Reformation Instability in Elastic Solids

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Abstract

The reformation of a body fundamentally involves the mapping of one natural reference configuration of it into another natural reference configuration. The mass and the constitutive properties of the material remain unaltered, but the overall shape of the reference configuration generally changes. If, when a natural reference configuration is distorted, there is a portion of the boundary of the body that is displacement controlled, then a reformation of the body must be such that the original displacement controlled part of the boundary and its reformation are identical. In common applications that involve reformation, the remainder of the boundary is traction-free and a reformation essentially involves a change of the morphology of this traction-free surface. For example, undulations are often a characteristic feature of the reformation of a free, plane boundary surface. Reformations are a result of a material instability and they may associate with a chemically induced diffusive processes in which particles of the body move into preferred places. Fundamentally, a reformation is generated in response to the drive to lower the total stored energy of the body. In this work we are not concerned with the physical processes that take place during reformation, but rather we are concerned with characterizing the onset of the instability. We develop a variational characterization of the reformation instability for a nonlinear elastic body and we include the effect of surface energy. As an example, we consider the axial deformation of a circular cylinder and argue that small scale nano-wires, for which the diameter—to—length ratio is sufficiently small, are expected to be stable with respect to spatial variations when extended. Moreover, we observe that if the surfacial energy function is sufficiently convex at the undistorted state such wires may also be stable with respect to spatial variations when compressed. We then show that such small scale nano-wires are unstable with respect to reformation when extended.

Keywords: Reformation, Instability, Nonlinear Elasticity.

Contents

1 Introduction

In material science the occurrence of morphological instabilities is widely reported (see for instance [3], [14], [18], [16], [19]). Indeed, a typical experimental observation reveals that a strained solid that is in contact with a more compliant phase, such as its own melt or vapor, can partially release its elastic energy by a sudden morphological transition at the interface. This occurs, for instance, when the strained solid has a surface at which material can be redistributed by a transport mechanism and thus it may reduce its elastic energy via surface undulations. While these kinds of instability relate to the studies of Asaro and Tiller [1] concerning corrosion phenomena, a recognition of the general nature of these instabilities can be traced to the studies of Grinfeld ([11], [12]) and Srolovitz ([20]). These works concern stress-induced instabilities in the context of linear elasticity and they show that beyond some critical stress a planar traction-free face of an elastic solid is unstable against undulations. Recently, an interesting analytical analysis of this problem has been given by Fusco, Fonseca, Leoni and Morini in [7], and by Fusco and Morini in [8].

The problem of morphological instability rests at the core of epitaxy [17], [21], [6], i.e., the over-growth of a crystalline material onto a single crystal surface of a different material. In this case the two lattice planes in contact accommodate the misfit, which is the origin of a certain amount of elastic energy, and this leads to the major problem of the subject;

the study of stress-induced crystal growth. This requires a detailed description of the interaction between bulk and surface energies, where the role of surface stress and strain, though recognized since the works of Gibbs [9], is still not completely clear, especially in the case of a genuine three-dimensional model, [17], [22]. The inadequacy of linear theories when misfits are too large and when elastic relaxation is taken into account suggests the need for a nonlinear theory where the reference and deformed configurations are distinguished. To our knowledge, the published works on morphological instabilities in solids applies to the linearized theory of elasticity. Thus, in an attempt to contribute to the broader issues, in this work we consider the morphological instability of stressed solids within the framework of finite elasticity theory. Our approach uses variational methods and we focus on the mechanical nature of the factors that regulate morphological instabilities.

In Section 2, we illustrate some of the main features of the problem addressed in this paper through an elementary approach in which surface energy is neglected and linearized elasticity is assumed. We use this setting to describe how a comparison argument may be used to detect unstable configurations.

In Section 3, we formulate a variational characterization of *material reformation* in which the ability to change relative positions of material particles is taken into account through volume preserving mappings. In this framework the investigation of morphological instabilities involves the analysis of the second variation of the total stored energy with respect to domain variations. This leads to the study of variational problems for varying domains—problems for which several techniques have been proposed in the literature ([4], [5], [13]), but problems which yet involve many open and challenging mathematical issues. Here, we use variational methods to establish decreasing energy directions in the space of admissible perturbations in order to detect morphological instabilities. The inclusion of surfacial energy in addition to bulk energy is important from a physical point of view, but it complicates the story. For example, the second spatial variation shown in (3.16), which governs the local stability of a minimizing state, contains many surface contributions which are potential sources of instability, especially when the surface area-to-volume scale differential becomes important. We find that for instability with respect to reformation, the condition (3.24) and the negativeness of the quadratic form (3.28) is sufficient.

Finally, in Section 4, we investigate, as an example, the morphological instability of a cylindrical solid under axial load. Nonlinear elasticity and both volume and surface energies are considered. This investigation is thought to be relevant to the area of small scale physics e.g., quantum (i.e., nano-) wires.

2 Heuristic approach to morphological instabilities

Let $\mathcal{B}_0 := [0, l] \times [0, \gamma_0(x_1)]$, where $\gamma_0(x_1) > 0$ for all $x_1 \in [0, l]$, be the natural reference configuration of a elastic body in a two dimensional rectangular frame in $\mathbb{E}^2$, and let $\mathbf{x} =$

$(x_1, x_2) \in \mathcal{B}_0 \subset \mathbb{E}^2$ denote the position of its particles. Suppose that $\mathbf{y} = \mathbf{y}(\mathbf{x})$ is the deformation of $\mathcal{B}_0$ to an equilibrium configuration $\mathcal{B} := \mathbf{y}(\mathcal{B}_0)$ such that $\mathbf{y}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial_1\mathcal{B}_0 := \partial\mathcal{B}_0 \setminus \partial_2\mathcal{B}_0$, where $\partial_2\mathcal{B}_0$ represents the upper edge of $\mathcal{B}_0$ at $x_2 = \gamma_0(x_1)$ for $x_1 \in [0,1]$, which is supposed to be traction-free. Thus, the distortion of $\mathcal{B}_0$ is due to the fact that the left, the right, and the bottom edges of $\mathcal{B}_0$ are subject to the given placement $\mathbf{y}^*(\mathbf{x})$. Suppose that for this material the function $g\big(\nabla\mathbf{y}(\cdot)\big) : \mathcal{B}_0 \mapsto \mathbb{R}$ represents the specific strain energy of the distorted configuration $\mathcal{B}$ measured per unit reference area. For example, within linearized elasticity theory for an isotropic material one may take

$$g\big(\nabla\mathbf{y}(\mathbf{x})\big) := \mu\mathrm{tr}\big(\mathbf{E}^2(\mathbf{x})\big) + \frac{\lambda}{2}\big(\mathrm{tr}(\mathbf{E}(\mathbf{x}))\big)^2, \tag{2.1}$$

where $\mathbf{E} = \frac{1}{2}\big(\nabla\mathbf{y} + (\nabla\mathbf{y})^{\mathrm{T}}\big) - \mathbf{1}$ is the linearized strain tensor and λ, μ are the Lamé coefficients, chosen such that the quadratic form in (2.1) is positive definite. Suppose, also, that the configuration $\mathcal{B} = \mathbf{y}(\mathcal{B}_0)$ is a local minimum of the potential energy

$$\mathsf{E}(\mathbf{z}; \mathcal{B}_0) = \int_0^l \int_0^{\gamma_0(x_1)} g\big(\nabla\mathbf{z}(\mathbf{x})\big) dx_2 dx_1$$

corresponding to zero traction on $\partial_2\mathcal{B}_0$ for the body $\mathcal{B}_0$ among all placements $\mathbf{z}(\mathbf{x})$ that lie in an admissible class of placements, $\mathcal{A}$, that satisfy $\mathbf{z}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ on $\partial_1\mathcal{B}_0$. Thus,

$$\mathsf{E}(\mathbf{y}; \mathcal{B}_0) = \min_{\mathbf{z}(\cdot)\in\mathcal{A}} \mathsf{E}(\mathbf{z}; \mathcal{B}_0).$$

For later technical reasons, we suppose that both $\mathbf{y}(\mathbf{x})$ and $g\left(\nabla\mathbf{y}(\mathbf{x})\right)$ have smooth extensions to points $\mathbf{x} \in \mathbb{E}^2$ in a neighborhood $\mathcal{N}(\mathcal{B}_0)$ of $\mathcal{B}_0$. Let $\mathbf{y}^{\mathrm{E}}(\cdot)$ denote such an extension for $\mathbf{y}(\cdot)$, so that $\mathbf{y}^{\mathrm{E}}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{B}_0$.

The distorted equilibrium configuration $\mathcal{B} = \mathbf{y}(\mathcal{B}_0)$ is assumed to be given as characterized above, and our interest here is in the local stability of $\mathcal{B}$ with respect to a *reformation* of the reference configuration $\mathcal{B}_0$. A *reformation* is defined as a mapping of $\mathcal{B}_0$ to $\mathcal{B}_0(\varepsilon)$ with $\mathcal{B}_0(0) = \mathcal{B}_0$ such that $\partial_1\mathcal{B}_0 = \partial_1\mathcal{B}_0(\varepsilon)$ remains fixed and the areas of $\mathcal{B}_0$ and $\mathcal{B}_0(\varepsilon)$ are equal. Thus, under a reformation, the top edge $\partial_2\mathcal{B}_0$ is mapped to $\partial_2\mathcal{B}_0(\varepsilon)$, which we shall characterize as $x_2 = \gamma_0^\varepsilon(x_1) := \gamma_0(x_1) + \varepsilon\eta(x_1)$, where $\varepsilon > 0$ is sufficiently small, $\eta(0) = \eta(l) = 0$, and

$$\int_0^l \eta(x_1) dx_1 = 0. \tag{2.2}$$

After a reformation of $\mathcal{B}_0 \mapsto \mathcal{B}_0(\varepsilon)$, suppose we consider a placement that is a local minimizer of the potential energy

$$\mathsf{E}\big(\mathbf{z};\mathcal{B}_0(\varepsilon)\big) = \int_0^l \int_0^{\gamma_0(x_1)+\varepsilon\eta(x_1)} g\big(\nabla\mathbf{z}(\mathbf{x})\big)dx_2dx_1$$

corresponding to zero traction on $\partial_2\mathcal{B}_0(\varepsilon)$ for the body $\mathcal{B}_0(\varepsilon)$ among all placements $\mathbf{z}(\mathbf{x})$ that lie in an admissible class of placements, $\mathcal{A}(\varepsilon)$, that satisfy $\mathbf{z}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial_1\mathcal{B}_0(\varepsilon) = \partial_1\mathcal{B}_0$. Of course, the constraint (2.2) also is supposed to hold. If

$$\min_{\mathbf{z}(\cdot)\in\mathcal{A}(\varepsilon)} \mathsf{E}\big(\mathbf{z};\mathcal{B}_0(\varepsilon)\big) < \min_{\mathbf{z}(\cdot)\in\mathcal{A}} \mathsf{E}(\mathbf{z};\mathcal{B}_0) \tag{2.3}$$

for sufficiently small $\varepsilon > 0$, then we say that $\mathcal{B}_0$ is locally unstable under a reformation. To investigate (2.3), one may set out to determine a minimizer $\mathbf{y}_\varepsilon(\cdot)$ of $\mathsf{E}\big(\mathbf{z};\mathcal{B}_0(\varepsilon)\big)$ for all sufficiently small ε and then attempt to compute the derivatives of $\mathsf{E}\big(\mathbf{y}_\varepsilon;\mathcal{B}_0(\varepsilon)\big)$ with respect to ε at $\varepsilon = 0$. However, in general this kind of computation cannot be made explicit because the sequence of minimizers $\mathbf{y}_\varepsilon(\cdot)$ is not explicit in its dependence on $\mathcal{B}_0(\varepsilon)$. To overcome this difficulty one is led to consider the theory of variational problems over varying domains, for which several techniques have been proposed in literature and for which many open problems are still under investigation. For an account of these issues we refer to [4, 5, 10, 13] and for related deep calculations to [8].

Although the reformation instability problem is generally difficult, a sufficient condition for determining unstable reference configurations is relatively straightforward and is based on simpler calculations. Indeed, if both $\mathbf{y}^{\mathrm{E}}(\cdot)$ and $g\big(\nabla\mathbf{y}^{\mathrm{E}}(\cdot)\big)$ denote the extensions to an open neighborhood of $\mathcal{B}_0$, as assumed earlier, then for sufficiently small $\varepsilon > 0$ it follows that $\mathbf{y}^{\mathrm{E}}(\mathbf{x}) \in \mathcal{A}(\varepsilon)$ for $\mathbf{x} \in \mathcal{B}_0(\varepsilon)$, and we may consider $\mathsf{E}\big(\mathbf{y}^{\mathrm{E}};\mathcal{B}_0(\varepsilon)\big)$ and compute its ε-derivatives. Suppose that the following conditions hold:

$$\frac{d}{d\varepsilon}\mathsf{E}\big(\mathbf{y}^{\mathrm{E}};\mathcal{B}_0(\varepsilon)\big)_{|\varepsilon=0} = 0, \quad \frac{d^2}{d\varepsilon^2}\mathsf{E}\big(\mathbf{y}^{\mathrm{E}};\mathcal{B}_0(\varepsilon)\big)_{|\varepsilon=0} < 0. \tag{2.4}$$

Then, it follows that $\mathcal{B}_0$ is not locally stable with respect to a reformation. Indeed,

$$\min_{\mathbf{z}(\cdot)\in\mathcal{A}(\varepsilon)} \mathsf{E}\big(\mathbf{z};\mathcal{B}_0(\varepsilon)\big) \le \mathsf{E}\big(\mathbf{y}^{\mathrm{E}};\mathcal{B}_0(\varepsilon)\big) < \mathsf{E}\big(\mathbf{y};\mathcal{B}_0\big) = \min_{\mathbf{z}(\cdot)\in\mathcal{A}} \mathsf{E}\big(\mathbf{z};\mathcal{B}_0\big).$$

Example. The following brief review is taken from [10], where the full calculations can be found: Consider the placement $\mathbf{x} \in \mathcal{B}_0 \mapsto \mathbf{y}(\mathbf{x}) := (x_1 + ax_1x_2 + bx_1, x_2 + cx_2^2 + dx_1^2 + ex_2)$, which, of course, is well-defined on all of $\mathbb{E}^2$. In [10] it is shown that three of the five coefficients (a, b, c, d, e) may be chosen such that $\mathbf{y}(x)$ is a local minimizer of $\mathsf{E}(\mathbf{z};\mathcal{B}_0)$ in the admissible class $\mathcal{A}$ defined earlier. For this choice, it follows that

$$g\big(\nabla\mathbf{y}(x)\big) = \mu(ax_2+b)^2 + \mu(2cx_2+e)^2 + \frac{\lambda}{2}\big((a+2c)x_2+b+e\big)^2,$$

which in turn is well defined on all of $\mathbb{E}^2$. Again, it is shown in [10] that the relations (2.4) can be satisfied by a selection of the remaining two coefficients. Therefore, $\mathcal{B}_0$ is not locally stable under reformations. □

In the following section we will apply a strategy similar to that described above to the study of reformation stability in the setting of finite elasticity including bulk and surface energy.

3 Variational approach to reformation

Let $\mathcal{B}_0 \subset \mathbb{E}^3$ denote the reference configuration of a homogeneous elastic body whose particles are located at $\mathbf{x} \in \mathcal{B}_0$. Given a smooth, injective deformation $\mathbf{y} = \mathbf{y}(\mathbf{x}) : \mathcal{B}_0 \mapsto \mathcal{B} := \mathbf{y}(\mathcal{B}_0) \subset \mathbb{E}^3$, the specific strain energy at $\mathbf{x} \in \mathcal{B}_0$ measured per unit volume of $\mathcal{B}_0$ is given by $W = W(\nabla\mathbf{y}(\mathbf{x}))$, where $W(\cdot)$ is defined on the set Lin of all linear transformation of $\mathbb{E}^3 \mapsto \mathbb{E}^3$. We assume that $W(\cdot)$ is objective in the sense that $W(\mathbf{F}) = W(\mathbf{QF})$ for all $\mathbf{Q} \in$ Orth, where Orth $\subset$ Lin denotes the set of all orthogonal linear transformations. We consider that the boundary surface $\partial\mathcal{B}_0$ of the body supports a surface energy $\sigma(\mu)$ measured per unit area of $\partial\mathcal{B}_0$. Here, $\mu := |(\text{cof }\nabla\mathbf{y})\mathbf{n}_0|$ denotes the surface stretch under the deformation $\mathbf{y}(\mathbf{x})$ and $\mathbf{n}_0$ is the outer unit normal to $\partial\mathcal{B}_0$. We shall let ρ_0 denote the *constant* mass density of the reference configuration $\mathcal{B}_0$.

The body is subject to a given placement $\mathbf{y}^*(\mathbf{x})$ for all $\mathbf{x} \in \partial_1\mathcal{B}_0$ and the remainder of the boundary $\partial_2\mathcal{B}_0 = \partial\mathcal{B}_0 \setminus \partial_1\mathcal{B}_0$ is considered to be traction-free. Throughout this work we assume that there is no body force action. Thus, for any admissible deformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$, the total stored energy in $\mathcal{B}$ is[1]

$$\mathsf{E}(\mathbf{y};\mathcal{B}_0) = \int_{\mathcal{B}_0} W(\nabla\mathbf{y})dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}_0} \sigma(\mu)da(\mathbf{x}) \,. \tag{3.1}$$

Suppose $\mathbf{y}(\mathbf{x})$ is a smooth admissible minimizer of the functional $\mathsf{E}(\cdot;\mathcal{B}_0)$. Then, for any

[1]Clearly, if $\sigma(\mu) = \sigma_0\mu$, where $\sigma_0 > 0$ is constant, then because $\mu da(\mathbf{x}) = da(\mathbf{y})$ the surface integral here may be replace with $\sigma_0\times$area$(\partial_2\mathcal{B})$, where $\partial_2\mathcal{B} := \mathbf{y}(\partial_2\mathcal{B}_0)$. In this case σ_0 is interpreted as the surface tension and the surface is a classical fluid-like surface. For a solid-like surface, it would be reasonable to assume that $\sigma(1) = 0$, i.e., that the surface energy vanishes when the surface is not deformed from its reference state $\partial\mathcal{B}_0$, in which case $\mu = 1$. Of course, more general and physically improved forms of the surface energy function for solid surfaces are possible. We have in mind the inclusion of the full surfacial deformation gradient in place of the surface stretch μ. However, we shall not investigate this added complication in the present work.

spatial variation $\mathbf{y}_\lambda(\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \lambda\mathbf{u}(\mathbf{x})$, $\lambda \in \mathbb{R}$, with $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \partial_1\mathcal{B}_0$, the first spatial variation of

$$\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0) = \int_{\mathcal{B}_0} W(\nabla\mathbf{y}_\lambda)dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}_0} \sigma(\mu_\lambda)da(\mathbf{x}) \tag{3.2}$$

must vanish at $\lambda = 0$ for all admissible $\mathbf{u}(\mathbf{x})$, and the second spatial variation at $\lambda = 0$ must be non-negative for all such $\mathbf{u}(\mathbf{x})$. Of course, here, $\mu_\lambda := |(\mathrm{cof}\ \nabla\mathbf{y}_\lambda)\mathbf{n}_0|$ with $\mu_\lambda da(\mathbf{x}) = da(\mathbf{y}_\lambda)$. In the following, we shall use the notation $\mathcal{D}_\lambda := \mathbf{y}_\lambda(\mathcal{B}_0)$ and then identify $\mathcal{D}_0 = \mathbf{y}_0(\mathcal{B}_0) = \mathbf{y}(\mathcal{B}_0) = \mathcal{B}$. Also, we define $\partial_2\mathcal{D}_\lambda := \mathbf{y}_\lambda(\partial_2\mathcal{B}_0)$.

Later in this section we shall introduce the notion of a 'reformation' of the reference configuration and define related first and second referential variations. This will allow us to investigate the local stability of the field $\mathbf{y}(\cdot)$ with respect to reformation. First, however, we attend to spatial variational issues.

3.1 The First Spatial Variation

In the following, it will be convenient to use the shorthand notation $\dot{\overline{(\cdot)}}$ to denote $\partial(\cdot)/\partial\lambda$ with $\mathbf{x}$ held fixed. Thus, for the first spatial variation we need to calculate

$$\frac{d}{d\lambda}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0) = \int_{\mathcal{B}_0} \dot{\overline{W(\nabla\mathbf{y}_\lambda)}}dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}_0} \dot{\overline{\sigma(\mu_\lambda)}}da(\mathbf{x})\ , \tag{3.3}$$

and evaluate at $\lambda = 0$. Clearly, as integrand of the volume integral we have (noting that $\dot{\mathbf{y}}_\lambda = \mathbf{u}$)

$$\dot{\overline{W(\nabla\mathbf{y}_\lambda)}} = W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\cdot\nabla\mathbf{u}\ ,$$

where the subscript $\mathbf{F}$ denotes gradient of $W(\mathbf{F})$ with respect to $\mathbf{F}$. For the integrand of the surface integral, we shall need the following elementary identity from the differential geometry of surfaces:

$$\dot{\mu}_\lambda = \mu_\lambda\,\mathrm{div}_s\,\dot{\mathbf{y}}_\lambda\ , \tag{3.4}$$

where 'div_s' denotes the surface divergence on $\partial_2\mathcal{D}_\lambda$ and where the independent variable in $\dot{\mathbf{y}}_\lambda$ is understood to be $\mathbf{y}_\lambda$, via the inversion $\mathbf{x} = \mathbf{y}_\lambda^{-1}(\mathbf{y}_\lambda)$. Thus, using $\dot{\mathbf{y}}_\lambda = \mathbf{u}$, for the surface integral in (3.3) we have

$$
\begin{aligned}
\int_{\partial_2\mathcal{B}_0} \dot{\overline{\sigma(\mu_\lambda)}} da(\mathbf{x}) &= \int_{\partial_2\mathcal{D}_\lambda} \sigma'(\mu_\lambda)\,\mathrm{div}_s\,\mathbf{u}\,da(\mathbf{y}_\lambda)\\
&= -\int_{\partial_2\mathcal{D}_\lambda} \mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)\cdot\mathbf{u}\,da(\mathbf{y}_\lambda)\ ,
\end{aligned}
$$

where we have introduced $\mathbb{I}_\lambda := \mathbf{1} - \mathbf{m}_\lambda \otimes \mathbf{m}_\lambda$ with $\mathbf{m}_\lambda = (\mathrm{cof}\,\nabla\mathbf{y}_\lambda)\mathbf{n}_0/\mu_\lambda$ denoting the outer unit normal to $\partial_2\mathcal{D}_\lambda$, applied the identity

$$
\begin{aligned}
\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)\cdot\mathbf{u} &= \mathrm{grad}_s\sigma'(\mu_\lambda)\cdot\mathbf{u} - \sigma'(\mu_\lambda)\mathrm{div}_s\,\mathbf{m}_\lambda\cdot\mathbf{u}\\
&= \mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbf{u}^{\tan}\big) - \sigma'(\mu_\lambda)\,\mathrm{div}_s\,\mathbf{u}
\end{aligned}
$$

with the decomposition $\mathbf{u} = \mathbf{u}^{\tan} + (\mathbf{u}\cdot\mathbf{m}_\lambda)\mathbf{m}_\lambda$ on $\partial_2\mathcal{D}_\lambda$, and called upon the surface divergence theorem with $\mathbf{u} = \mathbf{0}$ on the border of $\partial_2\mathcal{D}_\lambda$, i.e., on $\partial_1\mathcal{D}_\lambda$.

The volume integral in (3.3) may be integrated by parts, and, because $\mathbf{u} = \mathbf{0}$ on $\partial_1\mathcal{B}_0$, we obtain

$$
\begin{aligned}
\int_{\mathcal{B}_0} \dot{\overline{W(\nabla\mathbf{y}_\lambda)}} dv(\mathbf{x}) &= \int_{\mathcal{B}_0} W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\cdot\nabla\mathbf{u}\,dv(\mathbf{x})\\
&= -\int_{\mathcal{B}_0} \mathrm{Div}\,W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\cdot\mathbf{u}\,dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}_0} W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\mathbf{n}_0\cdot\mathbf{u}\,da(\mathbf{x})\\
&= -\int_{\mathcal{B}_0} \mathrm{Div}\,W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\cdot\mathbf{u}\,dv(\mathbf{x}) + \int_{\partial_2\mathcal{D}_\lambda} \mathbf{T}_\lambda^{\mathrm{T}}\mathbf{m}_\lambda\cdot\mathbf{u}\,da(\mathbf{y}_\lambda)\ ,
\end{aligned}
$$

where we have used 'Div' to denote the divergence operator in $\mathcal{B}_0$, and in the last line we have introduced a change of domain of integration using the relation $\mathbf{m}_\lambda = (\mathrm{cof}\,\nabla\mathbf{y}_\lambda)\mathbf{n}_0/\mu_\lambda$ and the definition of the (symmetric) Cauchy stress in $\mathcal{D}_\lambda$, i.e.,

$$
\mathbf{T}_\lambda := \frac{1}{\det\nabla\mathbf{y}_\lambda}\nabla\mathbf{y}_\lambda W_{\mathbf{F}}^{\mathrm{T}}(\nabla\mathbf{y}_\lambda)\ . \tag{3.5}
$$

Thus, we have

$$
\frac{d}{d\lambda}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0) = \int_{\mathcal{B}_0} W_{\mathbf{F}}(\nabla\mathbf{y}_\lambda)\cdot\nabla\mathbf{u}\,dv(\mathbf{x}) - \int_{\partial_2\mathcal{D}_\lambda} \mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)\cdot\mathbf{u}\,da(\mathbf{y}_\lambda)\ , \tag{3.6}
$$

and we may conclude that

$$\frac{d}{d\lambda}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0)_{|\lambda=0} = -\int_{\mathcal{B}_0} \operatorname{Div} W_{\mathbf{F}}(\nabla\mathbf{y})\cdot\mathbf{u}\,dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}}\Big(\mathbf{T}^{\mathrm{T}}\mathbf{n} - \operatorname{div}_s\big(\sigma'(\mu)\mathbb{I}\big)\Big)\cdot\mathbf{u}\,da(\mathbf{y})\ .$$

Here, $\mathbf{n} = (\operatorname{cof}\nabla\mathbf{y})\mathbf{n}_0/\mu$ is the outer unit normal to $\partial_2\mathcal{B} = \mathbf{y}(\partial_2\mathcal{B}_0)$, $\mathbf{T}$ is the value of $\mathbf{T}_\lambda$ in (3.5) at $\lambda = 0$ and represents the Cauchy stress in $\mathcal{B}$, and $\mathbb{I} := \mathbf{1} - \mathbf{n}\otimes\mathbf{n}$. Because this last equation must hold for all $\mathbf{u}$ that vanish on $\partial_1\mathcal{B}_0$, we arrive at the first spatial variation conditions

$$\operatorname{Div} W_{\mathbf{F}}(\nabla\mathbf{y}) = \mathbf{0} \quad \forall\mathbf{x}\in\mathcal{B}_0 \tag{3.7a}$$

and

$$\mathbf{T}^T\mathbf{n} - \operatorname{div}_s(\sigma'(\mu)\mathbb{I}) = 0 \quad \forall\mathbf{y}\in\partial_2\mathcal{B}\ . \tag{3.7b}$$

In addition, recall that

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}^*(\mathbf{x}) \quad \forall\mathbf{x}\in\partial_1\mathcal{B}_0\ . \tag{3.7c}$$

3.2 The Second Spatial Variation

We now use (3.6) to determine the second spatial variation. Thus, we begin with

$$\frac{d^2}{d\lambda^2}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0) = \int_{\mathcal{B}_0} W_{\mathbf{FF}}(\nabla\mathbf{y}_\lambda)[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}dv(\mathbf{x}) - \underbrace{\int_{\partial_2\mathcal{B}_0}\overline{\Big(\operatorname{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)\mu_\lambda\Big)}^{\boldsymbol{\cdot}}\cdot\mathbf{u}\,da(\mathbf{x})}_{\mathsf{J}(\lambda)}\ , \tag{3.8}$$

where the domain of integration of the second surface integral in (3.6) has been mapped to $\partial_2\mathcal{B}_0$ for the convenience of interchanging differentiation and integration. Note that the surface integral term in (3.8) has been denoted as $\mathsf{J}(\lambda)$ for easy reference. Note, also, that with the aid of (3.4) and $\mu_\lambda da(\mathbf{x}) = da(\mathbf{y}_\lambda)$ we may write

$$\mathsf{J}(\lambda) = \int_{\partial_2\mathcal{D}_\lambda}\Big(\overline{\operatorname{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)}^{\boldsymbol{\cdot}}\cdot\mathbf{u} + \operatorname{div}_s\dot{\mathbf{y}}_\lambda\big(\operatorname{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)\big)\cdot\mathbf{u}\Big)da(\mathbf{y}_\lambda)\ . \tag{3.9}$$

In (3.8), we are interested in $(d^2/d\lambda^2)\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0)$ at $\lambda = 0$, so below we begin to develop $\mathsf{J}(0)$ by recording a few key steps in the process of reaching this goal. First, we obtain the following identity:

$$\overline{\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)}^{\boldsymbol{\cdot}}_{|\lambda=0} = \mathrm{div}_s\big(\sigma''(\mu)\mu(\mathrm{div}_s\mathbf{u})\mathbb{I} + \sigma'(\mu)\mathbf{n}\otimes(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\big)$$
$$+\sigma'(\mu)\mathrm{div}_s\mathbf{n}(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n} - (\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathrm{grad}_s\sigma'(\mu)$$
$$+\sigma'(\mu)(\mathrm{grad}_s\mathbf{n}\cdot\mathrm{grad}_s\mathbf{u})\mathbf{n} - \sigma'(\mu)(\mathrm{grad}_s\mathbf{n})(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\,. \tag{3.10}$$

To see this, we find it convenient to use

$$\overline{\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)}^{\boldsymbol{\cdot}}\cdot\mathbf{a} = \overline{\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\mathbf{a}\big)}^{\boldsymbol{\cdot}} \quad \forall \text{ constant } \mathbf{a}\in\mathbb{E}^3\,, \tag{3.11}$$

and to introduce the substitution variable $\mathbf{v} := \sigma'(\mu_\lambda)\mathbb{I}_\lambda\mathbf{a}$. Then, using

$$\overline{\mathrm{grad}\mathbf{v}}^{\boldsymbol{\cdot}} = \mathrm{grad}\dot{\mathbf{v}} - \mathrm{grad}\mathbf{v}(\mathrm{grad}\dot{\mathbf{y}}_\lambda)\,,$$

which follows from a 'chain-rule' calculation, and the identity $\mathrm{grad}_s\mathbf{v} = \mathrm{grad}\mathbf{v}\mathbb{I}_\lambda$ it readily follows that

$$\overline{\mathrm{grad}_s\mathbf{v}}^{\boldsymbol{\cdot}} = \mathrm{grad}_s\big(\sigma''(\mu_\lambda)\dot{\mu}_\lambda\mathbb{I}_\lambda\mathbf{a}\big) + \mathrm{grad}_s\big(\sigma'(\mu_\lambda)\dot{\mathbb{I}}_\lambda\mathbf{a}\big)$$
$$-\mathrm{grad}_s\mathbf{v}(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda) + \mathrm{grad}_s\mathbf{v}(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\otimes\mathbf{m}_\lambda\,.$$

Here, we also have used $\dot{\mathbb{I}}_\lambda = -\dot{\mathbf{m}}_\lambda\otimes\mathbf{m}_\lambda - \mathbf{m}_\lambda\otimes\dot{\mathbf{m}}_\lambda$ and the readily established identity $\dot{\mathbf{m}}_\lambda = -(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda$, which follows by differentiating $\mathbf{m}_\lambda = (\mathrm{cof}\,\nabla\mathbf{y}_\lambda)\mathbf{n}_0/\mu_\lambda$ and use of (3.4). With the additional calculation

$$\mathrm{grad}_s\mathbf{v} = \mathrm{grad}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\mathbf{a}\big) = \mathbb{I}_\lambda\mathbf{a}\otimes\mathrm{grad}_s\sigma'(\mu_\lambda) + \sigma'(\mu_\lambda)\mathrm{grad}_s(\mathbb{I}_\lambda\mathbf{a})\,,$$

where, for constant $\mathbf{a}$,

$$\mathrm{grad}_s(\mathbb{I}_\lambda\mathbf{a}) = \mathrm{grad}_s\big((\mathbf{1}-\mathbf{m}_\lambda\otimes\mathbf{m}_\lambda)\mathbf{a}\big) = -\mathbf{m}_\lambda\otimes(\mathrm{grad}_s\mathbf{m}_\lambda)\mathbf{a} - (\mathbf{m}_\lambda\cdot\mathbf{a})\mathrm{grad}_s\mathbf{m}_\lambda\,,$$

and the definition $\mathrm{div}_s(\cdot) := \mathrm{tr}\big(\mathrm{grad}_s(\cdot)\big)$, we have the intermediate conclusion that

$$\overline{\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\mathbf{a}\big)}^{\boldsymbol{\cdot}} = \mathrm{div}_s\big(\sigma''(\mu_\lambda)\dot{\mu}_\lambda\mathbb{I}_\lambda\mathbf{a}\big) + \mathrm{div}_s\big(\sigma'(\mu_\lambda)\dot{\mathbb{I}}_\lambda\mathbf{a}\big)$$
$$-\mathrm{tr}\big(\mathrm{grad}_s\mathbf{v}(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)\big) + (\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\cdot(\mathrm{grad}_s\mathbf{v})^{\mathrm{T}}\mathbf{m}_\lambda\,.$$

To go further, we need the following helpful identities:

$$(\mathrm{grad}_s\mathbf{v})^{\mathrm{T}}\mathbf{m}_\lambda = -\sigma'(\mu_\lambda)(\mathrm{grad}_s\mathbf{m}_\lambda)\mathbf{a}\ ,$$

$$\begin{aligned}\mathrm{tr}\big(\mathrm{grad}_s\mathbf{v}(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)\big) &= \mathrm{tr}\Big(\big(\mathbb{I}_\lambda\mathbf{a}\otimes\mathrm{grad}_s\sigma'(\mu_\lambda)+\sigma'(\mu_\lambda)\mathrm{grad}_s(\mathbb{I}_\lambda\mathbf{a})\big)\mathrm{grad}_s\dot{\mathbf{y}}_\lambda\Big)\\ &= \Big(\big(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda\big)^{\mathrm{T}}\mathrm{grad}_s\sigma'(\mu_\lambda)-\sigma'(\mu_\lambda)\mathbf{m}_\lambda\big(\mathrm{grad}_s\mathbf{m}_\lambda\big)\cdot\mathrm{grad}_s\dot{\mathbf{y}}_\lambda\Big)\cdot\mathbf{a}\ ,\end{aligned}$$

$$\begin{aligned}\mathrm{div}_s\big(\sigma'(\mu_\lambda)\dot{\mathbb{I}}_\lambda\mathbf{a}\big) = \Big(&\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbf{m}_\lambda\otimes(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\big)\\ &+\sigma'(\mu_\lambda)\mathrm{div}_s\mathbf{m}_\lambda(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\Big)\cdot\mathbf{a}\ ,\end{aligned}$$

and

$$\mathrm{div}_s\big(\sigma''(\mu_\lambda)\dot{\mu}_\lambda\mathbb{I}_\lambda\mathbf{a}\big) = \mathrm{div}_s\big(\sigma''(\mu_\lambda)\dot{\mu}_\lambda\mathbb{I}_\lambda\big)\cdot\mathbf{a}\ .$$

Thus, using these and (3.11) we find that

$$\begin{aligned}&\overline{\mathrm{div}_s\big(\sigma'(\mu_\lambda)\mathbb{I}_\lambda\big)}^{\boldsymbol{\cdot}}\cdot\mathbf{a}\\ &\quad=\Big(\mathrm{div}_s\big(\sigma''(\mu_\lambda)\dot{\mu}_\lambda\mathbb{I}_\lambda+\sigma'(\mu_\lambda)\mathbf{m}_\lambda\otimes(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\big)\\ &\quad+\sigma'(\mu_\lambda)\mathrm{div}_s\mathbf{m}_\lambda(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda-(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathrm{grad}_s\sigma'(\mu_\lambda)\\ &\quad+\sigma'(\mu_\lambda)(\mathrm{grad}_s\mathbf{m}_\lambda\cdot\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)\mathbf{m}_\lambda-\sigma'(\mu_\lambda)(\mathrm{grad}_s\mathbf{m}_\lambda)(\mathrm{grad}_s\dot{\mathbf{y}}_\lambda)^{\mathrm{T}}\mathbf{m}_\lambda\Big)\cdot\mathbf{a}\ .\end{aligned}$$

Finally, substituting (3.4) into this last equation and evaluating at $\lambda = 0$ we find, because of the arbitrariness of $\mathbf{a}$, that the claim of (3.10) holds. Thus, recalling (3.9) we see that $\mathsf{J}(0)$ may be expressed as

$$\begin{aligned}\mathsf{J}(0) = \int_{\partial_2\mathcal{B}}\Big(&\mathrm{div}_s\big(\sigma''(\mu)\mu(\mathrm{div}_s\mathbf{u})\mathbb{I}+\sigma'(\mu)\mathbf{n}\otimes(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\big)\\ &+\sigma'(\mu)\mathrm{div}_s\mathbf{n}(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}-(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathrm{grad}_s\sigma'(\mu)\\ &+\sigma'(\mu)(\mathrm{grad}_s\mathbf{n}\cdot\mathrm{grad}_s\mathbf{u})\mathbf{n}-\sigma'(\mu)(\mathrm{grad}_s\mathbf{n})(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\\ &+\mathrm{div}_s\mathbf{u}\big(\mathrm{div}_s\big(\sigma'(\mu)\mathbb{I}\big)\big)\Big)\cdot\mathbf{u}\,da(\mathbf{y})\ .\end{aligned}\tag{3.12}$$

In anticipation of three further rearrangements in (3.12), we note that the surface divergence theorem and the fact that $\mathbf{u} = \mathbf{0}$ on the border of $\partial_2\mathcal{B}$ implies

$$
\begin{aligned}
\int_{\partial_2\mathcal{B}} \Big(\operatorname{div}_s\big(\sigma''(\mu)\mu(\operatorname{div}_s\mathbf{u})\mathbb{I} + \sigma'(\mu)\mathbf{n}\otimes(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\big)\Big)\cdot\mathbf{u}\,da(\mathbf{y})& \\
= -\int_{\partial_2\mathcal{B}} \Big(\sigma''(\mu)\mu(\operatorname{div}_s\mathbf{u})^2 + \sigma'(\mu)|\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}|^2\Big)\,da(\mathbf{y})&\ ,
\end{aligned}
$$

which we shall use for the first line of integration in (3.12). Before making this substitution, however, it is convenient to observe that $(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n} = \operatorname{grad}_s(\mathbf{u}\cdot\mathbf{n}) - (\operatorname{grad}_s\mathbf{n})\mathbf{u}$ and to use $\mathbf{u} = \mathbf{u}^{\tan} + (\mathbf{u}\cdot\mathbf{u})\mathbf{n}$ on $\partial_2\mathcal{B}$ to obtain the identity

$$
|(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}|^2 = |\operatorname{grad}_s(\mathbf{u}\cdot\mathbf{n})|^2 - 2(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\cdot(\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan} - |(\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan}|^2\ .
$$

Thus, in (3.12) we may make the following substitution:

$$
\begin{aligned}
&\int_{\partial_2\mathcal{B}} \Big(\operatorname{div}_s\big(\sigma''(\mu)\mu(\operatorname{div}_s\mathbf{u})\mathbb{I} + \sigma'(\mu)\mathbf{n}\otimes(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\big)\Big)\cdot\mathbf{u}\,da(\mathbf{y}) \\
&\quad= -\int_{\partial_2\mathcal{B}} \sigma'(\mu)\Big(|\operatorname{grad}_s(\mathbf{u}\cdot\mathbf{n})|^2 - 2(\operatorname{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}\cdot(\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan} - |(\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan}|^2\Big)\,da(\mathbf{y}) \\
&\qquad\qquad - \int_{\partial_2\mathcal{B}} \sigma''(\mu)\mu(\operatorname{div}_s\mathbf{u})^2\,da(\mathbf{y})\ . \qquad (3.13a)
\end{aligned}
$$

We now note that the integrand corresponding to the third line of (3.12) may be written as

$$
\begin{aligned}
\big(\operatorname{grad}_s\mathbf{n}\cdot\operatorname{grad}_s\mathbf{u}\big)\mathbf{u}\cdot\mathbf{n} &= \Big(\operatorname{div}_s\big((\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan}\big) - \operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big)\cdot\mathbf{u}\Big)\mathbf{u}\cdot\mathbf{n} \\
&= \Big(\operatorname{div}_s\big((\operatorname{grad}_s\mathbf{n})\mathbf{u}^{\tan}\big)\Big)\mathbf{u}\cdot\mathbf{n} - \Big(\operatorname{grad}_s\big(\operatorname{div}_s\mathbf{n}\big)\cdot\mathbf{u}^{\tan}\Big)\mathbf{u}\cdot\mathbf{n} + |\operatorname{grad}_s\mathbf{n}|^2(\mathbf{n}\cdot\mathbf{u})^2\ ,
\end{aligned}
$$

wherein we have used the symmetry of $\operatorname{grad}_s\mathbf{n}$, and the identity[2] $\operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big) = \operatorname{grad}_s\big(\operatorname{div}_s\mathbf{n}\big) - |\operatorname{grad}_s\mathbf{n}|^2\mathbf{n}$. Further reorganization leads to

[2]To see this, first note that

$$
\operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big) = \mathbb{I}\operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big) + \Big(\big(\operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big)\big)\cdot\mathbf{n}\Big)\mathbf{n}\ .
$$

But, we also have that

$$
\big(\operatorname{div}_s\big(\operatorname{grad}_s\mathbf{n}\big)\big)\cdot\mathbf{n} = \operatorname{div}_s\big((\operatorname{grad}_s^{\mathrm{T}}\mathbf{n})\mathbf{n}\big) - |\operatorname{grad}_s\mathbf{n}|^2\ ,
$$

$$\big(\mathrm{grad}_s\mathbf{n}\cdot\mathrm{grad}_s\mathbf{u}\big)\mathbf{u}\cdot\mathbf{n}=\mathrm{div}_s\Big((\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}(\mathbf{u}\cdot\mathbf{n})-(\mathrm{div}_s\mathbf{n})(\mathbf{u}\cdot\mathbf{n})\mathbf{u}^{\tan}\Big)$$
$$-|(\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}|^2-(\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}\cdot(\mathrm{grad}_s\mathbf{u})^{\mathrm{T}}\mathbf{n}$$
$$+(\mathrm{div}_s\mathbf{n})\mathrm{div}_s\big((\mathbf{u}\cdot\mathbf{n})\mathbf{u}^{\tan}\big)+|\mathrm{grad}_s\mathbf{n}|^2(\mathbf{n}\cdot\mathbf{u})^2\ . \tag{3.13b}$$

As a third observation in the rearrangement of (3.12), it is helpful to rewrite the integrand in the last line of (3.12) as

$$\mathrm{div}_s\mathbf{u}\big(\mathrm{div}_s\big(\sigma'(\mu)\mathbb{I}\big)\big)\cdot\mathbf{u}=(\mathrm{div}_s\mathbf{u})\mathrm{grad}_s\sigma'(\mu)\cdot\mathbf{u}-\sigma'(\mu)(\mathrm{div}_s\mathbf{n})(\mathrm{div}_s\mathbf{u})\mathbf{n}\cdot\mathbf{u}\ . \tag{3.13c}$$

If we now substitute (3.13a,b,c) into (3.12), again recognize the symmetry of $\mathrm{grad}_s\mathbf{n}$, and combine terms, we obtain

$$\mathsf{J}(0)=-\int_{\partial_2\mathcal{B}}\Big(\sigma''(\mu)\mu(\mathrm{div}_s\mathbf{u})^2+\sigma'(\mu)\Big(|\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})|^2-2(\mathrm{div}_s\mathbf{n})\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})\cdot\mathbf{u}^{\tan}$$
$$+(\mathrm{div}_s\mathbf{n})(\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}\cdot\mathbf{u}^{\tan}+\big((\mathrm{div}_s\mathbf{n})^2-|\mathrm{grad}_s\mathbf{n}|^2\big)(\mathbf{u}\cdot\mathbf{n})^2$$
$$-\mathrm{div}_s\big((\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}(\mathbf{u}\cdot\mathbf{n})-(\mathrm{div}_s\mathbf{n})(\mathbf{u}\cdot\mathbf{n})\mathbf{u}^{\tan}\big)\Big)$$
$$-\mathrm{grad}_s\sigma'(\mu)\cdot\big((\mathrm{div}_s\mathbf{u})\mathbf{u}+(\mathrm{grad}_s\mathbf{u})\mathbf{u}\big)\Big)\,da(\mathbf{y})\ . \tag{3.14}$$

There are two final rearrangements of (3.14) that are worth observing. First, we note that the Gaussian curvature $\det(\mathrm{grad}_s\mathbf{n})$ satisfies the well-known relation $(\mathrm{div}_s\mathbf{n})^2-|\mathrm{grad}_s\mathbf{n}|^2=2\det(\mathrm{grad}_s\mathbf{n})$, and this may be used to simplify a term in the second line of (3.14). Second, we note that by a reorganization of terms the explicit dependence on $\mathrm{grad}_s\sigma'(\mu)$ in the last line of (3.14) may be eliminated. To see this, we first recall the elementary identities

$$\mathrm{grad}_s\mathbf{u}=\mathrm{grad}_s\big(\mathbf{u}^{\tan}+(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\big)=\mathrm{grad}_s\mathbf{u}^{\tan}+(\mathbf{u}\cdot\mathbf{n})\mathrm{grad}_s\mathbf{n}+\mathbf{n}\otimes\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})$$

---

where the first term on the right hand side vanishes because $\mathbf{n}$ is a unit vector field. Moreover, a straightforward calculation using the symmetry of $\mathrm{grad}_s\mathbf{n}$ shows that

$$\mathbb{I}\mathrm{div}_s\big(\mathrm{grad}_s\mathbf{n}\big)=\mathrm{grad}_s\big(\mathrm{div}_s\mathbf{n}\big)\ ,$$

which then gives

$$\mathrm{div}_s\big(\mathrm{grad}_s\mathbf{n}\big)=\mathrm{grad}_s\big(\mathrm{div}_s\mathbf{n}\big)-|\mathrm{grad}_s\mathbf{n}|^2\mathbf{n}\ .$$

and

$$\mathrm{div}_s\mathbf{u} = \mathrm{tr}(\mathrm{grad}_s\mathbf{u}) = \mathrm{div}_s\mathbf{u}^{\tan} + (\mathbf{u}\cdot\mathbf{n})\mathrm{div}_s\mathbf{n}\ .$$

Then, it follows that

$$\begin{aligned}
&\mathrm{grad}_s\sigma'(\mu)\cdot\big((\mathrm{div}_s\mathbf{u})\mathbf{u} + (\mathrm{grad}_s\mathbf{u})\mathbf{u}\big)\\
&\quad = \mathrm{grad}_s\sigma'(\mu)\cdot\big((\mathrm{div}_s\mathbf{u})\mathbf{u}^{\tan} + (\mathrm{grad}_s\mathbf{u})\mathbf{u}^{\tan}\big)\\
&\quad = \mathrm{grad}_s\sigma'(\mu)\cdot\big(\mathbb{I}(\mathrm{grad}_s\mathbf{u}^{\tan})\mathbf{u}^{\tan} + (\mathbf{u}\cdot\mathbf{n})(\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}\\
&\qquad\qquad - \mathbf{u}^{\tan}\mathrm{div}_s\mathbf{u}^{\tan} - \mathbf{u}^{\tan}(\mathbf{u}\cdot\mathbf{n})\mathrm{div}_s\mathbf{n}\big)\ .
\end{aligned}$$

When these last two observations are substituted into (3.14) and the surface divergence theorem is used (with $\mathbf{u} = \mathbf{0}$ on the border of $\partial_2\mathcal{B}$), we find the following final form for the second spatial variation (3.8):

$$\begin{aligned}
\frac{d^2}{d\lambda^2}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0)_{|\lambda=0} \;=\;& \int_{\mathcal{B}_0} W_{\mathbf{FF}}(\nabla\mathbf{y})[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}\,dv(\mathbf{x}) + \int_{\partial_2\mathcal{B}}\sigma''(\mu)\mu(\mathrm{div}_s\mathbf{u})^2 da(\mathbf{y})\\
&+ \int_{\partial_2\mathcal{B}}\sigma'(\mu)\Big(|\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})|^2 + 2\det(\mathrm{grad}_s\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^2\\
&\qquad +\mathrm{div}_s\mathbf{n}\left((\mathrm{grad}_s\mathbf{n})\,\mathbf{u}^{\tan}\cdot\mathbf{u}^{\tan} - 2\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})\cdot\mathbf{u}^{\tan}\right)\Big)da(\mathbf{y})\\
&+ \int_{\partial_2\mathcal{B}}\sigma'(\mu)\mathrm{div}_s\Big(2(\mathrm{div}_s\mathbf{n})(\mathbf{u}\cdot\mathbf{n})\mathbf{u}^{\tan} - 2(\mathbf{u}\cdot\mathbf{n})(\mathrm{grad}_s\mathbf{n})\mathbf{u}^{\tan}\\
&\qquad -\mathbb{I}(\mathrm{grad}_s\mathbf{u}^{\tan})\mathbf{u}^{\tan} + \mathbf{u}^{\tan}\mathrm{div}_s\mathbf{u}^{\tan}\Big)da(\mathbf{y})\ . \qquad (3.16)
\end{aligned}$$

Notice that if the surface stretch μ is constant on $\partial_2\mathcal{B}$, then, because of the surface divergence theorem and the fact that $\mathbf{u}$ vanishes on the border of $\partial_2\mathcal{B}$, i.e., on $\partial_1\mathcal{B}$, the last integral in (3.16) is zero. Later, we give an example where this condition applies. The field $\mathbf{y}(\mathbf{x})$ that satisfies the first variation conditions (3.7), with $\mu(\mathbf{x}) = |(\mathrm{cof}\ \nabla\mathbf{y})(\mathbf{x})\mathbf{n}_0(\mathbf{x})|$, is said to be stable with respect to spatial variations if the second variation (3.16) is positive. We shall assume that there exists a smooth solution $\mathbf{y}(\cdot)$ of (3.7) that is stable in this sense.

3.3 The First Reformation Variation

We now consider the question of the stability of the field $\mathbf{y}(\cdot)$ with respect to 'reformation' of the reference configuration. In rough terms, if the natural reference configuration of a body is suitably reformed into a neighboring natural reference configuration, is the reformed body more or less stable to the action of 'equivalent' boundary conditions?

First, we define the meaning of a 'suitable' reformation: Suppose that the reference configuration is changed from $\mathcal{B}_0$ to a neighboring reference configuration $\mathcal{B}_0(\varepsilon)$ having the same homogeneous mass density ρ_0 via a mapping

$$\mathbf{z} = \mathbf{z}_\varepsilon(\mathbf{x}) := \mathbf{x} + \varepsilon \mathbf{v}(\mathbf{x}) \ , \tag{3.17a}$$

where $\mathbf{v}(\mathbf{x})$ satisfies

$$\mathbf{v}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \partial_1 \mathcal{B}_0 \tag{3.17b}$$

and[3]

$$\mathrm{vol}(\mathcal{B}_0(\varepsilon)) = \int_{\mathcal{B}_0(\varepsilon)} dv(\mathbf{z}) = \int_{\mathcal{B}_0} |\det \nabla \mathbf{z}_\varepsilon| dv(\mathbf{x}) = \int_{\mathcal{B}_0} dv(\mathbf{x}) = \mathrm{vol}(\mathcal{B}_0) \ . \tag{3.17c}$$

We refer to such a variation as a 'reformation' of the body from $\mathcal{B}_0$ to $\mathcal{B}_0(\varepsilon) = \mathbf{z}_\varepsilon(\mathcal{B}_0)$ because the particles of the two bodies have the same mass density but are reorganized in a way compatible with the fixed placement boundary condition on $\partial_1 \mathcal{B}_0$. A 'reformation' of $\mathcal{B}_0$ is not to be considered as a *deformation* of $\mathcal{B}_0$ in the sense of mechanics. It is simply a mapping between two natural reference configurations of two bodies made out of the same material: The reference placement of part of the boundary of $\mathcal{B}_0(\varepsilon)$ is specified, i.e., $\partial_1 \mathcal{B}_0(\varepsilon) = \partial_1 \mathcal{B}_0$, and the remainder of the boundary, as well as the interior of $\mathcal{B}_0(\varepsilon)$ is governed by the variation (3.17a).

Now, to define 'equivalent' boundary conditions, suppose we let $\boldsymbol{\xi}(\mathbf{z})$ denote a smooth injective deformation of $\mathcal{B}_0(\varepsilon) \mapsto \mathcal{B}(\varepsilon) = \boldsymbol{\xi}(\mathcal{B}_0(\varepsilon))$ such that $\partial_1 \mathcal{B}_0(\varepsilon) = \mathbf{z}_\varepsilon(\partial_1 \mathcal{B}_0) = \partial_1 \mathcal{B}_0$ has the same given placement as $\partial_1 \mathcal{B}_0$, i.e.,

$$\boldsymbol{\xi}(\mathbf{z}) = \mathbf{y}^*(\mathbf{z}) \quad \forall \mathbf{z} \in \partial_1 \mathcal{B}_0(\varepsilon) = \partial_1 \mathcal{B}_0 \ . \tag{3.17d}$$

We let $\mathcal{A}(\varepsilon)$ denote the set of all such admissible fields $\boldsymbol{\xi}(\cdot) : \mathcal{B}_0(\varepsilon) \mapsto \mathcal{B}(\varepsilon)$. The remainder of the boundary $\partial_2 \mathcal{B}_0(\varepsilon) = \partial \mathcal{B}_0(\varepsilon) \setminus \partial_1 \mathcal{B}_0(\varepsilon)$ is supposed to be traction-free, as was $\partial_2 \mathcal{B}_0$. Again, with neglect of body force actions, for any admissible deformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{z})$, the total stored energy in $\mathcal{B}(\varepsilon)$ is

[3]Note that because the mass densities of both reference configurations are equal to the constant value ρ_0, both being constructed of the same homogeneous material, the requirement (3.17c) states that in a 'reformation' we must have $\mathrm{mass}(\mathcal{B}_0(\varepsilon)) = \mathrm{mass}(\mathcal{B}_0)$.

$$\mathsf{E}(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)) = \int_{\mathcal{B}_0(\varepsilon)} W(\mathrm{D}\boldsymbol{\xi})dv(\mathbf{z}) + \int_{\partial_2\mathcal{B}_0(\varepsilon)} \sigma(\nu)da(\mathbf{z}) \ , \tag{3.18}$$

where D denotes the gradient with respect to $\mathbf{z}$, $\nu = \nu(\mathbf{z}) := |(\mathrm{cof}\ \mathrm{D}\boldsymbol{\xi})\mathbf{m}_0|$ is the surface stretch under the deformation $\boldsymbol{\xi}(\mathbf{z})$ and $\mathbf{m}_0(= \mathbf{m}_0(\varepsilon))$ denotes the outer unit normal to $\partial_2\mathcal{B}_0(\varepsilon)$. Clearly, $\mathbf{m}_0(0) = \mathbf{n}_0$.

Now suppose, for the moment, that ε is fixed and that $\bar{\boldsymbol{\xi}}(\mathbf{z})$ is a smooth admissible minimizer of the functional $\mathsf{E}(\cdot;\mathcal{B}_0(\varepsilon))$ in (3.18) in the sense of spatial variational calculus, as discussed above in the two previous sub-sections. Then, conditions similar to (3.7) apply for this minimizer and the second variation is calculated similar to (3.16), and is supposed to be positive. One need only replace $\mathbf{x}$, $\mathbf{y}(\mathbf{x})$, $\mathbf{n}_0$ and $\mathcal{B}_0$ in (3.7) and (3.16) with $\mathbf{z}$, $\bar{\boldsymbol{\xi}}(\mathbf{z})$, $\mathbf{m}_0(\varepsilon)$ and $\mathcal{B}_0(\varepsilon)$, respectively. In addition, the operator ∇ must be replaced by D and grad_s and div_s must be interpreted as the surface gradient and surface divergence on $\partial_2\mathcal{B}(\varepsilon) = \bar{\boldsymbol{\xi}}(\partial_2\mathcal{B}_0(\varepsilon))$. We assume that such a minimizing field $\bar{\boldsymbol{\xi}}(\mathbf{z})$, for $\mathbf{z} \in \mathcal{B}_0(\varepsilon)$, exists for each ε in a neighborhood of 0. Of course, if $\varepsilon = 0$ then $\mathcal{B}_0(0) = \mathcal{B}_0$ and $\mathcal{B}(0) = \mathcal{B}$ and the results of (3.7) and the positiveness of (3.16) are recovered. Thus, we know that

$$\bar{\boldsymbol{\xi}}(\mathbf{z}_\varepsilon(\mathbf{x}))_{|\varepsilon=0} = \bar{\boldsymbol{\xi}}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) \ , \quad \nu(\mathbf{z}_\varepsilon(\mathbf{x}))_{|\varepsilon=0} = \mu(\mathbf{x}) \ .$$

Now, following a line of investigation outlined in Section 2, it is convenient to introduce $\mathbf{y}^{\mathrm{E}}(\mathbf{z})$ as a smooth extension of the minimizer $\mathbf{y}(\cdot)$ of (3.1) for all $\mathbf{z}$ in an open neighborhood $\mathcal{N}(\mathcal{B}_0)$ of $\mathcal{B}_0$, such that $\mathbf{y}^{\mathrm{E}}(\mathbf{z}) = \mathbf{y}(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{B}_0 \cup \partial\mathcal{B}_0$. Then, we have that *for each sufficiently small* $\varepsilon > 0$,

$$\boldsymbol{\xi}(\mathbf{z}) := \mathbf{y}^{\mathrm{E}}(\mathbf{z}) \in \mathcal{A}(\varepsilon) \quad \forall\, \mathbf{z} \in \mathcal{B}_0(\varepsilon) \subset \mathcal{N}(\mathcal{B}_0). \tag{3.19}$$

Then, because $\boldsymbol{\xi}(\mathbf{z})$ is in the admissible class $\mathcal{A}(\varepsilon)$, we see that

$$\mathsf{E}\big(\bar{\boldsymbol{\xi}};\mathcal{B}_0(\varepsilon)\big) = \min_{\boldsymbol{\phi}(\cdot)\in\mathcal{A}(\varepsilon)} \mathsf{E}\big(\boldsymbol{\phi};\mathcal{B}_0(\varepsilon)\big) \le \mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big),$$

and from (3.18) we see that

$$\mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big)_{|\varepsilon=0} = \mathsf{E}\big(\mathbf{y}^{\mathrm{E}};\mathcal{B}_0\big) = \mathsf{E}\big(\mathbf{y};\mathcal{B}_0\big).$$

Thus, considering $\mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big)$ as constructed from (3.19) and (3.18), we see that *if*

$$\frac{d}{d\varepsilon}\mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big)_{|\varepsilon=0}=0 \quad\text{and}\quad \frac{d^2}{d\varepsilon^2}\mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big)_{|\varepsilon=0}<0, \tag{3.20a}$$

then

$$\mathsf{E}\big(\bar{\boldsymbol{\xi}};\mathcal{B}_0(\varepsilon)\big)<\mathsf{E}\big(\boldsymbol{\xi};\mathcal{B}_0(\varepsilon)\big)\le\mathsf{E}\big(\mathbf{y};\mathcal{B}_0\big), \tag{3.20b}$$

for sufficiently small $\varepsilon>0$. This implies that the reference configuration $\mathcal{B}_0$ is not stable with respect to reformation and it provides a sufficient condition for reformation instability. In the remainder of work, we follow this approach and develop a sufficient condition for reformation instability.

We now suppose, for any given ε and $\mathbf{z}_\varepsilon(\mathbf{x})$ as characterized in (3.17) and for $\boldsymbol{\xi}(\cdot)$ as defined in (3.19), that $\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon(\mathbf{x}):=\boldsymbol{\xi}(\mathbf{z})_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}$ is determined as outlined above. Then, to investigate the effect of the 'reformation' variation (3.17) on the stability of the minimizing solution $\mathbf{y}(\mathbf{x})$, it is convenient to first rewrite the functional (3.18) in the form

$$\begin{aligned}\mathsf{E}(\xi\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))=&\int_{\mathcal{B}_0}W(\mathrm{D}\boldsymbol{\xi}(\mathbf{z}))_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}(\det\nabla\mathbf{z}_\varepsilon)dv(\mathbf{x})\\&+\int_{\partial_2\mathcal{B}_0}\sigma(\nu(\mathbf{z}))_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}|(\operatorname{cof}\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0|da(\mathbf{x})\ ,\end{aligned}$$

where, recall, $\nu(\mathbf{z})=|(\operatorname{cof}\mathrm{D}\boldsymbol{\xi}(\mathbf{z})\mathbf{m}_0(\mathbf{z})|$, and where from the mapping $\mathcal{B}_0\mapsto\mathcal{B}_0(\varepsilon)$ we have used $da(\mathbf{z})=|(\operatorname{cof}\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0|da(\mathbf{x})$. This is in a form that most clearly exposes how the total stored energy (3.18) depends on the reformation parameter ε for the field $\boldsymbol{\xi}(\cdot)$ of (3.19). Our approach is to develop the conditions that require the first (reformation) variation of $\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))$ to vanish and the second (reformation) variation to be negative at $\varepsilon=0$. Satisfaction of these conditions then implies that the minimizing solution $\mathbf{y}(\mathbf{x})$, discussed earlier, is unstable with respect to referential reformation.

In the following we use $\dot{\overline{(\cdot)}}$ to denote the operation $\partial(\cdot)/\partial\varepsilon$ holding $\mathbf{x}$ fixed. Thus, assuming smoothness for all fields so that the order of differentiation and integration can be interchanged, we find that the first reformation variation has the form

$$\begin{aligned}\frac{d}{d\varepsilon}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))=&\int_{\mathcal{B}_0(\varepsilon)}\big(\mathrm{D}W(\mathrm{D}\boldsymbol{\xi})\cdot\dot{\mathbf{z}}_\varepsilon+W(\mathrm{D}\boldsymbol{\xi})\,\mathrm{D}\cdot\dot{\mathbf{z}}_\varepsilon\big)\,dv(\mathbf{z})\\&+\int_{\partial_2\mathcal{B}_0(\varepsilon)}\big(\mathrm{D}_s\sigma(\nu)\cdot\dot{\mathbf{z}}_\varepsilon+\sigma(\nu)\mathrm{D}_s\cdot\dot{\mathbf{z}}_\varepsilon\big)\,da(\mathbf{z})\ .\end{aligned}$$

Here, we have used the well-known identities

$$\overline{\det \nabla \mathbf{z}_\varepsilon}^{\boldsymbol{\cdot}} = (\det \nabla \mathbf{z}_\varepsilon)\mathrm{D} \cdot \dot{\mathbf{z}}_\varepsilon \quad \text{and} \quad \overline{|(\mathrm{cof}\, \nabla \mathbf{z}_\varepsilon)\mathbf{n}_0|}^{\boldsymbol{\cdot}} = |(\mathrm{cof}\, \nabla \mathbf{z}_\varepsilon)\mathbf{n}_0| \mathrm{D}_s \cdot \dot{\mathbf{z}}_\varepsilon \ ,$$

where $\mathrm{D} \cdot (\cdot)$ is the divergence in $\mathcal{B}_0(\varepsilon)$ and $\mathrm{D}_s \cdot (\cdot)$ denotes the surface divergence on $\partial\mathcal{B}_0(\varepsilon)$. Clearly, the integrand of the volume integral above is equivalent to the divergence $\mathrm{D} \cdot \big(W(\boldsymbol{\xi})\dot{\mathbf{z}}_\varepsilon\big)$ and, noting that $\dot{\mathbf{z}}_\varepsilon = \dot{\mathbf{z}}_\varepsilon^{\tan} + (\dot{\mathbf{z}}_\varepsilon \cdot \mathbf{m}_0)\mathbf{m}_0$, the integrand of the surface integral above is equivalent to $\mathrm{D}_s \cdot (\sigma(\nu)\dot{\mathbf{z}}_\varepsilon^{\tan}) + \sigma(\nu)(\mathrm{D}_s \cdot \mathbf{m}_0)(\dot{\mathbf{z}}_\varepsilon \cdot \mathbf{m}_0)$. Thus, applying the volume and surface divergence theorems and using the fact that $\dot{\mathbf{z}}_\varepsilon \equiv \mathbf{v}$ and $\mathbf{v}(\mathbf{x}) = \mathbf{0}\ \forall\ \ \mathbf{x} \in \partial_1\mathcal{B}_0$, we obtain

$$\frac{d}{d\varepsilon}\mathsf{E}(\boldsymbol{\xi} \circ \mathbf{z}_\varepsilon; \mathcal{B}_0(\varepsilon)) = \int_{\partial_2\mathcal{B}_0(\varepsilon)} \big(W(\mathrm{D}\boldsymbol{\xi}) + \sigma(\nu)(\mathrm{D}_s \cdot \mathbf{m}_0)\big)(\mathbf{v} \cdot \mathbf{m}_0) da(\mathbf{z}) \ . \tag{3.21}$$

In addition to this, it is straightforward to see that the 'reformation' constraint of (3.17b,c), together with $\dot{\mathbf{z}}_\varepsilon \equiv \mathbf{v}$, requires

$$\frac{d}{d\varepsilon}\mathrm{vol}(\mathcal{B}_0(\varepsilon)) = \int_{\partial_2\mathcal{B}_0(\varepsilon)} (\mathbf{v} \cdot \mathbf{m}_0) da(\mathbf{z}) = 0 \ . \tag{3.22}$$

Now, according to the second of (3.20a), the first reformation variation given by (3.21) evaluated at $\varepsilon = 0$ must vanish for all admissible $\mathbf{v}$, i.e.,

$$\frac{d}{d\varepsilon}\mathsf{E}(\boldsymbol{\xi} \circ \mathbf{z}_\varepsilon; \mathcal{B}_0(\varepsilon))_{|\varepsilon=0} = \int_{\partial_2\mathcal{B}_0} \big(W(\nabla\mathbf{y}) + \sigma(\mu)(\mathrm{Div}_s\mathbf{n}_0)\big)(\mathbf{v} \cdot \mathbf{n}_0) da(\mathbf{x}) = 0 \ , \tag{3.23a}$$

for all $\mathbf{v}(\mathbf{x})$ that vanish for all $\mathbf{x} \in \partial_1\mathcal{B}_0$ and, according to (3.22) evaluated at $\varepsilon = 0$, also meet

$$\int_{\partial_2\mathcal{B}_0} (\mathbf{v} \cdot \mathbf{n}_0) da(\mathbf{x}) = 0 \ . \tag{3.23b}$$

In the evaluation of (3.21) and (3.22) at $\varepsilon = 0$, it is clear from earlier observations how the various symbols and operators are mapped. In particular, note that in (3.23a), and in the sequel, we use Div_s to denote the surface divergence operator on $\partial_2\mathcal{B}_0$.

Clearly, (3.23) implies that

$$W(\nabla\mathbf{y}) + \sigma(\mu)(\mathrm{Div}_s\mathbf{n}_0) = \text{const.} \quad \forall\mathbf{x} \in \partial_2\mathcal{B}_0 \ . \tag{3.24}$$

We shall return to the first reformation variation condition (3.24) in an example later in Section 4.

3.4 The Second Reformation Variation

Now, to investigate the second variation condition (3.20b) for the instability of $\mathbf{y}(\mathbf{x})$ with respect to reformation of $\mathcal{B}_0$, we first observe, from (3.21) with a domain of integration change, that

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon)) = &\int_{\partial_2\mathcal{B}_0} \overset{\cdot}{\overline{\big(W(\mathrm{D}\boldsymbol{\xi}) + \sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}}}\mathbf{v}\cdot(\mathrm{cof}\,\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0\,da(\mathbf{x})\\ &+\int_{\partial_2\mathcal{B}_0}\big(W(\mathrm{D}\boldsymbol{\xi}) + \sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}\mathbf{v}\cdot\overset{\cdot}{\overline{(\mathrm{cof}\,\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0}}\,da(\mathbf{x}) \ .\end{aligned} \tag{3.25}$$

Analogously, the constraint (3.22), yields

$$\frac{d^2}{d\varepsilon^2}\mathrm{vol}(\mathcal{B}_0(\varepsilon)) = \int_{\partial_2\mathcal{B}_0}\mathbf{v}\cdot\overset{\cdot}{\overline{(\mathrm{cof}\,\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0}}\,da(\mathbf{x}) = 0 \ .$$

However, because we are eventually interested in the second variation evaluated at $\varepsilon = 0$, and because (3.24) must hold, we see that the second line in (3.25) will cancel after this evaluation and, therefore, we may concentrate our attention solely on the first line in (3.25). Thus, defining

$$\mathsf{K}(\varepsilon) := \int_{\partial_2\mathcal{B}_0} \overset{\cdot}{\overline{\big(W(\mathrm{D}\boldsymbol{\xi}) + \sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)_{|\mathbf{z}=\mathbf{z}_\varepsilon(\mathbf{x})}}}\mathbf{v}\cdot(\mathrm{cof}\,\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0\,da(\mathbf{x}) \tag{3.26}$$

we see that

$$\frac{d^2}{d\varepsilon^2}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))_{|\varepsilon=0} = \mathsf{K}(0) \ , \tag{3.27}$$

and we shall be interested in determining $\mathsf{K}(0)$.

With the view toward simplifying $\mathsf{K}(\varepsilon)$, we first observe that a change of domain of integration using $(\mathrm{cof}\,\nabla\mathbf{z}_\varepsilon)\mathbf{n}_0\,da(\mathbf{x}) = \mathbf{m}_0\,da(\mathbf{z})$ supports the representation

$$\mathsf{K}(\varepsilon) = \int_{\partial_2 \mathcal{B}_0(\varepsilon)} \Big(\mathrm{D}W(\mathrm{D}\boldsymbol{\xi}) \cdot \dot{\mathbf{z}}_\varepsilon + \mathrm{D}_s \sigma(\nu) \cdot \dot{\mathbf{z}}_\varepsilon \big(\mathrm{D}_s \cdot \mathbf{m}_0 \big)$$
$$+ \sigma(\nu) \big(-\mathrm{D}_s \cdot \big(\mathrm{D}_s (\dot{\mathbf{z}}_\varepsilon \cdot \mathbf{m}_0) \big) + \big(\mathrm{D}_s \cdot (\mathrm{D}_s \mathbf{m}_0) \big) \cdot \dot{\mathbf{z}}_\varepsilon \big) \Big) \mathbf{v} \cdot \mathbf{m}_0 \, da(\mathbf{z}) \; .$$

Here, we have used $\dot{W}(\mathrm{D}\boldsymbol{\xi}) = \mathrm{D}W(\mathrm{D}\boldsymbol{\xi}) \cdot \dot{\mathbf{z}}_\varepsilon$ and $\dot{\sigma}(\nu) = \mathrm{D}_s\sigma(\nu) \cdot \dot{\mathbf{z}}_\varepsilon$, and we have employed the identity

$$\dot{\overline{\mathrm{D}_s \cdot \mathbf{m}_0}} = -\mathrm{D}_s \cdot \big(\mathrm{D}_s (\dot{\mathbf{z}}_\varepsilon \cdot \mathbf{m}_0) \big) + \big(\mathrm{D}_s \cdot (\mathrm{D}_s \mathbf{m}_0) \big) \cdot \dot{\mathbf{z}}_\varepsilon \; ,$$

which results from the following argument: First, we recall the well known identity $\dot{\mathbf{m}}_0 = -(\mathrm{D}_s \dot{\mathbf{z}}_\varepsilon)^{\mathrm{T}} \mathbf{m}_0$ for the unit normal field $\mathbf{m}_0 = \mathbf{m}_0(\varepsilon)$ on $\partial \mathcal{B}_0(\varepsilon)$, and the chain rule $\nabla_s \mathbf{m}_0 = (\mathrm{D}_s \mathbf{m}_0) \nabla \mathbf{z}_\varepsilon$, where ∇_s denotes the surface gradient in $\partial \mathcal{B}_0$. Then,

$$\dot{\overline{\nabla_s \mathbf{m}_0}} = \dot{\overline{\mathrm{D}_s \mathbf{m}_0}} \nabla \mathbf{z}_\varepsilon + (\mathrm{D}_s \mathbf{m}_0) \dot{\overline{\nabla \mathbf{z}_\varepsilon}} \; ,$$

wherein we may use $\dot{\overline{\nabla \mathbf{z}_\varepsilon}} = (\mathrm{D}\dot{\mathbf{z}}_\varepsilon) \nabla \mathbf{z}_\varepsilon$ to obtain

$$\dot{\overline{\mathrm{D}_s \mathbf{m}_0}} = \dot{\overline{\nabla_s \mathbf{m}_0}} (\nabla \mathbf{z}_\varepsilon)^{-1} - (\mathrm{D}_s \mathbf{m}_0)(\mathrm{D}\dot{\mathbf{z}}_\varepsilon) \; .$$

But, because $\dot{\overline{\nabla_s \mathbf{m}_0}} = \nabla_s \dot{\mathbf{m}}_0$ and because the chain rule yields

$$\nabla_s \dot{\mathbf{m}}_0 = -\nabla_s \big((\mathrm{D}_s \dot{\mathbf{z}}_\varepsilon)^{\mathrm{T}} \mathbf{m}_0(\varepsilon) \big) = -\mathrm{D}_s \big((\mathrm{D}_s \dot{\mathbf{z}}_\varepsilon)^{\mathrm{T}} \mathbf{m}_0(\varepsilon) \big) \nabla \mathbf{z}_\varepsilon \; ,$$

we see that

$$\dot{\overline{\mathrm{D}_s \mathbf{m}_0}} = -\mathrm{D}_s \big((\mathrm{D}_s \dot{\mathbf{z}}_\varepsilon)^{\mathrm{T}} \mathbf{m}_0(\varepsilon) \big) - (\mathrm{D}_s \mathbf{m}_0)(\mathrm{D}\dot{\mathbf{z}}_\varepsilon) \; .$$

Finally, using $(\mathrm{D}_s \dot{\mathbf{z}}_\varepsilon) = (\mathrm{D}\dot{\mathbf{z}}_\varepsilon)(\mathbf{1} - \mathbf{m}_0 \otimes \mathbf{m}_0)$ and $\mathrm{D}_s \cdot \mathbf{m}_0 = \mathrm{tr}\,(\mathrm{D}_s \mathbf{m}_0)$, we readily find the identity

$$\dot{\overline{\mathrm{D}_s \cdot \mathbf{m}_0}} = -\mathrm{D}_s \cdot \big(\mathrm{D}_s (\dot{\mathbf{z}}_\varepsilon \cdot \mathbf{m}_0) \big) + \big(\mathrm{D}_s \cdot (\mathrm{D}_s \mathbf{m}_0) \big) \cdot \dot{\mathbf{z}}_\varepsilon \; .$$

Now, according to the identity[4] $\mathrm{D}_s \cdot (\mathrm{D}_s \mathbf{m}_0) = \mathrm{D}_s(\mathrm{D}_s \cdot \mathbf{m}_0) - |\mathrm{D}_s \mathbf{m}_0|^2 \mathbf{m}_0$ and the replacement

[4]See footnote 2 for the development of this identity in a similar context.

$$\mathrm{D}_s\sigma(\nu)\cdot\dot{\mathbf{z}}_\varepsilon\big(\mathrm{D}_s\cdot\mathbf{m}_0\big)=\Big(\mathrm{D}_s\big(\sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)-\sigma(\nu)\mathrm{D}_s(\mathrm{D}_s\cdot\mathbf{m}_0)\Big)\cdot\dot{\mathbf{z}}_\varepsilon\ ,$$

we see that $\mathsf{K}(\varepsilon)$ may be written as

$$\begin{aligned}\mathsf{K}(\varepsilon)=\int_{\partial_2\mathcal{B}_0(\varepsilon)}&\Big(\mathrm{D}W(\mathrm{D}\boldsymbol{\xi})\cdot\dot{\mathbf{z}}_\varepsilon+\mathrm{D}_s\big(\sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)\cdot\dot{\mathbf{z}}_\varepsilon\\&-\sigma(\nu)\mathrm{D}_s\cdot\big(\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)\big)-\sigma(\nu)|\mathrm{D}_s\mathbf{m}_0|^2\mathbf{m}_0\cdot\dot{\mathbf{z}}_\varepsilon\Big)\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0\,da(\mathbf{z})\ .\end{aligned}$$

This can be further simplified by first noting that $\mathrm{D}_sW=(\mathbf{1}-\mathbf{m}_0\otimes\mathbf{m}_0)\mathrm{D}W=\mathrm{D}W-\big((\mathrm{D}W)\cdot\mathbf{m}_0\big)\mathbf{m}_0$, so that the first two terms in the integrand above can be rewritten as

$$\begin{aligned}&\mathrm{D}W(\mathrm{D}\boldsymbol{\xi})\cdot\dot{\mathbf{z}}_\varepsilon+\mathrm{D}_s\big(\sigma(\nu)(\mathrm{D}_s\cdot\mathbf{m}_0)\big)\cdot\dot{\mathbf{z}}_\varepsilon\\&\quad=\mathrm{D}_s\big(W+\sigma(\nu)(\mathrm{D}_s\mathbf{m}_0)\big)\cdot\dot{\mathbf{z}}_\varepsilon+\big((\mathrm{D}W)\cdot\mathbf{m}_0\big)\mathbf{m}_0\cdot\dot{\mathbf{z}}_\varepsilon\ .\end{aligned}$$

Then, noting that

$$\begin{aligned}\Big(\mathrm{D}_s\cdot\big(\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)\big)\Big)(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)&=\mathrm{D}_s\cdot\Big(\big(\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)\big)(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)\Big)-|\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)|^2\\&=\frac{1}{2}\mathrm{D}_s\cdot\big(\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)^2\big)-|\mathrm{D}_s(\dot{\mathbf{z}}_\varepsilon\cdot\mathbf{m}_0)|^2\ ,\end{aligned}$$

recalling (3.24) and (3.27), and evaluating at $\varepsilon=0$, we find the final form of the second reformation variation:

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))_{|\varepsilon=0}\ =\ &\int_{\partial_2\mathcal{B}_0}\Big(\nabla W(\nabla\mathbf{y})\cdot\mathbf{n}_0(\mathbf{v}\cdot\mathbf{n}_0)^2+\sigma(\mu)\Big(|\nabla_s(\mathbf{v}\cdot\mathbf{n}_0)|^2\\&-|\nabla_s\mathbf{n}_0|^2(\mathbf{v}\cdot\mathbf{n}_0)^2-\frac{1}{2}\mathrm{Div}_s\cdot\big(\nabla_s(\mathbf{v}\cdot\mathbf{n}_0)^2\big)\Big)\Big)da(\mathbf{x})\ .\end{aligned}\tag{3.28}$$

Thus, $\mathbf{y}(\mathbf{x})$ is said to be unstable with respect to 'reformation' if, in addition to (3.24), we require that (3.28) is negative for all $\mathbf{v}(\mathbf{x})$ that vanish on $\partial_1\mathcal{B}_0$ and meet the condition (3.23b). Notice that if $\mu=$ const. on $\partial_2\mathcal{B}_0$ then the surface divergence theorem and the fact that $\mathbf{v}(\mathbf{x})=\mathbf{0}$ on $\partial_1\mathcal{B}_0$ shows that the $\mathrm{Div}_s(\cdot)$ term in (3.28) integrates to zero. In this case, if the curvature of $\partial_2\mathcal{B}_0$ is 'large enough' (in the sense that $|\nabla_s\mathbf{n}_0|$ is sufficiently large on $\partial_2\mathcal{B}_0$), then the second reformation variation (3.28) is possibly negative for some admissible fields $\mathbf{v}$ and $\mathbf{y}(\mathbf{x})$ would then be unstable with respect to reformation. We shall consider an example for which this happens below.

4 Example

Consider an isotropic cylindrical wire which in its natural reference configuration $\mathcal{B}_0$ has length l_0 and radius b_0. A fixed right-handed orthonormal coordinate basis $\{\mathbf{i}_i \;\; i = 1, 2, 3\}$ and corresponding right-handed orthonormal cylindrical basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{i}_3\}$ is given such that the axis of the cylinder corresponds to the $\mathbf{i}_3$ base vector and corresponding coordinate line. In $\mathcal{B}_0$, $x_3 = 0$ is one end of the cylinder while $x_3 = l_0$ is the other. The deformation $\mathbf{y}(\mathbf{x})$ is given by

$$y_1 = \beta x_1 \;, \quad y_2 = \beta x_2 \;, \quad y_3 = \alpha x_3 \;,$$

the ends $\partial_1\mathcal{B}_0$ corresponding to that part of the boundary on which the displacement is assumed to be given, i.e., α is considered to be a given constant and β will be determined below in terms of α, and fixed. The lateral surface $\partial_2\mathcal{B}_0$ corresponds to the remainder of the boundary of $\mathcal{B}_0$, which is supposed to be traction-free. This deformation field maps $\mathcal{B}_0$ to $\mathcal{B} := \mathbf{y}(\mathcal{B}_0)$ which is a cylinder of radius $b = \beta b_0$ and length $l = \alpha l_0$. For an isotropic elastic body, it is well-known that the Cauchy stress $\mathbf{T}$ appearing in (3.7b) is given by

$$\mathbf{T} = \frac{2}{\det \nabla\mathbf{y}} \left[\mathrm{III}\frac{\partial \bar{W}}{\partial \mathrm{III}}\mathbf{1} + \left(\frac{\partial \bar{W}}{\partial \mathrm{I}} + \mathrm{I}\frac{\partial \bar{W}}{\partial \mathrm{II}} \right) \mathbf{B} - \frac{\partial \bar{W}}{\partial \mathrm{II}}\mathbf{B}^2 \right] \;,$$

where $\mathrm{I} := \mathrm{tr}\mathbf{B}$, $\mathrm{II} := (\mathrm{I}^2 - \mathrm{tr}\mathbf{B}^2)/2$, $\mathrm{III} \equiv \det \mathbf{B}$, $W = \bar{W}(\mathrm{I}, \mathrm{II}, \mathrm{III})$, and $\mathbf{B} \equiv \nabla\mathbf{y}(\nabla\mathbf{y})^{\mathrm{T}}$. Thus, with the homogeneous deformation above, it follows after some calculation that

$$\mathbf{T} = \frac{1}{2\alpha\beta}\frac{\partial \bar{\bar{W}}}{\partial \beta}\big(\mathbf{i}_1 \otimes \mathbf{i}_1 + \mathbf{i}_2 \otimes \mathbf{i}_2\big) + \frac{1}{\beta^2}\frac{\partial \bar{\bar{W}}}{\partial \alpha}\mathbf{i}_3 \otimes \mathbf{i}_3 \;, \tag{4.1}$$

where, now, by substitution we have written $W = \bar{\bar{W}}(\alpha, \beta)$. Of course, because the deformation is homogeneous the equilibrium equation (3.7a)is satisfied.

We now consider the traction-free boundary condition expressed in (3.7b). First, we note that the surface stretch $\mu = |(\mathrm{cof}\,\nabla\mathbf{y})\mathbf{n}_0|$ on $\partial_2\mathcal{B}_0$ is easily calculated as $\mu = \alpha\beta$, and is constant. Here, of course, we have used $\mathbf{n}_0 = \mathbf{e}_r$. Because $\mu = \mathrm{const.}$, it follows that $\mathrm{div}_s(\sigma'(\mu)\mathbb{I}) = -\sigma'(\mu)(\mathrm{div}_s\mathbf{n})\mathbf{n}$, in which we must recall that $\mathbf{n}$ is the outer unit normal to the lateral surface $\partial_2\mathcal{B}$ of the deformed cylinder. Thus, again we have $\mathbf{n} = \mathbf{e}_r$, but, here, $\mathbf{e}_r$ is evaluated on the deformed cylindrical surface $r = b$. For this surface, we have

$$\mathrm{grad}_s\mathbf{n} = \frac{\mathbf{e}_\theta \otimes \mathbf{e}_\theta}{b} \ , \qquad \mathrm{div}_s\mathbf{n} = \frac{1}{b} \ . \tag{4.2}$$

With (4.1), the boundary condition (3.7b) is expressed as

$$\frac{1}{2\alpha\beta}\frac{\partial \bar{\bar{W}}}{\partial \beta} + \frac{\sigma'(\mu)}{b} = 0 \quad \text{where } \mu = \alpha\beta \ . \tag{4.3}$$

Given the volume and surface energy functions $\bar{\bar{W}}(\alpha, \beta)$ and $\sigma(\mu)$, this equation is used to determine β in terms of the given constant axial extension α. Of course, we may set $b = \beta b_0$ in (4.3). We emphasize that for a given α and β thus determined, the ends of the cylinder $\mathcal{B}_0$ are considered to represent the displacement part of the boundary $\partial_1\mathcal{B}_0$.

Because $\mu = \text{const.}$, we may now write the second spatial variation (3.16) as

$$\begin{aligned} \frac{d^2}{d\lambda^2}\mathsf{E}(\mathbf{y}_\lambda; \mathcal{B}_0)_{|\lambda=0} \ = \ & \int_{\mathcal{B}_0} W_{\mathbf{FF}}(\nabla\mathbf{y})[\nabla\mathbf{u}] \cdot \nabla\mathbf{u} dv(\mathbf{x}) + \sigma''(\mu)\mu \int_{\partial_2\mathcal{B}} (\mathrm{div}_s\mathbf{u})^2 da(\mathbf{y}) \\ + \ & \sigma'(\mu) \int_{\partial_2\mathcal{B}} \Big(|\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})|^2 + 2\det(\mathrm{grad}_s\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^2 \\ & + \mathrm{div}_s\mathbf{n}\big((\mathrm{grad}_s\mathbf{n})\,\mathbf{u}^{\tan}\cdot\mathbf{u}^{\tan} - 2\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n})\cdot\mathbf{u}^{\tan}\big)\Big) da(\mathbf{y}) \ , \end{aligned}$$

which, for the stability with respect to spatial variations of $\mathbf{y}(\mathbf{x})$, should be positive for all admissible $\mathbf{u}$. To analyze this possibility we first concentrate on the surface integral terms in which we use (4.2) and make the replacements

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_3\mathbf{i}_3 \ , \qquad \mathbf{u}^{\tan} = u_\theta\mathbf{e}_\theta + u_3\mathbf{i}_3 \ ,$$

where, on $\partial_2\mathcal{B}$ the fields u_r, u_θ and u_3 are functions of θ and y_3. It is clear that on $\partial_2\mathcal{B}$ the element of area is $da(\mathbf{y}) = bd\theta dy_3$, and it is easy to show that

$$\mathrm{div}_s\mathbf{u} = \frac{1}{b}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right) + \frac{\partial u_3}{\partial y_3} \ , \tag{4.4a}$$

$$\mathrm{grad}_s(\mathbf{u}\cdot\mathbf{n}) = \frac{1}{b}\frac{\partial u_r}{\partial\theta}\mathbf{e}_\theta + \frac{\partial u_r}{\partial y_3}\mathbf{i}_3 \ . \tag{4.4b}$$

Thus, for sufficiently small $b = \beta b_0$, (small, say, relative to $l = \alpha l_0$) the second spatial variation is approximately given by

$$
\begin{aligned}
\frac{d^2}{d\lambda^2}\mathsf{E}(\mathbf{y}_\lambda;\mathcal{B}_0)_{|\lambda=0} &= \int_{\mathcal{B}_0} W_{\mathbf{FF}}(\nabla\mathbf{y})[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}\,dv(\mathbf{x}) \\
&+ \frac{\sigma''(\mu)\mu}{b}\int_0^l\int_0^{2\pi}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)^2 d\theta dy_3 \\
&+ \frac{\sigma'(\mu)}{b}\int_0^l\int_0^{2\pi}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)^2 d\theta dy_3 \\
&+ \text{terms of order } O(1) \text{ in } b \text{ as } b\to 0\,. \qquad (4.5)
\end{aligned}
$$

We expect that physically 'reasonable' surface energy functions $\sigma(\cdot)$ for solid-like bodies will be convex for a non-trivial range of surface stretches μ around the undistorted state where $\mu = 1$, and, thus, satisfy $\sigma''(\mu) > 0$ for $\mu \neq 1$, with $\sigma'(\mu) = 0$ for $\mu = 1$. In addition, for extension of the wire (i.e., $\alpha > 1$), if we assume that (4.3) yields a value of β (this quantity governs the lateral contraction or expansion of the wire) such that $\mu = \alpha\beta > 1$ for $\alpha > 1$, then, for uniform wires of sufficiently small radius, (4.5) shows that a homogeneous extension is stable with respect to spatial variations. Moreover, if the surface energy 'well' at $\mu = 1$ is strongly convex it may be that uniform wires of sufficiently small radius are also stable in this sense under homogeneous *compression*. Of course, definite conclusions are dictated by the specific forms of the volumetric and surfacial energy functions $\bar{\bar{W}}(\alpha,\beta)$ and $\sigma(\mu)$. This may have a bearing on understanding a possible novel characteristic behavior of special 'quantum (or nano-) wires'.

We now turn to the question of the stability of $\mathbf{y}(\mathbf{x})$ for the above homogeneous deformation with respect to 'reformation' of the reference configuration $\mathcal{B}_0$. This requires a consideration of (3.24) and (3.28). Clearly (3.24) holds because the deformation is homogeneous and, similar to (4.2), we have $\mathrm{Div}_s\mathbf{n}_0 = 1/b_0$ on the reference lateral boundary $\partial_2\mathcal{B}_0$. Note, also, that the first term in (3.28) containing $\nabla W((\nabla\mathbf{y}(\mathbf{x}))$ is zero because the deformation is homogeneous, and that $\sigma(\mu)$ is constant so that we may use the surface divergence theorem to eliminate the $\mathrm{Div}_s(\cdot)$ term in (3.28). Thus, for this example (3.28) becomes

$$
\frac{d^2}{d\varepsilon^2}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))_{|\varepsilon=0} = \sigma(\mu)\int_{\partial_2\mathcal{B}_0}\Big(|\nabla_s(\mathbf{v}\cdot\mathbf{n}_0)|^2 - |\nabla_s\mathbf{n}_0|^2(\mathbf{v}\cdot\mathbf{n}_0)^2\Big)da(\mathbf{x})\,,
$$

in which we may set $|\nabla_s\mathbf{n}_0|^2 = 1/b_0^2$ and $da(\mathbf{x}) = b_0 d\theta dx_3$. Knowing that $\mathbf{v}\cdot\mathbf{n}_0 = v_r(\theta, x_3)$ on $\partial_2\mathcal{B}_0$, we finally write this as

$$\frac{d^2}{d\varepsilon^2}\mathsf{E}(\boldsymbol{\xi}\circ\mathbf{z}_\varepsilon;\mathcal{B}_0(\varepsilon))_{|\varepsilon=0} = \frac{\sigma(\mu)}{b_0}\int_0^{l_0}\int_0^{2\pi}\Big[\Big(\frac{\partial v_r}{\partial\theta}\Big)^2 - {v_r}^2 + b_0^2\Big(\frac{\partial v_r}{\partial x_3}\Big)^2\Big]d\theta dx_3\ , \tag{4.6}$$

which, as a sufficient condition for instability with respect to 'reformation', should be negative for all $v_r(\theta, x_3)$ that vanish at $x_3 = 0$ and $x_3 = l_0$; in addition, according to (3.23b), $v_r(\theta, x_3)$ also must satisfy

$$\int_0^{l_0}\int_0^{2\pi} v_r d\theta dx_3 = 0\ . \tag{4.7}$$

Clearly, the Poincarè inequality implies that

$$\int_0^{l_0} v_r^2 dx_3 \leq C\int_0^{l_0}\Big(\frac{\partial v_r}{\partial x_3}\Big)^2 dx_3\ ,$$

for some positive constant C. Thus, we see that (4.6) will be positive for sufficiently large b_0. It is well-known that the choice of $C = 2/l_0^2$ is valid for the Poincarè inequality (though not the optimal choice); this implies that (4.6) is positive for $b_0 > l_0/\sqrt{2}$. Of course, the value $l_0/\sqrt{2}$ on the right hand side of this inequality can be made even lower by choosing the optimal Poincarè constant C. Notice, also, that if b_0 is sufficiently small, then (4.6) can be made negative by choice of an admissible function $v_r(x_3)$, which is independent of θ. Thus, for quantum (i.e., nano-) wires of small b_0/l_0 the homogeneous extension $\mathbf{y}(\mathbf{x})$ of this example application is not stable with respect to referential 'reformation'.

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