

An Anzellotti type pairing for divergence-measure fields and a notion of weakly super-1-harmonic functions

Christoph Scheven* Thomas Schmidt^{†‡}

Abstract

We study generalized products of divergence-measure fields and gradient measures of BV functions. These products depend on the choice of a representative of the BV function, and here we single out a specific choice which is suitable in order to define and investigate a notion of weak supersolutions for the 1-Laplace operator.

1 Introduction

For a positive integer n and an open set Ω in \mathbb{R}^n , we consider the 1-Laplace equation

$$\operatorname{div} \frac{Du}{|Du|} \equiv 0 \quad \text{on } \Omega. \quad (1)$$

This equation formally arises as the Euler equation of the total variation and is naturally posed for functions $u: \Omega \rightarrow \mathbb{R}$ of locally bounded variation whose gradient Du is merely a measure. In order to make sense of (1) in this setting it has become standard [12, 9, 10, 5, 13, 14] to work with a generalized product, which has been studied systematically by Anzellotti [2, 3]. The product is defined for $u \in \operatorname{BV}_{\operatorname{loc}}(\Omega)$ and $\sigma \in L^{\infty}_{\operatorname{loc}}(\Omega, \mathbb{R}^n)$ with vanishing distributional divergence $\operatorname{div} \sigma \equiv 0$ as the distribution

$$\llbracket \sigma, Du \rrbracket := \operatorname{div}(u\sigma) \in \mathcal{D}'(\Omega),$$

and in fact the pairing $\llbracket \sigma, Du \rrbracket$ turns out to be a signed Radon measure on Ω . By requiring $\|\sigma\|_{L^{\infty}(\Omega, \mathbb{R}^n)} \leq 1$ and $\llbracket \sigma, Du \rrbracket = |Du|$ one can now phrase precisely what it means that σ takes over the role of $\frac{Du}{|Du|}$, and whenever there exists some σ with all these properties (including $\operatorname{div} \sigma \equiv 0$), one calls $u \in \operatorname{BV}_{\operatorname{loc}}(\Omega)$ a BV solution of (1) or a weakly 1-harmonic function on Ω . In a similar vein, variants of the pairing $\llbracket \sigma, Du \rrbracket$ can be used to define BV solutions u of $\operatorname{div} \frac{Du}{|Du|} = f$ for right-hand sides $f \in L^1_{\operatorname{loc}}(\Omega)$ and to explain BV $\cap L^{\infty}$ solutions u of $\operatorname{div} \frac{Du}{|Du|} = f$ even for arbitrary $f \in L^1_{\operatorname{loc}}(\Omega)$.

In this note we deal with a notion of *supersolutions* of (1) or — this is essentially equivalent — of solutions of $\operatorname{div} \frac{Du}{|Du|} = -\mu$ with a Radon measure μ on Ω . To this end, we first collect some preliminaries in Section 2. Then, in Section 3, we consider generalized pairings, which make sense

*Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany.
Email address: christoph.scheven@uni-due.de. URL: http://www.uni-due.de/mathematik/ag_scheven.

[†]Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany.

Email address: thomas.schmidt@math.uni-hamburg.de. URL: <http://www.math.uni-hamburg.de/home/schmidt>.

[‡]Former institution: Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg.

even for L^∞ divergence-measure fields σ , but require precise evaluations of $u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ up to sets of zero $(n-1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} . We mainly investigate a pairing $\llbracket \sigma, Du^+ \rrbracket$, which is built with a specific \mathcal{H}^{n-1} -a.e. defined representative u^+ of u and which does not seem to have been considered before (while a similar pairing with the mean-value representative u^* already occurred in [7, Theorem 3.2] and [14, Appendix A]). Adapting the approach in [4, Section 5], we moreover deal with an up-to-the-boundary pairing $\llbracket \sigma, Du^+ \rrbracket_{u_0}$, which accounts for a boundary datum u_0 . In Section 4 we employ the local pairing $\llbracket \sigma, Du^+ \rrbracket$ in order to introduce a notion of weakly super-1-harmonic functions, and we prove a compactness statement which crucially depends on the choice of the representative u^+ . Finally, Section 5 is concerned with a refined notion of super-1-harmonicity which incorporates Dirichlet boundary values. This last notion is based on (a modification of) the pairing $\llbracket \sigma, Du^+ \rrbracket_{u_0}$.

We emphasize that the proofs, which are omitted in this announcement, can be found in the companion paper [15], where we also provide a more detailed study of pairings and supersolutions together with adaptations to the case of the minimal surface equation. Furthermore, in our forthcoming work [16], we will discuss connections with obstacle problems and convex duality.

2 Preliminaries

L^∞ divergence-measure fields. We record two results related to the classes

$$\begin{aligned} \mathcal{DM}_{\text{loc}}^\infty(\Omega, \mathbb{R}^n) &:= \{ \sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n) : \text{div } \sigma \text{ exists as a signed Radon measure on } \Omega \}, \\ \mathcal{DM}^\infty(\Omega, \mathbb{R}^n) &:= \{ \sigma \in L^\infty(\Omega, \mathbb{R}^n) : \text{div } \sigma \text{ exists as a finite signed Borel measure on } \Omega \}. \end{aligned}$$

Lemma 2.1 (absolute-continuity property for divergences of L^∞ vector fields). *Consider $\sigma \in \mathcal{DM}_{\text{loc}}^\infty(\Omega, \mathbb{R}^n)$. Then, for every Borel set $A \subset \Omega$ with $\mathcal{H}^{n-1}(A) = 0$, we have $|\text{div } \sigma|(A) = 0$.*

Lemma 2.1 has been proved by Chen & Frid [7, Proposition 3.1].

Lemma 2.2 (finiteness of divergences with a sign). *If Ω is bounded with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ satisfies $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$, then we necessarily have $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$. Moreover, there holds $(-\text{div } \sigma)(\Omega) \leq \frac{n\omega_n}{\omega_{n-1}} \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} \mathcal{H}^{n-1}(\partial\Omega)$ with the volume ω_n of the unit ball in \mathbb{R}^n .*

Indeed, it follows from the Riesz representation theorem that $\text{div } \sigma$ in Lemma 2.2 is a Radon measure. The finiteness of this measure will be established in [15] by a reasoning based on the divergence theorem.

BV-functions. We mostly follow the terminology of [1], but briefly comment on additional conventions and results. For $u \in \text{BV}(\Omega)$, we recall that \mathcal{H}^{n-1} -a.e. point in Ω is either a Lebesgue point (also called an approximate continuity point) or an approximate jump point of u ; compare [1, Sections 3.6, 3.7]. We write u^+ for the \mathcal{H}^{n-1} -a.e. defined representative of u which takes the Lebesgue values in the Lebesgue points and the larger of the two jump values in the approximate jump points. Correspondingly, u^- takes on the lesser jump values, and we set $u^* := \frac{1}{2}(u^+ + u^-)$. Finally, if Ω has finite perimeter in \mathbb{R}^n , we write $u_{\partial^* \Omega}^{\text{int}}$ for the \mathcal{H}^{n-1} -a.e. defined interior trace of u on the reduced boundary $\partial^* \Omega$ of Ω (compare [1, Sections 3.3, 3.5, 3.7]), and in the case that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^* \Omega) = 0$ we also denote the same trace by $u_{\partial\Omega}^{\text{int}}$.

The following two lemmas are crucial for our purposes. The first one extends [1, Proposition 3.62] and is obtained by essentially the same reasoning. The second one follows by combining [6, Theorem 2.5] and [8, Lemma 1.5, Section 6]; compare also [11, Sections 4, 10].

Lemma 2.3 (BV extension by zero). *If we have $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then for every $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$ we have $\mathbb{1}_\Omega u \in \text{BV}(\mathbb{R}^n)$ and $|\text{D}(\mathbb{1}_\Omega u)|(\partial\Omega) \leq \frac{n\omega_n}{\omega_{n-1}} \|u\|_{L^\infty(\Omega)} \mathcal{H}^{n-1}(\partial\Omega)$. In particular, Ω is a set of finite perimeter in \mathbb{R}^n , and $u_{\partial^*\Omega}^{\text{int}}$ is well-defined.*

Lemma 2.4 (\mathcal{H}^{n-1} -a.e. approximation of a BV function from above). *For every $u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ there exist $v_\ell \in W_{\text{loc}}^{1,1}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ such that $v_1 \geq v_\ell \geq u$ holds \mathcal{L}^n -a.e. on Ω for every $\ell \in \mathbb{N}$ and such that v_ℓ^* converges \mathcal{H}^{n-1} -a.e. on Ω to u^+ .*

3 Anzellotti type pairings for L^∞ divergence-measure fields

We first introduce a local pairing of divergence-measure fields and gradient measures.

Definition 3.1 (local pairing). *Consider $u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $\sigma \in \mathcal{DM}_{\text{loc}}^\infty(\Omega, \mathbb{R}^n)$. Then — since Lemma 2.1 guarantees that u^+ is $|\text{div } \sigma|$ -a.e. defined — we can define the distribution*

$$\llbracket \sigma, Du^+ \rrbracket := \text{div}(u\sigma) - u^+ \text{div } \sigma \in \mathcal{D}'(\Omega).$$

Written out this definition means

$$\llbracket \sigma, Du^+ \rrbracket(\varphi) = - \int_{\Omega} u\sigma \cdot \text{D}\varphi \, dx - \int_{\Omega} \varphi u^+ \, d(\text{div } \sigma) \quad \text{for } \varphi \in \mathcal{D}(\Omega). \quad (2)$$

Next we define a global pairing, which incorporates Dirichlet boundary values given by a function u_0 .

Definition 3.2 (up-to-the-boundary pairing). *Consider $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$, and $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$. Then we define the distribution $\llbracket \sigma, Du^+ \rrbracket_{u_0} \in \mathcal{D}'(\mathbb{R}^n)$ by setting*

$$\llbracket \sigma, Du^+ \rrbracket_{u_0}(\varphi) := - \int_{\Omega} (u-u_0)\sigma \cdot \text{D}\varphi \, dx - \int_{\Omega} \varphi(u^+ - u_0^*) \, d(\text{div } \sigma) + \int_{\Omega} \varphi \sigma \cdot \text{D}u_0 \, dx \quad (3)$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

We emphasize that the pairings in Definitions 3.1 and 3.2 coincide on φ with compact support in Ω (since an integration-by-parts then eliminates u_0 in (3)). However, the up-to-the-boundary pairing stays well-defined even if φ does *not* vanish on $\partial\Omega$. In addition, we remark that both pairings can be explained analogously with other representatives of u .

In some of the following statements we impose a mild regularity assumption on $\partial\Omega$, namely we require

$$\mathcal{H}^{n-1}(\partial\Omega) = \mathbf{P}(\Omega) < \infty, \quad (4)$$

where \mathbf{P} stands for the perimeter. We remark that the condition (4) is equivalent to having $\mathbf{P}(\Omega) < \infty$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ and also to having $\mathbb{1}_\Omega \in \text{BV}(\mathbb{R}^n)$ and $|\text{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$. For a more refined discussion we refer to [17], where the relevance of (4) for certain approximation results is pointed out.

Two vital properties of the pairing are recorded in the next statements. The proofs will appear in [15].

Lemma 3.3 (the pairing trivializes on $W^{1,1}$ -functions).

- (local statement) For $u \in W_{\text{loc}}^{1,1}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$, $\sigma \in \mathcal{DM}_{\text{loc}}^\infty(\Omega, \mathbb{R}^n)$, and $\varphi \in \mathcal{D}(\Omega)$, we have

$$\llbracket \sigma, Du^+ \rrbracket(\varphi) = \int_{\Omega} \varphi \sigma \cdot Du \, dx.$$

- (global statement with traces) If Ω is bounded with (4), then for every $\sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n)$ there exists a uniquely determined normal trace $\sigma_n^* \in L^\infty(\partial\Omega; \mathcal{H}^{n-1})$ with

$$\|\sigma_n^*\|_{L^\infty(\partial\Omega; \mathcal{H}^{n-1})} \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)}$$

such that for all $u, u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ there holds

$$\llbracket \sigma, Du^+ \rrbracket_{u_0}(\varphi) = \int_{\Omega} \varphi(\sigma \cdot Du) \, dx + \int_{\partial\Omega} \varphi(u-u_0)_{\partial\Omega}^{\text{int}} \sigma_n^* \, d\mathcal{H}^{n-1}.$$

Next we focus on bounded σ with $\text{div } \sigma \leq 0$. By Lemma 2.2 the pairings stay well-defined in this case.

Proposition 3.4 (the pairing is a bounded measure). Fix $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$.

- (local estimate) For $u \in \text{BV}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$, the distribution $\llbracket \sigma, Du^+ \rrbracket$ is a signed Radon measure on Ω with

$$\|\llbracket \sigma, Du^+ \rrbracket\| \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} |Du| \quad \text{on } \Omega.$$

- (global estimate with equality at the boundary) If Ω is bounded with (4), for $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$ the pairing $\llbracket \sigma, Du^+ \rrbracket_{u_0}$ is a finite signed Borel measure on \mathbb{R}^n with

$$\|\llbracket \sigma, Du^+ \rrbracket_{u_0} - (u-u_0)_{\partial\Omega}^{\text{int}} \sigma_n^* \mathcal{H}^{n-1} \llcorner \partial\Omega\| \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} |Du| \llcorner \Omega. \quad (5)$$

4 Weakly super-1-harmonic functions

We now give a definition of super-1-harmonic functions, which employs the convenient notation

$$S^\infty(\Omega, \mathbb{R}^n) := \{\sigma \in L^\infty(\Omega, \mathbb{R}^n) : |\sigma| \leq 1 \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega\}.$$

Definition 4.1 (weakly super-1-harmonic). We call $u \in \text{BV}_{\text{loc}}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$ weakly super-1-harmonic on Ω if there exists some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$ and $\llbracket \sigma, Du^+ \rrbracket = |Du|$ on Ω .

We next provide a compactness result for super-1-harmonic functions. We emphasize that this result does not hold anymore if one replaces u^+ by any other representative of u in the definition. We also remark that the assumed type of convergence is very natural, and indeed the statement applies to every *increasing* sequence of super-1-harmonic functions which is bounded in $\text{BV}_{\text{loc}}(\Omega)$ and $L_{\text{loc}}^\infty(\Omega)$.

Theorem 4.2 (convergence from below preserves super-1-harmonicity). Consider a sequence of weakly super-1-harmonic functions u_k on Ω . If u_k locally weak* converges to a limit u both in $\text{BV}_{\text{loc}}(\Omega)$ and $L_{\text{loc}}^\infty(\Omega)$ and if $u_k \leq u$ holds on Ω for all $k \in \mathbb{N}$, then u is weakly super-1-harmonic on Ω .

Proof. In view of Definition 4.1 there exist $\sigma_k \in S^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma_k \leq 0$ in $\mathcal{D}'(\Omega)$ and $\llbracket \sigma_k, Du_k^+ \rrbracket = |Du_k|$ on Ω . Possibly passing to a subsequence, we assume that σ_k weak* converges in $L^\infty(\Omega, \mathbb{R}^n)$ to $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div} \sigma \leq 0$ in $\mathcal{D}'(\Omega)$, and as before we regard $\operatorname{div} \sigma_k$ and $\operatorname{div} \sigma$ as non-positive measures on Ω . We fix a non-negative $\varphi \in \mathcal{D}(\Omega)$ and approximations v_ℓ of u with the properties of Lemma 2.4. Relying on (2), Lemma 2.1, and the dominated convergence theorem, we then infer

$$\begin{aligned} \llbracket \sigma, Du^+ \rrbracket(\varphi) &= - \int_{\Omega} u\sigma \cdot D\varphi \, dx + \int_{\Omega} \varphi u^+ \, d(-\operatorname{div} \sigma) \\ &= \lim_{\ell \rightarrow \infty} \left[- \int_{\Omega} v_\ell \sigma \cdot D\varphi \, dx + \int_{\Omega} \varphi v_\ell^* \, d(-\operatorname{div} \sigma) \right] = \lim_{\ell \rightarrow \infty} \llbracket \sigma, Dv_\ell^+ \rrbracket(\varphi). \end{aligned}$$

Since the pairing trivializes on $v_\ell \in W_{\operatorname{loc}}^{1,1}(\Omega)$, we can exploit the local weak* convergence of σ_k in $L_{\operatorname{loc}}^\infty(\Omega, \mathbb{R}^n)$ and the inequalities $v_\ell^* \geq u^+ \geq u_k^+$ to arrive at

$$\begin{aligned} \llbracket \sigma, Dv_\ell^+ \rrbracket(\varphi) &= \lim_{k \rightarrow \infty} \llbracket \sigma_k, Dv_\ell^+ \rrbracket(\varphi) = - \lim_{k \rightarrow \infty} \int_{\Omega} v_\ell \sigma_k \cdot D\varphi \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} \varphi v_\ell^* \, d(-\operatorname{div} \sigma_k) \\ &\geq - \int_{\Omega} v_\ell \sigma \cdot D\varphi \, dx + \liminf_{k \rightarrow \infty} \int_{\Omega} \varphi u_k^+ \, d(-\operatorname{div} \sigma_k). \end{aligned}$$

Next we rely in turn on the dominated convergence theorem, on the observation that $u_k \sigma_k$ locally weakly converges to $u\sigma$ in $L_{\operatorname{loc}}^1(\Omega, \mathbb{R}^n)$, on the definition in (2), on the coupling $\llbracket \sigma_k, Du_k^+ \rrbracket = |Du_k|$, and finally on a lower semicontinuity property of the total variation. In this way, we deduce

$$\begin{aligned} &\lim_{\ell \rightarrow \infty} \left[- \int_{\Omega} v_\ell \sigma \cdot D\varphi \, dx + \liminf_{k \rightarrow \infty} \int_{\Omega} \varphi u_k^+ \, d(-\operatorname{div} \sigma_k) \right] \\ &= - \int_{\Omega} u\sigma \cdot D\varphi \, dx + \liminf_{k \rightarrow \infty} \int_{\Omega} \varphi u_k^+ \, d(-\operatorname{div} \sigma_k) \\ &= \liminf_{k \rightarrow \infty} \left[- \int_{\Omega} u_k \sigma_k \cdot D\varphi \, dx + \int_{\Omega} \varphi u_k^+ \, d(-\operatorname{div} \sigma_k) \right] \\ &= \liminf_{k \rightarrow \infty} \llbracket \sigma_k, Du_k^+ \rrbracket(\varphi) = \liminf_{k \rightarrow \infty} \int_{\Omega} \varphi \, d|Du_k| \\ &\geq \int_{\Omega} \varphi \, d|Du|. \end{aligned}$$

Collecting the estimates, we arrive at the inequality $\llbracket \sigma, Du^+ \rrbracket \geq |Du|$ of measures on Ω . Since the opposite inequality is generally valid by Proposition 3.4, we infer that u is weakly super-1-harmonic on Ω . \square

5 Super-1-harmonic functions with respect to Dirichlet data

Finally, we introduce a concept of super-1-harmonic functions with respect to a generalized Dirichlet boundary datum. In [16] we will show that this notion is useful in connection with obstacle problems.

Concretely, consider a bounded Ω with (4), $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, and $u \in \operatorname{BV}(\Omega) \cap L^\infty(\Omega)$. We then extend the measure Du on Ω to a measure $D_{u_0}u$ on $\bar{\Omega}$ which takes into account the possible

deviation of $u_{\partial\Omega}^{\text{int}}$ from the boundary datum $(u_0)_{\partial\Omega}^{\text{int}}$. To this end, writing ν_Ω for the inward unit normal of Ω , we set

$$D_{u_0}u := Du \llcorner \Omega + (u - u_0)_{\partial\Omega}^{\text{int}} \nu_\Omega \mathcal{H}^{n-1} \llcorner \partial\Omega.$$

Now, for $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$ — which is meant to potentially satisfy a coupling like $\llbracket \sigma, Du^+ \rrbracket_{u_0} = |D_{u_0}u|$ on $\bar{\Omega}$ — we adopt the viewpoint that σ_n^* should typically equal the constant 1. If this is not the case, we compensate for this defect by extending $(-\text{div } \sigma)$ to a measure on $\bar{\Omega}$ with

$$(-\text{div } \sigma) \llcorner \partial\Omega := (1 - \sigma_n^*) \mathcal{H}^{n-1} \llcorner \partial\Omega. \quad (6)$$

Then we define a modified pairing $\llbracket \sigma, Du^+ \rrbracket_{u_0}^*$ by interpreting u^+ as $\max\{u_{\partial\Omega}^{\text{int}}, (u_0)_{\partial\Omega}^{\text{int}}\}$ on $\partial\Omega$ and extending the $(\text{div } \sigma)$ -integral in (3) from Ω to $\bar{\Omega}$. In other words, we define the measure $\llbracket \sigma, Du^+ \rrbracket_{u_0}^*$ by setting

$$\llbracket \sigma, Du^+ \rrbracket_{u_0}^* := \llbracket \sigma, Du^+ \rrbracket_{u_0} + [(u - u_0)_{\partial\Omega}^{\text{int}}]_+ (-\text{div } \sigma) \llcorner \partial\Omega. \quad (7)$$

With these conventions, we now complement Definition 4.1 as follows.

Definition 5.1 (super-1-harmonic function with respect to a Dirichlet datum). *For bounded Ω with (4) and $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, we say that $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$ is weakly super-1-harmonic on $\bar{\Omega}$ with respect to u_0 if there exists some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$ such that the equality of measures $\llbracket \sigma, Du^+ \rrbracket_{u_0}^* = |D_{u_0}u|$ holds on $\bar{\Omega}$.*

For $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$ and $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$, we get from (5), (6), (7)

$$\llbracket \sigma, Du^+ \rrbracket_{u_0}^* \llcorner \partial\Omega = \left([(u - u_0)_{\partial\Omega}^{\text{int}}]_+ - [(u - u_0)_{\partial\Omega}^{\text{int}}]_- \sigma_n^* \right) \mathcal{H}^{n-1} \llcorner \partial\Omega.$$

Thus, the boundary condition in Definition 5.1 is equivalent to the \mathcal{H}^{n-1} -a.e. equality $\sigma_n^* \equiv -1$ on the boundary portion $\{u_{\partial\Omega}^{\text{int}} < (u_0)_{\partial\Omega}^{\text{int}}\} \cap \partial\Omega$, while no requirement is made on $\{u_{\partial\Omega}^{\text{int}} \geq (u_0)_{\partial\Omega}^{\text{int}}\} \cap \partial\Omega$. We believe that this is very natural, in particular in the case $n = 1$, where super-1-harmonicity of u on an interval $[a, b]$ just means that u is increasing up to a certain point and decreasing afterwards, and where σ_n^* can take the value -1 at most at *one* endpoint and only if u is monotone on the open interval (a, b) .

Another indication that Definition 5.1 is meaningful is provided by the next statement, which will also be proved in [15]. We emphasize that the statement does not hold anymore (not even for $n=1$, $u_{0;k} = u_0 \equiv 0$, and $u_k \in W_0^{1,1}(\Omega)$) if one replaces $\llbracket \sigma, Du^+ \rrbracket_{u_0}^*$ with $\llbracket \sigma, Du^+ \rrbracket_{u_0}$ in the definition.

Theorem 5.2. *Suppose that Ω is bounded with (4), and consider weakly super-1-harmonic functions $u_k \in \text{BV}(\Omega) \cap L^\infty(\Omega)$ on $\bar{\Omega}$ with respect to boundary data $u_{0;k} \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. If $u_{0;k}$ converges strongly in $W^{1,1}(\Omega)$ and weakly* in $L^\infty(\Omega)$ to some u_0 , and if u_k weak* converges to a limit u in $\text{BV}(\Omega)$ and $L^\infty(\Omega)$ such that $u_k \leq u$ holds on Ω for all $k \in \mathbb{N}$, then u is weakly super-1-harmonic on $\bar{\Omega}$ with respect to u_0 .*

Remark. *In the situation of the theorem, it follows from the previously recorded reformulation of the boundary condition that u is also weakly super-1-harmonic on $\bar{\Omega}$ with respect to every $\tilde{u}_0 \in W^{1,1}(\mathbb{R}^n) \cap L^\infty(\Omega)$ such that $\mathcal{H}^{n-1}(\{u_0^* \leq u_{\partial\Omega}^{\text{int}} < \tilde{u}_0^*\} \cap \partial\Omega) = 0$. Roughly speaking, this means that the boundary values can always be decreased and that they can even be increased as long as the trace of u is not traversed. In view of the 1-dimensional case, we believe that this behavior is very reasonable.*

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