HARDY INEQUALITIES FOR p−LAPLACIANS WITH ROBIN BOUNDARY CONDITIONS

TOMAS EKHOLM, HYNEK KOVAŘÍK, AND ARI LAPTEV

ABSTRACT. In this paper we study the best constant in a Hardy inequality for the $p-$Laplace operator on convex domains with Robin boundary conditions. We show, in particular, that the best constant equals $((p-1)/p)^p$ whenever Dirichlet boundary conditions are imposed on a subset of the boundary of non-zero measure. We also discuss some generalizations to non-convex domains.

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1. **Introduction**

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and denote by

$$\delta(x) = \min_{y \in \partial\Omega} |x-y| \tag{1.1}$$

the distance between a given $x \in \Omega$ and the boundary of Ω. The Hardy inequality for the p-Laplace operator with Dirichlet boundary conditions on $\partial\Omega$:

$$\int_\Omega |\nabla u(x)|^p \, dx \ \geq \ K \int_\Omega \frac{|u(x)|^p}{\delta(x)^p} \, dx, \qquad \forall \, u \in W_0^{1,p}(\Omega), \quad p > 1, \tag{1.2}$$

is closely related to the variational problem

$$\mu_p(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u(x)|^p \, dx}{\int_\Omega \, |u(x)/\delta(x)|^p \, dx}. \tag{1.3}$$

Hence $\mu_p(\Omega)$ is the best possible value of the constant K in (1.2). Hardy showed in [H] that inequality (1.2) holds with some $K > 0$ in dimension one. In higher dimensions it is known, see [OK], that if Ω has Lipschitz continuous boundary, then $\mu_p(\Omega) > 0$. In general, $\mu_p(\Omega)$ depends on the domain Ω and satisfies the upper bound

$$\mu_p(\Omega) \leq C_p := \Big(\frac{p-1}{p}\Big)^p,$$

see [MMP]. However, if Ω is convex, then $\mu_p(\Omega) = C_p$. The latter was first proved for $p = n = 2$, see [D3, Sec. 5.3] or [D1, Sec. 1.5], then in [MS1] for $n = 2$ and any $p > 1$, and finally in [MMP] for any n and any $p > 1$. Moreover, it was shown in [MMP] that $\mu_2(\Omega) = C_2$ if and only if the variational problem (1.3) has no minimiser. The fact that for convex domains there is no minimiser of (1.3) opens a possibility to improve inequality

(1.2), even with the sharp constant $K = C_p$, by adding to its right hand side a positive contribution. Such improvements, with various forms of the remainder terms, have been obtained in [A1, A2, AW, BM, FMT, HHL] for $p = 2$ and later in [T] for $p \neq 2$. As for non-convex domains, it is known, due to [A], that in the case $n = p = 2$ for simply connected domains one has $\mu_2(\Omega) \geq 1/16$, see also [LS]. For a throughout discussion of various Hardy inequalities for $p = 2$ we refer to [D2] and references therein.

In [EHR] the authors have obtained a Hardy's inequality with the distance to the part of the boundary where functions satisfy Dirichlet boundary conditions. The main result of [EHR] is not related to our main results as it does not contain estimates of the Hardy constant and does not include the contribution of the boundary where functions do not satisfy the Dirichlet boundary conditions.

In this paper we consider an analogue of the variational problem (1.3) for a Robin Laplacian. This means that we replace the numerator of (1.3) by the functional

$$\mathcal{Q}_p[\sigma, u] = \int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} \sigma \, |u|^p \, d\nu, \qquad u \in \mathcal{F}(\Omega), \tag{1.4}$$

where $d\nu$ denotes the surface measure on $\partial\Omega$, $\sigma : \partial\Omega \to [0, +\infty]$ is a function which defines the boundary conditions and $\mathcal{F}(\Omega)$ is a suitable family of test functions. The function space $\mathcal{F}(\Omega)$ clearly depends on the choice of σ. Notice that with the choice $\sigma = +\infty$, and consequently $\mathcal{F}(\Omega) = W_0^{1,p}(\Omega)$, we arrive at the Dirichlet boundary conditions and hence at problem (1.3).

To pass from Dirichlet boundary conditions to Robin boundary conditions means to take $\sigma \neq +\infty$. In order to make the choice of σ as general as possible we will impose the Dirichlet boundary on a part of the boundary $\Gamma \subseteq \partial\Omega$, which might be empty, and Robin boundary conditions on the remaining part $\partial\Omega \setminus \Gamma$;

$$\sigma \in \Sigma_\Gamma := \big\{ f : \partial\Omega \to [0, +\infty] \, , \, f = +\infty \text{ on } \Gamma, \; 0 < \|f\|_{L^\infty(\partial\Omega \setminus \overline{\Gamma})} < \infty \big\}. \tag{1.5}$$

Consequently, we choose

$$\mathcal{F}(\Omega) = W^{1,p}_{0,\Gamma}(\Omega) := \overline{\big\{ u \in C^1(\overline{\Omega}) \, : \, u|_\Gamma = 0 \big\}}^{\, \|\cdot\|_{W^{1,p}(\Omega)}}.$$

Obviously, the weight function in the denominator of (1.3) has to be modified accordingly, since the test functions from $W^{1,p}_{0,\Gamma}(\Omega)$ do not vanish on the whole $\partial\Omega$.

In order to define our variational problem we need to introduce some notations. Let S be the singular set of Ω, i.e. the set of points in Ω for which there exist at least two points $y_1, y_2 \in \partial\Omega$ where the minimum in (1.1) is achieved. Hence, for $x \in \Omega \setminus S$ let $\pi(x) = y$, where y is the unique point on $\partial\Omega$ satisfying $\delta(x) = |x - y|$. In analogy with the case $p = 2$, see [KL], we then define the function $\alpha : \Omega \setminus S \to [0, +\infty]$ by

$$\alpha(x) = \begin{cases} \frac{p-1}{p} \, \sigma\big(\pi(x)\big)^{\frac{1}{1-p}} & \text{if} \quad \sigma(\pi(x)) > 0, \\ & \\ +\infty & \text{otherwise} \; . \end{cases} \tag{1.6}$$

We now pass from the weight function $\delta(x)^{-p}$ in (1.3) to the weight function

$$(\delta(x)+\alpha(x))^{-p},$$

which takes into account the boundary conditions defined in term of σ. For example, if $\sigma = +\infty$, then $\alpha = 0$ as expected. Note also that the function is defined almost everywhere in Ω since the set S has Lebesgue measure zero, see [LN]. Hence we are led to the variational problem

$$\lambda_p(\Omega,\sigma) := \inf_{u\in W^{1,p}_{0,\Gamma}(\Omega)} \frac{\mathcal{Q}_p[\sigma,u]}{\|u\|^p_{p,\sigma}}\,, \tag{1.7}$$

with

$$\|u\|_{p,\sigma} = \left(\int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha(x))^p}\,dx\right)^{\frac{1}{p}}, \qquad u \in W^{1,p}_{0,\Gamma}(\Omega). \tag{1.8}$$

Remark 1.1. Note that the integral weight on the right hand side of (1.8) is not identically zero in view of the definition of Σ_Γ, see (1.5). Hence the variational problem (1.7) is well-posed.

We are going to establish a relation between $\lambda_p(\Omega,\sigma)$ on one hand, and the function σ and geometry of Ω on the other hand. The main results of this paper are the following:

2. Main results

Theorem 2.1. *Let $\Omega\subset\mathbb{R}^n$ be open bounded and convex with $\partial\Omega$ of class C^2. Let $\Gamma\subseteq\partial\Omega$. Then for any $\sigma\in\Sigma_\Gamma$ and $u\in W^{1,p}_{0,\Gamma}(\Omega)$ we have*

$$\int_\Omega |\nabla u|^p\,dx + \int_{\partial\Omega} \sigma\,|u|^p\,d\nu \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha(x))^p}\,dx. \tag{2.1}$$

Moreover,

$$\lambda_p(\Omega,\sigma) = \Big(\frac{p-1}{p}\Big)^p \quad \Leftrightarrow \quad \Gamma\neq\emptyset. \tag{2.2}$$

Remark 2.2. Note that Theorem 2.1 includes also the extreme cases $\Gamma=\emptyset$ and $\Gamma=\partial\Omega$. The first part of the statement, i.e. the inequality $\lambda_p(\Omega,\sigma)\geq C_p$ is proven in Proposition 3.4 which provides a generalization of the Hardy inequality obtained in [KL] for $p=2$.

The second part of the claim is a consequence of Proposition 4.2. Equivalence (2.2) is closely related to the question of the existence of a minimiser for the variational problem (1.7), see Proposition 4.1.

Remark 2.3. Let us comment on the sharpness of the lower bound $\lambda_p(\Omega,\sigma)\geq C_p$. The bound is sharp in the sense that the constant C_p cannot replaced by a bigger one and remain independent of σ, see section 4.1 for details. However, if $\Gamma=\emptyset$, then for a given $\sigma\in\Sigma_\Gamma$ Theorem 2.1 implies that $\lambda_p(\Omega,\sigma)>C_p$. The following Theorem quantifies the gap between $\lambda_p(\Omega,\sigma)$ and C_p in terms of the $\|\sigma\|_{L^\infty(\partial\Omega)}$.

Theorem 2.4. *Let Ω be as in Theorem 2.1. If $\Gamma = \emptyset$, then for any $\sigma \in \Sigma_\Gamma$ it holds*

$$\lambda_p(\Omega, \sigma) \; \geq \; C_p \, \left(1 + (p-1)^{p+1} \Big(p - 1 + p\, R_{in} \, \|\sigma\|_{L^\infty(\partial\Omega)}^{\frac{1}{p-1}}\Big)^{-p}\right) \tag{2.3}$$

where

$$R_{in} = \sup_{x\in\Omega} \delta(x)$$

is the in-radius of Ω.

2.1. **Outline of the paper.** We start by the proof of an L^p version of the Hardy inequality for Robin Laplacians, see section 3. Then we provide the proofs of our main results; this is done in section 4. In section 5 we study the behavior of the minimising sequences of the variational problem (1.7) in the case when $\lambda_p(\Omega, \sigma) = C_p$, which corresponds to $\Gamma \neq \emptyset$. In particular, we show that minimising sequences, under certain conditions, concentrate on Γ. Finally, section 6 is dedicated to the analysis of a hardy-type inequality on a particular non-convex domain, namely on a complement of a ball.

3. **A Hardy inequality**

Similarly as in the case $p = 2$, see [KL], we first establish an appropriate one-dimensional estimate.

Lemma 3.1. *Let $b > 0$ and assume that u belongs to $AC[0, b]$, the space of absolutely continuous functions on $[0, b]$. Then for any $\sigma \geq 0$ we have*

$$\int_0^b |u'(t)|^p \, dt + \sigma \, |u(0)|^p \; \geq \; C_p \int_0^b \frac{|u(t)|^p}{(t+\alpha)^p} \, dt + \frac{(p-1)\, C_p}{(b+\alpha)^p} \int_0^b |u(t)|^p \, dt, \tag{3.1}$$

where

$$\alpha = \frac{p-1}{p} \, \sigma^{\frac{1}{1-p}}. \tag{3.2}$$

Proof. It suffices to prove the inequality for $u > 0$. We may assume that $\sigma > 0$. Let

$$f(t) = -(p-1)^{p-1} \, (t+\alpha)^{1-p}$$

and define

$$A := \Big| \int_0^b f'(t) \, u^p \, dt - (f(b) - f(0)) \, u(0)^p \Big|, \qquad B := \int_0^b |f(b) - f(t)|^{\frac{p}{p-1}} \, u^p(t) \, dt.$$

Integration by parts and Hölder inequality show that

$$A^p \leq p^p \Big(\int_0^b |f(b) - f(t)| \, u^{p-1} |u'| \, dt \Big)^p \leq p^p \, B^{p-1} \int_0^b |u'|^p \, dt \tag{3.3}$$

On the other hand, the Young inequality gives

$$A^p \geq p \, A \, B^{p-1} - (p-1) \, B^p. \tag{3.4}$$

Using the fact that f is negative increasing and that

$$(1-s)^{\frac{p}{p-1}} \leq 1 - s \leq 1 - s^{\frac{p}{p-1}} \qquad \forall \, s \in [0,1]$$

we obtain

$$B = \int_0^b |f(t)|^{\frac{p}{p-1}} \left(1 - \frac{|f(b)|}{|f(t)|}\right)^{\frac{p}{p-1}} u^p(t)\, dt \le \int_0^b \left(|f(t)|^{\frac{p}{p-1}} - |f(b)|^{\frac{p}{p-1}}\right) u^p(t)\, dt.$$

Moreover, since $u > 0$, from the definition of A we get

$$A \ge \int_0^b f'(t)\, u^p\, dt + f(0)\, u(0)^p.$$

The above inequalities in combination with (3.4) and (3.3) then imply that

$$p^p \int_0^b |u'(t)|^p\, dt - pf(0)\, u(0)^p \ge (p-1)^p \int_0^b \frac{u(t)^p}{(t+\alpha)^p}\, dt + (p-1)^{p+1} \int_0^b \frac{u(t)^p}{(b+\alpha)^p}\, dt.$$

This implies (3.1). $\square$

If $\Gamma = \emptyset$, then $\sigma \in L^\infty(\partial\Omega)$ and it is easily seen that $W^{1,p}_{0,\Gamma}(\Omega) = W^{1,p}(\Omega)$. Mimicking the approach of [KL] we deduce from Lemma 3.1 the following version of the Hardy inequality for Robin Laplacians on $W^{1,p}(\Omega)$.

Proposition 3.2. *Let Ω satisfy the hypothesis of Theorem 2.1. Then for any $\sigma \in L^\infty(\partial\Omega)$ and all $u \in W^{1,p}(\Omega)$ it holds*

$$\mathcal{Q}_p[\sigma, u] \ge C_p \int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha(x))^p}\, dx + (p-1)\, C_p \int_\Omega \frac{|u(x)|^p}{(R_{in}+\alpha(x))^p}\, dx. \tag{3.5}$$

Proof. As in [KL] we first prove inequality (3.5) for $u \in C^1(\overline{\Omega})$ and σ continuous. By Tietze extension theorem then there exists a continuous function $\zeta : \mathbb{R}^n \to \mathbb{R}$ such that

$$\zeta\big|_{\partial\Omega} = \sigma. \tag{3.6}$$

Now let $Q \subset \Omega$ be an open convex polytop with N sides Γ_j, $1 \le j \le N$. Let n_j be the inner normal vector of the side Γ_j.

Let $\delta(x;Q)$ be the distance from $x \in Q$ to the boundary ∂Q and let

$$P_j = \big\{x \in Q \,:\, \exists\, y \in \Gamma_j,\ \delta(x;Q) = |x-y|\big\}.$$

For each $x \in P_j$ there is a unique $y \in \Gamma_j$ and $t \in [0, t_y]$ for which

$$x = y + t\, n_j, \tag{3.7}$$

where t_y is chosen in such a way that $y + t_y\, n_j \in \partial P_j$. Moreover, we have

$$R_{in}(Q) := \sup_{x\in Q} \delta(x;Q) = \max_{1\le j\le N} \sup_{x\in P_j} \delta(x;Q). \tag{3.8}$$

Using Lemma 3.1 and (3.7) we get for each $y \in \Gamma_j$ the lower bound

$$\begin{aligned} \int_0^{t_y} |u'_{n_j}(x)|^p dt + \zeta(y)|u(y)|^p \ge\ & C_p \int_0^{t_y} \frac{|u(x)|^p}{(t+\alpha(x;Q))^p} dt \\ & + (p-1)C_p \int_0^{t_y} \frac{|u(x)|^p}{(t_y+\alpha(x;Q))^p} dt, \end{aligned} \tag{3.9}$$

where

$$\alpha(x;Q) = \frac{p-1}{p}\,\zeta\big(\pi\big(x;Q\big)\big)^{\frac{1}{1-p}} \tag{3.10}$$

and for x in the interior of some P_j we define $\pi(x;Q) = y \in \Gamma_j$, such that $\delta(x;Q) = |x-y|$. Note that $\pi(\cdot;Q)$ is densely defined in Q.

By integrating (3.9) over the boundary Γ_j and then summing the resulting inequality over $j = 1,..,N$ we arrive at

$$\begin{aligned}\int_Q |\nabla u|^p\,dx + \int_{\partial Q} \zeta(y)|u(y)|^p\,d\nu(y) \geq\ & C_p \int_Q \frac{|u(x)|^p}{(\delta(x;Q)+\alpha(x;Q))^p}dx \\ & + (p-1)\,C_p \int_Q \frac{|u(x)|^p}{(R_{in}(Q)+\alpha(x;Q))^p}dx.\end{aligned} \tag{3.11}$$

From the convexity of Ω it follows that there exits a sequence of convex polytops $Q_m \subset \Omega$, $m \in \mathbb{N}$, which approximates Ω. More precisely, for every ε there exists an m_ε such that the Hausdorf distance between Ω and Q_{m_ε} satisfies $d_H(\Omega, Q_{m_\varepsilon}) < \varepsilon$. Similarly as in [KL] we then conclude, using the continuity of ζ in combination with (3.6), that

$$\zeta\big(\pi\big(x;Q_m\big)\big) \;\to\; \sigma(\pi(x)) \quad m\to\infty, \qquad \text{a. e.} \quad x\in\Omega.$$

Hence by the continuity of u

$$\int_{\partial Q_m} \zeta(y)\,|u(y)|^p\,dy \;\to\; \int_{\partial\Omega} \sigma(y)\,|u(y)|^p\,d\nu(y)$$

as $m \to \infty$. The last two equations together with (3.10), dominated convergence theorem and the fact that $R_{in}(Q_m) \leq R_{in}$ for every m imply that

$$\mathcal{Q}_p[\sigma,u] \geq C_p \int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha(x))^p}dx + (p-1)\,C_p \int_\Omega \frac{|u(x)|^p}{(R_{in}+\alpha(x))^p}dx, \quad u\in C^1(\overline{\Omega}) \tag{3.12}$$

holds for all σ continuous.

Now if $\sigma \in L^\infty(\partial\Omega)$, then in view of the regularity of $\partial\Omega$ there exists a sequence of continuous functions σ_k on $\partial\Omega$ which converges to σ in $L^\infty(\partial\Omega)$ as $k\to\infty$. From inequality (3.12) it follows that (3.5) holds for all σ_k. Since $u|_{\partial\Omega} \in L^p(\partial\Omega, d\nu)$ for any $u \in C^1(\overline{\Omega})$, using the dominated convergence we obtain (3.5) for any $\sigma \in L^\infty(\partial\Omega)$ and all $u\in C^1(\overline{\Omega})$.

Finally, let $u \in W^{1,p}(\Omega)$. By density there exists a sequence $u_j \in C^1(\overline{\Omega})$ such that $u_j \to u$ in $W^{1,p}(\Omega)$ as $j\to\infty$. In view of the regularity of Ω it follows that $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ with compact imbedding, see [Ad, Sect.7.5]. Hence, after applying inequality (3.12) to u_j and letting $j\to\infty$ we conclude that (3.5) holds for all $u\in W^{1,p}(\Omega)$. □

Remark 3.3. In the situation when σ is constant, a simpler proof of (3.5), without the second term on the right hand side, can be given, see [K, Lem. 4.4] for the case $p=2$ and [DPG, Lem. 3.1] for the case $p>1$.

As an immediate consequence of the above Proposition we obtain

Proposition 3.4. *Let Ω satisfy the hypothesis of Theorem 2.1. Then for any $\sigma \in \Sigma_\Gamma$ and all $u \in W^{1,p}_{0,\Gamma}(\Omega)$ it holds*

$$\mathcal{Q}_p[\sigma, u] \geq \ C_p \int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha(x))^p}\, dx + (p-1)\, C_p \int_\Omega \frac{|u(x)|^p}{(R_{in}+\alpha(x))^p}\, dx. \tag{3.13}$$

Proof. Let $u \in W^{1,p}_{0,\Gamma}(\Omega)$ and define the sequence $\{\sigma_n\}_{n\in\mathbb{N}} \subset L^\infty(\partial\Omega)$ by

$$\sigma_n(y) = \begin{cases} \sigma(y) & \text{if } \ y \in \ \partial\Omega \setminus \Gamma, \\ n & \text{if } \ y \in \Gamma. \end{cases}$$

Proposition 3.2 now implies

$$\mathcal{Q}_p[\sigma_n, u] \geq C_p \int_\Omega \frac{|u(x)|^p}{(\delta(x)+\alpha_n(x))^p}\, dx + (p-1)\, C_p \int_\Omega \frac{|u(x)|^p}{(R_{in}+\alpha_n(x))^p}\, dx, \tag{3.14}$$

where

$$\alpha_n(x) = \frac{p-1}{p}\, \sigma_n\big(\pi(x)\big)^{\frac{1}{1-p}}, \qquad x \in \Omega.$$

Since $\mathcal{Q}_p[\sigma, u] = \mathcal{Q}_p[\sigma_n, u]$ and

$$\alpha_n(x) \geq \alpha_{n+1}(x) \qquad \forall\, n \in \mathbb{N}, \quad x \in \Omega,$$

the statement follows from (3.14) by monotone convergence. $\square$

The following corollary of Proposition 3.4 provides yet another improvement of the Hardy inequality (1.2) with the sharp constant $K = C_p$.

Corollary 3.5. *For any $u \in W^{1,p}_0(\Omega)$ it holds*

$$\int_\Omega |\nabla u(x)|^p\, dx \ \geq \ C_p \int_\Omega \frac{|u(x)|^p}{\delta(x)^p}\, dx + \frac{(p-1)\, C_p}{R^p_{in}} \int_\Omega |u(x)|^p\, dx. \tag{3.15}$$

Proof. It suffices to apply Proposition 3.4 with $\Gamma = \partial\Omega$. $\square$

4. **Proofs of the main results**

We start with the following Proposition which provides sufficient conditions for the existence of a minimizer of the variational problem (1.7).

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ of class C^2. Assume that $\sigma \in L^\infty(\partial\Omega)$ and hence $\Gamma = \emptyset$. Then* (1.7) *admits a minimiser. In other words, there exists $\psi \in W^{1,p}(\Omega),\ \psi \neq 0$, such that*

$$\lambda_p(\Omega, \sigma) = \frac{\mathcal{Q}_p[\sigma, \psi]}{\|\psi\|^p_{p,\sigma}}\ . \tag{4.1}$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimising sequence for $\lambda(\Omega,\sigma)$. Assume that

$$\|u_j\|_{p,\sigma}^p = \int_\Omega (\delta(x)+\alpha(x))^{-p}\, |u_j(x)|^p\, dx = 1 \quad \forall\, j \in \mathbb{N}. \tag{4.2}$$

Since $\{u_j\}$ is bounded in $W^{1,p}(\Omega)$, there exists a subsequence, which we still denote by u_j and a function $\psi \in W^{1,p}(\Omega)$ such that $u_j \to \psi$ weakly in $W^{1,p}(\Omega)$. In view of the regularity of Ω and the compactness of the imbedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ we may suppose (by passing to a subsequence if necessary) that u_j converges strongly to ψ in $L^p(\Omega)$. Moreover, since $W^{1,p}(\Omega)$ is compactly imbedded also in $L^p(\partial\Omega)$, see e.g. [Ad, Thm.5.22], it follows that we can find a subsequence $\{v_j\} \subset \{u_j\}$ such that $v_j|_{\partial\Omega} \to \psi|_{\partial\Omega}$ almost everywhere on $\partial\Omega$. By the weak lower semicontinuity of $\int_\Omega |\nabla u|^p$ and the Fatou Lemma we thus obtain

$$\liminf_{j\to\infty} \mathcal{Q}_p[\sigma, v_j] \geq \mathcal{Q}_p[\sigma,\psi]. \tag{4.3}$$

On the other hand,

$$\|(\delta+\alpha)^{-p}\|_{L^\infty(\Omega)} = \left(\frac{p}{p-1}\right)^p \|\sigma\|_{L^\infty(\partial\Omega)}^{\frac{p}{p-1}} \, < \, \infty,$$

see (1.6). The strong convergence of $v_j \to \psi$ in $L^p(\Omega)$ thus implies that

$$\|\psi\|_{p,\sigma}^p = \int_\Omega (\delta(x)+\alpha(x))^{-p}\, |\psi(x)|^p\, dx = 1.$$

Hence $\psi \neq 0$ and in view of (4.3) we have

$$\mathcal{Q}_p[\sigma,\psi] \geq \lambda(\Omega,\sigma) = \liminf_{j\to\infty} \mathcal{Q}_p[\sigma, v_j] \geq \mathcal{Q}_p[\sigma,\psi].$$

This implies (4.1). □

Proof of Theorem 2.4. Let ψ be a minimiser for $\lambda(\Omega,\sigma)$ and denote

$$\Omega_0 = \{x \in \Omega \,:\, \alpha(x) < \infty\}\,.$$

Note that Ω_0 is not empty in view of (1.5) and (1.6). Moreover,

$$\begin{aligned}\int_\Omega \frac{|\psi(x)|^p\, dx}{(R_{in}+\alpha(x))^p} &= \int_{\Omega_0} \frac{|\psi(x)|^p\, dx}{(R_{in}+\alpha(x))^p} \,\geq\, \inf_{x\in\Omega_0}\left(\frac{\delta(x)+\alpha(x)}{R_{in}+\alpha(x)}\right)^p \int_{\Omega_0} \frac{|\psi(x)|^p\, dx}{(\delta(x)+\alpha(x))^p} \\ &= \inf_{x\in\Omega_0}\left(\frac{\delta(x)+\alpha(x)}{R_{in}+\alpha(x)}\right)^p \|\psi\|_{p,\sigma}^p\,. \end{aligned} \tag{4.4}$$

By inserting the above lower bound into (3.13) we obtain

$$\begin{aligned}\lambda_p(\Omega,\sigma) = \frac{\mathcal{Q}_p[\sigma,\psi]}{\|\psi\|_{p,\sigma}^p} &\geq C_p\, \left(1+(p-1)\, \inf_{x\in\Omega_0}\left(\frac{\delta(x)+\alpha(x)}{R_{in}+\alpha(x)}\right)^p\right) \\ &\geq C_p\, \left(1+(p-1)\, \left(\frac{\inf_{x\in\Omega_0}\alpha(x)}{R_{in}+\inf_{x\in\Omega_0}\alpha(x)}\right)^p\right) \\ &= C_p\, \left(1+(p-1)^{p+1}\Big(p-1+p\, R_{in}\, \|\sigma\|_{L^\infty(\partial\Omega)}^{\frac{1}{p-1}}\Big)^{-p}\right),\end{aligned}$$

where, in the last step, we have used (1.6). □

In order to give a proof of Theorem 2.1 we need the following

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ of class C^2. If $\Gamma \neq \emptyset$, then $\lambda_p(\Omega, \sigma) \leq C_p$.*

Proof. By assumption there exists $y_0 \in \Gamma$ and an $r > 0$ such that $\alpha(x) = 0$ on $B(y_0, r) \cap \Omega$. Let $\varepsilon > 0$ and introduce the following continuous functions

$$f_\varepsilon(x) = \begin{cases} \varepsilon & \text{if } |x - y_0| \leq r, \\ \text{linear in } |x| & \text{if } r \leq |x - y_0| \leq r + \varepsilon, \\ \frac{1}{p} & \text{if } r + \varepsilon \leq |x - y_0|, \end{cases}$$

and

$$u_\varepsilon(x) = \begin{cases} \delta(x)^{f_\varepsilon(x)+1-\frac{1}{p}} & \text{if } 0 \leq \delta(x) \leq \varepsilon, \\ \text{linear in } \delta(x) & \text{if } \varepsilon \leq \delta(x) \leq 2\varepsilon, \\ 0 & \text{if } 2\varepsilon \leq \delta(x). \end{cases} \tag{4.5}$$

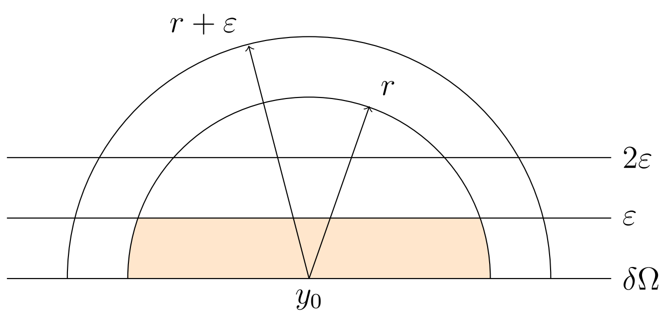

FIGURE 1

To proceed we introduce the following notation:

$$\Omega_\varepsilon := \{x \in \Omega \, : \, \delta(x) \leq \varepsilon\}, \quad E(\varepsilon, r) := B(y_0, r) \cap \Omega_\varepsilon, \quad D(y_0, r) := B(y_0, r) \cap \partial\Omega.$$

Notice that $E(\varepsilon, r)$ is the set in Figure 1 marked in grey. By [Se, Sec. I.3] there exists a set of coordinates $(\delta, \omega) \in \mathbb{R}^n$ such that the transformation $x \to (\delta(x), \omega(x))$ is C^1 on Ω_ε for ε sufficiently small. Moreover, the Jacobian $J(\delta, \omega)$ of this transformation satisfies

$$\lim_{\delta \to 0} J(\delta, \omega) = 1. \tag{4.6}$$

From (4.5) and (4.6) we obtain

$$\begin{aligned} \varepsilon \int_{E(\varepsilon,r)} |\nabla u_\varepsilon(x)|^p \, dx &= \left(\varepsilon + 1 - \frac{1}{p}\right)^p \varepsilon \int_{E(\varepsilon,r)} \delta(x)^{p\varepsilon - 1} \, dx \\ &= \left(\varepsilon + 1 - \frac{1}{p}\right)^p \varepsilon \int_{D(y_0,r)} \int_0^\varepsilon \delta^{p\varepsilon - 1} \, J(\delta, \omega) \, d\delta \, d\omega \qquad (4.7) \\ &= \frac{C_p}{p} \, \nu(D(y_0, r)) \, (1 + o_\varepsilon(1)), \end{aligned}$$

where $o_\varepsilon(1)$ denotes a quantity which tends to zero as $\varepsilon \to 0$. Similarly we find for $\varepsilon \to 0$

$$\varepsilon \int_{E(\varepsilon,r)} \frac{|u_\varepsilon(x)|^p}{(\delta(x)+\alpha(x))^p}\, dx = \varepsilon \int_{E(\varepsilon,r)} \frac{|u_\varepsilon(x)|^p}{\delta(x)^p}\, dx = \varepsilon \int_{E(\varepsilon,r)} \delta(x)^{p\varepsilon-1}\, dx$$
$$= \frac{1}{p}\, \nu(D(y_0,r))\,(1+o_\varepsilon(1)), \tag{4.8}$$

On the other hand, using the fact that $|\nabla f_\varepsilon| \leq C/\varepsilon$ for some $C>0$ in combination with (4.6) it is straightforward to verify that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Omega\setminus E(\varepsilon,r)} |\nabla u_\varepsilon(x)|^p\, dx \;=\; \lim_{\varepsilon\to 0} \varepsilon \int_{\Omega\setminus E(\varepsilon,r)} \frac{|u_\varepsilon(x)|^p}{(\delta(x)+\alpha(x))^p}\, dx = 0.$$

Hence by collecting the above results we arrive at

$$\lambda_p(\Omega,\sigma) \leq \lim_{\varepsilon\to 0} \frac{\mathcal{Q}_p[\sigma,u_\varepsilon]}{\|u_\varepsilon\|^p_{p,\sigma}} = C_p,$$

and the claim follows. $\square$

Proof of Theorem 2.1. The inequality $\lambda_p(\Omega,\sigma) \geq C_p$ follows from Proposition 3.4. The equivalence (2.2) follows from Theorem 2.4 and Proposition 4.2. $\square$

4.1. **The case of constant σ.** Here we provide a more detailed information about the quantity $\lambda_p(\Omega,\sigma)$ in the case when σ is a positive constant.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be convex and bounded. Then*

$$\lim_{\sigma\to 0+} \lambda_p(\Omega,\sigma) = +\infty, \tag{4.9}$$
$$\lim_{\sigma\to\infty} \lambda_p(\Omega,\sigma) = C_p. \tag{4.10}$$
$$\inf_{\Omega \text{ convex}} \lambda_p(\Omega,\sigma) = C_p. \tag{4.11}$$

Proof. To prove (4.9) we first note that that there exists a constant c, depending only on R_{in}, such that for all $\sigma \leq 1$ and all $x\in\Omega$ we have $(\delta(x)+\alpha)^p \geq c\,\sigma^{\frac{p}{1-p}}$. Hence

$$\lambda_p(\Omega,\sigma) \geq c\,\sigma^{\frac{p}{1-p}} \inf_{u\in W^{1,p}(\Omega)} \frac{\mathcal{Q}_p[\sigma,u]}{\|u\|^p_{L^p(\Omega)}} \;\geq\; c\,\sigma^{\frac{1}{1-p}} \inf_{u\in W^{1,p}(\Omega)} \frac{\mathcal{Q}_p[1,u]}{\|u\|^p_{L^p(\Omega)}} \;\geq\; \tilde{c}\,\sigma^{\frac{1}{1-p}}.$$

holds for all $\sigma \leq 1$. This proves (4.9). To prove (4.10) let $u_j \in W^{1,p}_0(\Omega)$ be a minimising sequence for the variational problem (1.3). Since $\alpha \to 0$ as $\sigma\to\infty$, the monotone convergence shows that

$$\limsup_{\sigma\to\infty} \lambda_p(\Omega,\sigma) \leq \limsup_{\sigma\to\infty} \frac{\mathcal{Q}_p[\sigma,u_j]}{\int_\Omega (\delta(x)+\alpha)^{-p}\, |u_j(x)|^p\, dx} = \frac{\int_\Omega |\nabla u_j(x)|^p\, dx}{\int_\Omega\; |u_j(x)/\delta(x)|^p\, dx}$$

holds for all j. By letting $j\to\infty$ we get

$$\limsup_{\sigma\to\infty} \lambda_p(\Omega,\sigma) \leq C_p.$$

This in combination with (2.3) implies (4.10).

Finally, to prove (4.11) we consider the example $\Omega = B_R$, i.e. the the ball centered at origin with radius R. Let

$$u_R(x) = (R + \alpha - |x|)^{(p-1)/p}.$$

Then

$$\lambda_p(B_R, \sigma) \le C_p + \frac{\sigma \alpha^{p-1} R^{n-1}}{\int_0^R r^{n-1}(R+\alpha-r)^{-1} dr}.$$

Since

$$\lim_{R\to\infty} \frac{\sigma \alpha^{p-1} R^{n-1}}{\int_0^R r^{n-1}(R+\alpha-r)^{-1} dr} = 0,$$

this shows that

$$\inf_{\Omega \text{ convex}} \lambda_p(\Omega, \sigma) \le C_p.$$

The opposite inequality follows from Theorem 2.4. $\square$

5. **Concentration effect**

In this section we are going to study the properties of the minimizing sequences of the problem (1.7) in the case $\Gamma \neq \emptyset$. Consider first the (normalized) minimizing sequence constructed in the proof of Proposition 4.2. More precisely, let

$$v_n = n^{-\frac{1}{p}}\, u_{1/n}, \qquad n \in \mathbb{N},$$

where u_ε is given by (4.5). In view of (4.7) and (4.8) it is straightforward to verify that

$$v_n \xrightarrow{w} 0 \quad \text{in } W^{1,p}_{0,\Gamma}(\Omega), \quad \text{and} \quad \liminf_{n\to\infty} \|v_n\|_{p,\sigma} > 0. \tag{5.1}$$

Moreover, we observe that v_n concentrates at Γ. Indeed, we have

$$\nabla v_n \to 0 \quad \text{in} \quad L^p_{loc}(\Omega). \tag{5.2}$$

Below we are going to show that *any* minimizing sequence satisfying (5.1) concentrates at Γ in the sense of (5.2).

Theorem 5.1. *Let v_n be a minimizing sequence for the problem* (1.7). *Assume that v_n satisfies* (5.1). *Then*

$$\int_M |\nabla v_n|^p \to 0 \tag{5.3}$$

for any compact set $M \subset \overline{\Omega} \setminus \overline{\Gamma}$.

Proof. Let n_y denote the inner normal vector to $\partial\Omega$ at a point $y \in \partial\Omega$. For a given $\varepsilon > 0$ we define

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \exists\, t \in [0, \varepsilon],\ \exists\, y \in \overline{\Gamma} \,:\, x = y + t\, n_y\}. \tag{5.4}$$

From the regularity assumptions on $\partial\Omega$ it follows that Ω_ε is not self-intersecting for ε small enough.

Suppose now that (5.3) is false. Then there exists a compact set $K \subset \overline{\Omega} \setminus \overline{\Gamma}$ and a number γ such that

$$\liminf_{n\to\infty} \int_K |\nabla v_n|^p \, \geq \, \gamma. \tag{5.5}$$

Let us now take ε small enough such that $K \subset \Omega'_\varepsilon := \overline{\Omega} \setminus \Omega_\varepsilon$. This is possible due to the assumption on K. From the boundedness of v_n in $W^{1,p}(\Omega)$ and from the Hardy inequality we infer that

$$\sup_n \|v_n\|_{p,\sigma} < \infty. \tag{5.6}$$

By the Rellich-Kondrashov theorem and the first part of (5.1)

$$v_n \to 0 \quad \text{in} \;\; L^p_{loc}(\Omega).$$

Moreover, $(\delta+\alpha)^{-1} \in L^\infty(\Omega'_\varepsilon)$. Hence in view of (5.6) we have

$$a_n := \int_{\Omega'_\varepsilon} \Big| \frac{v_n}{\delta+\alpha} \Big|^p \to 0 \tag{5.7}$$

We thus obtain the following lower bound:

$$\frac{\mathcal{Q}_p[\sigma, v_n]}{\|v_n\|^p_{p,\sigma}} \, \geq \, \frac{\int_{\Omega_\varepsilon} |\nabla v_n|^p + \gamma}{\int_{\Omega_\varepsilon} \big|\frac{v_n}{\delta+\alpha}\big|^p + a_n}.$$

Following [MMP] we now pass to the coordinates (δ, ω) in Ω_ε. Using the one-dimensional Hardy inequality and (4.6) we find that

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla v_n|^p &\geq \int_\Gamma \int_0^\varepsilon |\partial_\delta \, v_n|^p \, J(\delta, \omega) \, d\delta \, d\omega \\
&\geq (1+o(1)) \, C_p \int_\Gamma \int_0^\varepsilon |v_n/\delta|^p \, J(\delta,\omega) \, d\delta \, d\omega \\
&= (1+o(1)) \, C_p \int_{\Omega_\varepsilon} |v_n/\delta|^p,
\end{aligned}$$

where $o(1)$ denotes a quantity which tends to zero as $\varepsilon \to 0$. Hence for ε small enough we have

$$\liminf_{n\to\infty} \frac{\mathcal{Q}_p[\sigma, v_n]}{\|v_n\|^p_{p,\sigma}} \, \geq \, (1+o(1)) \, C_p + \frac{\gamma}{\sup_n \|v_n\|^p_{p,\sigma}} \, > \, C_p \, ,$$

see (5.6) and (5.7). This is in contradiction with the fact that v_n is a minimizing sequence. $\square$

Remark 5.2. The concentration effect in the case $\Gamma = \partial\Omega$ was proved in [MMP].

6. **Hardy inequality on a complement of a ball**

In this section we are going to discuss the validity of a Hardy-type inequality for the functional (1.4) on a particular non-convex domain, namely on a complement of a ball in $\mathbb{R}^n$. Let us denote by B^c_R the complement in $\mathbb{R}^n$ of the ball of radius R centered in the origin. The following result is certainly not new, but we prefer to give its proof for the sake of completeness.

Proposition 6.1. *Assume that $n > p$. Then the inequality*

$$\int_{B_R^c} |\nabla u|^p \;\geq\; \left(\frac{n-p}{p}\right)^p \int_{B_R^c} \frac{|u|^p}{|x|^p} \tag{6.1}$$

holds true for all $u \in W^{1,p}(B_R^c)$ and any $R > 0$.

Proof. By density and by the inequality $|\nabla u(x)| \geq |\nabla |u(x)||$, which holds for almost every $x \in B_R^c$, see e.g. [LL, Thm. 6.17], it suffices to prove the inequality for all positive functions $u \in C^\infty(B_R^c)$ supported in a compact set containing B_R. Moreover, in view of the rearrangement inequalities, see [LL, Thm. 3.4], we may assume without loss of generality that u is radial, i.e. $u(x) = f(|x|)$, where $f \in C^\infty([R,\infty)$ is non-negative and such that for some $\rho > R$ we have

$$r \geq \rho \quad \Rightarrow \quad f(r) = 0.$$

Integration by parts together with the Hölder inequality then imply

$$\begin{aligned}
\int_R^\infty \frac{f(r)^p}{r^p}\, r^{n-1}\, dr &= \frac{1}{n-p}\left[f(r)^p\, r^{n-p}\right]_R^\rho - \frac{p}{n-p}\int_R^\infty f(r)^{p-1}\, f'(r)\, r^{n-p}\, dr \\
&\leq \frac{p}{n-p}\int_R^\infty f(r)^{p-1}\, r^{\frac{(n-p-1)(p-1)}{p}}\, |f'(r)|\, r^{\frac{n-1}{p}}\, dr \\
&\leq \frac{p}{n-p}\left(\int_R^\infty \frac{f(r)^p}{r^p}\, r^{n-1}\, dr\right)^{\frac{p-1}{p}} \left(\int_R^\infty |f'(r)|^p\, r^{n-1}\, dr\right)^{\frac{1}{p}},
\end{aligned}$$

where we have used the positivity of f in the second line. Hence

$$\int_R^\infty \frac{f(r)^p}{r^p}\, r^{n-1}\, dr \;\leq\; \left(\frac{p}{n-p}\right)^p \int_R^\infty |f'(r)|^p\, r^{n-1}\, dr,$$

and the claim follows. $\square$

It is not difficult to verify that the constant $\left(\frac{n-p}{p}\right)^p$ cannot be improved and that inequality (6.1) fails if $p \geq n$. It turns out that when we replace the left hand side by the functional (1.4) with σ constant and positive, then (6.1), with a different constant, extends also to the case $p > n$.

Theorem 6.2. *Assume that $p > n$ and that $\sigma > 0$. Then the inequality*

$$\int_{B_R^c} |\nabla u|^p + \sigma \int_{\partial B_R} |u|^p \;\geq\; C(\sigma, R) \int_{B_R^c} \frac{|u|^p}{|x|^p}, \tag{6.2}$$

with

$$C(\sigma, R) = \min\left\{ \left(\frac{p-n}{p}\right)^p,\; R^p\, \sigma^{\frac{p}{p-1}} \right\}. \tag{6.3}$$

holds for all $u \in W^{1,p}(B_R^c)$.

Proof. Let $\delta(x) = |x| - R$ and let

$$\gamma := \frac{p-n}{p}\, \sigma^{\frac{1}{p-1}}.$$

From the convexity of the function $|x|^p$ in $\mathbb{R}^n$ it follows that

$$|\xi_1|^p \;\geq\; |\xi_2|^p + p\,|\xi_2|^{p-2}\,\xi_2\cdot(\xi_1-\xi_2) \tag{6.4}$$

holds for all $\xi_1, \xi_2 \in \mathbb{R}^n$. We apply (6.4) with

$$\xi_1 = \nabla u, \qquad \xi_2 = \frac{\beta\, u\,\nabla\delta}{\delta+\gamma},$$

where $\beta > 0$ is a parameter whose value will be specified later. Hence

$$\int_{B_R^c} |\nabla u|^p \geq \int_{B_R^c} \frac{\beta^p\,|u|^p}{(\delta+\gamma)^p} + p\int_{B_R^c} \frac{\beta^{p-1}|u|^{p-1}}{(\delta+\gamma)^{p-1}}\,\nabla\delta\cdot\left(\nabla u - \frac{\beta\,u\,\nabla\delta}{\delta+\gamma}\right) \tag{6.5}$$

Since $|\nabla\delta| = 1,\ \Delta\delta = \frac{n-1}{|x|}$ and since the normal derivative of δ is equal to -1 on ∂B_R, an integration by parts gives

$$\begin{aligned}\int_{B_R^c} \frac{|u|^{p-1}}{(\delta+\gamma)^{p-1}}\,\nabla\delta\cdot\nabla u = -\gamma^{1-p}\int_{\partial B_R} |u|^p - (n-1)\int_{B_R^c} \frac{|u|^p}{|x|\,(\delta+\gamma)^{p-1}}\\ + (1-p)\int_{B_R^c} \frac{|u|^{p-1}}{(\delta+\gamma)^{p-1}}\,\nabla\delta\cdot\nabla u + (p-1)\int_{B_R^c} \frac{|u|^p}{(\delta+\gamma)^p}.\end{aligned}$$

This in combination with (6.5) yields

$$\begin{aligned}\int_{B_R^c} |\nabla u|^p + \gamma^{1-p}\beta^{p-1}\int_{\partial B_R} |u|^p \geq (p-1)(\beta^{p-1}-\beta^p)\int_{B_R^c} \frac{|u|^p}{(\delta+\gamma)^p}\\ - (n-1)\,\beta^{p-1}\int_{B_R^c} \frac{|u|^p}{|x|\,(\delta+\gamma)^{p-1}}\end{aligned} \tag{6.6}$$

Assume now that $R > \gamma$ in which case $\left(\frac{p}{p-n}\right)^p R^p\,\sigma^{\frac{p}{p-1}} < 1$. Then

$$\delta(x) + \gamma = |x| - R + \gamma < |x|$$

and the above inequality gives

$$\int_{B_R^c} |\nabla u|^p + \gamma^{1-p}\beta^{p-1}\int_{\partial B_R} |u|^p \;\geq\; \Big((p-n)\beta^{p-1} - (p-1)\beta^p\Big)\int_{B_R^c} \frac{|u|^p}{|x|^p}\,. \tag{6.7}$$

The constant in front of the integral on the right hand side attains its maximum for

$$\beta = \frac{p-n}{p}. \tag{6.8}$$

Inserting this value of β into (6.7) we obtain (6.2) in the case $R < \gamma$.

If $R \geq \gamma$, then we have $\left(\frac{p}{p-n}\right)^p R^p\,\sigma^{\frac{p}{p-1}} \geq 1$ and

$$\delta(x) + \gamma = |x| - R + \gamma \;\leq\; \frac{\gamma}{R}\,|x|.$$

Inequality (6.6) then implies that

$$\int_{B_R^c} |\nabla u|^p + \gamma^{1-p}\beta^{p-1}\int_{\partial B_R} |u|^p \;\geq\; \Big((p-n)\beta^{p-1} - (p-1)\beta^p\Big)\left(\frac{R}{\gamma}\right)^p\int_{B_R^c} \frac{|u|^p}{|x|^p}.$$

Choosing β as in (6.8) we thus arrive again at (6.2). $\square$

Remark 6.3. **(i)**. Note that for σ large enough we have $C(\sigma, R) = \left(\frac{p-n}{p}\right)^p$ which is in modulus equal to the sharp constant in the inequality (6.1) valid in the case $n > p$. On the other hand, for σ small enough we have $C(\sigma, R) = R^p\, \sigma^{\frac{p}{p-1}}$ which vanishes in the limit $\sigma \to 0$, as expected.

(ii). In the case $n = p$ we have $C(\sigma, R) = 0$ which is natural since the inequality

$$\int_{B_R^c} |\nabla u|^n + \sigma \int_{\partial B_R} |u|^n \ \geq\ C \int_{B_R^c} \frac{|u|^n}{|x|^n}\,, \qquad u \in W^{1,n}(B_R^c) \tag{6.9}$$

fails for any $C > 0$ independently of R and σ. To see this, consider the family of test functions

$$u_k(x) = \left(1 - \frac{\log(|x|/R)}{\log k}\right)_+ \qquad k \in \mathbb{N}, \quad x \in B_R^c.$$

By inserting u_k into (6.9) and letting $k \to \infty$ it follows that (6.9) must fail whenever $C > 0$. This is closely related to [MMP, Ex. 2] which shows that if $\Omega = B_R^c$ and $p = n$, then the best constant in the hardy inequality (1.2) is zero, i.e. $\mu_p(B_R^c) = 0$, see equation (1.3).

(iii). Hardy's inequality for complements of bounded domains with Dirichlet boundary conditions were for the first time studied in [MS2].

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Tomas Ekholm, Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden

E-mail address: tomase@math.kth.se

Hynek Kovařík, DICATAM, Sezione di Matematica, Università degli studi di Brescia, Via Branze, 38 - 25123 Brescia, Italy

E-mail address: hynek.kovarik@unibs.it

Ari Laptev, Imperial College London, Huxley Building, 180 Queen's Gate, London SW7 2AZ, UK

E-mail address: a.laptev@imperial.ac.uk