BV supersolutions to equations of 1-Laplace and minimal surface type

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Abstract

We propose notions of BV supersolutions to (the Dirichlet problem for) the 1-Laplace equation, the minimal surface equation, and equations of similar type. We then establish some related compactness and consistency results.

Our main technical tool is a generalized product of L^∞ divergence-measure fields and gradient measures of BV functions. This product crucially depends on the choice of a representative of the BV function, and the proofs of its basic properties involve results on one-sided approximation and fine (semi)continuity in the BV context.

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1 Introduction

This paper deals with weak supersolutions to the 1-Laplace and minimal surface equations

$$\operatorname{div}\frac{\mathrm{D}u}{|\mathrm{D}u|}\equiv 0 \qquad\text{and}\qquad \operatorname{div}\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\equiv 0 \qquad \text{on } \Omega\,, \tag{1.1}$$

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where Ω always denotes an open subset of $\mathbb{R}^n$ with an arbitrary positive integer n, and where the unknown u is a real-valued function on Ω. The equations in (1.1) make sense for functions u of class $\mathrm{W}^{1,1}$ (in the first case for functions without zeroes of $\mathrm{D}u$) and can then be considered as the Euler-Lagrange equations of the total variation and the non-parametric area, given for $u \in \mathrm{W}^{1,1}(\Omega)$ by

$$\int_\Omega |\mathrm{D}u| \,\mathrm{d}x \qquad \text{and} \qquad \int_\Omega \sqrt{1+|\mathrm{D}u|^2} \,\mathrm{d}x \,, \tag{1.2}$$

respectively.

However, due to a lack of compactness, the existence problems for both weak solutions to the equations in (1.1) and minimizers of the functionals in (1.2) cannot be solved, in general, in the non-reflexive and non-dual space $\mathrm{W}^{1,1}(\Omega)$. Therefore, one usually treats the existence problem in the larger space of functions of (locally) bounded variation on Ω, that is, in the space $\mathrm{BV}_{(\mathrm{loc})}(\Omega)$ of functions on Ω whose gradients are merely (finite) $\mathbb{R}^n$-valued Radon measures. Clearly, this approach to existence results requires a suitable reformulation of the problem at hand, and in this paper we actually focus on the BV reformulation of the equations in (1.1).

In case of the 1-Laplace equation, the basic idea can be understood by rewriting the equation as the two equalities

$$\sigma \cdot \mathrm{D}u = |\mathrm{D}u| \qquad \text{and} \qquad \operatorname{div} \sigma \equiv 0 \qquad \text{on } \Omega \,, \tag{1.3}$$

for some auxiliary sub-unit vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$. Then the second equality is directly meaningful in $\mathscr{D}'(\Omega)$ (i.e. in the sense of distributions), and the first equality can be extended to the BV setup, in the spirit of Kohn & Temam [16, 17] and Anzellotti [2], as follows. For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with vanishing distributional divergence $\operatorname{div} \sigma$, one defines the distribution

$$[\![\sigma, \mathrm{D}u]\!] := \operatorname{div}(u\sigma) \in \mathscr{D}'(\Omega) \,, \tag{1.4}$$

which, in fact, turns out to be a signed Radon measure on Ω. One then regards $[\![\sigma, \mathrm{D}u]\!]$ as a generalized product of σ and $\mathrm{D}u$ (since, in the $\mathrm{W}^{1,1}$ case, it reduces to the ordinary pointwise product $\sigma \cdot \mathrm{D}u$ by the product rule), and consequently one uses $[\![\sigma, \mathrm{D}u]\!]$ as the BV substitute for the product in (1.3). All in all, one thus calls $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ a BV solution of the 1-Laplace equation or a weakly 1-harmonic function on Ω if there exists a sub-unit vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with

$$[\![\sigma, \mathrm{D}u]\!] = |\mathrm{D}u| \text{ as measures on } \Omega \qquad \text{and} \qquad \operatorname{div} \sigma \equiv 0 \text{ in } \mathscr{D}'(\Omega)$$

(where $|\mathrm{D}u|$ stands for the variation measure of the gradient measure $\mathrm{D}u$ of u).

More generally, the pairing

$$[\![\sigma, \mathrm{D}u]\!] := \operatorname{div}(u\sigma) - u \operatorname{div}\sigma \in \mathscr{D}'(\Omega) \tag{1.5}$$

has been studied systematically by Anzellotti [2] and is still meaningful in each of the three cases

- $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $\operatorname{div} \sigma \in \mathrm{L}^n_{\mathrm{loc}}(\Omega)$ (since $u \in \mathrm{L}^{\frac{n}{n-1}}_{\mathrm{loc}}(\Omega)$ by Sobolev's embedding),
- $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $\operatorname{div} \sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$,
- $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}^0(\Omega)$, $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} \sigma$ is a signed Radon measure on Ω.

Therefore, the same approach allows to explain BV solutions of

$$\operatorname{div} \frac{\mathrm{D}u}{|\mathrm{D}u|} = f \qquad \text{on } \Omega \tag{1.6}$$

for every $f \in \mathrm{L}^n_{\mathrm{loc}}(\Omega)$, to explain $\mathrm{BV} \cap \mathrm{L}^\infty$ solutions of (1.6) for every $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, and to explain $\mathrm{BV} \cap \mathrm{C}^0$ solutions of (1.6) even when f is a measure; compare with [10, 11, 12, 5, 18, 19, 23], for instance.

In this paper, which explicates the considerations of our announcement [20], we are concerned with notions of BV *super*solutions (and BV *sub*solutions) to the 1-Laplace equation. Thus we are naturally led to the study of pairings of the type $[\![\sigma, \mathrm{D}u]\!]$ when u is possibly discontinuous and $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ merely satisfies $\operatorname{div} \sigma \le 0$ in $\mathscr{D}'(\Omega)$ — and hence the non-positive distribution $\operatorname{div} \sigma$ is a measure, but not necessarily an L^1 function. In this case, in order to make sense of the last term in (1.5), we need to work with a $(-\operatorname{div} \sigma)$-a.e. defined representative of $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, and indeed it turns out that in general there is more than just one reasonable choice of such a representative. In recent literature [19], related problems have been tackled by working with a standard representative u^* of u, but here we will demonstrate that the choice of a *different* representative u^+ is more suitable for the definition of supersolutions. However, in working with our corresponding pairing $[\![\sigma, \mathrm{D}u^+]\!]$ and establishing its most basic properties (as, for instance, the fact that it is a Radon measure) we then encounter the difficulty that u^+, in contrast to u^*, is not the pointwise limit of standard mollifications of u. This is overcome in the present paper by involving some tools from the theory of BV capacity and some results on one-sided approximation of BV-functions [15, 9, 6]. With the pairing $[\![\sigma, \mathrm{D}u^+]\!]$ at hand it is natural to call $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ a BV *super*solution to the 1-Laplace equation or a weakly *super*-1-harmonic function on Ω if there exists a sub-unit vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with

$$[\![\sigma, \mathrm{D}u^+]\!] = |\mathrm{D}u| \text{ as measures on } \Omega \qquad \text{and} \qquad \operatorname{div} \sigma \le 0 \text{ in } \mathscr{D}'(\Omega).$$

Starting from this definition we then establish basic properties of supersolutions. Specifically, a compactness statement is already contained in the announcement [20], and here we additionally deal with the (surprisingly non-trivial) question whether functions $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ which are both super-1-harmonic and sub-1-harmonic are necessarily 1-harmonic. We will prove that the latter property is true if the critical set of u is not too diffuse and thus in particular if u is C^1. Our proof of this fact is related to some ideas of [23] and rests on an application of convex duality, which — so we believe — may be of some interest in itself.

Following the approach of [4, Definition 5.1] we also introduce a global variant $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ of our pairing, which accounts (in a generalized way) for a Dirichlet boundary datum $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and for $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \le 0$ in $\mathscr{D}'(\Omega)$, we will show that the global pairing yields a finite Borel measure on $\overline{\Omega}$. Following once more classical ideas of Anzellotti [2, 3], we also employ the global pairing to define a normal trace of σ on the boundary of Ω, which can in turn be used to state a version of the divergence theorem and to describe the behavior of $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ on $\partial\Omega$ in terms of traces. Notably, all these considerations near the boundary require only a very mild regularity hypothesis on $\partial\Omega$, which has been introduced in [22] and is crucial in order to apply the approximation results obtained there. With regard to the 1-Laplace equation, we finally discuss the suitability of yet another modification $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0}$ of the pairing in connection with supersolutions with Dirichlet data, and we establish a corresponding compactness result.

Towards the end of the paper, we also come back to the case of the minimal surface equation, and we comment on more general equations which arise as Euler equations of convex variational integrals $\int_\Omega f(\,\cdot\,,\nabla u)\,\mathrm{d}x$. Postponing the details to the final Sections 5 and 6, at this point we only briefly mention that most of the previously described results extend without difficulty — and in case of C^1 integrands even with slight simplifications — to BV supersolutions to these equations.

Finally, we stress that a major motivation for our interest in BV supersolutions stems from the fact that they arise as solutions to obstacle problems for the integrals in (1.2). This issue is discussed at length in our forthcoming preprint [21], where also a systematic connection with convex duality will be made and the need for using the representative u^+ and the modified up-to-the-boundary pairing $[\![\sigma,\mathrm{D}u^+]\!]_{u_0}^*$ will be further clarified.

2 Preliminaries

2.1 General notation

In the sequel, $\mathrm{B}_r(x)$ stands for the open ball in $\mathbb{R}^n$ with radius $r>0$ and center $x\in\mathbb{R}^n$, and we write ∂A, $\overline{A}$, and $\mathbb{1}_A$ for boundary, closure, and characteristic function of a set A in $\mathbb{R}^n$. By $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ we denote the Lebesgue measure and the $(n{-}1)$-dimensional spherical[1] Hausdorff measure on $\mathbb{R}^n$, and we set $\omega_n\coloneqq\mathcal{L}^n(\mathrm{B}_1(0))$. The measure-theoretic closure A^+ of a Borel set A in $\mathbb{R}^n$ is

$$A^+\coloneqq\left\{x\in\mathbb{R}^n\,:\,\limsup_{r\searrow 0}r^{-n}\mathcal{L}^n(\mathrm{B}_r(x)\cap A)>0\right\}.$$

Moreover, if ν is a (possibly signed or vector-valued) measure, $g\nu$ stands for the weighted measure with (suitably measurable) weight function g and base measure ν, and we abbreviate $\nu\llcorner A\coloneqq\mathbb{1}_A\nu$. If ν is a Radon measure on a subset of $\mathbb{R}^n$, we also write $|\nu|$ for the variation measure of ν and ν^{a}, ν^{s} for the absolutely continuous and singular parts in the Lebesgue decomposition $\nu=\nu^{\mathrm{a}}+\nu^{\mathrm{s}}$ of ν with respect to $\mathcal{L}^n$. Finally, we use some standard terminology for numbers, functions, derivatives, integrals, and function spaces, which we do not introduce in detail. We only briefly mention that $\mathscr{D}(\Omega)$ stands for the space of smooth and compactly supported real-valued functions on Ω, while $\mathscr{D}'(\Omega)$ denotes the corresponding space of distributions on Ω.

2.2 L^∞ divergence-measure fields

The spaces of local and global L^∞ divergence-measure fields on Ω are given by

$$\begin{aligned}\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^n)&\coloneqq\{\sigma\in\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^n)\,:\,\operatorname{div}\sigma\text{ exists as a signed Radon measure on }\Omega\}\,,\\ \mathcal{DM}^\infty(\Omega,\mathbb{R}^n)&\coloneqq\{\sigma\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)\,:\,\operatorname{div}\sigma\text{ exists as a finite signed Borel measure on }\Omega\}\,.\end{aligned}$$

We next record two related lemmas, the first one proved by Chen & Frid [7, Proposition 3.1].

Lemma 2.1 (absolute-continuity property for divergences of L^∞ vector fields)**.** *Consider $\sigma\in\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. Then, for every Borel set $A\subset\Omega$ with $\mathcal{H}^{n-1}(A)=0$, we have $|\operatorname{div}\sigma|(A)=0$.*

[1] Several results in the literature, on which we eventually rely, are usually stated for the standard Hausdorff measure instead of the spherical one. However, it can be checked that analogous statements are valid for the spherical measure, and thus there is no inconsistency in our reasoning. In addition, we mostly evaluate these measures on $\mathcal{H}^{n-1}$-rectifiable sets, where they coincide anyway by [14, Theorem 3.2.26].

Lemma 2.2 (finiteness of divergences with a sign)**.** *If Ω is bounded with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ satisfies $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$, then we necessarily have $\sigma \in \mathcal{DM}^\infty(\Omega,\mathbb{R}^n)$. Moreover, there holds $(-\operatorname{div}\sigma)(\Omega) \le \frac{n\omega_n}{\omega_{n-1}}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1}(\partial\Omega)$.*

Proof. Fix $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$. The Riesz representation theorem implies that the non-negative distribution $-\operatorname{div}\sigma$ is actually a Radon measure on Ω. Hence it only remains to establish the estimate for $(-\operatorname{div}\sigma)(\Omega)$. To this end, consider an arbitrary $\delta > 0$. Following a classical argument (compare [1, Proof of Proposition 3.62]), we exploit the definition of $\mathcal{H}^{n-1}$ in order to cover $\partial\Omega$ by a countable family $(\mathrm{B}_{r_k}(x_k))_{k\in\mathbb{N}}$ of balls with radii $r_k \le \delta$ such that $\sum_{k=1}^\infty \omega_{n-1}r_k^{n-1} \le \mathcal{H}^{n-1}(\partial\Omega)+\delta$. In view of its compactness, $\partial\Omega$ is already covered by finitely many balls $\mathrm{B}_{r_{k_1}}(x_{k_1})$, $\mathrm{B}_{r_{k_2}}(x_{k_2})$, …, $\mathrm{B}_{r_{k_m}}(x_{k_m})$, thus the set $\Omega_\delta := \Omega\setminus\bigcup_{i=1}^m \mathrm{B}_{r_{k_i}}(x_{k_i})$ is compactly contained in Ω, and we have $\mathbb{1}_{\Omega_\delta} \in \mathrm{BV}(\mathbb{R}^n)$ with

$$|\mathrm{D}\mathbb{1}_{\Omega_\delta}|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial\Omega_\delta) \le \sum_{i=1}^m n\omega_n r_{k_i}^{n-1} \le \frac{n\omega_n}{\omega_{n-1}}(\mathcal{H}^{n-1}(\partial\Omega)+\delta)\,.$$

Consequently, a mollification φ_δ of $\mathbb{1}_{\Omega_\delta}$ with a suitably small mollification radius $\le \delta$ satisfies $0 \le \varphi_\delta \in \mathscr{D}(\Omega)$ and

$$\int_\Omega |\mathrm{D}\varphi_\delta|\,\mathrm{d}x \le |\mathrm{D}\mathbb{1}_{\Omega_\delta}|(\mathbb{R}^n) \le \frac{n\omega_n}{\omega_{n-1}}(\mathcal{H}^{n-1}(\partial\Omega)+\delta)\,. \tag{2.1}$$

Therefore, we have

$$\int_\Omega \varphi_\delta\,\mathrm{d}(-\operatorname{div}\sigma) = \int_\Omega \sigma\cdot\mathrm{D}\varphi_\delta\,\mathrm{d}x \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\int_\Omega |\mathrm{D}\varphi_\delta|\,\mathrm{d}x \le \frac{n\omega_n}{\omega_{n-1}}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}(\mathcal{H}^{n-1}(\partial\Omega)+2\delta)\,.$$

Finally, we recall that δ is arbitrary, and we observe that $\lim_{\delta\searrow 0}\varphi_\delta = 1$ pointwisely on Ω. Thus, passing to the limit via Fatou's lemma, we arrive at

$$(-\operatorname{div}\sigma)(\Omega) \le \frac{n\omega_n}{\omega_{n-1}}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1}(\partial\Omega) < \infty\,,$$

and the proof of Lemma 2.2 is complete. □

2.3 BV-functions

In connection with BV-functions and sets of finite perimeter, we mostly adopt the terminology of [1], but we briefly comment on additional conventions and results. For $u \in \mathrm{BV}(\Omega)$, we recall that $\mathcal{H}^{n-1}$-a.e. point in Ω is either a Lebesgue point (also called an approximate continuity point) or an approximate jump point of u; compare [1, Sections 3.6, 3.7]. We write u^+ for the $\mathcal{H}^{n-1}$-a.e. defined representative of u which takes the Lebesgue values in the Lebesgue points and the larger of the two jump values in the approximate jump points. Correspondingly, u^- takes on the lesser jump values, and we set $u^* := \frac{1}{2}(u^++u^-)$. We emphasize in this connection that we strictly distinguish between superscripts $^\pm$ and subscripts $_\pm$, since the latter are employed, as usual, for the non-negative and non-positive parts $g_\pm$ of real-valued functions g. Moreover, with regard to the Lebesgue decomposition of the gradient measure $\mathrm{D}u$, we stick to the convention that $\mathrm{D}^{\mathrm{a}}u$ stands for the density of $(\mathrm{D}u)^{\mathrm{a}}$, while $\mathrm{D}^{\mathrm{s}}u := (\mathrm{D}u)^{\mathrm{s}}$ denotes the singular measure. Thus, the decomposition is given by $\mathrm{D}u = (\mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n + \mathrm{D}^{\mathrm{s}}u$ on Ω.

Finally, if Σ is a set of finite perimeter in $\mathbb{R}^n$, we observe that $(\mathbb{1}_\Sigma)^+ = \mathbb{1}_{\Sigma^+}$ holds $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$. In the case $\Sigma \subset \Omega$, for $u \in \mathrm{BV}(\Omega)$, we moreover write $u^{\mathrm{int}}_{\partial^*\Sigma}$ for the $\mathcal{H}^{n-1}$-a.e. defined interior trace of u on the reduced boundary $\partial^*\Sigma$ of Σ; compare [1, Sections 3.3, 3.5, 3.7]. If $\mathcal{H}^{n-1}(\partial\Sigma \setminus \partial^*\Sigma)$ happens to vanish, we also denote this trace by $u^{\mathrm{int}}_{\partial\Sigma}$ (and this will often be used with $\Sigma = \Omega$).

The following two lemmas are crucial for our purposes. The first one extends [1, Proposition 3.62] and is obtained by essentially the same reasoning.

Lemma 2.3 (BV extension by zero)**.** *If we have $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then for every $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ we have $\mathbb{1}_\Omega u \in \mathrm{BV}(\mathbb{R}^n)$ and $|\mathrm{D}(\mathbb{1}_\Omega u)|(\partial\Omega) \le \frac{n\omega_n}{\omega_{n-1}} \|u\|_{\mathrm{L}^\infty(\Omega)} \mathcal{H}^{n-1}(\partial\Omega)$. In particular, Ω is a set of finite perimeter in $\mathbb{R}^n$, and $u^{\mathrm{int}}_{\partial^*\Omega}$ is well-defined.*

Proof. Assuming that Ω is bounded, we rely once more on the functions $\varphi_\delta \in \mathscr{D}(\Omega)$ constructed in the proof of Lemma 2.2. Then $\varphi_\delta u$ converges for $\delta \searrow 0$ to $\mathbb{1}_\Omega u$ in $\mathrm{L}^1(\mathbb{R}^n)$, and via the product rule and (2.1) we get the bound

$$|\mathrm{D}(\varphi_\delta u) - \varphi_\delta \mathrm{D}u|(\mathbb{R}^n) \le \frac{n\omega_n}{\omega_{n-1}} \|u\|_{\mathrm{L}^\infty(\Omega)} (\mathcal{H}^{n-1}(\partial\Omega) + \delta) \,.$$

By lower semicontinuity of the variation we infer $\mathbb{1}_\Omega u \in \mathrm{BV}(\mathbb{R}^n)$ and

$$|\mathrm{D}(\mathbb{1}_\Omega u) - \mathrm{D}u \llcorner \Omega|(\mathbb{R}^n) \le \frac{n\omega_n}{\omega_{n-1}} \|u\|_{\mathrm{L}^\infty(\Omega)} \mathcal{H}^{n-1}(\partial\Omega) \,.$$

Since $\mathrm{D}u \llcorner \Omega$ vanishes on $\partial\Omega$, the claimed estimate follows immediately. □

The next lemma follows by combining [6, Theorem 2.5] and [9, Lemma 1.5, Section 6]; compare also [15, Sections 4, 10].

Lemma 2.4 ($\mathcal{H}^{n-1}$-a.e. approximation of a BV function from above)**.** *For every $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ there exist $v_\ell \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ such that $v_1 \ge v_\ell \ge u$ holds $\mathcal{L}^n$-a.e. on Ω for every $\ell \in \mathbb{N}$ and such that v_ℓ^* converges $\mathcal{H}^{n-1}$-a.e. on Ω to u^+.*

2.4 1-capacity

In the sequel we adopt the following notion of 1-capacity, which is equivalent to [6, Definition 2.1].

Definition 2.5 (1-capacity)**.** *The 1-capacity of an arbitrary set $E \subset \mathbb{R}^n$ is defined as the number*

$$\inf \left\{ \int_{\mathbb{R}^n} |\mathrm{D}u| \,\mathrm{d}x \,:\, u \in \mathrm{W}^{1,1}(\mathbb{R}^n)\,,\, u \ge 1 \textit{ holds } \mathcal{L}^n\textit{-a.e. on an open set } U \textit{ with } E \subset U \right\} \in [0, \infty] \,.$$

In the next three auxiliary lemmas we summarize properties of 1-capacity which are relevant for the proof of the later Lemma 3.3. The three auxiliary statements can be inferred, for instance, from [6, Theorem 2.1, Proposition 2.2, Theorem 2.5]; for the second one see also [15, Section 4].

Lemma 2.6 (perimeter characterization of 1-capacity)**.** *The 1-capacity of an arbitrary set $E \subset \mathbb{R}^n$ is equal to*

$$\inf \left\{ \mathbf{P}(H) \,:\, H \textit{ is a Borel set in } \mathbb{R}^n \textit{ with } \mathcal{L}^n(H) < \infty \textit{ and } E \subset H^+ \right\} .$$

Lemma 2.7 (zero capacity and Hausdorff measure)**.** *The* 1*-capacity of a set* $E \subset \mathbb{R}^n$ *vanishes if and only if* E *is a null set for* $\mathcal{H}^{n-1}$*.*

Lemma 2.8 (u^+ is 1-capacity quasi upper semicontinuous)**.** *For every* $u \in \mathrm{BV}(\mathbb{R}^n)$*, the representative* u^+ *is* 1*-capacity quasi upper semicontinuous on* $\mathbb{R}^n$*, that is, for every* $\varepsilon > 0$ *there exists an open set* $E_\varepsilon \subset \mathbb{R}^n$ *of* 1*-capacity smaller than* ε *such that* u^+ *is pointwisely well-defined and lower semicontinuous on* $\mathbb{R}^n \setminus E_\varepsilon$*. Clearly,* u^- *is* 1*-capacity quasi lower semicontinuous in the same sense.*

2.5 Strict approximation results

Next we collect some preliminaries, which are later relevant for the global statement of Lemma 3.3 and for Proposition 3.5. Given $u \in \mathrm{BV}(\Omega)$, we write $|(\mathcal{L}^n, \mathrm{D}u)|$ for the variation measure of the $\mathbb{R}^{1+n}$-valued Radon measure $(\mathcal{L}^n, \mathrm{D}u)$ on Ω. For $u \in \mathrm{W}^{1,1}(\Omega)$, this measure coincides with $\sqrt{1+|\mathrm{D}u|^2}\mathcal{L}^n$, so that one may view $|(\mathcal{L}^n, \mathrm{D}u)|(\Omega)$ as the generalized area of the graph of u. In the sequel, we use a corresponding type of convergence.

Definition 2.9 (strict convergence in BV)**.** *Assume* $\mathcal{L}^n(\Omega) < \infty$*, and consider functions* $u_k, u \in \mathrm{BV}(\Omega)$*. We say that* u_k *converges strictly to* u *in* $\mathrm{BV}(\Omega)$ *if we have*

$$\lim_{k\to\infty} \left[\|u_k - u\|_{\mathrm{L}^1(\Omega)} + \big| |(\mathcal{L}^n, \mathrm{D}u_k)|(\Omega) - |(\mathcal{L}^n, \mathrm{D}u)|(\Omega) \big| \right] = 0\,.$$

In many subsequent statements we impose a mild regularity assumption on $\partial\Omega$, namely we require

$$\mathcal{H}^{n-1}(\partial\Omega) = \mathbf{P}(\Omega) < \infty\,, \tag{2.2}$$

where $\mathbf{P}$ stands for the perimeter in $\mathbb{R}^n$. For a detailed discussion of (2.2) we refer to [22], where the relevance of this condition for certain approximation results has been pointed out. Here, we briefly remark that the condition (2.2) is equivalent to having $\mathbf{P}(\Omega) < \infty$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ and also to having $\mathbb{1}_\Omega \in \mathrm{BV}(\mathbb{R}^n)$ and $|\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$. In particular, (2.2) thus implies that $\partial\Omega$ is $\mathcal{H}^{n-1}$-finite and countably $\mathcal{H}^{n-1}$-rectifiable. Beyond that we only need (2.2) in connection with the subsequent lemma, which is a slight variant of [22, Proposition 4.1].

Lemma 2.10 (strict interior approximation of a BV function)**.** *If* Ω *is bounded with* (2.2)*, then, for* $u \in \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^n, \mathbb{R}^N)$*, there exist open sets of finite perimeter* $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \ldots$ *in* $\mathbb{R}^n$ *with* $\bigcup_{k=1}^\infty \Omega_k = \Omega$ *such that* $\mathbb{1}_{\Omega_k} u$ *converges strictly to* $\mathbb{1}_\Omega u$ *in* $\mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N)$*.*

Notice, in particular, that we have $\mathbb{1}_{\Omega_k} u, \mathbb{1}_\Omega u \in \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^n)$ by a standard estimate for the product of two $(\mathrm{BV} \cap \mathrm{L}^\infty)$-functions; see [1, formula (3.10)], for instance.

Lemma 2.10 is a direct consequence of [22, Proposition 4.1] and [1, Theorem 3.84] except for the two additional claims that the subsets Ω_k form an *increasing* sequence and that their union is all of Ω. Since these additional claims require only marginal modifications of the arguments in [22], we do not discuss the proof of the lemma in full detail, but we only point out these relevant modifications.

Indeed, the reasoning in [22, Section 4] is based on coverings of the compactum $\partial\Omega$ with suitable open balls and cylinders and obtains each Ω_k by removing the closure of a finite subcover from Ω. For our purposes, carrying out this construction iteratively (first for Ω_1, then for Ω_2, Ω_3, and so on), it suffices to choose the diameters of all relevant balls and cylinders in the construction of Ω_k smaller than $\min\{1/k, \mathrm{dist}(\Omega_{k-1}, \partial\Omega)/2\}$. Such an additional smallness condition for the diameters does not at all affect the line of argument in [22, Section 4]. Since this has already been worked out in case of the closely related reasoning in [22, Section 3], we here omit all further details. $\square$

The following one-sided approximation result is due to Carriero & Dal Maso & Leaci & Pascali [6, Theorem 3.3].

Lemma 2.11 (strict $\mathrm{W}^{1,1}$ approximation of a BV function from above)**.** *For bounded Ω and $u \in \mathrm{BV}(\Omega)$, there exist $u_k \in \mathrm{W}^{1,1}(\Omega)$ such that u_k converges strictly to u in $\mathrm{BV}(\Omega)$ and such that $u_k \geq u$ holds $\mathcal{L}^n$-a.e. on Ω for every $k \in \mathbb{N}$.*

We remark that the strict convergence of u_k to u in the above sense implies also that $|\mathrm{D}u_k|(\Omega)$ converges to $|\mathrm{D}u|(\Omega)$. Notice moreover that the $\mathcal{L}^n$-a.e. inequality $u_k \geq u$ implies the $\mathcal{H}^{n-1}$-a.e. inequality $u_k^* \geq u^+$. Finally, observe that in case of a bounded function u one can also find uniformly bounded approximations u_k (just replace possibly unbounded ones by $\min\{u_k, \sup_\Omega u\}$ and rely on a short reasoning with the lower semicontinuity of the total variation).

For Lipschitz domains Ω, it is well known that the trace operator on $\mathrm{BV}(\Omega)$ is continuous with respect to strict convergence; see [1, Theorem 3.88]. The following lemma establishes a similar continuity property — only for (uniformly) bounded functions, but under a much weaker assumption on Ω.

Lemma 2.12 (strict continuity of the trace operator)**.** *Suppose that Ω is bounded with (2.2), and consider $u_k, u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. If u_k converges strictly in $\mathrm{BV}(\Omega)$ and weakly$*$ in $\mathrm{L}^\infty(\Omega)$ to u, then $(u_k)^{\mathrm{int}}_{\partial\Omega}$ weak$*$ converges in $\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$ to $u^{\mathrm{int}}_{\partial\Omega}$.*

Proof. Via Lemma 2.3 we deduce that $\mathbb{1}_\Omega u_k$ weak$*$ converges to $\mathbb{1}_\Omega u$ in $\mathrm{BV}(\mathbb{R}^n)$ and thus $\mathrm{D}(\mathbb{1}_\Omega u_k)$ weak$*$ converges to $\mathrm{D}(\mathbb{1}_\Omega u)$ in the space of finite $\mathbb{R}^n$-valued Borel measures on $\mathbb{R}^n$. Taking into account the assumed strict convergence, we also know that $\mathrm{D}u_k \llcorner \Omega$ weak$*$ converges to $\mathrm{D}u \llcorner \Omega$ in the same sense, i.e. as measures on all of $\mathbb{R}^n$. Thus we infer that $\mathrm{D}(\mathbb{1}_\Omega u_k) \llcorner \partial\Omega = (u_k)^{\mathrm{int}}_{\partial\Omega} \mathrm{D}\mathbb{1}_\Omega$ weak$*$ converges to $\mathrm{D}(\mathbb{1}_\Omega u) \llcorner \partial\Omega = u^{\mathrm{int}}_{\partial\Omega} \mathrm{D}\mathbb{1}_\Omega$ (where the equalities result from [1, Theorem 3.84]). Since the u_k are uniformly bounded, we can now conclude that $(u_k)^{\mathrm{int}}_{\partial\Omega}$ weak$*$ converges to $u^{\mathrm{int}}_{\partial\Omega}$ in $\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$. $\square$

3 Anzellotti type pairings for L^∞ divergence-measure fields

3.1 Definitions

We first introduce a local pairing of divergence-measure fields and gradient measures.

Definition 3.1 (local pairing)**.** *Consider $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and $\sigma \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Then — since Lemma 2.1 guarantees that u^+ is $|\mathrm{div}\,\sigma|$-a.e. defined — we can define the distribution*

$$[\![\sigma, \mathrm{D}u^+]\!] := \mathrm{div}(u\sigma) - u^+ \mathrm{div}\,\sigma \in \mathscr{D}'(\Omega)\,.$$

Written out this definition means

$$[\![\sigma, \mathrm{D}u^+]\!](\varphi) = -\int_\Omega u\sigma \cdot \mathrm{D}\varphi \,\mathrm{d}x - \int_\Omega \varphi u^+ \,\mathrm{d}(\mathrm{div}\,\sigma) \qquad \text{for } \varphi \in \mathscr{D}(\Omega)\,. \tag{3.1}$$

We find it worth remarking that $[\![\sigma, \mathrm{D}u^+]\!]$ is bilinear in (σ, u^+), but does *not* depend linearly on u itself (since already the mapping of BV-functions u to their u^+-representative is non-linear).

Next we define a global version of this pairing, which incorporates Dirichlet boundary values given by a function u_0.

Definition 3.2 (up-to-the-boundary pairing)**.** *Consider $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$. Then we define the distribution $[\![\sigma, \mathrm{D}u^+]\!]_{u_0} \in \mathscr{D}'(\mathbb{R}^n)$ by setting*

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_0}(\varphi) := -\int_\Omega (u - u_0)\sigma \cdot \mathrm{D}\varphi \,\mathrm{d}x - \int_\Omega \varphi(u^+ - u_0^*)\,\mathrm{d}(\operatorname{div}\sigma) + \int_\Omega \varphi\sigma \cdot \mathrm{D}u_0 \,\mathrm{d}x \qquad \text{for } \varphi \in \mathscr{D}(\mathbb{R}^n)\,. \tag{3.2}$$

We emphasize that the pairings in Definitions 3.1 and 3.2 coincide on φ with compact support in Ω (since an integration-by-parts then eliminates u_0 in (3.2)). However, the up-to-the-boundary pairing stays well-defined even if φ does *not* vanish on $\partial\Omega$. Both pairings can be explained analogously for other representatives of u. Though we mostly stick to $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ in the sequel, for later usage we record that in the case $\operatorname{div}\sigma \le 0$ we have $[\![\sigma, \mathrm{D}u^-]\!]_{u_0} \le [\![\sigma, \mathrm{D}u^*]\!]_{u_0} \le [\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ (in the sense that these inequalities hold whenever a non-negative test function is plugged in).

In the sequel we show that our pairing exhibits essentially the same basic properties, which Anzellotti [2] obtained for the more classical pairing in (1.5).

3.2 The pairing trivializes on $\mathrm{W}^{1,1}$-functions

A first vital property of the pairing is recorded in the next statement.

Lemma 3.3 (the pairing trivializes on $\mathrm{W}^{1,1}$-functions)**.**

- **(local statement)** *For $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and $\sigma \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, we have*

$$[\![\sigma, \mathrm{D}u^+]\!](\varphi) = \int_\Omega \varphi\sigma \cdot \mathrm{D}u \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathscr{D}(\Omega)\,. \tag{3.3}$$

- **(global statement with fixed boundary values)** *For $u, u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $u - u_0 \in \mathrm{W}^{1,1}_0(\Omega)$ and $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$, we have*

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_0}(\varphi) = \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^n)\,. \tag{3.4}$$

- **(global statement with general traces)** *If Ω is bounded with (2.2), then for every $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$ there exists a uniquely determined normal trace $\sigma_{\mathrm{n}}^* \in \mathrm{L}^\infty(\partial\Omega; \mathcal{H}^{n-1})$ with*

$$\|\sigma_{\mathrm{n}}^*\|_{\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})} \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \tag{3.5}$$

such that for $u, u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ there holds

$$[\![\sigma, \mathrm{D}u^+]\!]_{u_0}(\varphi) = \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x + \int_{\partial\Omega} \varphi(u - u_0)^{\mathrm{int}}_{\partial\Omega}\sigma_{\mathrm{n}}^*\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^n)\,. \tag{3.6}$$

In particular, by taking $u_0 \equiv 0$ and $\varphi \equiv 1$ in the last statement, we infer the following identity. If Ω is bounded with (2.2), then, for all $u \in \mathrm{W}^{1,1}(\Omega)$ and $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$, we have

$$\int_\Omega \sigma \cdot \mathrm{D}u \,\mathrm{d}x + \int_\Omega u^*\,\mathrm{d}(\operatorname{div}\sigma) + \int_{\partial\Omega} u^{\mathrm{int}}_{\partial\Omega}\sigma_{\mathrm{n}}^*\,\mathrm{d}\mathcal{H}^{n-1} = 0\,. \tag{3.7}$$

Before proving the preceding claims, we record a basic lemma, which has been established in [15, Sections 4, 10] by a capacity method. The statements in [15] are made for $\Omega = \mathbb{R}^n$, but by localization one easily passes over to any open subset.

Lemma 3.4 (strong convergence in $\mathrm{W}^{1,1}$ implies convergence $\mathcal{H}^{n-1}$-a.e.)**.** *Suppose that u_k converges to u strongly in $\mathrm{W}^{1,1}(\Omega)$. Then, some subsequence $u^*_{k_\ell}$ converges $\mathcal{H}^{n-1}$-a.e. on Ω to u^*.*

Now we are ready to prove (3.3)–(3.6).

Proof of Lemma 3.3. We first treat the case of fixed boundary values, which clearly comprises the local statement, and in view of $[\![\sigma, \mathrm{D}u^+]\!]_{u_0} = [\![\sigma, \mathrm{D}(u{-}u_0)^+]\!]_0 + (\sigma \cdot \mathrm{D}u_0)\mathcal{L}^n \llcorner \Omega$, we assume $u_0 \equiv 0$. Then we have $u \in \mathrm{W}^{1,1}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and there exist uniformly bounded approximations $u_k \in \mathscr{D}(\Omega)$ such that u_k converges to u in $\mathrm{W}^{1,1}(\Omega)$. By Lemma 3.4 we can assume that u_k converges to u^* also $\mathcal{H}^{n-1}$-a.e. on Ω, and then via Lemma 2.1, the dominated convergence theorem, and integration by parts we get

$$
\begin{aligned}
[\![\sigma, \mathrm{D}u^+]\!](\varphi) &= -\int_\Omega u(\sigma \cdot \mathrm{D}\varphi)\,\mathrm{d}x - \int_\Omega u^*\varphi\,\mathrm{d}(\operatorname{div}\sigma)\\
&= \lim_{k\to\infty}\left[-\int_\Omega u_k(\sigma\cdot\mathrm{D}\varphi)\,\mathrm{d}x - \int_\Omega u_k\varphi\,\mathrm{d}(\operatorname{div}\sigma)\right]\\
&= \lim_{k\to\infty}\int_\Omega (\sigma\cdot\mathrm{D}u_k)\varphi\,\mathrm{d}x = \int_\Omega (\sigma\cdot\mathrm{D}u)\varphi\,\mathrm{d}x\,.
\end{aligned}
$$

This establishes the claim (3.4).

Now we turn to the statement for general boundary values, and we assume once more $u_0 \equiv 0$. We start with a construction of σ^*_{n} which satisfies (3.6) up to an ε-error for a given u and ε and which also satisfies (3.6) for the constant 1 in place of u without any error. We directly observe that σ^*_{n} is already determined by the latter requirement and is thus unique and independent of u and ε. For the construction we now fix $u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\varepsilon > 0$. Relying on Lemma 2.8, we choose an open subset E_ε of $\mathbb{R}^n$ with 1-capacity smaller than ε such that u, when understood as $u^{\mathrm{int}}_{\partial\Omega}$ on $\partial\Omega$, is pointwisely defined and continuous on $\overline{\Omega} \setminus E_\varepsilon$. (In more detail, this claim on 1-capacity quasi continuity up to the boundary can be justified by applying Lemma 2.8 to the extensions of u with values $\sup_\Omega u$ and $\inf_\Omega u$ on $\mathbb{R}^n \setminus \Omega$; by Lemma 2.3 these extensions are in $\mathrm{BV}(\mathbb{R}^n)$ and $u^{\mathrm{int}}_{\partial\Omega}$ coincides on $\partial\Omega$ with their 1-capacity quasi lower and upper semicontinuous representative, respectively.) By the characterization of 1-capacity in Lemma 2.6 we can also find a Borel set $H_\varepsilon \subset \mathbb{R}^n$ with $E_\varepsilon \subset H^+_\varepsilon$, $\mathcal{L}^n(H_\varepsilon) < \infty$, and $\mathbf{P}(H_\varepsilon) < \varepsilon$. Now we apply Lemma 2.10 with the $\mathbb{R}^2$-valued function $(1, \mathbb{1}_{H_\varepsilon})$ in place of u. We thus find open sets of finite perimeter $\Omega_\ell \Subset \Omega$ with $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \ldots$ such that $(\mathbb{1}_{\Omega_\ell}, \mathbb{1}_{\Omega_\ell}\mathbb{1}_{H_\varepsilon})$ converges strictly in $\mathrm{BV}(\mathbb{R}^n, \mathbb{R}^2)$ and pointwisely to $(\mathbb{1}_\Omega, \mathbb{1}_\Omega\mathbb{1}_{H_\varepsilon})$. By (a simple case of) Reshetnyak continuity it follows that the single components also converge strictly in $\mathrm{BV}(\mathbb{R}^n)$, and in particular $|\mathrm{D}\mathbb{1}_{\Omega_\ell}| = \mathcal{H}^{n-1} \llcorner \partial^*\Omega_\ell$ weak$*$ converges to $|\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$ in the space of signed Radon measures on $\mathbb{R}^n$. Next we consider mollifications $\eta_{\ell,m} \in \mathscr{D}(\Omega)$ which converge $\mathcal{H}^{n-1}$-a.e. on Ω to $(\mathbb{1}_{\Omega_\ell})^*$ and strictly in $\mathrm{BV}(\mathbb{R}^n)$ to $\mathbb{1}_{\Omega_\ell}$ so that in particular $|\mathrm{D}\eta_{\ell,m}|\mathcal{L}^n$ converges to $\mathcal{H}^{n-1} \llcorner \partial^*\Omega_\ell$. Possibly passing to subsequences, it follows that also $(\sigma \cdot \mathrm{D}\eta_{\ell,m})\mathcal{L}^n$ weak$*$ converges to a limit ν_ℓ with $|\nu_\ell| \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1} \llcorner \partial^*\Omega_\ell$ and that ν_ℓ weak$*$ converges to a limit ν with $|\nu| \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1} \llcorner \partial\Omega$ in the space of finite signed Borel measures on $\mathbb{R}^n$. By the Radon-Nikodým theorem we can write $\nu = \sigma^*_{\mathrm{n}}\mathcal{H}^{n-1} \llcorner \partial\Omega$ for some $\sigma^*_{\mathrm{n}} \in \mathrm{L}^\infty(\partial\Omega; \mathcal{H}^{n-1})$ with (3.5).

Next we establish (3.6), momentarily still for $u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Testing the definition of the distributional divergence with $\eta_{\ell,m}\varphi u \in \mathscr{D}(\Omega)$, we initially find

$$-\int_\Omega \eta_{\ell,m} u(\sigma \cdot \mathrm{D}\varphi)\,\mathrm{d}x - \int_\Omega \eta_{\ell,m}\varphi u \,\mathrm{d}(\operatorname{div}\sigma) = \int_\Omega \eta_{\ell,m}\varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x + \int_{\mathbb{R}^n} \varphi u \sigma \cdot \mathrm{D}\eta_{\ell,m}\,\mathrm{d}x\,.$$

Now we first send $m \to \infty$. Relying heavily on the fact that u is continuous in the interior of Ω, we then infer

$$-\int_\Omega \mathbb{1}_{\Omega_\ell} u(\sigma \cdot \mathrm{D}\varphi)\,\mathrm{d}x - \int_\Omega (\mathbb{1}_{\Omega_\ell})^*\varphi u \,\mathrm{d}(\operatorname{div}\sigma) = \int_\Omega \mathbb{1}_{\Omega_\ell}\varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x + \int_{\mathbb{R}^n} \varphi u \,\mathrm{d}\nu_\ell\,. \tag{3.8}$$

In order to send $\ell \to \infty$, we take a closer look at the limit behavior of the last term. Since ν_ℓ weak$*$ converges to ν and since u coincides outside $E_\varepsilon \subset H_\varepsilon^+$ with some $\widetilde{u}_\varepsilon \in \mathrm{C}^0(\overline{\Omega})$ such that $\|\widetilde{u}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \le \|u\|_{\mathrm{L}^\infty(\Omega)}$, we find

$$\begin{aligned}
&\limsup_{\ell\to\infty} \left| \int_{\mathbb{R}^n} \varphi u \,\mathrm{d}\nu_\ell - \int_{\mathbb{R}^n} \varphi u^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}\nu \right| \\
&\qquad \le \limsup_{\ell\to\infty} \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u\|_{\mathrm{L}^\infty(\Omega)}\big[|\nu|(H_\varepsilon^+){+}|\nu_\ell|(H_\varepsilon^+)\big] \\
&\qquad \le \limsup_{\ell\to\infty} \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u\|_{\mathrm{L}^\infty(\Omega)}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\big[\mathcal{H}^{n-1}(H_\varepsilon^+ \cap \partial\Omega) + \mathcal{H}^{n-1}(H_\varepsilon^+ \cap \partial^*\Omega_\ell)\big]\,.
\end{aligned}$$

Now we first take into account $\mathbb{1}_{H_\varepsilon^+} = (\mathbb{1}_{H_\varepsilon})^+ \le (\mathbb{1}_{H_\varepsilon})^{\mathrm{int}}_{\partial^*\Omega_\ell} + \mathbb{1}_{\partial^* H_\varepsilon}$ on $\partial^*\Omega_\ell$, and then we exploit that, as a consequence of the strict convergence of $\mathbb{1}_{\Omega_\ell}\mathbb{1}_{H_\varepsilon}$, also $(\mathbb{1}_{H_\varepsilon})^{\mathrm{int}}_{\partial^*\Omega_\ell}\mathcal{H}^{n-1}(\partial^*\Omega_\ell)$ converges to $(\mathbb{1}_{H_\varepsilon})^{\mathrm{int}}_{\partial\Omega}\mathcal{H}^{n-1}(\partial\Omega)$. Finally, we also use $(\mathbb{1}_{H_\varepsilon})^{\mathrm{int}}_{\partial\Omega} \le (\mathbb{1}_{H_\varepsilon})^+ = \mathbb{1}_{H_\varepsilon^+}$ on $\partial\Omega$ in order to infer

$$\begin{aligned}
&\limsup_{\ell\to\infty} \left| \int_{\mathbb{R}^n} \varphi u \,\mathrm{d}\nu_\ell - \int_{\partial\Omega} \varphi u^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1} \right| \\
&\qquad \le \|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u\|_{\mathrm{L}^\infty(\Omega)}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\Big[2\mathcal{H}^{n-1}(H_\varepsilon^+ \cap \partial\Omega) + \limsup_{\ell\to\infty}\mathcal{H}^{n-1}(\partial^* H_\varepsilon \cap \partial^*\Omega_\ell)\Big]\,.
\end{aligned}$$

Here, in view of $\sum_{\ell=1}^\infty \mathcal{H}^{n-1}(\partial^* H_\varepsilon \cap \partial^*\Omega_\ell) \le \mathcal{H}^{n-1}(\partial^* H_\varepsilon)$, the last $\limsup$ is actually zero, and then, using the last estimate to pass $\ell \to \infty$ in (3.8), we arrive at

$$\begin{aligned}
&\left| [\![\sigma, \mathrm{D}u^+]\!]_0(\varphi) - \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x - \int_{\partial\Omega} \varphi u^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1} \right| \\
&\qquad\qquad \le 2\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u\|_{\mathrm{L}^\infty(\Omega)}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1}(H_\varepsilon^+ \cap \partial\Omega)\,.
\end{aligned}$$

Now we observe that for $u \equiv 1$ the same argument actually works in a simpler way (without using the exceptional sets E_ε and H_ε) and gives the claimed equality. Thus, as explained above, we can assume that σ^*_{n} is independent of u and ε. Using this fact, we deduce from the last estimate

$$\begin{aligned}
&\left| [\![\sigma, \mathrm{D}u^+]\!]_0(\varphi) - \int_\Omega \varphi(\sigma \cdot \mathrm{D}u)\,\mathrm{d}x - \int_{\partial\Omega} \varphi u^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1} \right| \\
&\qquad\qquad \le 2\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\|u\|_{\mathrm{L}^\infty(\Omega)}\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\mathcal{H}^{n-1}\bigg(\bigcap_{i=1}^\infty\bigcup_{j=i}^\infty H^+_{1/j^2}\bigg)\,.
\end{aligned}$$

Finally, observing that $\bigcup_{j=i}^{\infty} H^{+}_{1/j^2}$ has 1-capacity smaller than $\sum_{j=i}^{\infty} 1/j^2$ and recalling that 1-capacity zero implies $\mathcal{H}^{n-1}$-measure zero by Lemma 2.7, we arrive at (3.6) (with $u_0 \equiv 0$).

Finally, in order to drop the assumption $u \in \mathrm{C}^\infty(\Omega)$ and assume only $u \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, we use uniformly bounded Meyers-Serrin approximations $u_k \in \mathrm{C}^\infty(\Omega)$ which converge strongly in $\mathrm{W}^{1,1}(\Omega)$ and by Lemma 3.4 also $\mathcal{H}^{n-1}$-a.e. on Ω to u (possibly after passing to a subsequence). It follows via Lemma 2.12 that moreover $(u_k)^{\mathrm{int}}_{\partial\Omega}$ weak$*$ converges to $u^{\mathrm{int}}_{\partial\Omega}$ in $\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$. In view of all these convergences it is now straightforward to carry over (3.6) (with $u_0 \equiv 0$) from u_k to u. ☐

3.3 The pairing is a bounded measure

Next we focus on bounded vector fields σ with non-positive (distributional) divergences, and we record that in view of Lemma 2.2 the pairings stay well-defined in this case. We can then state the most important property of the pairing on arbitrary BV functions.

Proposition 3.5 (the pairing is a bounded measure)**.** *Fix $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$.*

- **(local estimate)** *For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, the distribution $[\![\sigma, \mathrm{D}u^+]\!]$ is a signed Radon measure on Ω with*
$$|[\![\sigma, \mathrm{D}u^+]\!]| \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} |\mathrm{D}u| \qquad \text{on } \Omega\,. \tag{3.9}$$
- **(global estimate with equality at the boundary)** *If Ω is bounded with (2.2), then for $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, the distribution $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ is a finite signed Borel measure on $\mathbb{R}^n$ with*
$$\big|[\![\sigma, \mathrm{D}u^+]\!]_{u_0} - (u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\mathcal{H}^{n-1} \llcorner \partial\Omega\big| \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} |\mathrm{D}u| \llcorner \Omega\,. \tag{3.10}$$

Here, the estimate (3.10) contains the interior inequality (3.9), the equality $[\![\sigma, \mathrm{D}u^+]\!]_{u_0} \llcorner \partial\Omega = (u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\mathcal{H}^{n-1} \llcorner \partial\Omega$ at the boundary, and the fact that $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ is supported in $\overline{\Omega}$.

Proof of Proposition 3.5. We only prove the global statement, which then implies the local one. By Lemma 2.11 and the remark following it, we can find uniformly bounded approximations $u_k \in \mathrm{W}^{1,1}(\Omega)$ such that u_k converges to u strictly in $\mathrm{BV}(\Omega)$ and such that $u_k \ge u$ holds $\mathcal{L}^n$-a.e. on Ω for every $k \in \mathbb{N}$. Fixing a non-negative test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$, we then rely on the definition in (3.2). We control the first term in the definition via the L^1-convergence of u_k and the second one via the assumption $\operatorname{div}\sigma \le 0$ and the $\mathcal{H}^{n-1}$-a.e. inequality $u_k^* \ge u^+$ on Ω. In this way we infer
$$[\![\sigma, \mathrm{D}u^+]\!]_{u_0}(\varphi) \le \limsup_{k\to\infty} [\![\sigma, \mathrm{D}u_k^+]\!]_{u_0}(\varphi)\,.$$

Moreover, from (3.6) we get
$$\begin{aligned}
[\![\sigma, \mathrm{D}u_k^+]\!]_{u_0}(\varphi) &= \int_\Omega \varphi(\sigma \cdot \mathrm{D}u_k)\,\mathrm{d}x + \int_{\partial\Omega} \varphi(u_k{-}u_0)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1}\\
&\le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega \varphi|\mathrm{D}u_k|\,\mathrm{d}x + \int_{\partial\Omega} \varphi(u_k{-}u_0)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned}$$

The first term on the right-hand side is now controlled via the strict convergence of u_k in $\mathrm{BV}(\Omega)$. In order to deal with the second term, we involve Lemma 2.12 and record that $(u_k)^{\mathrm{int}}_{\partial\Omega}$ weak$*$ converges to $u^{\mathrm{int}}_{\partial\Omega}$ in $\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$. All in all, we thus conclude
$$[\![\sigma, \mathrm{D}u^+]\!]_{u_0}(\varphi) \le \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega \varphi\,\mathrm{d}|\mathrm{D}u| + \int_{\partial\Omega} \varphi(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\sigma^*_{\mathrm{n}}\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{whenever } 0 \le \varphi \in \mathscr{D}(\mathbb{R}^n)\,.$$

Applying the last estimate with $-u$ and $-u_0$ in place of u and u_0, we also get

$$[\![\sigma, \mathrm{D}u^-]\!]_{u_0}(\varphi) \geq -\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega \varphi \,\mathrm{d}|\mathrm{D}u| + \int_{\partial\Omega} \varphi(u-u_0)^{\mathrm{int}}_{\partial\Omega} \sigma_{\mathrm{n}}^* \,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{whenever } 0 \leq \varphi \in \mathscr{D}(\mathbb{R}^n)\,,$$

and taking into account $[\![\sigma, \mathrm{D}u^-]\!]_{u_0} \leq [\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ the last two inequalities yield indeed

$$\left| [\![\sigma, \mathrm{D}u^\pm]\!]_{u_0}(\varphi) - \int_{\partial\Omega} \varphi(u-u_0)^{\mathrm{int}}_{\partial\Omega} \sigma_{\mathrm{n}}^* \,\mathrm{d}\mathcal{H}^{n-1} \right| \leq \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega |\varphi| \,\mathrm{d}|\mathrm{D}u| \qquad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^n)\,.$$

Consequently, the Riesz-Markov representation theorem implies that $[\![\sigma, \mathrm{D}u^+]\!]_{u_0}$ is a finite signed Borel measure with (3.9) and (3.10) (and the same for $[\![\sigma, \mathrm{D}u^-]\!]_{u_0}$ and $[\![\sigma, \mathrm{D}u^*]\!]_{u_0}$). □

3.4 The absolutely continuous part of the pairing is a pointwise product

Finally, we record a last property of the pairing, which is later exploited in the proofs of Theorem 4.5 and Lemma 5.5.

Lemma 3.6 (the absolutely continuous part of the pairing is a pointwise product)**.** *For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and $\sigma \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \leq 0$ in $\mathscr{D}'(\Omega)$, we have*

$$[\![\sigma, \mathrm{D}u^+]\!]^{\mathrm{a}} = (\sigma \cdot \mathrm{D}^{\mathrm{a}} u)\mathcal{L}^n \qquad \text{on } \Omega\,.$$

The following argument is an adaption of the proof of [2, Theorem 2.4], but also partially resembles the preceding reasoning.

Proof of Lemma 3.6*.* Since the claim is local, we can assume that Ω is bounded and that we have $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$. Via Lemma 2.11 we can then find uniformly bounded approximations $u_k \in \mathrm{W}^{1,1}(\Omega)$ such that u_k converges to u strictly in $\mathrm{BV}(\Omega)$ and such that $u_k \geq u$ holds $\mathcal{L}^n$-a.e. on Ω for every $k \in \mathbb{N}$. By a version of the Reshetnyak continuity theorem we infer

$$\lim_{k\to\infty} \int_\Omega \varphi |\mathrm{D}u_k - \mathrm{D}^{\mathrm{a}} u| \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}|\mathrm{D}^{\mathrm{s}} u| \qquad \text{whenever } 0 \leq \varphi \in \mathscr{D}(\Omega)\,. \tag{3.11}$$

Indeed, if $\mathrm{D}^{\mathrm{a}} u$ is locally bounded on Ω, (3.11) follows directly from [4, Theorem 3.10], applied with the integrand $f(x,z) := \varphi(x)|z - \mathrm{D}^{\mathrm{a}} u(x)|$. In the general case $\mathrm{D}^{\mathrm{a}} u \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$, the same statement is not directly applicable, since the continuity assumption in [4] need not be satisfied. However, even in this generality, we can still apply [4, Theorem 3.10] with the cut-off integrands $f_\ell(x,z) := \varphi(x)|z - \psi_\ell(x)|$, where ψ_ℓ are bounded functions with $\lim_{\ell\to\infty} \|\psi_\ell - \mathrm{D}^{\mathrm{a}} u\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} = 0$. Passing $\ell \to \infty$ we then conclude that (3.11) is generally valid.

In the next estimate we make use of the L^1-convergence $u_k \to u$ and the $\mathcal{H}^{n-1}$-a.e. inequality $u^+ \leq u_k^*$ to control the first and second terms in the definition (3.1), respectively. Also involving Lemma 3.3 and (3.11), for $0 \leq \varphi \in \mathscr{D}(\Omega)$, we hence find

$$\begin{aligned}
[\![\sigma, \mathrm{D}u^+]\!](\varphi) - \int_\Omega \varphi(\sigma \cdot \mathrm{D}^{\mathrm{a}} u) \,\mathrm{d}x &\leq \limsup_{k\to\infty} [\![\sigma, \mathrm{D}u_k^+]\!](\varphi) - \int_\Omega \varphi(\sigma \cdot \mathrm{D}^{\mathrm{a}} u) \,\mathrm{d}x \\
&= \limsup_{k\to\infty} \int_\Omega \varphi\sigma \cdot (\mathrm{D}u_k - \mathrm{D}^{\mathrm{a}} u) \,\mathrm{d}x \\
&\leq \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \limsup_{k\to\infty} \int_\Omega \varphi |\mathrm{D}u_k - \mathrm{D}^{\mathrm{a}} u| \,\mathrm{d}x \\
&= \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \int_\Omega \varphi \,\mathrm{d}|\mathrm{D}^{\mathrm{s}} u|\,.
\end{aligned}$$

Applying this estimate with $-u$ in place of u, we also deduce

$$\llbracket\sigma,\mathrm{D}u^-\rrbracket(\varphi)-\int_\Omega\varphi(\sigma\cdot\mathrm{D}^{\mathrm{a}}u)\,\mathrm{d}x\ge-\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\int_\Omega\varphi\,\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|\qquad\text{whenever }0\le\varphi\in\mathscr{D}(\Omega)\,,$$

and combining the last two estimates with the observation $\llbracket\sigma,\mathrm{D}u^-\rrbracket\le\llbracket\sigma,\mathrm{D}u^+\rrbracket$ we arrive at

$$\left|\llbracket\sigma,\mathrm{D}u^+\rrbracket(\varphi)-\int_\Omega\varphi(\sigma\cdot\mathrm{D}^{\mathrm{a}}u)\,\mathrm{d}x\right|\le\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\int_\Omega|\varphi|\,\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|\qquad\text{for all }\varphi\in\mathscr{D}(\Omega)\,.$$

Since Proposition 3.5 guarantees that $\llbracket\sigma,\mathrm{D}u^+\rrbracket$ is a Radon measure, we can rewrite this estimate as the inequality of measures

$$|\llbracket\sigma,\mathrm{D}u^+\rrbracket-(\sigma\cdot\mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n|\le\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}|\mathrm{D}^{\mathrm{s}}u|\qquad\text{on }\Omega\,.$$

By Lebesgue decomposition of the measure on the left-hand side, we then read off that the absolutely continuous part $\left|\llbracket\sigma,\mathrm{D}u^+\rrbracket^{\mathrm{a}}-(\sigma\cdot\mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n\right|$ necessarily vanishes on Ω. □

4 Weakly super-1-harmonic functions

4.1 Weakly super-1-harmonic functions

We now give a definition of weakly super-1-harmonic functions, which employs the convenient notation

$$S^\infty(\Omega,\mathbb{R}^n):=\{\sigma\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)\,:\,|\sigma|\le1\text{ holds }\mathcal{L}^n\text{-a.e. on }\Omega\}$$

for the class of sub-unit vector fields on Ω.

Definition 4.1 (weakly super-1-harmonic function). *We call $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ weakly super-1-harmonic on Ω if there exists some $\sigma\in S^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma\le0$ in $\mathscr{D}'(\Omega)$ and $\llbracket\sigma,\mathrm{D}u^+\rrbracket=|\mathrm{D}u|$ on Ω.*

We next provide a compactness result for weakly super-1-harmonic functions. We directly remark that the assumed type of convergence is very natural, and indeed the statement applies to every *increasing* sequence of weakly super-1-harmonic functions which is bounded in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$.

Theorem 4.2 (convergence from below preserves super-1-harmonicity). *Consider a sequence of weakly super-1-harmonic functions u_k on Ω. If u_k locally weak$*$ converges to a limit u both in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and if $u_k\le u$ holds on Ω for all $k\in\mathbb{N}$, then u is weakly super-1-harmonic on Ω.*

Theorem 4.2 has already been established in [20, Theorem 4.2], and we do not repeat the proof here. However, later on we adapt the same line of argument in order to establish the more general Theorem 4.7.

We emphasize that Theorem 4.2 does not hold anymore if one replaces $\llbracket\sigma,\mathrm{D}u^+\rrbracket$ with the analogous pairing $\llbracket\sigma,\mathrm{D}u^\sharp\rrbracket$ which involves another representative $u^\sharp$ of u — at least if the choice of $u^\sharp$ is *reasonable* in the sense that $u^-\le u^\sharp\le u^+$. This can be seen, already in dimension $n=1$ (where weak super-1-harmonicity on an interval means that the function is increasing up to a certain point and decreasing afterwards), from the following example. For $\Omega := (-2,2)$ and $u_k(x) := (\min\{kx, 1-x\})_+$, it is easily checked that $u_k \in \mathrm{W}_0^{1,\infty}(\Omega)$ are weakly super-1-harmonic on Ω and that the limit $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ satisfies $u(x) = (1-x)\mathbb{1}_{(0,1)}(x)$ and $|\mathrm{D}u| = \mathcal{L}^1 \llcorner (0,1) + \delta_0$. If a non-increasing $\sigma \in S^\infty(\Omega)$ satisfies $[\![\sigma, \mathrm{D}u^\sharp]\!] = |\mathrm{D}u|$ on Ω, one first infers $\sigma \equiv -1$ on $(0,2)$, and then an integration by parts on $(0,2)$ shows

$$[\![\sigma, \mathrm{D}u^\sharp]\!] = \mathcal{L}^1 \llcorner (0,1) + u^\sharp(0)(-\sigma' \llcorner \{0\}) - \delta_0 \,.$$

Consequently, the equality $[\![\sigma, \mathrm{D}u^\sharp]\!] \llcorner \{0\} = |\mathrm{D}u| \llcorner \{0\}$ requires $u^\sharp(0)(-\sigma'(\{0\})) = 2$, and in view of $-\sigma'(\{0\}) \leq 2$ this can only hold if the reasonable representative $u^\sharp$ satisfies $u^\sharp(0) = 1$ and thus coincides with u^+.

4.2 Simultaneously super-1-harmonic and sub-1-harmonic functions are 1-harmonic

Next we aim at establishing a consistency result for the notion introduced in Definition 4.1. To this end we need the following duality result, which is essentially a restatement of [13, Theorem III.4.1]. More precisely, [13] covers the non-trivial case that the infimum in (4.1) does not equal $-\infty$; the other case and indeed the inequality '$\geq$' in (4.1) follow directly from the definitions of F^* and T^*.

Theorem 4.3 (abstract convex duality)**.** *Consider normed spaces X and Y, a bounded linear operator $T\colon X \to Y$, and a convex functional $F\colon X \times Y \to (-\infty,\infty]$. If there exists some $u_0 \in X$ such that $F[u_0, Tu_0]$ is finite and $F[u_0,\,\cdot\,]$ is continuous at Tu_0, then there holds*

$$\inf_{u \in X} F[u, Tu] = \sup_{\zeta \in Y^*} \big(-F^*[T^*\zeta, -\zeta] \big) \in [-\infty, \infty) \,, \tag{4.1}$$

where $T^*\colon Y^* \to X^*$ denotes the adjoint of T and where the lower semicontinuous and convex functional $F^*\colon X^* \times Y^* \to (-\infty,\infty]$ is given by $F^*[\tau,-\zeta] := \sup_{(u,w)\in X\times Y} \big(\langle \tau, u\rangle - \langle \zeta, w\rangle - F[u,w]\big)$. Moreover, if the common target value in (4.1) is finite, then the supremum on the right-hand side is, in fact, a maximum.

Next we specialize Theorem 4.3 in order to deduce a duality correspondence in a less abstract setting. This statement has been partially inspired by the considerations in [23], and in the case $\Gamma = \partial\Omega$ it is essentially contained in [23, Theorem 1.1].

Theorem 4.4 (a concrete duality result)**.** *Suppose that Ω is bounded with Lipschitz boundary, and consider a boundary datum $\beta \in \mathrm{L}^\infty(\Gamma; \mathcal{H}^{n-1})$ on a Borel boundary portion $\Gamma \subset \partial\Omega$. Abbreviating*

$$\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega) := \{u \in \mathrm{W}^{1,1}(\Omega) \,:\, u \text{ has zero trace } \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \setminus \Gamma\}$$
$$\mathrm{L}^\infty_{\mathrm{div};\Gamma,\beta}(\Omega,\mathbb{R}^n) := \{\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n) \,:\, \operatorname{div}\sigma \equiv 0 \text{ in } \mathscr{D}'(\Omega) \text{ and } \sigma_{\mathrm{n}}^* = \beta \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma\}\,,$$

we then have

$$\sup_{\substack{u \in \mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)\\ \mathrm{D}u \not\equiv 0}} \frac{1}{\|\mathrm{D}u\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)}} \int_\Gamma u\beta \,\mathrm{d}\mathcal{H}^{n-1} = \inf_{\mathrm{L}^\infty_{\mathrm{div};\Gamma,\beta}(\Omega,\mathbb{R}^n)} \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \in (-\infty,\infty] \,, \tag{4.2}$$

where u is evaluated on Γ in the sense of trace. If, moreover, we have $\int_{\partial C} \beta \,\mathrm{d}\mathcal{H}^{n-1} = 0$ for every connected component C of Ω with $\mathcal{H}^{n-1}(\partial C \setminus \Gamma) = 0$, then the common target value in (4.2) is finite (in particular $\mathrm{L}^\infty_{\mathrm{div};\Gamma,\beta}(\Omega,\mathbb{R}^n) \neq \emptyset$), and the infimum on the right-hand side of (4.2) is, in fact, a minimum.

Proof. We will identify the equality

$$\inf_{\substack{u\in \mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)\\ \|\mathrm{D}u\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)}\le 1}} \left(-\int_\Gamma u\beta\,\mathrm{d}\mathcal{H}^{n-1}\right) = \sup_{\mathrm{L}^\infty_{\mathrm{div};\Gamma,\beta}(\Omega,\mathbb{R}^n)} \left(-\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)}\right) \tag{4.3}$$

as a special case of (4.1) in Theorem 4.3. Once (4.3) is available, all the claims of Theorem 4.4 follow quickly. Indeed, if the extra assumption fails and thus there exists some connected component C of Ω with $\mathcal{H}^{n-1}(\partial C\setminus\Gamma)=0$ and $\int_{\partial C}\beta\,\mathrm{d}\mathcal{H}^{n-1}\neq 0$, then one readily checks that the supremum in (4.2) and the infimum in (4.3) equal ∞ and $-\infty$, respectively (just consider functions u which are constant on ∂C). Thus, in this case, (4.2) follows trivially from (4.3). If, however, the extra assumption is at hand, then it suffices to observe that on the left-hand side of (4.3) we can restrict[2] to functions u with $\mathrm{D}u\not\equiv 0$, and (4.2) follows from (4.3) by exploiting the 1-homogeneity of the left-hand functionals in u. Moreover, in order to deduce the finiteness of the left-hand side in (4.3), respectively (4.2), we note that due to the extra assumption, we may restrict ourselves to functions u with $\int_C u\,\mathrm{d}x=0$ on every connected component C of Ω with $\mathcal{H}^{n-1}(\partial C\setminus\Gamma)=0$. Applying the Poincaré inequality for vanishing mean values on these components and a version of Poincaré's inequality for zero boundary values on the remaining components (which can be established, for instance, by the usual contradiction argument), we deduce $\|u\|_{\mathrm{W}^{1,1}(\Omega)}\le c\|\mathrm{D}u\|_{L^1(\Omega,\mathbb{R}^n)}$ for some constant $c=c(\Omega,\Gamma)$. Then the boundedness of the trace operator $\mathrm{W}^{1,1}(\Omega)\to\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$ yields the finiteness of the target value in (4.3) and in (4.2). The attainment of the supremum in (4.3) and the infimum in (4.2) follows from the corresponding statement in Theorem 4.3.

In order to verify (4.3) we now take $X=\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)$, $Y=\mathrm{L}^1(\Omega,\mathbb{R}^n)\times\mathrm{L}^1(\Gamma;\mathcal{H}^{n-1})$ and identify Y^* with $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)\times\mathrm{L}^\infty(\Gamma;\mathcal{H}^{n-1})$ via the Riesz representation theorem. In addition, we choose T as the bounded linear operator $X\to Y$ whose components are the gradient and the trace on Γ (where the boundedness of the trace operator results from the Lipschitz assumption on $\partial\Omega$; compare [1, Theorem 3.87]). We then observe that for $(\sigma,\chi)\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)\times\mathrm{L}^\infty(\Gamma;\mathcal{H}^{n-1})$ we have

$$\begin{aligned} T^*(\sigma,\chi)=0 \text{ in } \mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)^* &\iff \int_\Omega \sigma\cdot\mathrm{D}u\,\mathrm{d}x+\int_\Gamma \chi u\,\mathrm{d}\mathcal{H}^{n-1}=0 \text{ for all } u\in\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)\\ &\iff \operatorname{div}\sigma\equiv 0 \text{ in } \mathscr{D}'(\Omega) \text{ and } \sigma_{\mathrm{n}}^*=\chi \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma\,. \end{aligned} \tag{4.4}$$

Setting

$$F[u,(v,w)] := \begin{cases} -\int_\Gamma w\beta\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } \|v\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)}\le 1\\ \infty & \text{otherwise}\end{cases}$$

for $u\in\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)$ and $(v,w)\in\mathrm{L}^1(\Omega,\mathbb{R}^n)\times\mathrm{L}^1(\Gamma;\mathcal{H}^{n-1})$ we find that the left-hand sides of (4.1) and (4.3) coincide. In addition from the definition of F^* one readily checks, for $\tau\in\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)^*$ and $(\sigma,\chi)\in\mathrm{L}^\infty(\Omega,\mathbb{R}^n)\times\mathrm{L}^\infty(\Gamma;\mathcal{H}^{n-1})$,

$$F^*[\tau,-(\sigma,\chi)] = \begin{cases} \|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} & \text{if } \tau \text{ equals } 0 \text{ and } \chi=\beta \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma\\ \infty & \text{otherwise}\end{cases}\,.$$

[2] Indeed, for $\widehat{u}\in\mathrm{W}^{1,1}_{\partial\Omega\setminus\Gamma}(\Omega)$ with $\mathrm{D}\widehat{u}\equiv 0$ on Ω, consider any connected component C of Ω. Then we have $\widehat{u}\equiv c$ on C for some $c\in\mathbb{R}$, in the case $\mathcal{H}^{n-1}(\partial C\setminus\Gamma)>0$ we infer $c=0$, and in the case $\mathcal{H}^{n-1}(\partial C\setminus\Gamma)=0$ we have assumed $\int_{\partial C}\beta\,\mathrm{d}\mathcal{H}^{n-1}=0$. Thus, in both cases we arrive at $-\int_{\Gamma\cap\partial C}\widehat{u}\beta\,\mathrm{d}\mathcal{H}^{n-1}=0$ while clearly there exist also $u\in\mathrm{W}^{1,1}(\Omega)$ with $\mathrm{D}u\not\equiv 0$ on C and $-\int_{\Gamma\cap\partial C}u\beta\,\mathrm{d}\mathcal{H}^{n-1}\le 0$.

Taking into account (4.4), from this formula for F^* we infer that also the right-hand sides of (4.1) and (4.3) coincide, and thus we have identified (4.3) as a special case of (4.1). $\square$

Now we are ready to prove that — under an extra assumption which is generally satisfied at least for C^1 functions — simultaneous super-1-harmonicity and sub-1-harmonicity imply 1-harmonicity. Somewhat surprisingly this seems to require a non-trivial argument, and in particular we will rely on the preceding Theorem 4.4. We remark that the same issue becomes much simpler in case of the minimal surface equation; see Section 5 and, in particular, Proposition 5.3.

Theorem 4.5 (simultaneously super-1-harmonic and sub-1-harmonic functions are 1-harmonic)**.** *Consider $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and suppose that there exists an open set $U \subset \Omega$ with $\mathrm{D}^{\mathrm{a}}u \neq 0$ $\mathcal{L}^n$-a.e. on U and $|\mathrm{D}u|(\Omega \setminus U) = 0$. If both u and $-u$ are weakly super-1-harmonic on Ω, then u is also weakly 1-harmonic on Ω (i.e. there exists some $\sigma \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \equiv 0$ in $\mathscr{D}'(\Omega)$ and $[\![\sigma, \mathrm{D}u]\!] = |\mathrm{D}u|$ on Ω, where $[\![\sigma, \mathrm{D}u]\!]$ denotes the classical Anzellotti pairing from* (1.4)*).*

We remark that the set U in the assumptions of Theorem 4.5 is essentially determined by u, and in fact these assumptions are equivalent to the requirements that, for some pointwisely defined representative of the density $\mathrm{D}^a u$, the set $U = \Omega \cap \{\mathrm{D}^a u \neq 0\}$ is open with $|\mathrm{D}^{\mathrm{s}}u|(\Omega \setminus U) = 0$.

Proof of Theorem 4.5*.* By assumption there exist $\overline{\sigma}, \underline{\sigma} \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with

$$\operatorname{div}\overline{\sigma} \leq 0 \leq \operatorname{div}\underline{\sigma} \qquad \text{in } \mathscr{D}'(\Omega) \tag{4.5}$$

such that $[\![\overline{\sigma}, \mathrm{D}u^+]\!] = |\mathrm{D}u| = [\![\underline{\sigma}, \mathrm{D}u^-]\!]$ holds on Ω. Via Lemma 3.6 we infer that $\overline{\sigma} = \frac{\mathrm{D}^{\mathrm{a}}u}{|\mathrm{D}^{\mathrm{a}}u|} = \underline{\sigma}$ holds $\mathcal{L}^n$-a.e. on U. We now choose increasing sequences of smooth open sets $\Omega_k \Subset \Omega$ and $U_k \Subset U \cap \Omega_k$ with $\bigcup_{k=1}^\infty \Omega_k = \Omega$ and $\bigcup_{k=1}^\infty U_k = U$. We set $\beta_k := -\overline{\sigma}_{\mathrm{n}}^* = -\underline{\sigma}_{\mathrm{n}}^* \in \mathrm{L}^\infty(\partial U_k;\mathcal{H}^{n-1})$ (where the normal traces are taken with respect to U_k). Then, if C is a connected component of $\Omega_k \setminus \overline{U_k}$ with $\mathcal{H}^{n-1}(\partial C \setminus \partial U_k) = 0$, it follows that $\int_{\partial C} \beta_k \,\mathrm{d}\mathcal{H}^{n-1}$ vanishes (since the application of (3.7) with the constant 1 in place of u, once with the domain $\Omega_k \setminus \overline{C}$ and once with Ω_k, shows that this integral equals both the non-negative quantity $(-\operatorname{div}\overline{\sigma})(\overline{C})$ and the non-positive quantity $(-\operatorname{div}\underline{\sigma})(\overline{C})$). Moreover, exploiting (3.7), (4.5), and $\overline{\sigma}, \underline{\sigma} \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$, for $w \in \mathrm{W}^{1,1}_{\partial\Omega_k}(\Omega_k \setminus \overline{U_k})$ we find

$$\begin{aligned}
&\int_{\partial U_k} w\beta_k \,\mathrm{d}\mathcal{H}^{n-1}\\
&= \int_{\partial U_k} w_+\beta_k \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial U_k} w_-\beta_k \,\mathrm{d}\mathcal{H}^{n-1}\\
&= -\int_{\Omega_k\setminus\overline{U_k}} (w_+)^* \,\mathrm{d}(\operatorname{div}\underline{\sigma}) - \int_{\Omega_k\setminus\overline{U_k}} \underline{\sigma}\cdot \mathrm{D}w_+ \,\mathrm{d}x - \int_{\Omega_k\setminus\overline{U_k}} (w_-)^* \,\mathrm{d}(-\operatorname{div}\overline{\sigma}) + \int_{\Omega_k\setminus\overline{U_k}} \overline{\sigma}\cdot \mathrm{D}w_- \,\mathrm{d}x\\
&\leq -\int_{\Omega_k\setminus\overline{U_k}} \underline{\sigma}\cdot \mathrm{D}w_+ \,\mathrm{d}x + \int_{\Omega_k\setminus\overline{U_k}} \overline{\sigma}\cdot \mathrm{D}w_- \,\mathrm{d}x\\
&\leq \|\mathrm{D}w_+\|_{\mathrm{L}^1(\Omega_k\setminus\overline{U_k},\mathbb{R}^n)} + \|\mathrm{D}w_-\|_{\mathrm{L}^1(\Omega_k\setminus\overline{U_k},\mathbb{R}^n)} = \|\mathrm{D}w\|_{\mathrm{L}^1(\Omega_k\setminus\overline{U_k},\mathbb{R}^n)}\,,
\end{aligned}$$

and thus we can apply Theorem 4.4 on $\Omega_k \setminus \overline{U_k}$ in order to deduce

$$\min_{\tau\in \mathrm{L}^\infty_{\operatorname{div};\partial U_k,\beta_k}(\Omega_k\setminus\overline{U_k},\mathbb{R}^n)} \|\tau\|_{\mathrm{L}^\infty(\Omega_k\setminus\overline{U_k},\mathbb{R}^n)} \leq 1\,. \tag{4.6}$$

In particular, if we set $\sigma_k := \overline{\sigma} = \underline{\sigma}$ on U_k and extend σ_k to all of Ω_k by using on $\Omega_k \setminus \overline{U_k}$ the values of a minimizing τ in (4.6), then we have $\sigma_k \in S^\infty(\Omega_k, \mathbb{R}^n)$, and we claim that $\operatorname{div} \sigma_k$ vanishes on Ω_k. Indeed, keeping in mind the preceding choices and applying (3.7) on the smooth bounded open sets $\Omega_k \setminus \overline{U_k}$ and U_k, for every $\varphi \in \mathscr{D}(\Omega_k)$, we find

$$\int_{\Omega_k} \sigma_k \cdot \mathrm{D}\varphi \,\mathrm{d}x = \int_{\Omega_k \setminus \overline{U_k}} \tau \cdot \mathrm{D}\varphi \,\mathrm{d}x + \int_{U_k} \overline{\sigma} \cdot \mathrm{D}\varphi \,\mathrm{d}x = -\int_{\partial U_k} \varphi \beta_k \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial U_k} \varphi \overline{\sigma}^*_{\mathrm{n}} \,\mathrm{d}\mathcal{H}^{n-1} = 0\,.$$

Thus, we have verified $\operatorname{div} \sigma_k \equiv 0$ in $\mathscr{D}'(\Omega_k)$. Possibly passing to a subsequence and recalling that Ω is the increasing union of the Ω_k, it now follows that σ_k locally weak$*$ converges in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ to some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \equiv 0$ in $\mathscr{D}'(\Omega)$. In order to finalize the reasoning we recall that $\sigma = \sigma_k = \overline{\sigma}$ holds $\mathcal{L}^n$-a.e. on U_k, and as a consequence we observe $[\![\sigma, \mathrm{D}u]\!] \llcorner U_k = [\![\overline{\sigma}, \mathrm{D}u^+]\!] \llcorner U_k = |\mathrm{D}u| \llcorner U_k$. Since the open set U is the union of the U_k and since (3.9) and the assumption on U give $|[\![\sigma, \mathrm{D}u]\!]| \llcorner (\Omega \setminus U) \le |\mathrm{D}u| \llcorner (\Omega \setminus U) = 0$, this suffices to establish the equality $[\![\sigma, \mathrm{D}u]\!] = |\mathrm{D}u|$ on all of Ω. $\square$

4.3 Weakly super-1-harmonic functions with respect to boundary data

Finally, we introduce a concept of weakly super-1-harmonic functions with respect to generalized Dirichlet boundary data. In [21] we will show that this notion is useful in connection with obstacle problems.

Concretely, consider a bounded Ω with (2.2), $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. We then extend the measure $\mathrm{D}u$ on Ω to a measure $\mathrm{D}_{u_0} u$ on $\overline{\Omega}$ which takes into account the possible deviation of $u^{\mathrm{int}}_{\partial\Omega}$ from the boundary datum $(u_0)^{\mathrm{int}}_{\partial\Omega}$. To this end, writing ν_Ω for the inward unit normal of Ω (which is defined on $\partial^*\Omega$ and thus $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$), we set

$$\mathrm{D}_{u_0} u := \mathrm{D}u \llcorner \Omega + (u{-}u_0)^{\mathrm{int}}_{\partial\Omega} \nu_\Omega \mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{4.7}$$

Now, for some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \le 0$ in $\mathscr{D}'(\Omega)$, one may first attempt to work with the up-to-the-boundary coupling condition $[\![\sigma, \mathrm{D}u^+]\!]_{u_0} = |\mathrm{D}_{u_0} u|$ on $\overline{\Omega}$. Since by Lemma 3.5 this condition contains the $\mathcal{H}^{n-1}$-a.e. equality $(u{-}u_0)^{\mathrm{int}}_{\partial\Omega} \sigma^*_{\mathrm{n}} = |(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}|$ on the boundary $\partial\Omega$, it is then natural to take the viewpoint that σ^*_{n} should typically equal the constant 1 (notice that the constant -1 is incompatible with $\operatorname{div} \sigma \le 0$). However, we can even allow that σ^*_{n} differs from 1, if we compensate for this defect by extending $(-\operatorname{div} \sigma)$ to a non-negative measure on $\overline{\Omega}$ with

$$(-\operatorname{div} \sigma) \llcorner \partial\Omega := (1{-}\sigma^*_{\mathrm{n}}) \mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{4.8}$$

Then we define a modified pairing $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0}$ by interpreting u^+ as $\max\{u^{\mathrm{int}}_{\partial\Omega}, (u_0)^{\mathrm{int}}_{\partial\Omega}\}$ on $\partial\Omega$ and extending the $(\operatorname{div} \sigma)$-integral in (3.2) from Ω to $\overline{\Omega}$. In other words, this means that we define the signed measure $[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0}$ by setting

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0} := [\![\sigma, \mathrm{D}u^+]\!]_{u_0} + \big[(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\big]_+ (-\operatorname{div} \sigma) \llcorner \partial\Omega\,. \tag{4.9}$$

For $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \le 0$ in $\mathscr{D}'(\Omega)$, $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, and $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, we get from (3.10), (4.8), (4.9)

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0} \llcorner \partial\Omega = \Big(\big[(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\big]_+ - \big[(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\big]_- \sigma^*_{\mathrm{n}}\Big) \mathcal{H}^{n-1} \llcorner \partial\Omega\,, \tag{4.10}$$

and in particular we read off the basic estimate

$$\left|[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}\right| \le |\mathrm{D}_{u_0}u| \qquad \text{on } \overline{\Omega}\,. \tag{4.11}$$

With these conventions, we now complement Definition 4.1 as follows.

Definition 4.6 (super-1-harmonic function with respect to a Dirichlet datum)**.** *For bounded Ω with (2.2) and $u_0 \in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$, we say that $u \in \mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ is weakly super-1-harmonic on $\overline{\Omega}$ with respect to u_0 if there exists some $\sigma \in S^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$ such that the equality of measures $[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0} = |\mathrm{D}_{u_0}u|$ holds on $\overline{\Omega}$.*

From (4.10) we infer that the boundary condition in Definition 4.6 is equivalent to the $\mathcal{H}^{n-1}$-a.e. equality $\sigma^*_{\mathrm{n}} \equiv -1$ on the boundary portion $\{u^{\mathrm{int}}_{\partial\Omega} < (u_0)^{\mathrm{int}}_{\partial\Omega}\}\cap\partial\Omega$, while no requirement is made on $\{u^{\mathrm{int}}_{\partial\Omega} \ge (u_0)^{\mathrm{int}}_{\partial\Omega}\}\cap\partial\Omega$. We believe that this is very natural, in particular in the case $n=1$, where super-1-harmonicity of u on an interval $[a,b]$ just means that u is increasing up to a certain point and decreasing afterwards, and where σ^*_{n} can take the value -1 at most at *one* endpoint and only if u is monotone on the full interval (a,b).

Another indication that Definition 4.6 is meaningful is provided by the next statement. We emphasize that the statement does not hold anymore (not even for $n{=}1$, $u_{0;k} = u_0 \equiv 0$, and $u_k \in \mathrm{W}^{1,\infty}_0(\Omega)$) if one replaces $[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}$ with $[\![\sigma,\mathrm{D}u^+]\!]_{u_0}$ in the definition; this can be seen by considering once more the example presented after Theorem 4.2 — but now on the domain $\Omega=(0,2)$, for which the critical point 0 is at the boundary.

Theorem 4.7 (convergence from below preserves super-1-harmonicity)**.** *Suppose that Ω is bounded with (2.2), and consider weakly super-1-harmonic functions $u_k \in \mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ on $\overline{\Omega}$ with respect to boundary data $u_{0;k} \in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$. If $u_{0;k}$ converges strongly in $\mathrm{W}^{1,1}(\Omega)$ and weakly$*$ in $\mathrm{L}^\infty(\Omega)$ to some u_0, and if u_k weak$*$ converges to a limit u in $\mathrm{BV}(\Omega)$ and $\mathrm{L}^\infty(\Omega)$ such that $u_k \le u$ holds on Ω for all $k\in\mathbb{N}$, then u is weakly super-1-harmonic on $\overline{\Omega}$ with respect to u_0.*

Remark 4.8. *In the situation of the theorem, it follows from the previously recorded reformulation of the boundary condition that u is also weakly super-1-harmonic on $\overline{\Omega}$ with respect to every $\widetilde{u}_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)\cap\mathrm{L}^\infty(\Omega)$ such that $\mathcal{H}^{n-1}(\{u_0^* \le u^{\mathrm{int}}_{\partial\Omega} < \widetilde{u}_0^*\}\cap\partial\Omega)=0$. Roughly speaking, this means that the boundary values can always be decreased and that they can even be increased as long as the trace of u is not traversed. In view of the 1-dimensional case, we believe that this behavior is very reasonable.*

In order to prove Theorem 4.7 we will use the following global variant of Lemma 2.4.

Lemma 4.9 ($\mathcal{H}^{n-1}$-a.e. approximation of a BV function from above)**.** *For bounded Ω with (2.2) and $u \in \mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ there exist $v_\ell \in \mathrm{W}^{1,1}(\Omega)$ such that $v_\ell \ge u$ holds $\mathcal{L}^n$-a.e. on Ω and such that v_ℓ^* converges $\mathcal{H}^{n-1}$-a.e. on Ω to u^+. If also $u_0 \in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ is given, one can additionally achieve that $(v_\ell)^{\mathrm{int}}_{\partial\Omega} \ge (u_0)^{\mathrm{int}}_{\partial\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ and that $(v_\ell)^{\mathrm{int}}_{\partial\Omega}$ converges $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ to $\max\{u^{\mathrm{int}}_{\partial\Omega},(u_0)^{\mathrm{int}}_{\partial\Omega}\}$.*

Proof. As a consequence of Lemma 2.3 we have $\mathbb{1}_\Omega(u{-}u_0) \in \mathrm{BV}(\mathbb{R}^n)\cap\mathrm{L}^\infty(\mathbb{R}^n)$. By Lemma 2.4 and a cut-off procedure far away from Ω we can thus find $w_\ell \in \mathrm{W}^{1,1}(\mathbb{R}^n)\cap\mathrm{L}^\infty(\mathbb{R}^n)$ such that $w_\ell \ge \mathbb{1}_\Omega(u{-}u_0)$ holds $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$ and such that w_ℓ^* converges $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$ to $[\mathbb{1}_\Omega(u{-}u_0)]^+$. We now define $v_\ell \coloneqq u_0{+}w_\ell$ on Ω, and since $[\mathbb{1}_\Omega(u{-}u_0)]^+$ coincides with $(u^+{-}u_0^*)$ on Ω and with $\big[(u{-}u_0)^{\mathrm{int}}_{\partial\Omega}\big]_+$ on $\partial\Omega$, we straightforwardly deduce the claims. ∎

Proof of Theorem 4.7. By Definition 4.6 there exist $\sigma_k \in S^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma_k \le 0$ in $\mathscr{D}'(\Omega)$ and $[\![\sigma_k, \mathrm{D}u_k^+]\!]^*_{u_{0;k}} = |\mathrm{D}_{u_{0;k}} u_k|$ on $\overline{\Omega}$. Possibly passing to a subsequence, we can assume that σ_k weak$*$ converges in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ to $\sigma \in S^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$. We now work with the approximations v_ℓ of Lemma 4.9, and after a cut-off argument we also assume that these approximations are uniformly bounded. Relying on (3.2), Lemma 2.1, and the dominated convergence theorem, we then infer

$$\begin{aligned}
[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0}(\overline{\Omega}) &= \int_\Omega (u^+ - u_0^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_{\partial\Omega} \big[(u-u_0)^{\mathrm{int}}_{\partial\Omega}\big]_+\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma\cdot \mathrm{D}u_0\,\mathrm{d}x \\
&= \lim_{\ell\to\infty}\bigg[\int_\Omega (v_\ell^* - u_0^*)\,\mathrm{d}(-\operatorname{div}\sigma) + \int_{\partial\Omega} (v_\ell-u_0)^{\mathrm{int}}_{\partial\Omega}\,\mathrm{d}(-\operatorname{div}\sigma) + \int_\Omega \sigma\cdot\mathrm{D}u_0\,\mathrm{d}x\bigg] \\
&= \lim_{\ell\to\infty} [\![\sigma, \mathrm{D}v_\ell^+]\!]^*_{u_0}(\overline{\Omega})\,.
\end{aligned}$$

Since the pairing exhibits the boundary behavior (4.10) and since we have $(v_\ell)^{\mathrm{int}}_{\partial\Omega} \ge (u_0)^{\mathrm{int}}_{\partial\Omega}$ and $(\sigma_k)^*_{\mathrm{n}} \ge -1$ on $\partial\Omega$, at the boundary we moreover get

$$\begin{aligned}
[\![\sigma, \mathrm{D}v_\ell^+]\!]^*_{u_0}(\partial\Omega) &\ge [\![\sigma_k, \mathrm{D}v_\ell^+]\!]^*_{u_{0;k}}(\partial\Omega) \\
&\qquad + \int_{\partial\Omega} \Big(\big[(v_\ell-u_0)^{\mathrm{int}}_{\partial\Omega}\big]_+ - \big[(v_\ell-u_{0;k})^{\mathrm{int}}_{\partial\Omega}\big]_+ - \big[(v_\ell-u_{0;k})^{\mathrm{int}}_{\partial\Omega}\big]_-\Big)\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned}$$

Since $|u_{0;k}-u_0|$ converges to 0 strongly in $\mathrm{W}^{1,1}(\Omega)$ and weakly$*$ in $\mathrm{L}^\infty(\Omega)$, we infer from Lemma 2.12 that $|(u_{0;k}-u_0)^{\mathrm{int}}_{\partial\Omega}|$ weak$*$ converges to 0 in $\mathrm{L}^\infty(\partial\Omega;\mathcal{H}^{n-1})$, and then it also follows that $(u_{0;k})^{\mathrm{int}}_{\partial\Omega}$ converges to $(u_0)^{\mathrm{int}}_{\partial\Omega}$ strongly in $\mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$. Therefore, the $\mathcal{H}^{n-1}$-integral in the last formula vanishes in the limit $k\to\infty$. We next exploit Lemma 3.3 (on Ω) and the weak$*$ convergence of σ_k in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ to arrive at

$$\begin{aligned}
[\![\sigma, \mathrm{D}v_\ell^+]\!]^*_{u_0}(\overline{\Omega}) &= \int_\Omega \sigma\cdot\mathrm{D}v_\ell\,\mathrm{d}x + [\![\sigma, \mathrm{D}v_\ell^+]\!]^*_{u_0}(\partial\Omega) \\
&\ge \liminf_{k\to\infty}\bigg[\int_\Omega \sigma_k\cdot\mathrm{D}v_\ell\,\mathrm{d}x + [\![\sigma_k, \mathrm{D}v_\ell^+]\!]^*_{u_{0;k}}(\partial\Omega)\bigg] = \liminf_{k\to\infty} [\![\sigma_k, \mathrm{D}v_\ell^+]\!]^*_{u_{0;k}}(\overline{\Omega}).
\end{aligned}$$

Via the definition of the modified pairing and the inequalities $v_\ell^* \ge u^+ \ge u_k^+$ on Ω and $(v_\ell)^{\mathrm{int}}_{\partial\Omega} \ge u^{\mathrm{int}}_{\partial\Omega} \ge (u_k)^{\mathrm{int}}_{\partial\Omega}$ on $\partial\Omega$, we moreover get

$$\begin{aligned}
&\liminf_{k\to\infty} [\![\sigma_k, \mathrm{D}v_\ell^+]\!]^*_{u_{0;k}}(\overline{\Omega}) \\
&\quad= \liminf_{k\to\infty}\bigg[\int_\Omega (v_\ell-u_{0;k})^*\,\mathrm{d}(-\operatorname{div}\sigma_k) + \int_{\partial\Omega}\big[(v_\ell-u_{0;k})^{\mathrm{int}}_{\partial\Omega}\big]_+\,\mathrm{d}(-\operatorname{div}\sigma_k) + \int_\Omega \sigma_k\cdot\mathrm{D}u_{0;k}\,\mathrm{d}x\bigg] \\
&\quad\ge \liminf_{k\to\infty}\bigg[\int_\Omega (u_k^+-u_{0;k}^*)\,\mathrm{d}(-\operatorname{div}\sigma_k) + \int_{\partial\Omega}\big[(u_k-u_{0;k})^{\mathrm{int}}_{\partial\Omega}\big]_+\,\mathrm{d}(-\operatorname{div}\sigma_k) + \int_\Omega \sigma_k\cdot\mathrm{D}u_{0;k}\,\mathrm{d}x\bigg] \\
&\quad= \liminf_{k\to\infty} [\![\sigma_k, \mathrm{D}u_k^+]\!]^*_{u_{0;k}}(\overline{\Omega})\,.
\end{aligned}$$

Next we record that, since $\mathbb{1}_\Omega(u_k-u_{0;k})$ converges to $\mathbb{1}_\Omega(u-u_0)$ in $\mathrm{L}^1(\mathbb{R}^n)$ and the estimate of Lemma 2.3 is at hand, also $\mathrm{D}\big[\mathbb{1}_\Omega(u_k-u_{0;k})\big]$ weak$*$ converges to $\mathrm{D}\big[\mathbb{1}_\Omega(u-u_0)\big]$ in the space of finite $\mathbb{R}^n$-valued Borel measures on $\mathbb{R}^n$. Involving [1, Theorem 3.84] we thus find that also $\mathrm{D}_{u_{0;k}}u_k =$

$\mathrm{D}\big[\mathbb{1}_\Omega(u_k-u_{0;k})\big]+\mathrm{D}u_{0;k}\mathcal{L}^n$ weak$*$ converges to $\mathrm{D}_{u_0}u=\mathrm{D}\big[\mathbb{1}_\Omega(u-u_0)\big]+\mathrm{D}u_0\mathcal{L}^n$ in the same space. Relying on the coupling $[\![\sigma_k,\mathrm{D}u_k^+]\!]_{u_{0;k}}^*=|\mathrm{D}_{u_{0;k}}u_k|$ and on a lower semicontinuity property of the total variation, we hence deduce

$$\liminf_{k\to\infty}[\![\sigma_k,\mathrm{D}u_k^+]\!]_{u_{0;k}}^*\big(\overline{\Omega}\big)=\liminf_{k\to\infty}|\mathrm{D}_{u_{0;k}}u_k|\big(\overline{\Omega}\big)\ge|\mathrm{D}_{u_0}u|\big(\overline{\Omega}\big)\,.$$

Collecting the preceding estimates, we arrive at the inequality $[\![\sigma,\mathrm{D}u^+]\!]_{u_0}^*\big(\overline{\Omega}\big)\ge|\mathrm{D}_{u_0}u|\big(\overline{\Omega}\big)$. In view of (4.11), this gives in fact equality $[\![\sigma,\mathrm{D}u^+]\!]_{u_0}^*=|\mathrm{D}_{u_0}u|$ on $\overline{\Omega}$, and thus u is weakly super-1-harmonic on $\overline{\Omega}$ with respect to u_0. $\square$

5 Weak supersolutions to the minimal surface equation

In this section we turn to the minimal surface equation

$$\operatorname{div}\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\equiv 0\qquad\text{on }\Omega\,,\tag{5.1}$$

and we discuss the adaption of our ideas to this case. In order to mimic the approach of the previous section, in the following we aim at introducing a dual quantity σ which takes over the role of $\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}$. To this end, we rely on the inequality

$$z^*\cdot z\le\sqrt{1+|z|^2}-\sqrt{1-|z^*|^2}\qquad\text{for all }z,z^*\in\mathbb{R}^n\text{ with }|z^*|\le 1\,,\tag{5.2}$$

which becomes an equality if and only if z^* equals $\frac{z}{\sqrt{1+|z|^2}}$. Here, (5.2) and the characterization of its cases of equality can either be checked by an elementary computation or can be obtained by specializing the general inequality $z^*\cdot z\le f(z)+f^*(z)$ with the convex conjugate f^* of a real-valued function f on $\mathbb{R}^n$. In any case, for $u\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ and $\sigma\in\mathcal{S}^\infty(\Omega,\mathbb{R}^n)$, one infers

$$\sigma=\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\iff\sigma\cdot\mathrm{D}u=\sqrt{1+|\mathrm{D}u|^2}-\sqrt{1-|\sigma|^2}\,.\tag{5.3}$$

With the help of the pairing $[\![\sigma,\mathrm{D}u^+]\!]$, the relation on the right-hand side of (5.3) can be formulated for functions $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$, and then one may introduce BV supersolutions of (5.1) as follows.

Definition 5.1 (supersolution to the minimal surface equation)**.** *We call $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ a weak supersolution to the minimal surface equation on Ω if there exists some $\sigma\in\mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with* $\operatorname{div}\sigma\le 0$ *in $\mathscr{D}'(\Omega)$ and*

$$[\![\sigma,\mathrm{D}u^+]\!]=|(\mathcal{L}^n,\mathrm{D}u)|-\sqrt{1-|\sigma|^2}\mathcal{L}^n\qquad\text{on }\Omega\tag{5.4}$$

(*where $|(\mathcal{L}^n,\mathrm{D}u)|$ is understood as in Section* 2.5).

As in Definition 4.6 we can also employ the modified pairing $[\![\sigma,\mathrm{D}u^+]\!]_{u_0}^*$ from (4.9) to define BV supersolutions to the Dirichlet problem for the minimal surface equation.

Definition 5.2 (supersolution to the minimal surface equation with respect to a boundary datum)**.** *For bounded Ω with* (2.2) *and $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, we say that $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a weak supersolution to the minimal surface equation on $\overline{\Omega}$ with respect to u_0 if there exists some $\sigma \in \mathcal{S}^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$ such that we have*

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0} = |(\mathcal{L}^n, \mathrm{D}_{u_0}u)| - \sqrt{1-|\sigma|^2}\mathcal{L}^n \qquad \text{on } \overline{\Omega}\,. \tag{5.5}$$

It follows from Lemma 3.3 and the equivalence (5.3) that the requirements of Definitions 5.1 and 5.2 are consistently formulated in the sense that they are certainly satisfied for $\mathrm{W}^{1,1}$ supersolutions u, that is for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in (u_0 + \mathrm{W}^{1,1}_0(\Omega)) \cap \mathrm{L}^\infty(\Omega)$, respectively, with

$$\operatorname{div} \frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}} \le 0 \qquad \text{in } \mathscr{D}'(\Omega)\,.$$

However, the minimal surface case crucially differs from the 1-Laplace case insofar as $\sigma \in \mathcal{S}^\infty(\Omega, \mathbb{R}^n)$ is uniquely determined by u (and even by $\mathrm{D}^{\mathrm{a}}u$) via (5.4). Indeed, relying on Lemma 3.6 and the Lebesgue decomposition $|(\mathcal{L}^n, \mathrm{D}u)| = \sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2}\mathcal{L}^n + |\mathrm{D}^{\mathrm{s}}u|$, we can split (5.4) into the two relations

$$\sigma \cdot \mathrm{D}^{\mathrm{a}}u = \sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2} - \sqrt{1-|\sigma|^2} \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,,$$
$$[\![\sigma, \mathrm{D}u^+]\!]^{\mathrm{s}} = |\mathrm{D}^{\mathrm{s}}u| \qquad \text{as measures on } \Omega\,,$$

and then the first relation uniquely determines

$$\sigma = \frac{\mathrm{D}^{\mathrm{a}}u}{\sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2}} \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,.$$

In view of this observation, an analog of Theorem 4.5 in the minimal surface case can now be established by much simpler means.

Proposition 5.3 (simultaneous supersolutions and subsolutions are solutions)**.** *If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ is such that both u and $-u$ are weak supersolutions to the minimal surface equation on Ω, then u is a* BV *solution to the minimal surface equation on Ω (i.e. $\sigma := \mathrm{D}^{\mathrm{a}}u/\sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2}$ satisfies $[\![\sigma, \mathrm{D}u]\!]^{\mathrm{s}} = |\mathrm{D}^{\mathrm{s}}u|$ on Ω and $\operatorname{div}\sigma \equiv 0$ in $\mathscr{D}'(\Omega)$).*

Proof. The supersolution property of u means that $\sigma = \mathrm{D}^{\mathrm{a}}u/\sqrt{1+|\mathrm{D}^{\mathrm{a}}u|^2}$ satisfies $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$ and $[\![\sigma, \mathrm{D}u^+]\!]^{\mathrm{s}} = |\mathrm{D}^{\mathrm{s}}u|$ on Ω. The supersolution property of $-u$ means that the same σ also satisfies $\operatorname{div}\sigma \ge 0$ in $\mathscr{D}'(\Omega)$ and $[\![\sigma, \mathrm{D}u^-]\!]^{\mathrm{s}} = |\mathrm{D}^{\mathrm{s}}u|$ on Ω. Thus, we have $\operatorname{div}\sigma \equiv 0$ in $\mathscr{D}'(\Omega)$, and the pairings $[\![\sigma, \mathrm{D}u^\pm]\!]$ reduce to the classical Anzellotti pairing $[\![\sigma, \mathrm{D}u]\!]$ in (1.4), so that we arrive at the claim. □

Moreover, we have compactness results which resemble Theorems 4.2 and 4.7. We only state the global version which contains the local one as a special case.

Theorem 5.4 (convergence from below preserves the supersolution property)**.** *Suppose that Ω is bounded with* (2.2)*, and consider weak supersolutions $u_k \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ to the minimal surface equation on $\overline{\Omega}$ with respect to boundary data $u_{0;k} \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. If $u_{0;k}$ converges strongly in $\mathrm{W}^{1,1}(\Omega)$ and weakly$*$ in $\mathrm{L}^\infty(\Omega)$ to some u_0, and if u_k weak$*$ converges to a limit u in $\mathrm{BV}(\Omega)$ and $\mathrm{L}^\infty(\Omega)$ such that $u_k \le u$ holds on Ω for all $k \in \mathbb{N}$, then u is a weak supersolution to the minimal surface equation on $\overline{\Omega}$ with respect to u_0.*

In order to prove Theorem 5.4 we first record that (5.2) leads to estimates on the level of the pairing which (partially) substitute Proposition 3.5 in the present situation.

Lemma 5.5. *Fix $\sigma \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$.*

- **(local estimate)** *For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)\cap \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ we have*
$$|[\![\sigma,\mathrm{D}u^+]\!]| \le |(\mathcal{L}^n,\mathrm{D}u)| - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n \qquad \text{on } \Omega\,.$$
- **(global estimate)** *If Ω is bounded with (2.2), then for $u_0 \in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and $u \in \mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ we have*
$$\big|[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}\big| \le |(\mathcal{L}^n,\mathrm{D}_{u_0}u)| - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n \qquad \text{on } \overline{\Omega}\,.$$

Proof. We employ in turn Lebesgue decomposition, Lemma 3.6, Proposition 3.5, and (5.2). Arguing in this way, we get the following (in)equalities of measures on Ω:
$$\begin{aligned}
|[\![\sigma,\mathrm{D}u^+]\!]| &= |[\![\sigma,\mathrm{D}u^+]\!]^{\mathrm{s}}| + |[\![\sigma,\mathrm{D}u^+]\!]^{\mathrm{a}}|\\
&= |[\![\sigma,\mathrm{D}u^+]\!]|^{\mathrm{s}} + (\sigma\cdot\mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n\\
&\le |\mathrm{D}u|^{\mathrm{s}} + \sqrt{1{+}|\mathrm{D}^{\mathrm{a}}u|^2}\mathcal{L}^n - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n\\
&= |(\mathcal{L}^n,\mathrm{D}u)^{\mathrm{s}}| + |(\mathcal{L}^n,\mathrm{D}u)^{\mathrm{a}}| - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n\\
&= |(\mathcal{L}^n,\mathrm{D}u)| - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n\,.
\end{aligned}$$
Thus, we have established the local estimate. The global estimate then follows by taking into account $[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0} = [\![\sigma,\mathrm{D}u^+]\!]$ on Ω and (4.11) on $\partial\Omega$. ☐

Proof of Theorem 5.4. By Definition 5.2 there exist $\sigma_k \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma_k \le 0$ in $\mathscr{D}'(\Omega)$ such that the coupling condition (5.5) holds with $(\sigma_k,u_k,u_{0;k})$ in place of (σ,u,u_0). Possibly passing to a subsequence, we assume that σ_k weak$*$ converges in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ to some $\sigma \in \mathcal{S}^\infty(\Omega,\mathbb{R}^n)$ with $\operatorname{div}\sigma \le 0$ in $\mathscr{D}'(\Omega)$. Arguing exactly as in the proof of Theorem 4.7 and then using the modified coupling condition, we end up with
$$[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}(\overline{\Omega}) \ge \liminf_{k\to\infty}[\![\sigma_k,\mathrm{D}u_k^+]\!]^*_{u_{0;k}}(\overline{\Omega}) = \liminf_{k\to\infty}\left[|(\mathcal{L}^n,\mathrm{D}_{u_{0;k}}u_k)|(\overline{\Omega}) - \int_\Omega\sqrt{1{-}|\sigma_k|^2}\,\mathrm{d}x\right].$$
Since $\mathrm{D}_{u_{0;k}}u_k$ weak$*$ converges to $\mathrm{D}_{u_0}u$ for the reasons also detailed in the proof of Theorem 4.7, by lower semicontinuity of the total variation, we get
$$\liminf_{k\to\infty}|(\mathcal{L}^n,\mathrm{D}_{u_{0;k}}u_k)|(\overline{\Omega}) \ge |(\mathcal{L}^n,\mathrm{D}_{u_0}u)|(\overline{\Omega})\,.$$
By weak$*$ lower semicontinuity of the convex functional $\tau \mapsto -\int_\Omega\sqrt{1{-}|\tau|^2}\,\mathrm{d}x$ (see [8, Theorem 3.20, Remark 3.25(ii)]), we moreover have
$$\limsup_{k\to\infty}\int_\Omega\sqrt{1{-}|\sigma_k|^2}\,\mathrm{d}x \le \int_\Omega\sqrt{1{-}|\sigma|^2}\,\mathrm{d}x\,.$$
Thus, all in all we end up with
$$[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}(\overline{\Omega}) \ge \Big[|(\mathcal{L}^n,\mathrm{D}_{u_0}u)| - \sqrt{1{-}|\sigma|^2}\mathcal{L}^n\Big](\overline{\Omega})\,.$$
This suffices to guarantee that equality occurs in the global estimate of Lemma 5.5 and that $[\![\sigma,\mathrm{D}u^+]\!]^*_{u_0}$ is non-negative. Hence the coupling condition (5.5) holds, and u is a weak supersolution to the minimal surface equation on $\overline{\Omega}$ with respect to u_0. ☐

6 Brief remarks on more general variational equations

Last but not least we point out that our approach can be adapted in order to define BV supersolutions of more general equations. Specifically, we can deal with Euler equations of functionals

$$F[u] := \int_\Omega f(\,\cdot\,, \mathrm{D}u)\,\mathrm{d}x\,,$$

where the Borel integrand $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is convex in the second variable and satisfies the growth condition

$$|f(x,z)| \le \Psi(x) + L|z| \qquad \text{for } (x,z) \in \Omega \times \mathbb{R}^n$$

with an L^1 function Ψ on Ω and a constant $L \in [0,\infty)$.

In this situation, if Ω is bounded with (2.2), it makes sense to call $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ a weak supersolution to the Euler equation of F with respect to a Dirichlet datum $u_0 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ if there exists some $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \le 0$ in $\mathscr{D}'(\Omega)$ which satisfies the coupling condition

$$[\![\sigma, \mathrm{D}u^+]\!]^*_{u_0} = f(\,\cdot\,, \mathrm{D}_{u_0}u) + f^*(\,\cdot\,, \sigma)\mathcal{L}^n \qquad \text{on } \overline{\Omega}\,. \tag{6.1}$$

Here the first term on the right-hand side is understood as a convex functional of a measure on $\overline{\Omega}$, that is $f(\,\cdot\,, \mathrm{D}_{u_0}u) := f(\,\cdot\,, \mathrm{D}^{\mathrm{a}}u)\mathcal{L}^n + f^\infty\big(\,\cdot\,, \frac{\mathrm{d}(\mathrm{D}_{u_0}u)^{\mathrm{s}}}{\mathrm{d}|\mathrm{D}_{u_0}u|^{\mathrm{s}}}\big)|\mathrm{D}_{u_0}u|^{\mathrm{s}}$ with the recession function[3] f^∞ of f. Moreover, the convex conjugate $f^*\colon \Omega \times \mathbb{R}^n \to (-\infty,\infty]$ of f is given by $f^*(x,z^*) := \sup_{z\in\mathbb{R}^n} \big[z^*\cdot z - f(x,z)\big]$. If f is C^1 in the second variable, a pointwise connection between f, f^*, $\mathrm{D}_z f$ (see [4, formula (3.1)]) implies that the vector field σ in (6.1) is fully determined by u (and f) and in fact is given by $\sigma = \mathrm{D}_z f(\,\cdot\,, \mathrm{D}^{\mathrm{a}}u)$ on Ω. Moreover, still under the differentiability assumption on f, it follows via Lemma 3.3 that the above definition is satisfied for every $u \in (u_0 + \mathrm{W}^{1,1}_0(\Omega)) \cap \mathrm{L}^\infty(\Omega)$ with $\operatorname{div}\big[\mathrm{D}_z f(\,\cdot\,, \mathrm{D}u)\big] \le 0$ in $\mathscr{D}'(\Omega)$. Thus, our definition is consistent with the standard notion of $\mathrm{W}^{1,1}$ supersolutions. We find it worth remarking, however, that — thanks to convex duality — our notion is still meaningful if f is not C^1 in the second variable and the usual formulation of the Euler equation cannot be written down even for smooth solutions u.

Finally, we suppose that f is convex in z, lower semicontinuous in (x,z) and admits, in addition to the above growth condition, a lower bound

$$f(x,z) \ge -\ell(z) \qquad \text{for } (x,z) \in \Omega \times \mathbb{R}^n$$

with a linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$. Then the Reshetnyak semicontinuity theorem applies to the quantity $f(\,\cdot\,, \mathrm{D}_{u_0}u)$ (compare [4, Appendix B]), and most of our results extend to supersolutions defined via (6.1), with analogous proofs. Specifically, we believe that this is true for Theorems 4.2, 4.7, 5.4, and when f is C^1 in z also for Proposition 5.3. Analoga to Theorem 4.5 for non-differentiable integrands f may require assumptions on the set $\Omega \cap \{\mathrm{D}^{\mathrm{a}}u \in \mathcal{N}_f\}$, with the non-differentiability set $\mathcal{N}_f$ of f. Anyway, this last issue is likely to be more delicate and will not be discussed here.

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