COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **15**, Number **4**, July **2016**

doi:10.3934/cpaa.2016.15.1251

pp. **1251–1263**

ON A PARABOLIC HAMILTON-JACOBI-BELLMAN EQUATION DEGENERATING AT THE BOUNDARY

Daniele Castorina*

Dipartimento di Matematica, Università di Padova
Via Trieste 63, 35121 Padova, Italy

Annalisa Cesaroni

Dipartimento di Scienze Statistiche, Università di Padova
Via Cesare Battisti 141, 35121 Padova, Italy

Luca Rossi

Dipartimento di Matematica, Università di Padova
Via Trieste 63, 35121 Padova, Italy

(Communicated by Martino Bardi)

Abstract. We derive the long time asymptotic of solutions to an evolutive Hamilton-Jacobi-Bellman equation in a bounded smooth domain, in connection with ergodic problems recently studied in [1]. Our main assumption is an appropriate degeneracy condition on the operator at the boundary. This condition is related to the characteristic boundary points for linear operators as well as to the irrelevant points for the generalized Dirichlet problem, and implies in particular that no boundary datum has to be imposed. We prove that there exists a constant c such that the solutions of the evolutive problem converge uniformly, in the reference frame moving with constant velocity c, to a unique steady state solving a suitable ergodic problem.

1. Introduction.

We are concerned with the asymptotic behavior as $t \to +\infty$ of solutions of the evolutive Hamilton-Jacobi-Bellman equation

$$u_t + \sup_{\alpha \in A} \left(-b(x,\alpha) \cdot Du - \operatorname{tr}(a(x,\alpha) D^2 u) - l(x,\alpha) \right) = 0, \quad x \in \Omega,\ t > 0, \tag{1}$$

with bounded initial data $u(x,0) = u_0(x)$. The domain $\Omega \subset \mathbb{R}^N$ is assumed to be bounded and smooth; no boundary condition is imposed, but just the following control on the growth:

$$\exists\, \lambda > 0, \quad \lim_{x \to \partial\Omega} u(x,t) d(x)^\lambda = 0 \qquad \text{loc. unif. in } t \geq 0 \tag{2}$$

where $d(x)$ is the distance of $x \in \Omega$ from $\partial\Omega$.

2000 *Mathematics Subject Classification.* Primary: 35K55; Secondary: 35B40, 35B45, 35K65.
Key words and phrases. Large time behavior, Hamilton-Jacobi-Bellman operators, ergodic problem, characteristic boundary points, initial boundary value problems.
D.C. was partially supported by project *Bando Giovani Studiosi 2013 - Università di Padova - GRIC131695*. A.C. and L.R. were partially supported by the GNAMPA Project 2015 Processi di diffusione degeneri o singolari legati al controllo di dinamiche stocastiche.
* Corresponding author: Daniele Castorina.

The main assumption is that the fully nonlinear elliptic operator

$$F[u] := \sup_{\alpha \in A} \left(-b(x,\alpha) \cdot Du(x) - \operatorname{tr}\left(a(x,\alpha)D^2u(x)\right)\right), \tag{3}$$

degenerates in the normal direction to the boundary $\partial\Omega$, for all $\overline{\alpha} \in A$, and the quantity $b(x,\overline{\alpha}) \cdot Dd(x) + \operatorname{tr}(a(x,\overline{\alpha})D^2d(x))$ is positive near $\partial\Omega$ (see Assumption (11)).

This condition is related to the invariance of the set Ω for the controlled diffusion process associated with the operator F (see [1]). It allows us to prove existence and uniqueness of a smooth solution to the Cauchy problem associated with (1), without imposing any boundary condition on $\partial\Omega$, but just the control on the growth (see Theorem 4.2).

Once the well posedness of the Cauchy problem is established, we investigate the large time behavior. Our main result states that there exists a unique constant c, called the ergodic constant, which governs the large time behavior of solutions, in the following sense: for every bounded continuous initial datum u_0, there exists a constant K, depending only on u_0, such that the unique smooth solution to the Cauchy problem satisfies

$$u(x,t) + ct - \chi(x) + K \to 0 \qquad \text{as } t \to +\infty, \text{ uniformly in } x \in \Omega.$$

For the precise statement we refer to Corollary 1 at the end of the paper. The constant c and the function χ are uniquely defined as the solution of the so called ergodic problem (or additive eigenvalue problem), that is,

$$\begin{cases} \sup_{\alpha \in A} \left(-b(x,\alpha) \cdot D\chi(x) - \operatorname{tr}(a(x,\alpha)D^2\chi(x)) - l(x,\alpha)\right) = c, & x \in \Omega \\ \chi \in L^\infty(\Omega), \ \sup \chi = 0. \end{cases} \tag{4}$$

Recently in [1] it has been proved that there exists a unique c such that the first equation in (4) admits a smooth solution (unique up to additive constants) satisfying the growth condition

$$\lim_{x \to \partial\Omega} \frac{\chi(x)}{\log d(x)} = 0.$$

In this paper, we refine this result, showing that actually the solution χ is bounded in Ω by using appropriate bounded barriers at the boundary of Ω (see Proposition 2). Moreover, in Proposition 2 we derive some regularity estimates of the solution χ to (4) up to the boundary of Ω, which in particular imply Hölder regularity of χ up to the boundary in the 1D case. An interesting open problem is to determine under which conditions χ is Lipschitz-continuous up to the boundary. Such regularity cannot be expected in general under our assumptions, as shown in Remark 1 in Section 5. Analogous regularity results for solutions of ergodic problems for linear operators with singular drift in bounded domains have been obtained in [14]. Moreover we recall that the generalized Dirichlet problem and the state constraint problem for Hamilton-Jacobi-Bellman operators in bounded domains has been studied in [2, 11, 12].

Our methods are mainly based on comparison principle, strong maximum principle and careful estimates of solutions to (1) up to the boundary of Ω (see Proposition 1).

Finally we recall that large time asymptotic of solutions to fully nonlinear parabolic equations have been studied in the periodic setting by Barles and Souganidis in [4]. More recently, the large time behavior in the periodic setting for possibly degenerate Hamilton-Jacobi equations has been treated in [6, 13]. Results on the large time behaviour of solutions in bounded domains have been obtained by Da Lio with Neumann boundary conditions in [8], and with state constraint boundary conditions by Barles, Porretta and Tchamba in [3].

The paper is organized as follows. In Section 2 we specify our assumptions and set up convenient notations for the development of our study. Section 3 is devoted to the explicit construction of Lyapunov functions and bounded barriers. In Section 4 we study the well posedness of the Cauchy problem associated with (1) as well as some *ad hoc* comparison principles for sub/super solutions satisfying mild growth conditions at the boundary. Next, in Section 5 we apply these results to the study of the boundary behavior and the sharp regularity of the solution to (1). Finally, we establish in Section 6 our main result about the large time convergence towards a steady state solving a suitable additive eigenvalue problem.

2. **Assumptions and notations.** Throughout the paper we will assume, if not otherwise stated, that Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. Let $d(x)$ be the signed distance function to $\partial\Omega$, i.e.

$$d(x) := \mathrm{dist}(x, \mathbb{R}^n \backslash \Omega) - \mathrm{dist}(x, \overline{\Omega}).$$

We know, from e.g. [10, Lemma 14.16], that d is of class C^2 in some neighborhood $\overline{\Omega}_\delta$ of the boundary, where, here and in the sequel,

$$\Omega_\delta := \{x \in \Omega \mid d(x) < \delta\}. \tag{5}$$

We introduce the fully nonlinear homogeneous operator

$$F[u] := \sup_{\alpha \in A} \left(-b(x,\alpha) \cdot Du(x) - \mathrm{tr}\left(a(x,\alpha) D^2 u(x)\right)\right). \tag{6}$$

and the Hamilton-Jacobi-Bellman operator

$$H[u] := \sup_{\alpha \in A} \left(-b(x,\alpha) \cdot Du(x) - \mathrm{tr}(a(x,\alpha) D^2 u(x)) - l(x,\alpha)\right). \tag{7}$$

where A is a complete metric space and

$$b, l : \overline{\Omega} \times A \to \mathbb{R}^N, \qquad a : \overline{\Omega} \times A \to \mathbf{M}_{N \times N}$$

are bounded and continuous, $\mathbf{M}_{N\times N}$ being the space of $N \times N$ real matrices. We further assume $a(x, \alpha)$ to be symmetric and nonnegative definite for all x, α. This implies that $a \equiv \sigma\sigma^T$ for some $\sigma : \overline{\Omega} \times A \to \mathbf{M}_{N\times r}$, $r \in \mathbb{N}$.

The main regularity assumptions on the coefficients of the operator are the following: there exist $B > 0$, $\eta \in (0, 1]$, $\beta \in (1/2, 1]$ such that, for all $x, y \in \overline{\Omega}$ and $\alpha \in A$,

$$|b(x,\alpha) - b(y,\alpha)|, |l(x,\alpha) - l(y,\alpha)| \leq B|x - y|^\eta \tag{8}$$

$$|\sigma(x,\alpha) - \sigma(y,\alpha)| \leq B|x - y|^\beta, \tag{9}$$

where, even for matrices, $|\cdot|$ stands for the standard Euclidean norm. The regularity assumption on a is given in terms of its square root σ as it is natural for applications to stochastic control problems. We recall that if $a(\cdot, \alpha) \in W^{2,p}(\Omega)$ with $\|a(\cdot, \alpha)\|_{W^{2,p}} \leq C$ for some $p > 2N$ and $C > 0$ independent of $\alpha \in A$, then it has a square root σ satisfying (9) with $\beta = 1 - \frac{N}{p}$.

We assume that the operator is elliptic in the interior of Ω, in the strong sense

$$a(x,\alpha) > 0 \quad \text{for all } x \in \Omega \text{ and } \alpha \in A \tag{10}$$

and that it degenerates at the boundary according to the following condition:

$$\begin{array}{l}\exists\, \delta, k>0,\ \gamma<2\beta-1\leq 1, \quad \text{such that for all } \overline{x}\in\partial\Omega \text{ and } \alpha\in A,\\ \begin{cases}\sigma^T(\overline{x},\alpha)Dd(\overline{x})=0\\ b(x,\alpha)\cdot Dd(x)+\operatorname{tr}\left(a(x,\alpha)D^2d(x)\right)\geq k\, d^\gamma(x), \quad x\in\Omega\cap B_\delta(\overline{x}).\end{cases}\end{array} \tag{11}$$

The first condition in (11) means that at any boundary point, the normal is a direction of degeneracy for F. The second condition can be rewritten as: $F[d]\leq -k\, d^\gamma$ in a neighborhood of $\partial\Omega$; it is guaranteed if at the boundary the normal component of the drift points inward and is sufficiently large. Notice however that condition (11) does not prevent the function $b(\cdot,\alpha)\cdot Dd(\cdot)+\operatorname{tr}(a(\cdot,\alpha)D^2d(\cdot))$ from vanishing at the boundary.

We recall that (11) is a sufficient condition for the invariance of the domain Ω for the stochastic control system with drift b and diffusion σ (see Prop. 6.5 in [1]).

3. **Lyapunov functions and barriers at the boundary.** In this section we show that under condition (11) the function $V(x)=d(x)^{-\lambda}$, for $\lambda>0$, plays the role of a Lyapunov function for the system (see also [1]).

Lemma 3.1. *Assume that (8), (9), (11) hold. Then for every $M\geq 0$ and every $\lambda>0$, there exists $\delta>0$ such that d is of class C^2 in the set Ω_δ defined by (5) and there holds*

$$-F[d^{-\lambda}]\leq F[-d^{-\lambda}]\leq -M \qquad \text{in } \Omega_\delta.$$

Proof. The first inequality immediately follows from the definition of F. For the second one we take δ small enough so that $d\in C^2(\Omega_\delta)$ and we compute, for $x\in\Omega_\delta$,

$$F[-d^{-\lambda}]=\frac{\lambda}{d^{\lambda+1}}\sup_{\alpha\in A}\left(-b(x,\alpha)\cdot Dd-\operatorname{tr}\left(a(x,\alpha)D^2d\right)+\frac{\lambda+1}{d}|\sigma(x,\alpha)Dd|^2\right).$$

Using (9), (11) and choosing $\bar{x}\in\partial\Omega$ such that $Dd(x)=Dd(\bar{x})$, we get for every α

$$|\sigma(x,\alpha)Dd(x)|^2=|(\sigma(x,\alpha)-\sigma(\bar{x},\alpha))Dd(\bar{x})|^2\leq B^2|x-\bar{x}|^{2\beta}=B^2d^{2\beta}(x). \tag{12}$$

Then we obtain, using (11) and recalling that $\gamma<2\beta-1$,

$$F[-d^{-\lambda}]\leq\frac{\lambda}{d^{\lambda+1}}\left(F[d]+B^2(\lambda+1)d^{2\beta-1}\right)\leq\frac{\lambda}{d^{\lambda+1}}\left(-kd^\gamma+B^2(\lambda+1)d^{2\beta-1}\right),$$

which is smaller than any given $-M$ in Ω_δ, provided δ is sufficiently small. □

In the following we will also need the existence of strict supersolutions to $F=0$ in a neighborhood of the boundary of Ω which are not explosive at the boundary.

Lemma 3.2. *Assume that (8), (9), (11) hold. Let $\rho\in(0,1-\gamma)$, where γ is the constant appearing in condition (11). Then for every $M>0$ there exists δ small enough such that the function $1-d^\rho$ is $C^2(\Omega_\delta)$ and satisfies*

$$-F[1-d^\rho]\leq F[d^\rho-1]\leq -M \qquad \text{in } \Omega_\delta.$$

Proof. As before, we prove the second inequality, since the first comes from the definition of F. For $\delta\in(0,1)$ small enough we have that the function $d^\rho-1$ is of class C^2 in Ω_δ, where it satisfies

$$F[d^\rho-1]=\rho d^{\rho-1}\sup_{\alpha}\left(-b(x,\alpha)\cdot Dd-\frac{\rho-1}{d}|\sigma(x,\alpha)Dd|^2-\operatorname{tr}(a(x,\alpha)D^2d)\right).$$

Then, recalling that $0<\rho<1$ and the computation (12), we obtain

$$F[d^\rho-1]\leq\rho d^{\rho-1}\left(F[d]+(\rho-1)B^2d^{2\beta-1}\right).$$

Hence, using (11) and choosing $\rho < 1-\gamma$, we see that

$$F[d^\rho - 1] \le \rho d^{\gamma+\rho-1}\left(-k + (\rho-1)B^2 d^{2\beta-\gamma-1}\right) \le -M \qquad \text{in } \ \Omega_\delta,$$

provided δ is sufficiently small (depending in particular on $k, B, \beta, \gamma, \rho, M$). □

4. **The Cauchy problem.** We study the following Cauchy problem in the cylinder $\Omega \times (0, +\infty)$:

$$\begin{cases} u_t + H[u] = 0, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{13}$$

The initial condition is understood to hold in the classical sense, with $u_0 \in C(\Omega) \cap L^\infty(\Omega)$. Notice that no boundary condition on $\partial\Omega$ is imposed.

The first result is a comparison principle between smooth sub and supersolution which satisfy an appropriate growth condition at the boundary.

Lemma 4.1. *Let $\underline{u}, \overline{u} \in C^{2,1}(\Omega \times (0,T]) \cap C(\Omega \times [0,T])$ be respectively a sub and a supersolution to (13) such that*

$$\exists\, \lambda > 0, \quad \limsup_{x\to\partial\Omega} \underline{u}(x,t) d(x)^\lambda \le 0 \le \liminf_{x\to\partial\Omega} \overline{u}(x,t) d(x)^\lambda \quad \textit{uniformly in } t \in [0,T].$$

Then $\underline{u} \le \overline{u}$ in $\Omega \times (0,T]$.

Proof. We start with observing that

$$\forall u, w \in C^2, \quad H[u] - F[w] \le H[u-w] \le H[u] + F[-w]. \tag{14}$$

The first inequality implies that the function $w := \overline{u} - \underline{u}$ satisfies

$$w_t + F[w] \ge \overline{u}_t + H[\overline{u}] - \underline{u}_t - H[\underline{u}] \ge 0, \qquad x \in \Omega,\ t > 0.$$

Furthermore, w is nonnegative at $t = 0$ and fulfills the same condition as $\overline{u}$ at $\partial\Omega$. We know from Lemma 3.1 that $F[-d(x)^{-\lambda}] < -1$ in some Ω_δ. Fix $T > 0$ and call

$$m := \min\left\{ \min_{(\Omega\setminus\Omega_\delta)\times[0,T]} w\,,\ 0 \right\}.$$

For $\varepsilon > 0$, the function $V_\varepsilon(x) := m - \varepsilon d(x)^{-\lambda}$ satisfies $F[V_\varepsilon] < 0$ in Ω_δ, as well as $V_\varepsilon(x) < m \le w(x,t)$ if $d(x) = \delta$, and also

$$\liminf_{x\to\partial\Omega} (w(x,t) - V_\varepsilon(x)) = +\infty \qquad \text{uniformly in } t \in [0,T].$$

The latter implies that, for $\delta' \in (0,\delta)$ small enough, $w(x,t) > V_\varepsilon(x)$ if $d(x) = \delta'$ and $t \in [0,T]$. Finally, observe that $w \ge V_\varepsilon$ at initial time. We can therefore apply the standard parabolic comparison principle (see, e.g., [15]) in the cylinder $(\Omega_\delta \setminus \Omega_{\delta'}) \times [0,T]$ and infer that $w \ge V_\varepsilon$ there. Due to the arbitrariness of δ' and ε, this implies that $w \ge m$ in $\Omega_\delta \times [0,T]$, whence

$$\inf_{\Omega\times[0,T]} w \ge \min\left\{ \min_{(\Omega\setminus\Omega_\delta)\times[0,T]} w\,,\ 0 \right\}.$$

If the above right-hand side were negative, since $w \ge 0$ at $t = 0$, it would be reached at some $(\underline{x}, \underline{t}) \in (\Omega \setminus \Omega_\delta) \times (0,T]$, and therefore, by the parabolic strong maximum principle (see [15]), w would coincide with a negative constant for $t \le \underline{t}$, which is impossible. This shows that $w \ge 0$ in $\Omega \times [0,T]$. □

Theorem 4.2. *For any $u_0 \in L^\infty(\Omega) \cap C(\Omega)$, problem (13) admits a unique solution $u \in C^{2,1}(\Omega \times (0,+\infty)) \cap C(\Omega \times [0,+\infty))$ satisfying (2). Moreover, $u \in L^\infty(\Omega \times (0,T))$ for every $T > 0$.*

Proof. We start by proving existence and interior regularity. Let $\Omega^n := \Omega \setminus \overline{\Omega}_{1/n}$ (according to definition (5)), for n sufficiently large, so that Ω_n is smooth. Let ξ be a standard mollifier with support contained in the unit ball and $\xi_n(x) := n^N \xi(nx)$ for $n \in \mathbb{N}$. We define $u_{0,n} := u_0 * \xi_n$ and consider the following Cauchy-Dirichlet problem

$$\begin{cases} v_t + H[v] = 0, & x \in \Omega_n,\ t > 0 \\ v(x,t) = u_{0,n}(x), & x \in \partial\Omega_n, t > 0 \\ v(x,0) = u_{0,n}(x), & x \in \Omega_n \end{cases} \tag{15}$$

By [15, Theorem 14.18], problem (15) admits a unique solution $u_n \in C^{2,1}(\Omega_n \times (0,+\infty)) \cap C(\overline{\Omega}_n \times [0,+\infty))$.

Let us now fix $T > 0$ and a compact set $Q \subset \Omega \times (0,T)$. Then $Q \subset \Omega_n \times (0,T)$ for n larger than some $\overline{n}$. Thanks to our assumptions and a covering argument, we can apply Theorem 1.1 in [18] (see also Remark 1.1 parts (a) and (b) for the regularity issues regarding u and H respectively) in order to see that there exist some constants $\theta \in (0,1]$ and $C > 0$ not depending on n such that

$$\forall n > \overline{n}, \quad \|D^2 u_n\|_{C^{\theta,\theta/2}(Q)} \leq C. \tag{16}$$

Notice that in principle C depends also on n through the uniform bound $\|u_n\|_{L^\infty(\Omega_n \times [0,T])}$, but we can actually substitute this bound with the uniform bound $\|u_0\|_{L^\infty(\Omega)}$, independent of n, because $\pm\|u_{0,n}\|_{L^\infty(\Omega_n)} \pm \|l\|_\infty t$ are sub/super solutions of (15) and thus the standard comparison principle yields

$$\|u_n\|_{L^\infty(\Omega_n \times [0,T])} \leq \|u_{0,n}\|_{L^\infty(\Omega_n)} + \|l\|_\infty T \leq \|u_0\|_{L^\infty(\Omega)} + \|l\|_\infty T. \tag{17}$$

Now, from (15), (16) and the regularity of the coefficients, it follows that the $(\partial_t u_n)_{n > \overline{n}}$ are uniformly Hölder-continuous in Q in both the x, t variables. The Ascoli-Arzelà theorem eventually implies that there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging in $C^{2+\zeta,1+\zeta/2}(Q)$, $\zeta < \theta$, to a function u satisfying the first equation of (13) in Q. Finally, by a diagonal argument, we find a subsequence of $(u_n)_n$ for which this convergence holds true in any compact subset of $\Omega_n \times (0,T)$. Notice that, the limit being just local, we lose the information about the boundary and the initial behavior of the solution. Observe nevertheless that by (17), we get that u is bounded in $\Omega \times [0,T]$.

We claim now that $u \in C(\Omega \times [0,+\infty))$ and satisfies $u(x,0) = u_0(x)$, that is, it solves (13). Fix $x_0 \in \Omega$. For $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$ consider the function $u^\varepsilon_{0,n} \in C(\Omega)$ defined by

$$u^\varepsilon_{0,n}(x) = \max_{|z-x_0|\leq\varepsilon} u_{0,n}(z) + \frac{2}{\varepsilon^2}|x-x_0|^2 \|u_{0,n}\|_\infty.$$

We compute

$$H[u^\varepsilon_{0,n}] \geq -\frac{4\|u_{0,n}\|_\infty}{\varepsilon^2}\left(\|b\|_\infty \operatorname{diam}(\Omega) + N\|a\|_\infty\right) - \|l\|_\infty.$$

Then, we take $M \geq \|b\|_\infty \operatorname{diam}(\Omega) + N\|a\|_\infty$ and define the function

$$u^\varepsilon_n(x,t) = u^\varepsilon_{0,n}(x) + \frac{4M}{\varepsilon^2}\|u_{0,n}\|_\infty t + \|l\|_\infty t.$$

It is easy to check, using our choice of M and noticing that $u^\varepsilon_n(x,t) \geq u^\varepsilon_{0,n}(x) \geq u_{0,n}(x)$ in $\overline{\Omega}_n \times [0,T]$, that $u^\varepsilon_n(x,t)$ is a supersolution to (15). Then, by comparison, we get

$$u_n(x,t) \leq u^\varepsilon_n(x,t) \qquad x \in \Omega_n,\ t \geq 0.$$

Computing the previous inequality at $x = x_0$ and letting $n \to +\infty$, we obtain

$$u(x_0, t) \le \max_{|z-x_0|\le\varepsilon} u_0(z) + \frac{4M}{\varepsilon^2}\|u_0\|_\infty t + \|l\|_\infty t \qquad t \ge 0.$$

Now, letting $t \to 0$, we get

$$\limsup_{t\to 0} u(x_0, t) \le \max_{|z-x_0|\le\varepsilon} u_0(z).$$

Finally letting $\varepsilon \to 0$, the continuity of u_0 yields

$$\limsup_{t\to 0} u(x_0, t) \le u_0(x_0).$$

Arguing in an analogous way, with the function

$$u_n^\varepsilon(x, t) = \min_{|z-x_0|\le\varepsilon} u_{0,n}(z) - \frac{2}{\varepsilon^2}|x - x_0|^2\|u_{0,n}\|_\infty - \frac{4M}{\varepsilon^2}\|u_{0,n}\|_\infty t - \|l\|_\infty t$$

we also obtain that

$$\liminf_{t\to 0} u(x_0, t) \ge u_0(x_0).$$

We conclude by the arbitrariness of x_0. Finally, uniqueness follows from Lemma 4.1. □

5. **Boundary behavior of the solution.** We investigate now the behavior of the solution u to (13) at the boundary of Ω.

Proposition 1. *Let u be the solution to (13), (2) provided by Theorem 4.2. Then for every $\rho \in (0, 1-\gamma)$, there exists $\bar\delta \in (0, 1)$ such that for $\delta < \bar\delta$ it holds*

$$\forall\, x \in \Omega_\delta,\ t \ge 1, \quad -\delta^\rho + d(x)^\rho + \min_{(\partial\Omega_\delta\setminus\partial\Omega)\times[0,t]} u \le u(x, t) \le \delta^\rho - d(x)^\rho + \max_{(\partial\Omega_\delta\setminus\partial\Omega)\times[0,t]} u. \tag{18}$$

Proof. By Lemma 3.2 there exists $\bar\delta > 0$ such that, for $x \in \Omega_{\bar\delta}$,

$$F[1 - d(x)^\rho] \ge M, \quad F[d(x)^\rho - 1] \le -M, \quad \text{with } M := 2\|u_0\|_\infty + \|l\|_\infty.$$

Owing to Lemma 3.1, up to reducing $\bar\delta$ if needed, we also have that $F[-d(x)^{-1}] \le 0$ and $F[d(x)^{-1}] \ge 0$ in $\Omega_{\bar\delta}$. Fix $t \ge 1$ and, for $\delta < \bar\delta$ and $\varepsilon > 0$, let us define in $\Omega_{\bar\delta} \times [0, t]$ the following function:

$$v^\varepsilon(x, s) := \max_{\partial\Omega_\delta\setminus\partial\Omega\times[0,t]} u + (1 - d(x)^\rho) - 1 + \delta^\rho + \varepsilon d(x)^{-1} - 2(s - t)\frac{\|u_0\|_\infty}{t}.$$

This is a $C^{2,1}$ function which, by (14), satisfies in $\Omega_\delta \times (0, t]$,

$$v_s^\varepsilon + H[v^\varepsilon] \ge -2\frac{\|u_0\|_\infty}{t} + F[1-d(x)^\rho] - \|l\|_\infty - \varepsilon F[-d(x)^{-1}] > -2\|u_0\|_\infty + M - \|l\|_\infty = 0.$$

Moreover v^ε fulfills the boundary condition

$$\liminf_{x\to\partial\Omega} v^\varepsilon(x, s)d(x) = \varepsilon \quad \text{ uniformly in } s \in [0, t],$$

and, for $x \in \partial\Omega_\delta \setminus \partial\Omega$ and $s \in [0, t]$,

$$v^\varepsilon(x, s) = \max_{\partial\Omega_\delta\setminus\partial\Omega\times[0,t]} u + \varepsilon\delta^{-1} + 2(t - s)\frac{\|u_0\|_\infty}{t} \ge \max_{\partial\Omega_\delta\setminus\partial\Omega\times[0,t]} u \ge u(x, s),$$

as well as the initial condition

$$v^\varepsilon(x, 0) \ge \max_{\partial\Omega_\delta\setminus\partial\Omega\times[0,t]} u + 2\|u_0\|_\infty \ge \max_{\partial\Omega_\delta\setminus\partial\Omega} u_0 + 2\|u_0\|_\infty \ge u_0(x).$$

Then by the comparison principle given by Lemma 4.1 applied with $\lambda = 1$ (recall that u is bounded in $\Omega \times [0,t]$) we obtain

$$u(x,s) \le v^\varepsilon(x,s) \qquad \text{for } x \in \Omega_\delta,\ s \in [0,t],$$

which, computed at $s = t$ and with $\varepsilon \to 0$, eventually implies the second inequality in (18).

To obtain the first inequality in (18) one argues analogously: define the function

$$w^\varepsilon(x,s) = \min_{\partial\Omega_\delta \setminus \partial\Omega \times [0,t]} u - 1 + d(x)^\rho + 1 - \delta^\rho - \varepsilon d(x)^{-1} + 2(s-t)\frac{\|u_0\|_\infty}{t},$$

and, after observing that

$$w^\varepsilon_s + H[w^\varepsilon] \le 2\frac{\|u_0\|_\infty}{t} + H[d(x)^\rho - 1] + \varepsilon F[-d(x)^{-1}] \le 2\|u_0\|_\infty - M \le 0,$$

one concludes as before. $\square$

We now turn to the ergodic problem

$$H[\chi] = c, \qquad \text{in } \Omega. \tag{19}$$

The solution χ will be used to derive the large time behavior for the Cauchy problem (13).

It has been proved in [1] that, under the same standing assumptions as here, there exists a unique constant c for which (19) admits a solution $\chi \in C^2(\Omega)$ satisfying (2). Moreover χ is unique up to additive constants and actually satisfies the stronger condition

$$\chi(x) = o(-\log(d(x))) \quad \text{as } x \to \partial\Omega. \tag{20}$$

However, such condition is not sufficient for our purpose, but we need boundedness. This is provided by the following.

Proposition 2. *Let $c \in \mathbb{R}$ and $\chi \in C^2(\Omega)$ be the solution to (19) satisfying (20). Then $\chi \in L^\infty(\Omega)$. In particular, for every $\rho \in (0, 1-\gamma)$, there exists $\bar\delta \in (0,1)$ such that, for every $\delta \le \bar\delta$ and $x \in \Omega_\delta$, there holds*

$$\min_{\partial\Omega_\delta \setminus \partial\Omega} \chi - \delta^\rho + d(x)^\rho \le \chi(x) \le \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi + \delta^\rho - d(x)^\rho. \tag{21}$$

Proof. First of all notice that $\chi(x) - ct$ is an entire solution to the first equation in (13). Set $M = 2|c| + \|l\|_\infty$ and consider the associated quantity δ given by Lemma 3.2. Then

$$-H[1 - d(x)^\rho] \le H[d(x)^\rho - 1] \le -2|c| \quad \text{for } x \in \Omega_\delta.$$

Possibly decreasing δ, we can also assume that $-F[d(x)^{-1}] \le F[-d(x)^{-1}] \le 0$ in Ω_δ by Lemma 3.1.

Take now $t_\varepsilon < 0$ such that

$$-|c|t_\varepsilon \ge \max_{x \in \overline{\Omega}_\delta} (\chi(x) - \varepsilon d(x)^{-1}) - \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi.$$

Note that the first maximum above exists due to the fact that χ satisfies (20).

For $\varepsilon > 0$ define in $\overline{\Omega}_\delta \times [t_\varepsilon, 0]$ the function

$$v^\varepsilon(x,s) = \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi + (1 - d(x)^\rho) - 1 + \delta^\rho + \varepsilon d(x)^{-1} - 2|c|s.$$

Then v^ε is in $C^{2,1}(\Omega_\delta \times [t_\varepsilon, 0])$ and satisfies the boundary condition

$$\liminf_{x \to \partial\Omega} v^\varepsilon(x,t) d(x) = \varepsilon \ge 0$$

uniformly in $t \in [t_\varepsilon, 0]$. Moreover

$$v_s^\varepsilon + H[v^\varepsilon] \geq -2|c| + H[1 - d(x)^\rho] - \varepsilon F[-d(x)^{-1}] \geq 0.$$

Finally at $x \in \partial\Omega_\delta \setminus \partial\Omega$, for $s \in [t_\varepsilon, 0]$,

$$v^\varepsilon(x, s) = \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi + \varepsilon\delta^{-1} - 2|c|s \geq \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi - |c|s \geq \chi(x) - cs$$

and for all $x \in \overline{\Omega}_\delta$

$$v^\varepsilon(x, t_\varepsilon) \geq \max_{\partial\Omega_\delta \setminus \partial\Omega} \chi + \varepsilon d(x)^{-1} - 2|c|t_\varepsilon \geq \chi(x) - ct_\varepsilon$$

by our choice of t_ε. Then by the parabolic comparison principle (see Lemma 4.1), we get that for every $\varepsilon > 0$

$$\chi(x) - cs \leq v^\varepsilon(x, s) \qquad x \in \Omega_\delta,\ s \in [t_\varepsilon, 0].$$

Computing the previous inequality at $s = 0$, and letting $\varepsilon \to 0$, we get the right hand side of the inequality (21).

The other side is obtained with similar arguments, by considering the function

$$w^\varepsilon(x, s) = \min_{\partial\Omega_\delta \setminus \partial\Omega} \chi - (1 - d(x)^\rho) + 1 - \delta^\rho - \varepsilon d(x)^{-1} + 2|c|s$$

for $s \in [t^\varepsilon, 0]$, where $t^\varepsilon < 0$ satisfies

$$|c|t^\varepsilon \leq \min_{x \in \overline{\Omega}_\delta} (\chi(x) + \varepsilon d(x)^{-1}) - \min_{\partial\Omega_\delta \setminus \partial\Omega} \chi.$$

□

Remark 1. In dimension $N = 1$ condition (21) implies that χ is Hölder-continuous in Ω with Hölder exponent up to $1 - \gamma$. Indeed, if $\Omega = (a, b)$ then the sets Ω_δ are composed by two disjoint intervals and this allows one to split the estimates (21) into the following two sets of estimates:

$$\forall\ a < x < y < a + \overline{\delta}, \quad \chi(y) - (y - a)^\rho + (x - a)^\rho \leq \chi(x) \leq \chi(y) + (y - a)^\rho - (x - a)^\rho,$$

$$\forall\ b - \overline{\delta} < y < x < b, \quad \chi(y) - (b - y)^\rho + (b - x)^\rho \leq \chi(x) \leq \chi(y) + (b - y)^\rho - (b - x)^\rho.$$

Combining this with standard interior regularity for elliptic equations, we get that $\chi \in C^\rho(\Omega)$ for every $\rho \in (0, 1 - \gamma)$.

In general we cannot expect more than Hölder regularity for χ under the hypothesis (11). In particular, if the constant γ there is strictly positive, χ may not be Lipschitz-continuous in Ω, as shown by the following example. Let Ω be the interval $(0, 1)$, the set A to be a singleton, a to be a smooth function such that $a > 0$ for $x \in (0, 1)$, $a(x) = x^{2\beta}$ for x in a neighborhood of 0 and $a(x) = (1 - x)^{2\beta}$ for x in a neighborhood of 1 with $2\beta > 1$, b be a smooth function such that $b(0) = b(1) = 0$, and l be a smooth function such that $l(0) \neq l(1)$. Assume moreover that there exist δ, k, γ, with $0 < \gamma < 2\beta - 1$ such that

$$\forall\ x \in (0, \delta), \quad b(x) \geq kx^\gamma \qquad \text{and} \qquad \forall\ x \in (1 - \delta, 1), \quad b(x) \leq -k(1 - x)^\gamma.$$

Note that this condition is exactly condition (11), and that is compatible with the assumption $b(0) = b(1) = 0$, since $\gamma > 0$. The solution χ of (19) solves

$$\chi''(x) = a(x)^{-1}(-b(x)\chi'(x) - l(x) - c), \qquad x \in (0, 1).$$

Define $G(x) = -b(x)\chi'(x) - l(x) - c$. Assume by contradiction that χ is Lipschitz-continuous in $(0, 1)$, so in particular there exists $C > 0$ such that $|\chi'| \leq C$. By our assumptions on the coefficients,

$$\lim_{x\to 0^+} G(x) = -l(0) - c \neq \lim_{x\to 1^-} G(x) = -l(1) - c.$$

Then necessarily, either $\lim_{x\to 0^+} G(x) \neq 0$ or $\lim_{x\to 1^-} G(x) \neq 0$. Assume to fix the ideas that $\lim_{x\to 0^+} G(x) \neq 0$ (the other case is completely analogous), then in a neighborhood of 0 we get that $\chi''(x) \approx a(x)^{-1} = x^{-2\beta}$. So $\chi'(x) \approx x^{-2\beta+1} + C$, and this is in contradiction with the Lipschitz-continuity of χ because $1 - 2\beta < 0$.

6. **Convergence result.** In this last section we show that solutions of the Cauchy problem share the same large time behavior. This will imply our main result.

Theorem 6.1. *Let $u_1, u_2 \in C^{2,1}(\Omega\times(0, +\infty))\cap C(\Omega\times[0, +\infty))$ be two solutions to the first equation in (13) which satisfy the boundary control (2), such that $u_1 - u_2$ is bounded and*

$$\forall\tau > 0,\ K \subset\subset \Omega, \quad \exists C > 0,\ \theta \in (0, 1], \quad \|u_1 - u_2\|_{C^{2+\theta,1+\theta/2}(K\times(\tau,+\infty))} \leq C. \tag{22}$$

Then, as $t \to +\infty$, $u_1 - u_2$ converges to a constant uniformly in Ω.

Proof. Consider the difference function $w := u_1 - u_2$, which is bounded by hypothesis. By Lemmas 3.1 and 3.2, there exist $\delta \in (0, 1)$ and $\rho \in (0, 1 - \gamma)$ such that $F[-d^{-1}] \leq -1$ and $F[d^\rho - 1] \leq -2\|w\|_\infty$ in Ω_δ. For $t > 0$ define

$$m(t) := \min_{x\in\Omega\setminus\Omega_\delta} w(x, t).$$

Let $(t_n)_{n\in\mathbb{N}}$ be such that

$$\lim_{n\to\infty} t_n = +\infty, \qquad \lim_{n\to\infty} m(t_n) = \liminf_{t\to+\infty} m(t) =: \tilde{m}\,.$$

Our aim is to show that $w = u_1 - u_2 \to \tilde{m}$ uniformly in Ω as $t \to +\infty$.

Step 1. *$w(\cdot, \cdot + t_n) \to \tilde{m}$ as $n \to \infty$, locally uniformly in $\Omega \times (-\infty, 0]$.*

By assumption (22), the functions $(w(\cdot, \cdot + t_n))_{n\in\mathbb{N}}$ and their derivatives ∂_t, D, D^2 are locally uniformly bounded in $\Omega\times\mathbb{R}$. Moreover, by (14), w solves $w_t + F[w] \geq 0$. Thus, as $n \to \infty$, $w(\cdot, \cdot + t_n)$ converges locally uniformly (up to subsequences) to a supersolution $\tilde{w}$ of the same equation in $\Omega \times \mathbb{R}$, which satisfies in addition

$$\min_{(\Omega\setminus\Omega_\delta)\times\mathbb{R}} \tilde{w} = \min_{(\Omega\setminus\Omega_\delta)\times\{0\}} \tilde{w} = \tilde{m}\,. \tag{23}$$

For given $T \in \mathbb{R}$ and $\varepsilon > 0$, the function v defined by

$$v(x, t) := \varepsilon(t - T - d(x)^{-1}) + \tilde{m}$$

satisfies $v_t + F[v] \leq 0$ in $\Omega_\delta \times \mathbb{R}$. Moreover, for $x \in \Omega_\delta$, $v(x, t) < \tilde{m}$ if $t \leq T$, and $v(x, t) < \inf \tilde{w}$ if t is less than some $t_\varepsilon \in \mathbb{R}$, that we can suppose to be smaller than T. We can therefore apply the comparison principle between v and $\tilde{w}$ in $\Omega_\delta \times [t_\varepsilon, T]$ and deduce in particular that

$$\forall\, x \in \Omega_\delta, \quad \tilde{w}(x, T) \geq v(x, T) = -\varepsilon d(x)^{-1} + \tilde{m}\,.$$

By the arbitrariness of ε and T, we then infer that $\tilde{w} \geq \tilde{m}$ in $\Omega_\delta \times \mathbb{R}$. Eventually, by (23), $\tilde{w}$ attains its global minimum $\tilde{m}$ somewhere in $\Omega \setminus \Omega_\delta$ at time 0. The parabolic strong maximum principle then implies that $\tilde{w} = \tilde{m}$ for all $t \leq 0$. We have shown that $w(\cdot, \cdot + t_n) \to \tilde{m}$ as $n \to \infty$, locally uniformly in $\Omega \times (-\infty, 0]$.

Step 2. *$w(\cdot, t_n) \to \tilde{m}$ uniformly in Ω as $n \to \infty$.*

Define $m_{n,\delta} := \min_{(y,s)\in(\Omega\setminus\Omega_\delta)\times[-1,0]} w(y, t_n + s)$. Observe that by Step 1, $\lim_n m_{n,\delta} = \tilde{m}$. For given $n \in \mathbb{N}$ consider the function

$$v(x,t) = m_{n,\delta} - 1 + d(x)^\rho + 1 - \delta^\rho - \varepsilon d(x)^{-1} + 2t\|w\|_\infty.$$

Using (14) and recalling our definition of δ, we find that

$$v_t + F[v] \le 2\|w\|_\infty + F[d(x)^\rho - 1] + \varepsilon F[-d(x)^{-1}] \le 0, \qquad x \in \Omega_\delta,\ t \in (-1,0).$$

Moreover $\limsup_{x\to\partial\Omega} v(x,t)d(x) \le 0$, uniformly in $t \in (-1,0)$, and

$$\forall x \in \Omega_\delta, \quad v(x,-1) \le m_{n,\delta} - 2\|w\|_\infty \le -\|w\|_\infty \le w(x, t_n - 1).$$

Finally, if $d(x) = \delta$ and $t \in [-1,0]$, then $v(x,t) \le m_{n,\delta}$. Then the standard comparison principle yields

$$\forall x \in \Omega_\delta,\ s \in [-1,0], \quad w(x, t + t_n) \ge m_{n,\delta} + d(x)^\rho - \delta^\rho - \varepsilon d(x)^{-1} + 2s\|w\|_\infty.$$

Letting $\varepsilon \to 0$ and computing the inequality at $s = 0$, we obtain

$$w(x,t_n) \ge m_{n,\delta} + d(x)^\rho - \delta^\rho \ge m_{n,\delta} - \delta^\rho \qquad x \in \Omega_\delta.$$

This implies that

$$m_{n,\delta} - \delta^\rho \le \inf_{x\in\Omega} w(x,t_n) \le m_{n,\delta}.$$

So, letting $n \to +\infty$, by the arbitrariness of δ we get

$$\liminf_n \inf_{x\in\Omega} w(x,t_n) = \tilde{m}.$$

Again by Step 1, as $n \to +\infty$ we have

$$M_{n,\delta} := -\max_{(y,s)\in(\Omega\setminus\Omega_\delta)\times[-1,0]} w(y,t_n+s) = \min_{(y,s)\in(\Omega\setminus\Omega_\delta)\times[-1,0]} (-w(y,t_n+s)) \to -\tilde{m}.$$

Hence repeating the same argument as above for $-w$ (exchanging the role of u_1, u_2) we get

$$\liminf_n \inf_{x\in\Omega} (-w(x,t_n)) = -\tilde{m}, \qquad \text{i.e.} \qquad \limsup_n \sup_{x\in\Omega} w(x,t_n) = \tilde{m}.$$

This concludes the proof of the step.

Step 3. *$w(\cdot,t) \to \tilde{m}$ uniformly in Ω as $t \to +\infty$.*
Define

$$\underline{m}(t) := \inf_{x\in\Omega} w(x,t), \qquad \overline{m}(t) := \sup_{x\in\Omega} w(x,t).$$

For any $s > 0$, the function $u_2 + \underline{m}(s)$ is a subsolution to the first equation in (13) and lies below u_1 at time $t = s$. Hence, Lemma 4.1 implies that $u_2 + \underline{m}(s) \le u_1$ for all $t \ge s$, from which we deduce that $t \mapsto \underline{m}(t)$ is nondecreasing. Analogously, $u_2 + \overline{m}(s) \ge u_1$ for all $t \ge s$, whence $t \mapsto \overline{m}(t)$ is nonincreasing. It follows that

$$\lim_{t\to+\infty} \underline{m}(t) = \lim_{n\to\infty} \underline{m}(t_n) = \tilde{m} = \lim_{n\to\infty} \overline{m}(t_n) = \lim_{t\to+\infty} \overline{m}(t).$$

This concludes the proof. $\square$

Corollary 1. *Let $u_0 \in L^\infty(\Omega) \cap C(\Omega)$ and let u be the unique solution to (13) satisfying the boundary control (2). Then there exists a constant K, depending only on $\|u_0\|_\infty$, such that*

$$u(x,t) + ct - \chi(x) + K \to 0 \qquad \text{as } t \to +\infty, \ \text{uniformly in } x \in \Omega,$$

where (c,χ) is the bounded solution to (19) normalized by $\sup\chi = 0$.

Proof. The result is a straightforward application of Theorem 6.1 to $u_1(x,t) := u(x,t)$ and $u_2(x,t) := \chi(x) - ct$, once we check the assumptions.

We recall that χ is bounded, by Proposition 2. Then, by the comparison principle of Lemma 4.1, we have that

$$-\|u_0\|_\infty \leq u(x,t) - (\chi(x) - ct) \leq \|u_0\|_\infty - \inf \chi.$$

This implies that $u_1 - u_2$ is bounded. The same inequality also implies that $u(x,t) + ct$ is bounded in $\Omega \times (0,+\infty)$. So, both $u(x,t) + ct$ and $\chi(x)$ are globally bounded solutions to $\tilde{u}_t + H[\tilde{u}] = c$. Let $\tilde{u}$ be a solution to $\tilde{u}_t + H[\tilde{u}] = c$, such that $\tilde{u} \in L^\infty(\Omega \times (0,\infty))$. Arguing as in Theorem 4.2, by Theorem 1.1 in [18], we infer that, for all $\tau > 0$ and all $\Omega' \subset\subset \Omega$, there exist $\theta, C > 0$ such that $\|D^2\tilde{u}\|_{C^{\theta,\theta/2}(\Omega' \times (\tau,+\infty))} \leq C$. Hence, using the equation, we eventually find that $\tilde{u} \in C^{2+\theta,1+\theta/2}(\Omega' \times (\tau,+\infty))$. In particular, we have that this estimate holds for both $u(x,t) + ct$ and χ, which implies that the hypothesis (22) is satisfied. □

REFERENCES

[1] M. Bardi, A. Cesaroni and L. Rossi, Nonexistence of nonconstant solutions of some degenerate Bellman equations and applications to stochastic control, *ESAIM Control Optim. Calc. Var.*, to appear.

[2] G. Barles and J. Burdeau, The Dirichlet problem for semilinear second-order degenerate elliptic equations and applications to stochastic exit control problems, *Comm. Part. Diff. Eq.*, **20** (1995), 129–178.

[3] G. Barles, A. Porretta and T.T. Tchamba, On the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton-Jacobi equations, *J. Math. Pures Appl.*, **94** (2010), 497–519.

[4] G. Barles and P.E. Souganidis, Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations, *SIAM J. Math. Anal.*, **32** (2001), 1311–1323.

[5] H. Berestycki, I. Capuzzo Dolcetta, A. Porretta and L. Rossi, Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators, *J. Math. Pures Appl.*, **103** (2015), 1276–1293.

[6] F. Cagnetti, D. Gomes, H. Mitake and H.V. Tran, A new method for large time behavior of degenerate viscous Hamilton–Jacobi equations with convex Hamiltonians, *Ann. Inst. H. Poincare Anal. Non Lineaire*, **32** (2015), 183–200.

[7] M.G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)*, **27** (1992), 1–67.

[8] F. Da Lio, Large time behavior of solutions to parabolic equations with Neumann boundary conditions, *J. Math. Anal. Appl.*, **339** (2008), 384–398.

[9] G. Fichera, *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I.*, **5** (1956), 1–30.

[10] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1988.

[11] H. Ishii and P. Loreti, A class of stochastic optimal control problems with state constraints, *Indiana Univ. Math. J.*, **51** (2002), 1167–1196.

[12] J.M. Lasry and P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, *Math. Ann.*, **283** (1989), 583–630.

[13] O. Ley and V.D. Nguyen, Large time behavior for some nonlinear degenerate parabolic equations, *J. Math. Pures Appl.*, **102** (2014), 293–314.

[14] T. Leonori and A. Porretta, Gradient bounds for elliptic problems singular at the boundary, *Arch. Ration. Mech. Anal.*, **202** (2011), 663–705.

[15] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ 1996.

[16] M.V. Safonov, On the classical solution of Bellman's elliptic equation, *Sov. Math. Dokl.*, **30** (1984), 482–485.

[17] N.S. Trudinger, On regularity and existence of viscosity solutions of nonlinear second order, elliptic equations, Partial differential equations and the calculus of variations, Vol. II, 939–957, *Progr. Nonlinear Differential Equations Appl.*, 2, Birkhauser Boston, Boston, MA, 1989.
[18] G. Tian and X.J. Wang, A priori estimates for fully nonlinear parabolic equations, *Int. Math. Res. Notes*, **169** (2012), 1–21.

Received September 2015; revised January 2016.

E-mail address: castorin@math.unipd.it
E-mail address: annalisa.cesaroni@unipd.it
E-mail address: lucar@math.unipd.it