Convergence of alternate minimization schemes for phase field fracture and damage

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Abstract. We consider time-discrete evolutions for a phase field model (for fracture and damage) obtained by alternate minimization schemes. First, we characterize their time-continuous limit in terms of parametrized BV-evolutions, introducing a suitable family of "intrinsic energy norms". Further, we show that the limit evolution satisfies Griffith's criterion, for a phase field energy release, and that the irreversibility constraint is thermodynamically consistent.

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1 Introduction

In this paper we study a class of approximation schemes originally introduced by Bourdin, Francfort and Marigo in [5] and nowadays widely employed in quasi-static phase field models for fracture and damage, e.g. [29, 16, 24, 1]. Following [5] (and the many later developments) we consider a phase-field energy of Ambrosio-Tortorelli type [2] and the corresponding time-discrete evolutions obtained by a class of (constrained) alternate minimization algorithms. Our first and main goal is to prove the convergence of discrete solutions and to derive a precise characterization of the limit evolution; our second task is the study of its mechanical properties. By our analysis we obtained the following results: the time-continuous evolution is characterized "mathematically" in terms of a (non-degenerate, parametrized) BV-evolution [17, 19, 23] while "mechanically" it satisfies a phase-field version of Griffith's criterion, at least in the regime of stable propagation.

In detail, we employ the energy

$$\mathcal{F}(t,u,z) = \tfrac12 \int_\Omega (z^2+\eta)\mathbf{C}\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(u)\,dx + G_c \int_\Omega \frac{(z-1)^2}{4\varepsilon} + \varepsilon\,|\nabla z|^2\,dx - \langle b(t), u\rangle, \qquad (1)$$

where $z \in \mathcal{Z}$ is the scalar phase field/damage variable (with $0 \le z \le 1$, $z = 1$ corresponding to no damage and $z = 0$ corresponding to maximum damage), $u \in \mathcal{U}$ is the displacement field with linearized strain $\boldsymbol{\varepsilon}(u)$ and elasticity tensor $\mathbf{C}$, $b(t)$ is a linear functional, G_c is fracture toughness while ε and η are positive parameters (typically very small). It is well known that (1) provides an approximation of Griffith's energy for brittle fracture; indeed, the Γ-limit [8] of $\mathcal{F}(t,\cdot,\cdot)$ for $\varepsilon \to 0$, and $\eta = o(\varepsilon)$, turns out to be of the form

$$\tfrac12 \int_\Omega \mathbf{C}\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(u)\,dx + G_c\mathcal{H}^{n-1}(S_u) - \langle b(t), u\rangle$$

for $u \in SBD(\Omega)$ [6] and $GSBD(\Omega)$ [12, 9]. In our work, we are not going to study any limit as $\varepsilon \to 0$ (and thus we set for convenience $\varepsilon = 1/2$ for the rest of the paper); our interest, motivated by applications, is instead the evolution obtained by alternate minimization for the energy $\mathcal{F}$. We remark that Γ-convergence by itself is not enough to provide convergence of BV-evolutions, as ours; being crafted for global energy minimizers it is more suitable for the convergence of "energetic evolutions" [10].

Alternate minimization algorithms are descent algorithms which exploit the separate convexity of the energy functional $\mathcal{F}(t,\cdot,\cdot)$, with respect to u and z, to find critical points. In the context of fracture/damage this idea, applied to approximate a quasi-static evolution, goes back to [5]. Among the many possible implementations of alternate minimization we employ the following. Let the time interval $[0,T]$ be discretized by the time-steps t^n_k with $k = 0, ..., T/n$. Known (u^n_{k-1}, z^n_{k-1}), at time t^n_{k-1}, in order to calculate the solution (u^n_k, z^n_k) at time t^n_k let $u^n_{k,0} := u^n_{k-1}$, $z^n_{k,0} := z^n_{k-1}$ and calculate recursively for $i \in \mathbb{N}$ (until a certain stopping criterion is satisfied leading to the index i_{max}):

$$u^n_{k,i} \in \operatorname{argmin} \{ \mathcal{F}(t^n_k, u, z^n_{k,i-1}) \,:\, u \in \mathcal{U} \}, \tag{2}$$

$$z^n_{k,i} \in \operatorname{argmin} \{ \mathcal{F}(t^n_k, u^n_{k,i}, z) \,:\, z \le z^n_{k,i-1} \}. \tag{3}$$

Then we define $u^n_k = u^n_{k,i_{max}}$, $z^n_k = z^n_{k,i_{max}}$ (possibly including the case $i_{max} = \infty$ with a limit). We study the limit as the number of time steps n discretizing the interval $[0,T]$ tends to infinity. For that purpose, we switch to a parametrized picture and introduce interpolating curves $(t_n, u_n, z_n) : [0, S_n] \to [0,T] \times \mathcal{U} \times \mathcal{Z}$ depending on an artificial arc-length parameter s and study the limiting behavior of these trajectories. This is a convenient "mathematical description" which allows to easily take into account possible discontinuities in time of the limit BV-evolutions [18]. We show that the lengths S_n are uniformly bounded and that (subsequences of) the curves (t_n, u_n, z_n) converge to a Lipschitz continuous limit curve $(t, u, z) : [0, S] \to [0,T] \times \mathcal{U} \times \mathcal{Z}$ with $S < \infty$, $t(0) = 0$, $t(S) = T$, $s \to t(s)$ non-decreasing and $0 < C \le t'(s) + \|z'(s)\|_{\mathcal{Z}} + \|u'(s)\|_{\mathcal{U}}$ for almost all s.

Moreover, the limit (parametrized) evolution satisfies the following equilibrium criterion and an energy balance, respectively:

(S') for every $s \in [0, S]$ with $t'(s) > 0$

$$|\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} = 0, \qquad |\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} = 0, \tag{4}$$

(E') for every $s \in [0, S]$

$$\begin{aligned} \mathcal{F}(t(s), u(s), z(s)) = \mathcal{F}(0, u(0), z(0)) &- \int_0^s |\partial_u \mathcal{F}(t(r), u(r), z(r))|_{z(r)} \, \|u'(r)\|_{z(r)} \, dr \\ &- \int_0^s |\partial_z \mathcal{F}(t(r), u(r), z(r))|_{u(r)} \, \|z'(r)\|_{u(r)} \, dr \\ &+ \int_0^s \partial_t \mathcal{F}(t(r), u(r), z(r)) \, t'(r) \, dr \,. \end{aligned} \tag{5}$$

Here, $|\partial_z \mathcal{F}(t,u,z)|_u$ and $|\partial_u \mathcal{F}(t,u,z)|_z$ denote the partial weighted slopes of $\mathcal{F}$ with respect to u and z, respectively, while $\|u'\|_z$ and $\|v'\|_u$ are suitable weighted norms (for the definition see §2). For delicate technical reasons, the analysis is carried out for the two dimensional setting.

It turns out that our solutions are close to the one for vanishing viscosity limits for rate independent systems [22] and derive an equivalent characterization of (E') and (S') in terms of differential inclusions (a detailed discussion is contained in §6.3). Moreover, we show in §6.3 that functions satisfying (E') and (S') are local solutions, see for instance [18, 31], with respect to the physical time t. Finally, we show that for every $s \in [0, S]$ with $t'(s) > 0$ it holds the (parametrized) phase field Griffith's criterion

$$\mathcal{G}(t(s), z(s)) \leq G_c \qquad (\mathcal{G}(t(s), z(s)) - G_c)\, \mathcal{L}'(v(s)) = 0,$$

where $\mathcal{G}(t, z)$ is a phase-field energy release rate (defined in §6) while $\mathcal{L}'$ denotes the derivative of the length term $\mathcal{L}(z) = \frac{1}{2} \int_\Omega (z-1)^2 + |\nabla z|^2 \, dx$.

The paper is organized as follows: in §2 the notation, the energy $\mathcal{F}$, its slopes, the relevant function spaces and the weighted norms are introduced. In §3 we study the properties of the alternate minimization scheme at a fixed time step (with $i_{max} = \infty$) , derive the convergence properties of the sequence $(u_{k,i}, z_{k,i})_{i \in \mathbb{N}}$ and prove energy identities valid for a single time-step. §4 is devoted to the analysis of the full time-discrete problem. There, the artificial arc-length parameter and the corresponding interpolating curves are introduced, uniform estimates for the arc-length parameters and a uniform non-degeneracy condition for the curves $(t_n, u_n, z_n)_n$ are derived and the limits for $n \to \infty$ are investigated. By lower semicontinuity the limiting curves satisfy an energy inequality which, by using a chain rule argument, turns out to give the energy equality. In §5, we review the analysis of §3–§4 under the assumption that the maximum number of iterations in (2)–(3) is a-priori fixed to a finite value M including the case $M = 1$. In §6, we give different alternative formulations of the limit model and compare it with other notions of solutions to rate-independent processes. Finally, in §7 we give a numerical example showing for a finite dimensional toy problem the different solution curves obtained with: our algorithm, Bourdin's algorithm with backtracking, a one-step algorithm, see for instance [27], and the algorithm based on global minimization, leading to global energetic solutions.

Contents

2 Notation and preliminaries

Assume that Ω is a bounded Lipschitz connected open set in $\mathbb{R}^2$. Its boundary $\partial\Omega$ is split into $\partial_D\Omega$ and $\partial_N\Omega$ which are regular in the sense of Gröger [11]; this technical assumptions allows us to obtain a uniform higher integrability of the strain, independent of the phase field state, and is satisfied for instance when $\partial_D\Omega$ is relatively open with a finite number of connected components. In the following we assume that $\partial_D\Omega$ has positive measure. The space for the phase field variable and the displacement will be respectively

$$\mathcal{Z} = H^1(\Omega) \qquad \mathcal{U} = \{u \in H^1(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega\}.$$

Denote also

$$\mathcal{U}^p = W^{1,p}_{\partial_D\Omega} = \{u \in W^{1,p}(\Omega, \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega\}.$$

For $\eta > 0$ the *phase field elastic energy* $\mathcal{E} : \mathcal{U} \times \mathcal{Z} \to [0, +\infty]$ is defined as

$$\mathcal{E}(u, z) = \tfrac{1}{2} \int_\Omega (z^2 + \eta) W(Du)\, dx. \tag{6}$$

Here Du denotes the gradient of u and $W(Du) = \boldsymbol{\varepsilon}(u) : \mathbf{C}\boldsymbol{\varepsilon}(u) = \boldsymbol{\varepsilon}(u) : \boldsymbol{\sigma}(u)$ where $\mathbf{C}$ is symmetric and positive definite on symmetric 2×2 tensors, $\boldsymbol{\varepsilon}(u)$ is the symmetrized gradient and $\boldsymbol{\sigma}(u) = \mathbf{C}\boldsymbol{\varepsilon}(u)$ is the stress tensor.

Assume that the loading term

$$b \in C^{1,1}([0, T]; W^{-1,q}_{\partial_D\Omega}) \text{ for some } q > 2, \tag{7}$$

where $W^{-1,q}_{\partial_D\Omega}$ is the dual of $W^{1,q'}_{\partial_D\Omega}$. For $G_c > 0$ we define the *total energy functional* $\mathcal{F}$: $[0, T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ to be

$$\begin{aligned} \mathcal{F}(t, u, z) &= \tfrac{1}{2} \int_\Omega (z^2 + \eta) W(Du)\, dx + \tfrac{1}{2} G_c \int_\Omega (z - 1)^2 + |\nabla z|^2\, dx - \langle b(t), u\rangle \\ &= \mathcal{E}(u, z) + G_c \mathcal{L}(z) - \langle b(t), u\rangle. \end{aligned} \tag{8}$$

Note that the pairing $\langle b(t), u\rangle$ is well defined since, being $q > 2$, $\mathcal{U} \subset W^{1,q'}_{\partial_D\Omega}$ and thus $b \in \mathcal{U}'$. The choice $q > 2$ is due to the fact that we will often need higher integrability of the strains. This is provided by the following result [13].

Lemma 2.1. *Let $q > 2$. Denote*

$$\mathcal{A}_z(u, \phi) = \int_\Omega (z^2 + \eta)\boldsymbol{\varepsilon}(u) : \mathbf{C}\boldsymbol{\varepsilon}(\phi)\, dx. \tag{9}$$

For every $M > 0$ there exists $p \in (2, q]$ and $C_M > 0$ such that for every $\tilde{p} \in [2, p]$, every $z \in \mathcal{Z}$ with $\|z\|_{L^\infty(\Omega)} \le M$ and every $f \in W^{-1,\tilde{p}}_{\partial_D\Omega}$ there exists a unique $u \in W^{1,\tilde{p}}_{\partial_D\Omega}$ such that

$$\mathcal{A}_z(u, \phi) = \langle f, \phi\rangle \qquad \text{for every } \phi \in W^{1,\tilde{p}'}_{\partial_D\Omega}.$$

Moreover $\|u\|_{W^{1,\tilde{p}}} \le C_M \|f\|_{W^{-1,\tilde{p}}_{\partial_D\Omega}}$.

Let us fix for the rest of the paper $p > 2$ as the exponent obtained for $M = 1$ in the previous Lemma; the choice $M = 1$ is related to the fact that in the evolution the phase field variable z will always satisfy $\|z\|_{L^\infty} \le 1$.

By [7, Theorem 3.4] and direct calculations the following properties of $\mathcal{F}$ hold true.

Lemma 2.2. *The functional $\mathcal{F}(t,\cdot,\cdot)$ is sequentially weakly lower semicontinuous w.r.t. $\mathcal{U}\times\mathcal{Z}$. Moreover, for $z \in \mathcal{Z} \cap L^\infty$ and $u \in \mathcal{U}$ the following partial derivative is well defined*

$$\partial_u \mathcal{F}(t,u,z)[\phi] = \int_\Omega (z^2+\eta)\boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(\phi)\,dx - \langle b(t),\phi\rangle \qquad \text{for every } \phi \in \mathcal{U}. \tag{10}$$

If $z \in \mathcal{Z}$ and $u \in \mathcal{U}^{\tilde p}$ for $\tilde p > 2$ then

$$\partial_z \mathcal{F}(t,u,z)\,[\xi] = \int_\Omega z\xi\, W(Du)\,dx + G_c \int_\Omega (z-1)\xi + \nabla z\cdot\nabla\xi\,dx \qquad \text{for every } \xi \in \mathcal{Z}. \tag{11}$$

Finally, for all $u \in \mathcal{U}$ we have $\partial_t \mathcal{F}(t,u,z) = -\langle \dot b(t),u\rangle$.

Note that the first term in (11) is well defined thanks to the fact that $z,\xi \in L^r$ for every $1 \le r < \infty$ and $W(Du) \in L^{\tilde p/2}$.

Throughout the paper we will need the following norms and scalar product in the space $\mathcal{Z}$

$$\|z\|_{\mathcal{Z}} = \left(\int_\Omega z^2 + |\nabla z|^2\,dx\right)^{1/2}, \qquad \|z\|_u = \left(\int_\Omega z^2\big(G_c + W(Du)\big) + G_c|\nabla z|^2\,dx\right)^{1/2}, \tag{12}$$

$$\langle z,\xi\rangle_u = \int_\Omega z\xi\big(G_c + W(Du)\big) + G_c\nabla z\cdot\nabla\xi\,dx\,. \tag{13}$$

In the space $\mathcal{U}$ we introduce the following norms and scalar product

$$\|u\|_{\mathcal{U}} = \left(\int_\Omega u^2 + |Du|^2\,dx\right)^{1/2}, \qquad \|u\|_z = \left(\int_\Omega (z^2+\eta)W(Du)\,dx\right)^{1/2}, \tag{14}$$

$$\langle u,\phi\rangle_z = \int_\Omega (z^2+\eta)\boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(\phi)\,dx\,. \tag{15}$$

Under the conditions of Lemma 2.2 the above expressions are all well defined. Moreover, by the lower semicontinuity of $\mathcal{F}$ we obtain the following Corollary.

Corollary 2.3. *If $u_n \rightharpoonup u$ in $\mathcal{U}$ and $z_n \rightharpoonup z$ in $\mathcal{Z}$, then $\|u\|_z \le \liminf_n \|u_n\|_{z_n}$ and $\|z\|_u \le \liminf_n \|z_n\|_{u_n}$.*

Denoting $\ell(z) = \int_\Omega z\,dx$ we can rewrite $\mathcal{F}$ as quadratic functionals in u and z:

$$\mathcal{F}(t,u,z) = \tfrac12\|u\|_z^2 - \langle b(t),u\rangle + c_z, \quad \mathcal{F}(t,u,z) = \tfrac12\|z\|_u^2 - \ell(z) + c_{t,u}, \tag{16}$$

where

$$c_z = G_c\mathcal{L}(z), \qquad c_{t,u} = \tfrac12\int_\Omega \eta W(Du) + G_c\,dx - \langle b(t),u\rangle.$$

Moreover

$$\partial_u\mathcal{F}(t,u,z)[\phi] = \langle u,\phi\rangle_z - \langle b(t),\phi\rangle, \qquad \partial_z\mathcal{F}(t,u,z)[\xi] = \langle z,\xi\rangle_u - \ell(\xi).$$

Lemma 2.4. *Provided $z \in \mathcal{Z}\cap L^\infty$ and $u \in \mathcal{U}^{\tilde p}$ for $\tilde p > 2$, the norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{Z}}$ are equivalent respectively to $\|\cdot\|_z$ and $\|\cdot\|_u$.*

Proof. For $u \in \mathcal{U}^{\widetilde p}$ and $\bar z \in \mathcal{Z}$ it holds

$$\begin{aligned} G_c\,\|\bar z\|_{\mathcal{Z}}^2 \le \|\bar z\|_u^2 &= \int_\Omega \bar z^2(G_c + W(Du))\,dx + G_c\,\|\nabla\bar z\|_{L^2}^2 \\ &\le C_1\,\|u\|_{\mathcal{U}^{\tilde p}}^2\,\|\bar z\|_{L^r}^2 + G_c\,\|\bar z\|_{\mathcal{Z}}^2 \le C_2\left(1 + \|u\|_{\mathcal{U}^{\tilde p}}^2\right)\|\bar z\|_{\mathcal{Z}}^2\,, \end{aligned} \tag{17}$$

with $r = 2\widetilde p/(\widetilde p - 2)$ and C_i independent of u. This shows equivalence of norms in $\mathcal{Z}$. Finally, for $z \in L^\infty$ by Korn's inequality follows the equivalence of the norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_z$ in $\mathcal{U}$. $\square$

Due to the irreversibility constraint, the natural set of admissible variations for the phase field variable is

$$\Xi = \{\xi \in H^1 : \xi \le 0\}.$$

Next, under the assumptions of Lemma 2.2 we define the slopes

$$|\partial_u \mathcal{F}(t,u,z)|_z = \max\{-\partial_u \mathcal{F}(t,u,z)[\phi] : \phi \in \mathcal{U},\ \|\phi\|_z \le 1\},$$
$$|\partial_z \mathcal{F}(t,u,z)|_u = \max\{-\partial_z \mathcal{F}(t,u,z)[\xi] : \xi \in \Xi,\ \|\xi\|_u \le 1\}.$$

Lemma 2.5. *If $t_n \to t$, $u_n \to u$ in $\mathcal{U}$ and $z_n \rightharpoonup z$ in $\mathcal{Z}$ with $0 \le z_n \le 1$, then*

$$\lim_{n\to\infty} |\partial_u \mathcal{F}(t_n,u_n,z_n)|_{z_n} = |\partial_u \mathcal{F}(t,u,z)|_z.$$

If $t_n \to t$, $u_n \to u$ in $\mathcal{U}^{\tilde p}$ with $\tilde p > 2$ and $z_n \rightharpoonup z$ in $\mathcal{Z}$, then

$$\liminf_{n\to\infty} |\partial_z \mathcal{F}(t_n,u_n,z_n)|_{u_n} \ge |\partial_z \mathcal{F}(t,u,z)|_u$$

Proof. Consider a subsequence with

$$\limsup_{n\to\infty} |\partial_u \mathcal{F}(t_n,u_n,z_n)|_{z_n} = \lim_{n_m\to\infty} |\partial_u \mathcal{F}(t_{n_m},u_{n_m},z_{n_m})|_{z_{n_m}}.$$

In the following we write m instead of n_m. Let $\phi_m \in \operatorname{argmax}\{-\partial_u \mathcal{F}(t_m,u_m,z_m)[\phi] : \phi \in \mathcal{U},\ \|\phi\|_{z_m} \le 1\}$. Then ϕ_m is bounded in H^1 and thus there exists a subsequence (not relabeled) s.t. $\phi_m \rightharpoonup \phi_\infty$. By Corollary 2.3, $\|\phi_\infty\|_z \le 1$, as well. Extract a subsequence s.t. Du_m and z_m converge a.e. to Du and z respectively. Then $(z_m^2+\eta)\boldsymbol{\sigma}(u_m) \to (z^2+\eta)\boldsymbol{\sigma}(u)$ a.e. and is dominated by $c(1+\eta)|Du_m|$ which converges in L^2. By generalized dominated convergence it follows that $(z_m^2+\eta)\boldsymbol{\sigma}(u_m) \to (z^2+\eta)\boldsymbol{\sigma}(u)$ in L^2. Since $\phi_m \rightharpoonup \phi_\infty$ in $\mathcal{U}$ we have

$$\begin{aligned}\partial_u \mathcal{F}(t_m,u_m,z_m)[\phi_m] &= \int_\Omega (z_m^2+\eta)\boldsymbol{\sigma}(u_m) : \boldsymbol{\varepsilon}(\phi_m)\,dx - \langle b(t_m),\phi_m\rangle \\ &\to \int_\Omega (z^2+\eta)\boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(\phi_\infty)\,dx - \langle b(t),\phi_\infty\rangle = \partial_u \mathcal{F}(t,u,z)[\phi_\infty].\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} |\partial_u \mathcal{F}(t_n,u_n,z_n)|_{z_n} &= \lim_{m\to\infty} |\partial_u \mathcal{F}(t_m,u_m,z_m)|_{z_m} = -\lim_{m\to\infty} \partial_u \mathcal{F}(t_m,u_m,z_m)[\phi_m] \\ &= -\partial_u \mathcal{F}(t,u,z)[\phi_\infty] \le |\partial_u \mathcal{F}(t,u,z)|_z.\end{aligned}$$

Let $\phi_* \in \operatorname{argmax}\{-\partial_u \mathcal{F}(t,u,z)[\phi] : \phi \in \mathcal{U}, \|\phi\|_z \le 1\}$. For every subsequence n_k there exists a further subsequence n_i s.t. $z_{n_i} \to z$ a.e. in Ω. By dominated convergence

$$\int_\Omega (z_{n_i}^2+\eta) W(D\phi_*)\,dx \to \int_\Omega (z^2+\eta) W(D\phi_*)\,dx$$

By arbitrariness of the subsequence n_k it follows that $\|\phi_*\|_{z_n} \to \|\phi_*\|_z \le 1$.

Now, extract a subsequence n_m s.t. $\liminf_{n\to\infty} |\partial_u \mathcal{F}(t_n,u_n,z_n)|_{z_n} = \lim_{m\to\infty} |\partial_u \mathcal{F}(t_m,u_m,z_m)|_{z_m}$ and such that $(z_m^2+\eta)\boldsymbol{\sigma}(u_m) \to (z^2+\eta)\boldsymbol{\sigma}(u)$ strongly in L^2, see above. Then

$$\partial_u \mathcal{F}(t_m,u_m,z_m)[-\phi_*] \to \partial_u \mathcal{F}(t,u,z)[-\phi_*]$$

and finally

$$\begin{aligned}|\partial_u \mathcal{F}(t,u,z)|_z &= \partial_u \mathcal{F}(t,u,z)[-\phi_*] = \lim_{m\to\infty} \partial_u \mathcal{F}(t_m,u_m,z_m)[-\phi_*] \\ &\le \liminf_{m\to\infty} |\partial_u \mathcal{F}(t_m,u_m,z_m)|_{z_m} \|\phi_*\|_{z_m} = \liminf_{n\to\infty} |\partial_u \mathcal{F}(t_n,u_n,z_n)|_{z_n}.\end{aligned}$$

Next, we discuss the slope with respect to z. Let $\xi_* \in \operatorname{argmax}\{-\partial_z \mathcal{F}(t,u,z)[\xi] : \xi \in \Xi,\ \|\xi\|_u \le 1\}$. Remember that $t_n \to t$ and that $z_n \rightharpoonup z$ in H^1 and thus strongly in L^r for every $r < +\infty$. Note also that $u_n \in \mathcal{U}^{\tilde p}$ for some $\tilde p > 2$. Then $W(Du_n) \to W(Du)$ strongly in $L^{\tilde p/2}$ for $\tilde p > 2$ while $z_n \xi_* \to z \xi_*$ in L^r for every $r < +\infty$. It follows that

$$\int_\Omega z_n \xi_* W(Du_n)\,dx + G_c \int_\Omega (z_n - 1)\xi_* + \nabla z_n \cdot \nabla \xi_*\,dx \\ \to \int_\Omega z \xi_* W(Du)\,dx + G_c \int_\Omega (z-1)\xi_* + \nabla z \cdot \nabla \xi_*\,dx$$

and thus $\partial_z \mathcal{F}(t_n,u_n,z_n)[\xi_*] \to \partial_z \mathcal{F}(t,u,z)[\xi_*]$. Moreover $\int_\Omega \xi_*^2 W(Du_n)\,dx \to \int_\Omega \xi_*^2 W(Du)\,dx$ and thus $\|\xi_*\|_{u_n} \to \|\xi_*\|_u \le 1$. Then,

$$\begin{aligned} |\partial_z \mathcal{F}(t,u,z)|_u &= -\partial_z \mathcal{F}(t,u,z)[\xi_*] = -\lim_{n\to\infty} \partial_u \mathcal{F}(t_n,u_n,z_n)[\xi_*] \\ &\le \liminf_{n\to\infty} |\partial_z \mathcal{F}(t_n,u_n,z_n)|_{u_n} \|\xi_*\|_{u_n} = \liminf_{n\to\infty} |\partial_z \mathcal{F}(t_n,u_n,z_n)|_{u_n} \end{aligned}$$

and the proof is concluded. □

3 Alternate minimization scheme

In this section we will collect all the properties of the alternate minimization scheme at a fixed time t. The time incremental problem will be discussed in §4.

Given $t \in [0,T]$, $u \in \mathcal{U}$ and $z \in \mathcal{Z}$ with $0 \le z \le 1$, let us consider the sequences u_i and z_i generated recursively by the following alternate minimization scheme. Set $u_0 = u$, $z_0 = z$ and define

$$\begin{cases} u_{i+1} \in \operatorname{argmin}\{\mathcal{F}(t,u,z_i) : u \in \mathcal{U}\}, \\ z_{i+1} \in \operatorname{argmin}\{\mathcal{F}(t,u_{i+1},z) : z \in \mathcal{Z} \text{ with } z \le z_i\}. \end{cases} \tag{18}$$

Standard arguments guarantee that for every index $i \in \mathbb{N}$ there exist unique minimizers $u_{i+1} \in \mathcal{U}$ and $z_{i+1} \in \mathcal{Z}$ with $0 \le z_{i+1} \le 1$. Clearly it holds $\mathcal{F}(t,u_{i+1},z_{i+1}) \le \mathcal{F}(t,u_{i+1},z_i) \le \mathcal{F}(t,u_i,z_i)$.

Lemma 3.1. *There exists a constant C independent of t, u_0 and z_0 such that for every $i \ge 1$ and for every $\tilde p \in [2,p]$ it holds $u_i \in \mathcal{U}^{\tilde p}$ with $\|u_i\|_{\mathcal{U}^{\tilde p}} \le C$ and $\|z_i\|_{\mathcal{Z}} \le C(1 + \|z_0\|_{\mathcal{Z}})$.*

For $i \ge 0$ we have $\partial_u \mathcal{F}(t,u_{i+1},z_i)[\phi] = 0$ for every $\phi \in \mathcal{U}$, $\partial_z \mathcal{F}(t,u_{i+1},z_{i+1})[\xi] \ge 0$ for every $\xi \in \Xi$ and $\partial_z \mathcal{F}(t,u_{i+1},z_{i+1})[z_{i+1} - z_i] = 0$. These conditions imply that the slopes satisfy

$$|\partial_u \mathcal{F}(t,u_{i+1},z_i)|_{z_i} = 0, \qquad |\partial_z \mathcal{F}(t,u_{i+1},z_{i+1})|_{u_{i+1}} = 0.$$

Proof. By Lemma 2.1 we know that $\|u_i\|_{\mathcal{U}^{\tilde p}} \le C \|b(t)\|_{W^{-1,\tilde p}_{\partial_D \Omega}}$ for every $i \ge 1$. From (16)

$$\tfrac12 \|z_i\|^2_{u_i} - \ell(z_i) + c_{t,u_i} = \mathcal{F}(t,u_i,z_i) \le \mathcal{F}(t,u_1,z_0) \le \tfrac12 \|z_0\|^2_{u_1} - \ell(z_0) + c_{t,u_1}.$$

By (17) and by the previous estimate for $\|u_i\|_{\mathcal{U}^{\tilde p}}$, the right hand side is bounded from above by $C(1 + \|z_0\|^2_{\mathcal{Z}})$. Again by (17) and by the previous estimate for $\|u_i\|_{\mathcal{U}^{\tilde p}}$, the left hand side is bounded from below by $C(\|z_i\|^2_{\mathcal{Z}} - 1)$. This proves the uniform estimates.

Since Ξ is a closed, convex cone, minimality of z_{i+1} implies the variational inequality and the equality for $\partial_z \mathcal{F}$ stated in the Lemma. □

Lemma 3.2. *The sequence u_i converges to u_∞ strongly in $\mathcal{U}^{\tilde p}$ for $\tilde p \in [2,p)$; the sequence z_i converges to z_∞ strongly in $\mathcal{Z}$ with $0 \le z_\infty \le 1$. Moreover, $\partial_u \mathcal{F}(t,u_\infty,z_\infty)[\phi] = 0$ for every $\phi \in \mathcal{U}$ and $\partial_z \mathcal{F}(t,u_\infty,z_\infty)[\xi] \ge 0$ for every $\xi \in \Xi$. These conditions are equivalent to $|\partial_u \mathcal{F}(t,u_\infty,z_\infty)|_{z_\infty} = 0$ and $|\partial_z \mathcal{F}(t,u_\infty,z_\infty)|_{u_\infty} = 0$.*

Proof. Relying on the uniform bound of Lemma 3.1, up to subsequences z_i converge a.e. to z_∞. Since the sequence z_i is monotone non-increasing, it turns out that the whole sequence converges a.e. to z_∞ and thus in L^r for every $1 \le r < \infty$. By standard arguments it follows that the whole sequence $z_i \rightharpoonup z_\infty$ in $\mathcal{Z}$. By Lemma A.1, u_i is a Cauchy sequence in $\mathcal{U}^{\tilde{p}}$ for every $\tilde{p} < p$ with limit u_∞. Thanks to Lemma A.2, z_i is again a Cauchy sequence in $\mathcal{Z}$ with $z_i \to z_\infty$.

By the same arguments of Lemma 2.5 the properties of the slopes follow. □

Remark 3.3. *Note that, given the initial data, the sequence $\{(u_i, z_i)\}_i$ and its limit (u_∞, z_∞) are unique.*

Lemma 3.4. *Consider the sequences $\{u_i\}_i$ and $\{z_i\}_i$ generated by the alternate minimization scheme* (18)*. Let $\widetilde{p} \in (2, p)$ be arbitrary and $\mu^{-1} = \widetilde{p}^{-1} - p^{-1}$. Then for $i \ge 1$*

$$\|z_{i+1} - z_i\|_{\mathcal{Z}} \le C\big(\|u_i\|_{\mathcal{U}^{\tilde{p}}} + \|u_{i+1}\|_{\mathcal{U}^{\tilde{p}}}\big) \|z_i - z_{i-1}\|_{L^\mu(\Omega)} . \tag{19}$$

The constant C is independent of $i \ge 1$ and $t \in [0, T]$.

Proof. Since $u_i, u_{i+1} \in \mathcal{U}^{\tilde{p}}$, Lemma A.2 with $z_0 = z_{i-1}, z_1 = z_i$, $z_2 = z_{i+1}$, $u_1 = u_i$ and $u_2 = u_{i+1}$ implies

$$\|z_{i+1} - z_i\|^2_{\mathcal{Z}} \le C \|u_{i+1} - u_i\|_{\mathcal{U}^{\tilde{p}}} \left(\|u_{i+1}\|_{\mathcal{U}^{\tilde{p}}} + \|u_i\|_{\mathcal{U}^{\tilde{p}}} \right) \|z_{i+1} - z_i\|_{\mathcal{Z}} ,$$

where the continuous embedding $\mathcal{Z} \subset L^r(\Omega)$ is exploited. Lemma A.1 yields

$$\|u_{i+1} - u_i\|_{\mathcal{U}^{\tilde{p}}} \le C \|z_i - z_{i-1}\|_{L^\mu(\Omega)}$$

with $\mu^{-1} = \widetilde{p}^{-1} - p^{-1}$. Combining these estimates leads to (19). □

Using the separate quadratic structure of the energy we will prove the following energy identities (of normalized gradient flow type).

Lemma 3.5. *For every $i \ge 0$ and every $\bar{r} \in [0, 1]$ it holds*

$$\mathcal{F}(t, u_{i+\bar{r}}, z_i) = \mathcal{F}(t, u_i, z_i) - \int_0^{\bar{r}} |\partial_u \mathcal{F}(t, u_{i+r}, z_i)|_{z_i} \ \|u_{i+1} - u_i\|_{z_i} \, dr, \tag{20}$$

where $u_{i+r} = u_i + r(u_{i+1} - u_i)$. Similarly

$$\mathcal{F}(t, u_{i+1}, z_{i+\bar{r}}) = \mathcal{F}(t, u_{i+1}, z_i) - \int_0^{\bar{r}} |\partial_z \mathcal{F}(t, u_{i+1}, z_{i+r})|_{u_{i+1}} \ \|z_{i+1} - z_i\|_{u_{i+1}} \, dr, \tag{21}$$

where $z_{i+r} = z_i + r(z_{i+1} - z_i)$.

Proof. Without loss of generality we can assume that $u_{i+1} \ne u_i$. Since u_{i+1} is a minimizer of $\mathcal{F}(t, \cdot, z_i)$ we have $\langle u_{i+1}, \phi\rangle_{z_i} = \langle b(t), \phi\rangle$ for every $\phi \in \mathcal{U}$. Write

$$\langle u_{i+1} - u_i, \phi\rangle_{z_i} = -\langle u_i, \phi\rangle_{z_i} + \langle b(t), \phi\rangle = -\partial_u \mathcal{F}(t, u_i, z_i)[\phi] \ \forall \phi \in \mathcal{U}.$$

Thus

$$\max\{\langle u_{i+1} - u_i, \phi\rangle_{z_i} : \phi \in \mathcal{U}, \ \|\phi\|_{z_i} \le 1\} = \max\{-\partial_u \mathcal{F}(t, u_i, z_i)[\phi] : \phi \in \mathcal{U}, \ \|\phi\|_{z_i} \le 1\}$$

Since $(u_{i+1} - u_i)/\|u_{i+1} - u_i\|_{z_i} \in \mathcal{U}$ is a maximizer, we get

$$\|u_{i+1} - u_i\|_{z_i} = |\partial_u \mathcal{F}(t, u_i, z_i)|_{z_i} = -\partial_u \mathcal{F}(t, u_i, z_i)[u_{i+1} - u_i]/\|u_{i+1} - u_i\|_{z_i}.$$

Now we will show that from the quadratic structure it will follow that for $s \in [0, 1]$

$$\|u_{i+1} - u_{i+s}\|_{z_i} = |\partial_u \mathcal{F}(t, u_{i+s}, z_i)|_{z_i} = -\partial_u \mathcal{F}(t, u_{i+s}, z_i)[u_{i+1} - u_i]/\|u_{i+1} - u_i\|_{z_i}. \tag{22}$$

Indeed, keeping in mind the Euler-Lagrange equation for u_{i+1},

$$\begin{aligned}\partial_u \mathcal{F}(t, u_{i+s}, z_i)[\phi] &= \langle u_{i+s}, \phi \rangle_{z_i} - \langle b(t), \phi \rangle = \langle u_i + s(u_{i+1} - u_i), \phi \rangle_{z_i} - \langle b(t), \phi \rangle \\ &= (1-s)(\langle u_i, \phi \rangle_{z_i} - \langle b(t), \phi \rangle) \\ &= (1-s)\partial_u \mathcal{F}(t, u_i, z_i)[\phi].\end{aligned}$$

Thus $(u_{i+1} - u_i)/\|u_{i+1} - u_i\|_{z_i} \in \mathcal{U}$ is again a maximizer and

$$|\partial_u \mathcal{F}(t, u_{i+s}, z_i)|_{z_i} = -\partial_u \mathcal{F}(t, u_{i+s}, z_i)[u_{i+1} - u_i]/\|u_{i+1} - u_i\|_{z_i}$$

by definition of the slope. Thus, by the chain rule, for $0 \le \bar{r} \le 1$,

$$\begin{aligned}\mathcal{F}(t, u_{i+\bar{r}}, z_i) &= \mathcal{F}(t, u_i, z_i) + \int_0^{\bar{r}} \frac{d}{dr} \mathcal{F}(t, u_{i+r}, z_i)\, dr \\ &= \mathcal{F}(t, u_i, z_i) + \int_0^{\bar{r}} \partial_u \mathcal{F}(t, u_{i+r}, z_i)[u_{i+1} - u_i]\, dr \\ &= \mathcal{F}(t, u_i, z_i) - \int_0^{\bar{r}} |\partial_u \mathcal{F}(t, u_{i+r}, z_i)|_{z_i}\ \|u_{i+1} - u_i\|_{z_i}\, dr.\end{aligned}$$

For the phase field variable z the argument is similar but not identical due to the irreversibility constraint. Let $\mathcal{Z}_i = \{z \in \mathcal{Z} : z \le z_i\}$. By minimality, for all $\xi \in \Xi$

$$\partial_z \mathcal{F}(t, u_{i+1}, z_{i+1})[\xi] \ge 0, \qquad \partial_z \mathcal{F}(t, u_{i+1}, z_{i+1})[z_i - z_{i+1}] = 0. \tag{23}$$

Let us write

$$\begin{aligned}\partial_z \mathcal{F}(t, u_{i+1}, z_i)[\xi] &= \langle z_i, \xi \rangle_{u_{i+1}} - \ell(\xi) = \langle z_{i+1}, \xi \rangle_{u_{i+1}} - \ell(\xi) + \langle z_i - z_{i+1}, \xi \rangle_{u_{i+1}} \\ &= \partial_z \mathcal{F}(t, u_{i+1}, z_{i+1})[\xi] + \langle z_i - z_{i+1}, \xi \rangle_{u_{i+1}}\end{aligned}$$

By (23) and Cauchy-Schwarz inequality

$$\max\{-\partial_z \mathcal{F}(t, u_{i+1}, z_i)[\xi] : \xi \in \Xi,\ \|\xi\|_{u_{i+1}} \le 1\} \le \|z_{i+1} - z_i\|_{u_{i+1}}$$

Choosing $\xi = (z_{i+1} - z_i)/\|z_{i+1} - z_i\|_{u_{i+1}}$, again by (23), yields

$$\partial_z \mathcal{F}(t, u_{i+1}, z_i)[z_{i+1} - z_i]/\|z_{i+1} - z_i\|_{u_{i+1}} = -\|z_{i+1} - z_i\|_{u_{i+1}}$$

Therefore $\xi = (z_{i+1} - z_i)/\|z_{i+1} - z_i\|_{u_{i+1}}$ is again a maximizer for the slope in (t, u_{i+1}, z_i).

Next we will show that for every $s \in [0, 1]$ we have

$$\|z_{i+1} - z_{i+s}\|_{u_{i+1}} = |\partial_z \mathcal{F}(t, u_{i+1}, z_{i+s})|_{u_{i+1}} = -\partial_z \mathcal{F}(t, u_{i+1}, z_{i+s})[z_{i+1} - z_i]/\|z_{i+1} - z_i\|_{u_{i+1}}.$$

Indeed, again by the quadratic structure of $\mathcal{F}(, u, \cdot)$, we find for all $\xi \in \Xi$

$$\begin{aligned}\partial_z \mathcal{F}(t, u_{i+1}, z_{i+s})[\xi] &= \langle z_{i+s}, \xi \rangle_{u_{i+1}} - \ell(\xi) = \langle z_i + s(z_{i+1} - z_i), \xi \rangle_{u_{i+1}} - \ell(\xi) \\ &= s\left[\langle z_{i+1}, \xi \rangle_{u_{i+1}} - \ell(\xi)\right] + (1-s)\left[\langle z_i, \xi \rangle_{u_{i+1}} - \ell(\xi)\right] \\ &= s\, \partial_z \mathcal{F}(t, u_{i+1}, z_{i+1})[\xi] + (1-s)\, \partial_z \mathcal{F}(t, u_{i+1}, z_i)[\xi]\end{aligned}$$

For $\xi = (z_{i+1} - z_i)/\|z_{i+1} - z_i\|_{u_{i+1}}$, by Lemma 3.1,

$$\partial_z \mathcal{F}(t, u_{i+1}, z_{i+s})[\xi] = (1-s)\, \partial_z \mathcal{F}(t, u_{i+1}, z_i)[\xi].$$

Since $\xi = (z_{i+1} - z_i)/\|z_{i+1} - z_i\|_{u_{i+1}}$ is a maximizer of the slope in (t, u_{i+1}, z_i) we get

$$|\partial_z \mathcal{F}(t, u_{i+1}, z_{i+s})|_{u_{i+1}} = -\partial_u \mathcal{F}(t, u_{i+1}, z_{i+s})[z_{i+1} - z_i]/\|z_{i+1} - z_i\|_{u_{i+1}}$$

Thus, by the chain rule,

$$\begin{aligned}\mathcal{F}(t, u_{i+1}, z_{i+\tau}) &= \mathcal{F}(t, u_{i+1}, z_i) + \int_0^{\tau} \partial_z \mathcal{F}(t, u_{i+1}, z_{i+r})[z_{i+1} - z_i]\, dr \\ &= \mathcal{F}(t, u_{i+1}, z_i) - \int_0^{\tau} |\partial_z \mathcal{F}(t, u_{i+1}, z_{i+r})|_{u_{i+1}}\ \|z_{i+1} - z_i\|_{u_{i+1}}\, dr.\end{aligned}$$

and the proof is concluded. □

4 Quasi-static evolution

4.1 Discrete evolution by the alternate minimization scheme

We now turn to the time discrete evolution, where each incremental update is provided by the alternate minimization scheme. Given $z(0) = z_0 \in \mathcal{Z}$, with $0 \le z_0 \le 1$, and $u(0) = u_0 \in \mathcal{U}$ we assume that

$$u_0 \in \operatorname{argmin} \{ \mathcal{F}(0, u, z_0) \, : \, u \in \mathcal{U} \}, \qquad z_0 \in \operatorname{argmin} \{ \mathcal{F}(0, u_0, z) \, : \, z \in \mathcal{Z}, \; z \le z_0 \}. \tag{24}$$

Observe that initial data that are stable in the sense of energetic solutions, i.e. with $(u_0, z_0) \in \operatorname{Argmin}\{ \mathcal{F}(0, u, z) \, : \, u \in \mathcal{U}, z \in \mathcal{Z}, z \le z_0 \}$, in particular satisfy (24).

Let $\Delta t^n = T/n$ and set $t^n_k = k\Delta t^n$ for $k = 0, ..., n$. Set $u^n_0 = u_0$ and $z^n_0 = z_0$. Given u^n_{k-1} and z^n_{k-1} consider the alternate minimization scheme at time $t^n_k = t^n_{k-1} + \Delta t^n$ with initial conditions u^n_{k-1} and z^n_{k-1}, viz. set $u^n_{k,0} = u^n_{k-1}$ and $z^n_{k,0} = z^n_{k-1}$ and then define by induction (for $i \ge 0$)

$$\begin{cases} u^n_{k,i+1} \in \operatorname{argmin} \{\mathcal{F}(t^n_k, u, z^n_{k,i}) : u \in \mathcal{U}\} \\ z^n_{k,i+1} \in \operatorname{argmin} \{\mathcal{F}(t^n_k, u^n_{k,i+1}, z) : z \in \mathcal{Z} \text{ with } z \le z^n_{k,i}\}. \end{cases}$$

The updates u^n_k and z^n_k are respectively the limits of the above sequences, i.e.

$$u^n_k = u^n_{k,\infty} = \lim_{i\to\infty} u^n_{k,i}, \qquad z^n_{k+1} = z^n_{k,\infty} = \lim_{i\to\infty} z^n_{k,i}.$$

Existence of such limits has been proven in Lemma 3.2. Observe that the alternate minimization procedure might lead to a fixed point already after a finite number of iterations. In this case the sequences $(u^n_{k,i}, z^n_{k,i})_i$ will be constant starting form a certain index. The case in which the algorithm is stopped at a finite index i_{max} will be treated in §5.

4.2 Length of the alternate minimizing path

For convenience we introduce, for $n \in \mathbb{N}$ and $0 \le k \le n$, the notation

$$\gamma^n_k := \sum_{i=0}^{\infty} \Big(\big\| u^n_{k,i+1} - u^n_{k,i} \big\|_{\mathcal{U}} + \big\| z^n_{k,i+1} - z^n_{k,i} \big\|_{\mathcal{Z}} \Big), \tag{25}$$

which corresponds to the length of the alternate minimizing path at time t^n_k with respect to the standard $\mathcal{U} \times \mathcal{Z}$-norm.

Theorem 4.1. *There exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$\sum_{k=0}^{n} \gamma^n_k \le C. \tag{26}$$

Proof. We first estimate γ^n_k from (25). By Lemma 3.1 there exists a constant $C > 0$ such that for every $\widetilde{p} \in [2, p)$ and for all $n \ge 1$, $0 \le k \le n$, $i \ge 0$ it holds

$$\big\| u^n_{k,i} \big\|_{\mathcal{U}^{\tilde{p}}} \le C. \tag{27}$$

By Lemma 3.4 there exists a further constant $C_1 > 0$ such that for all $n \ge 1$, $0 \le k \le n$, $i \ge 1$ we have

$$\big\| z^n_{k,i+1} - z^n_{k,i} \big\|_{\mathcal{Z}} \le C_1 \big\| z^n_{k,i-1} - z^n_{k,i} \big\|_{L^\mu(\Omega)} \tag{28}$$

with $\mu^{-1} = \widetilde{p}^{-1} - p^{-1}$. Since $\mathcal{Z} \subset L^\mu(\Omega) \subset L^1(\Omega)$ with continuous embeddings, by the Gagliardo-Nirenberg inequality the right hand side can further be estimated as follows: fixing

$\theta \in (0,1)$ according to the Gagliardo-Nirenberg conditions and choosing $\delta > 0$ a posteriori it holds

$$\begin{aligned} C_1 \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^\mu(\Omega)} &\leq \left(\delta \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{\mathcal{Z}}\right)^{1-\theta} C_\mu \delta^{\theta-1} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|^{\theta}_{L^1(\Omega)} \\ &\leq (1-\theta)\delta \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{\mathcal{Z}} + \theta C_\mu^{1/\theta} \delta^{1-1/\theta} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}, \end{aligned}$$

where in the second line we applied Young's inequality. With $\delta = 1/(2(1-\theta))$, we arrive at

$$C_1 \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^\mu(\Omega)} \leq \frac{1}{2} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{\mathcal{Z}} + C_2 \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}$$

with $C_2 = \theta C_\mu^{1/\theta} (2(1-\theta))^{-1+1/\theta}$. Joining this estimate with (28) yields

$$\left\| z^n_{k,i+1} - z^n_{k,i} \right\|_{\mathcal{Z}} \leq \frac{1}{2} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{\mathcal{Z}} + C_2 \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}$$

for $n \geq 1$, $0 \leq k \leq n$, $i \geq 1$. Summing up this inequality for $m \leq i \leq N$ $(m \geq 1)$ leads to

$$\sum_{i=m}^{N} \left\| z^n_{k,i+1} - z^n_{k,i} \right\|_{\mathcal{Z}} \leq \frac{1}{2} \sum_{i=m}^{N} \left\| z^n_{k,i} - z^n_{k,i-1} \right\|_{\mathcal{Z}} + C_2 \sum_{i=m}^{N} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}. \tag{29}$$

Absorbing the first term on the right hand side into the left hand side (except for the term $i = m$) and neglecting the term $i = N$ on the left hand side we get for all $N \in \mathbb{N}$:

$$\frac{1}{2} \sum_{i=m+1}^{N} \left\| z^n_{k,i} - z^n_{k,i-1} \right\|_{\mathcal{Z}} \leq \frac{1}{2} \left\| z^n_{k,m} - z^n_{k,m-1} \right\|_{\mathcal{Z}} + C_2 \sum_{i=m}^{N} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}. \tag{30}$$

Observe that due to the relations $0 \leq z^n_{k,i} \leq z^n_{k,i-1} \leq 1$ the following series is finite

$$\sum_{i=1}^{\infty} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)} = \int_\Omega z^n_{k,0} - z^n_{k,\infty} \, dx \leq |\Omega|. \tag{31}$$

Let now $m = 1$ in (30). The first term on the right hand side of (30) can be estimated with Lemma A.2 by choosing $z_0 = z_1 = z^n_{k,0}$, $z_2 = z^n_{k,1}$ and $u_1 = u^n_{k,0}$, $u_2 = u^n_{k,1}$ and by taking into account (27):

$$\left\| z^n_{k,0} - z^n_{k,1} \right\|^2_{\mathcal{Z}} \leq C \left\| u^n_{k,0} - u^n_{k,1} \right\|_{\mathcal{U}^{\tilde{p}}} \left\| z^n_{k,0} - z^n_{k,1} \right\|_{L^r} \leq C \left\| u^n_{k,0} - u^n_{k,1} \right\|_{\mathcal{U}^{\tilde{p}}} \left\| z^n_{k,0} - z^n_{k,1} \right\|_{\mathcal{Z}}.$$

Applying Lemma A.1 with $u^n_{k,0} = u_{\min}(t^n_{k-1}, z^n_{k,0})$ and $u^n_{k,1} = u_{\min}(t^n_k, z^n_{k,0})$ gives

$$\left\| z^n_{k,0} - z^n_{k,1} \right\|_{\mathcal{Z}} \leq C \left| t^n_{k-1} - t^n_k \right|.$$

Adding this inequality to (30) finally yields (with $m = 1$)

$$\sum_{i=1}^{\infty} \left\| z^n_{k,i} - z^n_{k,i-1} \right\|_{\mathcal{Z}} \leq C \left| t^n_{k-1} - t^n_k \right| + C_2 \sum_{i=1}^{\infty} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}. \tag{32}$$

Again thanks to Lemma A.1, this estimate immediately carries over to the corresponding estimate for the displacement fields:

$$\begin{aligned} \sum_{i=1}^{\infty} \left\| u^n_{k,i} - u^n_{k,i-1} \right\|_{\mathcal{U}} &\leq C_3 \left(\left| t^n_{k-1} - t^n_k \right| + \sum_{i=1}^{\infty} \left\| z^n_{k,i} - z^n_{k,i-1} \right\|_{\mathcal{Z}} \right) \\ &\leq C_4 \left(\left| t^n_{k-1} - t^n_k \right| + \sum_{i=1}^{\infty} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)} \right). \end{aligned} \tag{33}$$

This implies that γ^n_k from (25) is finite and that

$$\sum_{k=0}^{n} \gamma^n_k \leq C(T + \sum_{k=0}^{n} \sum_{i=1}^{\infty} \left\| z^n_{k,i-1} - z^n_{k,i} \right\|_{L^1(\Omega)}) \leq C(T + |\Omega|),$$

where the last term has been estimated as in (31). ☐

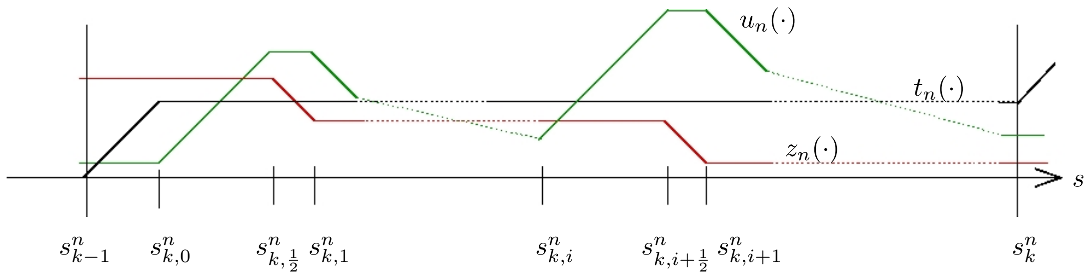

Figure 1: Notation and interpolating curves

4.3 Arc-length parametrization of the graph

Now we provide the interpolation and parametrization (with respect to the arc length parameter s) of the discrete solutions $u^n_{k,i}$ and $z^n_{k,i}$ given by the alternate minimization scheme. Remember that $u^n_{k-1,\infty} = u^n_{k-1} = u^n_{k,0}$ and similarly for z^n_{k-1}.

Let $t^n_0 = 0$ and define $s^n_0 = 0$. For each $k \geq 1$, given s^n_{k-1} and $i \geq 0$ we define

$$s^n_{k,-1} = s^n_{k-1}, \quad s^n_{k,0} = s^n_{k,-1} + \Delta t^n = s^n_{k-1} + \Delta t^n, \tag{34}$$

$$s^n_{k,i+1/2} = s^n_{k,i} + \|u^n_{k,i+1} - u^n_{k,i}\|_{z^n_{k,i}}, \qquad s^n_{k,i+1} = s^n_{k,i+1/2} + \|z^n_{k,i+1} - z^n_{k,i}\|_{u^n_{k,i+1}}. \tag{35}$$

We define $s^n_k = \lim_{i\to\infty} s^n_{k,i}$ and know from Theorem 4.1 that this quantity is finite. Note that it may happen that $s^n_{k,i+1/2} = s^n_{k,i}$ and $s^n_{k,i+1} = s^n_{k,i+1/2}$. We refer to Figure 1 for the notation.

In the time update interval $[s^n_{k,-1}, s^n_{k,0}] = [s^n_{k-1}, s^n_{k-1} + \Delta t^n]$ we define

$$\begin{aligned} u_n(s) &= u^n_{k,0} && \text{for } s \in [s^n_{k-1}, s^n_{k,0}], \\ z_n(s) &= z^n_{k,0} && \text{for } s \in [s^n_{k-1}, s^n_{k,0}], \\ t_n(s) &= t^n_{k-1} + (s - s^n_{k-1}) && \text{for } s \in [s^n_{k-1}, s^n_{k,0}], \end{aligned}$$

so that $t^n(s^n_{k,0}) = t^n_k$. Next, for $s \in [s^n_{k,i}, s^n_{k,i+1/2}]$ for $i \geq 0$ we define the affine interpolates as follows:

$$\begin{aligned} u_n(s) &= \begin{cases} u^n_{k,i} + (s - s^n_{k,i}) \dfrac{u^n_{k,i+1} - u^n_{k,i}}{s^n_{k,i+1/2} - s^n_{k,i}} & \text{for } s \in [s^n_{k,i}, s^n_{k,i+1/2}] \text{ if } s^n_{k,i+1/2} - s^n_{k,i} > 0 \\ u^n_{k,i} = u^n_{k,i+1} & \text{for } s = s^n_{k,i+1/2} = s^n_{k,i} \end{cases}, \\ z_n(s) &= z^n_{k,i}, \\ t_n(s) &= t^n_k. \end{aligned}$$

Similarly, for $s \in [s^n_{k,i+1/2}, s^n_{k,i+1}]$ and $i \geq 0$

$$\begin{aligned} u_n(s) &= u^n_{k,i+1}, \\ z_n(s) &= \begin{cases} z^n_{k,i} = z^n_{k,i+1} & \text{for } s = s^n_{k,i} = s^n_{k,i+1/2} \\ z^n_{k,i} + (s - s^n_{k,i+1/2}) \dfrac{z^n_{k,i+1} - z^n_{k,i}}{s^n_{k,i+1} - s^n_{k,i+1/2}} & \text{for } s \in [s^n_{k,i+1/2}, s^n_{k,i+1}] \text{ if } s^n_{k,i+1} > s^n_{k,i+1/2} \end{cases} \\ t_n(s) &= t^n_k. \end{aligned}$$

Denoting $S_n = s^n_n$ the total length of the parametrization interval, we have constructed a curve $(t_n, u_n, z_n) : [0, S_n] \to [0, T] \times \mathcal{U} \times \mathcal{Z}$ with $t_n(S_n) = T$. In Theorem 4.1 we proved that S_n is uniformly bounded.

Note that with this definition in the time update interval $(s^n_{k-1}, s^n_{k,0})$ we have

$$u'_n(s) = 0, \quad z'_n(s) = 0, \quad t'_n(s) = 1 \qquad \text{for } s \in (s^n_{k-1}, s^n_{k,0}).$$

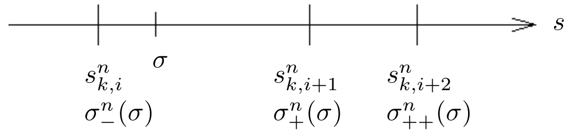

Figure 2: Definition of $\sigma^n_\pm(\sigma)$.

In each interval $(s^n_{k,i}, s^n_{k,i+1})$ for $i \geq 0$ we have

$$\|u_n'(s)\|_{z_n(s)} = \begin{cases} 1 & \text{for } s \in (s^n_{k,i}, s^n_{k,i+1/2}) \\ 0 & \text{for } s \in (s^n_{k,i+1/2}, s^n_{k,i+1}) \end{cases}$$

$$\|z_n'(s)\|_{u_n(s)} = \begin{cases} 0 & \text{for } s \in (s^n_{k,i}, s^n_{k,i+1/2}) \\ 1 & \text{for } s \in (s^n_{k,i+1/2}, s^n_{k,i+1}) \end{cases}$$

$$t_n'(s) = 0 \quad \text{for } s \in (s^n_{k,i}, s^n_{k,i+1}) \,.$$

Observe that $(s^n_{k,i}, s^n_{k,i+1/2})$ might be the empty set. Observe further that thanks to the uniform equivalence of the norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{z_n(s)}$ on $\mathcal{U}$ and of the norms $\|\cdot\|_{\mathcal{Z}}$ and $\|\cdot\|_{u_n(s)}$ on $\mathcal{Z}$ there exists a constant $C \in (0,1]$ (independently of n, s) such that

$$\|u_n'(s)\|_{\mathcal{U}} > C \text{ for } s \in (s^n_{k,i}, s^n_{k,i+1/2}) \tag{36}$$

$$\|z_n'(s)\|_{\mathcal{Z}} > C \text{ for } s \in (s^n_{k,i+1/2}, s^n_{k,i+1}) \tag{37}$$

provided that the respective interval is not empty.

Corollary 4.2. *There exists $C > 0$ such that for all $n \in \mathbb{N}$ the interpolating functions u_n, z_n, t_n satisfy $S_n \leq C$ and*

$$\|u_n\|_{W^{1,\infty}((0,S_n),\mathcal{U})} + \|z_n\|_{W^{1,\infty}((0,S_n),\,\mathcal{Z})} + \|t_n\|_{W^{1,\infty}((0,S_n),[0,T])} \leq C. \tag{38}$$

Estimate $S_n < C$ and (38) are a direct consequence of (26) in combination with the fact that by (27) the standard norms in $\mathcal{U}$ and $\mathcal{Z}$ and the corresponding weighted norms $\|\cdot\|_{z_n}$ on $\mathcal{U}$ and $\|\cdot\|_{u_n}$ on $\mathcal{Z}$ are uniformly equivalent, see Lemma 2.4.

4.4 Uniform non degeneracy of the interpolating curves

The next technical Lemma shows that the curves $s \mapsto (t_n(s), u_n(s), z_n(s))$ enjoy a uniform non-degeneracy property. The proof uses estimates from the previous Theorem and in particular exploits the unidirectionality of the curve $s \mapsto z_n(s)$. The following definitions will be used for $\sigma \in [0, S_n)$;

$$\sigma^n_+(\sigma) = \min\{\, s^n_{k,i} \,:\, s^n_{k,i} \geq \sigma, 0 \leq k \leq n, i \in \mathbb{N} \cup \{-1,0\} \,\}\,,$$
$$\sigma^n_-(\sigma) = \max\{\, s^n_{k,i} \,:\, s^n_{k,i} \leq \sigma, 0 \leq k \leq n, i \in \mathbb{N} \cup \{-1,0\} \,\}\,.$$

In other words, among the discrete points $s^n_{k,i}$, $\sigma^n_+(\sigma)$ is the smallest that is greater or equal to σ, while $\sigma^n_-(\sigma)$ is the largest that is less than or equal to σ. Let $\sigma \in [0, S_n)$. By definition, there exist unique indices $k \in \{0, \dots, n\}$, $m \in \mathbb{N} \cup \{-1,0\}$ such that $\sigma^n_+(\sigma) = s^n_{k,m}$. We define $\sigma^n_{++}(\sigma)$ to be the next discrete point, i.e. $\sigma^n_{++}(\sigma) = s^n_{k,m+1}$, see Figure 2. Moreover, the Heaviside function $H : \mathbb{R} \to \{0,1\}$ is defined by $H(s) = 1$ if $s \geq 0$ and $H(s) = 0$ otherwise.

Lemma 4.3. *There exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in [0, S_n)$ with $\sigma_1 < \sigma_2$ it holds*

$$
\begin{aligned}
C(\sigma_2 - \sigma_1) \leq (t_n(\sigma_2) - t_n(\sigma_1)) + \|z_n(\sigma_2) - z_n(\sigma_1)\|_{\mathcal{Z}} + \|u_n(\sigma_2) - u_n(\sigma_1)\|_{\mathcal{U}} \\
+ H(\sigma_-^n(\sigma_2) - \sigma_+^n(\sigma_1))\Big(\big\|z_n(\sigma_2) - z_n(\sigma_-^n(\sigma_2))\big\|_{\mathcal{Z}} + \big\|z_n(\sigma_1) - z_n(\sigma_+^n(\sigma_1))\big\|_{\mathcal{Z}} \\
+ \big\|u_n(\sigma_2) - u_n(\sigma_-^n(\sigma_2))\big\|_{\mathcal{U}} + \big\|u_n(\sigma_1) - u_n(\sigma_+^n(\sigma_1))\big\|_{\mathcal{U}} \Big) \\
+ H(\sigma_-^n(\sigma_2) - \sigma_{++}^n(\sigma_1))\Big(\big\|z_n(\sigma_+^n(\sigma_1)) - z_n(\sigma_{++}^n(\sigma_1))\big\|_{\mathcal{Z}} + \big\|u_n(\sigma_+^n(\sigma_1)) - u_n(\sigma_{++}^n(\sigma_1))\big\|_{\mathcal{U}} \Big).
\end{aligned} \tag{39}
$$

Proof. In this proof we assume that for each $k \in \{0, \ldots, n\}$ the alternate minimization algorithm does not have a fixed point at finite indices i, the latter case is very similar and is discussed in Remark 4.4.

Let $\sigma_1 < \sigma_2 \in [0, S_n]$. By definition there exist indices $k \leq \ell$ and minimal indices $m, N \in \mathbb{N} \cup \{-1, 0\}$ such that

$$
\sigma_+^n(\sigma_1) = s_{k,m}^n, \sigma_-^n(\sigma_2) = s_{\ell,N}^n. \tag{40}
$$

Remember that for $N = -1$ we have $s_{\ell,-1}^n = s_{\ell-1,\infty}^n = s_{\ell-1}^n$, cf. (34), and similar for the case $m = -1$. Observe that $m = -1$ may only occur if $\sigma_1 \in \{s_{k,-1}^n : 0 \leq k \leq n\}$.

We distinguish two cases.

Case 1, $\sigma_+^n(\sigma_1) > \sigma_-^n(\sigma_2)$: In this case, $\sigma_+^n(\sigma_1) > \sigma_2$, because otherwise, by the definition of σ_-^n, we would have $\sigma_-^n(\sigma_2) \geq \sigma_+^n(\sigma_1)$, which is excluded in Case 1. Similarly, $\sigma_-^n(\sigma_2) < \sigma_1$. Moreover, $m > -1$, because otherwise $\sigma_1 = s_{k,-1}^n$ and $\sigma_-^n(\sigma_2) \geq \sigma_1$, which is a contradiction. Hence, $\sigma_-^n(\sigma_2) = s_{k,m-1}^n$ and $(\sigma_1, \sigma_2) \subset (s_{k,m-1}^n, s_{k,m}^n)$.

If now $m = 0$, then $t_n'(r) = 1$ on $(s_{k,m-1}^n, s_{k,m}^n)$, while $s \mapsto z_n(s)$, $s \mapsto u_n(s)$ are constant on this interval. Hence, we have

$$
C(\sigma_2 - \sigma_1) \leq t_n(\sigma_2) - t_n(\sigma_1) + \|z_n(\sigma_2) - z_n(\sigma_1)\|_{\mathcal{Z}} + \|u_n(\sigma_2) - u_n(\sigma_1)\|_{\mathcal{U}} \tag{41}
$$

with the constant C from (36)–(37). This is (39). If $m > 0$, then by (36)–(37) the estimate (41) is valid, as well.

Case 2, $\sigma_+^n(\sigma_1) \leq \sigma_-^n(\sigma_2)$: Having in mind (40) we write $\sigma_2 - \sigma_1 = (\sigma_2 - \sigma_-^n(\sigma_2)) + (s_{\ell,N}^n - s_{k,m}^n) + (\sigma_+^n(\sigma_1) - \sigma_1)$ and estimate each of the differences separately. Observe that $(\sigma_-^n(\sigma_2), \sigma_2) \subset (s_{l,N}^n, s_{l,N+1}^n)$. Hence, arguing like in the first case we obtain

$$
\begin{aligned}
C(\sigma_2 - \sigma_-^n(\sigma_2)) \leq t_n(\sigma_2) - t_n(\sigma_-^n(\sigma_2)) + \big\|z_n(\sigma_2) - z_n(\sigma_-^n(\sigma_2))\big\|_{\mathcal{Z}} + \\
+ \big\|u_n(\sigma_2) - u_n(\sigma_-^n(\sigma_2))\big\|_{\mathcal{U}},
\end{aligned} \tag{42}
$$

again with C from (36)–(37). Moreover, with the same constant C we have

$$
\begin{aligned}
C(\sigma_+^n(\sigma_1) - \sigma_1) \leq t_n(\sigma_+^n(\sigma_1)) - t_n(\sigma_1) + \big\|z_n(\sigma_+^n(\sigma_1)) - z_n(\sigma_1)\big\|_{\mathcal{Z}} + \\
+ \big\|u_n(\sigma_+^n(\sigma_1)) - u_n(\sigma_1)\big\|_{\mathcal{U}}.
\end{aligned} \tag{43}
$$

Indeed, if $\sigma_+^n(\sigma_1) > \sigma_1$, then (see above) $m > -1$ and $(\sigma_1, \sigma_+^n(\sigma_1)) \subset (s_{k,m-1}^n, s_{k,m}^n)$. Now we can argue similarly to Case 1.

In order to estimate the missing term $s_{\ell,N}^n - s_{k,m}^n$ we assume that $k < \ell$; the modifications for the case $k = \ell$ are then obvious. Observe first that with C from (36)–(37) it holds (with

$z^n_{k,-1} := z^n_{k,0}$, $z^n_{k,-1} := z^n_{k,0}$)

$$\begin{aligned}
C(s^n_{\ell,N} - s^n_{k,m}) &= C\Big(\int_{s^n_{k,m}}^{s^n_{\ell,N}} |t_n'(s)| + \|z_n'(s)\|_{u_n(s)} + \|u_n'(s)\|_{z_n(s)} \, ds \Big) \\
&\leq \int_{s^n_{k,m}}^{s^n_{\ell,N}} |t_n'(s)| + \|z_n'(s)\|_{\mathcal{Z}} + \|u_n'(s)\|_{\mathcal{U}} \, ds \\
&= t_n(s^n_{\ell,N}) - t_n(s^n_{k,m}) + \sum_{i=m+1}^{\infty} \big(\|z^n_{k,i} - z^n_{k,i-1}\|_{\mathcal{Z}} + \|u^n_{k,i} - u^n_{k,i-1}\|_{\mathcal{U}} \big) \\
&\quad + \left(\sum_{\mu=k+1}^{\ell-1} \sum_{i=1}^{\infty} \big(\|z^n_{\mu,i} - z^n_{\mu,i-1}\|_{\mathcal{Z}} + \|u^n_{\mu,i} - u^n_{\mu,i-1}\|_{\mathcal{U}} \big) \right) \\
&\quad + \sum_{i=1}^{N} \big(\|z^n_{\ell,i} - z^n_{\ell,i-1}\|_{\mathcal{Z}} + \|u^n_{\ell,i} - u^n_{\ell,i-1}\|_{\mathcal{U}} \big) \\
&=: t_n(s^n_{\ell,N}) - t_n(s^n_{k,m}) + M_1 + M_2 + M_3.
\end{aligned}$$

By (32) and (33), we have

$$\begin{aligned}
M_2 &\leq \big|t_n(s^n_{\ell,N}) - t_n(s^n_{k,m})\big| + C_2 \sum_{\mu=k+1}^{\ell-1} \sum_{i=1}^{\infty} \|z^n_{k,i-1} - z^n_{k,i}\|_{L^1(\Omega)} \\
&\leq |t_n(\sigma_2) - t_n(\sigma_1)| + C_2 \|z_n(\sigma_1) - z_n(\sigma_2)\|_{L^1(\Omega)} ,
\end{aligned}$$

where for the last estimate we exploited the monotonicity of z_n and t_n, compare also (31) and the fact that $z_n(\sigma_2) \leq z^n_{\ell,0} = z^n_{\ell-1,\infty} \leq z^n_{k+1,0} = z^n_{k,\infty} \leq z_n(\sigma_1)$. Since $\mathcal{Z} \subset L^1(\Omega)$, we finally obtain

$$M_2 \leq |t_n(\sigma_2) - t_n(\sigma_1)| + C_3 \|z_n(\sigma_1) - z_n(\sigma_2)\|_{\mathcal{Z}}$$

with a constant C_3 that is independent of σ_1, σ_2 and n.

Since the arguments for estimating M_1 and M_3 are similar, we give here the estimate for M_1, only. We start from estimate (30) with $i = m+2$ on the left hand side and $N = \infty$, add the term $\|z^n_{k,m+1} - z^n_{k,m}\|_{\mathcal{Z}}$ on both sides and obtain after multiplying by 2:

$$\sum_{i=m+1}^{\infty} \|z^n_{k,i} - z^n_{k,i-1}\|_{\mathcal{Z}} \leq 2 \|z^n_{k,m+1} - z^n_{k,m}\|_{\mathcal{Z}} + 2C_2 \sum_{i=m+1}^{\infty} \|z^n_{k,i-1} - z^n_{k,i}\|_{L^1(\Omega)} .$$

Again thanks to Lemma A.1, in a similar way as in (33), we obtain

$$\sum_{i=m+1}^{\infty} \|u^n_{k,i} - u^n_{k,i-1}\|_{\mathcal{U}} \leq \|u^n_{k,m+1} - u^n_{k,m}\|_{\mathcal{U}} + C\big((t_n(\sigma_2) - t_n(\sigma_1)) + \sum_{i=m+1}^{\infty} \|z^n_{k,i-1} - z^n_{k,i}\|_{L^1(\Omega)} \big) .$$

Adding both estimates, with arguments similar to those for M_2 we finally arrive at

$$\begin{aligned}
M_1 \leq C_4 \big(&(t_n(\sigma_2) - t_n(\sigma_1)) + \|z_n(\sigma_2) - z_n(\sigma_1)\|_{\mathcal{Z}} \\
&+ \|z_n(\sigma^n_{+}(\sigma_1)) - z_n(\sigma^n_{++}(\sigma_1))\|_{\mathcal{Z}} + \|u_n(\sigma^n_{+}(\sigma_1)) - u_n(\sigma^n_{++}(\sigma_1))\|_{\mathcal{U}} \big) .
\end{aligned}$$

Joining all estimates finishes the proof of the Lemma. □

Remark 4.4. *If for some t^n_κ the alternate minimization algorithm arrives at a fixed point at some finite index i_{max}, the above proof has to be modified as we will describe below. However, this modification is basically a matter of notation rather than a mathematical issue.*

Let (n, κ) be a pair with $0 \leq \kappa \leq n$ such that there exists $i_{max} \in \mathbb{N}_0$ with one of the following three properties: If $i_{max} = 0$, then $(u^n_{k,0}, z^n_{k,0})$ is a fixed point of the alternate minimization algorithm with $t = t^n_k$. If $i_{max} \geq 1$, then either $(u^n_{\kappa,i_{max}}, z^n_{\kappa,i_{max}-1})$ is a fixed point of the alternate minimization algorithm with $u^n_{\kappa,i_{max}-1} \neq u^n_{\kappa,i_{max}}$, or $(u^n_{\kappa,i_{max}}, z^n_{\kappa,i_{max}})$ is a fixed point with $z^n_{\kappa,i_{max}-1} \neq z^n_{\kappa,i_{max}}$ and $u^n_{\kappa,i_{max}-1} \neq u^n_{\kappa,i_{max}}$. (For the latter observe that $u^n_{k,i-1} = u^n_{k,i}$ implies that $z^n_{k,i-1} = z^n_{k,i}$.)

Let $(u^n_{\kappa,i_1}, z^n_{\kappa,i_2})$ denote this fixed point (with the indices i_1, i_2 minimal). Observe that the identities $u^n_{\kappa+1,0} = u^n_{\kappa,\infty} = u^n_{\kappa,i_1}$, $z^n_{\kappa+1,0} = z^n_{\kappa,i_2}$ hold. Moreover, the slopes of $\mathcal{F}$ satisfy

$$\left|\partial_u \mathcal{F}(t^n_\kappa, u^n_{\kappa,i_1}, z^n_{\kappa,i_2})\right|_{z^n_{\kappa,i_2}} = 0, \quad \left|\partial_z \mathcal{F}(t^n_\kappa, u^n_{\kappa,i_1}, z^n_{\kappa,i_2})\right|_{u^n_{\kappa,i_1}} = 0.$$

Coming back to the proof of Lemma 4.3 we have to be more careful with the cases $m = -1$ or $N = i_1 - 1$. Let $\sigma_1, \sigma_2 \in [0, S_n)$ with $\sigma_1 < \sigma_2$. Let further the indices $\kappa, \ell, m, \mathbb{N}$ be given according to (40) *with (κ, m) and (ℓ, N) being minimal for σ^n_+, σ^n_- in lexicographical order. Now, the case $m = -1$ can occur if either (A) or (B) hold, where*

- *(A) $\sigma_1 = s^n_{k,-1}$ and the alternate minimization at time t^n_{k-1} does not reach a fixed point after a finite number of iterations,*
- *(B) the alternate minimization procedure at t^n_{k-1} reaches a fixed point after a finite number of steps.*

In case (B), $\sigma_+(\sigma_1) = s^n_{k,-1}$ implies $\sigma_1 \in (s^n_{k-1,i_1-1}, s^n_{k,-1}]$.

Case 1, $\sigma^n_-(\sigma_2) < \sigma^n_+(\sigma_1)$: *As in the proof of Lemma 4.3 we conclude that $\sigma^n_-(\sigma_2) < \sigma_1 < \sigma_2 < \sigma^n_+(\sigma_1)$. In case (A) we conclude as in the proof of Lemma 4.3, while in case (B) we have $\sigma^n_-(\sigma_2) = s^n_{k-1,i_1-1}$. But now we can conclude in the same way as in case (A) and obtain* (41).

Case 2, $\sigma^n_-(\sigma_2) \geq \sigma^n_+(\sigma_1)$: *Similarly to the modified Case 1 here above and the arguments of Case 2 in the proof of Lemma 4.3, we find* (43). *Concerning estimate* (42) *observe that $\sigma^n_-(\sigma_2) \in \{s^n_{\ell,N+1}, s^n_{\ell+1,-1}\}$ depending on whether $N = i_1 - 1$ (in case (B)) or not (i.e. in case (A)). Hence, estimate* (42) *now can be concluded in the usual way. The estimates for $s^n_{\ell,N} - s^n_{k,m}$ do not have to be modified.*

4.5 Discrete stationarity and energy balance

Lemma 3.1 and Lemma 3.2 prove the following stationarity and energy equality for the parametrized evolution (cf. Lemma 3.1 and 3.5).

Lemma 4.5. *For every index $n \in \mathbb{N}$, $0 \leq k \leq n$, it holds*

$$|\partial_u \mathcal{F}(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k))|_{z_n(s^n_k)} = 0\,, \qquad |\partial_z \mathcal{F}(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k))|_{u_n(s^n_k)} = 0\,.$$

Lemma 4.6. *For every $\bar{s} \in [s^n_k, s^n_{k+1}]$ it holds*

$$\begin{aligned}
\mathcal{F}\big(t_n(\bar{s}), u_n(\bar{s}), z_n(\bar{s})\big) = {} & \mathcal{F}\big(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k)\big) + \\
& - \int_{s^n_k}^{\bar{s}} |\partial_u \mathcal{F}(t_n(s), u_n(s), z_n(s))|_{z_n(s)} \; \|u'_n(s)\|_{z_n(s)} \, ds + \\
& - \int_{s^n_k}^{\bar{s}} |\partial_z \mathcal{F}(t_n(s), u_n(s), z_n(s))|_{u_n(s)} \; \|z'_n(s)\|_{u_n(s)} \, ds + \\
& + \int_{s^n_k}^{\bar{s}} \partial_t \mathcal{F}(t_n(s), u_n(s), z_n(s)) \; t'_n(s) \, ds.
\end{aligned}$$

Proof. Since $r \mapsto \mathcal{F}(t_n(r), u_n(r), z_n(r))$ is absolutely continuous, we have

$$\mathcal{F}\big(t_n(s), u_n(s), z_n(s)\big) = \mathcal{F}\big(t_n(s_k^n), u_n(s_k^n), z_n(s_k^n)\big) + \int_{s_k^n}^{s} \frac{d}{dr}\mathcal{F}(t_n(r), u_n(r), z_n(r))\, dr$$

and by the chain rule

$$\begin{aligned}\frac{d}{dr}\mathcal{F}(t_n(r), u_n(r), z_n(r)) = {}& \partial_t\mathcal{F}(t_n(r), u_n(r), z_n(r))\, t_n'(r)) + \partial_u\mathcal{F}(t_n(r), u_n(r), z_n(r))[u_n'(r)] \\ & + \partial_z\mathcal{F}(t_n(r), u_n(r), z_n(r))[z_n'(r)].\end{aligned}$$

The following arguments are valid for intervals $[s^n_{k,i}, s^n_{k,i+\frac{1}{2}}]$ and $[s^n_{k,i+\frac{1}{2}}, s^n_{k,i+1}]$ with positive length.

Consider an interval $[s^n_{k,i}, s^n_{k,i+\frac{1}{2}}]$ where $t_n' = 0$ and $z_n' = 0$. We will invoke (20) with the change of variable $r = (s - s^n_{k,i})/(s^n_{k,i+\frac{1}{2}} - s^n_{k,i})$ and with

$$u_i = u_n(s^n_{k,i}), \quad u_{i+1} = u_n(s^n_{k,i+\frac{1}{2}}), \quad u_{i+r} = u_n(s).$$

Note that (since they are constant) we have $t = t_n(s)$, $z_i = z_n(s)$ for every $s \in [s^n_{k,i}, s^n_{k,i+\frac{1}{2}}]$. Thus we obtain for $\bar{s} \in [s^n_{k,i}, s^n_{k,i+\frac{1}{2}}]$

$$\begin{aligned}\mathcal{F}(t_n(\bar{s}), u_n(\bar{s}), z_n(\bar{s})) &= \mathcal{F}(t_n(s^n_{k,i}), u_n(s^n_{k,i}), z_n(s^n_{k,i})) - \\ &\quad + \int_{s^n_{k,i}}^{\bar{s}} |\partial_u\mathcal{F}(t_n(s), u_n(s), z_n(s))|_{z_n(s)} \left\| \frac{u_n(s^n_{k,i+1}) - u_n(s^n_{k,i+1})}{s^n_{k,i+\frac{1}{2}} - s^n_{k,i}} \right\|_{z_n(s)} ds \\ &= \mathcal{F}(t_n(s^n_{k,i}), u_n(s^n_{k,i}), z_n(s^n_{k,i})) + \\ &\quad - \int_{s_k^n}^{\bar{s}} |\partial_u\mathcal{F}(t_n(s), u_n(s), z_n(s))|_{z_n(s)}\ \|u_n'(s)\|_{z_n(s)}\, ds + \\ &\quad - \int_{s_k^n}^{\bar{s}} |\partial_z\mathcal{F}(t_n(s), u_n(s), z_n(s))|_{u_n(s)}\ \|z_n'(s)\|_{u_n(s)}\, ds + \\ &\quad + \int_{s_k^n}^{\bar{s}} \partial_t\mathcal{F}(t_n(s), u_n(s), z_n(s))\ t_n'(s)\, ds.\end{aligned}$$

Invoking (21) leads to the same energy equality in the intervals $[s^n_{k,i+\frac{1}{2}}, s^n_{k,i+1}]$ where $t_n' = 0$ and $u_n' = 0$.

It remains to consider the time update interval $[s^n_{k,-1}, s^n_{k,-1} + \Delta t^n]$ where $u_n' = 0$ and $z_n' = 0$. In this case it sufficient to apply the chain rule remembering that z_n and u_n are constant.

□

4.6 Compactness

By Theorem 4.1 we know that $(t_n, u_n, z_n) \in W^{1,\infty}([0, S_n], [0, T] \times \mathcal{U} \times \mathcal{Z})$ where S_n is uniformly bounded. Denote $S = \liminf_{n\to\infty} S_n$ and consider that $t_n \in W^{1,\infty}([0, S], [0, T])$, $u_n \in W^{1,\infty}([0, S], \mathcal{U})$ and $z_n \in W^{1,\infty}([0, S], \mathcal{Z})$ using (if necessary) a constant extension in the set $(S_n, S]$. Note that $0 < T \le S < +\infty$.

Lemma 4.7. *Up to subsequences (not relabeled) (t_n, u_n, z_n) converge to (t, u, z) weakly* in $W^{1,\infty}([0, S], [0, T] \times \mathcal{U} \times \mathcal{Z})$. Moreover, if $s_n \to s$ then $t_n(s_n) \to t(s)$, $u_n(s_n) \to u(s)$ in $\mathcal{U}^{\tilde{p}}$ (for $\tilde{p} \in [2, p]$) and $z_n(s_n) \rightharpoonup z(s)$ in $\mathcal{Z}$. Finally, $t(S) = T$ and $s \mapsto z(s)$ is monotone non-increasing.*

Proof. By Corollary 4.2, weak* convergence in $W^{1,\infty}([0,S],[0,T]\times\mathcal{U}\times\mathcal{Z})$ is standard, as well as the convergence of $t_n(s_n)$ and the weak convergence of $u_n(s_n)$ in $\mathcal{U}$ and $z_n(s_n)$ in $\mathcal{Z}$. Since $z_n(s)\to z(s)$ strongly in $L^1(\Omega)$ and thus a.e. (up to subsequences) it follows that $z(\cdot)$ is monotone non-increasing. Moreover, by definition $t_n(S)=T$ and thus $t(S)=T$.

It remains to show the strong convergence of $u_n(s_n)$ to $u(s)$ in $\mathcal{U}^{\tilde p}$ if $s_n\to s$. Let us pick up a subsequence (not relabeled) of $u_n(s_n)$. Choose k,i (depending on n) such that $s_n\in[s^n_{k,i},s^n_{k,i+1}]$. Assume that $i\geq 1$ (the cases $i=-1,0$ are slightly different). Then in the point s_n for some $\lambda^n_{k,i}\in[0,1]$ we have

$$u_n(s_n)=\lambda^n_{k,i}u_n(s^n_{k,i})+(1-\lambda^n_{k,i})u_n(s^n_{k,i+1})\,.$$

Remember that $u_n(s^n_{k,i+1})=u_n(s^n_{k,i+1/2})$ and that

$$u_n(s^n_{k,i+1/2})\in\operatorname{argmin}\big\{\mathcal{F}\big(t^n(s^n_{k,i+1/2}),u,z^n(s^n_{k,i+1/2})\big):u\in\mathcal{U}\big\}.$$

Similarly, $u_n(s^n_{k,i})=u_n(s^n_{k,i-1/2})$ with

$$u_n(s^n_{k,i-1/2})\in\operatorname{argmin}\big\{\mathcal{F}\big(t^n(s^n_{k,i-1/2}),u,z^n(s^n_{k,i-1/2})\big):u\in\mathcal{U}\big\}.$$

Since $0\leq s^n_{k,i-1/2}\leq s^n_{k,i+1/2}\leq S_n$ is uniformly bounded, cf. Theorem 4.1, we can extract a subsequence (not relabeled) s.t.

$$s^n_{k,i-1/2}\to\underline{s},\qquad s^n_{k,i+1/2}\to\bar{s}.$$

As a consequence $t_n(s^n_{k,i-1/2})\to t(\underline{s})$ and $z_n(s^n_{k,i-1/2})\to z(\underline{s})$ strongly in L^q for every $q<\infty$ (by compact embedding). Hence $t_n(s^n_{k,i-1/2})$ and $z_n(s^n_{k,i-1/2})$ are Cauchy sequences with respect to n and thus by Lemma A.1 $u_n(s^n_{k,i})=u_n(s^n_{k,i-1/2})$ is a Cauchy sequence in $\mathcal{U}^{\tilde p}$. The same holds for $u_n(s^n_{k,i+1})=u_n(s^n_{k,i+1/2})$.

Clearly $s_n\to s\in[\underline{s},\bar{s}]$. Up to subsequences $\lambda^n_{k,i}\to\lambda$ and thus $u_n(s_n)\to\lambda u(\underline{s})+(1-\lambda)u(\bar{s})$ strongly in $\mathcal{U}^{\tilde p}$. Since $u_n(s_n)\rightharpoonup u(s)$ in $\mathcal{U}$ it follows that $u_n(s_n)\to u(r)=\lambda u(\underline{s})+(1-\lambda)u(\bar{s})$ in $\mathcal{U}^{\tilde p}$. Since the limit is independent of the subsequences the whole sequence converges.

Finally, in the cases $s\in[s^n_{k,i},s^n_{k,i+1}]$ for $i=-1,0$ it is sufficient to use the above argument, replacing $s^n_{k,i-1/2}$ with $s^n_{k,-1}$ and $s^n_{k,i+1/2}$ with $s^n_{k,1/2}$.

□

4.7 Convergence to a quasi-static parametrized evolution

Theorem 4.8. *Let (t,u,z) be a weak* limit of (t_n,u_n,z_n) provided by Lemma 4.7. Then (t,u,z) satisfies $(u(0),z(0))=(u_0,z_0)$, $t(0)=0$, $t(S)=T$ and*

(S') *for every $s\in[0,S]$ with $t'(s)>0$*

$$|\partial_u\mathcal{F}(t(s),u(s),z(s))|_{z(s)}=0,\qquad |\partial_z\mathcal{F}(t(s),u(s),z(s))|_{u(s)}=0,\tag{44}$$

(E') *for every $s\in[0,S]$*

$$\begin{aligned}\mathcal{F}(t(s),u(s),z(s))=&\mathcal{F}(0,u(0),z(0))-\int_0^s|\partial_u\mathcal{F}(t(r),u(r),z(r))|_{z(r)}\;\|u'(r)\|_{z(r)}\,dr\;+\\&-\int_0^s|\partial_z\mathcal{F}(t(r),u(r),z(r))|_{u(r)}\;\|z'(r)\|_{u(r)}\,dr\;+\\&+\int_0^s\partial_t\mathcal{F}(t(r),u(r),z(r))\,t'(r)\,dr\,.\end{aligned}\tag{45}$$

Moreover, for almost all $s \in [0, S]$ it holds

$$|\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} \left\| z'(s) \right\|_{u(s)} = -\partial_z \mathcal{F}(t(s), u(s), z(s))[z'(s)], \tag{46}$$

$$|\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} \left\| u'(s) \right\|_{z(s)} = -\partial_u \mathcal{F}(t(s), u(s), z(s))[u'(s)]. \tag{47}$$

Definition 4.9. *A triple $(t, u, z) \in W^{1,\infty}([0, S]; [0, T] \times \mathcal{U} \times \mathcal{Z})$ is called a parametrized BV-solution for the functional $\mathcal{F} : [0, T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ with initial data $(u_0, z_0) \in \mathcal{U} \times \mathcal{Z}$ if $s \mapsto t(s)$ is non decreasing with $t(0) = 0$, $t(S) = T$, $u(s) \in W^{1,\widetilde{p}}(\Omega)$ uniformly in $s \in [0, S]$ for some $\widetilde{p} > 2$, $s \mapsto z(s)$ is non increasing and (S') and (E') (i.e.* (44) *and* (45)*) are satisfied.*

Proof. Let us start with property (S'). Let $t'(r) > 0$. Then, by [23, Theorem 3.4], there exists a sequence s^n_k (with k depending on n) such that $s^n_k \to s$ and with

$$\left|\partial_u \mathcal{F}(t_n(s^n_k), u(s^n_k), z(s^n_k))\right|_{z(s^n_k)} = 0, \qquad \left|\partial_z \mathcal{F}(t_n(s^n_k), u(s^n_k), z(s^n_k))\right|_{u(s^n_k)} = 0.$$

By Lemma 4.7 we have: $t_n(s^n_k) \to t(s)$, $u_n(s^n_k) \to u(s)$ in $\mathcal{U}^{\tilde{p}}$ and $z_n(s^n_k) \rightharpoonup z(s)$ in $\mathcal{Z}$. It is then sufficient to pass the limit and use the lower semi-continuity of the slopes provided in Lemma 2.5.

Let us prove (E'). Taking the sum over all the subintervals of $[0, s]$ from Lemma 4.6 we obtain

$$\begin{aligned}\mathcal{F}\big(t_n(s), &u_n(s), z_n(s)\big) = \mathcal{F}\big(0, u(0), z(0)\big) + \\ &- \int_0^s |\partial_u \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{z_n(r)} \, \|u'_n(r)\|_{z_n(r)} \, dr + \\ &- \int_0^s |\partial_z \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{u_n(r)} \, \|z'_n(r)\|_{u_n(r)} \, dr + \\ &+ \int_0^s \partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r)) \; t'_n(r) \, dr.\end{aligned}$$

Taking the limsup we obtain

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{F}\big(t_n(s), &u_n(s), z_n(s)\big) \le \mathcal{F}\big(0, u(0), z(0)\big) + \\ &- \liminf_{n\to\infty} \int_0^s |\partial_u \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{z_n(r)} \, \|u'_n(r)\|_{z_n(r)} \, dr + \\ &- \liminf_{n\to\infty} \int_0^s |\partial_z \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{u_n(r)} \, \|z'_n(r)\|_{u_n(r)} \, dr + \\ &+ \limsup_{n\to\infty} \int_0^s \partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r)) \; t'_n(r) \, dr.\end{aligned}$$

Now, by the lower semi-continuity of the integrand, cf. Appendix A.1, we obtain

$$\begin{aligned}\int_0^s |\partial_u \mathcal{F}(t(r), u(r), z(r))|_{z(r)} \, \|u'(r)\|_{z(r)} \, dr \le \\ \le \liminf_{n\to\infty} \int_0^s |\partial_u \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{z_n(r)} \, \|u'_n(r)\|_{z_n(r)} \, dr.\end{aligned} \tag{48}$$

$$\begin{aligned}\int_0^s |\partial_z \mathcal{F}(t(r), u(r), z(r))|_{u(r)} \, \|z'(r)\|_{u(r)} \, dr \le \\ \le \liminf_{n\to\infty} \int_0^s |\partial_z \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{u_n(r)} \, \|z'_n(r)\|_{u_n(r)} \, dr.\end{aligned} \tag{49}$$

Let us see the convergence of the power. Remember that $\partial_t \mathcal{F}(t, u, z) = -\langle \dot{b}(t), u \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the duality between $W^{-1,q}_{\partial_D \Omega}$ and $W^{1,q'}_{\partial_D \Omega}$ for $q > 2$. By the regularity (7) in time of

b we have $\dot{b}(t_n(r)) \to \dot{b}(t(r))$ in $W^{-1,q}_{\partial_D\Omega}$. By Lemma 4.7 $u_n(r) \to u(r)$ in $\mathcal{U}^{\tilde{p}}$ and thus in $W^{1,q'}_{\partial_D\Omega}$ for $q' < 2$.

Hence $\partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r)) \to \partial_t \mathcal{F}(t(r), u(r), z(r))$ pointwise in $[0, S]$. Moreover

$$|\partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r))| = |\langle \dot{b}(t_n(r)), u_n(r)\rangle| \le \|\dot{b}(t_n(r))\|_{W^{-1,q}_{\partial_D\Omega}} \|u_n(r)\|_{W^{1,q'}_{\partial_D\Omega}} \le C$$

again by (7) and by (38). Thus by dominated convergence, $\partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r))$ converge to $\partial_t \mathcal{F}(t(r), u(r), z(r))$ in $L^1(0, S)$. Since $t_n' \overset{*}{\rightharpoonup} t'$ in $L^\infty(0, S)$ we obtain the convergence of the work, i.e.

$$\lim_{n\to\infty} \int_0^s \partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r)) \; t_n'(r)\, dr = \int_0^s \partial_t \mathcal{F}(t(r), u(r), z(r)) \; t'(r)\, dr.$$

Therefore

$$\begin{aligned} \limsup_{n\to\infty} \mathcal{F}\big(t_n(s), u_n(s), z_n(s)\big) \le \mathcal{F}(0, u(0), z(0)) + & \\ & - \int_0^s |\partial_u \mathcal{F}(t(r), u(r), z(r))|_{z(r)} \; \|u'(r)\|_{z(r)}\, dr \; + \\ & - \int_0^s |\partial_z \mathcal{F}(t(r), u(r), z(r))|_{u(r)} \; \|z'(r)\|_{u(r)}\, dr \; + \\ & + \int_0^s \partial_t \mathcal{F}(t(r), u(r), z(r))\, t'(r)\, dr\,. \end{aligned} \tag{50}$$

By definition of the slope

$$-|\partial_u \mathcal{F}(t(r), u(r), z(r))|_{z(r)} \; \|u'(r)\|_{z(r)} \le \partial_u \mathcal{F}(t(r), u(r), z(r))[u'(r)], \tag{51}$$

$$-|\partial_z \mathcal{F}(t(r), u(r), z(r))|_{u(r)} \; \|z'(r)\|_{u(r)} \le \partial_z \mathcal{F}(t(r), u(r), z(r)))[z'(r)]. \tag{52}$$

Hence by the chain rule

$$\begin{aligned} \limsup_{n\to\infty} \mathcal{F}\big(t_n(s), u_n(s), z_n(s)\big) \le \; & \mathcal{F}\big(0, u(0), z(0)\big) + \\ & + \int_0^s \partial_u \mathcal{F}(t(r), u(r), z(r))[u'(r)] + \partial_z \mathcal{F}(t(r), u(r), z(r))[z'(r)] dr \\ & + \int_0^s \partial_t \mathcal{F}(t(r), u(r), z(r))\, t'(r) dr \\ & \le \mathcal{F}\big(0, u(0), z(0)\big) + \int_0^s \frac{d}{dr} \mathcal{F}(t(r), u(r), z(r))\, dr \\ & = \mathcal{F}(t(r), u(r), z(s)). \end{aligned}$$

By Lemma 2.2 $\mathcal{F}$ is also sequentially lower semi-continuous, thus

$$\mathcal{F}(t(s), u(s), z(s)) \le \liminf_{n\to\infty} \mathcal{F}\big(t_n(s), u_n(s), z_n(s)\big).$$

Thus (E') holds. Finally, a proof by contradiction using (E') shows that (51)–(52) in fact hold with equality. □

The next Corollary is a consequence of the proof of the previous Theorem.

Corollary 4.10. *$\mathcal{F}(t_n, u_n, z_n) \to \mathcal{F}(t, u, z)$ pointwise in $[0, S]$ and thus $z_n(s)$ converges to $z(s)$ strongly in $\mathcal{Z}$ for every $s \in [0, S]$.*

Proposition 4.11. *Under the assumptions of Theorem 4.8 the limit curve (t, u, z) is non degenerate, i.e. there exists a constant $C > 0$ such that for almost every $s \in [0, S]$ we have*

$$0 < C \le t'(s) + \big\|z'(s)\big\|_{\mathcal{Z}} + \big\|u'(s)\big\|_{\mathcal{U}}\,. \tag{53}$$

Proof. Let $\sigma_1 < \sigma_2 \in [0, S]$. Due to the strong convergence stated in Corollary 4.10 and the upper semicontinuity of the function H, passing to the limit in estimate (39) we obtain for the limit curve

$$\begin{aligned} C \leq (\sigma_2 - \sigma_1)^{-1}\Big(&(t(\sigma_2) - t(\sigma_1)) + \|z(\sigma_2) - z(\sigma_1)\|_{\mathcal{Z}} + \|u(\sigma_2) - u(\sigma_1)\|_{\mathcal{U}} \\ &+ H(\sigma_{2,-} - \sigma_{1,+})\big(\|z(\sigma_2) - z(\sigma_{2,-})\|_{\mathcal{Z}} + \|z(\sigma_1) - z(\sigma_{1,+})\|_{\mathcal{Z}} \\ &\qquad + \|u(\sigma_2) - u(\sigma_{2,-})\|_{\mathcal{U}} + \|u(\sigma_1) - u(\sigma_{1,+})\|_{\mathcal{U}} \big) \\ &+ H(\sigma_{2,-} - \sigma_{1,++})\big(\|z(\sigma_{1,+}) - z(\sigma_{1,++})\|_{\mathcal{Z}} + \|u(\sigma_{1,+}) - u(\sigma_{1,++})\|_{\mathcal{U}} \big)\Big) . \end{aligned} \tag{54}$$

Here, $\sigma_{1,+}, \sigma_{1,++}, \sigma_{2,-}$ are cluster points of the sequences $\sigma^n_+(\sigma_1), \sigma^n_{++}(\sigma_1), \sigma^n_-(\sigma_2)$. Let now $s \in [0, S)$ be a point of differentiability of the triple (t, u, z) and let $(h_\nu)_{\nu\in\mathbb{N}}$ be a sequence with $h_\nu \searrow 0$ for $\nu \to \infty$. With $\sigma_1 = s$, $\sigma_{2,\nu} = s + h_\nu$, the first three terms on the right hand side of (54) tend to $t'(r) + \|z'(r)\|_{\mathcal{Z}} + \|u'(r)\|_{\mathcal{U}}$.

Moreover, let $\sigma_-(\sigma_{2,\nu})$ be a cluster point of the sequence $(\sigma^n_-(\sigma_{2,\nu}))_{n\in\mathbb{N}}$. Assume first that $\sigma_{1,+} > \sigma_1 = s$. Then, for ν large enough, we have $\sigma_{2,\nu} = s + h_\nu < \sigma_{1,+} \leq \sigma_{1,++}$ and hence the term involving the factor $H(\sigma_{2,-} - \sigma_{1,+})$ in (54) vanishes.

If $\sigma_{1,+} = \sigma_1 = s$, then

$$H(\sigma_-(\sigma_{2,\nu}) - \sigma_{1,+})\, |\sigma_{2,\nu} - \sigma_-(\sigma_{2,\nu})| \leq h_\nu. \tag{55}$$

Indeed, if $\sigma_-(\sigma_{2,\nu}) < \sigma_{1,+}$, then $H(\sigma_-(\sigma_{2,\nu}) - \sigma_{1,+}) = 0$. If $\sigma_-(\sigma_{2,\nu}) \geq \sigma_{1,+}$, then the following chain of inequalities holds: $s = \sigma_{1,+} \leq \sigma_-(\sigma_{2,\nu}) = \sigma_-(s + h_\nu) \leq s + h_\nu = \sigma_{2,\nu}$ from which we deduce (55). Altogether we obtain

$$\begin{aligned} \limsup_{\nu\to\infty} H(\sigma_-(\sigma_{2,\nu}) - \sigma_{1,+})h_\nu^{-1}\Big(&\|z(\sigma_{2,\nu}) - z(\sigma_-(\sigma_{2,\nu}))\|_{\mathcal{Z}} + \|z(\sigma_1) - z(\sigma_{1,+})\|_{\mathcal{Z}} \\ &+ \|u(\sigma_{2,\nu}) - u(\sigma_-(\sigma_{2,\nu}))\|_{\mathcal{U}} + \|u(\sigma_1) - u(\sigma_{1,+})\|_{\mathcal{U}} \Big) \\ \leq \|z'(r)\|_{\mathcal{Z}} + \|u'(r)\|_{\mathcal{U}} . \end{aligned}$$

With the same arguments the term involving $H(\sigma_{2,-} - \sigma_{1,++})$ can be treated. This finishes the proof. □

5 A variant: the M-step algorithm

In this section we analyze the convergence of curves $(t_n, u_n, z_n)_{n\in\mathbb{N}}$ that, contrarily to the previous section, are generated by a time incremental M-step algorithm with iteration number $M \in \mathbb{N}$ fixed for all $n \in \mathbb{N}$.

Given $M \in \mathbb{N}$, $z(0) = z_0 \in \mathcal{Z}$, with $0 \leq z_0 \leq 1$, and $u(0) = u_0 \in \mathcal{U}$, let for $n \in \mathbb{N}$ $\Delta t^n = T/n$ and set $t^n_k = k\Delta t^n$ for $k = 0, ..., n$. Set $u^n_0 = u_0$ and $z^n_0 = z_0$. Given u^n_{k-1} and z^n_{k-1} consider the M-step alternate minimization scheme at time $t^n_k = t^n_{k-1} + \Delta t^n$ with initial conditions u^n_{k-1} and z^n_{k-1}, viz. set $u^n_{k,0} = u^n_{k-1}$ and $z^n_{k,0} = z^n_{k-1}$ and then define by induction (for $0 \leq i \leq M - 1$)

$$\begin{cases} u^n_{k,i+1} \in \operatorname{argmin}\{\mathcal{F}(t^n_k, u, z^n_{k,i}) : u \in \mathcal{U}\} \\ z^n_{k,i+1} \in \operatorname{argmin}\{\mathcal{F}(t^n_k, u^n_{k,i+1}, z) : z \in \mathcal{Z} \text{ with } z \leq z^n_{k,i}\}. \end{cases}$$

The updates u^n_k and z^n_k are respectively defined as $u^n_k = u^n_{k,M}$, $z^n_k = z^n_{k,M}$. As in the case $M = \infty$ discussed in the previous sections, the algorithm might reach a fixed point after $i \leq M - 1$ iterations. In this case, similar arguments as were described in detail in the previous sections apply.

As before, for $1 \le k \le n$ let

$$\gamma_{k,M}^n := \sum_{i=1}^{M} \Big(\left\| u_{k,i}^n - u_{k,i-1}^n \right\|_{\mathcal{U}} + \left\| z_{k,i}^n - z_{k,i-1}^n \right\|_{\mathcal{Z}} \Big), \tag{56}$$

which again corresponds to the length of the alternate minimizing path at time t_k^n.

Similarly to Theorem 4.1 it holds

Theorem 5.1. *Suppose the initial data satisfy* (24).
There exists $C > 0$ such that for all $n, M \in \mathbb{N}$

$$\sum_{k=0}^{n} \gamma_{k,M}^n \le C. \tag{57}$$

Proof. The proof is very similar to the one of Theorem 4.1 and we describe here the necessary changes, only. Let us start with an estimate for the first iteration step. For convenience, we also define $z_{0,M-1}^n := z_{0,M}^n := z_0^n$. For $1 \le k \le n$ we have due to Lemma A.2 and taking into account (27), which is valid also for the M-step algorithm:

$$\left\| z_{k,1}^n - z_{k,0}^n \right\|_{\mathcal{Z}} \le C \left\| z_{k,0}^n - z_{k,1}^n \right\|_{L^r(\Omega} \left\| u_{k,0}^n - u_{k,1}^n \right\|_{\mathcal{U}^{\tilde{p}}}$$

with a suitable $r \in [2, \infty)$ independently of n, M. Since $\mathcal{Z}$ is embedded in $L^r(\Omega)$ this reduces to

$$\left\| z_{k,0}^n - z_{k,1}^n \right\|_{\mathcal{Z}} \le C \left\| u_{k,0}^n - u_{k,1}^n \right\|_{\mathcal{U}^{\tilde{p}}} . \tag{58}$$

Next, applying Lemma A.1 with $u_{k,0}^n = u_{\min}(t_{k-1}^n, z_{k-1,M-1}^n)$ and $u_{k,1}^n = u_{\min}(t_k^n, z_{k-1,M}^n)$ gives

$$\left\| u_{k,1}^n - u_{k,0}^n \right\|_{\mathcal{U}^{\tilde{p}}} \le C \Big(\left| t_k^n - t_{k-1}^n \right| + \left\| z_{k-1,M}^n - z_{k-1,M-1}^n \right\|_{L^{r_1}(\Omega)} \Big)$$

with a suitable $r_1 \in [2, \infty)$ that again is independent of n, M. Applying Gagliardo-Nirenberg estimates in the same way as in the proof oh Theorem 4.1 we arrive at

$$\left\| z_{k,0}^n - z_{k,1}^n \right\|_{\mathcal{Z}} \le C \big(\left| t_k^n - t_{k-1}^n \right| + \left\| z_{k,0}^n - z_{k-1,M-1}^n \right\|_{L^1(\Omega)} \big) + \frac{1}{2} \left\| z_{k-1,M}^n - z_{k-1,M-1}^n \right\|_{\mathcal{Z}} . \tag{59}$$

If $M = 1$, this can be rewritten as follows for $1 \le k \le n$ (with $z_{-2} := z_{-1} := z_0$):

$$\left\| z_{k-1}^n - z_k^n \right\|_{\mathcal{Z}} \le C \big(\left| t_k^n - t_{k-1}^n \right| + \left\| z_{k-1}^n - z_{k-2}^n \right\|_{L^1(\Omega)} \big) + \frac{1}{2} \left\| z_{k-1}^n - z_{k-2}^n \right\|_{\mathcal{Z}} . \tag{60}$$

Summing up this estimate with respect to k we find

$$\sum_{k=1}^{n} \left\| z_{k-1}^n - z_k^n \right\|_{\mathcal{Z}} \le C \big(T + \sum_{k=1}^{n} \left\| z_{k-1}^n - z_{k-2}^n \right\|_{L^1(\Omega)} \big) + \frac{1}{2} \sum_{k=1}^{n} \left\| z_{k-1}^n - z_{k-2}^n \right\|_{\mathcal{Z}} . \tag{61}$$

Absorbing the last term on the right hand side into the left hand side and taking into account the monotonicity of the sequence $(z_k^n)_{k \in \mathbb{N}}$ we finally arrive at

$$\frac{1}{2} \sum_{k=1}^{n} \left\| z_{k-1}^n - z_k^n \right\|_{\mathcal{Z}} \le C \big(T + \| z_n^n - z_0^n \|_{L^1(\Omega)} \big) \le C.. \tag{62}$$

Now, (57) is an immediate consequence.

If $M \ge 2$ we proceed similarly to the arguments leading to (30). We obtain for $1 \le k \le n$

$$\left\| z_{k,M}^n - z_{k,M-1}^n \right\|_{\mathcal{Z}} + \frac{1}{2} \sum_{i=2}^{M-1} \left\| z_{k,i}^n - z_{k,i-1}^n \right\|_{\mathcal{Z}} \le \frac{1}{2} \left\| z_{k,1}^n - z_{k,0}^n \right\|_{\mathcal{Z}} + C_2 \sum_{i=1}^{M-1} \left\| z_{k,i-1}^n - z_{k,i}^n \right\|_{L^1(\Omega)} , \tag{63}$$

where the constant C_2 is independent of n, M. Adding (59) yields after rearranging the terms

$$\big\| z^n_{k,M} - z^n_{k,M-1} \big\|_{\mathcal{Z}} + \frac{1}{2} \sum_{i=1}^{M-1} \big\| z^n_{k,i} - z^n_{k,i-1} \big\|_{\mathcal{Z}} \le \frac{1}{2} \big\| z^n_{k-1,M} - z^n_{k-1,M-1} \big\|_{\mathcal{Z}} + C \left(\big| t^n_k - t^n_{k-1} \big| + \big\| z^n_{k-1,M} - z^n_{k-1,M-1} \big\|_{L^1(\Omega)} + \sum_{i=1}^{M-1} \big\| z^n_{k,i-1} - z^n_{k,i} \big\|_{L^1(\Omega)} \right).$$

Summing up this estimate with respect to k we finally obtain

$$\sum_{k=1}^{n} \sum_{i=1}^{M} \big\| z^n_{k,i} - z^n_{k,i-1} \big\|_{\mathcal{Z}} \le \frac{1}{2} \big\| z^n_{0,M} - z^n_{0,M-1} \big\|_{\mathcal{Z}} + C \left(T + \sum_{k=1}^{n} \sum_{i=1}^{M} \big\| z^n_{k,i-1} - z^n_{k,i} \big\|_{L^1(\Omega)} \right).$$

Again, (57) now is an immediate consequence. □

Next, following §4.3 let us define the discrete values s^n_k and $s^n_{k,i}$ (for $i = -1, ..., M$) which will provide the interpolation points in the parametrization interval $[0, S_n]$ whit S_n uniformly bounded. More precisely, let $s^n_0 = 0$ and $s^n_k = s^n_{k,M}$ (for $k \ge 1$) where for $0 \le i \le M-1$

$$s^n_{k,-1} = s^n_{k-1}, \quad s^n_{k,0} = s^n_{k-1} + \Delta t^n,$$
$$s^n_{k,i+1/2} = s^n_{k,i} + \| u^n_{k,i+1} - u^n_{k,i} \|_{z^n_{k,i}}, \qquad s^n_{k,i+1} = s^n_{k,i+1/2} + \| z^n_{k,i+1} - z^n_{k,i} \|_{u^n_{k,i+1}}.$$

Then, we can define the piecewise affine interpolation $(t_n, u_n, z_n) : [0, S_n] \to [0, T] \times \mathcal{U} \times \mathcal{Z}$ in the discrete points $s^n_{k,i}$ exactly as in §4.3.

The next two Lemmas provide equilibrium and energy balance, for the proof see respectively Lemma 4.5 and 4.6.

Lemma 5.2. *For every $n \in \mathbb{N}$ and $0 \le k \le n$ denote $\grave{s}^n_k = s^n_{k,M-1/2}$, then it holds*

$$|\partial_u \mathcal{F}(t_n(s^n_k), u_n(s^n_k), z_n(\grave{s}^n_k))|_{z_n(\grave{s}^n_k)} = 0, \qquad |\partial_z \mathcal{F}(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k))|_{u_n(s^n_k)} = 0. \tag{64}$$

Lemma 5.3. *For every $\bar{s} \in [s^n_k, s^n_{k+1}]$ it holds*

$$\begin{aligned} \mathcal{F}\big(t_n(\bar{s}), u_n(\bar{s}), z_n(\bar{s})\big) = {} & \mathcal{F}\big(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k)\big) + \\ & - \int_{s^n_k}^{\bar{s}} |\partial_u \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{z_n(r)} \; \|u'_n(r)\|_{z_n(r)} \, dr + \\ & - \int_{s^n_k}^{\bar{s}} |\partial_z \mathcal{F}(t_n(r), u_n(r), z_n(r))|_{u_n(r)} \; \|z'_n(r)\|_{u_n(r)} \, dr + \\ & + \int_{s^n_k}^{\bar{s}} \partial_t \mathcal{F}(t_n(r), u_n(r), z_n(r)) \; t'_n(r) \, dr. \end{aligned}$$

It is important to remark that in Lemma 5.2 equilibrium of the displacement field does not hold in $z_n(s^n_k)$ since the number of iterations is finite; as we will see, this will not affect the limit evolution.

Theorem 5.1 guarantees similar uniform bounds for the interpolating curves as in the case $M = \infty$. The resulting compactness of the sequence (t_n, u_n, z_n) is stated in the next Lemma, whose proof substantially coincides with that of Lemma 4.7.

Lemma 5.4. *Up to subsequences (not relabeled) (t_n, u_n, z_n) converge to (t, u, z) weakly* in $W^{1,\infty}([0, S], [0, T] \times \mathcal{U} \times \mathcal{Z})$. In particular if $s_n \to s$ then $t_n(s_n) \to t(s)$, $u_n(s_n) \to u(s)$ in $\mathcal{U}^{\tilde{p}}$ (for $\tilde{p} \in [2, p]$) and $z_n(s_n) \rightharpoonup z(s)$ in $\mathcal{Z}$. Finally, $t(S) = T$ and $s \mapsto z(s)$ is monotone non-increasing.*

Finally, let us see that the limit parametrized evolution $(t, u, z) : [0, S] \to [0, T] \times \mathcal{U} \times \mathcal{Z}$ is a parametrized BV-evolution in the sense of Definition 4.9.

Theorem 5.5. *Let (t, u, z) be a weak* limit of (t_n, u_n, z_n) provided by Lemma 5.4. Then (t, u, z) satisfies $(u(0), z(0)) = (u_0, z_0)$, $t(0) = 0$, $t(S) = T$ and*

(S') *for every $s \in [0, S]$ with $t'(s) > 0$*

$$|\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} = 0, \qquad |\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} = 0, \tag{65}$$

(E') *for every $s \in [0, S]$*

$$\begin{aligned} \mathcal{F}(t(s), u(s), z(s)) =& \mathcal{F}(0, u(0), z(0)) - \int_0^s |\partial_u \mathcal{F}(t(r), u(r), z(r))|_{z(r)} \, \|u'(r)\|_{z(r)} \, dr \; + \\ & - \int_0^s |\partial_z \mathcal{F}(t(r), u(r), z(r))|_{u(r)} \, \|z'(r)\|_{u(r)} \, dr \; + \\ & + \int_0^s \partial_t \mathcal{F}(t(r), u(r), z(r)) \, t'(r) \, dr \, . \end{aligned} \tag{66}$$

Moreover, for almost all $s \in [0, S]$ it holds

$$\begin{aligned} |\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} \left\| z'(s) \right\|_{u(s)} &= -\partial_z \mathcal{F}(t(s), u(s), z(s))[z'(s)], \\ |\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} \left\| u'(s) \right\|_{z(s)} &= -\partial_u \mathcal{F}(t(s), u(s), z(s))[u'(s)]. \end{aligned}$$

Proof. In this case the proof of (65) is slightly different from that of (44) because the configuration $(t_n(s^n_k), u_n(s^n_k), z_n(s^n_k))$ is not in equilibrium (cf. Lemma 5.4). Given $s \in [0, S)$ with $t'(s) > 0$ let $s \in [s^n_k, s^n_{k+1})$ (for k depending on n). Equivalently we can write $s \in [s^n_{k+1,-1}, s^n_{k+1,M})$. Since $s^n_{k,i} \in [0, S_n]$ with S_n uniformly bounded, we can extract a subsequence (not relabeled) such that $s^n_k \to \underline{s}$ and $s^n_{k,M} \to \overline{s}$. As a consequence $s^n_{k+1,0} = s^n_{k+1,-1} + \Delta t^n \to \underline{s}$. In general we will have $\underline{s} \le s \le \overline{s}$.

We claim that $\underline{s} = s = \overline{s}$. Assume by contradiction that $\underline{s} < \overline{s}$. Since t^n is constant (by definition) in the interval $[s^n_{k+1,0}, s^n_{k+1,M}]$ it follows that its limit t is constant in $[\underline{s}, \overline{s}]$ and thus $t'(s) = 0$, which contradicts the assumption $t'(s) > 0$.

It follows that $s^n_{k+1,i} \to s$ for every index $i \in \{-1, .., M\}$. In particular, by Lemma 5.4,

$$\begin{aligned} t_n(s^n_k) &\to t(s), & u_n(s^n_k) &\to u(s) \quad \text{in } \mathcal{U}^{\tilde{p}}, \\ z_n(\grave{s}^n_k) &\rightharpoonup z(s), & z_n(s^n_k) &\rightharpoonup z(s), \quad \text{in } \mathcal{Z}. \end{aligned}$$

Using the lower semi-continuity of the slopes (cf. Lemma 2.5) we can pass to the limit in (64) and obtain (65).

The rest of the proof is instead identical to that of Theorem 4.8. □

Remark 5.6. *All the results of this section remain valid if for each $n \in \mathbb{N}$ and time step $1 \le k \le n$ one chooses an individual $M^n_k \in \mathbb{N}$ that gives the number of iterations of the alternate minimization algorithm at time step t^n_k. Hence, the following "adaptive" algorithm leads to curves $(t_n, u_n, z_n)_{n \in \mathbb{N}}$, where for $n \to \infty$ subsequences converge to quasistatic parametrized solutions in the sense of Definition 4.9: for a given $n \in \mathbb{N}$ choose a stopping criterion $STOP_n$. Then, given (u^n_{k-1}, z^n_{k-1}) at the previous time step t^n_{k-1} determine (u^n_k, z^n_k) at time step t^n_k as follows: with $(u^n_{k,0}, z^n_{k,0}) = (u^n_{k-1}, z^n_{k-1})$ determine recursively for $i \ge 0$*

$$\begin{cases} u^n_{k,i+1} \in \operatorname{argmin} \{ \mathcal{F}(t^n_k, u, z^n_{k,i}) : u \in \mathcal{U} \} \\ z^n_{k,i+1} \in \operatorname{argmin} \{ \mathcal{F}(t^n_k, u^n_{k,i+1}, z) : z \in \mathcal{Z} \text{ with } z \le z^n_{k,i} \} \end{cases}$$

as long as $STOP_n$ is not satisfied. Then let $(u^n_k, z^n_k) = (u^n_{k,i_{max}+1}, z^n_{k,i_{max}+1})$, where i_{max} is the number of iterations that are carried out. Defining the interpolating curves $(t_n, u_n, z_n)_{n \in \mathbb{N}}$ as in §4.3 leads to the desired result.

6 Properties of solutions

6.1 Phase-field energy release rate

In analogy with the classic way of introducing the energy release rate for a sharp crack, let us define the reduced energy

$$\tilde{\mathcal{E}}(t,z) = \mathcal{E}(t,u_z,z) \qquad \text{for} \qquad u_z \in \operatorname{argmin}\{\mathcal{E}(t,\cdot,z) : u \in \mathcal{U}\}.$$

By [13] the energy $\tilde{\mathcal{E}}$ is differentiable (as a function of z) and for every $\xi \in \Xi$ it holds

$$\partial_z \tilde{\mathcal{E}}(t,z)[\xi] = \partial_z \mathcal{E}(t,u_z,z)[\xi] = \int_\Omega z\,\xi\, W(Du_z)\,dx.$$

It follows that $\partial_z \tilde{\mathcal{E}}(t,z)[\xi] \le 0$ for every $\xi \in \Xi$. In order to define the energy release, let us introduce the set of *normalized admissible variations* with respect to the "phase-field crack length", i.e.

$$\hat{\Xi} = \{\xi \in \Xi : d\mathcal{L}(z)[\xi] = 1\}.$$

First, note that $\mathcal{L}(z) = \frac{1}{2}\|z-1\|^2_{\mathcal{Z}}$ and thus $d\mathcal{L}(z)[z-1] = \langle z-1, z-1\rangle_{\mathcal{Z}} > 0$ unless $z \not\equiv 1$. In the sequel we will assume that $z \not\equiv 1$, so that $\hat{\Xi} \neq \emptyset$. Under this assumption we define the *phase-field energy release* $\mathcal{G}(t,z)$ as

$$\mathcal{G}(t,z) = \sup\big\{-\partial_z \tilde{\mathcal{E}}(t,z)[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}\big\} = -\inf\big\{\partial_z \tilde{\mathcal{E}}(t,z)[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}\big\}. \tag{67}$$

Clearly $\mathcal{G}(t,z) \ge 0$ since $\partial_z \tilde{\mathcal{E}}(t,z)[\xi] \le 0$ for every $\xi \in \Xi$

6.2 Behavior in continuity points

In the phase field approach the irreversibility of the crack is usually modeled by the monotonicity of the phase field variable z; this is a natural hypothesis which in general does not imply the monotonicity of the dissipated energy. Actually, for our evolution (obtained as the limit of alternate minimization) it turns out that $\mathcal{L}(z(\cdot))$ is monotone non-decreasing in the continuity points.

Moreover, even if the alternate minimization algorithm does not employ explicitly an energy release, the limit evolution satisfies some kind of Karush-Kuhn-Tucker conditions in terms of the phase-field energy release defined above.

Proposition 6.1. *If $t'(s) > 0$ then, with $\mathcal{L}$ from* (8) *and $\mathcal{G}$ from* (67)*, we have*

- $d\mathcal{L}(z(s))[z'(s)] \ge 0,$
- $\mathcal{G}(t(s), z(s)) \le G_c,$
- $(\mathcal{G}(t(s), z(s)) - G_c)\, d\mathcal{L}(z(s))[z'(s)] = 0.$

Proof. Since $t'(s) > 0$, equation (44) implies that

$$\partial_z \mathcal{F}(t(s), u(s), z(s))[\xi] \ge 0 \qquad \text{for every } \xi \in \Xi,$$

and thus $\partial_z \mathcal{F}(t(s), u(s), z(s))[z'(s)] \ge 0$, i.e.

$$\int_\Omega z(s)\, z'(s)\, W(Du(s))\,dx + G_c \int_\Omega (z(s)-1)z'(s) + \nabla z(s)\cdot\nabla z'(s)\,dx \ge 0\,.$$

Since the first integral is non-positive the second one, which is indeed the derivative of the "phase-field" length, must be non-negative.

Next, by (S') we know that

$$\partial_u \mathcal{F}(t(s), u(s), z(s))[\phi] = 0 \qquad \text{for every } \phi \in \mathcal{U}$$

and thus $u(s) \in \operatorname{argmin}\{\mathcal{F}(t(s), u, z(s)) : u \in \mathcal{U}\}$. Hence (S') reads

$$\partial_z \mathcal{E}(t(s), u_{z(s)}, z(s))[\xi] + G_c \, d\mathcal{L}(z(s))[\xi] \geq 0 \qquad \text{for every } \xi \in \Xi.$$

In particular,

$$G_c \geq -\partial_z \tilde{\mathcal{E}}(t(s), z(s))[\hat{\xi}] \qquad \text{for every } \hat{\xi} \in \hat{\Xi}$$

and thus

$$G_c \geq \sup\big\{ - \partial_z \tilde{\mathcal{E}}(t(s), z(s))[\hat{\xi}] : \hat{\xi} \in \hat{\Xi} \big\} = \mathcal{G}(t(s), z(s)).$$

Finally, assume now that $d\mathcal{L}(z(s))[z'(s)] > 0$. By the chain rule

$$\begin{aligned} \mathcal{F}'(t(s), u(s), z(s)) = {} & \partial_t \mathcal{F}(t(s), u(s), z(s)) \, t'(s) + \partial_u \mathcal{F}(t(s), u(s), z(s))[u'(s)] \\ & + \partial_z \mathcal{F}(t(s), u(s), z(s))[z'(s)] \end{aligned} \tag{68}$$

while by (45) for a.e. $s \in [0, S]$

$$\begin{aligned} \mathcal{F}'(t(s), u(s), z(s)) = {} & \partial_t \mathcal{F}(t(s), u(s), z(s)) \, t'(s) - |\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} \, \|u'(s)\|_{z(s)} \\ & - |\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} \, \|z'(s)\|_{u(s)}. \end{aligned} \tag{69}$$

Since $t'(s) > 0$ by (44) we have $|\partial_u \mathcal{F}(t(s), u(s), z(s))|_{z(s)} = 0$ and thus

$$\partial_u \mathcal{F}(t(s), u(s), z(s))[u'(s)] = 0.$$

Again by (44) we have $|\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} = 0$, thus comparing (68) and (69) we get

$$\partial_z \mathcal{F}(t(s), u(s), z(s))[z'(s)] = -|\partial_z \mathcal{F}(t(s), u(s), z(s))|_{u(s)} \, \|z'(s)\|_{u(s)} = 0\,.$$

Thus for every $\lambda > 0$

$$\partial_z \mathcal{F}(t(s), u(s), z(s))[\lambda z'(s)] = 0.$$

On the other hand

$$\partial_z \mathcal{F}(t(s), u(s), z(s))[\xi] \geq 0 \qquad \text{for every } \xi \in \Xi$$

and thus for every $\lambda > 0$

$$\begin{aligned} \lambda z'(s) & \in \operatorname{argmin}\{\partial_z \mathcal{F}(t(s), u(s), z(s))[\xi] : \xi \in \Xi\} \\ & \in \operatorname{argmin}\{\partial_z \tilde{\mathcal{E}}(t(s), z(s))[\xi] + G_c d\mathcal{L}(z(s))[\xi] : \xi \in \Xi\}. \end{aligned}$$

Choosing λ in such a way that $d\mathcal{L}(z)[\lambda z'] = 1$ we get

$$\lambda z'(s) \in \operatorname{argmin}\{\partial_z \tilde{\mathcal{E}}(t(s), z(s))[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}\}$$

and thus $\partial_z \tilde{\mathcal{E}}(t(s), z(s))[\lambda z'(s)] = -\mathcal{G}(t(s), z(s))$. Hence

$$\partial_z \mathcal{F}(t(s), u(s), z(s))[\lambda z'(s)] = -\mathcal{G}(t(s), z(s)) + G_c = 0,$$

which concludes the proof. □

6.3 Local characterization of the evolution as differential inclusion and connection with other types of solutions

We will now derive an equivalent characterization for functions (t, z, u) that satisfy (44) and (46)–(47). This opens the possibility to compare solutions obtained in Theorem 4.8 to other solution concepts in the literature.

Proposition 6.2. *Let $(t, u, z) \in W^{1,\infty}([0, S], [0, T] \times \mathcal{U} \times \mathcal{Z})$ with $u \in L^\infty(0, S; \mathcal{U}^{\widetilde{p}})$ for some $\widetilde{p} > 2$. For $\phi \in \mathcal{U}$, $\xi \in \mathcal{Z}$ define (cf. the notation introduced in (12)–(15) and (9))*

$$\mathcal{J}_1(\phi; \xi) = \tfrac{1}{2} \|\phi\|_\xi^2 = \tfrac{1}{2}\mathcal{A}_\xi(\phi, \phi), \qquad \mathcal{J}_2(\xi; \phi) = \tfrac{1}{2} \|\xi\|_\phi^2$$

(with $\partial_\phi \mathcal{J}_1(\phi; \xi)[\psi] = \langle \phi, \psi \rangle_\xi$). Then the following statements (a) and (b) are equivalent.

(a) For almost all $s \in [0, S]$, $(t(s), u(s), z(s))$ satisfies (44) and (46)–(47).

(b) There exist functions $\lambda_1, \lambda_2 : [0, S] \to [0, \infty)$ such that for almost all $s \in [0, S]$ we have

$$t'(s)(\lambda_1(s) + \lambda_2(s)) = 0, \quad (70)$$

$$(t'(s) + \|u'(s)\|_{z(s)})\Big(\partial_u \mathcal{F}(t(s), u(s), z(s)) + \lambda_1(s)\partial_\phi \mathcal{J}_1(u'(s); z(s))\Big) = 0, \quad (71)$$

$$(t'(s) + \|z'(s)\|_{u(s)})\big(\partial_z \mathcal{F}(t(s), u(s), z(s)) + \lambda_2(s)\partial_\xi \mathcal{J}_2(z'(s); u(s))\big)[\xi - z'(s)] \geq 0, \quad (72)$$

where the last inequality holds for every $\xi \in \Xi$.

Proof. Proposition 6.2 is an immediate application of the Karush-Kuhn-Tucker theory, see for instance [32, Theorem 47.E], to the functionals $\mathcal{I}_1(\phi) = \partial_u \mathcal{F}(t(s), u(s), z(s))[\phi]$ and $\mathcal{I}_2(\xi) = \partial_z \mathcal{F}(t(s), u(s), z(s))[\xi]$ that are minimized with respect to the constraints $\|\phi\|_{z(s)}^2 \leq 1$ and $\xi \in \Xi$ with $\|\xi\|_{u(s)}^2 \leq 1$, respectively. (S') and (46)–(47) imply that 0, $u'(s)/\|u'(s)\|_{z(s)}$ and $z'(s)/\|z'(s)\|_{u(s)}$ (when they make sense) are minimizers of the above mentioned minimization problems. □

Let us give an interpretation of (70)–(72). If $(t, u, z) \in W^{1,\infty}([0, S], [0, T] \times \mathcal{U} \times \mathcal{Z})$ is a solution that is obtained as a limit of time-discrete alternate minimization curves as in Theorem 4.8, then according to Proposition 4.11, (t, u, z) is a nondegenerate curve, i.e. (53) is valid for a.e. $s \in [0, S]$. From (70)–(72) we therefore deduce that for all $s \in [0, S]$, which are points of differentiability of (t, u, z):

if $t'(s) > 0$ then

$$\partial_u \mathcal{F}(t(s), u(s), z(s)) = 0 \text{ and } \partial_z \mathcal{F}(t(s), u(s), z(s))[\xi - z'(s)] \geq 0 \quad \forall \xi \in \Xi, \quad (73)$$

if $t'(s) = 0$ then $u'(s) \neq 0$ or $z'(s) \neq 0$ (due to the nondegeneracy) and

$$\partial_u \mathcal{F}(t(s), u(s), z(s)) + \lambda_1(s)\partial_\phi \mathcal{J}_1(u'(s); z(s)) = 0 \quad (\text{if } u'(s) \neq 0), \quad (74)$$

$$\partial_z \mathcal{F}(t(s), u(s), z(s)) + \lambda_2(s)\partial_\xi \mathcal{J}_2(z'(s); u(s))[\xi - z'(s)] \geq 0 \quad \forall \xi \in \Xi \ (\text{if } z'(s) \neq 0). \quad (75)$$

In order to give a mechanical interpretation and to compare (73)–(75) with results from [26, 14, 19], we split the functional $\mathcal{F}$ in the following way: for $v \in \mathcal{U}, \xi \in \mathcal{Z}$ let

$$\mathcal{I}(t, v, \xi) = \frac{1}{2}\mathcal{A}_\xi(v, v) + \frac{G_c}{2} \|\xi\|_{\mathcal{Z}}^2 - \langle b(t), v \rangle,$$

$$\mathcal{R}(\xi) = \begin{cases} G_c \|\xi\|_{L^1(\Omega)} & \text{if } \xi \in \Xi \\ \infty & \text{if } \xi \in \mathcal{Z} \backslash \Xi \end{cases}.$$

In [30, 14], $\mathcal{I}$ is interpreted as a stored energy functional with $\frac{G_c}{2} \|\xi\|_{\mathcal{Z}}^2$ as a regularization term, while $\mathcal{R}$ is interpreted as a dissipation pseudo potential. Observe that $\mathcal{R}$ is positively homogeneous of degree one. With these functionals, (73)–(75) are equivalent to the following conditions:

if $t'(s) > 0$ then

$$\partial_u \mathcal{I}(t(s), u(s), z(s)) = 0 \text{ and } 0 \in \partial\mathcal{R}(z'(s)) + \partial_z \mathcal{I}(t(s), u(s), z(s)), \tag{76}$$

if $t'(s) = 0$ then

$$\partial_u \mathcal{I}(t(s), u(s), z(s)) + \lambda_1(s)\mathbb{A}_{z(s)}(u'(s)) = 0 \quad (\text{if } u'(s) \neq 0), \tag{77}$$

$$0 \in \partial\mathcal{R}(z'(s)) + \lambda_2(s)\mathbb{B}_{u(s)}(z'(s)) + \partial_z \mathcal{I}(t(s), u(s), z(s)) \quad (\text{if } z'(s) \neq 0), \tag{78}$$

where $\mathbb{A}_{z(s)} : W^{1,2}_{\partial_D\Omega} \to W^{-1,2}_{\partial_D\Omega}$ denotes the operator associated to the bilinear form $\mathcal{A}_{z(s)}$ from Lemma 2.1, while $\mathbb{B}_{u(s)} : \mathcal{Z} \to \mathcal{Z}'$ is the operator defined by $\langle \mathbb{B}_{u(s)} z, \xi\rangle_{\mathcal{Z}} = \langle z, \xi\rangle_{u(s)}$ for all $z, \xi \in \mathcal{Z}$.

This shows that for $t'(s) > 0$ the damage component z of the solution given by Theorem 4.8 follows a rate-independent process, while the displacement field u satisfies the static balance of linear momentum. If $t'(s) = 0$, then (77) and (78) can also be rewritten more explicitly as follows in strong form (if $\langle b(t), v\rangle = \int_\Omega b(t) \cdot v \, dx$):

$$\operatorname{div}\big(\mathbb{D}(\lambda_1(s), z(s))\varepsilon(u'(s)) + \mathbb{C}(z(s))\varepsilon(u(s))\big) + b(t(s)) = 0 \quad (\text{if } u'(s) \neq 0),$$

$$0 \in \partial\mathcal{R}(z'(s)) + \lambda_2(s)\big(- \operatorname{div}(G_c \nabla z') + (G_c + W(Du))z'\big) + \partial_z \mathcal{I}(t(s), u(s), z(s)) \quad (\text{if } z'(s) \neq 0),$$

where we have set $\mathbb{C}(z) = (z^2 + \eta)\mathbf{C}$ and $\mathbb{D}(\lambda_1, z) = \lambda_1 \mathbb{C}(z)$. The first equation can be interpreted as a viscoelastic model of Kelvin-Voigt type with damage dependent elasticity tensor $\mathbb{C}(z)$ and viscosity tensor $\mathbb{D}(\lambda_1, z)$ together with corresponding Dirichlet and Neumann conditions on $\partial\Omega$. Relation (78) describes a damage evolution, where also a viscous (damage-) dissipation is active provided that $\lambda_2(s) > 0$. Observe that the viscous part of the dissipation involves spatial derivatives of z.

Now, let us compare with vanishing viscosity solutions and other solution concepts from literature. Exploiting the convexity of $\mathcal{I}$ with respect to the variable z, from (76) we obtain in the same way as for instance in [17] that the solution (t, u, z) satisfies the following semi-stability inequality for $t'(s) > 0$:

$$\mathcal{I}(t(s), u(s), z(s)) \leq \mathcal{I}(t(s), u(s), \xi) + \mathcal{R}(\xi - z(s)) \qquad \forall \xi \in \mathcal{Z}.$$

Moreover, from the energy identity (45), one obtains the following energy estimate (by neglecting the terms with the slope), rewritten in terms of $\mathcal{I}$ and $\mathcal{R}$ and taking into account the irreversibility of z for every $0 \leq s_1 \leq s_2 \leq S$

$$\mathcal{I}(t(s_2), u(s_2), z(s_2)) + \mathcal{R}(z(s_2) - z(s_1)) \leq \mathcal{I}(t(s_1), u(s_1), z(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{I}(t(s), u(s), z(s)) t'(s) \, ds.$$

This shows that the solution according to Theorem 4.8 is a local solution (see [31], where the notion of local solutions was introduced in the context of crack propagation) that satisfies an additional semi-stability property in the sense of [26] (so-called semi energetic solutions). In the papers [28, 25], for rate independent problems local solutions were obtained as the limit of time-discrete solutions that are constructed by an alternate minimization procedure at each time step but only with one minimization step for each variable (fractional step methods or semi-implicit Rothe methods). The paper [15] treats the same damage model but in the context of visco-elasticity with inertia terms. There it is shown that time discrete solutions obtained on the basis of a one-step alternate minimization algorithm in the limit, as the time step size tends to zero, converge to solutions (u, z) that are Lipschitz-continuous with respect to the physical time.

In [13, 14] the authors studied the vanishing viscosity limit for different generalizations of the Ambrosio-Tortorelli damage model. Adapted to the present two-dimensional setting with energy $\mathcal{I}$ and dissipation $\mathcal{R}$ from above, for every $\varepsilon > 0$ the existence of weak solutions $(u_\varepsilon, z_\varepsilon) \in H^1(0,T;\mathcal{U}\times\mathcal{Z})$ was shown for the system

$$\partial_u \mathcal{I}(t, u_\varepsilon(t), z_\varepsilon(t)) = 0 \text{ and } 0 \in \partial\mathcal{R}(z_\varepsilon'(t)) + \varepsilon z_\varepsilon'(t) + \partial_z \mathcal{I}(t, u_\varepsilon(t), z_\varepsilon(t)).$$

Furthermore it was shown that a subsequence of $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ converges to a parameterized curve $(t, u, z) \in C^0_{\text{lip}}([0,S];[0,T]\times\mathcal{U}\times\mathcal{Z})$, for which there exists a function $\lambda : [0,S] \to [0,\infty)$ such that for a.e. $s \in [0,S]$

$$t'(s)\lambda(s) = 0, \quad \partial_u \mathcal{I}(t(s), u(s), z(s)) = 0, \tag{79}$$

$$0 \in \partial\mathcal{R}(z'(s)) + \lambda(s)z'(s) + \partial_z \mathcal{I}(t(s), u(s), z(s)). \tag{80}$$

Comparing with (78), the viscous term in (80) is a linear and bounded operator on $L^2(\Omega)$, and for $t'(s) = 0$ the limit model contains viscous damage, only, whereas viscoelastic dissipation is not present.

The interplay between viscoelastic dissipation and viscous dissipation of the internal variables was investigated in [20] in the finite dimensional setting. There, for different scalings between the viscosity parameters the corresponding limit models were identified. With $\mathcal{U} = \mathbb{R}^n$, $\mathcal{Z} = \mathbb{R}^m$, $\mathcal{I} \in C^1([0,T]\times\mathcal{U}\times\mathcal{Z})$ and $\mathcal{R} \in C^0(\mathcal{Z},[0,\infty))$ convex and positively homogeneous of degree one, the authors studied the vanishing viscosity limit of a slightly more general version of the following system: For $\alpha > 0$, $\varepsilon > 0$

$$\varepsilon^\alpha u'(s) + \partial_u \mathcal{I}(t, u(t), z(t)) = 0\,,$$

$$\varepsilon z'(t) + \partial\mathcal{R}(z'(t)) + \partial_z \mathcal{I}(t, u(t), z(t)) \ni 0.$$

With suitable technical assumptions on $\mathcal{I}$ and $\mathcal{R}$, the authors proved that (subsequences of) solutions to this system converge for $\varepsilon \to 0$ to parametrized solutions of the following system, [20, Theorem 5.3]:

$$\theta_u(s)u'(s) + (1-\theta_u(s))\partial_u \mathcal{I}(t(s), u(s), z(s)) = 0, \tag{81}$$

$$\theta_z(s)z'(s) + (1-\theta_z(s))\Big(\partial\mathcal{R}(z'(s)) + \partial_z \mathcal{I}(t(s), u(s), z(s))\Big) \ni 0\,, \tag{82}$$

where $\theta_u, \theta_z : [0,S] \to [0,1]$ are Borel functions satisfying for a.e. $s \in [0,S]$

$$t'(s)(\theta_u(s) + \theta_z(s)) = 0, \tag{83}$$

$$\text{if } \alpha > 1: \ \theta_u(s)(1-\theta_z(s)) = 0; \tag{84}$$

$$\text{if } \alpha = 1: \ \theta_u(s) = \theta_z(s); \quad \text{if } 0 < \alpha < 1: \ \theta_z(s)(1-\theta_u(s)) = 0. \tag{85}$$

Observe the similarity between (81)–(82) and the system obtained in our case, i.e. (76)–(78). However, more information on possible strong couplings between the functions λ_1, λ_2 from (77)–(78) as in (85) is missing.

Summarizing, the solution approximated by the time-discrete alternate minimization scheme is closely related to solutions that are obtained as simultaneous viscosity limits of visco-elastic systems that are coupled to viscously regularized rate independent systems. However, it remains open whether also in our limit model the functions λ_1, λ_2 satisfy additional complementarity or compatibility conditions as it is the case in [20], see (85) from above. Moreover, it is not clear whether one should expect in the case $t'(s) = 0$ that (74) and (75) are valid also if $u'(s) = 0$ and $z'(s) = 0$ respectively, or whether it is possible to sharpen the nondegeneracy property such that $\min\{\|u'(s)\|_{z(s)}, \|z'(s)\|_{u(s)}\} \geq C > 0$ if $t'(s) = 0$.

7 A finite dimensional numerical example

The aim of this Section is to reveal similarities and differences between different solution concepts for damage models of Ambrosio–Tortorelli type for an explicit example. Since here the focus lies on the temporal behavior of the solutions the example deals with the finite dimensional case, i.e. $\mathcal{U} = \mathcal{Z} = \mathbb{R}$. For $t, u, z \in \mathbb{R}$, the energy functional is chosen as

$$F(t, u, z) = \frac{1}{2}(z^2 + \delta)u^2 + \frac{\alpha}{2}(\mu - z)^2 - tu$$

with $\delta = 0.1, \alpha = 10, \mu = 1, z_0 = 1, u_0 = 0$ and $T = 1.5$. The calculations documented below are carried out for different choices of n, which is the number of time steps, and of M, which is the number of steps in the alternate minimization problem at a fixed time step. To be more precise, for $\tau = T/n$ and initial values $u_0^n := u_0 = 0$, $z_0^n := z_0 = 1$, the time incremental solutions are obtained as follows: for $k \in \{0, \dots, n-1\}$ let u_k^n, z_k^n be given; then set $z_{k+1,0}^n := z_k^n$, $u_{k+1,0}^n := u_k^n$ and for $i \in \{1, \dots, M\}$ define recursively

$$u_{k+1,i}^n := \operatorname{argmin}\{ F(t_{k+1}^n, v, z_{k+1,i-1}^n) : v \in \mathbb{R} \}, \tag{86}$$

$$z_{k+1,i}^n := \operatorname{argmin}\{ F(t_{k+1}^n, u_{k+1,i}^n, z) : z \leq z_{k+1,i-1}^n) \} \tag{87}$$

$$u_{k+1}^n := u_{k+1,M}^n, \qquad z_{k+1}^n := z_{k+1,M}^n . \tag{88}$$

Observe that due to the special structure of the energy, the values $u_{k+1,i}^n$ can be calculated explicitly, while for the minimization in z, the Mathematica–routine ArgMin was used. The following algorithms were tested.

Alternate minimization Ia. This corresponds to the M-step alternate minimization algorithm analyzed in this paper. The alternate minimization loop is stopped at $M = 40$.

Alternate minimization Ib. This corresponds to an alternate minimization algorithm with a stopping criterion of the following type: at t_k^n define $res(u_{k,i}^n) := |\partial_u F(t_k^n, u_{k,i}^n, z_{k,i}^n)|$. If $res(u_{k,i}^n) \leq \delta_{\text{stop}}$ then terminate the internal minimization loop and go to the next time step. According to Remark 5.6, (a subsequence of) the corresponding interpolating curves converge (for $n \to \infty$) to a parametrized BV-solution. In the numerical example we chose $n = 100$ and $\delta_{\text{stop}} = 10^{-5}$. The maximum number of iterations that appeared within one time-step was 56, the total number of internal minimization steps was 622.

Alternate minimization II. Here, $M = 1$, which means that at each time step only one minimization with respect to u and one minimization with respect to z is done. This algorithm corresponds to those investigated for instance in [27, 25]. For $n = 600$, the numerical effort of this algorithm is comparable to the one described in Ib.

Alternate minimization III, with backtracking. Here, we chose $M = 40$ and introduced the backtracking criterion from [4, Theorem 3]. Adapted to our example the backtracking is initiated at time step t_k^n if there exists $t_r^n < t_k^n$ for which the inequality

$$F(t_r^n, u_r^n, z_r^n) > F(t_r^n, u_k^n, z_k^n)$$

is satisfied. In this case go back to the smallest t_r^n with this property and restart the time discrete algorithm replacing (u_r^n, z_r^n) with (u_k^n, z_k^n).

Algorithm IV: Global energetic solutions. Global energetic solutions are obtained as the limit of the following time incremental procedure

$$(u_{k+1}^n, z_{k+1}^n) \in \operatorname{Argmin}\{ F(t_{k+1}^n, v, z) : v \in \mathcal{U}, z \leq z_k^n \} .$$

See for instance [17, 21, 30], where global energetic solutions of damage models were investigated.

Table 1 summarizes the tests that were carried out. Since the results of Algorithm Ia and Ib with the values from the table are nearly identical, we plot here those of Ib, only.

Figure 3 shows the evolution of the damage-type variable z according to the different algorithms. The gray region contains points with $\partial_z F(t, u_{\text{opt}}(t, z), z) < 0$, i.e. stable points in the sense of the Griffith criterion. Here, $u_{\text{opt}}(t, z) = \operatorname{argmin}\{ F(t, v, z) : v \in \mathbb{R} \}$. In the white region we have $\partial_z F(t, u_{\text{opt}}(t, z), z) > 0$. According to the Griffith criterion, in

	n	M	color
Algorithm Ia	100	40	–
Algorithm Ib	100	$\delta_{\text{stop}} = 10^{-5}$	blue
Algorithm II	100	1	red
	600	1	green
Algorithm III (backtracking)	100	40	purple
Algorithm IV (global energetic solution)	100	–	dark red

Table 1: Table of performed tests

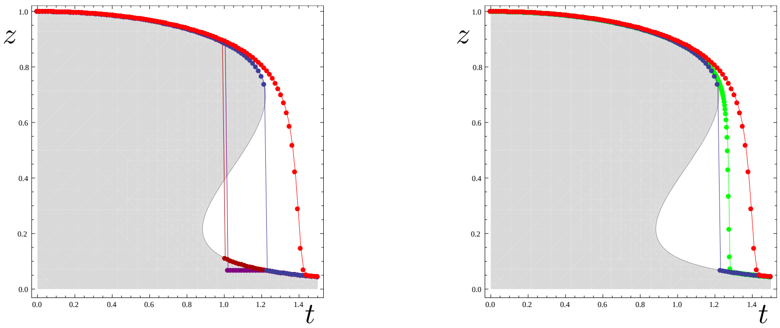

Figure 3: Left: Damage evolution for the different algorithms, all with $n = 100$. Blue: Algorithm Ib; Red: Algorithm II with $M = 1$; Dark red: Global energetic solution; Purple: Algorithm with backtracking. Right: Comparison of Algorithm Ib (blue, $n = 100$) with Algorithm II (red, $n = 100$; green, $n = 600$)

the interior of the gray region, the damage variable z should be constant in time, while the interior of the white region is not admissible. The calculations reveal that the global energetic solution (dark red) and the solution of the alternate minimization algorithm with backtracking (purple) develop a jump although a stable/ time-continuous damage propagation is still possible. The alternate minimization algorithms Ia/Ib discussed in this paper (blue curve) detect quite precisely the exact point in time where the jump should take place, while the alternate minimization algorithm II with $M = 1$ "smears out" the jump. In the right plot in Figure 3, the result of Algorithm II with $M = 1$ and number of time steps $n = 600$ (green) is plotted against the result of Algorithm Ib with $n = 100$. Now, the point of discontinuity predicted by Algorithm II is sharper.

Figure 4 shows the evolution of the energy $F(t, u(t), z(t))$ with respect to t. Here, the results for the global energetic solution (Alg. IV) and for Alg. III (backtracking) are nearly identical so that we plot those for the global energetic solution, only.

Finally, Figure 5 shows the trajectory of $s \mapsto (u(s), z(s))$ in the (u, z)-plane for Algorithm Ib. Blue points indicate the points (u_k^n, z_k^n), i.e. the limit points of the alternate minimization scheme at time t_k^n for $n = 500$, while red points mark also the intermediate points $(u_{k,i}^n, z_{k,i}^n)$ for $n = 500$, and green points mark the intermediate points $(u_{k,i}^n, z_{k,i}^n)$ for $n = 100$. In both cases we chose $\delta_{\text{stop}} = 10^{-5}$. For $n = 100$ the maximum number of iterations that appeared is 56, while for $n = 500$ the maximum number of iterations that appeared was 151. The starting point of the jump of the evolution of (u, z) with respect to the time t is approximately given by $(u_-, z_-) \approx (2.3, 0.7)$, while the end point is given by $(u_+, z_+) \approx (11.6, 0.08)$. It is not clear from the plot whether increasing the number of time-steps and decreasing δ_{stop} leads to a "finer resolution" of the trajectory connecting (u_-, z_-) with (u_+, z_+).

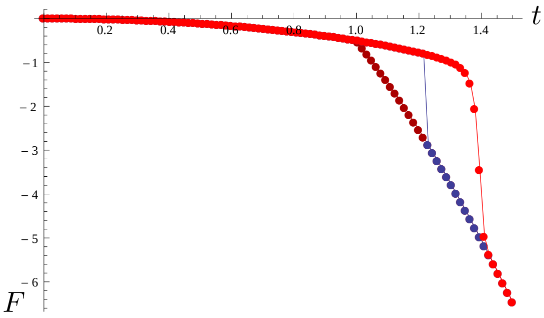

Figure 4: Evolution of the energy F for Algorithm Ib (blue), Algorithm II (red) and the global energetic solution (dark red). In all cases, $n = 100$.

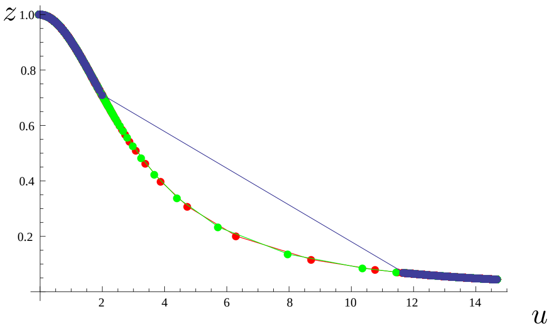

Figure 5: Plot of the trajectory of (u, z) in the (u, z)-plane for $n = 100$ (green) and $n = 500$ (red, blue) for Algorithm Ib with $\delta_{\text{stop}} = 10^{-5}$.

A Appendix

A.1 Lower semi-continuity

The lower semicontinuity inequalities (48) and (49) follow from [3, Theorem 3.1]. Let us see how our setting fits into the framework and notation of [3]. We set $X = [0, T] \times H^1 \times \{z \in H^1 : \|z\|_{H^1} \leq R\}_w$ where the ball in H^1 is endowed with the weak topology, $\Xi = H^1 \times H^1$, $T = (0, S)$, $H = \mathbb{R}$ and $\eta \equiv 1$. Note that X is a metric (actually metrizable) space. For $x = (t, u, z)$ and for $\xi = (u', z')$ the integrand is

$$l(\tau, x, \xi) = |\partial_u \mathcal{F}(t, u, z)| \; \|u'\|_z + |\partial_z \mathcal{F}(t, u, z)| \; \|z'\|_u$$

By Lemma 2.5 and Corollary 2.3 we know that $l(\tau\,\cdot, \cdot)$ is sequentially lower semi-continuous in $X \times \Xi$. Clearly $l(\tau, x, \cdot)$ is convex in Ξ and $l \geq 0$.

Then, by Lemma 4.7 we know that $x_n = (t_n, u_n, z_n)$ converges to $x = (t, u, z)$ in the topology of X, pointwise in $(0, S)$ and thus in measure (up to subsequences). Moreover, again by Lemma 4.7, we have that $\xi_n = (u'_n, z'_n)$ converges to $\xi = (u', z')$ weakly* in $L^\infty((0, S); H^1 \times H^1)$ and thus in $L^1((0, S); H^1 \times H^1)$.

A.2 Continuous dependence

Lemma A.1. *For $t \in [0, T]$ and $z \in \mathcal{Z}$ let $u_{min}(t, z)$ denote the minimizer of $\mathcal{F}(t, \cdot, z)$ over $\mathcal{U}$. There exists $C > 0$ s.t. for every $t_1, t_2 \in [0, T]$ and every $z_1, z_2 \in \mathcal{Z}$*

$$\|u_{min}(t_2, z_2) - u_{min}(t_1, z_1)\|_{1,\tilde{p}} \leq C|t_2 - t_1| + C\|z_2 - z_1\|_r$$

for $1/r = 1/\tilde{p} - 1/p$ and $\tilde{p} \in [2, p)$. In particular

$$\|u_{min}(t_2, z_2) - u_{min}(t_1, z_1)\|_{1,2} \leq C|t_2 - t_1| + C\|z_2 - z_1\|_{1,2}\,.$$

For the proof, see [13].

Lemma A.2. *Let $\widetilde{p} > 2$, $t \in [0,T]$, $v_1, v_2 \in W^{1,\widetilde{p}}(\Omega)$, $z_0 \in \mathcal{Z}$ with $0 \le z_0 \le 1$ and*

$$z_1 = \operatorname{argmin}\{\, \mathcal{F}(t, v_1, z) \,:\, z \in \mathcal{Z}, z \le z_0 \,\},$$
$$z_2 = \operatorname{argmin}\{\, \mathcal{F}(t, v_2, z) \,:\, z \in \mathcal{Z}, z \le z_1 \,\}.$$

Then for $r^{-1} = 1 - 2\widetilde{p}^{-1}$

$$\|z_1 - z_2\|^2_{\mathcal{Z}} \le C \,\|v_1 - v_2\|_{W^{1,\widetilde{p}}(\Omega)} \left(\|v_1\|_{W^{1,\widetilde{p}}(\Omega)} + \|v_2\|_{W^{1,\widetilde{p}}(\Omega)} \right) \|z_1 - z_2\|_{L^r(\Omega)} \tag{89}$$

and the constant C is independent of v_i, z_i, t.

Proof. Let $z_i = \operatorname{argmin}\{\, \mathcal{F}(t, v_i, z) \,:\, z \in \mathcal{Z}, z \le z_{i-1} \,\}$, $i \in \{1,2\}$. Then, by Lemma 3.1, z_i satisfies

$$\forall \xi \in \Xi : \quad 0 \ge -\partial_z \mathcal{F}(t, v_i, z_i)[\xi], \tag{90}$$
$$0 = -\partial_z \mathcal{F}(t, v_i, z_i)[z_i - z_{i-1}]. \tag{91}$$

Furthermore, it holds $z_i \ge 0$. Subtracting (91) with $i = 2$ from (90) with $i = 1$ and $\xi = z_2 - z_1$ we obtain

$$0 \ge \langle \partial_z \mathcal{F}(t, v_2, z_2) - \partial_z \mathcal{F}(t, v_1, z_1), z_2 - z_1 \rangle$$

which, after adding and subtracting the term $\partial_z \mathcal{F}(t, v_2, z_1)$ can be rewritten as

$$\langle \partial_z \mathcal{F}(t, v_2, z_2) - \partial_z \mathcal{F}(t, v_2, z_1), z_2 - z_1 \rangle \le \langle \partial_z \mathcal{F}(t, v_1, z_1) - \partial_z \mathcal{F}(t, v_2, z_1), z_2 - z_1 \rangle. \tag{92}$$

Observe that $\langle \partial_z \mathcal{F}(t, v_2, z_2) - \partial_z \mathcal{F}(t, v_2, z_1), z_2 - z_1 \rangle = \|z_2 - z_1\|^2_{t,v_2} \ge \|z_2 - z_1\|^2_{\mathcal{Z}}$, while for the right hand side of (92) we obtain

$$\begin{aligned} r.h.s. &= \int_\Omega z_1 (W(Dv_1) - W(Dv_2))(z_2 - z_1)\, dx \\ &\le C \,\|v_1 - v_2\|_{W^{1,\widetilde{p}}(\Omega)} (\|v_1\|_{W^{1,\widetilde{p}}(\Omega)} + \|v_1\|_{W^{1,\widetilde{p}}(\Omega)}) \,\|z_1 - z_2\|_{L^r(\Omega)}\,, \end{aligned}$$

where we exploited the quadratic structure of the density W and applied Hölder inequality with $2\widetilde{p}^{-1} + r^{-1} = 1$. This finishes the proof. □

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References

[1] A.Abdollahi and I.Arias, *Phase-field modeling of crack propagation in piezoelectric and ferroelectric materials with different electromechanical crack conditions*, J. Mech. Phys. Solids **60** (2012), no. 12, 2100–2126.

[2] L. Ambrosio and V.M. Tortorelli, *On the approximation of free discontinuity problems*, Boll. Un. Mat. Ital. B (7) **6** (1992), no. 1, 105–123.

[3] E. J. Balder, *An extension of Prohorov's theorem for transition probabilities with applications to infinite-dimensional lower closure problems*, Rend. Circ. Mat. Palermo (2) **34** (1985), no. 3, 427–447 (1986).

[4] B. Bourdin, *Numerical implementation of the variational formulation for quasi-static brittle fracture.*, Interfaces Free Bound. **9** (2007), no. 3, 411–430.

[5] B. Bourdin, G.A. Francfort, and J-J. Marigo, *Numerical experiments in revisited brittle fracture.*, J. Mech. Phys. Solids **48** (2000), no. 4, 797–826.

[6] A. Chambolle, *An approximation result for special functions with bounded deformation*, J. Math. Pures Appl. (9) **83** (2004), no. 7, 929–954.

[7] B. Dacorogna, *Direct methods in the calculus of variations*, Applied Mathematical Sciences, vol. 78, Springer-Verlag, Berlin, 1989.

[8] G. Dal Maso, *An introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[9] ———, *Generalised functions of bounded deformation*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 5, 1943–1997.

[10] A. Giacomini, *Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures.*, Calc. Var. Partial Differ. Equ. **22** (2005), no. 2, 129–172.

[11] K. Gröger, *A* $W^{1,p}$*-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann. **283** (1989), no. 4, 679–687.

[12] F. Iurlano, *A density result for gsbd and its application to the approximation of brittle fracture energies*, Calc. Var. Partial Differential Equation **51** (2014), 315–342.

[13] D. Knees, R. Rossi, and C. Zanini, *A vanishing viscosity approach to a rate-independent damage model*, Math. Models Methods Appl. Sci. **23** (2013), no. 4, 565–616.

[14] ———, *A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains*, Nonlinear Analysis Series B: Real World Applications **24** (2015), 126–162.

[15] C.J. Larsen, C. Ortner, and E. Süli, *Existence of solutions to a regularized model of dynamic fracture.*, Math. Models Methods Appl. Sci. **20** (2010), no. 7, 1021–1048.

[16] C. Miehe, F. Welschinger, and M. Hofacker, *Thermodynamically consistent phase-field models of fracture: variational principles and multi-field FE implementations*, Internat. J. Numer. Methods Engrg. **83** (2010), no. 10, 1273–1311.

[17] A. Mielke, *Evolution of rate-independent systems*, Handbook of differential equations: Evolutionary equations Vol. II, Amsterdam: Elsevier/North-Holland, 2005, pp. 461–559.

[18] ———, *Differential, energetic, and metric formulations for rate-independent processes.*, Nonlinear PDE's and applications. C.I.M.E. summer school, Cetraro, Italy, June 23–28, 2008, Berlin: Springer, 2011, pp. 87–170.

[19] A. Mielke, R. Rossi, and G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var. **18** (2012), no. 1, 3680.

[20] ———, *Balanced-viscosity solutions for multi-rate systems*, J. Phys., MURPHYS-HSFS Proc. (2015).

[21] A. Mielke and T. Roubíček, *Rate-independent damage processes in nonlinear elasticity.*, Math. Models Methods Appl. Sci. **16** (2006), no. 2, 177–209.

[22] A. Mielke and S. Zelik, *On the vanishing-viscosity limit in parabolic systems with rate-independent dissipation terms*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 1, 67–135.

[23] M. Negri, *Quasi-static rate-independent evolutions: characterization, existence, approximation and application to fracture mechanics*, ESAIM Control Optim. Calc. Var. **20** (2014), no. 4, 983–1008.

[24] K. Pham, H. Amor, J.J. Marigo, and C. Maurini, *Gradient damage models and their use to approximate brittle fracture*, International Journal of Damage Mechanics **20** (2011), no. 4, 618–652.

[25] T. Roubíček and J. Valdman, *Stress-driven solution to rate-independent elasto-plasticity with damage at small strains and its computer implementation*, Tech. Report 1506.01421v1, arXiv, 2015.

[26] T. Roubíček, *Rate-independent processes in viscous solids at small strains.*, Math. Methods Appl. Sci. **32** (2009), no. 7, 825–862.

[27] T. Roubíček, C. G. Panagiotopoulos, and V. Mantič, *Quasistatic adhesive contact of visco-elastic bodies and its numerical treatment for very small viscosity.*, ZAMM, Z. Angew. Math. Mech. **93** (2013), no. 10-11, 823–840.

[28] T. Roubíček, M. Thomas, and C.G. Panagiotopoulos, *Stress-driven local-solution approach to quasistatic brittle delamination.*, Nonlinear Anal., Real World Appl. **22** (2015), 645–663.

[29] T. Takaishi and M. Kimura, *Phase field model for mode III crack growth in two dimensional elasticity*, Kybernetika (Prague) **45** (2009), no. 4, 605–614.

[30] M. Thomas and A. Mielke, *Damage of nonlinearly elastic materials at small strain – existence and regularity results.*, ZAMM, Z. Angew. Math. Mech. **90** (2010), no. 2, 88–112.

[31] R. Toader and C. Zanini, *An artificial viscosity approach to quasistatic crack growth.*, Boll. Unione Mat. Ital. (9) **2** (2009), no. 1, 1–35.

[32] E. Zeidler, *Nonlinear functional analysis and its applications. III: Variational methods and optimization*, 1985.