result will be subdivided into a lower and an upper bound.

Remark 5.3 (non-homogeneous lower-order term)**.** In our hypotheses the lower-order term g depends on the fast variable k, which is integrated out in the limit. We may easily include a measurable dependence on the slow variable εk, by assuming $g = g(x, k, z)$ a Carathéodory function (this covers in particular the case $g = g(x, z)$) and substitute the last sum in (6) by

$$\sum_{k\in Z^\varepsilon(\Omega)} \varepsilon^d g(\varepsilon k, k, u_k).$$

Correspondingly, in Theorem 5.2 the integrand in the last term in (25) must be substituted by $\varphi(x, u^1, \dots, u^N)$, where the definition of this last function is the same but taking $g(x, k, z)$ in place of $g(k, z)$, so that x simply acts as a parameter.

5.1 Proof of the lower bound

Let $u^\varepsilon \to (u^1,\dots,u^N)$ be such that $F_\varepsilon(u^\varepsilon) \le c < +\infty$. Fixed $M \in \mathbb{N}$, we introduce the notation

$$J_M^\varepsilon = \{z \in \mathbb{Z}^d : \ Q_M + zM \subset \frac{1}{\varepsilon}\Omega\},$$
$$R^\varepsilon = \mathcal{N}_0^\varepsilon(\Omega) \setminus \bigcup_{z\in J_M^\varepsilon} \mathcal{N}_0^\varepsilon(Q_M + zM),$$
$$S^\varepsilon = Z^\varepsilon(\Omega) \setminus \bigcup_{z\in J_M^\varepsilon} Z(Q_M + zM),$$

and write

$$F_\varepsilon(u^\varepsilon) = \sum_{j=1}^N I_j^\varepsilon + II^\varepsilon + III^\varepsilon + IV^\varepsilon + V^\varepsilon,$$

where

$$I_j^\varepsilon = \mathcal{F}_\varepsilon^j(u),$$
$$II^\varepsilon = \sum_{j=1}^N \sum_{(k,k')\in\mathcal{N}_j^\varepsilon(\Omega)\setminus(C_j\times C_j)} \varepsilon^{d-1} a_{kk'}(v_k - v_{k'})^2,$$
$$III^\varepsilon = \sum_{z\in J_M^\varepsilon} \varepsilon^d \Bigg(\sum_{(k,k')\in\mathcal{N}_0^\varepsilon(Q_M+zM)} a_{kk'}(v_k - v_{k'})^2 + \sum_{k\in Z(Q_M+zM)} g(k,v_k) \Bigg),$$
$$IV^\varepsilon = \sum_{(k,k')\in R^\varepsilon} \varepsilon^d a_{kk'}(v_k - v_{k'})^2,$$

and

$$V^\varepsilon = \sum_{k\in S^\varepsilon} \varepsilon^d g(k,v_k).$$

Note that

$$II^\varepsilon \ge 0, \qquad IV^\varepsilon \ge -c/M + o(1), \qquad V^\varepsilon \ge -\max|g| \Bigg(\Big| \Omega \setminus \varepsilon^d \bigcup_{z\in J_M^\varepsilon} (Q_M + zM) \Big| + o(1) \Bigg), \quad (26)$$

where we have taken into account that the interactions in IV^ε may be negative. and

$$\liminf_{\varepsilon\to 0} \sum_{j=1}^N I_j^\varepsilon \ge \sum_{j=1}^N \liminf_{\varepsilon\to 0} I_j^\varepsilon \ge \sum_{j=1}^N \int_{\Omega\cap\partial^*\{u^j=1\}} f_{\rm hom}^j(\nu) \mathrm{d}\mathcal{H}^{d-1}. \quad (27)$$

It remains to estimate III^ε. To that end, we introduce the set of indices

$$\Lambda_M^\varepsilon = \{z \in J_M^\varepsilon : \ u^\varepsilon \text{ constant on every connected component of } A_j \cap (Q_{3M} + zM), j = 1,\dots,N\}.$$

Note that

$$\#(J_M^\varepsilon \setminus \Lambda_M^\varepsilon) \le \frac{c_M}{\varepsilon^{d-1}}. \quad (28)$$

We then write

$$III^\varepsilon \quad = \quad \sum_{z\in\Lambda_M^\varepsilon} \varepsilon^d \Bigg(\sum_{(k,k')\in\mathcal{N}_0^\varepsilon(Q_M+zM)} a_{kk'}(v_k - v_{k'})^2 + \sum_{k\in Z(Q_M+zM)} g(k,v_k) \Bigg)$$

$$
\begin{aligned}
&+\sum_{z\in J^\varepsilon_M\setminus\Lambda^\varepsilon_M}\varepsilon^d\Bigg(\sum_{(k,k')\in\mathcal{N}^\varepsilon_0(Q_M+zM)}a_{kk'}(v_k-v_{k'})^2+\sum_{k\in Z(Q_M+zM)}g(k,v_k)\Bigg)\\
\geq\ &\sum_{z\in\Lambda^\varepsilon_M}\varepsilon^dM^d\varphi_M(u^\varepsilon_1,\dots,u^\varepsilon_N)-c\varepsilon^dM^d\max(|g|+|a_{kk'}|)\#(J^\varepsilon_M\setminus\Lambda^\varepsilon_M),
\end{aligned}
$$

where u^ε_j is the constant value taken by u^ε on $(Q_M+zM)\cap C_j$. Here we suppose M large enough so that the connected component of C_j containing $(Q_M+zM)\cap C_j$ is connected in $Q_{3M}+zM$. We set

$$
U^\varepsilon=\sum_{z\in\Lambda^\varepsilon_M}(u^\varepsilon_1,\dots,u^\varepsilon_N)\chi_{Q_M+zM}
$$

and $\varphi_M(0,\dots,0)=0$. Note that $U^\varepsilon\to U:=(u^1,\dots,u^N)$ in $L^1(\Omega)^N$, so that

$$
\liminf_{\varepsilon\to0}III^\varepsilon\geq\liminf_{\varepsilon\to0}\Bigg(\int_\Omega\varphi_M(U^\varepsilon)\,dx-\varepsilon\max|g|c_MM^d\Bigg)=\int_\Omega\varphi_M(U)\,dx \tag{29}
$$

by the Lebesgue dominated convergence theorem and the estimate (28).

Summing up the inequalities (26), (27) and (29), we get

$$
\liminf_{\varepsilon\to0}F_\varepsilon(u^\varepsilon)\geq\sum_{j=1}^N\int_{\Omega\cap\partial^*\{u^j=1\}}f^j_{\rm hom}(\nu)\mathrm{d}\mathcal{H}^{d-1}+\int_\Omega\varphi_M(U)\,dx. \tag{30}
$$

The lower bound inequality then follows by taking the limit as $M\to+\infty$, using Proposition 4.1 and the Lebesgue dominated convergence theorem.

5.2 Proof of the upper bound

We fix $U=(u^1,\dots,u^N)\in BV(\Omega;\{1,-1\})^N$. For every $j=1,\dots,N$ we choose $u^{j,\varepsilon}\to u^j$ a recovery sequence for $\mathcal{F}^j_{\rm hom}(u^j)$. We tacitly extend all functions defined on $Z^\varepsilon(\Omega)$ to the whole $\mathbb{Z}^d$ with the value $+1$ outside $Z^\varepsilon(\Omega)$. This does not affect the value of the energies, but allows to rigorously define some sets of indices z in the sequel.

We fix $M\in\mathbb{N}$ large enough. Similarly as in the previous section, we introduce the sets of indices

$$
\widetilde{J}^\varepsilon_M=\Big\{z\in\mathbb{Z}^d:(Q_M+zM)\cap\frac{1}{\varepsilon}\Omega\neq\emptyset\Big\},
$$

$$
\Lambda^{j,\varepsilon}_M=\{z\in J^\varepsilon_M:\ u^\varepsilon\text{ constant on every connected component of }A_j\cap(Q_{3M}+zM)\},
$$

and remark the estimate

$$
\sum_{j=1}^N\#(\widetilde{J}^\varepsilon_M\setminus\Lambda^{j,\varepsilon}_M)\leq\frac{c_M}{\varepsilon^{d-1}}. \tag{31}
$$

Note that if $z\in\bigcap_{j=1}^N\Lambda^{j,\varepsilon}_M$ then $u^{j,\varepsilon}=:u^{j,\varepsilon,z}$ is constant on $C_j\cap(Q_M+zM)$ for $j=1,\dots,N$. Let $v^{\varepsilon,z}$ be a minimizer for $\widetilde{\varphi}_M(u^{1,\varepsilon,z},\dots,u^{N,\varepsilon,z})$.

We define

$$
u^\varepsilon_k=\begin{cases}u^{j,\varepsilon}_k & \text{if }k\in C_j,\ j=1,\dots,N\\ v^{\varepsilon,z}(k-zM) & \text{if }k\in Q_M+zM\text{ and }z\in\bigcap_{j=1}^N\Lambda^{j,\varepsilon}_M\\ 1 & \text{otherwise.}\end{cases}
$$

We first estimate the energy on the strong connections. By the definition of $u^{j,\varepsilon}$ we have, for all $j = 1, \ldots, N$

$$\lim_{\varepsilon\to 0} \sum_{(k,k')\in\mathcal{N}_j^\varepsilon(\Omega)\cap(C_j\times C_j)} \varepsilon^{d-1} a_{kk'}(u_k^\varepsilon - u_{k'}^\varepsilon)^2 = \mathcal{F}_{\rm hom}^j(u^j), \tag{32}$$

since $u^\varepsilon = u^{j,\varepsilon}$ on C_j. On the strong connections between points not in the infinite connected components C_j we have

$$\sum_{(k,k')\in\mathcal{N}_j^\varepsilon(\Omega)\setminus(C_j\times C_j)} \varepsilon^{d-1} a_{kk'}(u_k^\varepsilon - u_{k'}^\varepsilon)^2 = 0 \tag{33}$$

since u^ε is constant on every connected component of $A_j \setminus C_j$. Note that here we have used the condition that $v^{\varepsilon,z} = 1$ on D_M in the definition of $\widetilde{\varphi}_M$.

We then examine the contribution due to the interaction between weak connections and the term g. We first look at the contributions on the cubes in the sets $\Lambda_M^{j,\varepsilon}$, where we can use the definition of $\widetilde{\varphi}_M$: for every $z \in \bigcap_{j=1}^N \Lambda_M^{j,\varepsilon}$ we have

$$\sum_{(k,k')\in\mathcal{N}_0^\varepsilon(Q_M+zM)} a_{kk'}(u_k^\varepsilon - u_{k'}^\varepsilon)^2 + \sum_{k\in Z(Q_M+zM)} g(k, u_k^\varepsilon) = \widetilde{\varphi}_M(u^{1,\varepsilon,z}, \ldots, u^{N,\varepsilon,z}). \tag{34}$$

The contributions interior to all other cubes in $\widetilde{J}_M^\varepsilon$ sums up to

$$\begin{aligned}
&\sum_{z\notin\bigcap_{j=1}^N \Lambda_M^{j,\varepsilon}} \varepsilon^d \Big(\sum_{(k,k')\in\mathcal{N}_0^\varepsilon(Q_M+zM)} a_{kk'}(u_k^\varepsilon - u_{k'}^\varepsilon)^2 + \sum_{k\in Z(Q_M+zM)} g(k, u_k^\varepsilon) \Big) \\
\leq\ & \varepsilon^d M^d (\# P_0 \max a_{il} + \max |g|) \sum_{j=1}^N \#(J_M^\varepsilon \setminus \Lambda_M^{j,\varepsilon}) \\
\leq\ & \varepsilon M^d c'_M + o(1)
\end{aligned}$$

by (31) and the fact that the boundary of Ω has zero measure. Finally, the contribution due to the weak connection across the boundary of neighbouring cubes is given by

$$\begin{aligned}
&\sum_{z\neq z'\in\bigcap_{j=1}^N \Lambda_M^{j,\varepsilon}} \varepsilon^d \sum_{(k,k')\in\mathcal{N}_0^\varepsilon(\Omega), k\in Q_M+zM, k'\in Q_M+z'M} a_{kk'}(u_k^\varepsilon - u_{k'}^\varepsilon)^2 \\
\leq\ & \varepsilon^d M^{d-1} \# J_M^\varepsilon \# P_0 \max a_{il} \leq \# P_0 \max a_{il} \frac{|\Omega|}{M}.
\end{aligned}$$

From the inequalities above, we obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u^\varepsilon) \leq \sum_{j=1}^N \mathcal{F}_{\rm hom}^j(u^j) + \int_\Omega \widetilde{\varphi}_M(u^1, \ldots, u^N)\, dx + \# P_0 \max a_{il} \frac{|\Omega|}{M}$$

The thesis is then obtained by letting $M \to +\infty$ and using Propositions 4.2 and 4.1. □

6 Examples

In the pictures in the following examples weak connections are denoted by a dashed line, strong connections by a continuous line.

6.1 One-dimensional examples

In this section we consider very easy one-dimensional examples, highlighting the possibility of a double-porosity behaviour if long-range interactions are allowed, contrary to the continuum case. We use a slightly different notation that that followed hitherto, with the sums depending only on one index. The factor $1/4$ is just a normalization factor since $(u_i - u_j)^2$ is always a multiple of 4.

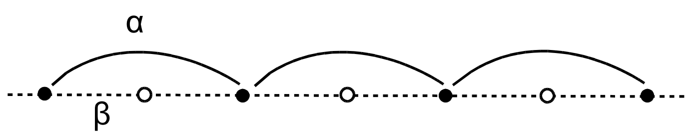

Figure 3: weak inclusions in one dimension

Example 6.1. We consider a system of weak nearest-neighbour interactions and strong next-to-nearest neighbour interactions on the odd lattice (see Fig. 3); namely,

$$F_\varepsilon(u) = \frac{\beta}{4}\sum_{i=1}^{N_\varepsilon} \varepsilon(u_i - u_{i-1})^2 + \frac{\alpha}{4}\sum_{j=1}^{N_\varepsilon/2-1} (u_{2j+1} - u_{2j-1})^2 + \sum_{i=1}^{N_\varepsilon} \varepsilon g(u_i)$$

where we assume that $\Omega = [0,1]$, $N_\varepsilon = 1/\varepsilon \in 2\mathbb{N}$. In this case $N = 1$, $A_1 = C_1 = 1 + 2\mathbb{N}$, $A_0 = 2\mathbb{N}$.

The Γ-limit is

$$\begin{aligned} F_{\rm hom}(u) &= \alpha\,\#S(u) + \frac{1}{2}\int_0^1 g(u)\,dx + \frac{1}{2}\int_0^1 \min\{g(u), g(-u) + 2\beta\}\,dx \\ &= \alpha\,\#S(u) + \int_0^1 g(u)\,dx - \frac{1}{2}\int_0^1 \max\{0, g(u) - g(-u) - 2\beta\}\,dx \end{aligned}$$

The last term favours states with the same value on A_0 and A_1 if the integrand is 0 and of opposite sign if the integrand is positive. Note that this is always the case if we have a strong enough 'antiferromagnetic' nearest-neighbour interaction; i.e., β is negative and $2|\beta| > |g(1) - g(-1)|$.

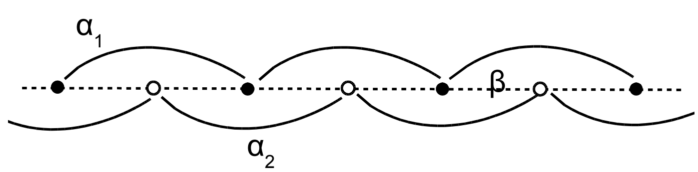

Figure 4: interacting sublattices in one dimension

Example 6.2. We consider a system of weak nearest-neighbour interactions and strong next-to-nearest neighbour interactions (see Fig. 4); namely,

$$F_\varepsilon(u) = \frac{\beta}{4}\sum_{i=1}^{N_\varepsilon} \varepsilon(u_i - u_{i-1})^2 + \frac{\alpha_1}{4}\sum_{j=1}^{N_\varepsilon/2-1} (u_{2j+1} - u_{2j-1})^2 + \frac{\alpha_2}{4}\sum_{j=0}^{N_\varepsilon/2-1} (u_{2j+2} - u_{2j})^2 + \sum_{i=1}^{N_\varepsilon} \varepsilon g(u_i)$$

where we assume that $N_\varepsilon = 1/\varepsilon \in 2\mathbb{N}$. In this case $N = 2$, $A_1 = C_1 = 1 + 2\mathbb{N}$, $A_2 = C_2 = 2\mathbb{N}$, $A_0 = \emptyset$.

The Γ-limit is

$$F_{\rm hom}(u^1, u^2) = \alpha_1 \#S(u^1) + \alpha_2 \#S(u^2) + \frac{1}{2}\int_0^1 g(u^1)\,dx + \frac{1}{2}\int_0^1 g(u^2)\,dx + \frac{\beta}{4}\int_0^1 (u^2 - u^1)^2.$$

Note that, since $A_0 = \emptyset$ we have no optimization in the interacting term, which then is just the pointwise limit of the nearest-neighbour interactions. Note moreover that in the case $\beta = 0$ the interactions are completely decoupled.

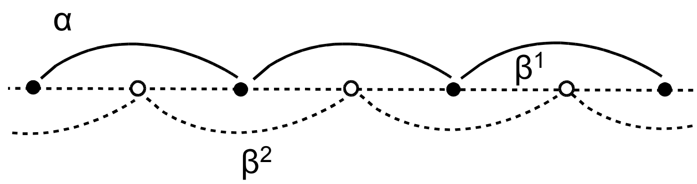

Figure 5: interacting weak and strong sublattices in one dimension

Example 6.3. We consider the same pattern of interactions as in the previous example, but with only strong connections on the odd lattice as in Example 6.1 (see Fig. 5); i.e., with

$$F_\varepsilon(u) = \frac{\beta^1}{4}\sum_{i=1}^{N_\varepsilon}\varepsilon(u_i - u_{i-1})^2 + \frac{\beta^2}{4}\sum_{j=0}^{N_\varepsilon/2-1}\varepsilon(u_{2j+2} - u_{2j})^2 + \frac{\alpha}{4}\sum_{j=1}^{N_\varepsilon/2-1}(u_{2j+1} - u_{2j-1})^2 + \sum_{i=1}^{N_\varepsilon}\varepsilon g(u_i).$$

In this case we have three possibilities:

1) the minimizing values on the even lattice agree with those on the odd lattice (ferromagnetic overall behaviour),

2) the minimizing values on the even lattice disagree with those on the odd lattice (antiferromagnetic overall behaviour),

3) the values on the even lattice alternate (antiferromagnetic behaviour on the weak lattice).

The value of φ is obtained by optimizing on these three possibilities; i.e.,

$$\varphi(u) = \min\Big\{g(u), \frac{g(u) + g(-u)}{2} + \beta^1, \frac{3g(u) + g(-u)}{4} + \frac{\beta^1 + \beta^2}{2}\Big\},$$

and we have

$$F_{\rm hom}(u) = \alpha\,\#S(u) + \int_0^1 \varphi(u)\,dx.$$

Example 6.4. In the system described in Fig. 6 involving strong third-neighbour interactions, we have two strong components, and a Γ-limit obtained by minimization of the nearest and next-to nearest neighbours. Using the same notation of the previous examples for the coefficients as in Fig. 6, we can write the limit as

$$F_{\rm hom}(u^1, u^2) = \alpha_1 \#S(u^1) + \alpha_2 \#S(u^2) + \int_0^1 \varphi(u^1, u^2)\,dx,$$

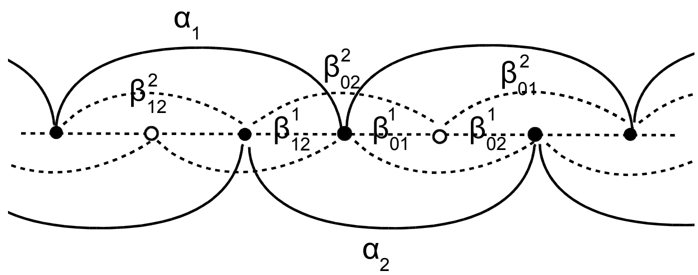

Figure 6: Third-neighbour hard phases

and

$$\begin{aligned}\varphi(u^1,u^2) &= \frac{1}{3}(g(u^1)+g(u^2))+\frac{1}{4}\beta^2_{12}(u^2-u^1)^2\\ &+\frac{1}{3}\min\Bigg\{\frac{1}{4}\Bigg((\beta^1_{01}+\beta^2_{01})(v-u^1)^2+(\beta^1_{02}+\beta^2_{02})(v-u^2)^2\Bigg)+g(v): v\in\{-1,1\}\Bigg\}.\end{aligned}$$

6.2 Higher-dimensional examples

In the following examples we go back to the notation used in the statement of the main result. The normalization factor 1/8 takes into account that each pair of nearest neighbours is accounted for twice.

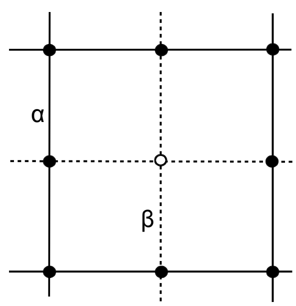

Figure 7: a nearest-neighbour system with soft inclusions

Example 6.5. We consider a nearest-neighbour system in two-dimension in which $A_0 = 2\mathbb{Z}^2$ and strong/weak interactions of the form

$$\frac{1}{8}\alpha(u_k-u_{k'})^2, \qquad \frac{1}{8}\varepsilon\beta(u_k-u_{k'})^2,$$

respectively (see Fig. 7). In this case we have

$$F_{\rm hom}(u)=\frac{1}{2}\alpha\int_{S(u)\cap\Omega}\|\nu_u\|_1 d\mathcal{H}^1+\int_\Omega\varphi(u)\,dx,$$

where

$$\varphi(u) = \min\left\{g(u), \frac{3g(u)+g(-u)}{4} + \beta\right\}.$$

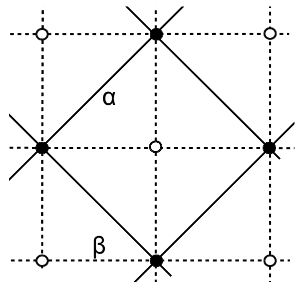

Figure 8: a lattice with weak nearest-neighbour interactions

Example 6.6. In this example we consider strong interactions on a lattice of next-to-nearest neighbours as in Fig. 8, and weak nearest-neighbour interactions on the square lattice, of the form

$$\frac{1}{8}\alpha(u_k - u_{k'})^2, \qquad \frac{1}{8}\varepsilon\beta(u_k - u_{k'})^2$$

respectively (the factor 8 taking into account that each pair is accounted for twice). We only have one strong component, and with this choice of coefficients we have

$$F_{\rm hom}(u) = \alpha\int_{\Omega\cap\partial\{u=1\}} \|\nu\|_\infty d\mathcal{H}^1 + \int_\Omega \varphi(u)\,dx,$$

where

$$\varphi(u) = \min\left\{g(u), \frac{1}{2}(g(u)+g(-u)) + \beta\right\}.$$

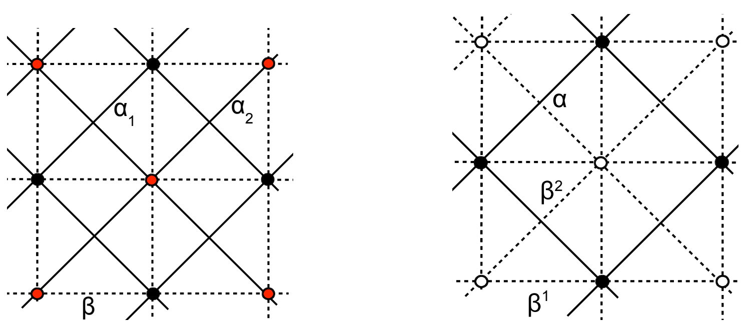

Figure 9: two-dimensional interacting sublattices

Example 6.7. We include just the pictorial description of two more two-dimensional systems with a limit with two parameters (the first one in Fig. 9), and with one limit parameter but with the possibilities of an oscillating behaviour on the weak lattice (the second one in Fig. 9), analogous to the one-dimensional Example 6.2 and Example 6.3, respectively.

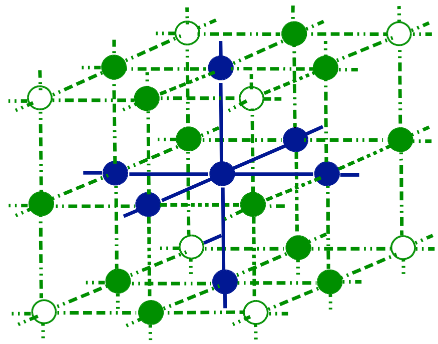

Figure 10: oscillations in the infinite weak component

Example 6.8. We finally consider a three-dimensional two-periodic geometry, with one strong connected component pictured in Fig. 10. Even in the absence of the forcing term g we may have several competing microstructures in the determination of φ. In Fig 10 we have represented the uniform data $u = +1$ on the strong component with solid circles, and a system of ferromagnetic connections between strong and weak sites (positive coefficients) and of anti-ferromagnetic connections between weak sites (a negative coefficient α). Correspondingly, the minimal states have the value $+1$ on weak sites connected with the strong component (represented by solid circles), and the value -1 on the other sites (represented by white circles). Note that in this case the contribution of the weak phase is a constant.

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