ON A CONSTRAINED SHAPE OPTIMIZATION PROBLEM

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Abstract. We study a classical scalar shape optimization problem adding the constraint that the design region is a manifold in $\mathbb{R}^n$. We prove an existence result for weak solutions using a connection with the Monge's optimal transport problem. The results applies to constrained and obstacle problems.
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1. INTRODUCTION

The goal of this paper is two-fold. The first aim is to prove an existence theorem for a classical shape optimization problem in a constrained or anisotropic version. Indeed we will assume that the design region is an imbedded Riemannian manifold. The second aim of the paper is to exploit the method introduced in [3], [4], [5], to prove existence results for this kind of problems. This method do not require any linear structure on the underlying design region while more classical methods do.

Let us describe the problem: Assume that we are given an heat source f and a certain amount m of a conductor C (conductor means positive conductivity coefficient). Which is the best way to distribute C in a certain assigned region Ω? (Of course one need to explain what "best" means). Taking as a model for the distribution of C a nonnegative bounded function θ in Ω such that $\int_\Omega \theta(x)dx = m$ (see [1]) the energy associated to a smooth distribution of temperature u is given by:

$$E(\theta, u) = \frac{1}{2}\int_\Omega |Du(x)|^2\theta(x)dx - \int_\Omega f(x)u(x)dx. \tag{1.1}$$

For an assigned distribution of material $\theta(x)$ the equilibrium temperature u_θ (not necessarily smooth) realize:

$$\overline{E}(\theta, u_\theta) = \inf_u E(\theta, u) := \mathfrak{C}(\theta), \tag{1.2}$$

where $\overline{E}$ is the relaxed energy of E which will be introduced in the next section. A distribution of material θ_1 is better then θ if

$$\mathfrak{C}(\theta_1) > \mathfrak{C}(\theta). \tag{1.3}$$

So we look for

$$\max_\theta \mathfrak{C}(\theta). \tag{1.4}$$

Using a function θ as a model for the distribution of C do not allow concentrate distributions, then we will use a positive measure μ such that $\int d\mu = m$ instead. Moreover using measures we can expect the problem to have a solution which usually does not happens in the class of "distributed mass". To deal with such kind of distributions of mass we need to adapt the instruments for Calculus of Variations with respect to a measure developed in [4], [5]. The use of the Riemannian setting permits to deal with non-convex open subsets of $\mathbb{R}^N$ (and then with obstacles) and with anisotropic situations The plan of the paper is the following: in section 2 we will fix the notations and summarize some background notions (i.e. A duality theorem, Sobolev spaces with respect to a measure, Monge-Kantorovich problem). In section 3 we will prove the equivalence of (2.3) with two problem related to the Monge-Kantorovich mass transportation problem (see [7], [10], [3]), and we will exploit this equivalence to obtain an existence result for (2.3) and an explicit formula for some solution of it. It is in section 3 that the difference of geometry between the manifold and $\mathbb{R}^n$ will show up in a multiplicity of solutions with very different structures. Finally in section 4 we will give two explicit and simple examples of application, we will make some remarks and we will address to an open problem.

2. NOTATION AND PRELIMINARIES

Through all the paper M will be a connected Riemannian manifold without boundary and not necessarily compact. The Riemannian metric on the tangent space at x will be denoted by $g_{ij}(x)$. By TM and T^*M we will denote the tangent and the cotangent bundles of M, $g^*_{ij}(x)$ will denote the dual metric of $g_{ij}(x)$. $D_M u$ denote the tangential gradient of u while $\mathrm{div}_M u$ will be the tangential divergence. We will often refer to M as a metric space, in this case it is intended that d is the geodesic distance on M. $\mathcal{M}(X,Y)$ will denote the space of Y-valued Radon measure on X, $Y=\mathbb{R}$ will be omitted. With these notations the energy associated to a smooth distribution of temperature u will be then:

$$E(\mu,u)=\frac{1}{2}\int_M |Du|^2_{g_{ij}}d\mu - < f,u > . \tag{2.1}$$

Then $\mathfrak{C}(\mu)$ is given by:

$$\mathfrak{C}(\mu)=\inf_u E(\mu,u) \tag{2.2}$$

and we look for:

$$\max_{|\mu|=m}\mathfrak{C}(\mu). \tag{2.3}$$

2.1. **Calculus of variations with respect to a measure and relaxed energy.**

In order to find a suitable representation of the relaxed energy of (2.1) we introduce some notion from [4]. Let $\mu\in\mathcal{M}(M)$ we consider the space

$$X_\mu := \{\varphi\in L^1_\mu(M,TM)|\ -div_M(\varphi\mu)\in\mathcal{M}(M)\}. \tag{2.4}$$

In some sense X_μ is the space of tangent field to μ. Then we define the tangent space to μ for μ-a.e. $x\in M$ as:

$$T_\mu(x) := \mu - ess\bigcup\{\varphi(x)|\ \varphi\in X_\mu\}. \tag{2.5}$$

The μ essential union is defined as a μ-measurable closed multifunction such that:

- $\varphi\in X_\mu \Rightarrow \varphi(x)\in T_\mu(x)$ for μ-a.a. $x\in M$.
- Between all the multifunction with the previous property the μ essential union is minimal with respect to the inclusion μ-a.e..

Property and applications of this definition of tangent space to a measure have been explored in various paper, among them, we address to [4], [5], [3], [8]. Once we have a notion of tangent space to μ it is natural to define the notion of μ-tangential gradient of a function u as:

$$D_\mu u(x)=P_\mu(x,D_M u(x)) \quad \mu-a.e.$$

where we denoted by $P_\mu(x,\cdot)$ the orthogonal projection on $T_\mu(x)$. It is possible to show that the operator D_μ is closable in L^p_μ and this take to a suitable definition of Sobolev space with respect to μ (see [4]):

Definition 2.1. *The Sobolev space $W^{1,p}_\mu(M)$ is the completion of $\mathcal{D}(M)$ with respect to the norm:*

$$\|u\|_{1,p} := \|u\|_{L^p_\mu} + \|D_\mu u\|_{L^p_\mu}.$$

Using this notions one obtain that if $f \in (W^{1,p}_\mu(M))'$ the relaxed energy of (2.1) is given (see [4]) by:

$$\frac{1}{2}\int_M |D_\mu u|^2_{g_{ij}} d\mu - < f, u > \qquad u \in W^{1,p}_\mu(M). \tag{2.6}$$

2.2. **The Monge mass transfer problem.** Given two measures μ_1, μ_2 on M of equal total mass, the transport problem consist in finding in the set of measurable maps $\varphi : M \to M$ such that $\varphi_\sharp \mu_1 = \mu_2$ (where $\varphi_\sharp$ is the push-forward of any measurable map $\varphi : M \to M$) the minimum of the "work" functional

$$\int_M d(x, \varphi(x)) d\mu_1. \tag{2.7}$$

where d is the distance on M induced by the metric. A weak formulation of this problem is the following:

$$\min\{\int_{M\times M} d(x,y) d\gamma(x,y) \;|\gamma \in \mathcal{M}^+(M\times M),\ \pi^1_\sharp\gamma = \mu_1,\ \pi^2_\sharp\gamma = \mu_2\}. \tag{2.8}$$

Actually, the minimum in (2.8) could be lower then the inf of (2.7), however in many important case this do not happens (see for example [9] or [2]). An optimal map φ for problem (2.7) will be called an optimal transport while an optimal measure for problem (2.8) will be called an optimal plan of transport. The relationship between the two problem is given by the fact that if φ is an optimal transport, then the measure associated to the graphs of φ is an optimal plan of transport. The name "optimal plan of transport" is due to the fact that if $(x,y) \in supp\gamma$ then part of the mass located in x should be moved to y in order to minimize the work.

2.3. **The Monge-Kantorovich-Wasserstein distance.** It is simple to verify that $\mathfrak{D}(\mu_1,\mu_2)$:=(2.8) define a distance on the set of probability measure on M. Analogously, rescaling in (2.8) one could obtain a distance on the set of Radon measures of total mass m. This distance is called Monge-Kantorovich-Wasserstein distance or Wasserstein[1] distance it induce a topology equivalent to the weak* topology on the set where is defined, and is very useful in various contests. In section 3 we will show an amazing relationship with the shape optimization problem we are considering.

2.4. **Monge-Kantorovich equation.** The name Monge-Kantorovich equation, is usually referred to the system:

$$(MK) = \begin{cases} -div_M a(x) Du(x) = f & on\ M \\ |Du|_{g_{ij}} = 1 & a(x) dg_{ij} - a.e.. \end{cases} \tag{2.9}$$

[1]In many contests the name Wasserstein distance is referred to the case in which the integrand in (2.8) is the distance to the power 2 and the name Monge's distance to the case of distance to the power 1

The integration by parts formula

$$\int_{\Omega} div_M \phi d\sqrt{g_{ij}} = \int_{\partial\Omega} \phi \nu_{\partial\Omega} d\mathcal{H}^{n-1},$$

which holds for each regular open subset Ω of M and each vector field $\phi \in C^1(M, TM)$ permits to give a weak formulation of this equation:

$$(WMK) = \begin{cases} -div_M \mu D_\mu u(x) = f & on\ M \\ |D_\mu u| = 1 & \mu - a.e.. \end{cases} \tag{2.10}$$

2.5. **A duality theorem.** We report here a reformulation useful in our settings of a classical duality theorem (see [6] proposition 2.3).

Theorem 2.2. *Let X, Y be two Banach Spaces and let X', Y' be respectively their dual spaces. Let $f : X \to]-\infty, +\infty]$ and $g : Y \to]-\infty, +\infty]$ be respectively a linear functional and a convex and l.s.c. functions. Let $A : X \to Y$ be a linear operator such that $D(A)$ is dense in X. Assume that there exists $x_0 \in D(A)$ such that*

$$y \to g(Ax_0 + y) + f(x_0)$$

is continuous at $0 \in Y$ and that the infimum in (2.11) is finite. Then

$$\inf_{D(A)} \{g(Ax) + f(x)\} = \sup_{Y'} \{-f^*(-A^*y^*) - g^*(y^*)\}. \tag{2.11}$$

Proof. Let $h(y) = \inf_{D(A)}\{g(Ax + y) + f(x)\}$, then $h(0)$ is the quantity on the left side of equation (2.11). Let us calculate $h^*(y^*)$

$$h^*(y^*) = \sup_Y \{< y, y^* > -h(y)\} =$$

$$= \sup_Y \{< y, y^* > - \inf_{D(A)} \{g(Ax + y) + f(x)\}\} =$$

$$= \sup_Y \sup_{D(A)} \{< y, y^* > -g(Ax + y) - f(x)\} =$$

$$= \sup_{D(A)} \sup_Y \{< y, y^* > -g(Ax + y) - f(x)\} =$$

$$= \sup_{D(A)} \sup_Y \{< y + Ax, y^* > -g(Ax + y) - < Ax, y^* > -f(x)\} =$$

$$= \sup_{D(A)} \{g^*(y^*) - < x, A^*y^* > -f(x)\} =$$

$$= g^*(y^*) + f^*(-A^*y^*). \tag{2.12}$$

Then $h^{**}(0) = \sup_{Y'}\{-g^*(y^*) - f^*(-A^*y^*)\}$ which is the quantity on the right side of equation (2.11). The Fenchel inequality imply $h^{**}(0) \leq h(0)$ to prove the reverse inequality it is enough to prove that h is bounded in a neighborhood of 0 (in fact h is convex). As the function

$$y \to g(Ax_0 + y) + f(x_0)$$

is continuous at $0 \in Y$ and $h(0)$ is finite, there exists a neighborhood U on $0 \in Y$ and a constant M such that

$$g(Ax_0 + y) + f(x_0) \leq M \quad \forall y \in U.$$

Then for all $y \in U$

$$h(y) \leq g(Ax_0 + y) + f(x_0) \leq M.$$

$\square$

3. MAIN RESULTS

Proposition 3.1. *Let M a connected Riemannian manifold, then the following duality holds:*

$$\inf\{\frac{1}{2}\int_M |D_M u|^2_{g_{ij}} d\mu - < f, u >: \qquad u \in \mathcal{D}(M)\} \tag{3.1}$$

is equal to

$$\sup\{-\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu : \sigma \in L^2_\mu(M, T^*M), \ -div_M \sigma\mu = f\}. \tag{3.2}$$

Moreover whenever (3.1) is finite (3.2) is a max.

Proof. Apply theorem 2.2 with $X = L^2_\mu(M)$, $A = D_M$, $Y = L^2_\mu(M, TM)$. □

Proposition 3.2. *Let M a connected Riemannian manifold, then the following duality holds:*

$$I(f, M) =: \min\{< -f, \phi >: \quad \phi \in Lip(M), \quad |D_M \phi|_{g_{ij}} \leq 1 \ a.e.\} \tag{3.3}$$

is equal to

$$\sup\{-\int_M d|\nu|_{g^*_{ij}} : \nu \in \mathcal{M}(M, T^*M), \ -div_M \nu = f\}. \tag{3.4}$$

Proof. Apply theorem 2.2 with $X = C(M)$, $A = D_M$, $Y = C(M, TM)$. □

Remark Notice that the total variation of ν depends on the Riemannian metric. In local coordinate (3.2) would involve a convex functional of ν in the sense of [11]. We do not need to work in local coordinate because our approach is global.

Theorem 3.3.

$$\mathfrak{C}(\mu) \leq \frac{1}{2m}(I(f, M))^2$$

Proof. Let $\phi \in Lip_1(M)$

$$\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu \geq \frac{1}{2m}(\int_M < \sigma, D_M \phi >_{T^*, T_*} d\mu)^2 = \frac{1}{2m}(< f, \varphi >)^2. \tag{3.5}$$

Then

$$\frac{1}{2}\int_M |\sigma|^2_{g^*_{ij}} d\mu \geq \frac{1}{2m}(I(f, M))^2. \tag{3.6}$$

Then using the definition of $\mathfrak{C}(\mu)$ and proposition (3.1) we obtain the following upper estimate

$$\mathfrak{C}(\mu) \leq \frac{1}{2m}(I(f, M))^2. \tag{3.7}$$

□

We will now prove that such estimate is optimal and establish an existence results for problem (2.3) and a relation between the Wasserstein distance of f^+ and f^- and the value of $\mathfrak{C}(\mu)$, which extend to our setting the results of [3].

Theorem 3.4. *There exists a couple (u, μ), $u \in Lip(M)$ and $\mu \in \mathcal{M}(M)$ solution of the problem:*

$$\begin{cases} -div_M \mu D_\mu u(x) = f & on\ M \\ |D_\mu u| = 1 & \mu - a.e.. \end{cases} \tag{3.8}$$

Proof. Let u and ν be respectively solution for problems (3.1) and (3.2). Applying proposition (3.2) we get:

$$-\int_M d|\nu|_{g^*_{ij}} =< -f, u > . \tag{3.9}$$

By Riesz representation theorem we may write $\nu = \theta\mu$ where $\mu = |\nu|$ and θ is an unitary cotangent vector defined μ-a.e. We get :

$$\int_M < \theta, D_\mu u >_{T^*,T_*} d\mu =< f, u >= \int_M d\mu, \tag{3.10}$$

which imply $\theta = D_\mu u$ μ-a.e. and $|D_\mu u(x)| = 1$ μ-a.e. . □

We now connect the previous theorem with the shape optimization problem.

Theorem 3.5. *Let (u, μ) a couple of solutions of (WMK) as obtained in theorem 3.4, define $\mu_0 = \frac{m}{I(f,M)}\mu$ then,*

$$C(\mu_0) = \frac{1}{2m}(I(f, M))^2.$$

Proof. Let us choose $u_0 = \frac{I(f,M)}{m}u$ and $\sigma_0 = du_0$ then:

$$-\frac{1}{2}\int_M |du_0|^2_{g^*_{ij}} d\mu_0 = -\frac{1}{2}\int_M |D_M u_0|^2_{g_{ij}} d\mu_0 = -\frac{1}{2} < u_0, f >= -\frac{1}{2m}I(f)^2.$$

□

The following theorem establish a relationship between the optimal value of $\mathfrak{C}(\mu)$ for assigned heat source f and the Monge-Wasserstein distance of f^+ and f^-, $\mathfrak{D}(f^+, f^-)$.

Theorem 3.6. $I(f, M) = \mathfrak{D}(f^+, f^-)$

Proof. We can write $I(f, M)$ in the following way:

$$I(f, M) = \inf\{< -f, u > | \quad u(x) - u(y) \le d(x, y)\}. \tag{3.11}$$

Let us consider $p \in C(M \times M)$ and

$$h(p) = \inf\{< -f, u > | \quad u(x) - u(y) + p(x, y) \le d(x, y)\}. \tag{3.12}$$

(In particular $I(f, M) = h(0)$.)

$$G(p, u) = \begin{cases} < -f, u > & if\ u(x) - u(y) + p(x, y) \le d(x, y) \\ +\infty & otherwise \end{cases} \tag{3.13}$$

G is convex, and $h(p) = \inf_u G(p, u)$ is convex too. Let us calculate $h^*(\gamma)$ for $\gamma \in \mathcal{M}(M \times M)$.

$$\begin{aligned} h^*(\gamma) &= \sup_p\{< \gamma, p > - \inf_u\{< -f, u > | \ u(x) - u(y) + p(x, y) \le d(x, y)\}\} = \\ &= \sup_u\{< -f, u > + \sup_p\{< \gamma, p > | \ u(x) - u(y) + p(x, y) \le d(x, y)\}\} = \\ &= \sup_u\{< -f, u > + \int d(x, y) d\gamma(x, y) + < \gamma, u(y) - u(x) >\} = \\ &= \begin{cases} +\infty & if\ f \ne \pi^1(\gamma) - \pi^2(\gamma), \\ \int\int d(x, y) d\gamma(x, y) & if\ f = \pi^1(\gamma) - \pi^2(\gamma). \end{cases} \end{aligned} \tag{3.14}$$

Then $h^{**}(0) = \mathfrak{D}(f^+, f^-)$. Then we just need to prove that h is lower semicontinuous at 0, it will follows that $h^{**}(0) = h(0)$ which will conclude the proof. Let $\{p_n\}$ be a sequence such that $p_n \to 0$ uniformly and

$$\varepsilon_n + h(p_n) \geq < -f, u_n > \tag{3.15}$$

for u_n such that $u_n(x) - u_n(y) + p_n(x,y) \leq d(x,y)$, applying the Ascoli-Arzelá's theorem $u_{n_k} \to u$ uniformly, then:

$$\liminf_{k\to\infty} h(p_{n_k}) = \liminf_{k\to\infty}\{h(p_{n_k}) + \varepsilon_k\} \geq \liminf_{k\to\infty} < -f, u_{n_k} > \geq h(0). \tag{3.16}$$

□

Remark 3.7. The Riemannian metric structure play a fundamental role in equation (3.11), it would be not possible to make such a reformulation with respect to a distance different from that induced by the metric g_{ij}.

Corollary 3.8. *The optimal value of $\mathfrak{C}(\mu_0)$ for assigned heat source f is :*

$$\mathfrak{C}(\mu_0) = \frac{1}{2m}(\mathfrak{D}(f^+, f^-))^2 \tag{3.17}$$

In ([3]) the authors introduced a method to produce explicit solutions of (3.4) and then of the problem of optimal mass distribution with or without Dirichlet conditions, this method can be extend to manifolds and thanks to the non uniqueness of minimal geodesic joining two points it produce a bigger set of solutions then in the Euclidean spaces.

Let γ be a minimizer for (2.8), fix $\epsilon > 0$ and for each $(x,y) \in supp\gamma$ select an almost-minimal curve $\Gamma_\epsilon(x,y)$ (almost minimal means $length(\gamma_\epsilon(x,y)) \leq d(x,y) + \epsilon$) joining x and y in such a way that the mapping $(x,y) \to \Gamma_\epsilon(x,y)$ is a measurable multivalued map. Let us define $\nu_\epsilon \in \mathcal{M}(M, T^*M)$ as follow

$$< \nu_\epsilon, \phi > := \int_{M\times M}\int_{\Gamma_\epsilon(x,y)} \phi \cdot \underline{t} d\mathcal{H}^1 d\gamma(x,y), \tag{3.18}$$

where $\underline{t}$ is the unitary tangent vector to Γ_ϵ.

Lemma 3.9. *Let ν_ϵ be defined as in (3.18), then*

$$-div_M \nu_\epsilon = f. \tag{3.19}$$

Proof. Let $\varphi \in \mathcal{D}(M)$, then

$$\begin{aligned} < \nu_\epsilon, D\varphi > &= \textstyle\int_{M\times M}\int_{\Gamma_\epsilon(x,y)} D_M\varphi \cdot \underline{t}\, d\mathcal{H}^1 d\gamma(x,y) = \\ &= \textstyle\int_{M\times M}\{\varphi(y) - \varphi(x)\} d\gamma(x,y) = \\ &= \textstyle\int_M \varphi df^+ - \int_M \varphi df^- = \int_M \varphi df. \end{aligned} \tag{3.20}$$

□

Moreover ν_ϵ has the following minimality property:

Theorem 3.10. *ν_ϵ is a minimizing sequence for the problem*

$$\inf\{\int_M d|\nu|_{g^*_{ij}} : \nu \in \mathcal{M}(M, T^*M),\ -div_M \nu = f\}. \tag{3.21}$$

Moreover it is weakly compact in $\mathcal{M}(M)$ and any cluster point is a solution of (3.21).*

Proof. Using theorem 3.6,

$$\sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f >= \int_{M\times M} d_M(y,x)d\gamma(y,x). \tag{3.22}$$

On the other hand, using equation (3.19) and the definition of total variation of ν_ϵ:

$$\begin{aligned} \sup_{|D\varphi|_{g_{ij}}\leq 1} < \nu_\epsilon, D\varphi > \leq \int d|\nu_\epsilon|_{g^*_{ij}} \leq \\ \leq \int_{M\times M} (d_M(y,x)+\epsilon)d\gamma(y,x) = \\ = \sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f > +\epsilon\gamma(M\times M). \end{aligned} \tag{3.23}$$

□

In general a cluster point ν of the set $\{\nu_\epsilon\}_\epsilon$ is not given by a formula like (3.18).

Remark 3.11. If M is geodesicaly complete, then choosing $\Gamma(x,y)$ to be a geodesic between (x,y) we obtain, by formula (3.18) an optimal measure for (3.4).

Then to each optimal plan of transport γ it is associated at least a solution of the shape optimization problem. We now show that using the same method one can construct a big family of solutions associated to each γ. Some remark about this family will be in order in the next section. Let γ be an optimal plan of transport. For each couple $(x,y)\in supp\gamma$ select a family of minimal geodesic $\mathfrak{B}(x,y)$ in such a way that the mapping $(x,y)\to\mathfrak{B}(x,y)$ is measurable. Consider a probability measure $\alpha_{(x,y)}$ on $\mathfrak{B}(x,y)$ (to be short we will write just α), and define $\nu\in\mathcal{M}(M,T^*M)$ as follows:

$$< \nu, \varphi >= \int_{M\times M}\int_{\mathfrak{B}}\int_{\Gamma(x,y)} \varphi\cdot \underline{t} d\mathcal{H}^1 \llcorner \Gamma(x,y) d\alpha(\Gamma)d\gamma, \tag{3.24}$$

with some minor variation on the previous proofs we obtain

Lemma 3.12. *Let ν be defined as in (3.24), then*

$$-div\nu = f. \tag{3.25}$$

Proof. Let $\varphi\in\mathcal{D}(M)$, then

$$\begin{aligned} < \nu, D\varphi > &= \textstyle\int_{M\times M}\int_{\mathfrak{B}}\int_{\Gamma(x,y)} D_M\varphi\cdot\underline{t} d\mathcal{H}^1 d\alpha d\gamma(x,y) = \\ &= \textstyle\int_{M\times M}\{\varphi(y)-\varphi(x)\}d\gamma(x,y) = \\ &= \textstyle\int_M \varphi df^+ - \int_M \varphi df^- = \int_M \varphi df. \end{aligned} \tag{3.26}$$

□

Theorem 3.13. *ν minimize the total variation in the class of measures satisfying equation (3.19) and it is optimal for (3.21).*

Proof. The proof is similar to that of theorem (3.10):

$$\sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f >= \int_{M\times M} d_M(y,x)d\gamma(y,x). \tag{3.27}$$

On the other hand, using equation (3.19) and the definition of total variation of ν:

$$
\begin{aligned}
\sup_{|D\varphi|_{g_{ij}}\leq 1} < \nu, D\varphi > &\leq \int d|\nu|_{g^*_{ij}} \leq \\
&\leq \int_{M\times M} d_M(y,x)d\gamma(y,x) = \\
&= \sup_{|D\varphi|_{g_{ij}}\leq 1} < \varphi, f > .
\end{aligned}
\tag{3.28}
$$

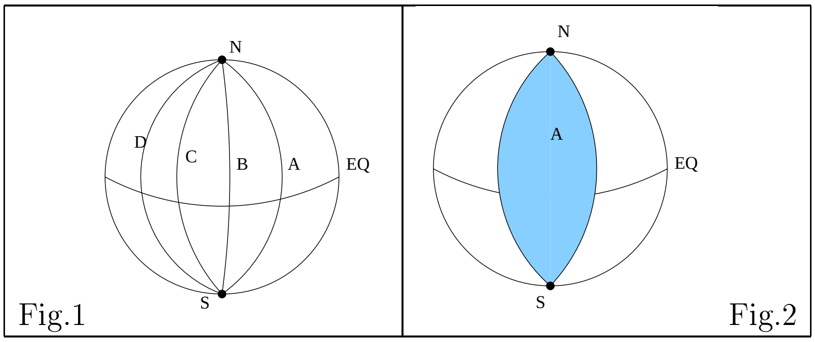
□

The solution of the shape optimization problem we generate by this last method are extremely various, for dimension and for qualitative behaviour, as we will show in the examples in the next section.

4. FURTHER REMARKS AND EXAMPLES

In this section we give two examples of solutions generated by formula (3.24). We compare the results with the convex Euclidean case and we address to an open problem. In the first example we show that formula (3.24) give rise to optimal shape of very different dimension. Notice that if a couple (u,μ) is solution of $-div\mu D_\mu u = f$ then the tangent space $T_\mu(x)$ has dimension at least one (indeed $D_\mu u(x) \in T_\mu(x)$ $\mu-$a.e.). The Hausdorff dimension of a measure is bigger then the dimension of the tangent space then μ has Hausdorff dimension at least one.

Example 4.1. Let $M = S^2$, $f^+ = \delta_N$ and $f^- = \delta_S$ where N and S are the northern and southern pole. In this case the unique (and the necessarily optimal) plan of transport is $\gamma = \delta_N \otimes \delta_S$, and there are infinitely many minimal geodesic curves connecting N and S. This set of geodesics can be parameterized by the intersections with the equatorial circle EQ, then a probability measure on this set is a probability measure on EQ. From figures (Fig.1) and (Fig.2) where we considered respectively a probability measure concentrated of four points or on an arc of EQ it is simple to see that formula (3.24) give rise to optimal shape of any dimension between 1 and 2.



Example 4.2. Let M be the complement of a ball in $\mathbb{R}^N$ and let $f^+ = \mathcal{H}^1 \llcorner A$ and $f^- = \mathcal{H}^1 \llcorner B$ where A and B are two segments of equal length as in figures (Fig.3) and (Fig.4). In this case any plan of transport γ is optimal, indeed, since for each

pair $(x, y) \in supp\gamma$ N is on a geodesic joining x and y, we have:

$$\begin{aligned}\int_{M\times M} d(x,y)d\gamma(x,y) = \int_{M\times M}\{d(x,N)+d(N,y)\}d\gamma(x,y) = \\ = \int_M d(x,N)df^+ + \int_M d(N,y)df^-.\end{aligned} \tag{4.1}$$

For each plan of transport γ and for each pair (x, y) in the support of γ there are two geodesics joining x and y and then the probability measures on such curves can be parameterized by the pairs (a, b) of numbers such that $0 \leq a, b \leq 1$ and $a + b = 1$. When inserted in formula (3.24) a denote the rate of mass in x transported to y via the northern geodesic while b is the remaining part which is transported along the southern one. In figures (Fig.3) and (Fig.4) we show two optimal shapes obtained from a plan of transport γ by two different (depending also on the point) choices of (a, b), in the first figure we chose $(a, b) \equiv (1, 0)$, in the second (a, b) is equal to $(1, 0)$ or $(\frac{1}{2}, \frac{1}{2})$ depending on the point. The level of gray indicate the mass density in a point.

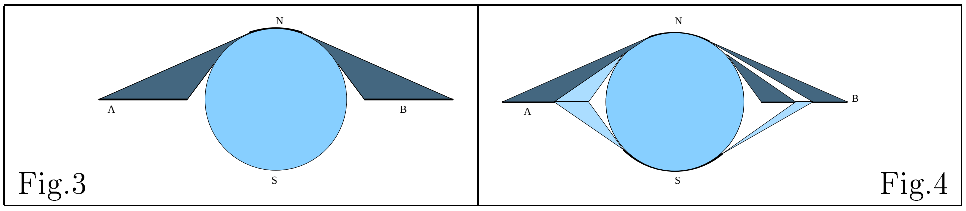


Fig.3 Fig.4

In [2] the author proved that in a convex open subset of $\mathbb{R}^N$ all the optimal shapes are obtained by formula (3.24) for a suitable optimal plan of transport γ (of course in that settings there is no need to introduce the probability measure α). It is an open problem to understand if this correspondence holds also on non convex settings.

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