

Contents

Preface	vii
Introduction to optimal transport	xiv
Acknowledgements	xix
Notation	xxv
1 Primal and dual problems	1
1.1 Kantorovich and Monge problems	1
1.2 Duality	9
1.3 The case $c(x, y) = h(x - y)$ for h strictly convex and the existence of an optimal T	13
1.3.1 The quadratic case in \mathbb{R}^d	16
1.3.2 The quadratic case on the flat torus	18
1.4 Counterexamples to existence	20
1.5 Kantorovich as a relaxation of Monge	21
1.6 c -concavity, duality and optimality	25
1.6.1 Convex and c -concave functions	25
1.6.2 c -Cyclical monotonicity and duality	28
1.6.3 A direct proof of duality	35
1.6.4 Sufficient conditions for optimality and stability	37
1.7 Discussion	41
1.7.1 Probabilistic interpretation	41
1.7.2 Polar factorization	42
1.7.3 Matching problems and economic interpretations	44
1.7.4 Multi-marginal transport problems	48
1.7.5 Martingale optimal transport and financial applications	51
1.7.6 Monge-Ampère equations and regularity	54
2 One-dimensional issues	59
2.1 Monotone transport maps and plans in 1D	59
2.2 The optimality of the monotone map	63
2.3 The Knothe transport	67

2.4	Knothe as a limit of Brenier maps	72
2.5	Discussion	79
2.5.1	Histogram equalization	79
2.5.2	Monotone maps from 1D constructions	81
2.5.3	Isoperimetric inequality via Knothe or Brenier maps	83
3	L^1 and L^∞ theory	87
3.1	The Monge case, with cost $ x - y $	87
3.1.1	Duality for distance costs	88
3.1.2	Secondary variational problem	89
3.1.3	Geometric properties of transport rays	91
3.1.4	Existence and nonexistence of an optimal transport map	99
3.1.5	Approximation of the monotone transport	101
3.2	The supremal case, L^∞	104
3.3	Discussion	109
3.3.1	Different norms and more general convex costs	109
3.3.2	Concave costs (L^p , with $0 < p < 1$)	113
4	Minimal flows	121
4.1	Eulerian and Lagrangian points of view	121
4.1.1	Static and dynamical models	121
4.1.2	The continuity equation	123
4.2	Beckmann's problem	127
4.2.1	Introduction, formal equivalences, and variants	127
4.2.2	Producing a minimizer for the Beckmann Problem	129
4.2.3	Traffic intensity and traffic flows for measures on curves	134
4.2.4	Beckman problem in one dimension	140
4.2.5	Characterization and uniqueness of the optimal w	142
4.3	Summability of the transport density	144
4.4	Discussion	151
4.4.1	Congested transport	151
4.4.2	Branched transport	162
5	Wasserstein spaces	177
5.1	Definition and triangle inequality	179
5.2	Topology induced by W_p	183
5.3	Curves in \mathbb{W}_p and continuity equation	187
5.3.1	The Benamou-Brenier functional \mathcal{B}_p	189
5.3.2	AC curves admit velocity fields	192
5.3.3	Regularization of the continuity equation	194
5.3.4	Velocity fields give Lipschitz behavior	197
5.3.5	Derivative of W_p^p along curves of measures	198
5.4	Constant-speed geodesics in \mathbb{W}_p	202
5.5	Discussion	207

5.5.1	The W_∞ distance	207
5.5.2	Wasserstein and negative Sobolev distances.....	209
5.5.3	Wasserstein and branched transport distances.....	211
5.5.4	The sliced Wasserstein distance.....	214
5.5.5	Barycenters in \mathbb{W}_2	215
6	Numerical methods	219
6.1	Benamou-Brenier	220
6.2	Aggenent-Hacker-Tannenbaum	225
6.3	Numerical solution of Monge-Ampère	232
6.4	Discussion	235
6.4.1	Discrete numerical methods.....	235
6.4.2	Semidiscrete numerical methods.....	242
7	Functionals over probabilities	249
7.1	Semi-continuity	250
7.1.1	Potential, interaction, transport costs, dual norms.....	250
7.1.2	Local functionals.....	254
7.2	Convexity, first variations, and subdifferentials	260
7.2.1	Dual norms.....	263
7.2.2	Transport costs	264
7.2.3	Optimality conditions.....	267
7.3	Displacement convexity	269
7.3.1	Displacement convexity of \mathcal{V} and \mathcal{W}	270
7.3.2	Displacement convexity of \mathcal{F}	271
7.3.3	Convexity on generalized geodesics	275
7.4	Discussion	276
7.4.1	A case study: $\min F(\varrho) + W_2^2(\varrho, v)$	276
7.4.2	Brunn-Minkowski inequality.....	279
7.4.3	Urban equilibria	280
8	Gradient flows	285
8.1	Gradient flows in \mathbb{R}^d and in metric spaces	285
8.2	Gradient flows in \mathbb{W}_2 , derivation of the PDE	290
8.3	Analysis of the Fokker-Planck case	293
8.4	Discussion	301
8.4.1	EVI, uniqueness, and geodesic convexity	301
8.4.2	Other gradient flow PDEs	304
8.4.3	Dirichlet boundary conditions	311
8.4.4	Evolution PDEs: not only gradient flows	313
Exercises	325
References	339
Index	351