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# On local and global minimizers of some non-convex variational problems

**Ph.D. Thesis**

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# Contents

# Introduction

Many physical systems are modeled mathematically as variational problems, where the observed configurations are expected to be local or global minimizers of a suitable energy. Such an energy can be very complicated, as well as the physical phenomenon under investigation. Thus, as a starting point, it is useful to focus on some simple models, which however capture the main features.

In this thesis we concentrate on two kinds of energies, that can be both viewed as nonlocal variants of the perimeter functional. The nonlocality consists in a bulk term, that in one case is given by an elastic energy, while in the other by a long-range interaction of Coulumbic type. The physical systems modeled by these energies displays a rich variety of observable patterns, as well as the formation of morphological instabilities of interfaces between different phases. These phenomena can be mathematically understood as the competition between the local geometric part of the energy, *i.e.*, the perimeter, and the nonlocal one. Indeed, while the first one prefers configurations in which the interfaces are regular and as small as possible, the latter, instead, favors more irregular and oscillating patterns. Thus, finding global or local minima of these energies is a highly nontrivial task, and indeed many big issues about them are still open. The aim of this thesis is to give a contribution to the investigation of such issues.

In particular, we study the following two energies:

- the Mumford-Shah functional, that is the prototype of the so called *free discontinuity problems*. Introduced for the first time in [**50**] in the context of image segmentation, nowadays is also used in the variational formulation of fracture mechanics;
- a model for diblock copolymer, where the energy is a nonlocal variant of the perimeter functional, where the nonlocality is given by a long-range repulsive interaction of Coulumbic type.

A common mathematical feature of these two energies is a deep lack of convexity. Thus, it is important to look for sufficient conditions for local and global minimality. In this thesis we undertake such investigation by adopting the point of view introduced by Cagnetti, Mora and Morini in [**9**], where a study of second order conditions for free-discontinuity problems has been initiated (in the case of the Mumford-Shah functional), and by Fusco and Morini in [**27**] (for a model of epitaxially growth). See also [**1**, **6**, **10**, **35**] for other related works.

We now describe in details the obtained results for the two energies we studied.

**A model for diblock copolymer.**

Diblock copolymers are a class of two-phase materials extensively used in the applications for their properties. They are composed by linear-chain macromolecules, each consisting of two thermodynamically incompatible subchains joined covalently. Due to this imcompatibility, the two phases try to separate as much as possible; on the other hand, because of the chemical bonds, only partial separation can occor at a suitable mesoscale. Such a partial segregation of the two ohases produces very complex patterns, that are experimentally observed to be (quasi) periodic at an intrinsic scale. The structure of these patterns depends strongly on the volume fraction of a phase with respect to the other, but they are seen to be very closed to *periodic surfaces with constant mean curvature*, as shown in Figure 1. All these diblock copolymers belong to a broad family of materials, usually called *soft materials*, which show a high degree of order at a suitable length scale, although their fluidlike disorder on the molecular scale. Their complex structures can give these materials many desiderable properties. It is thus useful to better understand the formation of these patterns.

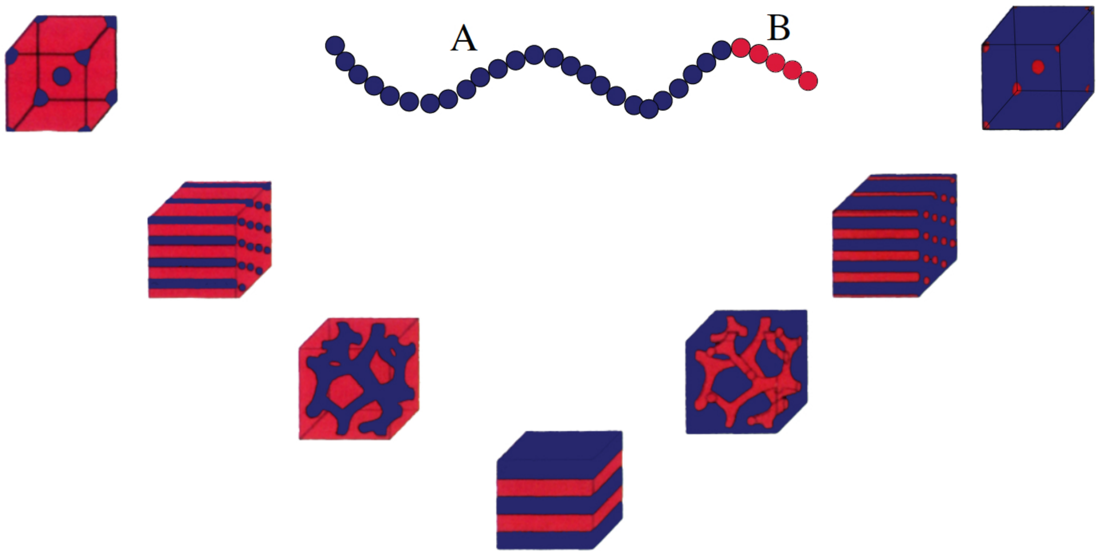


FIGURE 1. The typical patterns that are observed according to an increasing value of the volume fraction.

To model microphase separation of diblock copolymers, Ohta and Kawasaki proposed in [55] the following energy:

$$OK_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x \quad + \quad \frac{1}{\varepsilon}\int_\Omega (u^2-1)^2 \mathrm{d}x \\ + \quad \gamma \int_\Omega \int_\Omega G(x,y)\big(u(x)-m\big)\big(u(y)-m\big)\mathrm{d}x\mathrm{d}y\,, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set, $G$ is the Green's function for $-\triangle$, $u \in H^1(\Omega)$, and $m := \fint_\Omega u$. The function $u$ is a density distribution, and the two phases of the chain correspond to the regions where $u \approx -1$ and $u \approx +1$ respectively. See [13] for a rigorous derivation of the Ohta-Kawasaki energy from first principles, and [52] for a physical background on long-range interaction energies. According to the theory proposed by Ohta and Kawasaki, we expect observable configurations to be global (or local) minimizers of the energy (0.1).

Since the parameter $\varepsilon$ is usually small, from the mathematical point of view it is more convenient to consider the variational limit of the energy $OK_\varepsilon$. If we let the parameter $\varepsilon$ going to zero, we obtain a functional that, in the periodic setting, turns out to be (0.2). If we also consider the volume of one of the two phases to disappear, it has been proved in [11, 12] that the resulting variational limit is the functional (0.3). We studied both these functionals, from different perspectives.

**The periodic case.** We start by describing the results obtained in the first case. We study some properties of critical points of the functional

$$\mathcal{F}^\gamma(E) := \mathcal{P}_{\mathbb{T}^N}(E) + \gamma \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} G_{\mathbb{T}^N}(x,y)u^E(x)u^E(y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{0.2}$$

where $\gamma \geq 0$, $E$ is a subset of the $N$-dimensional flat torus $\mathbb{T}^N$, $\mathcal{P}_{\mathbb{T}^N}(E)$ denotes the perimeter of $E$ in $\mathbb{T}^N$, $u^E(x) := \chi_E(x) - \chi_{\mathbb{T}^N\setminus E}(x)$, and $G_{\mathbb{T}^N}$ is the unique solution of

$$-\triangle_y G_{\mathbb{T}^N}(x,\cdot) = \delta_x(\cdot) - 1 \quad \text{in } \mathbb{T}^N, \qquad \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} G_{\mathbb{T}^N}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0\,.$$

We will refer to the first term of (0.2) as the *local term*, while to the second one as the *nonlocal term*. The latter will be denoted with $\gamma\mathcal{NL}(E)$. We notice that the local term favours the formation of large regions of pure phase, while the nonlocal one prefers to break each phase into several connected components that tries to separate from each other as much as possible.

Proving analitically that global minimizers of (0.2) or (0.1) are (quasi) periodic is a formidable task. Indeed, so far, the best result in this direction is the work [3] by Alberti, Choksi and Otto, where it is proved that global minimizers of (0.2) in the whole $\mathbb{R}^N$ under a volume constraint, *i.e.* for a fixed $m$, present an uniform energy distribution of each component of the energy, on suitable big cubes. This result has been extended to the case of the functional (0.1) by Spadaro in [64]. Moreover, the structure of global minimizers has been investigated by many authors (see, for example, [11, 12, 18, 30, 31, 53, 65, 67]), but only in some asymptotic regimes, *i.e.*, when the parameter $\gamma$ is small or $m \approx \pm 1$.

A more reasonable, but still highly nontrivial, pourpose is to exhibit a class of local minimizers of the energies (0.2) and (0.1) that look like the observed configurations. Among the results in this direction we would like to recall the works by Ren and Wei ([60, 57, 56, 58, 59]), where they construct explicit critical configurations of the sharp interface energy, with lamellar, cylindrical and spherical patterns. They also provide a regime of the parameters that ensures the (linear) stability of such configurations. The natural notion of stability for (0.2) has been introduced by Choksi and Strernberg in [15], and it has been subsequently proved by Acerbi, Fusco and Morini in [1], that critical and strictly stable (namely with strictly positive second variation) configurations are local minimizers in the $L^1$ topology.

The aim of our work is to collect some new observations on critical points of the sharp interface energy (0.2).

We start by showing, in Proposition 2.32, that critical point are always local minimizers with respect to perturbations with sufficiently small support. This minimality-in-small-domains property of critical points is shared by many functional of the Calculus of Variations, but to the best of our knowledge it has been never been observed before for the Ohta-Kawasaki energy.

The second result (see Proposition 2.34) shows that the property of being critical and stable is preserved under small perturbations of the parameter $\gamma$. More precisely, we show that, given $\bar\gamma \geq 0$ and a strictly stable critical point $E$ of the functional $\mathcal{F}^{\bar\gamma}$, we can find a (unique) family $(E_\gamma)$ of smoothly varying uniform local minimizers of $\mathcal{F}^\gamma$ for $\gamma$ ranging in a small neighborhood of $\bar{\gamma}$. The procedure to construct such a family is purely variational and based on showing that the local minimality criterion provided in [1] can be made uniform with respect to the parameter $\gamma$ and with respect to critical sets ranging in a sufficiently small $C^1$-neighborhood of a given strictly stable set $E$. Such an observation, which has an independent interest, is proven in Proposition 2.34.

The above stability property is used to establish the main result of this paper (see Theorem 2.46): given $\bar{\gamma} > 0$ and $\varepsilon > 0$ and a subset $E$ of the torus $\mathbb{T}^N$ such that $\partial E$ is a strictly stable constant mean curvature hypersurface, we show that it is possible to find an integer $k = k(\bar{\gamma}, \varepsilon)$ and a $1/k$-periodic critical point of $\mathcal{F}^{\bar{\gamma}}_{\mathbb{T}^N}$, whose shape is $\varepsilon$-close (in a $C^1$-sense) to the $1/k$-rescaled version of $E$ and whose mean curvature is almost constant. Moreover, such a critical point is an isolated local minimizer with respect to $(1/k)$-periodic perturbations. In words, the above result says that it is possible to construct local minimizing periodic critical points of the energy (0.1), whit a shape closely resembling that of any given stictly stable periodic constant mean curvature surface.

This result is close in spirit to the aforementioned results by Ren and Wei. There are however some important differences: First of all, they work in the Neumann setting, while we are in the periodic one. Moreover, while their constructions are based on the Liapunov-Schmidt reduction method and require rather involved and (ad hoc for each specific example) spectral computations, we use a purely variational approach that works for all possible strictly stable patterns. However, the price to pay for such a generality is a less precise description of the parameter ranges for which the existence of the desired critical points can be established.

Another important consequence of our variational procedure is that it allows to show (see Proposition 2.47) that all the constructed critical points can be approximated by critical points of the $\varepsilon$-diffuse energy (0.1). This is done by $\Gamma$-convergence arguments in the spirit of the Kohn and Sternberg theory, see [40]. We conclude by remarking that numerical and experimental evidence suggest the following general structure for global minimizers: the nonlocal term determines an intrinsic scale of periodicity (the larger is $\gamma$ the smaller is the periodicity scale), while the shape of the global minimizer inside the periodicity cell is dictated by the perimeter term. Although we are very far from an analytical validation of such a picture, our result allows to construct a class of (locally minimizing) critical point that display the above structure.

**A nonlocal isoperimetric problem on $\mathbb{R}^N$.** We now describe the results obtained for the second type of variational limit of (0.1). For a parameter $\alpha \in (0, N-1)$, $N \geq 2$, we consider the following functional defined on measurable sets $E \subset \mathbb{R}^N$:

$$\mathcal{F}(E) := \mathcal{P}(E) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x \; \mathrm{d}y \,, \tag{0.3}$$

where $\mathcal{P}(E)$ is the perimeter of the set $E$ and the second term, the so called *nonlocal term*, will be hereafter denoted by $\mathcal{NL}_\alpha(E)$. We are interested in the study of the volume constrained minimization problem

$$\min\{\mathcal{F}(E) \; : \; |E| = m\} \,, \tag{0.4}$$

and in its dependence on the parameters $\alpha$ and $m > 0$.

Beyond its relation with the aforementioned model for diblock copolymers, the above problem is interesting because the energy (0.3) appears in the modeling of many other different physical phenomena. In particular, the most physically relevant case is in three dimensions with $\alpha = 1$, where the nonlocal term corresponds to a Coulombic repulsive interaction: one of the first examples is the celebrated Gamow's water-drop model for the constitution of the atomic nucleus (see [28]).

From a mathematical point of view, functionals of the form (0.3) recently drew the attention of many authors (see for example [1, 18, 22, 26, 30, 31, 34, 35, 36, 37, 38, 43, 53, 54]). The main feature of the energy (0.3) is the presence of two competing terms, the sharp interface energy and the long-range repulsive interaction. Indeed, while the first term is minimized by the ball (by the isoperimetric inequality), the nonlocal term is in fact maximized by the ball, as a consequence of the Riesz's rearrangement inequality (see [42, Theorem 3.7]), and favours scattered configurations.

In order to have an idea of the behaviour we would expect for such a functional, we notice that, calling $\widetilde{E} := \left(\frac{|B_1|}{|E|}\right)^{\frac{1}{N}} E$, where $B_1$ is the unit ball of $\mathbb{R}^N$, the functional reads as

$$\mathcal{F}(E) = \left(\frac{|E|}{|B_1|}\right)^{\frac{N-1}{N}} \left[\mathcal{P}(\widetilde{E}) + \left(\frac{m}{|B_1|}\right)^{\frac{N-\alpha+1}{N}} \mathcal{NL}_\alpha(\widetilde{E})\right].$$

Hence the parameter $m$ appearing in the volume constraint can be normalized and replaced by a coefficient $\gamma$ in front of the nonlocal energy: one can study the minimization problem, equivalent to (0.4),

$$\min\{\mathcal{F}_{\alpha,\gamma}(E) \,:\, |E| = |B_1|\}, \tag{0.5}$$

where we define $\mathcal{F}_{\alpha,\gamma}(E) := \mathcal{P}(E) + \gamma\mathcal{NL}_\alpha(E)$. It is clear from this expression that, for small masses, *i.e.*, small $\gamma$'s, the interfacial energy is the leading term and this suggests that in this case the functional should behave like the perimeter, namely we expect the ball to be the unique solution of the minimization problem, as in the isoperimetric problem; on the other hand, for large masses the nonlocal term becomes prevalent and should causes the existence of a solution to be not guaranteed. But this is just heuristic!

What was proved, in some particular cases, is that the functional $\mathcal{F}$ is uniquely minimized (up to translations) by the ball for every value of the volume below a critical threshold: in the planar case in [36], in the case $3 \leq N \leq 7$ in [37], and in any dimension $N$ with $\alpha = N-2$ in [34]. Moreover, the existence of a critical mass above which the minimum problem does not admit a solution was established in [36] in dimension $N=2$, in [37] for every dimension and for exponents $\alpha \in (0,2)$, and in [43] in the physical interesting case $N=3$, $\alpha=1$.

In [7] we provide a contribution to a more detailed picture of the nature of the minimization problem (0.4). In particular, we follow the approach used in [1] for the periodic case with $\alpha = N-2$, which is based on the positivity of the second variation of the functional, in order to obtain a local minimality criterion. This allows us to show the following new results: first, we prove that the ball is the unique global minimizer for small masses, for every values of the parameters $N$ and $\alpha$ (Theorem 1.10); moreover, for $\alpha$ small we also show that the ball is the unique global minimizer, as long as a minimizer exists (Theorem 1.11), and that in this regime we can write $(0,\infty) = \cup_k (m_k, m_{k+1}]$, with $m_{k+1} > m_k$, in such a way that for $m \in [m_{k-1}, m_k]$ a minimizing sequence for the functional is given by a configuration of at most $k$ disjoint balls with diverging mutual distance (Theorem 1.12).

Finally, we also investigate the issue of *local minimizers*, that is, sets which minimize the energy with respect to competitors sufficiently close in the $L^1$-sense (where we measure the distance between two sets by the quantity (1.5), which takes into account the translation invariance of the functional). We show the existence of a volume threshold below which the ball is an isolated local minimizer, determining it explicitly in the three dimensional case with a Newtonian potential (Theorem 1.9). The energy landscape of the functional $\mathcal{F}$, including the information coming from our analysis and from previous works, is illustrated in Figure 2.

After our work was completed, a deep analysis comprising also the case of the parameter $\alpha \in [N-1, N)$, and including the possibility for the perimeter term to be a nonlocal $s$-perimeter, has been performed in the paper [25].

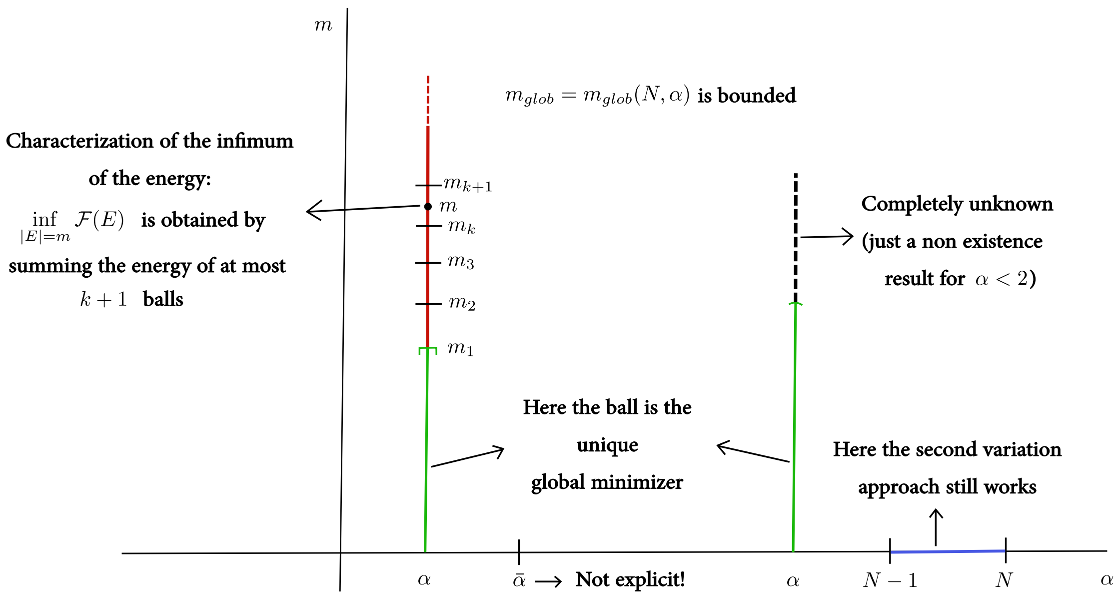


FIGURE 2. Energy landscape of the functional $\mathcal{F}_{\alpha,\gamma}$.

### The Mumford-Shah functional.

We present here the first part of an ongoing project aimed at providing a local minimality criterion, based on a second variation approach, for triple point configurations of the Mumford-Shah functional.

The (homogeneous) *Mumford-Shah functional* in the plane is defined as follows:

$$\mathcal{MS}(u,\Gamma) := \int_{\Omega\setminus\Gamma} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{N-1}(\Gamma\cap\Omega)\,, \tag{0.6}$$

where $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure, and $(u,\Gamma)$ is a pair where $\Gamma$ is a closed subset of $\mathbb{R}^2$ and $u \in H^1(\Omega\setminus\Gamma)$.

The existence of global minimizers in arbitrary dimension has been provided by De Giorgi, Carrieo and Leaci in [21] (for other proof see, for istance, [44] and, for dimension 2, [19, 48]) In the seminal paper [51] it has been conjectured that, if $(u,\Gamma)$ is a minimizing pair, then that the set $\Gamma$ is made by a finite union of $C^1$ arcs. Given this structure for grant, it is not difficult to prove (see [51]) that the only possible singularities of the set $\Gamma$ can be of the following two types:

- $\Gamma$ ends in an interior point (the so called *crack type*),
- three regular arcs meeting in an interion point $x_0$ with equal angles of $2\pi/3$ (the so called *triple point*).

Although several results on the regularity of the discountinuity set $\Gamma$ have been obtained (but we will not recall them here), the conjecture is still open.

As for the previous case, the deep lack of convexity of the functional (0.6) naturally leads one to ask what conditions imply that critical configurations as above are local minimizers.

The study of such a conditions has been initiated by Cagnetti, Mora and Morini in [9], where they deal with the regular part of the discontinuity set. In particular they introduce a suitable notion of second variation and prove that the strict positivity of the associated quadratic form is a sufficient condition for the local minimality with respect to small $C^2$ perturbations of the discontinuity set $\Gamma$.

Subsequently, the above result has been strongly improved by Bonacini and Morini in [8], where it is shown that if $(u,\Gamma)$ is a critical pair for (0.6) with strictly positive second variation, then it locally minimizes then it locally minimizes the functional with respet to small $L^1$-perturbations of $u$, namely that there exists $\delta > 0$ such that

$$MS(u,\Gamma) < MS(v,\Gamma')$$

for all admissible pairs $(v,\Gamma')$ satisfying $0 < \|u-v\|_{L^1} \leq \delta$.

Among other results on local and global minimality criterions, we would like to recall the important work [2] of Alberti, Bouchitté and Dal Maso, where they introduce a general calibration method for a family of non convex variational problems. In particular they applied this method to the case of the Mumford-Shah functional to obtain minimality results for some particular configurations. Moreover, Mora in [46] used that calibration technique to prove that a critical configuration $(u,\Gamma)$, where $\Gamma$ is made by three line segments meeting at the origin with equal angles, is a minimizer of the Mumford-Shah energy in a suitable neighborhood of the origin, with respect to its Dirichlet boundary conditions. Finally, we recall that the same method has been used by Mora and Morini in [47], and by Morini in [49] (in the case of the non homogeneous Mumford-Shah functional), to obtain local and global minimality results in the case of a regular curve $\Gamma$.

Our aim was to continue the investigation of second order sufficient conditions, by considering for the first time the case of configurations with a singularity, namely triple point configuration.

The plan is the following. In Section 3.3 we compute, as in [9], the second variation of the functional $\mathcal{MS}$ at a triple point configuration $(u,\Gamma)$, with respect to a one-parameter family of (sufficiently regular) diffeomorphisms $(\Phi_t)_{t\in(-1,1)}$, where each $\Phi_t$ equals the identity in the part of $\partial\Omega$ where we impose the Dirichlet condition and $\Phi_0 = \mathrm{Id}$. The idea is then to consider for each time $t \in (-1,1)$ the pair $(u_t,\Gamma_t)$, where $\Gamma_t := \Phi_t(\Gamma)$, and $u_t \in H^1(\Omega\backslash\Gamma_t)$ minimizes the Dirichlet energy with respect to the given boundary conditions.

We show that the second variation can be written as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{MS}(u_t,\Gamma_t)_{|t=0} = \partial^2\mathcal{MS}(u,\Gamma)\big[(X\cdot\nu^1, X\cdot\nu^2, X\cdot\nu^3)\big] + R\,, \tag{0.7}$$

where $\partial^2\mathcal{MS}(u,\Gamma)$ is a nonlocal (explicitly given) quadratic form, $X$ is the velocity field at time 0 of the flow $t \mapsto \Phi_t$ (see Definition 3.4), and $\nu^i$ is the normal vector field on $\Gamma^i$. Moreover, the remainder $R$ vanishes whenever $(u,\Gamma)$ is a critical triple point. Thus, in particular, if $(u,\Gamma)$ is a local minimizer with respect to smooth perturbations of $\Gamma$, then the quadratic form $\partial^2\mathcal{MS}(u,\Gamma)$ has to be nonnegative.

Next we address the question as to whether the strict positivity of $\partial^2\mathcal{MS}(u,\Gamma)$, with $(u,\Gamma)$ critical, is a sufficient condition for local minimality. The main result (see Theorem 3.24) is the following: if $(u,\Gamma)$ is a strictly stable critical pair, then there exists $\delta > 0$ such that

$$\mathcal{MS}(u,\Gamma) < \mathcal{MS}(v,\Phi(\Gamma))\,,$$

for any $W^{2,\infty}$-diffeomorphism $\Phi : \bar\Omega \to \bar\Omega$ and any function $v \in H^1\big(\Omega\setminus\Phi(\Gamma)\big)$ satisfying the proper boundary conditions, provided that $\|\Phi - \mathrm{Id}\|_{W^{2,\infty}} \leq \delta$, that $\Phi(\Gamma) \neq \Gamma$, and $D\Phi(x_0) = \lambda\mathrm{Id}$ for some $\lambda \neq 0$. Here $x_0$ denotes the triple point. We remark that the last assumption is just technical and we believe that it can be removed by refining our construction. The above result can be seen as the analog for triple points configurations of the minimality result established in [9] in the case of regular discontinuity sets.

From the technical point of view the presence of the singularity makes the problem considerably more challenging. The main difficulty lies in the construction of a suitable family of diffeomorphisms $(\Phi_t)_{t\in[0,1]}$ connecting the critical triple point configuration with the competitor, in such a way that the "tangential" part along $\Phi_t(\Gamma)$ of the velocity field $X_t$ of the flow $t\mapsto\Phi_t$ is controlled by its normal part. Moreover, one also has to make sure that the $C^2$-closeness to the identity is preserved along the way. This turns out to be a highly nontrivial task, due to the presence of the triple junction which poses severe regularity problems (see also [17] for a related problem in the context of area functional). Once such a construction is performed, one proceeds in the following way. Let $g(t) := \mathcal{MS}(u_t,\Gamma_t)$ and notice that, by criticality, we have $g'(0) = 0$. Thus, recalling (0.7), it is possible to write

$$\begin{aligned}\mathcal{MS}(v,\Phi(\Gamma)) - \mathcal{MS}(u,\Gamma) &= \int_0^1 (1-t)g''(t)\,\mathrm{d}t\\ &= \int_0^1 (1-t)\big(\partial^2\mathcal{MS}(u_t,\Gamma_t)[X_t\cdot\nu_t] + R_t\big)\,\mathrm{d}t. \qquad (0.8)\end{aligned}$$

If $\Gamma_t$ is sufficiently $C^2$-close to $\Gamma$, by the strict positivity assumption on $\partial^2\mathcal{MS}(u,\Gamma)$, we may conclude by continuity that

$$\partial^2\mathcal{MS}(u_t,\Gamma_t)[X_t\cdot\nu_t] \ge C\|X_t\cdot\nu_t\|^2\,.$$

Unfortunately, the remainder $R_t$ depends also on the tangential part of $X_t$. However, if the family $(\Phi_t)_t$ is properly constructed, on can ensure that such a tangential part is controlled by $X_t\cdot\nu_t$ and

$$|R_t| \le \varepsilon\|X_t\cdot\nu_t\|^2$$

for any $\varepsilon > 0$, provided that the $\Phi_t$'s are sufficiently $C^2$-close to the identity. Plugging the above two estimates into (0.8) one eventually concludes that, for a $\Phi$'s satisfying the above assumptions, $\mathcal{MS}(v,\Phi(\Gamma)) > \mathcal{MS}(u,\Gamma)$.

We conclude this introduction by observing that the above result represents just the first step of a more general strategy aimed at establishing the local minimality with respect to the $L^1$-topology in the spirit of [8], which will be the subject of future investigations.

**Organization of the thesis.** In Chapter 1 and Chapter 2 and we present the results about the functionals (0.5) and (0.2) respectively, while those relative to the Mumford-Shah functional can be found in Chapter 3. Some technical results needed in the presentation of the main contributions of this thesis will be proved in the Appendix at the end of each chapter. Moreover, at the beginning of a chapter, we will introduce the needed notations and the preliminaries.

Per me c'è solo il viaggio
su strade che hanno un cuore,
qualsiasi strada abbia un cuore.
Là io viaggio,
e l'unica sfida che valga
è attraversarla
in tutta la sua lunghezza.
Là io viaggio guardando,
guardando,
senza fiato.

*Carlos Castaneda,*
*Gli Insegnamenti di don Juan*

CHAPTER 1

# A nonlocal isoperimetric problem in $\mathbb{R}^N$

The aim of this chapter is to provide some information about the energy landscape of the family of functionals

$$\mathcal{F}(E) := \mathcal{P}(E) + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha}\, \mathrm{d}x\ \mathrm{d}y\,,$$

where $\mathcal{P}(E)$ is the perimeter of the set $E$ and the second term, the so called *nonlocal term*, will be hereafter denoted by $\mathcal{NL}_\alpha(E)$.

## 1.1. Statements of the results

We start our analysis with some preliminary observations about the features of the energy functional (0.3), before listing the main results of this work. For a finite perimeter set $E$, we will denote by $\nu_E$ the exterior generalized unit normal to $\partial^* E$, and we will not indicate the dependence on the set $E$ when no confusion is possible.

Given a measurable set $E \subset \mathbb{R}^N$, we introduce an auxiliary function $v_E$ by setting

$$v_E(x) := \int_E \frac{1}{|x-y|^\alpha}\,\mathrm{d}y \qquad \text{for } x \in \mathbb{R}^N. \tag{1.1}$$

The function $v_E$ can be characterized as the solution to the equation

$$(-\Delta)^s v_E = c_{N,s}\,\chi_E\,, \qquad s = \frac{N-\alpha}{2} \tag{1.2}$$

where $(-\Delta)^s$ denotes the fractional laplacian and $c_{N,s}$ is a constant depending on the dimension and on $s$ (see [23] for an introductory account on this operator and the references contained therein). Notice that we are interested in those values of $s$ which range in the interval $(\frac{1}{2}, \frac{N}{2})$. We collect in the following proposition some regularity properties of the function $v_E$.

Proposition 1.1. *Let $E \subset \mathbb{R}^N$ be a measurable set with $|E| \le m$. Then there exists a constant $C$, depending only on $N$, $\alpha$ and $m$, such that*

$$\|v_E\|_{W^{1,\infty}(\mathbb{R}^N)} \le C\,.$$

*Moreover, $v_E \in C^{1,\beta}(\mathbb{R}^N)$ for every $\beta < N-\alpha-1$ and*

$$\|v_E\|_{C^{1,\beta}(\mathbb{R}^N)} \le C'$$

*for some positive constant $C'$ depending only on $N$, $\alpha$, $m$ and $\beta$.*

Proof. The first part of the result is proved in [37, Lemma 4.4], but we repeat here the easy proof for the reader's convenience. By (1.1),

$$v_E(x) = \int_{B_1(x)\cap E} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y + \int_{E\setminus B_1(x)} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y \le \int_{B_1} \frac{1}{|y|^\alpha}\,\mathrm{d}y + m \le C.$$

By differentiating (1.1) in $x$ and arguing similarly, we obtain

$$|\nabla v_E(x)| \leq \alpha \int_E \frac{1}{|x-y|^{\alpha+1}}\,\mathrm{d}y \leq \alpha \int_{B_1} \frac{1}{|y|^{\alpha+1}}\,\mathrm{d}y + \alpha\, m \leq C.$$

Finally, by adding and subtracting the term $\frac{(x-y)|z-y|^\beta}{|x-y|^{\alpha+\beta+2}} - \frac{(z-y)|x-y|^\beta}{|z-y|^{\alpha+\beta+2}}$, we can write

$$\begin{aligned}
|\nabla v_E(x) - \nabla v_E(z)| &\leq \alpha \int_E \left| \frac{x-y}{|x-y|^{\alpha+2}} - \frac{z-y}{|z-y|^{\alpha+2}} \right| dy \\
&\leq \alpha \int_E \left( \frac{1}{|x-y|^{\alpha+\beta+1}} + \frac{1}{|z-y|^{\alpha+\beta+1}} \right) \big||x-y|^\beta - |z-y|^\beta\big|\,\mathrm{d}y \qquad (1.3)\\
&\quad + \alpha \int_E \left| \frac{(x-y)|z-y|^\beta}{|x-y|^{\alpha+\beta+2}} - \frac{(z-y)|x-y|^\beta}{|z-y|^{\alpha+\beta+2}} \right| \mathrm{d}y
\end{aligned}$$

Observe now that for every $v, w \in \mathbb{R}^N \setminus \{0\}$

$$\begin{aligned}
\left| \frac{v}{|v|}|w|^{\alpha+2\beta+1} - \frac{w}{|w|}|v|^{\alpha+2\beta+1} \right| &= \big| v\,|v|^{\alpha+2\beta} - w\,|w|^{\alpha+2\beta} \big| \leq C \max\{|v|,|w|\}^{\alpha+2\beta}|v-w| \\
&\leq C \max\{|v|,|w|\}^{\alpha+\beta+1}|v-w|^\beta
\end{aligned}$$

where $C$ depends on $N$, $\alpha$ and $\beta$. Using this inequality to estimate the second term in (1.3) we deduce

$$|\nabla v_E(x) - \nabla v_E(z)| \leq \alpha|x-z|^\beta \int_E \left( \frac{1}{|x-y|^{\alpha+\beta+1}} + \frac{1}{|z-y|^{\alpha+\beta+1}} + \frac{C}{\min\{|x-y|,|z-y|\}^{\alpha+\beta+1}} \right) \mathrm{d}y$$

which completes the proof of the proposition, since the last integral is bounded by a constant depending only on $N$, $\alpha$, $m$ and $\beta$. ☐

REMARK 1.2. In the case $\alpha = N-2$, the function $v_E$ solves the equation $-\Delta v_E = c_N\,\chi_E$, and the nonlocal term is exactly

$$\mathcal{NL}_{N-2}(E) = \int_{\mathbb{R}^N} |\nabla v_E(x)|^2\,\mathrm{d}x\,.$$

By standard elliptic regularity, $v_E \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p \in [1,+\infty)$.

The following proposition contains an auxiliary result which will be used frequently in the rest of this chapter.

PROPOSITION 1.3 (Lipschitzianity of the nonlocal term). *Given $\bar\alpha \in (0, N-1)$ and $m \in (0,+\infty)$, there exists a constant $c_0$, depending only on $N$, $\bar\alpha$ and $m$ such that if $E, F \subset \mathbb{R}^N$ are measurable sets with $|E|, |F| \leq m$ then*

$$|\mathcal{NL}_\alpha(E) - \mathcal{NL}_\alpha(F)| \leq c_0 |E \triangle F|$$

*for every $\alpha \leq \bar\alpha$, where $\triangle$ denotes the symmetric difference of two sets.*

PROOF. We have that

$$\begin{aligned}
\mathcal{NL}_\alpha(E) - \mathcal{NL}_\alpha(F) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left( \frac{\chi_E(x)(\chi_E(y)-\chi_F(y))}{|x-y|^\alpha} + \frac{\chi_F(y)(\chi_E(x)-\chi_F(x))}{|x-y|^\alpha} \right) \mathrm{d}x\mathrm{d}y \\
&= \int_{E\setminus F} \big(v_E(x) + v_F(x)\big)\,\mathrm{d}x - \int_{F\setminus E} \big(v_E(x) + v_F(x)\big)\,\mathrm{d}x \\
&\leq \int_{E\triangle F} \big(v_E(x) + v_F(x)\big)\,\mathrm{d}x \leq 2C\,|E\triangle F|,
\end{aligned}$$

where the constant $C$ is provided by Proposition 1.1, whose proof shows also that it can be chosen independently of $\alpha \leq \bar{\alpha}$. $\square$

The issue of existence and characterization of *global minimizers* of the problem

$$\min \left\{ \mathcal{F}(E) \,:\, E \subset \mathbb{R}^N,\, |E| = m \right\} , \tag{1.4}$$

for $m > 0$, is not at all an easy task. A principal source of difficulty in applying the direct method of the Calculus of Variations comes from the lack of compactness of the space with respect to $L^1$ convergence of sets (with respect to which the functional is lower semicontinuous). It is in fact well known that the minimum problem (1.4) does not admit a solution for certain ranges of masses.

Besides the notion of global minimality, we will address also the study of sets which minimize locally the functional with respect to small $L^1$-perturbations. By translation invariance, we measure the $L^1$-distance of two sets modulo translations by the quantity

$$\alpha(E,F) := \min_{x \in \mathbb{R}^N} |E \triangle (x + F)| \,. \tag{1.5}$$

Definition 1.4. We say that $E \subset \mathbb{R}^N$ is a *local minimizer* for the functional (0.3) if there exists $\delta > 0$ such that

$$\mathcal{F}(E) \leq \mathcal{F}(F)$$

for every $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\alpha(E,F) \leq \delta$. We say that $E$ is an *isolated local minimizer* if the previous inequality is strict whenever $\alpha(E,F) > 0$.

The first order condition for minimality, coming from the first variation of the functional (see (1.10), and also [15, Theorem 2.3]), requires a $C^2$-minimizer $E$ (local or global) to satisfy the Euler-Lagrange equation

$$H_{\partial E}(x) + 2\gamma v_E(x) = \lambda \qquad \text{for every } x \in \partial E \tag{1.6}$$

for some constant $\lambda$ which plays the role of a Lagrange multiplier associated with the volume constraint. Here $H_{\partial E} := \mathrm{div}_\tau \nu_E(x)$ denotes the sum of the principal curvatures of $\partial E$ ($\mathrm{div}_\tau$ is the tangential divergence on $\partial E$, see [5, Section 7.3]). Following [1], we define *critical sets* as those satisfying (1.6) in a weak sense, for which further regularity can be gained *a posteriori* (see Remark 1.6).

Definition 1.5. We say that $E \subset \mathbb{R}^N$ is a *regular critical set* for the functional (0.3) if $E$ is a bounded set of class $C^1$ and (1.6) holds weakly on $\partial E$, *i.e.*,

$$\int_{\partial E} \mathrm{div}_\tau \zeta \, \mathrm{d}\mathcal{H}^{N-1} = -2\gamma \int_{\partial E} v_E \, \langle \zeta, \nu_E \rangle \, \mathrm{d}\mathcal{H}^{N-1}$$

for every $\zeta \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ such that $\int_{\partial E} \langle \zeta, \nu_E \rangle \, \mathrm{d}\mathcal{H}^{N-1} = 0$.

Remark 1.6. By Proposition 1.1 and by standard regularity (see, *e.g.*, [5, Proposition 7.56 and Theorem 7.57]) a critical set $E$ is of class $W^{2,2}$ and $C^{1,\beta}$ for all $\beta \in (0,1)$. In turn, recalling Proposition 1.1, by Schauder estimates (see [29, Theorem 9.19]) we have that $E$ is of class $C^{3,\beta}$ for all $\beta \in (0, N - \alpha - 1)$.

We collect in the following theorem some regularity properties of local and global minimizers, which are mostly known (see, for instance, [37, 43, 65] for global minimizers, and [1] for local minimizers in a periodic setting). The basic idea is to show that a minimizer solves a suitable penalized minimum problem, where the volume constraint is replaced by a penalization term in the functional, and to deduce that a quasi-minimality property is satisfied (see Definition 1.26).

THEOREM 1.7. *Let $E \subset \mathbb{R}^N$ be a global or local minimizer for the functional (0.3) with volume $|E| = m$. Then the reduced boundary $\partial^* E$ is a $C^{3,\beta}$-manifold for all $\beta < N - \alpha - 1$, and the Hausdorff dimension of the singular set satisfies $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq N - 8$. Moreover, $E$ is (essentially) bounded. Finally, every global minimizer is connected, and every local minimizer has at most a finite number of connected components* [1].

PROOF. We divide the proof into three steps, following the ideas contained in [1, Proposition 2.7 and Theorem 2.8] in the first part.

*Step 1.* We claim that there exists $\Lambda > 0$ such that $E$ is a solution to the penalized minimum problem

$$\min \left\{ \mathcal{F}(F) + \Lambda \big| |F| - |E| \big| \,:\, F \subset \mathbb{R}^N,\, \alpha(F, E) \leq \frac{\delta}{2} \right\},$$

where $\delta$ is as in Definition 1.4 (the obstacle $\alpha(F, E) \leq \frac{\delta}{2}$ is not present in the case of a global minimizer). To obtain this, it is in fact sufficient to show that there exists $\Lambda > 0$ such that if $F \subset \mathbb{R}^N$ satisfies $\alpha(F, E) \leq \frac{\delta}{2}$ and $\mathcal{F}(F) + \Lambda \big| |F| - |E| \big| \leq \mathcal{F}(E)$, then $|F| = |E|$.

Assume by contradiction that there exist sequences $\Lambda_h \to +\infty$ and $E_h \subset \mathbb{R}^N$ such that $\alpha(E_h, E) \leq \frac{\delta}{2}$, $\mathcal{F}(E_h) + \Lambda_h \big| |E_h| - |E| \big| \leq \mathcal{F}(E)$, and $|E_h| \neq |E|$. Notice that, since $\Lambda_h \to +\infty$, we have $|E_h| \to |E|$.

We define new sets $F_h := \lambda_h E_h$, where $\lambda_h = \left( \frac{|E|}{|E_h|} \right)^{\frac{1}{N}} \to 1$, so that $|F_h| = |E|$. Then we have, for $h$ sufficiently large, that $\alpha(F_h, E) \leq \delta$ and

$$\begin{aligned}
\mathcal{F}(F_h) &= \mathcal{F}(E_h) + (\lambda_h^{N-1} - 1)\mathcal{P}(E_h) + \gamma(\lambda_h^{2N-\alpha} - 1)\mathcal{NL}_\alpha(E_h) \\
&\leq \mathcal{F}(E) + (\lambda_h^{N-1} - 1)\mathcal{P}(E_h) + \gamma(\lambda_h^{2N-\alpha} - 1)\mathcal{NL}_\alpha(E_h) - \Lambda_h \big| |E_h| - |E| \big| \\
&= \mathcal{F}(E) + |\lambda_h^N - 1|\, |E_h| \left( \frac{\lambda_h^{N-1} - 1}{|\lambda_h^N - 1|} \frac{\mathcal{P}(E_h)}{|E_h|} + \gamma \frac{\lambda_h^{2N-\alpha} - 1}{|\lambda_h^N - 1|} \frac{\mathcal{NL}_\alpha(E_h)}{|E_h|} - \Lambda_h \right) < \mathcal{F}(E),
\end{aligned}$$

which contradicts the local minimality of $E$ (notice that the same proof works also in the case of global minimizers).

*Step 2.* From the previous step, it follows that $E$ is an $(\omega, r_0)$-minimizer for the area functional for suitable $\omega > 0$ and $r_0 > 0$ (see Definition 1.26). Indeed, choose $r_0$ such that $\omega_N r_0^N \leq \frac{\delta}{2}$: then if $F$ is such that $F \triangle E \subset\subset B_r(x)$ with $r < r_0$, we clearly have that $\alpha(F, E) \leq \frac{\delta}{2}$ and by minimality of $E$ we deduce that

$$\begin{aligned}
\mathcal{P}(E) &\leq \mathcal{P}(F) + \gamma\big(\mathcal{NL}_\alpha(F) - \mathcal{NL}_\alpha(E)\big) + \Lambda \big| |F| - |E| \big| \\
&\leq \mathcal{P}(F) + \big(\gamma c_0 + \Lambda\big) |E \triangle F|
\end{aligned}$$

(using Proposition 1.3), and the claim follows with $\omega := \gamma c_0 + \Lambda$.

*Step 3.* The $C^{1,\frac{1}{2}}$-regularity of $\partial^* E$, as well as the condition on the Hausdorff dimension of the singular set, follows from classical regularity results for $(\omega, r_0)$-minimizers (see, *e.g.*, [66, Theorem 1]). In turn, the $C^{3,\beta}$-regularity follows from the Euler-Lagrange equation, as in Remark 1.6.

To show the essential boundedness, we use the density estimates for $(\omega, r_0)$-minimizers of the perimeter, which guarantee the existence of a positive constant $\vartheta_0 > 0$ (depending only

[1] Here and in the rest of this chapter *connectedness* is intended in a measure-theoretic sense: $E$ is said to be *connected* (or *indecomposable*) if $E = E_1 \cup E_2$, $|E| = |E_1| + |E_2|$ and $\mathcal{P}(E) = \mathcal{P}(E_1) + \mathcal{P}(E_2)$ imply $|E_1||E_2| = 0$. A *connected component* of $E$ is any connected subset $E_0 \subset E$ such that $|E_0| > 0$ and $\mathcal{P}(E) = \mathcal{P}(E_0) + \mathcal{P}(E \setminus E_0)$.

on $N$) such that for every point $y \in \partial^* E$ and $r < \min\{r_0, 1/(2N\omega)\}$

$$\mathcal{P}(E; B_r(y)) \geq \vartheta_0 r^{N-1} \tag{1.7}$$

(see, *e.g.*, [45, Theorem 21.11]). Assume by contradiction that there exists a sequence of points $x_n \in \mathbb{R}^N \setminus E^{(0)}$, where

$$E^{(0)} := \left\{ x \in \mathbb{R}^N : \limsup_{r \to 0^+} \frac{|E \cap B_r(x)|}{r^N} = 0 \right\},$$

such that $|x_n| \to +\infty$. Fix $r < \min\{r_0, 1/(2N\omega)\}$ and assume without loss of generality that $|x_n - x_m| > 4r$. It is easily seen that for infinitely many $n$ we can find $y_n \in \partial^* E \cap B_r(x_n)$; then

$$\mathcal{P}(E) \geq \sum_n \mathcal{P}(E, B_r(y_n)) \geq \sum_n \vartheta_0 r^{N-1} = +\infty,$$

which is a contradiction.

Connectedness of global minimizers follows easily from their boundedness, since if a global minimizer had at least two connected components one could move one of them far apart from the others without changing the perimeter but decreasing the nonlocal term in the energy (see [43, Lemma 3] for a formal argument).

Finally, let $E_0$ be a connected component of a local minimizer $E$: then, denoting by $B_r$ a ball with volume $|B_r| = |E_0|$, using the isoperimetric inequality and the fact that $E$ is a $(\omega, r_0)$-minimizer for the area functional, we obtain

$$\begin{aligned}\mathcal{P}(E \setminus E_0) + N\omega_N r^{N-1} &\leq \mathcal{P}(E \setminus E_0) + \mathcal{P}(E_0) = \mathcal{P}(E)\\ &\leq \mathcal{P}(E \setminus E_0) + \omega|E_0| = \mathcal{P}(E \setminus E_0) + \omega\omega_N r^N,\end{aligned}$$

which is a contradiction if $r$ is small enough. This shows an uniform lower bound on the volume of each connected component of $E$, from which we deduce that $E$ can have at most a finite number of connected components. □

We are now ready to state the main results of this chapter. The central theorem, whose proof lasts for Sections 1.2 and 1.3, provides a sufficiency local minimality criterion based on the second variation of the functional. Following [1] (see also [15]), we introduce a quadratic form associated with the second variation of the functional at a regular critical set (see Definition 1.18); then we show that its strict positivity (on the orthogonal complement to a suitable finite dimensional subspace of directions where the second variation degenerates, due to translation invariance) is a sufficient condition for isolated local minimality, according to Definition 1.4, by proving a *quantitative* stability inequality. The result reads as follows.

THEOREM 1.8. *Assume that $E$ is a regular critical set for $\mathcal{F}$ with positive second variation, in the sense of Definition 1.22. Then there exist $\delta > 0$ and $C > 0$ such that*

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C\big(\alpha(E,F)\big)^2 \tag{1.8}$$

*for every $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\alpha(E,F) < \delta$.*

The local minimality criterion in Theorem 1.8 can be applied to obtain information about local and global minimizers of the functional (0.3). In order to state the results more clearly, we will underline the dependence of the functional on the parameters $\alpha$ and $\gamma$ by writing $\mathcal{F}_{\alpha,\gamma}$ instead of $\mathcal{F}$. We start with the following theorem, which shows the existence of a critical mass $m_{\mathrm{loc}}$ such that the ball $B_R$ is an isolated local minimizer if $|B_R| < m_{\mathrm{loc}}$, but is no longer a local minimizer for larger masses. We also determine explicitly the volume threshold in the three-dimensional case. The result, which to the best of our knowledge provides the first characterization of the local minimality of the ball, will be proved in Section 1.4.

THEOREM 1.9 (Local minimality of the ball). *Given $N \geq 2$, $\alpha \in (0, N-1)$ and $\gamma > 0$, there exists a critical threshold $m_{\mathrm{loc}} = m_{\mathrm{loc}}(N, \alpha, \gamma) > 0$ such that the ball $B_R$ is an isolated local minimizer for $\mathcal{F}_{\alpha,\gamma}$, in the sense of Definition 1.4, if $0 < |B_R| < m_{\mathrm{loc}}$.*

*If $|B_R| > m_{\mathrm{loc}}$, there exists $E \subset \mathbb{R}^N$ with $|E| = |B_R|$ and $\alpha(E, B_R)$ arbitrarily small such that $\mathcal{F}_{\alpha,\gamma}(E) < \mathcal{F}_{\alpha,\gamma}(B_R)$.*

*Finally $m_{\mathrm{loc}}(N, \alpha, \gamma) \to \infty$ as $\alpha \to 0^+$, and in dimension $N = 3$ we have*

$$m_{\mathrm{loc}}(3, \alpha, \gamma) = \frac{4}{3}\pi \left( \frac{(6-\alpha)(4-\alpha)}{2^{3-\alpha}\gamma\alpha\pi} \right)^{\frac{3}{4-\alpha}}.$$

Our local minimality criterion allows us to deduce further properties about global minimizers, which will be proved in Section 1.5. The first result states that the ball is the unique global minimizer of the functional for small masses. We provide an alternative proof of this fact (which was already known in the literature in some particular cases, as explained in the introduction), removing the restrictions on the parameters $N$ and $\alpha$ which were present in the previous partial results (except for the upper bound $\alpha < N - 1$).

THEOREM 1.10 (Global minimality of the ball). *Given $N \geq 2$, $\alpha \in (0, N-1)$ and $\gamma > 0$, let $m_{\mathrm{glob}}(N, \alpha, \gamma)$ be the supremum of the masses $m > 0$ such that the ball of volume $m$ is a global minimizer of $\mathcal{F}_{\alpha,\gamma}$ in $\mathbb{R}^N$. Then $m_{\mathrm{glob}}(N, \alpha, \gamma)$ is positive and finite, and the ball of volume $m$ is a global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m \leq m_{\mathrm{glob}}(N, \alpha, \gamma)$. Moreover, it is the unique (up to translations) global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m < m_{\mathrm{glob}}(N, \alpha, \gamma)$.*

In the following theorems we analyze the global minimality issue for $\alpha$ close to 0, showing that in this case the unique minimizer, as long as a minimizer exists, is the ball, and characterizing the infimum of the energy when the problem does not have a solution. In particular, we recover the result already proved by different techniques in [36, Theorem 2.7] for $N = 2$, and we extend it to the general space dimension.

THEOREM 1.11 (Characterization of global minimizers for $\alpha$ small). *There exists $\bar{\alpha} = \bar{\alpha}(N, \gamma) > 0$ such that for every $\alpha < \bar{\alpha}$ the ball with volume $m$ is the unique (up to translations) global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m \leq m_{\mathrm{glob}}(N, \alpha, \gamma)$, while for $m > m_{\mathrm{glob}}(N, \alpha, \gamma)$ the minimum problem for $\mathcal{F}_{\alpha,\gamma}$ does not have a solution.*

THEOREM 1.12 (Characterization of minimizing sequences for $\alpha$ small). *Let $\alpha < \bar{\alpha}$ (where $\bar{\alpha}$ is given by Theorem 1.11) and let*

$$f_k(m) := \min_{\substack{\mu_1, \ldots, \mu_k \geq 0 \\ \mu_1 + \ldots + \mu_k = m}} \left\{ \sum_{i=1}^{k} \mathcal{F}(B^i) : B^i \text{ ball}, \, |B^i| = \mu_i \right\}.$$

*There exists an increasing sequence $(m_k)_k$, with $m_0 = 0$, $m_1 = m_{\mathrm{glob}}$, such that $\lim_k m_k = \infty$ and*

$$\inf_{|E|=m} \mathcal{F}(E) = f_k(m) \qquad \text{for every } m \in [m_{k-1}, m_k], \text{ for all } k \in \mathbb{N}, \tag{1.9}$$

*that is, for every $m \in [m_{k-1}, m_k]$ a minimizing sequence for the total energy is obtained by a configuration of at most $k$ disjoint balls with diverging mutual distance. Moreover, the number of non-degenerate balls tends to $+\infty$ as $m \to +\infty$.*

REMARK 1.13. Since $m_{\mathrm{loc}}(N, \alpha, \gamma) \to +\infty$ as $\alpha \to 0^+$ and the non-existence threshold is uniformly bounded for $\alpha \in (0, 1)$ (see Proposition 1.37), we immediately deduce that, for

$\alpha$ small, $m_{\rm glob}(N,\alpha,\gamma) < m_{\rm loc}(N,\alpha,\gamma)$. Moreover, denoting by $\bar m(N,\alpha,\gamma)$ the value of the mass for which the energy of a ball of volume $\bar m$ is equal to the energy of two disjoint balls of volume $\frac{\bar m}{2}$ "at an infinite distance" (that is, neglecting the interaction term between the two balls), we clearly have $m_{\rm glob}(N,\alpha,\gamma) \le \bar m(N,\alpha,\gamma)$. By a straightforward estimate on $\bar m$ (using $\mathcal{NL}_\alpha(B_1) \ge \omega_N^2 2^{-\alpha}$) we obtain the following upper bound for the global minimality threshold of the ball:

$$m_{\rm glob}(N,\alpha,\gamma) \le \bar m(N,\alpha,\gamma) < \omega_N \left( \frac{2^\alpha N(2^{\frac{1}{N}}-1)}{\omega_N \gamma (1-(\frac12)^{\frac{N-\alpha}{N}})} \right)^{\frac{N}{N+1-\alpha}}.$$

Hence, by comparing this value with the explicit expression of $m_{\rm loc}$ in the physical interesting case $N=3$, $\alpha=1$ (see Theorem 1.9), we deduce that $m_{\rm glob}(3,1,\gamma) < m_{\rm loc}(3,1,\gamma)$. Notice that a similar comparison between local and global stability is made in [36] for the two-dimensional case, where the explicit value of $\bar m$ is computed.

Remark 1.14. In the planar case, one can also consider a Newtonian potential in the nonlocal term, *i.e.*

$$\int_E \int_E \log \frac{1}{|x-y|}\,\mathrm{d}x\mathrm{d}y\,.$$

It is clear that the infimum of the corresponding functional on $\mathbb{R}^2$ is $-\infty$ (consider, for instance, a minimizing sequence obtained by sending to infinity the distance between the centers of two disjoint balls). Moreover, also the notion of local minimality considered in Definition 1.4 becomes meaningless in this situation, since, given any finite perimeter set $E$, it is always possible to find sets with total energy arbitrarily close to $-\infty$ in every $L^1$-neighbourhood of $E$. Nevertheless, by reproducing the arguments of this chapter one can show that, given a bounded regular critical set $E$ with positive second variation, and a radius $R>0$ such that $E \subset B_R$, there exists $\delta>0$ such that $E$ minimizes the energy with respect to competitors $F \subset B_R$ with $\alpha(F,E)<\delta$.

## 1.2. Second variation and $W^{2,p}$-local minimality

We start this section by introducing the notions of first and second variation of the functional $\mathcal{F}$ along families of deformations as in the following definition.

Definition 1.15. Let $X:\mathbb{R}^N \to \mathbb{R}^N$ be a $C^2$ vector field. The *admissible flow* associated with $X$ is the function $\Phi:\mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ defined by the equations

$$\frac{\partial \Phi}{\partial t} = X(\Phi)\,, \quad \Phi(x,0)=x\,.$$

Definition 1.16. Let $E \subset \mathbb{R}^N$ be a set of class $C^2$, and let $\Phi$ be an admissible flow. We define the *first and second variation of $\mathcal{F}$ at $E$ with respect to the flow* $\Phi$ to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(E_t)_{|t=0} \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)_{|t=0}$$

respectively, where we set $E_t := \Phi_t(E)$.

Given a regular set $E$, we denote by $X_\tau := X - \langle X, \nu_E\rangle \nu_E$ the tangential part to $\partial E$ of a vector field $X$. We recall that the tangential gradient $D_\tau$ is defined by $D_\tau\varphi := (D\varphi)_\tau$, and that $B_{\partial E} := D_\tau \nu_E$ is the second fundamental form of $\partial E$.

The following theorem contains the explicit formula for the first and second variation of $\mathcal{F}$. The computation, which is postponed to the Appendix, is performed by a regularization approach which is slightly different from the technique used, in the case $\alpha = N-2$, in [15] (for a critical set, see also [53]) and in [1] (for a general regular set): here we introduce a family of regularized potentials (depending on a small parameter $\delta \in \mathbb{R}$) to avoid the problems in the differentiation of the singularity in the nonlocal part, recovering the result by letting the parameter tend to 0.

THEOREM 1.17. *Let $E \subset \mathbb{R}^N$ be a bounded set of class $C^2$, and let $\Phi$ be the admissible flow associated with a $C^2$ vector field $X$. Then the first variation of $\mathcal{F}$ at $E$ with respect to the flow $\Phi$ is*

$$\frac{\mathrm{d}\mathcal{F}(E_t)}{\mathrm{d}t}_{|t=0} = \int_{\partial E} (H_{\partial E} + 2\gamma v_E)\langle X, \nu_E\rangle \,\mathrm{d}\mathcal{H}^{N-1}, \tag{1.10}$$

*and the second variation of $\mathcal{F}$ at $E$ with respect to the flow $\Phi$ is*

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0} &= \int_{\partial E} \left(|D_\tau\langle X, \nu_E\rangle|^2 - |B_{\partial E}|^2\langle X, \nu_E\rangle^2\right) \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + 2\gamma \int_{\partial E}\int_{\partial E} G(x,y)\langle X(x), \nu_E(x)\rangle\langle X(y), \nu_E(y)\rangle \,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) \\ &\quad + 2\gamma \int_{\partial E} \partial_{\nu_E} v_E \,\langle X, \nu_E\rangle^2 \,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E} (2\gamma v_E + H_{\partial E})\,\mathrm{div}_\tau\big(X_\tau\langle X, \nu_E\rangle\big)\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_{\partial E} (2\gamma v_E + H_{\partial E})(\mathrm{div}X)\langle X, \nu_E\rangle \,\mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

*where $G(x,y) := \frac{1}{|x-y|^\alpha}$ is the potential in the nonlocal part of the energy.*

If $E$ is a regular critical set (as in Definition 1.5) it holds

$$\int_{\partial E} (2\gamma v_E + H_{\partial E})\mathrm{div}_\tau\big(X_\tau\langle X, \nu_E\rangle\big)\,\mathrm{d}\mathcal{H}^{N-1} = 0\,.$$

Moreover if the admissible flow $\Phi$ preserves the volume of $E$, *i.e.* if $|\Phi_t(E)| = |E|$ for all $t \in (-1, 1)$, then (see [15, equation (2.30)])

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}|E_t|_{|t=0} = \int_{\partial E} (\mathrm{div}X)\langle X, \nu_E\rangle \,\mathrm{d}\mathcal{H}^{N-1}\,.$$

Hence we obtain the following expression for the second variation at a regular critical set with respect to a volume-preserving admissible flow:

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0} &= \int_{\partial E} \big(|D_\tau\langle X, \nu_E\rangle|^2 - |B_{\partial E}|^2\langle X, \nu_E\rangle^2\big)\,\mathrm{d}\mathcal{H}^{N-1} + 2\gamma\int_{\partial E} \partial_{\nu_E} v_E\langle X, \nu_E\rangle^2 \,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + 2\gamma \int_{\partial E}\int_{\partial E} G(x,y)\langle X(x), \nu_E(x)\rangle\langle X(y), \nu_E(y)\rangle \,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,.\end{aligned}$$

Following [1], we introduce the space

$$\widetilde{H}^1(\partial E) := \left\{\varphi \in H^1(\partial E)\ :\ \int_{\partial E} \varphi\,\mathrm{d}\mathcal{H}^{N-1} = 0\right\}$$

endowed with the norm $\|\varphi\|_{\widetilde{H}^1(\partial E)} := \|\nabla\varphi\|_{L^2(\partial E)}$, and we define on it the following quadratic form associated with the second variation.

DEFINITION 1.18. Let $E \subset \mathbb{R}^N$ be a regular critical set. We define the quadratic form $\partial^2\mathcal{F}(E) : \widetilde{H}^1(\partial E) \to \mathbb{R}$ by

$$\begin{aligned}\partial^2\mathcal{F}(E)[\varphi] &= \int_{\partial E} \big(|D_\tau\varphi|^2 - |B_{\partial E}|^2\varphi^2\big)\,\mathrm{d}\mathcal{H}^{N-1} + 2\gamma\int_{\partial E} (\partial_{\nu_E} v_E)\varphi^2 \,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + 2\gamma \int_{\partial E}\int_{\partial E} G(x,y)\varphi(x)\varphi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,.\end{aligned} \tag{1.11}$$

Notice that if $E$ is a regular critical set and $\Phi$ preserves the volume of $E$, then

$$\partial^2\mathcal{F}(E)[\langle X,\nu_E\rangle] = \frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0}. \tag{1.12}$$

We remark that the last integral in the expression of $\partial^2\mathcal{F}(E)$ is well defined for $\varphi\in\widetilde{H}^1(\partial E)$, thanks to the following result.

LEMMA 1.19. *Let $E$ be a bounded set of class $C^1$. There exists a constant $C>0$, depending only on $E$, $N$ and $\alpha$, such that for every $\varphi,\psi\in\widetilde{H}^1(\partial E)$*

$$\int_{\partial E}\int_{\partial E} G(x,y)\varphi(x)\psi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) \le C\|\varphi\|_{L^2}\|\psi\|_{L^2} \le C\|\varphi\|_{\widetilde{H}^1}\|\psi\|_{\widetilde{H}^1}. \tag{1.13}$$

PROOF. The proof lies on [29, Lemma 7.12], which states that if $\Omega\subset\mathbb{R}^n$ is a bounded domain and $\mu\in(0,1]$, the operator $f\mapsto V_\mu f$ defined by

$$(V_\mu f)(x) := \int_\Omega |x-y|^{n(\mu-1)} f(y)\,\mathrm{d}y$$

maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ provided that $0\le\delta := p^{-1}-q^{-1}<\mu$, and

$$\|V_\mu f\|_{L^q(\Omega)} \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_n^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_{L^p(\Omega)}.$$

In our case, from the fact that our set has compact boundary, we can simply reduce to the above case using local charts and partition of unity (notice that the hypothesis of compact boundary allows us to bound from above in the $L^\infty$-norm the area factor). In particular we have that $\mu = \frac{N-1-\alpha}{N-1}$, and applying this result with $p=q=2$ we easily obtain the estimate in the statement by the Sobolev Embedding Theorem. □

REMARK 1.20. Using the estimate contained in the previous lemma it is easily seen that $\partial^2\mathcal{F}(E)$ is continuous with respect to the strong convergence in $\widetilde{H}^1(\partial E)$ and lower semicontinuous with respect to the weak convergence in $\widetilde{H}^1(\partial E)$. Moreover, it is also clear from the proof that, given $\bar\alpha < N-1$, the constant $C$ in (1.13) can be chosen independently of $\alpha\in(0,\bar\alpha)$.

Equality (1.12) suggests that at a regular local minimizer the quadratic form (1.11) must be nonnegative on the space $\widetilde{H}^1(\partial E)$. This is the content of the following corollary, whose proof is analogous to [1, Corollary 3.4].

COROLLARY 1.21. *Let $E$ be a local minimizer of $\mathcal{F}$ of class $C^2$. Then*

$$\partial^2\mathcal{F}(E)[\varphi]\ge 0 \qquad \textit{for all } \varphi\in\widetilde{H}^1(\partial E).$$

Now we want to look for a sufficient condition for local minimality. First of all we notice that, since our functional is translation invariant, if we compute the second variation of $\mathcal{F}$ at a regular set $E$ with respect to a flow of the form $\Phi(x,t) := x + t\eta e_i$, where $\eta\in\mathbb{R}$ and $e_i$ is an element of the canonical basis of $\mathbb{R}^N$, setting $\nu_i := \langle\nu_E, e_i\rangle$ we obtain that

$$\partial^2\mathcal{F}(E)[\eta\nu_i] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)_{|t=0} = 0.$$

Following [1], since we aim to prove that the *strict* positivity of the second variation is a sufficient condition for local minimality, we shall exclude the finite dimensional subspace of $\widetilde{H}^1(\partial E)$ generated by the functions $\nu_i$, which we denote by $T(\partial E)$. Hence we split

$$\widetilde{H}^1(\partial E) = T^\perp(\partial E)\oplus T(\partial E),$$

where $T^\perp(\partial E)$ is the orthogonal complement to $T(\partial E)$ in the $L^2$-sense, *i.e.*,

$$T^\perp(\partial E) := \left\{ \varphi \in \widetilde{H}^1(\partial E) \,:\, \int_{\partial E} \varphi \nu_i \,\mathrm{d}\mathcal{H}^{N-1} = 0 \text{ for each } i = 1, \dots, N \right\}.$$

It can be shown (see [1, Equation (3.7)]) that there exists an orthonormal frame $(\varepsilon_1, \dots, \varepsilon_N)$ such that

$$\int_{\partial E} \langle \nu, \varepsilon_i \rangle \langle \nu, \varepsilon_j \rangle \,\mathrm{d}\mathcal{H}^{N-1} = 0 \qquad \text{for all } i \neq j\,,$$

so that the projection on $T^\perp(\partial E)$ of a function $\varphi \in \widetilde{H}^1(\partial E)$ is

$$\pi_{T^\perp(\partial E)}(\varphi) = \varphi - \sum_{i=1}^N \left( \int_{\partial E} \varphi \langle \nu, \varepsilon_i \rangle \,\mathrm{d}\mathcal{H}^{N-1} \right) \frac{\langle \nu, \varepsilon_i \rangle}{\|\langle \nu, \varepsilon_i \rangle\|^2_{L^2(\partial E)}}$$

(notice that $\langle \nu, \varepsilon_i \rangle \not\equiv 0$ for every $i$, since on the contrary the set $E$ would be translation invariant in the direction $\varepsilon_i$).

Definition 1.22. We say that $\mathcal{F}$ has *positive second variation* at the regular critical set $E$ if

$$\partial^2 \mathcal{F}(E)[\varphi] > 0 \qquad \text{for all } \varphi \in T^\perp(\partial E) \setminus \{0\}.$$

One could expect that the positiveness of the second variation implies also a sort of coercivity; this is shown in the following lemma.

Lemma 1.23. *Assume that $\mathcal{F}$ has positive second variation at a regular critical set $E$. Then*

$$m_0 := \inf \left\{ \partial^2 \mathcal{F}(E)[\varphi] \,:\, \varphi \in T^\perp(\partial E),\ \|\varphi\|_{\widetilde{H}^1(\partial E)} = 1 \right\} > 0\,,$$

*and*

$$\partial^2 \mathcal{F}(E)[\varphi] \geq m_0 \|\varphi\|^2_{\widetilde{H}^1(\partial E)} \qquad \textit{for all } \varphi \in T^\perp(\partial E)\,.$$

Proof. Let $(\varphi_h)_h$ be a minimizing sequence for $m_0$. Up to a subsequence we can suppose that $\varphi_h \rightharpoonup \varphi_0$ weakly in $H^1(\partial E)$, with $\varphi_0 \in T^\perp(\partial E)$. By the lower semicontinuity of $\partial^2 \mathcal{F}(E)$ with respect to the weak convergence in $H^1(\partial E)$ (see Remark 1.20), we have that if $\varphi_0 \neq 0$

$$m_0 = \lim_{h \to \infty} \partial^2 \mathcal{F}(E)[\varphi_h] \geq \partial^2 \mathcal{F}(E)[\varphi_0] > 0\,,$$

while if $\varphi_0 = 0$

$$m_0 = \lim_{h \to \infty} \partial^2 \mathcal{F}(E)[\varphi_h] = \lim_{h \to \infty} \int_{\partial E} |D_\tau \varphi_h|^2 \,\mathrm{d}\mathcal{H}^{N-1} = 1\,.$$

The second part of the statement follows from the fact that $\partial^2 \mathcal{F}(E)$ is a quadratic form. $\square$

We now come to the proof of the main result of this chapter, namely that the positivity of the second variation at a critical set $E$ is a sufficient condition for local minimality (Theorem 1.8). In the remaining part of this section we prove that a weaker minimality property holds, that is minimality with respect to sets whose boundaries are graphs over the boundary of $E$ with sufficiently small $W^{2,p}$-norm (Theorem 1.25). In order to do this, we start by recalling a technical result needed in the proof, namely [1, Theorem 3.7], which provides a construction of an admissible flow connecting a regular set $E \subset \mathbb{R}^N$ with an arbitrary set sufficiently close in the $W^{2,p}$-sense.

THEOREM 1.24. *Let $E \subset \mathbb{R}^N$ be a bounded set of class $C^3$ and let $p > N-1$. For all $\varepsilon > 0$ there exist a tubular neighbourhood $\mathcal{U}$ of $\partial E$ and two positive constants $\delta$, $C$ with the following properties: if $\psi \in C^2(\partial E)$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ then there exists a field $X \in C^2$ with $\mathrm{div} X = 0$ in $\mathcal{U}$ such that*

$$\|X - \psi\nu_E\|_{L^2(\partial E)} \leq \varepsilon \|\psi\|_{L^2(\partial E)}\,.$$

*Moreover the associated flow*

$$\Phi(x,0) = 0, \quad \frac{\partial \Phi}{\partial t} = X(\Phi)$$

*satisfies $\Phi(\partial E, 1) = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$, and for every $t \in [0,1]$*

$$\|\Phi(\cdot,t) - Id\|_{W^{2,p}} \leq C\|\psi\|_{W^{2,p}(\partial E)}\,,$$

*where $Id$ denotes the identity map. If in addition $E_1$ has the same volume as $E$, then for every $t$ we have $|E_t| = |E|$ and*

$$\int_{\partial E_t} \langle X, \nu_{E_t}\rangle \,\mathrm{d}\mathcal{H}^{N-1} = 0\,.$$

We are now in position to prove the following $W^{2,p}$-local minimality theorem, analogous to [1, Theorem 3.9]. The proof contained in [1] can be repeated here with minor changes, and we will only give a sketch of it for the reader's convenience.

THEOREM 1.25. *Let $p > \max\{2, N-1\}$ and let $E$ be a regular critical set for $\mathcal{F}$ with positive second variation, according to Definition 1.22. Then there exist $\delta$, $C_0 > 0$ such that*

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C_0(\alpha(E,F))^2\,,$$

*for each $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ with $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$.*

PROOF (SKETCH). We just describe the strategy of the proof, which is divided into two steps.

*Step 1.* There exists $\delta_1 > 0$ such that if $\partial F = \{x + \psi(x)\nu_E(x) \,:\, x \in \partial E\}$ with $|F| = |E|$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta_1$, then

$$\inf\left\{\partial^2\mathcal{F}(F)[\varphi] \,:\, \varphi \in \widetilde{H}^1(\partial F),\, \|\varphi\|_{\widetilde{H}^1(\partial F)} = 1,\, \left|\int_{\partial F} \varphi\nu_F \,\mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta_1\right\} \geq \frac{m_0}{2},$$

where $m_0$ is defined in Lemma 1.23. To prove this we suppose by contradiction that there exist a sequence $(F_n)_n$ of subsets of $\mathbb{R}^N$ such that $\partial F_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\}$, $|F_n| = |E|$, $\|\psi_n\|_{W^{2,p}(\partial E)} \to 0$, and a sequence of functions $\varphi_n \in \widetilde{H}^1(\partial F_n)$ with $\|\varphi_n\|_{\widetilde{H}^1(\partial F_n)} = 1$, $|\int_{\partial F_n} \varphi_n \nu_{F_n} \,\mathrm{d}\mathcal{H}^{N-1}| \to 0$, such that

$$\partial^2\mathcal{F}(F_n)[\varphi_n] < \frac{m_0}{2}.$$

We consider a sequence of diffeomorphisms $\Phi_n : E \to F_n$, with $\Phi_n \to Id$ in $W^{2,p}$, and we set

$$\tilde{\varphi}_n := \varphi_n \circ \Phi_n - a_n, \qquad a_n := \fint_{\partial E} \varphi_n \circ \Phi_n \,\mathrm{d}\mathcal{H}^{N-1}.$$

Hence $\tilde{\varphi}_n \in \widetilde{H}^1(\partial E)$, $a_n \to 0$, and since $\nu_{F_n} \circ \Phi_n - \nu_E \to 0$ in $C^{0,\beta}$ for some $\beta \in (0,1)$ and a similar convergence holds for the tangential vectors, we have that

$$\int_{\partial E} \tilde{\varphi}_n \langle \nu_E, \varepsilon_i \rangle \,\mathrm{d}\mathcal{H}^{N-1} \to 0$$

for every $i = 1, \dots, N$, so that $\|\pi_{T^\perp(\partial E)}(\tilde\varphi_n)\|_{\widetilde{H}^1(\partial E)} \to 1$. Moreover it can be proved that

$$\left|\partial^2\mathcal{F}(F_n)[\varphi_n] - \partial^2\mathcal{F}(E)[\tilde\varphi_n]\right| \to 0.$$

Indeed, the convergence of the first integral in the expression of the quadratic form follows easily from the fact that $B_{\partial F_n} \circ \Phi_n - B_{\partial E} \to 0$ in $L^p(\partial E)$, and from the Sobolev Embedding Theorem (recall that $p > \max\{2, N-1\}$). For the second integral, it is sufficient to observe that, as a consequence of Proposition 1.1, the functions $v_{F_h}$ are uniformly bounded in $C^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$ and hence they converge to $v_E$ in $C^{1,\gamma}(B_R)$ for all $\gamma < \beta$ and $R > 0$. Finally, the difference of the last integrals can be written as

$$\begin{aligned}
&\int_{\partial F_n}\int_{\partial F_n} G(x,y)\varphi_n(x)\varphi_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E}\int_{\partial E} G(x,y)\tilde\varphi_n(x)\tilde\varphi_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad = \int_{\partial E}\int_{\partial E} g_n(x,y)G(x,y)\tilde\varphi_n(x)\tilde\varphi_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad\qquad + a_n\int_{\partial E}\int_{\partial E} G(\Phi_n(x),\Phi_n(y))J_n(x)J_n(y)\big(\tilde\varphi_n(x)+\tilde\varphi_n(y)+a_n\big)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}
\end{aligned}$$

where $J_n(z) := J_{\partial E}^{N-1}\Phi_n(z)$ is the $(N-1)$-dimensional jacobian of $\Phi_n$ on $\partial E$, and

$$g_n(x,y) := \frac{|x-y|^\alpha}{|\Phi_n(x)-\Phi_n(y)|^\alpha}\,J_n(x)J_n(y) - 1\,.$$

Thus the desired convergence follows from the fact that $g_n \to 0$ uniformly, $a_n \to 0$, and from the estimate provided by Lemma 1.19.

Hence

$$\begin{aligned}
\frac{m_0}{2} \geq \lim_{n\to\infty}\partial^2\mathcal{F}(F_n)[\varphi_n] &= \lim_{n\to\infty}\partial^2\mathcal{F}(E)[\tilde\varphi_n] = \lim_{n\to\infty}\partial^2\mathcal{F}(E)[\pi_{T^\perp(\partial E)}(\tilde\varphi_n)]\\
&\geq m_0\lim_{n\to\infty}\|\pi_{T^\perp(\partial E)}(\tilde\varphi_n)\|_{\widetilde{H}^1(\partial E)} = m_0,
\end{aligned}$$

which is a contradiction.

*Step 2.* If $F$ is as in the statement of the theorem, we can use the vector field $X$ provided by Theorem 1.24 to generate a flow connecting $E$ to $F$ by a family of sets $E_t$, $t \in [0,1]$. Recalling that $E$ is critical and that $X$ is divergence free, we can write

$$\begin{aligned}
\mathcal{F}(F) - \mathcal{F}(E) &= \mathcal{F}(E_1) - \mathcal{F}(E_0) = \int_0^1 (1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)\,\mathrm{d}t\\
&= \int_0^1 (1-t)\Big(\partial^2\mathcal{F}(E_t)[\langle X,\nu_{E_t}\rangle] - \int_{\partial E_t}(2\gamma v_{E_t} + H_{\partial E_t})\mathrm{div}_{\tau_t}(X_{\tau_t}\langle X,\nu_{E_t}\rangle)\,\mathrm{d}\mathcal{H}^{N-1}\Big)\,\mathrm{d}t,
\end{aligned}$$

where $\mathrm{div}_{\tau_t}$ stands for the tangential divergence of $\partial E_t$. It is now possible to bound from below the previous integral in a quantitative fashion: to do this we use, in particular, the result proved in Step 1 for the first term, and we proceed as in Step 2 of [1, Theorem 3.9] for the second one. In this way we obtain the desired estimate. $\square$

## 1.3. $L^1$-local minimality

In this section we complete the proof of the main result of this chapter (Theorem 1.8), started in the previous section. The main argument of the proof relies on a regularity property of sequences of *quasi-minimizers* of the area functional, which has been observed by White in [68] and was implicitly contained in [4] (see also [62], [66]).

DEFINITION 1.26. A set $E \subset \mathbb{R}^N$ is said to be an $(\omega, r_0)$-*minimizer* for the area functional, with $\omega > 0$ and $r_0 > 0$, if for every ball $B_r(x)$ with $r \leq r_0$ and for every finite perimeter set $F \subset \mathbb{R}^N$ such that $E \triangle F \subset\subset B_r(x)$ we have

$$\mathcal{P}(E) \leq \mathcal{P}(F) + \omega |E \triangle F|.$$

THEOREM 1.27. *Let $E_n \subset \mathbb{R}^N$ be a sequence of $(\omega, r_0)$-minimizers of the area functional such that*

$$\sup_n \mathcal{P}(E_n) < +\infty \quad \text{and} \quad \chi_{E_n} \to \chi_E \text{ in } L^1(\mathbb{R}^N)$$

*for some bounded set $E$ of class $C^2$. Then for $n$ large enough $E_n$ is of class $C^{1,\frac{1}{2}}$ and*

$$\partial E_n = \{x + \psi_n(x)\nu_E(x) \,:\, x \in \partial E\},$$

*with $\psi_n \to 0$ in $C^{1,\beta}(\partial E)$ for all $\beta \in (0, \frac{1}{2})$.*

Another useful result is the following consequence of the classical elliptic regularity theory (see [1, Lemma 7.2] for a proof).

LEMMA 1.28. *Let $E$ be a bounded set of class $C^2$ and let $E_n$ be a sequence of sets of class $C^{1,\beta}$ for some $\beta \in (0,1)$ such that $\partial E_n = \{x + \psi_n(x)\nu_E(x) \,:\, x \in \partial E\}$, with $\psi_n \to 0$ in $C^{1,\beta}(\partial E)$. Assume also that $H_{\partial E_n} \in L^p(\partial E_n)$ for some $p \geq 1$. If*

$$H_{\partial E_n}(\cdot + \psi_n(\cdot)\nu_E(\cdot)) \to H_{\partial E} \qquad \text{in } L^p(\partial E),$$

*then $\psi_n \to 0$ in $W^{2,p}(\partial E)$.*

We recall also the following simple lemma from [1, Lemma 4.1].

LEMMA 1.29. *Let $E \subset \mathbb{R}^N$ be a bounded set of class $C^2$. Then there exists a constant $C_E > 0$, depending only on $E$, such that for every finite perimeter set $F \subset \mathbb{R}^N$*

$$\mathcal{P}(E) \leq \mathcal{P}(F) + C_E |E \triangle F|.$$

An intermediate step in the proof of Theorem 1.8 consists in showing that the $W^{2,p}$-local minimality proved in Theorem 1.25 implies local minimality with respect to competing sets which are sufficiently close in the Hausdorff distance. We omit the proof of this result, since it can be easily adapted from [1, Theorem 4.3] (notice, indeed, that the difficulties coming from the fact of working in the whole space $\mathbb{R}^N$ are not present, due to the constraint $F \subset \mathcal{I}_{\delta_0}(E)$).

THEOREM 1.30. *Let $E \subset \mathbb{R}^N$ be a bounded regular set, and assume that there exists $\delta > 0$ such that*

$$\mathcal{F}(E) \leq \mathcal{F}(F) \tag{1.14}$$

*for every set $F \subset \mathbb{R}^N$ with $|F| = |E|$ and $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$, for some function $\psi$ with $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$.*

*Then there exists $\delta_0 > 0$ such that (1.14) holds for every finite perimeter set $F$ with $|F| = |E|$ and such that $\mathcal{I}_{-\delta_0}(E) \subset F \subset \mathcal{I}_{\delta_0}(E)$, where for $\delta \in \mathbb{R}$ we set ($d$ denoting the signed distance to $E$)*

$$\mathcal{I}_\delta(E) := \{x : d(x) < \delta\}.$$

We are finally ready to complete the proof of the main result of this chapter. The strategy follows closely [1, Theorem 1.1], with the necessary technical modifications due to the fact that here we have to deal with a more general exponent $\alpha$ and with the lack of compactness of the ambient space.

PROOF OF THEOREM 1.8. We assume by contradiction that there exists a sequence of sets $E_h \subset \mathbb{R}^N$, with $|E_h| = |E|$ and $\alpha(E_h, E) > 0$, such that $\varepsilon_h := \alpha(E_h, E) \to 0$ and

$$\mathcal{F}(E_h) < \mathcal{F}(E) + \frac{C_0}{4}\big(\alpha(E_h, E)\big)^2, \tag{1.15}$$

where $C_0$ is the constant provided by Theorem 1.25. By approximation we can assume without loss of generality that each set of the sequence is bounded, that is, there exist $R_h > 0$ (which we can also take satisfying $R_h \to +\infty$) such that $E_h \subset B_{R_h}$.

We now define $F_h \subset \mathbb{R}^N$ as a solution to the penalization problem

$$\min\left\{\mathcal{J}_h(F) := \mathcal{F}(F) + \Lambda_1\sqrt{\big(\alpha(F,E) - \varepsilon_h\big)^2 + \varepsilon_h} + \Lambda_2\big||F| - |E|\big| \,:\, F \subset B_{R_h}\right\}, \tag{1.16}$$

where $\Lambda_1$ and $\Lambda_2$ are positive constant, to be chosen (notice that the constraint $F \subset B_{R_h}$ guarantees the existence of a solution). We first fix

$$\Lambda_1 > C_E + c_0\gamma\,. \tag{1.17}$$

Here $C_E$ is as in Lemma 1.29, while $c_0$ is the constant provided by Proposition 1.3 corresponding to the fixed values of $N$ and $\alpha$ and to $m := |E|+1$. We remark that with this choice $\Lambda_1$ depends only on the set $E$. We will consider also the sets $\widetilde{F}_h$ obtained by translating $F_h$ in such a way that $\alpha(F_h, E) = |\widetilde{F}_h \triangle E|$ (clearly $\mathcal{J}_h(\widetilde{F}_h) = \mathcal{J}_h(F_h)$).

*Step 1.* We claim that, if $\Lambda_2$ is sufficiently large (depending on $\Lambda_1$, but not on $h$), then $|F_h| = |E|$ for every $h$ large enough. This can be deduced by adapting an argument from [24, Section 2] (see also [1, Proposition 2.7]). Indeed, assume by contradiction that there exist $\Lambda_h \to \infty$ and $F_h$ solution to the minimum problem (1.16) with $\Lambda_2$ replaced by $\Lambda_h$ such that $|F_h| < |E|$ (a similar argument can be performed in the case $|F_h| > |E|$). Up to subsequences, we have that $F_h \to F_0$ in $L^1_{\mathrm{loc}}$ and $|F_h| \to |E|$.

As each set $F_h$ minimizes the functional

$$\mathcal{F}(F) + \Lambda_1\sqrt{\big(\alpha(F,E) - \varepsilon_h\big)^2 + \varepsilon_h}$$

in $B_{R_h}$ under the constraint $|F| = |F_h|$, it is easily seen that $F_h$ is a quasi-minimizer of the perimeter with volume constraint, so that by the regularity result contained in [61, Theorem 1.4.4] we have that the $(N-1)$-dimensional density of $\partial^* F_h$ is uniformly bounded from below by a constant independent of $h$. This observation implies that we can assume without loss of generality that the limit set $F_0$ is not empty and that there exists a point $x_0 \in \partial^* F_0$, so that, by repeating an argument contained in [24], we obtain that given $\varepsilon > 0$ we can find $r > 0$ and $\bar{x} \in \mathbb{R}^N$ such that

$$|F_h \cap B_{r/2}(\bar{x})| < \varepsilon r^N, \quad |F_h \cap B_r(\bar{x})| > \frac{\omega_N r^N}{2^{N+2}}$$

for every $h$ sufficiently large (and we assume $\bar{x} = 0$ for simplicity).

Now we modify $F_h$ in $B_r$ by setting $G_h := \Phi_h(F_h)$, where $\Phi_h$ is the bilipschitz map

$$\Phi_h(x) := \begin{cases} \big(1 - \sigma_h(2^N - 1)\big)x & \text{if } |x| \le \frac{r}{2}, \\ x + \sigma_h\big(1 - \frac{r^N}{|x|^N}\big)x & \text{if } \frac{r}{2} < x < r, \\ x & \text{if } |x| \ge r, \end{cases}$$

and $\sigma_h \in (0, \frac{1}{2^N})$. It can be shown (see [24, Section 2], [1, Proposition 2.7] for details) that $\varepsilon$ and $\sigma_h$ can be chosen in such a way that $|G_h| = |E|$, and moreover there exists a dimensional constant $C > 0$ such that

$$J_{\Lambda_h}(F_h) - J_{\Lambda_h}(G_h) \ge \sigma_h\Big(C\Lambda_h r^N - (2^N N + C\gamma + C\Lambda_1)\mathcal{P}(F_h; B_r)\Big)$$

(where $J_{\Lambda_h}$ denotes the functional in (1.16) with $\Lambda_2$ replaced by $\Lambda_h$). This contradicts the minimality of $F_h$ for $h$ sufficiently large.

*Step 2.* We now show that

$$\lim_{h\to+\infty} \alpha(F_h, E) = 0. \tag{1.18}$$

Indeed, by Lemma 1.29 we have that

$$\mathcal{P}(E) \le \mathcal{P}(\widetilde{F}_h) + C_E|\widetilde{F}_h \triangle E|,$$

while by Proposition 1.3

$$|\mathcal{NL}(E) - \mathcal{NL}(\widetilde{F}_h)| \le c_0|\widetilde{F}_h \triangle E|.$$

Combining the two estimates above, using the minimality of $F_h$ and recalling that $|F_h| = |E|$ we deduce

$$\begin{aligned}
\mathcal{P}(\widetilde{F}_h) + \gamma\mathcal{NL}(\widetilde{F}_h) + \Lambda_1\sqrt{\left(|\widetilde{F}_h\triangle E| - \varepsilon_h\right)^2 + \varepsilon_h} &= \mathcal{J}_h(F_h) \le \mathcal{J}_h(E)\\
&= \mathcal{P}(E) + \gamma\mathcal{NL}(E) + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h}\\
&\le \mathcal{P}(\widetilde{F}_h) + \gamma\mathcal{NL}(\widetilde{F}_h) + (C_E + c_0\gamma)|\widetilde{F}_h\triangle E| + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h},
\end{aligned}$$

which yields

$$\Lambda_1\sqrt{\left(|\widetilde{F}_h\triangle E| - \varepsilon_h\right)^2 + \varepsilon_h} \le (C_E + c_0\gamma)|\widetilde{F}_h\triangle E| + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h}.$$

Passing to the limit as $h \to +\infty$, we conclude that

$$\Lambda_1 \limsup_{h\to+\infty} |\widetilde{F}_h\triangle E| \le (C_E + c_0\gamma)\limsup_{h\to+\infty} |\widetilde{F}_h\triangle E|,$$

which implies $|\widetilde{F}_h\triangle E| \to 0$ by the choice of $\Lambda_1$ in (1.17). Hence (1.18) is proved, and this shows in particular that $\chi_{\widetilde{F}_h} \to \chi_E$ in $L^1(\mathbb{R}^N)$.

*Step 3.* Each set $F_h$ is an $(\omega, r_0)$-minimizer of the area functional (see Definition 1.26), for suitable $\omega > 0$ and $r_0 > 0$ independent of $h$. Indeed, choose $r_0$ such that $\omega_N {r_0}^N \le 1$, and consider any ball $B_r(x)$ with $r \le r_0$ and any finite perimeter set $F$ such that $F\triangle F_h \subset\subset B_r(x)$. We have

$$|\mathcal{NL}(F) - \mathcal{NL}(F_h)| \le c_0|F\triangle F_h|$$

by Proposition 1.3, where $c_0$ is the same constant as before since we can bound the volume of $F$ by $|F| \le |F_h| + \omega_N {r_0}^N \le |E| + 1$. Moreover

$$\begin{aligned}
\mathcal{P}(F) - \mathcal{P}(F\cap B_{R_h}) &= \int_{\partial^* F\setminus B_{R_h}} 1\,\mathrm{d}\mathcal{H}^{N-1}(x) - \int_{\partial^*(F\cap B_{R_h})\cap\partial B_{R_h}} 1\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&\ge \int_{\partial^* F\setminus B_{R_h}} \frac{x}{|x|}\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1}(x) - \int_{\partial^*(F\cap B_{R_h})\cap\partial B_{R_h}} \frac{x}{|x|}\cdot\nu_{F\cap B_{R_h}}\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&= \int_{\partial^*(F\setminus B_{R_h})} \frac{x}{|x|}\cdot\nu_{F\setminus B_{R_h}}\,\mathrm{d}\mathcal{H}^{N-1}(x) = \int_{F\setminus B_{R_h}} \mathrm{div}\,\frac{x}{|x|}\,\mathrm{d}x \ge 0.
\end{aligned}$$

Hence, as $F_h$ is a minimizer of $\mathcal{J}_h$ among sets contained in $B_{R_h}$, we deduce

$$\begin{aligned}
\mathcal{P}(F_h) &\le \mathcal{P}(F\cap B_{R_h}) + \gamma\big(\mathcal{NL}(F\cap B_{R_h}) - \mathcal{NL}(F_h)\big) + \Lambda_2\big||F\cap B_{R_h}| - |E|\big|\\
&\quad + \Lambda_1\sqrt{\left(\alpha(F\cap B_{R_h}, E) - \varepsilon_h\right)^2 + \varepsilon_h} - \Lambda_1\sqrt{\left(\alpha(F_h, E) - \varepsilon_h\right)^2 + \varepsilon_h}\\
&\le \mathcal{P}(F) + \big(c_0\gamma + \Lambda_1 + \Lambda_2\big)|(F\cap B_{R_h})\triangle F_h|\\
&\le \mathcal{P}(F) + \big(c_0\gamma + \Lambda_1 + \Lambda_2\big)|F\triangle F_h|
\end{aligned}$$

for $h$ large enough. This shows that $F_h$ is an $(\omega, r_0)$-minimizer of the area functional with $\omega = c_0\gamma + \Lambda_1 + \Lambda_2$ (and the same holds obviously also for $\widetilde{F}_h$).

Hence, by Theorem 1.27 and recalling that $\chi_{\widetilde{F}_h} \to \chi_E$ in $L^1$, we deduce that for $h$ sufficiently large $\widetilde{F}_h$ is a set of class $C^{1,\frac{1}{2}}$ and

$$\partial\widetilde{F}_h = \{x + \psi_h(x)\nu_E(x) \,:\, x \in \partial E\}$$

for some $\psi_h$ such that $\psi_h \to 0$ in $C^{1,\beta}(\partial E)$ for every $\beta \in (0, \frac{1}{2})$. We remark also that the sets $\widetilde{F}_h$ are uniformly bounded, and for $h$ large enough $\widetilde{F}_h \subset\subset B_{R_h}$: in particular, $\widetilde{F}_h$ solves the minimum problem (1.16).

*Step 4.* We now claim that

$$\lim_{h\to+\infty} \frac{\alpha(F_h, E)}{\varepsilon_h} = 1. \tag{1.19}$$

Indeed, assuming by contradiction that $|\alpha(F_h, E) - \varepsilon_h| \geq \sigma\varepsilon_h$ for some $\sigma > 0$ and for infinitely many $h$, we would obtain

$$\begin{aligned}
\mathcal{F}(F_h) + \Lambda_1\sqrt{\sigma^2\varepsilon_h^2 + \varepsilon_h} &\leq \mathcal{F}(F_h) + \Lambda_1\sqrt{\big(\alpha(F_h, E) - \varepsilon_h\big)^2 + \varepsilon_h} \\
&\leq \mathcal{F}(E_h) + \Lambda_1\sqrt{\varepsilon_h} < \mathcal{F}(E) + \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h} \\
&\leq \mathcal{F}(\widetilde{F}_h) + \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h}
\end{aligned}$$

where the second inequality follows from the minimality of $F_h$, the third one from (1.15) and the last one from Theorem 1.30. This shows that

$$\Lambda_1\sqrt{\sigma^2\varepsilon_h^2 + \varepsilon_h} \leq \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h}\,,$$

which is a contradiction for $h$ large enough.

*Step 5.* We now show the existence of constants $\lambda_h \in \mathbb{R}$ such that

$$\|H_{\partial\widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h} - \lambda_h\|_{L^\infty(\partial\widetilde{F}_h)} \leq 4\Lambda_1\sqrt{\varepsilon_h} \to 0. \tag{1.20}$$

We first observe that the function $f_h(t) := \sqrt{(t-\varepsilon_h)^2 + \varepsilon_h}$ satisfies

$$|f_h(t_1) - f_h(t_2)| \leq 2\sqrt{\varepsilon_h}\,|t_1 - t_2| \qquad \text{if} \quad |t_i - \varepsilon_h| \leq \varepsilon_h. \tag{1.21}$$

Hence for every set $F \subset \mathbb{R}^N$ with $|F| = |E|$, $F \subset B_{R_h}$ and $|\alpha(F, E) - \varepsilon_h| \leq \varepsilon_h$ we have

$$\begin{aligned}
\mathcal{F}(\widetilde{F}_h) &\leq \mathcal{F}(F) + \Lambda_1\Big(\sqrt{\big(\alpha(F, E) - \varepsilon_h\big)^2 + \varepsilon_h} - \sqrt{\big(\alpha(\widetilde{F}_h, E) - \varepsilon_h\big)^2 + \varepsilon_h}\Big) \\
&\leq \mathcal{F}(F) + 2\Lambda_1\sqrt{\varepsilon_h}\,|\alpha(F, E) - \alpha(\widetilde{F}_h, E)| \\
&\leq \mathcal{F}(F) + 2\Lambda_1\sqrt{\varepsilon_h}\,|F \triangle \widetilde{F}_h|
\end{aligned} \tag{1.22}$$

where we used the minimality of $\widetilde{F}_h$ in the first inequality, and (1.21) combined with the fact that $|\alpha(\widetilde{F}_h, E) - \varepsilon_h| \leq \varepsilon_h$ for $h$ large (which, in turn, follows by (1.19)) in the second one.

Consider now any variation $\Phi_t$, as in Definition 1.15, preserving the volume of the set $\widetilde{F}_h$, associated with a vector field $X$. For $|t|$ sufficiently small we can plug the set $\Phi_t(\widetilde{F}_h)$ in the inequality (1.22):

$$\mathcal{F}(\widetilde{F}_h) \leq \mathcal{F}(\Phi_t(\widetilde{F}_h)) + 2\Lambda_1\sqrt{\varepsilon_h}\,|\Phi_t(\widetilde{F}_h) \triangle \widetilde{F}_h|,$$

which gives

$$\mathcal{F}(\Phi_t(\widetilde{F}_h)) - \mathcal{F}(\widetilde{F}_h) + 2\Lambda_1\sqrt{\varepsilon_h}\,|t| \int_{\partial\widetilde{F}_h} |X \cdot \nu_{\widetilde{F}_h}|\,\mathrm{d}\mathcal{H}^{N-1} + o(t) \geq 0$$

for $|t|$ sufficiently small. Hence, dividing by $t$ and letting $t \to 0^+$ and $t \to 0^-$, we get

$$\left| \int_{\partial \widetilde{F}_h} (H_{\partial \widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h}) X \cdot \nu_{\widetilde{F}_h} \, d\mathcal{H}^{N-1} \right| \le 2\Lambda_1 \sqrt{\varepsilon_h} \int_{\partial \widetilde{F}_h} |X \cdot \nu_{\widetilde{F}_h}| \, d\mathcal{H}^{N-1},$$

and by density

$$\left| \int_{\partial \widetilde{F}_h} (H_{\partial \widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h}) \varphi \, d\mathcal{H}^{N-1} \right| \le 2\Lambda_1 \sqrt{\varepsilon_h} \int_{\partial \widetilde{F}_h} |\varphi| \, d\mathcal{H}^{N-1}$$

for every $\varphi \in C^\infty(\partial \widetilde{F}_h)$ with $\int_{\partial \widetilde{F}_h} \varphi \, d\mathcal{H}^{N-1} = 0$. In turn, this implies (1.20) by a simple functional analysis argument.

*Step 6.* We are now close to the end of the proof. Recall that on $\partial E$

$$H_{\partial E} = \lambda - 2\gamma v_E \tag{1.23}$$

for some constant $\lambda$, while by (1.20)

$$H_{\partial \widetilde{F}_h} = \lambda_h - 2\gamma v_{\widetilde{F}_h} + \rho_h, \qquad \text{with } \rho_h \to 0 \text{ uniformly.} \tag{1.24}$$

Observe now that, since the functions $v_{\widetilde{F}_h}$ are equibounded in $C^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$ (see Proposition 1.1) and they converge pointwise to $v_E$ since $\chi_{\widetilde{F}_h} \to \chi_E$ in $L^1$, we have that

$$v_{\widetilde{F}_h} \to v_E \qquad \text{in } C^1(\overline{B}_R) \text{ for every } R > 0. \tag{1.25}$$

We consider a cylinder $C = B' \times ]-L, L[$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centered at the origin, such that in a suitable coordinate system we have

$$\begin{aligned} \widetilde{F}_h \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g_h(x')\}, \\ E \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g(x')\} \end{aligned}$$

for some functions $g_h \to g$ in $C^{1,\beta}(\overline{B'})$ for every $\beta \in (0, \frac{1}{2})$. By integrating (1.24) on $B'$ we obtain

$$\begin{aligned} &\lambda_h \mathcal{L}^{N-1}(B') - 2\gamma \int_{B'} v_{\widetilde{F}_h}(x', g_h(x')) \, d\mathcal{L}^{N-1}(x') + \int_{B'} \rho_h(x', g_h(x')) \, d\mathcal{L}^{N-1}(x') \\ &= - \int_{B'} \operatorname{div}\left( \frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \right) d\mathcal{L}^{N-1}(x') = - \int_{\partial B'} \frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \cdot \frac{x'}{|x'|} \, d\mathcal{H}^{N-2}, \end{aligned}$$

and the last integral in the previous expression converges as $h \to 0$ to

$$\begin{aligned} - \int_{\partial B'} \frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \cdot \frac{x'}{|x'|} \, d\mathcal{H}^{N-2} &= - \int_{B'} \operatorname{div}\left( \frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \right) d\mathcal{L}^{N-1}(x') \\ &= \lambda \mathcal{L}^{N-1}(B') - 2\gamma \int_{B'} v_E(x', g(x')) \, d\mathcal{L}^{N-1}(x'), \end{aligned}$$

where the last equality follows by (1.23). This shows, recalling (1.25) and that $\rho_h$ tends to 0 uniformly, that $\lambda_h \to \lambda$, which in turn implies, by (1.23), (1.24) and (1.25),

$$H_{\partial \widetilde{F}_h}(\cdot + \psi_h(\cdot)\nu_E(\cdot)) \to H_{\partial E} \qquad \text{in } L^\infty(\partial E).$$

By Lemma 1.28 we conclude that $\psi_h \in W^{2,p}(\partial E)$ for every $p \ge 1$ and $\psi_h \to 0$ in $W^{2,p}(\partial E)$.

Finally, by minimality of $\widetilde{F}_h$ we have

$$\begin{aligned} \mathcal{F}(\widetilde{F}_h) &\le \mathcal{F}(\widetilde{F}_h) + \Lambda_1 \sqrt{\big(\alpha(\widetilde{F}_h, E) - \varepsilon_h\big)^2 + \varepsilon_h} - \Lambda_1 \sqrt{\varepsilon_h} \\ &\le \mathcal{F}(E_h) < \mathcal{F}(E) + \frac{C_0}{4} \varepsilon_h^2 \le \mathcal{F}(E) + \frac{C_0}{2} \big(\alpha(\widetilde{F}_h, E)\big)^2 \end{aligned}$$

where we used (1.15) in the third inequality and (1.19) in the last one. This is the desired contradiction with the conclusion of Theorem 1.25. □

REMARK 1.31. It is important to remark that in the arguments of this section we have not made use of the assumption of strict positivity of the second variation: the quantitative $L^1$-local minimality follows in fact just from the $W^{2,p}$-local minimality.

## 1.4. Local minimality of the ball

In this section we will obtain Theorem 1.9 as a consequence of Theorem 1.8, by computing the second variation of the ball and studying the sign of the associated quadratic form.

**1.4.1. Recalls on spherical harmonics.** We first recall some basic facts about spherical harmonics, referring to [33] for an account on this topic.

DEFINITION 1.32. A *spherical harmonic of dimension* $N$ is the restriction to $S^{N-1}$ of a *harmonic polynomial* in $N$ variables, *i.e.* a homogeneous polynomial $p$ with $\Delta p = 0$.

We will denote by $\mathcal{H}_d^N$ the set of all spherical harmonics of dimension $N$ that are obtained as restrictions to $S^{N-1}$ of homogeneous polynomials of degree $d$. In particular $\mathcal{H}_0^N$ is the space of constant functions, and $\mathcal{H}_1^N$ is generated by the coordinate functions. The basic properties of spherical harmonics that we need are listed in the following theorem (see [33, Chapter 3]).

THEOREM 1.33. *The following properties hold.*

*(1) For each $d \in \mathbb{N}$, $\mathcal{H}_d^N$ is a finite dimensional vector space.*
*(2) If $F \in \mathcal{H}_d^N$, $G \in \mathcal{H}_e^N$ and $d \neq e$, then $F$ and $G$ are orthogonal (in the $L^2$-sense).*
*(3) If $F \in \mathcal{H}_d^N$ and $d \neq 0$, then*

$$\int_{S^{N-1}} F \, \mathrm{d}\mathcal{H}^{N-1} = 0.$$

*(4) If $(H_d^1, \dots, H_d^{\dim(\mathcal{H}_d^N)})$ is an orthonormal basis of $\mathcal{H}_d^N$ for every $d \geq 0$, then this sequence is complete,* i.e. *every $F \in L^2(S^{N-1})$ can be written in the form*

$$F = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i , \tag{1.26}$$

*where $c_d^i := \langle F, H_d^i \rangle_{L^2}$.*
*(5) If $H_d^i$ are as in (4) and $F, G \in L^2(S^{N-1})$ are such that*

$$F = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i , \qquad G = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} e_d^i H_d^i ,$$

*then*

$$\langle F, G \rangle_{L^2} = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i e_d^i .$$

*(6) Spherical harmonics are eigenfunctions of the* Laplace-Beltrami operator $\Delta_{S^{N-1}}$. *More precisely, if $H \in \mathcal{H}_d^N$ then*

$$-\Delta_{S^{N-1}} H = d(d + N - 2)H.$$

*(7) If $F$ is a $C^2$ function on $S^{N-1}$ represented as in (1.26), then*

$$\int_{S^{N-1}} |D_\tau F|^2 \,\mathrm{d}\mathcal{H}^{N-1}(x) \;=\; \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} d(d+N-2)(c_d^i)^2 .$$

We recall also the following important result in the theory of spherical harmonics.

THEOREM 1.34 (Funk-Hecke Formula). *Let $f : (-1,1) \to \mathbb{R}$ such that*

$$\int_{-1}^{1} |f(t)|(1-t^2)^{\frac{N-3}{2}} \,\mathrm{d}t < \infty .$$

*Then if $H \in \mathcal{H}_d^N$ and $x_0 \in S^{N-1}$ it holds*

$$\int_{S^{N-1}} f(\langle x_0, x\rangle) H(x) \,\mathrm{d}\mathcal{H}^{N-1}(x) \;=\; \mu_d H(x_0) ,$$

*where the coefficient $\mu_d$ is given by*

$$\mu_d = (N-1)\omega_{N-1} \int_{-1}^{1} P_{N,d}(t) f(t)(1-t^2)^{\frac{N-3}{2}} \,\mathrm{d}t.$$

*Here $P_{N,d}$ is the* Legendre polynomial *of dimension $N$ and degree $d$ given by*

$$P_{N,d}(t) = (-1)^d \frac{\Gamma(\frac{N-1}{2})}{2^d \Gamma(d + \frac{N-1}{2})} (1-t^2)^{-\frac{N-3}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^d (1-t^2)^{d+\frac{N-3}{2}} ,$$

*where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$ is the Gamma function.*

**1.4.2. Second variation of the ball.** The quadratic form (1.11) associated with the second variation of $\mathcal{F}$ at the ball $B_R$, computed at a function $\tilde{\varphi} \in \widetilde{H}^1(\partial B_R)$ is

$$\begin{aligned}
\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] = &\int_{\partial B_R} \Big( |D_\tau \tilde{\varphi}(x)|^2 - \frac{N-1}{R^2}\, \tilde{\varphi}^2(x) \Big) \,\mathrm{d}\mathcal{H}^{N-1}(x) \\
&+ 2\gamma \int_{\partial B_R} \int_{\partial B_R} \frac{1}{|x-y|^\alpha} \tilde{\varphi}(x) \tilde{\varphi}(y) \,\mathrm{d}\mathcal{H}^{N-1}(x) \,\mathrm{d}\mathcal{H}^{N-1}(y) \\
&+ 2\gamma \int_{\partial B_R} \Big( \int_{B_R} -\alpha \frac{\langle x-y, \frac{x}{|x|} \rangle}{|x-y|^{\alpha+2}} \,\mathrm{d}y \Big) \tilde{\varphi}^2(x) \,\mathrm{d}\mathcal{H}^{N-1}(x).
\end{aligned}$$

Since we want to obtain a sign condition of $\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}]$ in terms of the radius $R$, we first make a change of variable:

$$\begin{aligned}
\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] = R^{N-3} &\int_{\partial B_1} (|D_\tau \varphi(x)|^2 - (N-1)\varphi^2(x)) \,\mathrm{d}\mathcal{H}^{N-1}(x) \\
&+ 2\gamma R^{2N-2-\alpha} \int_{\partial B_1} \int_{\partial B_1} \frac{1}{|x-y|^\alpha} \varphi(x) \varphi(x) \,\mathrm{d}\mathcal{H}^{N-1}(x) \,\mathrm{d}\mathcal{H}^{N-1}(y) \qquad (1.27) \\
&+ 2\gamma R^{2N-2-\alpha} \int_{\partial B_1} \Big( \int_{B_1} -\alpha \frac{\langle x-y, x \rangle}{|x-y|^{\alpha+2}} \,\mathrm{d}y \Big) \varphi^2(x) \,\mathrm{d}\mathcal{H}^{N-1}(x),
\end{aligned}$$

where the function $\varphi \in \widetilde{H}^1(S^{N-1})$ is defined as $\varphi(x) := \tilde{\varphi}(Rx)$. Since we are only interested in the sign of the second variation, which is continuous with respect to the strong convergence in $\widetilde{H}^1(S^{N-1})$, we can assume $\varphi \in C^2(S^{N-1}) \cap T^\perp(S^{N-1})$.

The idea to compute the second variation at the ball is to expand $\varphi$ with respect to an orthonormal basis of spherical harmonics, as in (1.26). First of all we notice that if $\varphi \in T^\perp(S^{N-1})$, then its harmonic expansion does not contain spherical harmonics of order 0 and

1. Indeed, harmonics of order 0 are constant functions, that are not allowed by the null average condition. Moreover $\mathcal{H}_1^N = T(S^{N-1})$, because $\nu_{S^{N-1}}(x) = x$, and the functions $x_i$ form an orthonormal basis of $\mathcal{H}_1^N$. Hence we can write the harmonic expansion of $\varphi \in C^2(S^{N-1}) \cap T^\perp(S^{N-1})$ as follows:

$$\varphi = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i \,,$$

where $(H_d^1, \ldots, H_d^{\dim(\mathcal{H}_d^N)})$ is an orthonormal basis of $\mathcal{H}_d^N$ for each $d \in \mathbb{N}$. We can now compute each term appearing in (1.27) as follows: the first term, by property (7) of Theorem 1.33, is

$$\int_{\partial B_1} (|D_\tau \varphi|^2 - (N-1)\varphi^2)\,\mathrm{d}\mathcal{H}^{N-1} = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \big(d(d+N-2) - (N-1)\big)(c_d^i)^2.$$

For the second term we want to use the Funk-Hecke Formula to compute the inner integral; so we define the function

$$f(t) := \Big(2(1-t)\Big)^{-\frac{\alpha}{2}}$$

and we notice that

$$|x-y|^{-\alpha} = f(\langle x, y\rangle) \qquad \text{for } x, y \in S^{N-1}\,,$$

and that, for $\alpha \in (0, N-1)$, $f$ satisfies the integrability assumptions of Theorem 1.34. Hence for each $y \in S^{N-1}$

$$\int_{\partial B_1} \frac{1}{|x-y|^\alpha} \varphi(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \;=\; \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \mu_d^{N,\alpha} c_d^i H_d^i(y)\,,$$

where the coefficient

$$\mu_d^{N,\alpha} \;:=\; 2^{N-1-\alpha} \frac{(N-1)\omega_{N-1}}{2} \left( \prod_{i=0}^{d-1} \left(\frac{\alpha}{2} + i\right)\right) \frac{\Gamma(\frac{N-1-\alpha}{2})\Gamma(\frac{N-1}{2})}{\Gamma(N-1-\frac{\alpha}{2}+d)} \tag{1.28}$$

is obtained by direct computation just integrating by parts. Therefore

$$\int_{\partial B_1} \int_{\partial B_1} \frac{1}{|x-y|^\alpha} \varphi(x)\varphi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \mu_d^{N,\alpha}(c_d^i)^2.$$

For the last term of (1.27), noticing that the integral

$$\mathcal{I}^{N,\alpha} \;:=\; \int_{B_1} \frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y$$

is independent of $x \in S^{N-1}$, we get

$$\int_{\partial B_1} \Big( \int_{B_1} -\alpha \frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y \Big) \varphi^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) = -\alpha \mathcal{I}^{N,\alpha} \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} (c_d^i)^2.$$

Combining all the previous equalities with (1.27) we obtain

$$\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} R^{N-3}(c_d^i)^2 \Big[ d(d+N-2) - (N-1) + 2\gamma R^{N+1-\alpha}\Big(\mu_d^{N,\alpha} - \alpha \mathcal{I}^{N,\alpha}\Big)\Big].$$

**1.4.3. Local minimality of the ball.** From the above expression we deduce that the quadratic form $\partial^2\mathcal{F}(B_R)$ is strictly positive on $T^\perp(\partial B_R)$, that is, the second variation of $\mathcal{F}$ at $B_R$ is positive according to Definition 1.22, if and only if

$$d(d+N-2)-(N-1)+2\gamma R^{N+1-\alpha}\Big(\mu_d^{N,\alpha}-\alpha\mathcal{I}^{N,\alpha}\Big)>0 \tag{1.29}$$

for all $d\geq 2$, where the "only if" part is due to the fact that $\mathcal{H}_d^N\subset T^\perp(S^{N-1})$ for each $d\geq 2$. On the contrary, $\partial^2\mathcal{F}(B_R)[\tilde\varphi]<0$ for some $\tilde\varphi\in T^\perp(\partial B_R)$ if and only if there exists $d\geq 2$ such that the left-hand side of (1.29) is negative.

We want to write (1.29) as a condition on $R$. Since $d(d+N-2)-(N-1)>0$ for $d\geq 2$, we have that (1.29) is certainly satisfied if $\mu_d^{N,\alpha}-\alpha\mathcal{I}^{N,\alpha}>0$. But this is not always the case, as the following lemma shows.

LEMMA 1.35. *The sequence $\mu_d^{N,\alpha}$ strictly decreases to 0 as $d\to\infty$.*

PROOF. First of all we note that

$$\mu_{d+1}^{N,\alpha}=\frac{\frac{\alpha}{2}+d}{N-1-\frac{\alpha}{2}+d}\,\mu_d^{N,\alpha}, \tag{1.30}$$

hence the sequence $(\mu_d^{N,\alpha})_{d\in\mathbb{N}}$ is decreasing since $\alpha<N-1$. Now

$$\begin{aligned}\mu_{d+1}^{N,\alpha}&=\Big(\prod_{k=1}^{d}\frac{\frac{\alpha}{2}+k}{N-1-\frac{\alpha}{2}+k}\Big)\mu_1^{N,\alpha}=\frac{\Gamma(N-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2}+d)}{\Gamma(1+\frac{\alpha}{2})\Gamma(N-\frac{\alpha}{2}+d)}\mu_1^{N,\alpha}\\&\sim_{d\to\infty}\frac{\Gamma(N-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})}\mu_1^{N,\alpha}\sqrt{\frac{\frac{\alpha}{2}+d}{N-1-\frac{\alpha}{2}+d}}\,\frac{e^{(\frac{\alpha}{2}+d)[\log(\frac{\alpha}{2}+d)-1]}}{e^{(N-1-\frac{\alpha}{2}+d)[\log(N-1-\frac{\alpha}{2}+d)-1]}},\end{aligned}$$

where in the second equality we used the well known property $\Gamma(x+1)=x\Gamma(x)$, and in the last step we used the Stirling's formula. Since the previous quantity is infinitesimal as $d\to\infty$, we conclude the proof of the lemma. $\square$

As a consequence of this lemma and of the fact that $\mathcal{I}^{N,\alpha}>0$, we have that the number

$$d_A^{N,\alpha}:=\min\{d\geq 2\,:\,\mu_d^{N,\alpha}<\alpha\mathcal{I}^{N,\alpha}\}$$

is well defined. This tells us that (1.29) is satisfied for every $R>0$ if $d<d_A^{N,\alpha}$, and for

$$R<\left(\frac{d(d+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_d^{N,\alpha}\big)}\right)^{\frac{1}{N+1-\alpha}}=:g^{N,\alpha}(d).$$

if $d\geq d_A^{N,\alpha}$. Moreover, by the previous lemma we get that $g^{N,\alpha}(d)\to\infty$ as $d\to\infty$. The following lemma tells us something more about the behaviour of the function $g^{N,\alpha}$.

LEMMA 1.36. *There exists a natural number $d_I^{N,\alpha}$ such that for $d<d_I^{N,\alpha}$ the function $g^{N,\alpha}$ is decreasing, while for $d>d_I^{N,\alpha}$ is increasing.*

PROOF. The condition $g^{N,\alpha}(d+1)>g^{N,\alpha}(d)$ is equivalent to

$$\frac{(d+1)(d+1+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_{d+1}^{N,\alpha}\big)}>\frac{d(d+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_d^{N,\alpha}\big)}.$$

Recalling (1.30), the above inequality can be rewritten, after some algebraic steps, as follows:

$$\alpha\mathcal{I}^{N,\alpha}>\frac{d^2(N-\alpha+1)+d(N^2-\alpha N+\alpha-1)+\frac{\alpha}{2}(N-1)}{(N-1-\frac{\alpha}{2}+d)(2d+N-1)}\,\mu_d^{N,\alpha}. \tag{1.31}$$

Using (1.30), it is easily seen that the right-hand side of the above inequality is decreasing and converges to 0 as $d \to \infty$. Hence the number

$$d_I^{N,\alpha} := \min\{d \in \mathbb{N} \,:\, (1.31) \text{ is satisfied}\}$$

is well defined and satisfies the requirement of the lemma. □

We are now in position to prove Theorem 1.9.

Proof of Theorem 1.9. Define

$$\overline{R}(N,\alpha,\gamma) := \min_{d \geq d_A^{N,\alpha}} g^{N,\alpha}(d)\,,$$

which can be characterized, by the previous lemmas, as

$$\overline{R}(N,\alpha,\gamma) := \begin{cases} g^{N,\alpha}(d_A^{N,\alpha}) & \text{if } d_A^{N,\alpha} > d_I^{N,\alpha}, \\[2ex] g^{N,\alpha}(d_I^{N,\alpha}) & \text{if } d_A^{N,\alpha} \leq d_I^{N,\alpha}. \end{cases}$$

Now, from (1.29), we have that

$$\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] > 0 \text{ for every } \tilde{\varphi} \in T^{\perp}(\partial B_R) \quad \Longleftrightarrow \quad R < \overline{R}(N,\alpha,\gamma),$$

while

$$\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] < 0 \text{ for some } \tilde{\varphi} \in T^{\perp}(\partial B_R) \quad \Longleftrightarrow \quad R > \overline{R}(N,\alpha,\gamma).$$

By virtue of Theorem 1.8 and Corollary 1.21, we obtain the first part of the theorem, where $m_{\mathrm{loc}}(N,\alpha,\gamma)$ is the volume of the ball of radius $\overline{R}(N,\alpha,\gamma)$.

In order to show that the critical radius tends to $\infty$ as $\alpha \to 0$, we notice that

$$\partial^2 \mathcal{F}(B_R)[\tilde{\varphi}] \geq \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} (c_d^i)^2 R^{N-3}\big(N+1-2\gamma\alpha \mathcal{I}^{N,\alpha} R^{N+1-\alpha}\big).$$

Since

$$\mathcal{I}^{N,\alpha} \overset{\alpha \to 0^+}{\longrightarrow} \int_{B_1} \frac{\langle x-y, x\rangle}{|x-y|^2}\,\mathrm{d}y < \infty\,,$$

we have that for each $R > 0$ there exists $\overline{\alpha}(N,\gamma,R) > 0$ such that for each $\alpha < \overline{\alpha}(N,\gamma,R)$

$$\alpha \mathcal{I}^{N,\alpha} < \frac{N+1}{2\gamma R^{N+1-\alpha}},$$

which immediately implies the claim. To conclude the proof we examine in more details the special case $N = 3$, determining explicitly the critical mass $m_{\mathrm{loc}}$. From (1.28) we have that

$$\mu_d^{3,\alpha} = 2^{2-\alpha}\pi\Big(\prod_{j=0}^{d-1}\Big(\frac{\alpha}{2}+j\Big)\Big)\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(2+d-\frac{\alpha}{2})} = 2^{2-\alpha}\pi\alpha\frac{\Big(\prod_{j=1}^{d-1}\Big(\frac{\alpha}{2}+j\Big)\Big)}{\prod_{j=1}^{d-1}\Big(1-\frac{\alpha}{2}+j\Big)}\frac{1}{d+1-\frac{\alpha}{2}}\frac{1}{2-\alpha}\,,$$

where we used the property $\Gamma(x+1) = x\Gamma(x)$. Moreover, we compute explicitly in the Appendix the integral $\mathcal{I}^{3,\alpha}$, obtaining (see (1.55))

$$\mathcal{I}^{3,\alpha} = 2\pi\frac{2^{2-\alpha}}{(4-\alpha)(2-\alpha)}.$$

It is now easily seen that $d_I^{3,\alpha} = d_A^{3,\alpha} = 2$ for every $\alpha \in (0,2)$. Hence

$$\overline{R}(3,\alpha,\gamma) = \left(\frac{(6-\alpha)(4-\alpha)}{2^{3-\alpha}\gamma\alpha\pi}\right)^{\frac{1}{4-\alpha}},$$

which completes the proof of the theorem. □

## 1.5. Global minimality

This section is devoted to the proof of the results concerning global minimality issues. We start by showing how the information gained in Theorem 1.9 can be used to prove the global minimality of the ball for small volumes.

Proof of Theorem 1.10. By scaling, we can equivalently prove that given $N \geq 2$ and $\alpha \in (0, N-1)$ and setting

$$\bar{\gamma} := \sup\big\{\gamma > 0 : B_1 \text{ is a global minimizer of } \mathcal{F}_{\alpha,\gamma} \text{ in } \mathbb{R}^N \text{ under volume constraint}\big\},$$

we have that $\bar{\gamma} \in (0, \infty)$ and $B_1$ is the unique global minimizer of $\mathcal{F}_{\alpha,\gamma}$ for every $\gamma < \bar{\gamma}$.

We start assuming by contradiction that there exist a sequence $\gamma_n \to 0$ and a sequence of sets $E_n$, with $|E_n| = |B_1|$ and $\alpha(E_n, B_1) > 0$, such that

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1). \tag{1.32}$$

By translating $E_n$ so that $\alpha(E_n, B_1) = |E_n \triangle B_1|$, from (1.32) one immediately gets

$$C(N)\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)\big) \leq \gamma_n c_0 |E_n \triangle B_1|$$

where the first inequality follows from the quantitative isoperimetric inequality and the last one from Proposition 1.3. Hence, as $\gamma_n \to 0$, we deduce that $\alpha(E_n, B_1) \to 0$.

From the results of Section 1.4 it follows that if $\gamma_0 > 0$ is sufficiently small then the functional $\mathcal{F}_{\alpha,\gamma_0}$ has positive second variation at $B_1$: by Theorem 1.8, this implies the existence of a positive $\delta$ such that

$$\mathcal{F}_{\alpha,\gamma_0}(B_1) < \mathcal{F}_{\alpha,\gamma_0}(E) \qquad \text{for every } E \text{ with } |E| = |B_1| \text{ and } 0 < \alpha(E, B_1) < \delta. \tag{1.33}$$

We now want to show that (1.33) holds for every $\gamma < \gamma_0$, with the same $\delta$. Indeed, assuming by contradiction the existence of $\gamma < \gamma_0$ and $E \subset \mathbb{R}^N$ such that $|E| = |B_1|$, $0 < \alpha(E, B_1) < \delta$ and

$$\mathcal{F}_{\alpha,\gamma}(E) \leq \mathcal{F}_{\alpha,\gamma}(B_1), \tag{1.34}$$

since $\mathcal{P}(B_1) < \mathcal{P}(E)$ we necessarily have $\mathcal{NL}_\alpha(E) < \mathcal{NL}_\alpha(B_1)$. Hence by (1.34)

$$\mathcal{P}(E) - \mathcal{P}(B_1) \leq \gamma\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big) < \gamma_0\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big)\,, \tag{1.35}$$

that is, $\mathcal{F}_{\alpha,\gamma_0}(E) < \mathcal{F}_{\alpha,\gamma_0}(B_1)$, which contradicts (1.33).

Now, since for $n$ large enough we have that $\gamma_n < \gamma_0$ and $0 < \alpha(E_n, B_1) < \delta$, the previous property is in contradiction with (1.32). This shows in particular that $\bar{\gamma} > 0$.

The fact that $\bar{\gamma}$ is finite follows from Theorem 1.9, which shows that for large masses the ball is not a local minimizer (and obviously not even a global minimizer).

Finally, assume by contradiction that for some $\gamma < \bar{\gamma}$ the ball is not the unique global minimizer, that is there exists a set $E$, with $|E| = |B_1|$ and $\alpha(E, B_1) > 0$, such that $\mathcal{F}_{\alpha,\gamma}(E) \leq \mathcal{F}_{\alpha,\gamma}(B_1)$. By definition of $\bar{\gamma}$, we can find $\gamma' \in (\gamma, \bar{\gamma})$ such that $B_1$ is a global minimizer of $\mathcal{F}_{\alpha,\gamma'}$. Arguing as before, we have that by the isoperimetric inequality $\mathcal{P}(B_1) < \mathcal{P}(E)$, which by our contradiction assumption implies that $\mathcal{NL}_\alpha(E) < \mathcal{NL}_\alpha(B_1)$; this yields

$$\mathcal{P}(E) - \mathcal{P}(B_1) \leq \gamma\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big) < \gamma'\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big)\,,$$

which contradicts the fact that $B_1$ is a global minimizer for $\mathcal{F}_{\alpha,\gamma'}$. $\square$

We now want to analyze what happens for small exponents $\alpha$. Since for $\alpha = 0$ the functional is just the perimeter, which is uniquely minimized by the ball, the intuition suggests that the unique minimizer of $\mathcal{F}_{\alpha,\gamma}$, for $\alpha$ close to 0, is the ball itself, as long as a minimizer exists. In order to prove the theorem, we need an auxiliary result: the non-existence volume threshold is uniformly bounded for $\alpha \in (0,1)$. The proof is a simple adaptation of the argument contained in [43, Section 2], where just the three-dimensional case with $\alpha = 1$ is considered.

PROPOSITION 1.37. *There exists $\bar{m} = \bar{m}(N,\gamma) < +\infty$ such that for every $m > \bar{m}$ the minimum problem*

$$I_m^\alpha := \inf\left\{\mathcal{F}_{\alpha,\gamma}(E) \,:\, E \subset \mathbb{R}^N,\, |E| = m\right\} \tag{1.36}$$

*does not have a solution for every $\alpha \in (0,1)$.*

PROOF. During the proof we will denote by $C$ a generic constant, depending only on $N$ and $\gamma$, which may change from line to line.

*Step 1.* We claim that there exists a constant $C_0$, depending only on $N$ and $\gamma$, such that

$$I_m^\alpha \le C_0\, m \qquad \text{for every } 0 < \alpha < N-1 \text{ and } m \ge 1. \tag{1.37}$$

Indeed, if $B$ is a ball of volume $m$, then

$$\mathcal{F}_{\alpha,\gamma}(B) = N{\omega_N}^{1/N} m^{(N-1)/N} + \gamma\, c_\alpha \left(\frac{m}{\omega_N}\right)^{\frac{2N-\alpha}{N}}, \qquad c_\alpha := \int_{B_1}\int_{B_1} \frac{1}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y\,.$$

It follows that for every $1 \le m < 2$ we have $I_m^\alpha \le C_0$, for some constant $C_0$ depending only on $N$ and $\gamma$. It is now easily seen that $I_m^\alpha \le I_{m_1}^\alpha + I_{m_2}^\alpha$ if $m = m_1 + m_2$ (see the proof of [43, Lemma 3]): hence by induction on $k$ we obtain $I_m^\alpha \le C_0 k$ for every $m \in [k, k+1)$.

*Step 2.* We claim that there exists a constant $C_1$, depending only on $N$ and $\gamma$, such that for every $0 < \alpha < N-1$ and $m \ge 1$, if $E$ is a solution to (1.36) then

$$|E \cap B_R(x)| \ge C_1 R^N \tag{1.38}$$

for every $R \le 1$ and for every $x \in E$ such that $|E \cap B_r(x)| > 0$ for all $r > 0$.

To prove the claim, assume without loss of generality that $x = 0$. It is clearly sufficient to show (1.38) for $\mathcal{L}^1$-a.e. $R < \varepsilon_0$, where $\varepsilon_0$ will be fixed later in the proof. In particular, from now on we can assume without loss of generality that $R$ is such that $\mathcal{H}^{N-1}(\partial E \cap \partial B_R) = 0$. We compare the energies of $E$ and $E' := \lambda(E \setminus B_R)$, where $\lambda > 1$ is such that $|E'| = m$: by minimality of $E$ we have $\mathcal{F}_{\alpha,\gamma}(E) \le \mathcal{F}_{\alpha,\gamma}(E')$, which gives after a direct computation

$$\mathcal{H}^{N-1}(\partial E \cap B_R) \le (\lambda^{2N-\alpha} - 1)\mathcal{F}_{\alpha,\gamma}(E) + \lambda^{N-1}\mathcal{H}^{N-1}(\partial B_R \cap E).$$

In turn this implies, by using $\mathcal{H}^{N-1}(\partial(E \cap B_R)) = \mathcal{H}^{N-1}(\partial E \cap B_R) + \mathcal{H}^{N-1}(\partial B_R \cap E)$ (recall that $\mathcal{H}^{N-1}(\partial E \cap \partial B_R) = 0$),

$$\mathcal{H}^{N-1}(\partial(E \cap B_R)) \le (\lambda^{2N-\alpha} - 1)\mathcal{F}_{\alpha,\gamma}(E) + (\lambda^{N-1} + 1)\mathcal{H}^{N-1}(\partial B_R \cap E).$$

Now, choosing $\varepsilon_0 > 0$ so small that $|E \setminus B_R| \ge \frac{1}{2}m$, we obtain the following estimates:

$$\lambda^{2N-\alpha} - 1 = \left(\frac{m}{|E\setminus B_R|}\right)^{\frac{2N-\alpha}{N}} - 1 \le C\left(\frac{m}{|E\setminus B_R|} - 1\right) \le C\frac{|E \cap B_R|}{m}, \quad \lambda^{N-1} \le C\,.$$

Hence from the isoperimetric inequality, (1.37), and from the above estimates we deduce that

$$|E \cap B_R|^{\frac{N-1}{N}} \le C|E \cap B_R| + C\mathcal{H}^{N-1}(\partial B_R \cap E).$$

Finally, observe that if $\varepsilon_0$ is sufficiently small we also have $|E \cap B_R| \le \frac{1}{2C}|E \cap B_R|^{\frac{N-1}{N}}$, hence we obtain

$$|E \cap B_R|^{\frac{N-1}{N}} \le C\mathcal{H}^{N-1}(\partial B_R \cap E) = C\frac{\mathrm{d}}{\mathrm{d}R}|E \cap B_R|,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}R}|E \cap B_R|^{\frac{1}{N}} \ge C \qquad \text{for } \mathcal{L}^1\text{-a.e. } R < \varepsilon_0.$$

By integrating the previous inequality we conclude the proof of the claim.

*Step 3.* We claim that there exists a constant $C_2$, depending only on $N$ and $\gamma$, such that for every $0<\alpha<1$ and $m\geq 1$, if $E$ is a solution to (1.36) then

$$\mathcal{NL}_\alpha(E)\geq C_2\, m\log m - C_2\, m \tag{1.39}$$

(notice that the conclusion of the proposition follows immediately from (1.37) and (1.39)).

In order to prove the claim, we first observe that

$$|E\cap B_R(x)|\geq CR \qquad \text{for every } x\in E \text{ and } 1<R<\tfrac{1}{2}\mathrm{diam}(E). \tag{1.40}$$

Indeed, as $E$ is not contained in $B_R(x)$ and $E$ is connected (see Theorem 1.7), we can find points $x_i\in E\cap\partial B_{R-i}(x)$ for $i=1,\dots,\lfloor R\rfloor$ such that $|E\cap B_r(x_i)|>0$ for every $r>0$. Then by (1.38)

$$|E\cap B_R(x)|\geq\sum_{i=1}^{\lfloor R\rfloor}|E\cap B_{\frac{1}{2}}(x_i)|\geq C_1\left(\frac{1}{2}\right)^N\lfloor R\rfloor\,.$$

Observe now that, if we set $E_R:=\{(x,y)\in E\times E : |x-y|<R\}$, we have by (1.40) that for every $1<R<\frac{1}{2}\mathrm{diam}(E)$

$$\mathcal{L}^{2N}(E_R)=\int_E|E\cap B_R(x)|\,\mathrm{d}x\geq C|E|R. \tag{1.41}$$

Hence

$$\begin{aligned}\mathcal{NL}_\alpha(E)&=\int_E\int_E\frac{1}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y=\frac{1}{\sqrt{2}}\int_0^{+\infty}\frac{1}{R^\alpha}\,\mathcal{H}^{2N-1}(\partial E_R)\,\mathrm{d}R\\&=\frac{1}{2}\int_0^{+\infty}\frac{1}{R^\alpha}\frac{\mathrm{d}}{\mathrm{d}R}\mathcal{L}^{2N}(E_R)\,\mathrm{d}R\geq\frac{1}{2}\int_1^{+\infty}\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\mathcal{L}^{2N}(E_R)\,\mathrm{d}R\\&=-\frac{1}{2}\,\mathcal{L}^{2N}(E_1)+\frac{1}{2}\int_1^{+\infty}\frac{1}{R^2}\,\mathcal{L}^{2N}(E_R)\,\mathrm{d}R\\&\geq -Cm+Cm\int_1^{\frac{1}{2}\mathrm{diam}(E)}\frac{1}{R}\,\mathrm{d}R\,,\end{aligned}$$

where in the first inequality we used the fact that $\alpha<1$, while the second one follows from (1.41). This completes the proof of the proposition. ☐

An essential remark for the proof of Theorem 1.11 is contained in the following lemma, where it is shown that the local minimality neighbourhood of the ball is in fact *uniform* with respect to $\gamma$ and $\alpha$.

LEMMA 1.38. *Given $\bar\gamma>0$, there exist $\bar\alpha>0$ and $\delta>0$ such that*

$$\mathcal{F}_{\alpha,\gamma}(B_1)<\mathcal{F}_{\alpha,\gamma}(E)$$

*for every $\alpha\leq\bar\alpha$, for every $\gamma\leq\bar\gamma$ and for every set $E$ with $|E|=|B_1|$ and $0<\alpha(E,B_1)<\delta$.*

PROOF (SKETCH). Notice that, by the same argument used in the proof of Theorem 1.10, it is sufficient to show that, given $\bar\gamma>0$, there exist $\bar\alpha>0$ and $\delta>0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(B_1)<\mathcal{F}_{\alpha,\bar\gamma}(E)$$

for every $\alpha\leq\bar\alpha$ and for every set $E$ with $|E|=|B_1|$ and $0<\alpha(E,B_1)<\delta$.

In order to prove this property, we start by observing that there exists $\alpha_1>0$ such that

$$m_0:=\inf_{\alpha\leq\alpha_1}\inf\left\{\partial^2\mathcal{F}_{\alpha,\bar\gamma}(B_1)[\varphi]\,:\,\varphi\in T^\perp(\partial B_1),\ \|\varphi\|_{\widetilde{H}^1(\partial B_1)}=1\right\}>0\,. \tag{1.42}$$

In fact, assuming by contradiction the existence of $\alpha_n \to 0$ and $\varphi_n \in T^\perp(\partial B_1)$, with $\|\varphi_n\|_{\widetilde{H}^1} = 1$, such that $\partial^2 \mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n] \to 0$, we have by compactness that, up to subsequences, $\varphi_n \rightharpoonup \varphi_0$ weakly in $H^1$ for some $\varphi_0 \in T^\perp(\partial B_1)$. It is now not hard to show that the last two integrals in the quadratic form $\partial^2 \mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n]$ converge to $0$ as $n \to \infty$: indeed, the second integral in (1.11) converges to $0$, since it is equal to

$$-\alpha_n \int_{\partial B_1} \Big( \int_{B_1} \frac{x-y}{|x-y|^{\alpha_n+2}} \, \mathrm{d}y \Big) \cdot x \, \varphi_n^2(x) \, \mathrm{d}\mathcal{H}^{N-1}(x) \leq C\alpha_n \int_{\partial B_1} \varphi_n^2 \, \mathrm{d}\mathcal{H}^{N-1} \to 0 \qquad \text{as } n \to \infty .$$

For the last integral in (1.11), denoting by $G_{\alpha_n}(x,y) := |x-y|^{-\alpha_n}$, we write

$$\begin{aligned}
&\int_{\partial B_1} \int_{\partial B_1} G_{\alpha_n}(x,y) \varphi_n(x) \varphi_n(y) \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad = \int_{\partial B_1} \int_{\partial B_1} G_{\alpha_n}(x,y) \varphi_n(x) (\varphi_n(y) - \varphi_0(y)) \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad + \int_{\partial B_1} \int_{\partial B_1} G_{\alpha_n}(x,y) (\varphi_n(x) - \varphi_0(x)) \varphi_0(y) \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad + \int_{\partial B_1} \int_{\partial B_1} G_{\alpha_n}(x,y) \varphi_0(x) \varphi_0(y) \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \, ;
\end{aligned}$$

the potential estimates provided by Lemma 1.19, where the constant can be chosen independently of $\alpha_n$ by Remark 1.20, guarantee that the first two integrals in the above expression converge to zero, while also the third one vanishes in the limit by the Lebesgue's Dominate Convergence Theorem, recalling that $\int_{\partial B_1} \varphi_0 = 0$ and $\alpha_n \to 0$. Moreover, for the first integral in the quadratic form $\partial^2 \mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n]$, we have that

$$\int_{\partial B_1} |D_\tau \varphi_0|^2 \leq \liminf_{n\to\infty} \int_{\partial B_1} |D_\tau \varphi_n|^2 , \qquad \int_{\partial B_1} |B_{\partial B_1}|^2 \varphi_n^2 \to \int_{\partial B_1} |B_{\partial B_1}|^2 \varphi_0^2 .$$

Hence, if $\varphi_0 = 0$ we conclude that $\int_{\partial B_1} |D_\tau \varphi_n|^2 \to 0$, which contradicts the fact that $\|\varphi_n\|_{\widetilde{H}^1} = 1$ for every $n$. On the other hand, if $\varphi_0 \neq 0$, we obtain

$$\int_{\partial B_1} |D_\tau \varphi_0|^2 \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial B_1} |B_{\partial B_1}|^2 \varphi_0^2 \, \mathrm{d}\mathcal{H}^{N-1} \leq 0 ,$$

that is, the second variation of the area functional computed at the ball $B_1$ is not strictly positive, which is again a contradiction.

With condition (1.42), it is straightforward to check that the proof of Theorem 1.25 provides the existence of $\delta_1 > 0$ and $C_1 > 0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(E) \geq \mathcal{F}_{\alpha,\bar\gamma}(B_1) + C_1 \big(\alpha(E, B_1)\big)^2$$

for every $\alpha \leq \alpha_1$ and for every $E \subset \mathbb{R}^N$ with $|E| = |B_1|$ and $\partial E = \{x + \psi(x)x : x \in \partial B_1\}$ for some $\psi$ with $\|\psi\|_{W^{2,p}(\partial B_1)} < \delta_1$.

In turn, having proved this property one can repeat the proofs of Theorem 1.30 and Theorem 1.8 to deduce that there exist $\alpha_2 > 0$, $\delta_2 > 0$ and $C_2 > 0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(E) \geq \mathcal{F}_{\alpha,\bar\gamma}(B_1) + C_2 \big(\alpha(E, B_1)\big)^2$$

for every $\alpha \leq \alpha_2$ and for every $E \subset \mathbb{R}^N$ with $|E| = |B_1|$ and $\alpha(E, B_1) < \delta_2$. The only small modifications consist in assuming, in the contradiction arguments, also the existence of sequences $\alpha_n \to 0$, instead of working with a fixed $\alpha$. Then the essential remark is that the constant $c_0$ provided by Proposition 1.3 is independent of $\alpha_n$. In addition, some small changes are required in the last part of the proof, since the functions $v_{F_n}$ associated, according to (1.1), with the sets $F_n$ constructed in the proof are defined with respect to different exponents $\alpha_n$, but observe that the bounds provided by Proposition 1.1 are still uniform. The easy details are left to the reader.

These observations complete the proof of the lemma. $\square$

We are now in position to complete the proof of Theorem 1.11.

PROOF OF THEOREM 1.11. We assume by contradiction that there exist $\alpha_n \to 0$, $m_n > 0$ and sets $E_n \subset \mathbb{R}^N$, with $|E_n| = m_n$, $\alpha(E_n, B_n) > 0$ (where we denote by $B_n$ a ball with volume $m_n$), such that $E_n$ is a global minimizer of $\mathcal{F}_{\alpha_n,\gamma}$ under volume constraint. Note that, as the non-existence threshold is uniformly bounded for $\alpha \in (0,1)$ (Proposition 1.37), we can assume without loss of generality that $m_n \leq \bar{m} < +\infty$.

By scaling, we can rephrase our contradiction assumption as follows: there exist $\alpha_n \to 0$, $\gamma_n > 0$ with $\bar{\gamma} := \sup_n \gamma_n < +\infty$, and $F_n \subset \mathbb{R}^N$ with $|F_n| = |B_1|$, $\alpha(F_n, B_1) > 0$ such that

$$\mathcal{F}_{\alpha_n,\gamma_n}(F_n) = \min\{\mathcal{F}_{\alpha_n,\gamma_n}(F) \,:\, |F| = |B_1|\}\,,$$

and in particular

$$\mathcal{F}_{\alpha_n,\gamma_n}(F_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1). \tag{1.43}$$

We now claim that, since $\alpha_n \to 0$,

$$\lim_{n\to+\infty} |\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)| = 0. \tag{1.44}$$

Indeed, we observe that by adapting the first step of the proof of Theorem 1.7, we have that there exists $\Lambda > 0$ (independent of $n$) such that $F_n$ is also a solution to the penalized minimum problem

$$\min\left\{\mathcal{F}_{\alpha_n,\gamma_n}(F) + \Lambda\big||F| - |B_1|\big| \,:\, F \subset \mathbb{R}^N\right\}$$

(for $n$ large enough). In turn, this implies that each set $F_n$ is an $(\omega, r_0)$-minimizer for the area functional (see Definition 1.26) for some positive $\omega$ and $r_0$ (independent of $n$): in fact for every finite perimeter set $F$ with $F \triangle F_n \subset\subset B_{r_0}(x)$ we have by minimality of $F_n$

$$\begin{aligned}\mathcal{P}(F_n) &\leq \mathcal{P}(F) + \gamma_n\big(\mathcal{NL}_{\alpha_n}(F) - \mathcal{NL}_{\alpha_n}(F_n)\big) + \Lambda\big||F| - |B_1|\big| \\ &\leq \mathcal{P}(F) + \big(\bar{\gamma}c_0 + \Lambda\big)|F \triangle F_n|,\end{aligned}$$

where we used Proposition 1.3 and the fact that the constant $c_0$ can be chosen independently of $\alpha_n$. We can now use the uniform density estimates for $(\omega, r_0)$-minimizers (see [45, Theorem 21.11]), combined with the connectedness of the sets $F_n$ (see Theorem 1.7), to deduce that (up to translations) they are equibounded: there exists $\bar{R} > 0$ such that $F_n \subset B_{\bar{R}}$ for every $n$. Using this information, it is now easily seen that, since $\alpha_n \to 0$,

$$\mathcal{NL}_{\alpha_n}(F_n) = \int_{F_n}\int_{F_n} \frac{1}{|x-y|^{\alpha_n}}\,dxdy \to |B_1|^2\,,$$

from which (1.44) follows.

By (1.43), (1.44) and using the quantitative isoperimetric inequality we finally deduce

$$\begin{aligned}C_N\big(\alpha(F_n, B_1)\big)^2 &\leq \mathcal{P}(F_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)\big) \\ &\leq \bar{\gamma}\,\big|\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)\big| \to 0,\end{aligned}$$

that is, $F_n$ converges to $B_1$ in $L^1$. Hence (1.43) is in contradiction with Lemma 1.38 for $n$ large enough. $\square$

We conclude this section with the proof of Theorem 1.12.

Proof of Theorem 1.12. First of all we notice that, since for masses smaller than $m_{\mathrm{glob}}$ the ball is the unique global minimizer, for each $m > 0$ there exists $k_m \in \mathbb{N}$ such that $f_{k_m}(m) = \min_i f_i(m)$. Setting $m_0 = 0$, $m_1 = m_{\mathrm{glob}}$, we have by Theorem 1.11 that (1.9) holds for $k = 1$. In the following, we denote by $E_R^m$ a solution to the constrained minimum problem

$$\min\{\mathcal{F}(E) : E \subset B_R,\, |E| = m\}.$$

We remark that

$$\mathcal{F}(E_R^m) \to \inf\left\{\mathcal{F}(E) : E \subset \mathbb{R}^N,\, |E| = m\right\} \quad \text{as } R \to \infty\,, \tag{1.45}$$

and that each set $E_R^m$ is an $(\omega, r_0)$-minimizer for some constant $\omega$ independent of $R$ (this conclusion can be obtained by arguing as in the proof of Theorem 1.7).

We now define

$$m_2 := \sup\left\{m \geq m_1 : f_2(m') = \inf_{|E|=m'}\mathcal{F}(E) \text{ for each } m' \in [m_1, m)\right\}$$

and we show that $m_2 > m_1$. Indeed, fix $\varepsilon > 0$ and $m \in (m_1, m_1 + \varepsilon)$. Observe that the sets $(E_R^m)_R$ cannot be equibounded, or otherwise they would converge (as $R \to \infty$) to a global minimizer of $\mathcal{F}$ with volume $m$, whose existence is excluded by Theorem 1.11. The fact that the diameter of $E_R^m$ tends to infinity, combined with the uniform density lower bound satisfied by $E_R^m$ (which, in turn, follows from the quasiminimality property), guarantees that for all $R$ large enough the set $E_R^m$ is not connected; moreover, if $\varepsilon$ is small enough, each of its connected component has mass smaller than $m_{\mathrm{glob}}$, again as a consequence of the lower bound. Then we can write $E_R^m = F_1 \cup F_2$, with $|F_1|, |F_2| < m_{\mathrm{glob}}$ and $F_1 \cap F_2 = \emptyset$, so that we can decrease the energy of $E_R^m$ by replacing each $F_i$ by a ball of the same volume, sufficiently far apart from each other, obtaining that $f_2(m) \leq \mathcal{F}(E_R^m)$. By (1.45) we easily conclude that $f_2(m) = \inf_{|E|=m} \mathcal{F}(E)$ for every $m \in (m_1, m_1 + \varepsilon)$, from which follows that $m_2 > m_1$. Moreover, by definition of $m_2$, we have that (1.9) holds for $k = 2$.

We now proceed by induction, defining

$$m_{k+1} := \sup\left\{m \geq m_k \,:\, f_{k+1}(m') = \inf_{|E|=m'}\mathcal{F}(E) \text{ for each } m' \in [m_k, m)\right\}$$

and showing that $m_k < m_{k+1}$. Arguing as before, we consider $m \in (m_k, m_k + \varepsilon)$, for some $\varepsilon > 0$ small enough, and we observe that for $R$ sufficiently large the set $E_R^m$ is not connected, and each of its connected components has volume belonging to an interval $(m_{i-1}, m_i]$ for some $i \leq k$. By the inductive hypothesis we can obtain a new set $F_R^m$, union of a finite number of disjoint balls, such that $\mathcal{F}(F_R^m) \leq \mathcal{F}(E_R^m)$, simply by replacing each connected component of $E_R^m$ by a disjoint union of balls. We can also assume that at least one of these balls, say $B$, has volume larger than $\varepsilon$ (if we choose for instance $\varepsilon < \frac{m_1}{2}$); in this way $|F_R^m \setminus B| < m_k$ and we can decrease the energy of $F_R^m$ by replacing $F_R^m \setminus B$ by a finite union of at most $k$ balls. With this procedure we find a disjoint union of at most $k+1$ balls whose energy is smaller than $\mathcal{F}(F_R^m)$, so that, recalling (1.45) and that $\mathcal{F}(F_R^m) \leq \mathcal{F}(E_R^m)$, we conclude that $f_{k+1}(m) = \inf_{|E|=m} \mathcal{F}(E)$ for every $m \in (m_k, m_k + \varepsilon)$. This completes the proof of the inequality $m_k < m_{k+1}$, and shows also, by definition of $m_k$, that (1.9) holds.

Now, assume by contradiction that $m_k \to \bar{m} < \infty$ as $k \to \infty$. Since each interval $(m_k, m_{k+1})$ is not degenerate, the definition of $m_k$ as a supremum ensures that we can find an increasing sequence of masses $\bar{m}_k \to \bar{m}$ such that an optimal configuration for $\min_i f_i(\bar{m}_k)$ is given by exactly $k+1$ balls. As, for $k$ large enough, at least two of these balls have volume smaller than $\frac{m_1}{2}$, they can be replaced by a single ball in a way that the energy decreases, contradicting the previous assertion and showing that $\lim_{k\to\infty} m_k = \infty$. Finally, it is clear that the number of non-degenerate balls tends to $\infty$ as $m \to \infty$, since the volume of each ball in an optimal configuration for $\min_i f_i(m)$ must be not larger than $m_1$. $\square$

## 1.6. Appendix

### 1.6.1. Computation of the first and second variations.

Proof of Theorem 1.17. The first and the second variations of the perimeter of a regular set $E$ are standard calculations (see, *e.g.*, [63]) and lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(E_t)_{|t=0} = \int_{\partial E} H_{\partial E}\langle X,\nu_E\rangle\,\mathcal{H}^{N-1} \tag{1.46}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{P}(E_t)_{|t=0} &= \int_{\partial E}\big(|D_\tau\langle X,\nu_E\rangle|^2 - |B_{\partial E}|^2\langle X,\nu_E\rangle^2\big)\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad+\int_{\partial E} H_{\partial E}\left(\langle X,\nu_E\rangle\,\mathrm{div}X - \mathrm{div}_\tau\big(X_\tau\langle X,\nu_E\rangle\big)\right)\mathrm{d}\mathcal{H}^{N-1}\,. \end{aligned}\tag{1.47}$$

This particular form of the second variation is in fact obtained in [9, Proposition 3.9], and we rewrote the last term according to [1, equation (7.5)].

So now on we will focus on the calculation of the first and the second variation of the nonlocal part. In order to compute these quantities we introduce the smoothed potential

$$G_\delta(a,b) := \frac{1}{(|a-b|^2+\delta^2)^{\frac{\alpha}{2}}}$$

for $\delta>0$, and the associated nonlocal energy

$$\mathcal{NL}_\delta(F) := \int_F\int_F G_\delta(a,b)\,\mathrm{d}a\mathrm{d}b.$$

We remark that the following identities hold:

$$\nabla_x\big(G_\delta(\Phi_t(x),\Phi_t(y))\big) = \nabla_a G_\delta(\Phi_t(x),\Phi_t(y))\cdot D\Phi_t(x), \tag{1.48}$$

$$\nabla_b G_\delta(a,b) = \nabla_a G_\delta(b,a). \tag{1.49}$$

*Step 1: first variation of the nonlocal term.* The idea to compute the first variation of the nonlocal part is to prove the following two steps:

(1) $\mathcal{NL}_\delta(E_t)\xrightarrow{\delta\to0}\mathcal{NL}(E_t)$ uniformly for $t\in(-t_0,t_0)$,
(2) $\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)$ converges uniformly for $t\in(-t_0,t_0)$ to some function $H(t)$ as $\delta\to0$,

where $t_0<1$ is a fixed number. From (1) and (2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{NL}(E_t)_{|t=0} = H(0) = \lim_{\delta\to0}\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)_{|t=0}. \tag{1.50}$$

We prove (1). We have that

$$|\mathcal{NL}_\delta(E_t)-\mathcal{NL}(E_t)| = \left|\int_{E_t}\int_{E_t}\big(G_\delta(x,y)-G(x,y)\big)\,\mathrm{d}x\mathrm{d}y\right| \le \int_{B_R}\int_{B_R}|G_\delta(x,y)-G(x,y)|\,\mathrm{d}x\mathrm{d}y\,,$$

where we have used the fact that $E$ is bounded and hence $E_t\subset B_R$ for some ball $B_R$. It is now easily seen that the last integral in the previous expression tends to 0 as $\delta\to0$, thanks to the Lebesgue's Dominated Convergence Theorem, hence

$$\sup_{t\in(-t_0,t_0)}|\mathcal{NL}_\delta(E_t)-\mathcal{NL}(E_t)|\to0 \qquad \text{as } \delta\to0.$$

We now prove (2). By a change of variables and using (1.48) and (1.49) we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t) &= 2\int_E\int_E \frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y\\ &\quad + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\langle\nabla_x\big(G_\delta(\Phi_t(x),\Phi_t(y))\big)\cdot(D\Phi_t(x))^{-1}, X(\Phi_t(x))\rangle\,\mathrm{d}x\mathrm{d}y\\ &= \int_E\int_E f(t,x,y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y\\ &\quad + \int_{\partial E}\left(\int_E g(t,x,y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}y\right)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,,\end{aligned}$$

where $J\Phi_t := \det(D\Phi_t)$ is the jacobian of the map $\Phi_t$,

$$f(t,x,y) := 2\frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y) - 2\mathrm{div}_x\big(J\Phi_t(x)J\Phi_t(y)X(\Phi_t(x))\cdot(D\Phi_t(x))^{-T}\big)\,,$$

$$g(t,x,y) := J\Phi_t(x)J\Phi_t(y)\langle X(\Phi_t(x))\cdot(D\Phi_t(x))^{-T},\nu(x)\rangle$$

and in the last step we used integration by parts and Fubini's Theorem. Now since $f$ and $g$ are uniformly bounded on $(-t_0,t_0)\times E\times E$ and $(-t_0,t_0)\times\partial E\times E$ respectively, it is easily seen that

$$\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)\overset{\delta\to 0}{\longrightarrow} H(t)\ \text{ uniformly for } t\in(-t_0,t_0)\,,$$

where

$$H(t) := \int_E\int_E f(t,x,y)G(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y + \int_{\partial E}\left(\int_E g(t,x,y)G(\Phi_t(x),\Phi_t(y))\,\mathrm{d}y\right)\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

We finally compute (1.50). Recalling that

$$\frac{\partial J\Phi_t}{\partial t}_{|t=0} = \mathrm{div}X\,, \tag{1.51}$$

we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)_{|t=0} &= 2\int_E\int_E\left(\frac{\mathrm{div}X(x)}{(|x-y|^2+\delta^2)^{\frac{\alpha}{2}}} - \alpha\frac{\langle X(x),x-y\rangle}{(|x-y|^2+\delta^2)^{\frac{\alpha+2}{2}}}\right)\mathrm{d}x\mathrm{d}y\\ &= 2\int_E\int_E \mathrm{div}_x\left(\frac{X(x)}{(|x-y|^2+\delta^2)^{\frac{\alpha}{2}}}\right)\mathrm{d}x\mathrm{d}y\\ &= 2\int_{\partial E}\left(\int_E\frac{\langle X(x),\nu(x)\rangle}{(|x-y|^2+\delta^2)^{\frac{\alpha}{2}}}\,\mathrm{d}y\right)\,\mathrm{d}\mathcal{H}^{N-1}(x)\end{aligned}$$

(where we used the divergence Theorem and Fubini's Theorem in the last equality), and hence by letting $\delta\to 0$ we conclude that

$$H(0) = 2\int_{\partial E}\left(\int_E\frac{\langle X(x),\nu(x)\rangle}{|x-y|^\alpha}\,\mathrm{d}y\right)\,\mathrm{d}\mathcal{H}^{N-1}(x) = 2\int_{\partial E} v_E\,\langle X,\nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}\,.$$

This, combined with (1.46), concludes the proof of the formula for the first variation of $\mathcal{F}$.

*Step 2: second variation of the nonlocal term.* We will compute the second variation of the nonlocal term by showing that

$$\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)\overset{\delta\to 0}{\longrightarrow} K(t)\ \text{ uniformly in } t\in(-t_0,t_0)$$

for some function $K$, hence getting

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{NL}(E_t)_{|t=0} = K(0) = \lim_{\delta\to 0}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)_{|t=0}\,. \tag{1.52}$$

First of all we have that

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) &= \frac{\partial}{\partial t}\Big[2\int_E\int_E \frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(x))\rangle\,\mathrm{d}x\mathrm{d}y\Big] \\
&= 2\int_E\int_E \frac{\partial}{\partial t}\Big(\frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y)\Big)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_E\int_E J\Phi_t(x)\frac{\partial}{\partial t}J\Phi_t(y)\Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(x))\rangle \\
&\qquad + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(y))\rangle\Big)\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_E\int_E \langle\frac{\partial}{\partial t}\Big(J\Phi_t(x)J\Phi_t(y)X(\Phi_t(x))\Big),\nabla_a G_\delta(\Phi_t(x),\Phi_t(y))\rangle\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\Big(\sum_{i,j=1}^N \frac{\partial^2 G_\delta}{\partial a_i\partial a_j}(\Phi_t(x),\Phi_t(y))X_i(\Phi_t(x))X_j(\Phi_t(x)) \\
&\qquad + \sum_{i,j=1}^N \frac{\partial^2 G_\delta}{\partial a_i\partial b_j}(\Phi_t(x),\Phi_t(y))X_i(\Phi_t(x))X_j(\Phi_t(y))\Big)\,\mathrm{d}x\mathrm{d}y\,. \qquad (1.53)
\end{aligned}
$$

Using identity (1.48) and integrating by parts, we can rewrite this expression as

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) = \int_E\int_E f(t,x,y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_E\int_E \Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),g_1(t,x,y)\rangle + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),g_2(t,x,y)\rangle\Big)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_E\int_{\partial E} \Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),h_1(t,x,y)\rangle + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),h_2(t,x,y)\rangle\Big)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}y\,,
\end{aligned}
$$

for some functions $f, g_1, g_2, h_1, h_2$ uniformly bounded in $(-t_0,t_0)\times\overline{E}\times\overline{E}$. It is then easily seen that

$$
\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) \xrightarrow{\delta\to 0} K(t) \ \text{ uniformly in } t\in(-t_0,t_0)\,,
$$

where $K(t)$ is simply obtained by replacing $G_\delta$ by $G$ in the previous expression.

We finally compute (1.52). Setting $Z := \frac{\partial^2\Phi}{\partial t^2}{}_{|t=0}$ we have that

$$
\frac{\partial^2 J\Phi_t}{\partial t^2}{}_{|t=0} = \mathrm{div}Z + (\mathrm{div}X)^2 - \sum_{i,j=1}^N \frac{\partial X_i}{\partial x_j}\frac{\partial X_j}{\partial x_i} = \mathrm{div}\big((\mathrm{div}X)X\big)\,.
$$

Therefore, computing (1.53) at $t=0$, from this identity and recalling (1.51) we obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)_{|t=0} &= 2\int_E\int_E \Big[\mathrm{div}\big((\mathrm{div}X)X\big)(x)G_\delta(x,y) + \mathrm{div}X(x)\mathrm{div}X(y)G_\delta(x,y)\Big]\,\mathrm{d}x\mathrm{d}y \\
&\quad + 4\int_E\int_E \mathrm{div}X(y)\sum_{i=1}^N\Big(\frac{\partial G_\delta}{\partial x_i}(x,y)X_i(x) + \frac{\partial G_\delta}{\partial y_i}(x,y)X_i(y)\Big)\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_E\int_E \sum_{i,j=1}^N\Big(\frac{\partial G_\delta}{\partial x_i}(x,y)\frac{\partial X_i}{\partial x_j}(x)X_j(x) + \frac{\partial^2 G_\delta}{\partial x_i\partial x_j}(x,y)X_i(x)X_j(x)
\end{aligned}
$$

$$+ \frac{\partial^2 G_\delta}{\partial x_i \partial y_j}(x,y) X_i(x) X_j(y) \Big) \, \mathrm{d}x\mathrm{d}y =: I_1 + I_2 + I_3 \,.$$

By integrating by parts in $I_1$, the sum of the first two integrals is equal to

$$\begin{aligned} I_1 + I_2 = 2 \int_E \int_E \langle \nabla_x G_\delta(x,y), X(x) \rangle \big( \mathrm{div} X(x) + \mathrm{div} X(y) \big) \, \mathrm{d}x\mathrm{d}y \\ + 2 \int_E \int_{\partial E} G_\delta(x,y) \big( \mathrm{div} X(x) + \mathrm{div} X(y) \big) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}y \,. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \mathcal{NL}_\delta(E_t)_{|t=0} &= 2 \int_E \int_{\partial E} G_\delta(x,y) \big( \mathrm{div} X(x) + \mathrm{div} X(y) \big) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}y \\ &\quad + 2 \int_E \int_E \Big( \mathrm{div}_x \big( \langle \nabla_x G_\delta(x,y), X(x) \rangle X(x) \big) + \mathrm{div}_y \big( \langle \nabla_x G_\delta(x,y), X(x) \rangle X(y) \big) \Big) \, \mathrm{d}x\mathrm{d}y \\ &= 2 \int_E \left( \int_{\partial E} \mathrm{div}_x \big( G_\delta(x,y) X(x) \big) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \right) \mathrm{d}y \\ &\quad + 2 \int_E \left( \int_{\partial E} \mathrm{div}_x \big( G_\delta(x,y) X(x) \big) \langle X(y), \nu(y) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(y) \right) \mathrm{d}x \\ &= 2 \int_{\partial E} \left( \int_E \mathrm{div}_x \big( G_\delta(x,y) X(x) \big) \, \mathrm{d}y \right) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ &\quad + 2 \int_{\partial E} \int_{\partial E} G_\delta(x,y) \langle X(x), \nu(x) \rangle \langle X(y), \nu(y) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \,, \end{aligned}$$

where the second equality follows after having applied the divergence theorem, and the last one by Fubini's Theorem and the divergence theorem. Thus, using the Lebesgue's Dominated Convergence Theorem to compute the limit of the previous quantity as $\delta \to 0$, and recalling that $\alpha \in (0, N-1)$, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \mathcal{NL}_\delta(E_t)_{|t=0} = 2 \int_{\partial E} \left( \int_E \mathrm{div}_x \big( G(x,y) X(x) \big) \, \mathrm{d}y \right) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ + 2 \int_{\partial E} \int_{\partial E} G(x,y) \langle X(x), \nu(x) \rangle \langle X(y), \nu(y) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}\mathcal{H}^{N-1}(y) \,. \end{aligned} \tag{1.54}$$

We can rewrite the first integral in the previous expression as

$$\begin{aligned} 2 \int_{\partial E} \left( \int_E \mathrm{div}_x \big( G(x,y) X(x) \big) \, \mathrm{d}y \right) \langle X(x), \nu(x) \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) &= 2 \int_{\partial E} \mathrm{div} \big( v_E X \big) \langle X, \nu \rangle \, \mathrm{d}\mathcal{H}^{N-1} \\ &= 2 \int_{\partial E} \Big( v_E (\mathrm{div} X) \langle X, \nu \rangle + \langle \nabla v_E, X_\tau \rangle \langle X, \nu \rangle + \partial_\nu v_E \, \langle X, \nu \rangle^2 \Big) \, \mathrm{d}\mathcal{H}^{N-1} \\ &= 2 \int_{\partial E} \Big( v_E (\mathrm{div} X) \langle X, \nu \rangle - v_E \, \mathrm{div}_\tau \big( X_\tau \langle X, \nu \rangle \big) + \partial_\nu v_E \, \langle X, \nu \rangle^2 \Big) \, \mathrm{d}\mathcal{H}^{N-1} \,. \end{aligned}$$

Finally, combining this expression with (1.54) and (1.47), we obtain the formula in the statement. □

**1.6.2. Computation of $\mathcal{I}^{N,\alpha}$.** Here we want to get an explicit expression of the integral

$$\mathcal{I}^{N,\alpha} := \int_{B_1} \frac{\langle x - y, x \rangle}{|x - y|^{\alpha+2}} \, \mathrm{d}y$$

appearing in Section 1.4, at least in the case $N = 3$. First of all, since $\mathcal{I}^{N,\alpha}$ is independent of $x \in S^{N-1}$, we fix $x = e_1$. By Fubini's Theorem we get

$$\mathcal{I}^{N,\alpha} = \int_{B_1} \frac{1 - y_1}{|e_1 - y|^{\alpha+2}}\,\mathrm{d}y = \int_{-1}^{1} \bigg( \int_{B_t} \frac{1 - t}{\big((1-t)^2 + |z|^2\big)^{\frac{\alpha+2}{2}}}\,\mathrm{d}\mathcal{L}^{N-1}(z) \bigg) \mathrm{d}t\,,$$

where $B_t := B^{N-1}(0, \sqrt{1-t^2})$ denotes a $(N-1)$-dimensional ball of radius $\sqrt{1-t^2}$ centered at the origin. To treat the inner integral, we apply the co-area formula (see [5, equation (2.74)]), by integrating on the level sets of the function $f_t(z) := \sqrt{(1-t)^2 + |z|^2}$, $z \in \mathbb{R}^{N-1}$: setting $\delta(r) = \sqrt{r^2 - (1-t)^2}$, we get

$$\begin{aligned} \int_{B_t} \frac{1}{\big((1-t)^2 + |z|^2\big)^{\frac{\alpha+2}{2}}}\,\mathrm{d}\mathcal{L}^{N-1}(z) &= \int_{1-t}^{\sqrt{2(1-t)}} \bigg( \int_{\partial B^{N-1}(0,\delta(r))} \frac{1}{r^{\alpha+1}\sqrt{r^2 - (1-t)^2}}\,\mathrm{d}\mathcal{H}^{N-2} \bigg) \mathrm{d}r \\ &= (N-1)\omega_{N-1} \int_{1-t}^{\sqrt{2(1-t)}} \frac{\big(r^2 - (1-t)^2\big)^{\frac{N-3}{2}}}{r^{\alpha+1}}\,\mathrm{d}r. \end{aligned}$$

Therefore

$$\mathcal{I}^{N,\alpha} = (N-1)\omega_{N-1} \int_{-1}^{1} (1-t) \bigg( \int_{1-t}^{\sqrt{2(1-t)}} \frac{\big(r^2 - (1-t)^2\big)^{\frac{N-3}{2}}}{r^{\alpha+1}}\,\mathrm{d}r \bigg) \mathrm{d}t.$$

From real analysis we know that we can write the inner integral in term of simple functions if and only if $N$ is odd or $\alpha$ is an integer. Since we are interested in the physical case ($N = 3$, $\alpha = 1$), we just compute the above integral for $N = 3$, obtaining

$$\mathcal{I}^{3,\alpha} = 2\pi \frac{2^{2-\alpha}}{(4-\alpha)(2-\alpha)}\,. \tag{1.55}$$

CHAPTER 2

# Periodic critical points of the Otha-Kawasaki functional

In this chapter we construct local minimizing periodic critical points of the sharp interface of the Otha-Kawasaki energy (0.1), whose shape closely resembles that of any give stictly stable periodic constant mean curvature surface. Moreover, we also establish some observations, of independent interest, about local minimziers of the sharp Otha-Kawasaki enery.

## 2.1. Preliminaries

In this section we introduce the objects and we fix the notation we will need in the following. Given $k \in \mathbb{N} \setminus \{0\}$, we will denote by $\mathbb{T}_k^N$ the $N$-dimensional flat torus rescaled by a factor $1/k$, *i.e.*, the quotient of $\mathbb{R}^N$ under the equivalence relation

$$\widehat{x} \sim_k \widehat{y} \Leftrightarrow k(\widehat{x} - \widehat{y}) \in \mathbb{Z}^N \,.$$

For simplicity, $\mathbb{T}_1^N$ will be denoted by $\mathbb{T}^N$. Points in $\mathbb{T}_k^N$ will be denoted by $x$, $y$. A set $E \subset \mathbb{T}_k^N$ can be naturally identify with the $1/k$-periodic set of $\mathbb{R}^N$ (or $\mathbb{T}^N$) that equals $E$ is a periodicity cell. When we speak about the regularity of a set $E \subset \mathbb{T}_k^N$, we will always refer to the regularity of the $1/k$-periodic set $E \subset \mathbb{R}^N$. Finally, for $\beta \in (0,1)$ and $r \in \mathbb{N}$, we define the functional space $C^{r,\beta}(\mathbb{T}_k^N)$ as the space of $1/k$-periodic functions in $C^{r,\beta}(\mathbb{R}^N)$.

Definition 2.1. Given a set $E \subset \mathbb{T}^N$ and $k \in \mathbb{N} \setminus \{0\}$, we define the set $E^k \subset \mathbb{T}_k^N$ as follows:

$$E^k := \{x \in \mathbb{T}_k^N \ : \ kx \in E\} \,.$$

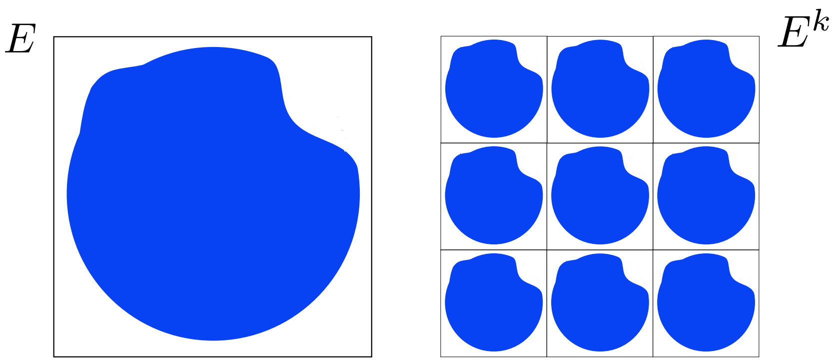


Figure 1. A set $E \subset \mathbb{T}^N$ on the left, and the set $E^k$, with $k = 3$, seen as a subset of $\mathbb{T}^N$, on the right.

Remark 2.2. Notice that $\int_{\mathbb{T}^N} u^E \, \mathrm{d}x = \fint_{\mathbb{T}_k^N} u_k^{E_k} \, \mathrm{d}x$, where we recall $u_k^F := \chi_F - \chi_{\mathbb{T}_k^N \setminus F}$.

We now introduce the notion of perimeter in $\mathbb{T}^N_k$.

Definition 2.3. Let $E \subset \mathbb{T}^N_k$. We say that $E$ is a set of *finite perimeter* in $\mathbb{T}^N_k$ if

$$\sup\Big\{\int_E \operatorname{div}\xi \,\mathrm{d}x \;:\; \xi \in C^1(\mathbb{T}^N_k;\mathbb{R}^N)\,, |\xi|\leq 1\Big\} < \infty\,.$$

In this case we denote by $\mathcal{P}_k(E)$ the above quantity.

We now introduce two kind of ways for saying how two sets in $\mathbb{T}^N$ are closed from each other. The first one takes into account the fact that our functional is invariant under translations.

Definition 2.4. Given two sets $E, F \subset \mathbb{T}^N_k$ we define the following distance between them:

$$\alpha(E,F) := \min_{x\in\mathbb{T}^N_k} |E\triangle(x+F)|\,.$$

Moreover, given $E\subset\mathbb{T}^N_k$ and $\beta\in(0,1)$, for sets $F\subset\mathbb{T}^N_k$ such that

$$\partial F = \{x+\psi(x)\nu_E(x) \;:\; x\in E\}\,,$$

for some function $\psi\in C^{r,\beta}(\partial E)$, we define

$$\mathrm{d}_{C^{r,\beta}}(E,F) := \|\psi\|_{C^{r,\beta}}\,.$$

Finally, to write the formulas for the first and the second variation of our functional $\mathcal{F}_\gamma$ (see Theorem 2.27), we need to reacall the following geometric definitions: given a set $E\subset\mathbb{T}^N$ of class $C^2$, we will denote by $D_\tau$ the tangential gradient operator, by $\operatorname{div}_\tau$ the tangential divergence, by $\nu_E$ the normal vector field on $\partial E$, by $B_{\partial E}$ its second fundamental form, and by $|B_{\partial E}|^2$ its Euclidean norm, that coincides with the sum of the squares of the principal curvatures of $\partial E$. Finally, $H_{\partial E}$ will denotes the mean curvature of $\partial E$.

**2.1.1. The area functional.** We recall some results about the area functional.

Definition 2.5. We say that a set $E\subset\mathbb{T}^N_k$ is a *local minimizer of the area functional* if there exists $\delta>0$ such that

$$\mathcal{P}_k(E)\leq\mathcal{P}_k(F)\,,$$

for all $F\subset\mathbb{T}^N_k$ with $|E|=|F|$, such that $\alpha(E,F)\leq\delta$.

Definition 2.6. A set $E\subset\mathbb{T}^N_k$ is said to be an $(\omega,r_0)$*-minimizer* for the area functional, with $\omega>0$ and $r_0>0$, if, for every ball $B_r(x)$ with $r\leq r_0$, we have

$$\mathcal{P}_k(E)\leq\mathcal{P}_k(F)+\omega|E\triangle F|,$$

whenever $F\subset\mathbb{T}^N_k$ is a set of finite perimeter such that $E\triangle F\subset\subset B_r(x)$.

We recall an improved convergence theorem for $(\omega,r_0)$-minimizers of the area functional. This result is well-known to the experts (see, for istance, [**68**]). One can find a complete proof of it in [**16**].

Theorem 2.7. *Let $E_n\subset\mathbb{T}^N_k$ be a sequence of $(\omega,r_0)$-minimizers of the area functional such that*

$$\sup_n \mathcal{P}_k(E_n)<+\infty \quad \textit{and} \quad \alpha(E_n,E)\to 0 \textit{ as } n\to\infty\,,$$

*for some bounded set $E$ of class $C^2$. Then for $n$ large enough $E_n$ is of class $C^{1,\beta}$ for all $\beta\in(0,1)$, and*

$$\partial E_n = \{x+\psi_n(x)\nu_E(x) \;:\; x\in\partial E\},$$

*with $\psi_n\to 0$ in $C^{1,\beta}(\partial E)$ for all $\beta\in(0,1)$.*

**2.1.2. The functional $\mathcal{F}_k^\gamma$.** We first define the functionals we are interested in.

DEFINITION 2.8. Given $\gamma \geq 0$ and $k \in \mathbb{N}$, we define, for sets $E \subset \mathbb{T}_k^N$, the functional

$$\begin{aligned}\mathcal{F}_k^\gamma(E) &:= \mathcal{P}_k(E) + \gamma \mathcal{NL}_k(E)\\ &:= \mathcal{P}_k(E) + \gamma \int_{\mathbb{T}_k^N}\int_{\mathbb{T}_k^N} G_k(x,y) u_k^E(x) u_k^E(y)\,\mathrm{d}x\,\mathrm{d}y\,,\end{aligned} \tag{2.1}$$

where $u_k^E(x) := \chi_E(x) - \chi_{\mathbb{T}_k^N \setminus E}(x)$ and $G_k$ is the unique solution of

$$-\triangle_y G_k(x,\cdot) = \delta_x(\cdot) - \frac{1}{|\mathbb{T}_k^N|} \quad \text{in } \mathbb{T}_k^N, \qquad \int_{\mathbb{T}_k^N} G_k(x,y)\,\mathrm{d}y = 0\,.$$

For simplicity, we will denote by $\mathcal{F}^\gamma$ and $u^E$ the functional $\mathcal{F}_1^\gamma$ and the function $u_1^E$ respectively.

REMARK 2.9. Notice that the area functional corresponds to the choice of $\gamma = 0$.

We now introduce the main objects under investigation in this paper: critical points and local minimizers.

DEFINITION 2.10. A set $E \subset \mathbb{T}^N$ will be called a *critical set* for the functional $\mathcal{F}^\gamma$ if it is a set of class $C^2$ satisfying

$$H_{\partial E} + 4\gamma v^E = \lambda\,,$$

for some constant $\lambda \in \mathbb{R}$.

REMARK 2.11. The above definition is motivated by the fact that (as one could expect) on critical sets, the first variation of the functional $\mathcal{F}$ vanishes (see Theorem 2.27).

DEFINITION 2.12. We say that a set $E \subset \mathbb{T}_k^N$ is a *local minimizer of the the functional* $\mathcal{F}_k^\gamma$, if there exists $\delta > 0$ such that

$$\mathcal{F}_k^\gamma(E) \leq \mathcal{F}_k^\gamma(F)\,,$$

for all $F \subset \mathbb{T}_k^N$ with $|E| = |F|$, such that $\alpha(E,F) \leq \delta$. Moreover, we say that $E$ is an *isolated local minimizer* if, in the above inequality, equality holds only when $F = E$.

We now want to derive some regularity properties of local minimizers of $\mathcal{F}_k^\gamma$. In order to do this, we observe that local minimizers of $\mathcal{F}_k^\gamma$ are in fact $(\omega,r)$-minimizer, and then we will rely on the well-known regularity theory for $(\omega,r)$-minimizer.

First of all one can see that the nonlocal term turns out to be Lipschitz (see [1, Lemma 2.6] for a proof).

PROPOSITION 2.13 (Lipschitzianity of the nonlocal term). *There exists a constant $c_0$, depending only on $N$, such that if $E, F \subset \mathbb{T}_k^N$ are measurable sets, then*

$$|\mathcal{NL}_k(E) - \mathcal{NL}_k(F)| \leq c_0 \alpha(E,F)\,.$$

The following lemma is a refinement of a result already present in [1] and [24].

LEMMA 2.14. *Fix constants $\bar\gamma > 0$, $\delta_0 > 0$, $m_0 \in (0, |\mathbb{T}_k^N|)$ and $M > 0$. Take a set $E \subset \mathbb{T}_k^N$, with $\mathcal{P}_k(E) \leq M$, solution of*

$$\min\Big\{\mathcal{P}_k(F) + \gamma\mathcal{NL}_k(F) \;:\; \fint_k u_k^F = m\,,\, \alpha(E,F) \leq \delta\Big\}\,, \tag{2.2}$$

*where* $\gamma \le \bar\gamma$, $\delta \in [\delta_0, +\infty]$ *and* $m \in [-m_0, |\mathbb{T}^N_k| - m_0]$. *Then we can find a constant* $\Lambda_0 = \Lambda_0(c_0, m_0, \bar\gamma, \delta_0, M) > 0$ *(where* $c_0$ *is the constant given by Proposition 2.13) such that* $E$ *is a solution of the* unconstrained minimum problem

$$\min\Big\{\mathcal{P}_k(F) + \gamma\mathcal{NL}_k(F) + \Lambda\Big|\fint_k u^F_k - m\Big| \ :\ \alpha(E,F) \le \delta/2\Big\},$$

*for all* $\Lambda \ge \Lambda_0$.

Proof. The idea is to prove that we can find a constant $\Lambda_0$ as in the statement of the lemma, such that if $\widetilde F$ solves

$$\min\Big\{\mathcal{P}_k(F) + \gamma\mathcal{NL}_k(F) + \Lambda\Big|\fint_k u^F_k - m\Big| \ :\ \alpha(E,F) \le \delta/2\Big\},$$

where $\gamma \le \bar\gamma$ and $\Lambda \ge \Lambda_0$, then $\alpha(\widetilde F, E) = 0$, where $E$ is a solution of (2.2). To prove it, suppose for the sake of contradiction that there exist sequences $\gamma_n \le \gamma$, $\Lambda_n \to \infty$, sets $E_n$ solutions of

$$\min\Big\{\mathcal{P}_k(F) + \gamma_n\mathcal{NL}_k(F) \ :\ \fint_k u^F_k = m_n\,,\ \alpha(E,F) \le \delta\Big\},$$

where $\delta \ge \delta_0$, $m_n := \fint_k u^{E_n}_k \in [-m_0, |\mathbb{T}^N_k| - m_0]$, $\mathcal{P}_k(E_n) \le M$, and sets $F_n$ solutions of

$$\min\Big\{\mathcal{P}_k(F) + \gamma_n\mathcal{NL}_k(F) + \Lambda_n\Big|\fint_k u^F_k - m_n\Big| \ :\ \alpha(E_n,F) \le \delta/2\Big\},$$

but with $m_n \ne \int_{\mathbb{T}^N_k} u^{F_n}_k$ (suppose $\int_{\mathbb{T}^N_k} u^{F_n}_k < m_n$). From now on we will suppose $|F_n \triangle E_n| = \alpha(E_n, F_n)$. The idea is to modify the sets $F_n$'s in such a way that $\int_{\mathbb{T}^N_k} u^{F_n}_k = m_n$ (notice that, since we are not working in the entire $\mathbb{R}^N$ but in $\mathbb{T}^N$, we need to modify the $F_n$'s in a more careful way than just rescaling them!). This idea has been developed in [24]. Set

$$\widetilde{\mathcal{F}}_n(F) := \mathcal{F}^{\gamma_n}_k(F) + \Lambda_n\Big|\fint_k u^F_k - m\Big|.$$

First of all we notice that $\sup_n \mathcal{P}_k(F_n) < \infty$. Indeed

$$\begin{aligned}\mathcal{P}_k(F_n) + \Lambda_n\Big|\fint_k u^{F_n}_k - m_n\Big| &\le \widetilde{\mathcal{F}}_n(E_n) - \gamma_n\mathcal{NL}_k(F_n)\\ &= \mathcal{P}_k(E_n) + \gamma_n\big(\mathcal{NL}_k(E_n) - \mathcal{NL}_k(F_n)\big) \le M + \bar\gamma c_0\,.\end{aligned}$$

Thus, up to a not relabelled subsequence, it is possible to find a set $F_0 \subset \mathbb{T}^N_k$ with $\fint_k v^{F_0}_k \in [-m_0, |\mathbb{T}^N_k| - m_0]$, such that $F_n \to F_0$ in $L^1$. Moreover $\alpha(E_n, F_n) \to 0$. We now sketch the argument presented in [24]. Given $\varepsilon > 0$, it is possible to find a radius $r > 0$ such that (up to translations)

$$|F_n \cap B_{r/2}| \le \varepsilon r^N\,, \qquad |F_n \cap B_r| \ge \frac{\mathbb{T}^N_k{}_N r^N}{2^{N+2}}\,,$$

for $n$ sufficiently large. Let $\sigma_n \in (0, 1/2^N)$, that will be choosen later, and define

$$\Phi_n(x) := \begin{cases} (1 - \sigma_n(2^N - 1))x & \text{if } |x| \le \frac{r}{2}\,,\\ x + \sigma_n\big(1 - \frac{r^N}{|x|^N}\big)x & \text{if } \frac{r}{2} \le |x| < r\,,\\ x & \text{if } |x| \ge r\,.\end{cases}$$

Let $\widetilde F_n := \Phi_n(F_n)$. It is possible to prove that

$$\mathcal{P}_k(F_n \cap B_r) - \mathcal{P}_k(\widetilde F_n \cap B_r) \ge -2^N N\sigma_n\mathcal{P}_k(F_n \cap B_r)\,,$$

and that, for $\varepsilon > 0$ sufficienlty small,

$$\fint_k u_k^{\widetilde{F}_n} - \fint_k u_k^{F_n} \geq \sigma_n r^N \Big[ c \frac{\mathbb{T}_k^N{}_N}{2^{N+2}} - \varepsilon\big(c + (2^N - 1)N\big) \Big] \geq c\sigma_n r^N \frac{\mathbb{T}_k^N{}_N}{2^{N+3}} =: C_1 \sigma_n r^N ,$$

where $c$ and $C_1$ are constants depending only on the dimension $N$. Then it is possible to choose the $\sigma_n$'s in such a way that $|F_n| = |E_n|$ for all $n$. In particular we obtain, from the above inequality, that $\sigma_n \to 0$. Finally, it is also possible to prove that

$$\alpha(\widetilde{F}_n, F_n) \leq C_2 \sigma_n \mathcal{P}_k(F_n \cap B_r) .$$

Combining all these estimates we have that

$$\widetilde{\mathcal{F}}_n(\widetilde{F}_n) \leq \widetilde{\mathcal{F}}_n(F_n) + \sigma_n \big[ (2^N N + C_2 c_0 \bar{\gamma}) \mathcal{P}_k(F_n \cap B_r) - \Lambda_n C_1 r^N \big] < \widetilde{\mathcal{F}}_n(F_n) \leq \widetilde{\mathcal{F}}_n(E_n) .$$

Since $\sigma_n \to 0$, we have that, for $n$ large enough, $\alpha(\widetilde{F}_n, E_n) \leq \delta_n$. Thus the above inequality is in contradiction with the local minimality property of $E_n$. □

Corollary 2.15. *Let $E \subset \mathbb{T}_k^N$ be a local minimizers of $\mathcal{F}_k^\gamma$. Then $E$ is an $(\omega, r)$-minimizer of the area functional. Moreover the parameter $\omega$ depends on the constants $c_0, m_0, \bar{\gamma}, \delta_0$ and $M$ of the previous lemma.*

Proof. From the above result, it follows that local minimizers of $\mathcal{F}_k^\gamma$ are in fact $(\omega, r)$-minimizer, providing we take $\omega := c_0 + \Lambda$ and we choose $r > 0$ such that $\omega_N r^N \leq \delta/2$. □

The regularity theory for $(\omega, r)$-minimizers allows us to say something about the regularity of local minimizers of $\mathcal{F}_k^\gamma$.

Proposition 2.16. *Let $E \subset \mathbb{T}_k^N$ be a local minimizer of $\mathcal{F}_k^\gamma$. Then we can write $\partial E = \partial^* E \cup \Sigma$, where the reduced boundary $\partial^* E$ is of class $C^{3,\alpha}$ for all $\alpha \in (0,1)$, and the Hausdorff dimension of $\Sigma$ is less than or equal to $N - 8$.*

Remark 2.17. Using the equation satisfied by a critical set $E$, it is also possible to prove (see [35]) the $C^\infty$ regularity of $\partial^* E$, in every dimension $N$. In particular, in dimension $N \leq 7$, we obtain the $C^\infty$-regularity for the entire boundary $\partial E$.

In the remaining part of this section we would like to investigate some properties of the nonlocal term, as well as the relation between the functionals $\mathcal{F}$ and $\mathcal{F}_k$.

Definition 2.18. For a set $E \subset \mathbb{T}_k^N$, we define the function:

$$v_k^E(x) := \int_{\mathbb{T}_k^N} G_k(x,y) u_k^E(y) \, \mathrm{d}y .$$

For simplicity, we wil denote by $v^E$ the function $v_1^E$.

Remark 2.19. We first want to investigate some properties of the nonlocal term. Notice that $v_k^E$ is the unique solution to

$$- \triangle v_k^E = u_k^E - m^E \quad \text{in } \mathbb{T}_k^N , \qquad \int_{\mathbb{T}_k^N} u_k^E \, \mathrm{d}x = 0 , \tag{2.3}$$

where we recall that $m^E := \int_{\mathbb{T}^N} u_{\mathbb{T}^N}^E \, \mathrm{d}x = \fint_k u_k^{E^k} \, \mathrm{d}x$. Moreover, $v_k^E$ is $1/k$-periodic. Thus, it is possible to rewrite the nonlocal in the following way:

$$\mathcal{NL}_k(E) = \int_{\mathbb{T}_k^N} u_k^E v_k^E \, \mathrm{d}x = - \int_{\mathbb{T}_k^N} v_k^E \triangle v_k^E \, \mathrm{d}x = \int_{\mathbb{T}_k^N} |\nabla v_k^E|^2 \, \mathrm{d}x .$$

By standard elliptic regularity we know that $v_k^E \in W^{2,p}(\mathbb{T}_k^N)$ for all $p \in [1, +\infty)$. In particular it holds that

$$\|v_k^E\|_{W^{2,p}(\mathbb{T}_k^N)} \leq C\,,$$

where $p > 1$ and $C$ is a constant depending only on $\mathbb{T}_k^N$.

Finally, we investigate the relation between the functionals $\mathcal{F}^\gamma$ and $\mathcal{F}_k^\gamma$.

LEMMA 2.20. *Let $E \subset \mathbb{T}^N$. Then it holds*

$$\mathcal{F}_k^\gamma(E^k) = k^{1-N}\Big[\mathcal{P}_{\mathbb{T}^N}(E) + \gamma k^{-3}\mathcal{NL}_{\mathbb{T}^N}(E)\Big]\,. \tag{2.4}$$

PROOF. We claim that, if $E \subset \mathbb{T}^N$ we have

$$v_k^{E^k}(x) = k^{-2}v^E(kx)\,.$$

Indeed, noticing that $\fint_{\mathbb{T}_k^N} u_k^{E_k} = \fint_{\mathbb{T}^N} u^E$, we have

$$-\triangle\big(k^{-2}v^E(kx)\big) = -\triangle v^E(kx) = u^E(kx) - m = u_k^{E^k}(x) - m\,,$$

and

$$\int_{\mathbb{T}_k^N} k^{-2}v^E(kx)\,\mathrm{d}x = k^{-N-2}\int_{\mathbb{T}^N} v^E(y)dy = 0\,.$$

By uniqueness of the solution of problem (2.3), we obtain our claim. Now, noticing that

$$\int_{\mathbb{T}_k^N} |\nabla v_k^{E^k}(x)|^2\,\mathrm{d}x = k^{-2-N}\int_{\mathbb{T}^N} |\nabla v^E(x)|^2\,\mathrm{d}x\,,$$

we conclude. □

REMARK 2.21. It is also easy to see that the function $v^{E^k}$ is $1/k$-periodic (where here we see $E_k$ as a subset of $\mathbb{T}^N$, *i.e.*, as $k$ copies of the $1/k$-rescalded of $E$). Thus

$$\mathcal{F}^\gamma(E^k) = k^N\mathcal{F}_k^\gamma(E^k)\,. \tag{2.5}$$

This means that the energy of $E^k$ in $\mathbb{T}^N$ is just the sum of the energies of each of its *pieces* in each $\mathbb{T}_k^N$.

**2.1.3. Results about $\Gamma$-convergence.** In this section we would like to recall an approximation theorem for isolated local minimizer of the area functional. For we need to write the functional $\mathcal{F}_{\mathbb{T}^N}^\gamma$ in the language of $\Gamma$-convergence.

DEFINITION 2.22. Let $(X, \mathrm{d})$ be a metric space, and let $F_n : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functionals. We say that the sequence $F_n$ ,$\Gamma(\mathrm{d})$*-converges* to the functional $F : X \to \mathbb{R} \cup \{+\infty\}$, $F_n \overset{\Gamma(d)}{\to} F$, if the following two conditions are satisfied

- for every $x_n \overset{\mathrm{d}}{\to} x$, $F(x) \leq \liminf_n F_n(x_n)$,
- for every $\bar{x} \in X$ there exists $x_n \overset{\mathrm{d}}{\to} \bar{x}$ such that $F(x) \geq \limsup_n F_n(x_n)$.

DEFINITION 2.23. Consider the space where $X := L^1(\mathbb{T}^N)/\sim$, where $f_1 \sim f_2 \Leftrightarrow$ there exists $v \in \mathbb{T}^N$ such that $f_1(x+v) = f_2(x)$, for each $x \in \mathbb{T}^N$. Endow this space with the distance

$$\alpha(u, v) := \min_{x\in\mathbb{T}^N} \|u - v(\cdot - x)\|_{L^1(\mathbb{T}^N)}\,.$$

Given $\gamma \in [0, +\infty)$ and fixed a constant $m \in (-1, 1)$, we define the functional $\widetilde{\mathcal{F}}_\gamma : X \to \mathbb{R} \cup \{+\infty\}$ as

$$\widetilde{\mathcal{F}}^\gamma(u) := \begin{cases} \mathcal{F}^\gamma(E) & \text{if } u = u^E, \text{ for some set } E \text{ with } \fint_{\mathbb{T}^N} u^E\,\mathrm{d}x = m\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

REMARK 2.24. Notice that the functionals $\widetilde{\mathcal{F}}^\gamma$ are equi-coercive and lower semicontinuous. Morever $\widetilde{\mathcal{F}}^\gamma \xrightarrow{\Gamma(\alpha)} \widetilde{\mathcal{F}}^0$ as $\gamma \to 0^+$.

Although the $\Gamma$-convergence has been designed for the convergence of *global* mininimizers, one can says also something about the convergence of *local* minimizers. The following result is a particular application of [40].

THEOREM 2.25. *Let $E \subset \mathbb{T}^N$ be a smooth isolated local minimizer of the area functional. Then there exists a sequence $(E_\gamma)_{\gamma>0}$, with $|E_\gamma| = |E|$, such that $E_\gamma$ is a local minimizer of $\mathcal{F}^\gamma$ in $\mathbb{T}^N$ and $\alpha(E_\gamma, E) \to 0$ as $\gamma \to 0^+$.*

## 2.2. Variations and local minimality

In the following we will use a local minimality criterion provided in [1], that we recall here for reader's convenience. This criterion is based on the positivity of the second variation. We thus need to introduce what do we mean by variation.

DEFINITION 2.26. Let $E \subset \mathbb{T}^N$ be a set of class $C^2$. Take a smooth vector field $X \in C^\infty(\mathbb{T}^N; \mathbb{R}^N)$ and consider the associated flow $\Phi : \mathbb{T}^N \times (-1,1) \to \mathbb{T}^N$ given by

$$\frac{\partial \Phi}{\partial t} = X(\Phi)\,,$$

such that $\Phi(x,0) = x$ for all $x \in \mathbb{T}^N$. Let $E_t := \Phi(E,t)$ and suppose $|E_t| = |E|$ for each time $t$. We define the *first* and the *second variation* of $\mathcal{F}^\gamma$ at a set $E$ with respect to the flow $\Phi$, respectively as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^\gamma(E_t)_{|t=0}\,, \qquad\qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}^\gamma(E_t)_{|t=0}\,.$$

We recall here the result present in [1, Theorem 3.1] for the computation of the first and the second variations.

THEOREM 2.27. *Let $E$, $X$ and $\Phi$ as above. Then the first variation of $\mathcal{F}^\gamma$ at $E$ with respect to the flow $\Phi$ is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^\gamma(E_t)_{|t=0} = \int_{\partial E} (H_{\partial E} + 4\gamma v^E)(X \cdot \nu_E)\ \mathrm{d}\mathcal{H}^{N-1}\,, \tag{2.6}$$

*while the second variation of $\mathcal{F}^\gamma$ at $E$ with respect to the flow $\Phi$ reads as*

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}^\gamma(E_t)_{|t=0} &= \int_{\partial E} \big(|D_\tau(X\cdot\nu_E)|^2 - |B_{\partial E}|^2 (X\cdot\nu_E)^2\big)\ \mathrm{d}\mathcal{H}^{N-1} \\ &+ 8\gamma \int_{\partial E}\int_{\partial E} G_{\mathbb{T}^N}(x,y)(X(x)\cdot\nu_E(x))(X(y)\cdot\nu_E(y))\ \mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\ &+ 4\gamma \int_{\partial E} \partial_{\nu_E} v^E\,(X\cdot\nu_E)^2\ \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E} (4\gamma v^E + H_{\partial E})\,\mathrm{div}_\tau\big(X_\tau (X\cdot\nu_E)\big)\ \mathrm{d}\mathcal{H}^{N-1}\,.\end{aligned}$$

REMARK 2.28. Notice that the last term of the second variation vanishes whenever $E$ is a critical set.

We now follow the ideas contatined in [1]. We introduce the space

$$\widetilde{H}^1(\partial E) := \left\{ \varphi \in H^1(\partial E)\ :\ \int_{\partial E} \varphi\ \mathrm{d}\mathcal{H}^{N-1} = 0 \right\}\,,$$

endowed with the norm $\|\varphi\|_{\widetilde{H}^1(\partial E)} := \|\nabla\varphi\|_{L^2(\partial E)}$, and we define on it the following quadratic form associated with the second variation.

DEFINITION 2.29. Let $E \subset \mathbb{T}^N$ be a regular critical set. We define the quadratic form $\partial^2\mathcal{F}^\gamma(E) : \widetilde{H}^1(\partial E) \to \mathbb{R}$ by

$$\begin{aligned}\partial^2\mathcal{F}^\gamma(E)[\varphi] &:= \int_{\partial E}\big(|D_\tau\varphi|^2 - |B_{\partial E}|^2\varphi^2\big)\ \mathrm{d}\mathcal{H}^{N-1} + 4\gamma\int_{\partial E}(\partial_{\nu_E}v^E)\varphi^2\ \mathrm{d}\mathcal{H}^{N-1}\\ &\quad + 8\gamma\int_{\partial E}\int_{\partial E} G_{\mathbb{T}^N}(x,y)\varphi(x)\varphi(y)\ \mathrm{d}\mathcal{H}^{N-1}(x)\ \mathrm{d}\mathcal{H}^{N-1}(y)\\ &=: \partial^2\mathcal{P}_{\mathbb{T}^N}(E)[\varphi] + \gamma\partial^2\mathcal{NL}_{\mathbb{T}^N}(E)[\varphi]\,,\end{aligned}\tag{2.7}$$

where $\partial^2\mathcal{P}_{\mathbb{T}^N}(E)$ denotes the first integral, while $\gamma\partial^2\mathcal{NL}_{\mathbb{T}^N}(E)$ the other two.

Since our functional is translation invariant, if we compute the second variation of $\mathcal{F}^\gamma$ at a regular set $E$ with respect to a flow of the form $\Phi(x,t) := x + t\eta e_i$, where $\eta \in \mathbb{R}$ and $e_i$ is an element of the canonical basis of $\mathbb{R}^N$, setting $\nu_i := \langle \nu_E, e_i\rangle$ we obtain that

$$\partial^2\mathcal{F}^\gamma(E)[\eta\nu_i] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}^\gamma(E_t)_{|_{t=0}} = 0\,.$$

Hence we split

$$\widetilde{H}^1(\partial E) = T^\perp(\partial E) \oplus T(\partial E)\,,$$

where $T^\perp(\partial E)$ is the orthogonal complement to $T(\partial E)$ in the $L^2$-sense, *i.e.*,

$$T^\perp(\partial E) := \left\{\varphi \in \widetilde{H}^1(\partial E)\ :\ \int_{\partial E}\varphi\nu_i\ \mathrm{d}\mathcal{H}^{N-1} = 0\ \text{ for each } i = 1,\dots,N\right\}\,.$$

It can be shown (see [1, Equation (3.7)]) that there exists an orthonormal frame $(\varepsilon_1,\dots,\varepsilon_N)$ such that

$$\int_{\partial E}(\nu\cdot\varepsilon_i)(\nu\cdot\varepsilon_j)\ \mathrm{d}\mathcal{H}^{N-1} = 0 \qquad \text{for all } i \neq j\,.\tag{2.8}$$

DEFINITION 2.30. We say that $\mathcal{F}^\gamma$ has *strictly positive second variation* at the regular critical set $E$ if

$$\partial^2\mathcal{F}^\gamma(E)[\varphi] > 0 \qquad \text{for all } \varphi \in T^\perp(\partial E)\setminus\{0\}.$$

Finally, we recall the local minimality result proved in [1].

THEOREM 2.31. *Let $E \subset \mathbb{T}^N$ be a regular critical set such that $\mathcal{F}^\gamma$ has strictly positive second variation at $E$. Then, there exist constants $C,\delta > 0$, such that*

$$\mathcal{F}^\gamma(F) \geq \mathcal{F}^\gamma(E) + C\big(\alpha(E,F)\big)^2\,,$$

*whenever $F \subset \mathbb{T}^N$ with $|F| = |E|$ is such that $\alpha(E,F) \leq \delta$.*

## 2.3. The results

**2.3.1. Minimality in small domains.** The first result we would like to prove is a local minimality property of critical points with respect to sufficiently small perturbations.

PROPOSITION 2.32. *Let $E \subset \mathbb{T}^N$ be a critical point for the functional $\mathcal{F}^\gamma$. Then there exists $\varepsilon > 0$ with the following property: for any set $F \subset \mathbb{T}^N$ different from $E$ we have that*

$$\mathcal{F}^\gamma(E) \leq \mathcal{F}^\gamma(F)\,,$$

*whenever $E\triangle F \Subset B_\varepsilon(x)$, for some $x \in \bar{E}$,* Sketch of the proof. *Part one.* We first want to prove that we can find $\widetilde{\varepsilon} > 0$ such that

$$\mathcal{F}^\gamma(E) \leq \mathcal{F}^\gamma(F)\,,$$

whenever $F \subset \mathbb{T}^N$ is a set different from $E$, having $E\triangle F \Subset B_{\widetilde{\varepsilon}}(x)$, for some $x \in \partial E$. So, fix $\bar{x} \in \partial E$. The idea is to follow the proofs of the various step leading to the proof of [1, Theorem 1.1], and adapt them to our case.

*Step 1.* For any $\varepsilon > 0$ sufficiently small, we have the following Poincaré inequality

$$\int_{\partial E\cap B_\varepsilon(\bar{x})} |D_\tau \varphi|^2 \, \mathrm{d}\mathcal{H}^{N-1} \geq C_\varepsilon \int_{\partial E\cap B_\varepsilon(\bar{x})} \varphi^2 \, \mathrm{d}\mathcal{H}^{N-1}\,,$$

for any $\varphi \in H^1(\partial E)$ with support contained in $B_\varepsilon(x)$. We know that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Let $M > 0$ such that

$$|B_{\partial E}| < M\,, \qquad |\partial_\nu v^E| < M\,,$$

and take $\varepsilon > 0$ such that $C_{2\varepsilon} > M(1+4\gamma)$. Notice that it is possible to write

$$\int_{\partial E}\int_{\partial E} G_{\mathbb{T}^N}(x,y)\varphi(x)\varphi(y) \; \mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) = \int_{\mathbb{T}^N} |\nabla z|^2 \, \mathrm{d}x\,,$$

where $-\triangle z = \varphi\mathcal{H}^{N-1} \llcorner \partial E$. Thus, we have that

$$\partial^2\mathcal{F}^\gamma(E)[\varphi] > 0\,, \tag{2.9}$$

for any $\varphi \in H^1(\partial E)\backslash\{0\}$ with support contained in $B_{2\varepsilon}(\bar{x})$.

*Step 2.* We claim that it is possible to find constants $\delta > 0$ and $C_0 > 0$ such that

$$\mathcal{F}^\gamma(E) + C_0\big(\alpha(E,F)\big)^2 \leq \mathcal{F}^\gamma(F)\,,$$

whenever $F \subset \mathbb{T}^N$, with $|F| = |E|$, is such that $\partial F = \{x + \psi(x)\nu_E(x) \;:\; x \in \partial E\}$, for some $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ with support contained in $B_{2\varepsilon}(\bar{x})$, for $p > \max\{2, N-1\}$. We use the two step technique of [1, Theorem 3.9]. We first prove that we can find constants $\delta > 0$ and $m > 0$ such that

$$\begin{aligned}\inf\Big\{\partial^2\mathcal{F}^\gamma(F)[\varphi] \;:\varphi \in \widetilde{H}^1(\partial F)\,,\, \|\varphi\|_{H^1(\partial F)} = 1\,,\\ \mathrm{supp}(\varphi) \subset B_{2\varepsilon}(x)\,,\, \Big|\int_{\partial F} \varphi\nu_F \, \mathrm{d}\mathcal{H}^{N-1}\Big| \leq \delta\Big\} \geq m\,,\end{aligned}$$

whenever $F \subset \mathbb{T}^N$, with $|F| = |E|$, is such that

$$\partial F = \{x + \psi(x)\nu_E(x) \,:\, x \in \partial E_\gamma\}\,,$$

for some $\psi \in W^{2,p}(\partial E)$ with $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$. To prove it, we reason by the sake of contradiction as in the first step of the proof of [1, Theorem 3.9].

Now consider the flow $\Phi$, given by Lemma 2.36, connecting the sets $E$ and $F$, and let $E_t := \Phi_t(E)$. Then it is possible to write

$$\mathcal{F}^\gamma(F) - \mathcal{F}^\gamma(E) = \int_0^1 (1-t)\Big(\partial^2\mathcal{F}(E_t)[X\cdot\nu_{E_t}] - \int_{\partial E_t} (4\gamma v^{E_t} + H_t)\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\Big)\mathrm{d}t\,,$$

where $\mathrm{div}_{\tau_t}$ is the tangential divergence on $\partial E_t$ and $X_{\tau_t} := (X\cdot\tau_{E_t})\tau_{E_t}$. It is possible to estimate from below of the integral, as it is done in the second step of the proof of [1, Theorem 3.9]. Namely, it is possible to find $\delta > 0$ such that

$$\Big|\int_{\partial E_t} (4\gamma v^{E_t} + H_t)\mathrm{div}_{\tau_t}(X_{\tau_t}(X\cdot\nu_{E_t}))\mathrm{d}t\Big| \leq \frac{m}{2}\|X\cdot\nu^{E_t}\|^2_{H(\partial E_t)}\,,$$

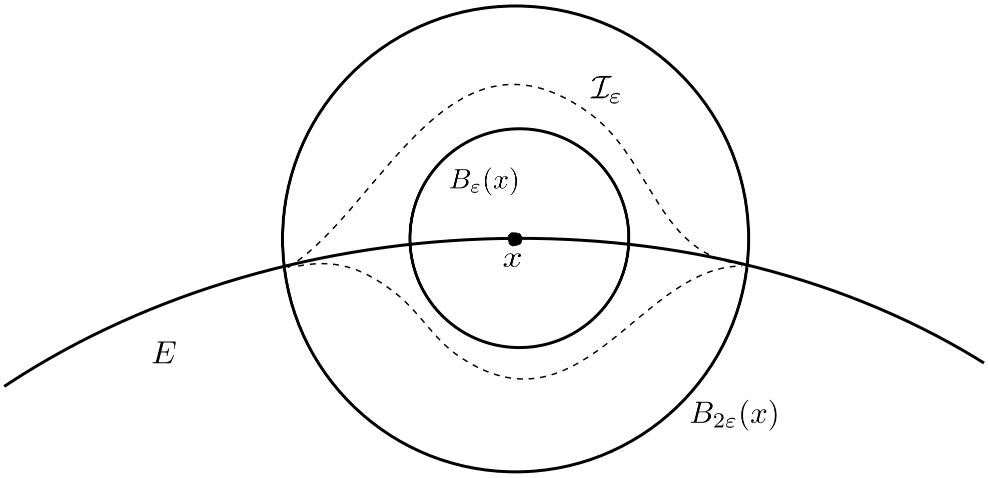


FIGURE 2. An example of the set $\mathcal{I}_\delta$.

for all $t \in [0,1]$. Thus, with the above uniform coercivity property of $\partial^2\mathcal{F}(E_t)$ in force, we conclude.

*Step 3.* For any $\varepsilon > 0$, let $\mathcal{I}_\varepsilon \subset B_{2\varepsilon}(\bar{x})$ be a smooth open set with the following properties: the curvature of $\mathcal{I}_\varepsilon$ are uniformly bounded with respect to $\varepsilon$, the sets $E \cup \mathcal{I}_\varepsilon$ and $E \setminus \mathcal{I}_\varepsilon$ are smooth, $B_\varepsilon(\bar{x}) \subset \mathcal{I}_\varepsilon$ (see Figure 2). We claim that it is possible to find $\varepsilon > 0$ such that

$$\mathcal{F}^\gamma(E) \leq \mathcal{F}^\gamma(F)\,,$$

for every set $F \subset \mathbb{T}^N$ with $|F| = |E|$, such that $E \triangle F \Subset \mathcal{I}_\varepsilon$. The proof of such a result is similar to those of [1, Theorem 4.3], where we reason by the sake of contradiction as follows: suppose there exist a sequence $\varepsilon_n \to 0$ and sets $F_n$ with $|F_n| = |E|$ and $E \setminus \mathcal{I}_{\varepsilon_n} \subset F_n \subset E \cup \mathcal{I}_{\varepsilon_n}$, such that

$$\mathcal{F}^\gamma(F_n) < \mathcal{F}^\gamma(E)\,.$$

Using the uniform bound on the curvatures of the $\mathcal{I}_{\varepsilon_n}$'s, it is possible to prove, as in the first step of the proof of [1, Theorem 4.3], that we can find a sequence of sets $E_n$ with $|E_n| = |E|$, and $E_n \triangle E \Subset \mathcal{I}_{\varepsilon_n}$, $\mathcal{F}^\gamma(E_n) < \mathcal{F}^\gamma(E)$ and the $E_n$'s are uniform $(\omega, r)$-minimizers of the area functional. Thus, the improved convergence result stated in Theorem 2.7 allows us to say that the $E_n$'s converge to $E$ in the $C^{1,\beta}$-topology. Finally, using the Euler-Lagrange equation satisfied by the $E_n$'s, it is also possible to prove that the $E_n$'s actually converge to $E$ in the $W^{2,p}$-topology. This is in contradiction with the result of the previous step.

*Step 4.* We now have to prove that the above constants can be made uniform with respect to $x \in \partial E$. Reason as follows: for any point $x \in \partial E$, consider the ball $B_{\varepsilon(x)}(x)$, where $\varepsilon(x) > 0$ is the radius found in Step 3 above. Then it is possible to cover $\partial E$ with a finite family of such a balls, let us say $(B_{\varepsilon(x_i)}(x_i))_{i=1}^L$. It is thus possible to find $\widetilde{\varepsilon} > 0$ with the following property: for any point $x \in \partial E$, there exists $i \in \{1, \ldots, L\}$ such that $B_{\widetilde{\varepsilon}}(x) \subset B_{\varepsilon(x_i)}(x_i)$. We can also suppose $\widetilde{\varepsilon} < \varepsilon(x_i)$ for each $i = 1, \ldots, L$.

*Second part.* We now want to prove that we can find $\varepsilon \in (0, \widetilde{\varepsilon}/2)$ such that

$$\mathcal{F}^\gamma(E) < \mathcal{F}^\gamma(F)\,, \tag{2.10}$$

whenever $F \subset \mathbb{T}^N$ is a set different from $E$, having $E \triangle F \Subset B_\varepsilon(x)$, for some $x \in E \setminus (\partial E)_{\widetilde{\varepsilon}/2}$. The key point is to observe that

$$|\mathcal{NL}(F) - \mathcal{NL}(E)| \le c_0 |E \triangle F| \le C\mathcal{P}(E \triangle F)^{\frac{N}{N-1}} = C\big(\mathcal{P}(F) - \mathcal{P}(E)\big)^{\frac{N}{N-1}}, \tag{2.11}$$

where we have used the Lipschitzianity of the nonlocal term (Proposition 2.13), the isoperimetric inequality, and the fact that $E \triangle F \Subset B_\varepsilon(x)$, with $x$ in the interior of $E$, respectively.

Now, (2.10) can be written as

$$\mathcal{P}(F) - \mathcal{P}(E) \ge \gamma\big(\mathcal{NL}(E) - \mathcal{NL}(F)\big).$$

Using (2.11) and the fact that $t^{\frac{N}{N-1}} < Ct$ for $t$ small, we know that the above inequality is satisfied if $\mathcal{P}(F) - \mathcal{P}(E) < \delta$, for some $\delta > 0$. If $\mathcal{P}(F) - \mathcal{P}(E) \ge \delta$, noticing that

$$|\mathcal{NL}(F) - \mathcal{NL}(E)| \le c_0 |E \triangle F| \le C\varepsilon^N,$$

we obtain the validity of (2.10) by taking $\varepsilon$ sufficiently small. This concludes the proof. □

**2.3.2. Uniform local minimizers.** We start by proving a lemma that will be used several times. The proof can be found in [1] (Step 4 of the proof of Theorem 3.4), but we prefer to report it here for reader's convenience.

LEMMA 2.33. *Let $E \subset \mathbb{T}^N$ be a critical set for $\mathcal{F}^{\bar\gamma}$, with $\bar\gamma \ge 0$. Then for any $\varepsilon > 0$ it is possible to find $\widetilde{\varepsilon} > 0$ with the following property: if $E_\gamma$ is a critical point of $\mathcal{F}^\gamma$, with $\gamma \in (\bar\gamma - \varepsilon, \bar\gamma + \varepsilon)$ such that $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon$, then $\mathrm{d}_{C^{3,\beta}}(E, E_\gamma) < \widetilde{\varepsilon}$, for all $\beta \in (0,1)$.*

PROOF. Suppose for the sake of contradiction that there exists a sequence $\gamma_n \to \bar\gamma$ and a sequence $(E_n)_n$ of critical points $\mathcal{F}^{\gamma_n}$ with $\mathrm{d}_{C^1}(E, E_\gamma) \to 0$ such that $\mathrm{d}_{C^{3,\beta}}(E, E_\gamma) \ge C > 0$. We recall that on $\partial E$

$$H_{\partial E} = \lambda - 4\bar\gamma v^E, \tag{2.12}$$

for some constant $\lambda$, while on $\partial E_{\gamma_n}$

$$H_{\partial E_{\gamma_n}} = \lambda_{\gamma_n} - 4\gamma_n v^{E_{\gamma_n}}. \tag{2.13}$$

Thanks to the $C^1$-convergence of $E_{\gamma_n}$ to $E$ and by standard elliptic estimates, it is easy to see that

$$v^{E_{\gamma_n}} \to v^E \qquad \text{in } C^{1,\beta}(\mathbb{T}^N), \tag{2.14}$$

for all $\beta \in (0,1)$. Now we would like to prove that $\lambda_{\gamma_n} \to \lambda$, thus obtaining the desired contradiction. We work locally, by considering a cylinder $C = B' \times (-L, L)$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centered at the origin, such that in a suitable coordinate system we have

$$\begin{aligned} E_{\gamma_n} \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g_{\gamma_n}(x')\}, \\ E \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g(x')\} \end{aligned}$$

for some functions $g_{\gamma_n} \to g$ in $C^{1,\beta}(\overline{B'})$. By integrating (2.13) on $B'$ we obtain

$$\begin{aligned} &\lambda_{\gamma_n} \mathcal{H}^{N-1}(B') - 4\gamma_n \int_{B'} v^{E_{\gamma_n}}(x', g_{\gamma_n}(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x') \\ &\quad = -\int_{B'} \operatorname{div}\bigg(\frac{\nabla g_{\gamma_n}}{\sqrt{1 + |\nabla g_{\gamma_n}|^2}}\bigg)\, \mathrm{d}\mathcal{H}^{N-1}(x') = -\int_{\partial B'} \frac{\nabla g_{\gamma_n}}{\sqrt{1 + |\nabla g_{\gamma_n}|^2}} \cdot \frac{x'}{|x'|}\, \mathrm{d}\mathcal{H}^{N-2}, \end{aligned}$$

and the last integral in the previous expression converges, as $n \to \infty$, to

$$
\begin{aligned}
&-\int_{\partial B'} \frac{\nabla g}{\sqrt{1+|\nabla g|^2}} \cdot \frac{x'}{|x'|} \, \mathrm{d}\mathcal{H}^{N-2} = -\int_{B'} \operatorname{div}\Big(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\Big) \, \mathrm{d}\mathcal{H}^{N-1}(x') \\
&= \lambda \mathcal{H}^{N-1}(B') - 4\bar{\gamma}_n \int_{B'} v^{E_{\bar{\gamma}_n}}(x', g_{\gamma_n}(x')) \, \mathrm{d}\mathcal{H}^{N-1}(x') \,,
\end{aligned}
$$

where the last equality follows by (2.12). This shows, recalling (2.14), that

$$\lambda_{\gamma_n} \to \lambda \,,$$

for $n \to \infty$. Thus, by standard elliptic estimates, we get that $E_{\gamma_n} \to E$ in $C^{3,\beta}$. □

We now present a uniform local minimality result for strictly stable critical points of $\mathcal{F}^\gamma$.

PROPOSITION 2.34. *Let $E \subset \mathbb{T}^N$ be a strictly stable critical point for $\mathcal{F}^{\bar{\gamma}}$, $\bar{\gamma} \geq 0$. Then there exist constants $\delta_2 > 0$, $\varepsilon_2 > 0$, $\gamma_2 > 0$ and $C_2 > 0$ with the following property: take $\gamma \in (\bar{\gamma} - \gamma_2, \bar{\gamma} + \gamma_2)$ and let $E_\gamma$ be a critical point for $\mathcal{F}^\gamma$ with $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_2$; then*

$$\mathcal{F}^\gamma(E_\gamma) + C_2\big(\alpha(E_\gamma, F)\big)^2 \leq \mathcal{F}^\gamma(F) \,,$$

*for every set $F \subset \mathbb{T}^N$ with $|F| = |E_\gamma|$, such that $\alpha(E_\gamma, F) \leq \delta_2$.*

The proof of Proposition2.34 follows the same strategy as [1]. The difficulty is to check that all the estimates permormed there can be made uniform with respct to the $C^1$ closeness of $E_\gamma$ to $E$. Checking this, we in fact simplify the general argument, by replacing [1, Lemma 3.8] by a penalization argument that was inspired to us by [20].

DEFINITION 2.35. Let $F \subset \mathbb{T}^N$ be a set of class $C^\infty$. We will denote by $\mathcal{N}_\mu(F)$, with $\mu > 0$, a tubular neighborhood of $F$ where the signed distance $\mathrm{d}_F$ from $F$ and the projection $\pi_F$ on $\partial F$ are smooth in $\mathcal{N}_\mu(F)$.

LEMMA 2.36. *Let $E \subset \mathbb{T}^N$ be a strictly stable critical point for $\mathcal{F}^{\bar{\gamma}}$, $\bar{\gamma} \geq 0$, and let $p > \max\{2, N-1\}$. Then there exist constants $\mu > 0$, $\gamma_3 > 0$, $\varepsilon_3 > 0$ and $C > 0$ with the following property: for any critical point $E_\gamma$ of $\mathcal{F}^\gamma$, with $\gamma \in (\bar{\gamma} - \gamma_3, \bar{\gamma} + \gamma_3)$ and $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_3$, and any $\psi \in C^\infty(E_\gamma)$ with $\|\psi\|_{W^{2,p}(\partial E_\gamma)} \leq \varepsilon_3$, there exists a vector field $X \in C^\infty$ with $\operatorname{div} X = 0$ in $\mathcal{N}_\mu(F)$ such that, if we consider its flow,* i.e., *the solution of*

$$\frac{\partial \Phi}{\partial t} = X(\Phi) \,, \qquad \Phi(0, x) = x \,, \tag{2.15}$$

*we have $\Phi(1, x) = x + \psi(x)\nu_{E_\gamma}(x)$, for any $x \in \partial E_\gamma$. Moreover, the following estimate holds true*

$$\|\Phi(t, \cdot) - \mathrm{Id}\|_{W^{2,p}(\partial E_\gamma)} \leq C\|\psi\|_{W^{2,p}(\partial E_\gamma)} \,.$$

*Finally, denote by $E^t_\gamma$ the set such that $\partial E^1_\gamma = \{x + t\psi(x)\nu_{E_\gamma}(x) \,:\, x \in \partial E_\gamma\}$. If $|E^1_\gamma| = |E_\gamma|$, then $|E^t_\gamma| = |E_\gamma|$ for all $t \in [0,1]$ and*

$$\int_{\partial E^t_\gamma} X \cdot \nu_{E^t_\gamma} \, \mathrm{d}\mathcal{H}^{N-1} = 0 \,.$$

PROOF. First of all take $0 < \varepsilon_3 < \varepsilon_0$, where $\varepsilon_0 > 0$ is the constant given by Lemma 2.33. Then, possibly reducing $\varepsilon_3$, we can find $\mu > 0$ such that $\mathcal{N}_\mu(E_\gamma)$ is a tubular neighborhood of $E_\gamma$ as in Definition 2.35, for every $E_\gamma$ critical point of $\mathcal{F}^\gamma$, with $\gamma \in (\bar{\gamma} - \gamma_3, \bar{\gamma} + \gamma_3)$, for $\gamma_3 \in (0, \varepsilon)$, and $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon$. Fix $\gamma \in (\bar{\gamma} - \gamma_3, \bar{\gamma} + \gamma_3)$.

For every $x \in \partial E_\gamma$ consider the function $f_x : (-\mu, \mu) \to \mathbb{R}$ solution of

$$
\begin{cases}
(f_x)'(t) + f_x(t) \triangle \mathrm{d}_{E_\gamma}(x + t\nu_{E_\gamma}(x)) = 0 \,, \\
f_x(0) = 1 \,.
\end{cases}
$$

Set

$$\xi(x+t\nu_{E_\gamma}(x)) := f_x(t) = \exp\Big(-\int_0^t \triangle \mathrm{d}_{E_\gamma}(x+s\nu_{E_\gamma}(x))\mathrm{d}s\Big)\,.$$

Using again the $C^{3,\beta}$-closeness of $E_\gamma$ to $E$, it is possible to find a constant $C>0$ such that $\|\psi\|_{L^\infty(\partial E_\gamma)} \leq C\|\psi\|_{W^{2,p}(\partial E_\gamma)} < C\varepsilon$ for any set $E_\gamma$ as above. Take $0<\varepsilon<\mu/C$. So, let $X$ be a smooth vector field such that

$$X(z) := \Big(\int_0^{\psi(\pi_{E_\gamma}(z))} \frac{\mathrm{d}s}{\xi\big(\pi_{E_\gamma}(x)+s\nu_{E_\gamma}(\pi_{E_\gamma}(z))\big)}\Big)\xi(z)\nabla d_{E_\gamma}(z) \qquad \text{for } z\in\mathcal{N}_\mu(E_\gamma)\,.$$

Notice that the above integral represents the time needed to go from a point $x\in\partial E_\gamma$ to the point $x+\Psi(x)\nu_{E_\gamma}(x)$ along the trajectory of the vector field $\xi\nabla \mathrm{d}_{E_\gamma}$. Thus, if we move along the trajectory of the vector field $X$, the time needed to go from a point $x\in\partial E_\gamma$ to the point $x+\Psi(x)\nu_{E_\gamma}(x)$ is always one. Moreover that integral does not change for points $z\in\mathcal{N}_\mu(E_\gamma)$ in the trajectory of the vector field $\xi\nabla \mathrm{d}_{E_\gamma}$. This ensure that $\mathrm{div}X=0$ in $\mathcal{N}_\mu(E_\gamma)$.

We now prove the estimates on $\Phi$. First of all notice that we can find a constant $C>0$ such that, for every set $E_\gamma$ as above, it holds

$$\|X\|_{W^{2,p}(\mathcal{N}_\mu(E_\gamma))} \leq C\|\psi\|_{W^{2,p}(\partial E_\gamma)}\,.$$

Thus, by the definition of the flow $\Phi$, we have that

$$\|\Phi-\mathrm{Id}\|_{C^0(\mathcal{N}_\mu(E_\gamma))} \leq C\|\psi\|_{W^{2,p}(\partial E_\gamma)}\,.$$

To estimate the other norms, we just differentiate in (2.15) to obtain

$$\|\nabla_x\Phi(t,\cdot)-\mathrm{Id}\|_{C^0(\mathcal{N}_\mu(E_\gamma))} \leq C_\mu\|\nabla X\|_{C^0(\mathcal{N}_\mu(E_\gamma))} \leq C_\mu\|\psi\|_{W^{2,p}(\partial E_\gamma)}\,.$$

Since this shows that the $(N-1)$-dimensional Jacobian of $\Phi(t,\cdot)$ is uniformly closed to 1 on $\partial E_\gamma$, deriving again in (2.15), we obtain also the following estimate:

$$\|\nabla_x^2\Phi(t,\cdot)\|_{L^p(\partial E_\gamma)} \leq C_\mu\|\nabla^2 X\|_{L^p(\mathcal{N}_\mu(E_\gamma))}\,.$$

Finally, if $|E^1_\gamma|=|E_\gamma|$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|E_t| = \int_{E^t_\gamma}(\mathrm{div}X)(X\cdot\nu_{E_\gamma})\,\mathrm{d}\mathcal{H}^{N-1} = 0 \qquad \text{for all } t\in[0,1]\,.$$

This follows from [15, Equation (2.30)]. Thus, the function $t\mapsto|E^t_\gamma|$ is affine in $[0,1]$, and since $|E_\gamma|=|E^t_\gamma|$, we have that it is constant. So

$$0=\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = \int_{E^t_\gamma}\mathrm{div}X\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E^t_\gamma} X\cdot\nu_{E^t_\gamma}dh\,.$$

This concludes the proof of the lemma. □

Definition 2.37. Let $E\subset\mathbb{T}^N$. Take a smooth function $f:\mathbb{T}^N\to\mathbb{R}^N$ such that $f=\nu_E$ on $\partial E$, and consider the functional

$$Pen_E(F) := \Big|\int_F f(x)\mathrm{d}x - \int_E f(x)\mathrm{d}x\Big|^2\,.$$

Moreover define the penalized functional

$$\mathcal{F}^\gamma_E(F) := \mathcal{F}^\gamma(F) + Pen_E(F)\,.$$

for sets $F\subset\mathbb{T}^N$.

LEMMA 2.38. *Let $E, F \subset \mathbb{T}^N$, and $(\Phi_t)_t$ be an admissible family of diffeomorphisms. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}s} Pen_E(F_s)_{|s=t} = 2\Big(\int_{F_t} f\mathrm{d}x - \int_E f\mathrm{d}x\Big) \cdot \int_{\partial F_t} f(X \cdot \nu_{F_t})\,\mathrm{d}\mathcal{H}^{N-1},$$

*and*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Pen_E(E_t)_{|t=0} = 2\Big|\int_{\partial E} \nu_E (X \cdot \nu_E)\,\mathrm{d}\mathcal{H}^{N-1}\Big|^2$$
$$+ 2\Big(\int_F f\mathrm{d}x - \int_E f\mathrm{d}x\Big) \cdot \int_{\partial F} f[(X\cdot\nu)\mathrm{div}X - \mathrm{div}_\tau(X_\tau(X\cdot\nu))]\,\mathrm{d}\mathcal{H}^{N-1}.$$

PROOF. Consider the vector function $F : (-1,1) \to \mathbb{R}$ given by

$$F(t) := \int_{E_t} f_i(x)\mathrm{d}x,$$

for some $i = 1, \dots, N$. Then

$$F'(t) = \int_{F_t} \big(\nabla f_i \cdot X + f_i \mathrm{div} X_t\big)\mathrm{d}x = \int_{\partial F_t} f_i (X \cdot \nu_{F_t})\,\mathrm{d}\mathcal{H}^{N-1}.$$

Moreover

$$\begin{aligned}
F''(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\partial F_t} f_i(X\cdot\nu_{F_t})\,\mathrm{d}\mathcal{H}^{N-1}\Big)_{t=0} \\
&= \int_{\partial F} (\nabla f_i \cdot X)(X\cdot\nu_F)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial F} f \frac{\mathrm{d}}{\mathrm{d}t}\big((X\circ\Phi_t)\cdot(\nu_{F_s}\circ\Phi_t)J^{N-1}\Phi_t\big)_{|t=0}\,\mathrm{d}\mathcal{H}^{N-1} \\
&= \int_{\partial F} (\nabla f_i \cdot X)(X\cdot\nu_F)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_{\partial F} f_i\big[\mathrm{div}_\tau(X(X\cdot\nu_F)) + Z\cdot\nu - 2X_\tau\cdot\nabla_\tau(X\cdot\nu) + D\nu_F[X_\tau, X_\tau]\big]\,\mathrm{d}\mathcal{H}^{N-1} \\
&= \int_{\partial F} f_i\big[(X\cdot\nu)\mathrm{div}X - \mathrm{div}_\tau(X\tau(X\cdot\nu))\big]\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

where in the last step we have used the same computations as in [1, Theorem 3.1]. $\square$

REMARK 2.39. Let $E$ be a stricly stable critical point for $\mathcal{F}^\gamma$. Then the quadratic form $\partial^2\mathcal{F}^\gamma_E$ associated with the second variation of $\mathcal{F}^\gamma_E$ computed at $E$ satisfies

$$\partial^2\mathcal{F}^\gamma_E(E)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial E)\backslash\{0\}.$$

Indeed, the non-negative term due to the second variation of the penalization vanishes only for $\varphi \in T^\perp(\partial E)$, where we know, by the strict stability of $E$, that $\partial^2\mathcal{F}^\gamma$ is strictly positive (except, of course, for $\varphi \equiv 0$).

We state a technical lemma, whose simle proof is left to the reader.

LEMMA 2.40. *Let $E \subset \mathbb{T}^N$ be a regular set, and let $M > \|\nu_E\|_{C^1(\partial E)}$. Then there exists a constant $\varepsilon > 0$ such that for every set $F \subset \mathbb{T}^N$ with $\mathrm{d}_{C^2}(E,F) < \varepsilon$, there exists a function $f_F : \mathbb{T}^N \to \mathbb{R}^N$ with $f_F = \nu_F$ on $\partial F$ and $\|f_F\|_{C^1(\mathbb{T}^N;\mathbb{R}^N)} < M$.*

*Moreover, for every $\delta > 0$ there exists $\eta > 0$ such that*

$$\Big|\int_{\partial F^\psi} f_F \varphi\,\mathrm{d}\mathcal{H}^{N-1}\Big| \le \delta \quad \Rightarrow \quad \Big|\int_{\partial F^\psi} \nu_{F^\psi}\varphi\,\mathrm{d}\mathcal{H}^{N-1}\Big| \le \eta,,$$

*for any set* $F^{\psi} \subset \mathbb{T}^N$ *with* $\partial F^{\psi} = \{x + \psi(x)\nu_F(x) \,:\, x \in \partial F\}$ *for some* $\|\psi\|_{W^{2,p}(\partial F)} \le \eta$, *where* $F \subset \mathbb{T}^N$ *is a set such that* $\mathrm{d}_{C^2}(E,F) < \varepsilon$, *and any function* $\varphi \in \widetilde{H}^1(\partial F^{\psi})$ *with* $\|\varphi\|_{H^1(\partial F^{\psi})} = 1$.

We now prove a uniform $W^{2,p}$-local minimality result for the penalized functional.

LEMMA 2.41. *Let* $p > \max\{2, N-1\}$, *and let* $E \subset \mathbb{T}^N$ *be a strictly stable critical point for* $\mathcal{F}^{\bar\gamma}$. *Then there exist constants* $\gamma_4 > 0$, $\delta_4 > 0$, $\varepsilon_4 > 0$ *and* $C_4 > 0$ *with the following property: take* $\gamma \in (\bar\gamma - \gamma_4, \bar\gamma + \gamma_4)$ *and let* $E_\gamma$ *be a critical point for* $\mathcal{F}^{\gamma}$ *with* $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_4$; *then*

$$\mathcal{F}^{\gamma}_{E_\gamma}(F) \ge \mathcal{F}^{\gamma}_{E_\gamma}(E_\gamma) + C_4 |E_\gamma \triangle F|^2 \,,$$

*for every set* $F \subset \mathbb{T}^N$ *with* $|F| = |E_\gamma|$ *and* $\partial F = \{x + \psi(x)\nu_{E_\gamma}(x) \,:\, x \in \partial E_\gamma\}$ *for some* $\|\psi\|_{W^{2,p}(\partial E_\gamma)} \le \delta_4$.

PROOF. *Step 1.* We claim that is possible to find a constants $\gamma_4 > 0$, $\delta_4 > 0$, $\varepsilon_4 > 0$ and $m > 0$ such that,for any $\gamma \in (\bar\gamma - \gamma_4, \bar\gamma + \gamma_4)$, any critical set $E_\gamma \subset \mathbb{T}^N$ for $\mathcal{F}^\gamma$, with $|E_\gamma| = |E|$ and $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_4$, we have that

$$\inf\Big\{ \partial^2 \mathcal{F}^{\gamma}_{E_\gamma}(F)[\varphi] \,:\, \varphi \in \widetilde{H}^1(\partial F)\,,\, \|\varphi\|_{H^1(\partial F)} = 1 \Big\} \ge m \,, \tag{2.16}$$

whenever $F \subset \mathbb{T}^N$, with $|F| = |E|$, is such that

$$\partial F = \{x + \psi(x)\nu_{E_\gamma}(x) \,:\, x \in \partial E_\gamma\} \,,$$

for some $\psi \in W^{2,p}(\partial E_\gamma)$ with $\|\psi\|_{W^{2,p}(\partial E_\gamma)} \le \delta_4$.

We first prove that (2.16) by supposing

$$\Big| \int_{\partial F} \nu_F \varphi \, \mathrm{d}\mathcal{H}^{N-1} \Big| < \delta_4 \,. \tag{2.17}$$

To prove it we reason as follows: suppose for the sake of contradiction that there exists a sequence $\gamma_n \to \bar\gamma$, a sequence of sets $E_{\gamma_n} \subset \mathbb{T}^N$ with $|E_{\gamma_n}| = |E|$ and $E_{\gamma_n} \to E$ in $C^1$ (and thus, by Lemma 2.33, in $C^{3,\beta}$), a sequence of sets $F_n \subset \mathbb{T}^N$ with $|F_n| = |E|$ and

$$\partial F_n = \{x + \psi_n(x)\nu_{E_{\gamma_n}}(x) \,:\, x \in \partial E_{\gamma_n}\} \,,$$

for $\psi_n \in W^{2,p}(\partial E_{\gamma_n})$ with $\|\psi_n\|_{W^{2,p}(\partial E_{\gamma_n})} \le 1/n$, and a sequence of functions $\varphi_n \in \widetilde{H}^1(\partial F_n)$ with $\|\varphi_n\|_{H^1(\partial F_n)} = 1$ and $\int_{\partial F_n} \varphi_n \nu_{F_n} \to 0$, such that

$$\partial^2 \mathcal{F}^{\gamma_n}(F_n)[\varphi_n] \to 0 \qquad \text{as } n \to \infty \,.$$

One can see that $E_{\gamma_n} \to E$ in $C^{3,\beta}$ implies that $F_n \to E$ in $W^{2,p}$. Then there exist diffeomorphisms $\Phi_n : E \to F_n$ converging to the identity in $W^{2,p}(\partial E)$. The idea is to consider the functions $\widetilde{\varphi}_n \in \widetilde{H}^1(\partial E)$ defined as

$$\widetilde{\varphi}_n := \varphi_n \circ \Phi_n - a_n \,,$$

where , $a_n := \int_{\partial E} \varphi_n \circ \Phi_n \, \mathrm{d}\mathcal{H}^{N-1}$, and to prove that

$$\partial^2 \mathcal{F}^{\gamma_n}(F_n)[\varphi_n] - \partial^2 \mathcal{F}^{\gamma_n}(E)[\widetilde{\varphi}_n] \to 0 \,, \tag{2.18}$$

and that

$$\partial^2 \mathcal{F}^{\gamma_n}(E)[(\widetilde{\varphi}_n)^{\perp}] - \partial^2 \mathcal{F}^{\gamma_n}(E)[\widetilde{\varphi}_n] \to 0 \,. \tag{2.19}$$

The above convergences are proved exactly as in Step 1 of [1, Theorem 3.9], where we notice that the convergence of the term of the quadratic form due to the penalization, is easily seen to converge.

This allows to conclude: indeed, from the fact that

$$\partial^2 \mathcal{F}^{\gamma_n}(E)[(\widetilde{\varphi}_n)^\perp] - \partial^2 \mathcal{F}^{\bar{\gamma}}(E)[(\widetilde{\varphi}_n)^\perp] \to 0\,, \tag{2.20}$$

we obtain a contradiction with

$$\inf\{\partial^2 \mathcal{F}^{\bar{\gamma}}(E)[\varphi] \,:\, \varphi \in T^\perp(\partial E) \setminus \{0\},\, \|\varphi\|_{H^1(\partial E)} = 1\} \geq C > 0\,.$$

This last fact follows from the strict positivity of the second variation (see [1, Lemma 3.6]). In order to prove (2.18) and (2.19) we have just to repeat the same computation as in step 1 of [1, Theorem 3.9]. Finally (2.20) is easily seen to be true.

Let $\eta$ be the constant given by Lemma 2.40 associated with $\delta_4$. Then, if

$$\Big| \int_{\partial F} f_{E_\gamma} \varphi \, \mathrm{d}\mathcal{H}^{N-1} \Big| > \eta\,,$$

where $f_{E_\gamma}$ is the function associated to the set $E_\gamma$ given by Lemma 2.40. Then, from the explicit expression of $\partial^2 \mathcal{F}^\gamma_{E_\gamma}(F)[\varphi]$, we get that (2.16) holds with $m = \eta$ Otherwise, thank to Lemma 2.40, we know that (2.17) holds, and thus we conclude by the the previous computations.

*Step 2.* To conclude, we have to check that all the estimates needed in the second step of [1, Theorem 3.9] can be made uniform with respect to $\gamma \in (\bar{\gamma} - \gamma_4, \bar{\gamma} + \gamma_4)$. For any pair of sets $E_\gamma$ and $F$ as in the statement, consider the vector field $X_\gamma$ and its flow $\Phi_\gamma(\cdot, t)$, provided by Lemma 2.36. Let $E^t_\gamma := \Phi_\gamma(E_\gamma, t)$. Fixed $\varepsilon > 0$, it is possible to find $\varepsilon_4 > 0$ and $\delta_4 > 0$ such that

$$\|\nu_{E_\gamma} - \nu_{E^t_\gamma}(\Phi_n(\cdot, t))\|_{L^\infty} < \varepsilon\,, \qquad \|J^{N-1}(\Phi_\gamma(\cdot, t)) - 1\|_{L^\infty} < \varepsilon\,.$$

Moreover, thanks to the $C^1$-closeness of $E^t_\gamma$ to $E$, we can also suppose

$$\|4\gamma v^{E^t_\gamma} + H_{E^t_\gamma} - \lambda_\gamma\|_{L^\infty} < \varepsilon\,,$$

where $4\gamma v^{E_\gamma} + H_{E_\gamma} = \lambda_\gamma$. Finally, thanks to the uniform control on the gradient of the functions $f_{E_\gamma}$, up to take smaller $\varepsilon_4 > 0$ and $\delta_4 > 0$, we have

$$\Big| \int_{E^t_\gamma} f_{E_\gamma} \mathrm{d}x - \int_{E_\gamma} f_{E_\gamma} \mathrm{d}x \Big| < \varepsilon\,,$$

for every $t \in [0, 1]$. Thus, we can write

$$\begin{aligned}
\mathcal{F}^\gamma_{E_\gamma}(F) - \mathcal{F}^\gamma_{E_\gamma}(E_\gamma) &= \int_0^1 (1-t) \Bigg[ \partial^2 \mathcal{F}_{E_\gamma}(E^t_\gamma)[X_\gamma \cdot \nu_{E^t_\gamma}] \\
&- \int_{\partial E^t_\gamma} (4\gamma v^{E^t_\gamma} + H_{E^t_\gamma}) \mathrm{div}_{\tau_t}(X^{\tau_t}_\gamma (X_\gamma \cdot \nu_{E^t_\gamma}))\, \mathrm{d}\mathcal{H}^{N-1} \\
&- 2\Big( \int_{E^\gamma_t} f_{E_\gamma} \mathrm{d}x - \int_{E_\gamma} f_{E_\gamma} \mathrm{d}x \Big) \cdot \int_{\partial E^t_\gamma} f_{E_\gamma} \mathrm{div}_{\tau_t}(X^{\tau_t}_\gamma (X_\gamma \cdot \nu_{E^t_\gamma}))\, \mathrm{d}\mathcal{H}^{N-1} \Bigg] \mathrm{d}t\,.
\end{aligned}$$

Since the vector fields $X_\gamma$'s are uniformly closed in the $C^1$-topology, it is possible to find a constant $C > 0$ such that

$$\|\mathrm{div}_{\tau_t}(X^{\tau_t}_\gamma (X_\gamma \cdot \nu_{E^t_\gamma}))\|_{L^{\frac{p}{p-1}}(\partial E^t_\gamma)} \leq C \|X_\gamma \cdot \nu_{E^t_\gamma}\|^2_{H^1(\partial E^t_\gamma)}\,,$$

for every $\gamma \in (\bar{\gamma} - \gamma_4, \bar{\gamma} + \gamma_4)$. Thus, the above uniform estimates allow us to conclude, as in [1, Theorem 3.9]. $\square$

LEMMA 2.42. *Let $E$ and $E_\gamma$ as in the statement of Lemma 2.41, and consider the functions $f_\gamma$ given by Lemma 2.40. Then there exists $\varepsilon > 0$ with the following property: for any $F \subset \mathbb{T}^N$ with $\mathrm{d}_{C^1}(E_\gamma, F) < \varepsilon$, there exists $v \in \mathbb{R}^N$ such that*

$$\int_{F+v} f_\gamma \mathrm{d}x = \int_{E_\gamma} f_\gamma \mathrm{d}x \,.$$

PROOF. Fix $\gamma \in (\bar\gamma - \gamma_2, \bar\gamma + \gamma_2)$. Consider the function $T_\gamma : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$T_\gamma(v) := \int_{E_\gamma} f_\gamma(x - v)\mathrm{d}x \,.$$

Then

$$DT_\gamma(0) = -\int_{E_\gamma} Df_\gamma(x)\mathrm{d}x \,.$$

In particular $\big(DT_\gamma(0)\big)_{ij} = -\int_{\partial E_\gamma} \nu_i \cdot \nu_j \,\mathrm{d}\mathcal{H}^{N-1}$. By (2.8), we know that there exists an orthonormal frame, where the expression of $DT_\gamma(0)$ is the identity. Thus, it is invertible. Now, take a set $F \subset \mathbb{T}^N$ such that there exists a diffeomorphism $\Phi : E_\gamma \to F$ of class $C^1$, and consider the map $T^\Phi_\gamma : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$T^\Phi_\gamma(v) := \int_{E_\gamma} f_\gamma\big(\Phi^{-1}(x) - v\big) J\Phi(x)\mathrm{d}x \,.$$

Then

$$DT^\Phi_\gamma(0) = -\int_{E_\gamma} Df_\gamma\big(\Phi^{-1}(x)\big) J\Phi(x)\mathrm{d}x \,.$$

So, fixed $\mu > 0$, there exists $\varepsilon > 0$ such that

$$\|DT^\Phi_\gamma(0) - DT_\gamma(0)\|_{C^0} \leq \mu \,,$$

whenever $\mathrm{d}_{C^1}(E_\gamma, F) < \varepsilon$, for all $\gamma \in (\bar\gamma - \gamma_2, \bar\gamma + \gamma_2)$ (thanks to the uniform control on the $C^1$-norm of the functions $f_\gamma$'s). Since the function $T^\Phi_\gamma$ is closed in $C^1$ to the function $T_\gamma$, it is possible to find $\varepsilon > 0$ and $\delta > 0$ such that for each $\Phi$ with $\|\Phi - \mathrm{Id}\| < \varepsilon$, it holds $T^\Psi_\gamma(B_\varepsilon) \supset B_\delta(T^\Psi_\gamma(0))$. This follows, for istance, from the proof of the Inverse Function Theorem. This allows to conclude. □

LEMMA 2.43. *Let $p > \max\{2, N-1\}$, and let $E \subset \mathbb{T}^N$ be a strictly stable critical point for $\mathcal{F}^{\bar\gamma}$. Then, for any $\gamma \in (\bar\gamma - \gamma_4, \bar\gamma + \gamma_4)$ and $E_\gamma$ critical point for $\mathcal{F}^\gamma$ with $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_4$, we have that*

$$\mathcal{F}^\gamma(F) \geq \mathcal{F}^\gamma(E_\gamma) + C_2\big(\alpha(E_\gamma, F)\big)^2 \,,$$

*for every set $F \subset \mathbb{T}^N$ with $|F| = |E_\gamma|$ and $\partial F = \{x + \psi(x)\nu_{E_\gamma}(x) \;:\; x \in \partial E_\gamma\}$ for some $\|\psi\|_{W^{2,p}(\partial E_\gamma)} \leq \delta_4$.*

PROOF. Let $\varepsilon_4 \in (0, \varepsilon)$, where $\varepsilon > 0$ is the constant given by the previous result. We know that we can find a vector $v \in \mathbb{R}^N$ such that

$$Pen_{E_\gamma}(F + v) = 0 \,.$$

Thus, by using the result of Lemma 2.41 we can write

$$\begin{aligned}\mathcal{F}^\gamma(F) = \mathcal{F}^\gamma(F+v) = \mathcal{F}^\gamma_{E_\gamma}(F+v) &\geq \mathcal{F}^\gamma_{E_\gamma}(E_\gamma) + C_2|E_\gamma \triangle F|^2 \\ &\geq \mathcal{F}^\gamma(E_\gamma) + C_2\big(\alpha(E_\gamma, F)\big)^2 \,.\end{aligned}$$

□

We now prove the uniform $L^\infty$-local minimality result, *i.e.*, the uniform version of [1, Theorem 4.3].

LEMMA 2.44. *Let $E \subset \mathbb{T}^N$ be a strictly stable critical point for $\mathcal{F}^{\bar\gamma}$. Then it is possible to find $\delta > 0$, $\gamma_5 > 0$ and $\varepsilon_5 > 0$ such that, for any $\gamma \in (\bar\gamma - \gamma_5, \bar\gamma + \gamma_5)$ and $E_\gamma$ critical point for $\mathcal{F}^\gamma$ with $\mathrm{d}_{C^1}(E, E_\gamma) < \varepsilon_5$, it holds*

$$\mathcal{F}^\gamma(E_\gamma) \leq \mathcal{F}^\gamma(F),$$

*for every set $F \subset \mathbb{T}^N$ with $|F| = |E_\gamma|$, such that $E_\gamma \triangle F \Subset \mathcal{N}_\delta(E_\delta)$, where $\mathcal{N}_\delta(E_\gamma)$ is a tubular neighborhood of $\partial E_\gamma$ of thickness $\delta$.*

PROOF. Suppose for the sake of contradiction that there exists a sequence $\gamma_n \to \bar\gamma$, $E_{\gamma_n} \to E$ in $C^1$, with $|E_\gamma| = |E|$, a sequence $\delta_n \to 0$ and a sequence of sets $F_n$ with $|F_n| = |E_{\gamma_n}|$, $E_\gamma \triangle F_n \Subset \mathcal{N}_\delta(E_{\delta_n})$, such that

$$\mathcal{F}^{\gamma_n}(E_{\gamma_n}) > \mathcal{F}^{\gamma_n}(F_n).$$

Let $E_n$ be a solution of the following constrained minimum problem

$$\min\big\{\mathcal{F}^{\gamma_n}(F) + \Lambda\big||F| - |E_\gamma|\big| \ : \ F \triangle E_\gamma \subset \mathcal{N}_\delta(E_{\delta_n})\big\}.$$

By using the $C^{3,\beta}$ convergence of the $E_{\gamma_n}$'s to $E$, and reasoning as in the proof of [1, Theorem 4.3], it is possible to find a constant $\Lambda > 0$ independent of $\gamma_n$ such that the sets $E_n$'s are $(4\Lambda, r_0)$-minimizers of the area functional, for some $r_0 > 0$ independent of $\gamma_n$, and $|E_n| = |E_\gamma|$. This is because, if we set $\nu_n := \nabla d_n$ (defined in $(\partial E)_\mu$, for some $\mu > 0$), where $d_n$ is the signed distance from $E_n$, we have that $\|\mathrm{div}\, \nu_n\|_{L^\infty} \leq C$ for some constant $C > 0$ independent of $n$.

Since $(E_n)_n$ is a sequence of uniform $(\omega, r)$-minimizers converging to $E$ in the $L^1$ topology, by Theorem 2.7 we have that indeed $E_n \to E$ in the $W^{2,p}$-topology. By using again the $C^{3,\beta}$ convergence of the $E_{\gamma_n}$'s to $E$ and the Euler-Lagrange equation satisfied by each $E_n$, we obtain that $\mathrm{d}_{W^{2,p}}(E_n, E_{\gamma_n}) \to 0$ as $n \to \infty$. Since, by definition, $\mathcal{F}^\gamma(E_n) < \mathcal{F}^\gamma(E_{\gamma_n})$ we obtain a contradiction with the result of Lemma 2.43. ☐

Finally, we prove the uniform $L^1$-local minimality result.

PROOF OF PROPOSITION 2.34. Suppose for the sake of contradiction that there exists a sequence $\gamma_n \to \bar\gamma$, $E_{\gamma_n} \to E$ in $C^1$, with $|E_\gamma| = |E|$, a sequence $\delta_n \to 0$ and a sequence of sets $F_n$ with $|F_n| = |E_{\gamma_n}|$, and $0 < \varepsilon_n \to 0$, where $\varepsilon_n := \alpha(F_n, E_{\gamma_n})$, such that

$$\mathcal{F}^{\gamma_n}(F_n) \leq \mathcal{F}^{\gamma_n}(E_{\gamma_n}) + \frac{C}{4}\big(\alpha(E_{\gamma_n}, F_n)\big)^2.$$

Let $E_n$ be a solution of the following constrained minimum problem

$$\min\Big\{\mathcal{F}^{\gamma_n}(F) + \Lambda\sqrt{\big(\alpha(F, E_{\gamma_n}) - \varepsilon_n\big)^2 + \varepsilon_n} \ : \ |F| = |E_\gamma|\Big\}.$$

Then, by usign a $\Gamma$-convergence argument it is possible to prove that the $E_n$'s converge (up to a subsequence) in the $L^1$ topology to a solution of the limiting problem

$$\min\big\{\mathcal{F}^{\bar\gamma}(F) + \Lambda|\alpha(F, E)| \ : \ |F| = |E|\big\}.$$

Reasoning as in the proof of [1, Theorem 1.1] and by using the $C^{3,\beta}$ convergence of the $E_{\gamma_n}$'s to $E$ (see Lemma 2.33), it is possible to prove that there exists a constant $\Lambda$, such that, the unique solution to the limiting problem is $E$ itself. Moreover, reasoning again as in the proof of [1, Theorem 1.1] and using Lemma 2.14 we can also infer that $E_n$ is a sequence of uniform $(\omega, r)$-minimizers, and that $E_n \to E$ in the $W^{2,p}$-topology, and thus $\mathrm{d}_{W^{2,p}}(E_n, E_{\gamma_n}) \to 0$ as

$n \to \infty$. Using the previous uniform $L^\infty$-local minimality result is it also possible to prove that $\frac{\alpha(E_n, E_{\gamma_n})}{\alpha(F_n, E_{\gamma_n})} \to 1$ (see [1, equation (4.17)]). Thus we may conclude

$$\mathcal{F}^{\gamma_n}(E_n) \le \mathcal{F}^{\gamma_n}(F_n) \le \mathcal{F}^{\gamma_n}(E_{\gamma_n}) + \frac{C}{4}\big(\alpha(E_{\gamma_n}, F_n)\big)^2 \le \mathcal{F}^{\gamma_n}(E_{\gamma_n}) + \frac{C}{2}\big(\alpha(E_{\gamma_n}, E_n)\big)^2 .$$

This yelds the contradiction with the result of Lemma 2.43.

□

**2.3.3. Continuous family of local minimizers.** We now prove a uniqueness result for critical points of $\mathcal{F}^\gamma$ close enough to a regular critical stable point of the area functional. We also prove that these critical points are isolated local minimizers.

Proposition 2.45. *Let $\bar\gamma \ge 0$ and let $E \subset \mathbb{T}^N$ be a strictly stable critical point for $\mathcal{F}^{\bar\gamma}$. Then there exist constants $\gamma_6 > 0$ and $\varepsilon_6 > 0$ and a unique family $\gamma \mapsto E_\gamma$, for $\gamma \in (\bar\gamma - \gamma_6, \bar\gamma + \gamma_6)$, with $|E_\gamma| = |E|$, such that*

- $\mathrm{d}_{C^1}(E_\gamma, E) < \varepsilon_6$,
- *$E_\gamma$ is a critical point for $\mathcal{F}^\gamma$.*

*Moreover $\gamma \mapsto E_\gamma$ is continuous in $C^{3,\beta}$, for all $\beta \in (0,1)$, and $E_\gamma$ is an isolated local minimizer of $\mathcal{F}^\gamma$.*

Proof. *Step 1.* Since $E$ is a strictly stable critical point for $\mathcal{F}^{\bar\gamma}$, by Theorem 2.31 we can infer that it is an isolated local minimizer of the same functional. Thus, by Theorem 2.25, we can find a sequence $(E_\gamma)_\gamma$, with $|E_\gamma| = |E|$, such that $E_\gamma$ is a local minimizer of $\mathcal{F}^\gamma$, and $\alpha(E_\gamma, E) \to 0$ as $\gamma \to \bar\gamma$. By Corollary 2.15, we know that the sequence $(E_\gamma)_\gamma$ is a sequence of $(\omega_0, r_0)$-minimizers, where the parameter $\omega$ can be choosen uniformly with respect to $\gamma$ (see Lemma 2.14). Hence, Theorem 2.7 allows to say that the $E_\gamma$'s converge to $E$ in the $C^{1,\beta}$-topology.

*Step 2.* Take $\varepsilon_6 < \varepsilon_2$ and $\gamma_6 < \gamma_2$ such that

$$\mathrm{d}_{C^1}(E_\gamma, E) < \varepsilon_6 ,$$

for any $\gamma \in (\bar\gamma - \gamma_6, \bar\gamma + \gamma_6)$. By Proposition 2.34, we know that the $E_\gamma$'s are uniform local minimizers with respect to sets $F$ with $|F| = |E_\gamma|$ such that $\alpha(F, E_\gamma) \le \delta_2$. In particular, we have that

$$\mathcal{F}^\gamma(E_\gamma) < \mathcal{F}^\gamma(F) ,$$

for any set $F \ne E_\gamma$ with $|F| = |E_\gamma|$ and $\alpha(F, E_\gamma) \le \delta_2$. By taking a smaller $\varepsilon_6$ (and a smaller $\gamma_6$) if necessary, we can assume that

$$\mathrm{d}_{C^1}(F, E) < \varepsilon_6 \Rightarrow \alpha(F, E_\gamma) \le \delta_2 ,$$

for any set $F \subset \mathbb{T}^N$ and any $\gamma \in (\bar\gamma - \gamma_6, \bar\gamma + \gamma_6)$. This allows to infer that $E_\gamma$ is the unique critical point of $\mathcal{F}^\gamma$ with $|E_\gamma| = |E|$ and $\mathrm{d}_{C^1}(E_\gamma, E) < \varepsilon_6$. Indeed, if $F$ is another critical point of $\mathcal{F}^\gamma$, with $|F| = |E|$ with $\mathrm{d}_{C^1}(F, E) < \varepsilon_6$, by using again Proposition 2.34, we would obtain that $F$ is an isolated local minimizer of $\mathcal{F}^\gamma$ with respect to sets $G$ with $|G| = |F|$ and $\alpha(G, F) \le \delta_2$. But this contradicts the isolated local minimality property of $E_\gamma$.

*Step 3.* Finally, we can deduce the continuity in the $C^{3,\beta}$-topology of the family $\gamma \mapsto E_\gamma$ as follows: fix $\tilde\gamma \in (\bar\gamma - \gamma, \bar\gamma + \gamma)$, and let $\gamma \to \tilde\gamma$. Then, up to a subsequence, the sets $E_\gamma \to F$ in the $L^1$ topology. By the uniqueness property just proved, we have that $F = E_{\tilde\gamma}$. Since $E_\gamma$ is a sequence of uniform $(\omega, r_0)$-minimizers, we infer from Lemma 2.7, that $E_\gamma \to F$ in the $C^{1,\alpha}$ topology. Then, by the Euler-Lagrange equation satisfied by the $E_\gamma$'s, we obtain the convergence of $E_\gamma$ to $E_{\tilde\gamma}$ in the $C^{3,\beta}$-topology. □

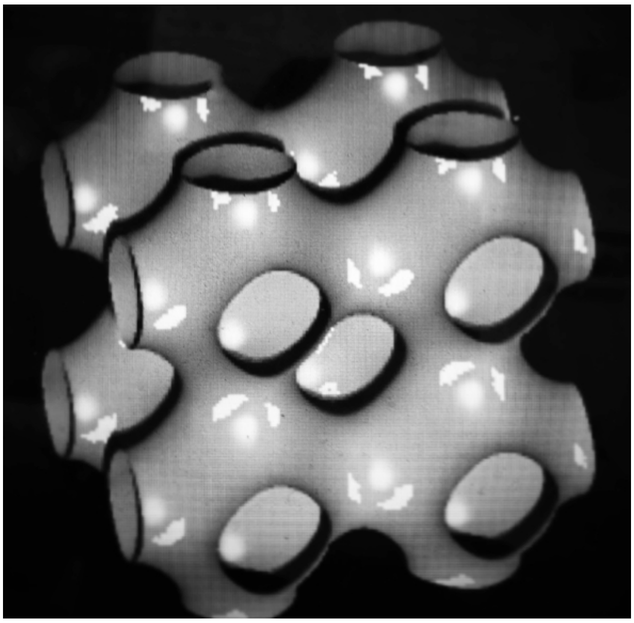

FIGURE 3. An example of strictly stable periodic surface with constant mean curvature.

**2.3.4. Periodic local minimizers with almost constant mean curvature.** The main result of this chapter is the following.

THEOREM 2.46. *Let $E \subset \mathbb{T}^N$ be a smooth set that is critical and strictly stable for the area functional,* i.e., *there exists $\lambda \in \mathbb{R}$ such that*

$$H_{\partial E} = \lambda \qquad \text{on } \partial E\,,$$

*and*

$$\int_{\partial E} \left(|D_\tau \varphi|^2 - |B_{\partial E}|^2 \varphi^2\right) \mathrm{d}\mathcal{H}^{N-1} > 0 \qquad \text{for every } \varphi \in T^\perp(\partial E)\setminus\{0\}\,.$$

*Fix constants $\bar\gamma > 0$, $\varepsilon > 0$. Then it is possible to find $\bar k = \bar k(\bar\gamma, \varepsilon) \in \mathbb{N}$ and $C = C(\bar\gamma) > 0$ such that for all $k \geq \bar k$ there exists a unique set $F \subset \mathbb{T}^N$ that is $1/k$-periodic and with*

- $\mathrm{d}_{C^0}(F, E^k) < \frac{\varepsilon}{k}$, *where $E^k$ is as Definition 2.1,*
- $\mathrm{d}_{C^1}(F, E^k) < \varepsilon$,
- $\|\nabla_\tau H_F\|_{L^\infty(\partial F)} < \frac{C}{k}$, *where $H_F$ is the mean curvature of $\partial F$.*

*Moreover $F$ is an isolated local minimizer of $\mathcal{F}^{\bar\gamma}$ with respect to $1/k$-periodic sets,* i.e., *there exists $\delta > 0$ such that, for any set $G \subset \mathbb{T}^N$ that is $1/k$-periodic and with $|G| = |F|$, it holds*

$$\mathcal{F}^{\bar\gamma}(F) < \mathcal{F}^{\bar\gamma}(G)\,,$$

*whenever $0 < \alpha(G, F) \leq \delta$.*

PROOF. Consider the sequence

$$(\gamma_k)_k := (\bar\gamma k^{-3})_{k \in \mathbb{N}\setminus\{0\}}\,.$$

Let $\gamma_k \mapsto E_{\gamma_k}$ be the unique family provided by Proposition 2.45 applied to $E$. Take $\bar k$ such that, for all $k \geq \bar k$, $\mathrm{d}_{C^1}(E_{\gamma_k}, E) < \varepsilon$ and $E_{\gamma_k}$ is an isolated local minimizer of $\mathcal{F}^\gamma$. This can be done by using the results of Proposition 2.45. Let $F := E^k_{\gamma_k}$. Now, it is easy to see that

$$\mathrm{d}_{C^0}(F, E^k) = \frac{1}{k}\mathrm{d}_{C^0}(E_{\gamma_k}, E) < \frac{\varepsilon}{k}\,, \qquad \mathrm{d}_{C^1}(F, E^k) = \mathrm{d}_{C^1}(E_{\gamma_k}, E) < \varepsilon\,.$$

Moreover, by (2.5) and (2.4), we have that

$$\mathcal{F}^{\bar{\gamma}}(F) = k^N \mathcal{F}^{\bar{\gamma}}_k(E_{\gamma_k}) = k\big[\mathcal{P}_{\mathbb{T}^N}(E_{\gamma_k}) + \gamma_k \mathcal{NL}_{\mathbb{T}^N}(E_{\gamma_k})\big] = k\mathcal{F}^{\gamma_k}_{\mathbb{T}^N}(E_{\gamma_k})\,.$$

Since $E_{\gamma_k}$ is an isolated local minimizer for $\mathcal{F}^{\gamma_k}$, we obtain that $F$ satisfied the isolated local minimimality property of the theorem.

Finally, we have that

$$H_{\partial F}(x) = kH_{\partial E_{\gamma_k}}(kx) = k\big(\lambda_k - 4\gamma_k v^{E_{\gamma_k}}(kx)\big)\,,$$

where in the last step we have used the Euler-Lagrange equation satisfied by $E_{\gamma_k}$. Thus, using the definition of $\gamma_k$, we obtain that

$$\|\nabla_\tau H_F\|_{L^\infty(\partial F)} \leq \frac{4\bar{\gamma}}{k}\|\nabla v^{E_{\gamma_k}}\|_{L^\infty(\partial E_{\gamma_k})}\,.$$

Since $v^{E_{\gamma_k}} \to v^E$ in $C^{1,\beta}$, up to choose a bigger $\bar{k}$, we also have the desired estimate for $\|\nabla_\tau H_F\|_{L^\infty(\partial F)}$. □

We finally show that the critical points constructed in the above theorem can be approximated with local minimizers of the $\varepsilon$-diffuse energy $OK^{\bar{\gamma}}_\varepsilon$.

Corollary 2.47. *Let $F$ be a periodic critical point constructed in the above theorem. Define the function $u := \chi_F - \chi_{\mathbb{T}^N \setminus F}$. Then it is possibile to find $\bar{\varepsilon} > 0$ and a family $(u_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ of local minimizers of the energy $OK^{\bar{\gamma}}_\varepsilon$ (see (0.1)) with prescibed volume $m := \int_{\mathbb{T}^N} u$, such that $u_\varepsilon \to u$ in $L^1(\mathbb{T}^N)$ as $\varepsilon \to 0$.*

Proof. The proof follows by the Kohn and Sternberg's theorem [40] (see also [14, Proposition 8]), thanks to the $\Gamma$-convergence of $OK^{\bar{\gamma}}_\varepsilon$ to $\mathcal{F}^{\bar{\gamma}}$ and thanks to the fact that $F$ is an isolated local minimizer with respect to $1/k$-periodic perturbations. □

CHAPTER 3

# A local minimality criterion for the triple point configuration of the Mumford-Shah functional

In this section we prove a local minimality result, based on a second variation approach, for triple point configurations of the Mumford-Shah functional. This is the first step of an ongoing project aimed at proving the local minimality result in the $L^1$-topology.

## 3.1. Setting

Here we collect the terminology and we introduce all the objects we will need in the rest of the chapter. First of all, we need to specify the class of triple points we are interested in.

DEFINITION 3.1. We say that a pair $(u, \Gamma)$ is *regular admissible triple point* (in brief *triple point*) if

- $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$, where the $\Gamma^i$'s are three disjoint relatively open curves in $\Omega$ that are of class $C^3$ and $C^{2,\alpha}$ up to their clousure. We also suppose $\partial\Gamma_i = \{x_0, x^i\}$, where $x_0 \in \Omega$ and $x^i \in \partial\Omega$ with $x^i \neq x^j$ for $i \neq j$,
- denting by $\nu^i$ the normal vector to $\Gamma_i$, we require the angle between $\nu^i(x_0)$ and $\nu_{(i+1)mod3}(x_0)$ to be less than or equal to $\pi$; moreover we require each $\bar{\Gamma}_i$ to does not intersect $\partial\Omega$ tangentially,
- there exists $\partial_D\Omega \subset\subset \partial\Omega \backslash \Gamma$, relatively open in $\partial\Omega$, such that $u$ solves
$$\int_{\Omega\backslash\Gamma} \nabla u \cdot \nabla z \, \mathrm{d}x = 0 \,, \tag{3.1}$$
for every $z \in H^1(\Omega\backslash\Gamma)$ with $z = 0$ on $\partial_D\Omega$.

REMARK 3.2. The regularity we impose on the curves $\Gamma^i$'s is not so restrictive as it may seems: indeed we will work with critical triple points (see Definition 3.14), and it was proved in [39] that, for critical configurations, each $\Gamma_i$ is analitic as soon as it is of class $C^{1,\alpha}$, and the regularity theory tells us that each curve is of class $C^{2,\alpha}$ up to its clousure. The second condition we asked for the $\Gamma^i$'s is to avoid pathological cases.

We would like to point out that the assumption that each curve $\Gamma_i$ is relatively open has been made just for convenience, and do not prevent the use of $(u, \Gamma)$, where $u \in H^1(\Omega\backslash\Gamma)$, as an admissible pair in which to compute the Mumford-Shah functional.

It is possible to rephrase the third condition above by saying that $u$ is a weak solition of
$$\begin{cases} \triangle v = 0 & \text{in } \Omega\backslash\bar{\Gamma}\,, \\ v = u & \text{on } \partial_D\Omega\,, \\ \partial_{\nu_{\partial\Omega}} v = 0 & \text{on } \partial\Omega\backslash\partial_D\Omega\,, \\ \partial_\nu v = 0 & \text{on } \bar{\Gamma}\,. \end{cases}$$
From the results on elliptic problems in domains with corners (see [32]) and from the regularity of $\partial\Omega$, we know that $u$ can have a singularity near $\mathcal{S}$, the relative boundary of $\partial_D\Omega$ in $\partial\Omega$:

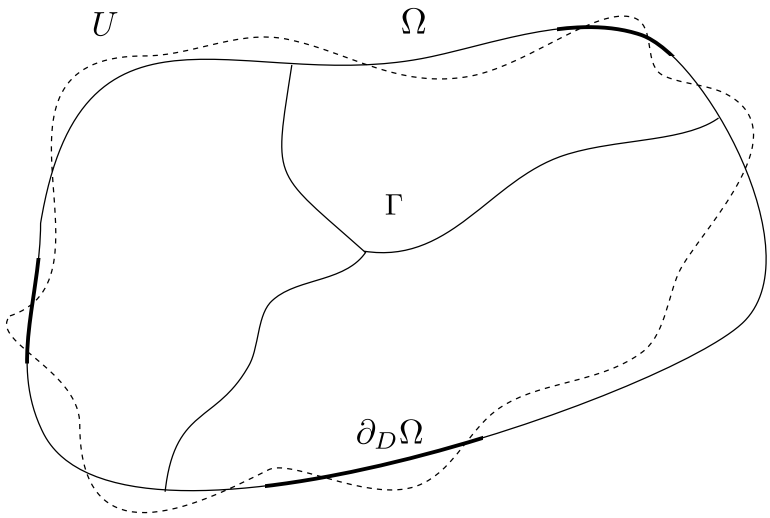


FIGURE 1. An admissible subdomain $U$. The bold part represents $\partial_D\Omega$.

namely, $u$ can be $H^1$ but not $H^2$ in a neighborhood of $\mathcal{S}$. Moreover, the gradient of $u$ may not be bounded in that region. In a future application of the present work we will need to impose a bound on the $L^\infty$-norm of the gradient of the admissible competitors. But this can be done only far from $\mathcal{S}$. So, we are forced to consider competitors equals to $u$ in a neighborhood of $\mathcal{S}$.

DEFINITION 3.3. Given a regular triple point $(u, \Gamma)$, we say that an open set $U \subset \Omega$ is an *admissible subdomain* if $\Gamma \subset U$ and $\overline{U} \cap S = \emptyset$. In this case we define

$$\mathcal{MS}\big((u,\Gamma);U\big) := \int_{U\setminus\Gamma} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{N-1}(\Gamma)\,.$$

Moreover, given an open set $A \subset \Omega$, we denote by $H^1_U(A)$ the space of functions $z \in H^1(A)$ such that $z = 0$ on $(\Omega\setminus U) \cup \partial_D\Omega$.

Our strategy requires to perform the first and the second variation of our functional $\mathcal{MS}$. So, we need to specify the perturbations of the set $\Gamma$ and of the function $u$ we want to consider.

DEFINITION 3.4. Let $(u, \Gamma)$ be a regular admissible triple point and let $U$ be an admissible subdomain. We say that a family of differomorphisms of $\overline{\Omega}$ onto itself, $(\Phi_t)_{t\in(-1,1)}$, is *admissible* for $(u, \Gamma)$ in $U$, if the following conditions are satisfied:

- $\Phi_0$ is the identity map Id,
- $\Phi_t = \mathrm{Id}$ in $(\Omega\setminus U) \cup \partial_D\Omega$,
- $\Phi$ is of class $C^2$ with respect to the variable $t$ and of class $C^3$ with respect to the variable $x$.

In this case, we define:

$$X_{\Phi_t} := \dot{\Phi}_t \circ \Phi_t^{-1}\,, \qquad Z_{\Phi_t} := \ddot{\Phi}_t \circ \Phi_t^{-1}\,,$$

where with $\dot{\Phi}_t$ we denote the derivative with respect to the variable $s$ of the map $(s, x) \to \Phi_s(x)$ computed at $(t, x)$. Moreover we also introduce the following abbreviations

$$X_t := X_{\Phi_t}\,, \qquad Z_t := Z_{\Phi_t}\,, \qquad X := X_0\,, \qquad Z := Z_0\,,$$

where no risk of confusion can occur.

The above variations will affect only the set $\Gamma$, *i.e.*, at every time $t$ we will consider the set $\Gamma_t := \Phi_t(\Gamma)$. Since our functional depends also on a function $u$, we have to choose, for each time $t$, a suitable function $u_t$ related to the set $\Gamma_t$ in which compute our functional $\mathcal{MS}$. The idea, as in [9], is to choose the function that minimizes the Dirichlet energy.

DEFINITION 3.5. Let $\Phi : \bar{\Omega} \to \bar{\Omega}$ be a diffeomorphism such that $\Phi = \mathrm{Id}$ on $(\Omega \backslash U) \cup \partial_D \Omega$, and set $\Gamma_\Phi := \Phi(\Gamma)$. We define $u_\Phi$ as the unique solution of:

$$\begin{cases} \int_{\Omega \backslash \Gamma_\Phi} \nabla u_\Phi \cdot \nabla z \,\mathrm{d}x = 0 & \text{for each } z \in H^1_U(\Omega \backslash \Gamma_\Phi)\,, \\ u_\Phi = u & \text{in } (\Omega \backslash U) \cup \partial_D \Omega\,, \\ u_\Phi \in H^1(\Omega \backslash \Gamma_\Phi)\,. \end{cases}$$

Moreover, given a family of admissible diffeomorphisms $(\Phi_t)_t$, we set $u_t := u_{\Phi_t}$, and we define the function $\dot{u}_t(x)$ as the derivative with respect to the variable $s$ of the map $(s, x) \mapsto u_s(x)$, computed in $(t, x)$. For simplicity, set $\dot{u} := \dot{u}_0$.

We are now in position to describe the admissible variations.

DEFINITION 3.6. We define the *first* and the *second variation* of the functional $\mathcal{MS}$ at a regular admissible triple point $(u, \Gamma)$ in $U$, with respect to the family of admissible diffeomorphisms $(\Phi_t)_{t \in (-1,1)}$, as

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{MS}\big((u_t, \Gamma_t); U\big)_{|t=0}\,, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{MS}\big((u_t, \Gamma_t); U\big)_{|t=0}\,,$$

respectively.

## 3.2. Preliminary results

**3.2.1. Geometric preliminaries.** We collect here some geometric definitions and identities that will be useful later. First of all, we will use the following matrix notation: if $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $v_1, v_2 \in \mathbb{R}^2$, we set

$$A[v_1, v_2] := A[v_1] \cdot v_2\,.$$

Let $\gamma \subset \mathbb{R}^2$ be a curve of class $C^2$ and let $\tau : \gamma \to \mathbb{S}^1$ be the tangent vector field on $\gamma$. Given an orientation on $\gamma$ it is possible to defined a signed distance function from $\gamma$ as follows:

$$\mathrm{d}_\gamma(x + t\nu(x)) := t\,,$$

where $\nu(x)$ is the normal vector to $\gamma$ at the point $x$. This signed distance turns out to be of class $C^2$ in a tubular neighborhood $\mathcal{U}$ of $\gamma$; moreover, its gradiend coincides with $\nu$ on $\gamma$. In the following we will use the extension of the normal vector field given by the gradient of the signed distance from $\gamma$, that we will denote by $\nu : \mathcal{U} \to \mathbb{S}^1$.

Given a smooth vector field $g : \mathcal{U} \to \mathbb{R}^k$, we define the *tangential differential* $D_\gamma g$ ($\nabla_\gamma g$ if $k = 1$) by $D_\gamma g(x) := dg(x) \circ \pi_x$, where $dg(x)$ is the classical differential of $g$ at $x$ and $\pi_x$ is the orthogonal projection on $T_x\gamma$, the tangent line to $\gamma$ at $x$. If $g : \mathcal{U} \to \mathbb{R}^2$ we define its *tangential divergence* as $\mathrm{div}_\gamma g := \tau \cdot \partial_\tau g$.

We define the *curvature* of $\gamma$ as the function $H : \mathcal{U} \to \mathbb{R}$ given by $H := \mathrm{div}\nu$. Notice that, since $\partial_\nu \nu = 0$ on $\Gamma$, we can write $H = \mathrm{div}_\gamma \nu = D\nu[\tau, \tau]$.

For every smooth vector field $g : \mathcal{U} \to \mathbb{R}^2$ the following *divergence formula holds*:

$$\int_\gamma \mathrm{div}_\gamma g \,\mathrm{d}\mathcal{H}^{N-1} = \int_\gamma H(g \cdot \nu) \,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\gamma} g \cdot \eta \,\mathrm{d}\mathcal{H}^0\,, \tag{3.2}$$

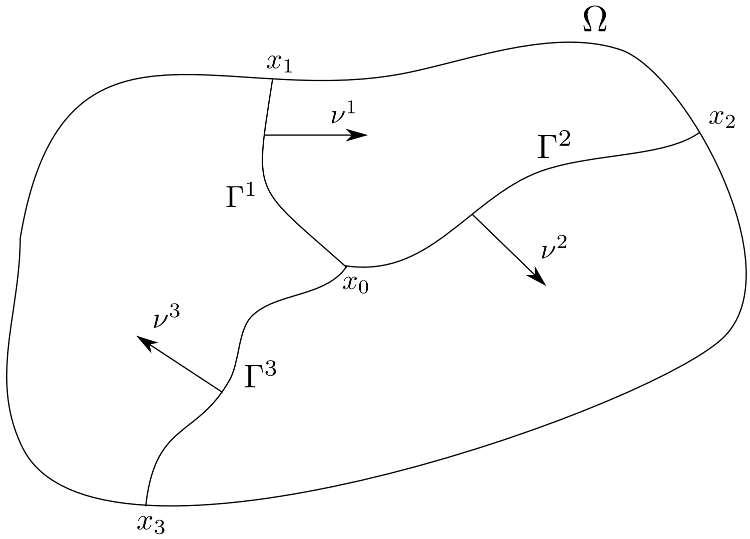


FIGURE 2. An admissible triple point with the choosen orientation.

where $\eta$ is a unit tangent vector pointing out of $\gamma$ in each point of $\partial\gamma$. Moreover, if $\Phi : \mathcal{U} \to \mathcal{U}$ is an orientation preserving diffeomorphism, and we denote by $\gamma_\Phi := \Phi(\gamma)$, a possible choice for the orientation of $\gamma_\Phi$ is given by:

$$\nu_\Phi := \frac{(D\Phi)^{-T}[\nu]}{\big|(D\Phi)^{-T}[\nu]\big|} \circ \Phi^{-1} .$$

In this case, the vector $\eta$ of the divergence formula (3.2) becomes

$$\eta_\Phi := \frac{D\Phi[\eta]}{\big|D\Phi[\eta]\big|} \circ \Phi^{-1} .$$

In particular, for an admissible flow $(\Phi_t)_{t\in(-1,1)}$, we will use the following notation: $\nu_t := \nu_{\Phi_t}$, $\eta_t := \eta_{\Phi_t}$, and we will denote by $H_t$ the curvature of $\gamma_t$.

Finally, setting $J_\Phi := \big|(D\Phi)^{-T}[\nu]\big| \det D\Phi$, for every $\Phi \in L^1(\gamma_\Phi)$ the following *area formula* holds (see [5, Theorem 2.91]):

$$\int_{\gamma_\Phi} \Phi \, \mathrm{d}\mathcal{H}^{N-1} = \int_\gamma (\Phi \circ \Phi)\, J_\Phi \, \mathrm{d}\mathcal{H}^{N-1} .$$

We now treat triple points. Fix for $\partial\Omega$ the clockwise orientation and orient the curvers $\Gamma_i$'s in such a way that $\nu^i(x^i) = \tau_{\partial\Omega}(x^i)$ for each $i = 1,2,3$ (see Figure 2), where $\nu^i$ is the normal vector on $\Gamma_i$.

For the sake of simplicity we will use the following notation: given $\varphi : \Gamma \to \mathbb{R}^k$, we will denote by $\varphi_i$ its restriction to $\Gamma_i$, and we will write

$$\int_{\partial\Gamma} \varphi \, \mathrm{d}\mathcal{H}^0 := \sum_{i=1}^{3} \big(\varphi_i(x_0) + \varphi_i(x^i)\big) .$$

In the following we will also need to use the trace of a function on $\Gamma$. Notice that, since each $\Gamma_i$ is relatively open, $x^i \notin \Gamma_i$, for each $i = 0,1,2,3$.

DEFINITION 3.7. Let $\Gamma$ be a regular admissible triple point, and let $z \in H^1(\Omega\setminus\bar{\Gamma})$. We define the *traces* $z^+, z^-$ of $z$ on $\Gamma$ as follows: let $x \in \Gamma$ and define

$$z^{\pm}(x) := \lim_{r\to 0^+} \frac{1}{|B_r(x)\cap V_x^{\pm}|}\int_{B_r(x)\cap V_x^{\pm}} z(y)\,\mathrm{d}y\,,$$

where $V_x^{\pm} := \{y \in \mathbb{R}^2 \,:\, \pm(y-x)\cdot\nu^i(x) \geq 0\}$, if $x \in \Gamma_i$.

In the computation of the second variation we will need some geometric identities, that we collect in the following lemma. The proofs of the first bloch of identities are the same as those of [9, Lemma 3.8], and hence we will not repeat them here. We just need to prove the last three.

LEMMA 3.8. *The following identities hold on each $\Gamma^i$:*

*(1)* $D^2u^{\pm}[\nu^i,\nu^i] = -\triangle_{\Gamma_i}u^{\pm}$*;*
*(2)* $D^2u^{\pm}[X,\nu^i] = -(X\cdot\nu^i)\triangle_{\Gamma_i}u^{\pm} - D\nu^i[\nabla_{\Gamma^i}u^{\pm},X]$*;*
*(3)* $\mathrm{div}_{\Gamma_i}[(X\cdot\nu^i)\nabla_{\Gamma^i}u^{\pm}] = (D_{\Gamma^i}X)^T[\nu^i,\nabla_{\Gamma^i}u^{\pm}] - \nabla^2u^{\pm}[X,\nu^i]$*;*
*(4)* $\partial_{\nu^i}H^i = -|D\nu^i|^2 = -(H^i)^2$*;*
*(5)* $D^2u^{\pm}[\nu^i,\nabla_{\Gamma^i}u^{\pm}] = -D\nu^i[\nabla_{\Gamma^i}u^{\pm},\nabla_{\Gamma^i}u^{\pm}] = -H_i|\nabla_{\Gamma_i}u^{\pm}|^2$*;*
*(6)* $\dot{\nu}^i = -(D_{\Gamma^i}X)^T[\nu^i] - D_{\Gamma^i}\nu^i[X] = -\nabla_{\Gamma}(X\cdot\nu)$*;*
*(7)* $\frac{\partial}{\partial t}\big(\dot{\Phi}_t\cdot(\nu_t^i\circ\Phi_t)J_{\Phi_t}\big)_{|t=0} = Z\cdot\nu^i - 2X^{||}\cdot\nabla_{\Gamma^i}(X\cdot\nu^i) + D\nu^i[X^{||},X^{||}] + \mathrm{div}_{\Gamma^i}((X\cdot\nu^i)X)$*.*

*Moreover, the following identities are satisfied:*

*(i)* $\frac{\partial}{\partial t}(\eta_t^i\circ\Phi_t)_{|t=0} = (D_{\Gamma_i}X)^T[\nu^i,\eta^i]\nu^i$*, on* $\partial\Gamma^i$*;*
*(ii)* $X\cdot\frac{\partial}{\partial t}(\eta_t^i\circ\Phi_t)_{|t=0} = -(X\cdot\nu^i)\dot{\nu}^i\cdot\eta^i - H^i(X\cdot\nu^i)(X\cdot\eta^i)$*, on* $\partial\Gamma^i$*;*
*(iii)* $Z\cdot\nu_{\partial\Omega} + D\nu_{\partial\Omega}[X,X] = 0$ *on* $\partial\Gamma^i\cap\partial\Omega$*.*

PROOF. *Proof of (i).* Let $w_t := D\Phi_t(x)[tau(x)]$. Then

$$\frac{\partial}{\partial t}(\eta_t^i\circ\Phi_t)_{|t=0} = \frac{\partial}{\partial t}\frac{w_t}{|w_t|}\,.$$

Since $\dot{w}_0 = D_{\Gamma^i}\dot{\Phi}[\tau^i] = D_{\Gamma^i}X[\tau^i]D_{\Gamma^i}\tau$, we obtain

$$\frac{\partial}{\partial t}(\eta_t^i\circ\Phi_t)_{|t=0} = D_{\Gamma^i}X[\tau] - (D_{\Gamma^i}X)^T[\tau,\tau]\,,$$

we conclude.
*Proof of (ii).* This identity follows by taking the scalar product of identity (6) with $(X\cdot\nu)\eta$, and by using (i).
*Proof of (iii).* This one follows by deriving with respect to the time the identity

$$(X_t\circ\Phi_t)\cdot(\nu_{\partial\Omega}\circ\Phi_t) = 0\,,$$

that holds on $\partial\Gamma^i\cap\partial\Omega$. □

**3.2.2. Properties of the function $\dot{u}$.** In the computations of the first and the second variation we need to know some properties of the sequence of functions $(u_t)_t$ that we state here. First of all we need to prove that the function $\dot{u}$ actualy exists. This is provided by the following result, whose proof is just the same as those of [9, Proposition 8.1], where the elliptic estimates in $W^{2,p}$ for $p < 4$, needed to prove the second part are, in our case, provided by Theorem 3.33.

PROPOSITION 3.9. *Let $(\Phi_t)_t$ be an admissible family of diffeomorphisms, and let $(u_t)_t$ be the functions defined in Definition 3.5. Set $\widetilde{u}_t := u_t\circ\Phi_t$ and $v_t := \widetilde{u}_t - u$. Then the following properties hold true:*

*(i) the map* $t \mapsto v_t$ *belongs to* $C^1\big((-1,1); H^1_U(\Omega\backslash\Gamma)\big)$*;*

*(ii) for every* $\bar{x} \in \bar{\Gamma}$*, let* $B$ *be a ball centered in* $\bar{x}$ *such that* $B\backslash\Gamma$ *has two (or, if* $\bar{x} = x_0$*, three) connected components* $B_1$*,* $B_2$ *(and* $B_3$*). For every* $t \in (-1,1)$*, let* $\widetilde{u}^i_t$ *be the restriction of* $\widetilde{u}_t$ *to* $B_i$*. Then we have that the map* $\widehat{u}^i(t,x) := \widetilde{u}^i_t(x)$ *belongs to* $C^1\big((-1,1)\times\bar{B}_i\big)$*.*

Using the above proposition it is possible to prove the following result, whose proof is just the same as those of [9, (3.6) of Theorem 3.6].

Proposition 3.10. *The function* $\dot{u}$ *exists, it is a well defined function of* $H^1_U(\Omega\backslash\Gamma)$*. Moreover, it is harmonic in* $\Omega\backslash\bar{\Gamma}$ *and satisfies the following Neumann boundary conditions:*

$$\partial_{\nu_{\partial\Omega}}\dot{u} = 0 \qquad \text{on } (\partial\Omega\backslash\partial_D\Omega)\cap U\,,$$

$$\partial_\nu \dot{u}^{\pm} = \mathrm{div}_\Gamma\big((X\cdot\nu)\nabla_\Gamma u^{\pm}\big) \quad \text{on } \Gamma\,. \tag{3.3}$$

*In particular, the following equation holds:*

$$\int_\Omega \nabla\dot{u}\cdot\nabla z\,\mathrm{d}x = \int_\Gamma \big[\mathrm{div}_\Gamma\big((X\cdot\nu)\nabla_\Gamma u^+\big)z^+ - \mathrm{div}_\Gamma\big((X\cdot\nu)\nabla_\Gamma u^-\big)z^-\big]\,\mathrm{d}\mathcal{H}^{N-1}\,, \tag{3.4}$$

*for each* $z \in H^1_U(\Omega\backslash\Gamma)$*.*

Remark 3.11. First of all we notice that the right-hand side of (3.3) is well defined. Indeed, by Theorem 3.33 that $u$ is of class $H^2$ in a neighborhood of $\Gamma$, and thus $\nabla_\Gamma u^{\pm} \in H^{\frac{1}{2}}(\bar{\Gamma})$. So, since $\Gamma$ and $X$ are regular, we get that $(X\cdot\nu)\nabla_\Gamma u^{\pm} \in H^{\frac{1}{2}}(\bar{\Gamma})$.

## 3.3. First and second variation

The aim of this section is to compute the first and the second variation of the functional $\mathcal{MS}$ at a regular admissible triple point $(u,\Gamma)$.

Theorem 3.12. *Let* $(u,\Gamma)$ *be a regular admissible triple point,* $U$ *an admissible subdomain and* $(\Phi_t)_{t\in(-1,1)}$ *an admissible family of diffeomorphisms for* $(u,\Gamma)$ *in* $U$*. Set* $f := |\nabla_\Gamma u^-|^2 - |\nabla_\Gamma u^+|^2 + H$*. Then the first variation of the functional* $\mathcal{MS}$ *computed at* $(u,\Gamma)$*, with respect to* $(\Phi_t)_{t\in(-1,1)}$*, is given by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{MS}\big((u_t,\Gamma_t);U\big)_{|t=0} = \int_\Gamma f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Gamma} X\cdot\eta\,\mathrm{d}\mathcal{H}^0\,, \tag{3.5}$$

*while the second variation reads as:*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{MS}\big((u_t,\Gamma_t);U\big)_{|t=0} = -2\int_U |\nabla\dot{u}|^2\,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^{N-1} + \int_\Gamma H^2(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^{N-1}$$
$$+ \int_\Gamma f\big[Z\cdot\nu - 2X^{\|}\cdot\nabla_\Gamma(X\cdot\nu) + D\nu[X^{\|},X^{\|}] - H(X\cdot\nu)^2\big]\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Gamma} Z\cdot\eta\,\mathrm{d}\mathcal{H}^0\,. \tag{3.6}$$

Proof. *Computation of the first variation.* In order to compute the first variation, we consider

$$\mathcal{MS}\big((u_t,\Gamma_t);U\big) = \int_{U\backslash\Gamma_t} |\nabla u_t|^2\,\mathrm{d}x + \mathcal{H}^{N-1}(\Gamma_t)\,,$$

and we treat the two terms separately. For the first one we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\biggl(\int_{U\setminus\Gamma_s}|\nabla u_s|^2\,\mathrm{d}x\biggr)_{|s=t} &= \frac{\mathrm{d}}{\mathrm{d}s}\biggl(\int_{U\setminus\Gamma}|\nabla u_s\circ\Phi_s|^2\det D\Phi_s\,\mathrm{d}x\biggr)_{|s=t}\\ &= \int_{U\setminus\Gamma}\Bigl[2(\nabla u_t\circ\Phi_t)\cdot\bigl((\nabla\dot u_t\circ\Phi_t)+(D^2u_t\circ\Phi_t)\dot\Phi_t\bigr)+|\nabla u_t\circ\Phi_t|^2\mathrm{div}X_t\circ\Phi_t\Bigr]\det D\Phi_t\,\mathrm{d}x\\ &= 2\int_{U\setminus\Gamma_t}\nabla u_t\cdot\nabla\dot u_t\,\mathrm{d}x+\int_{U\setminus\Gamma_t}\Bigl(2D^2u_t[\nabla u_t,X_t]+|\nabla u_t|^2\mathrm{div}X_t\Bigr)\,\mathrm{d}x\,.\end{aligned}$$

Recalling that $\dot u_t\in H^1_U(U\setminus\Gamma_t)$ by Proposition 3.10, from (3.1) we get that the first integral vanishes. Moreover, since it is possible to write

$$2D^2u_t[\nabla u_t,X_t]+|\nabla u_t|^2\mathrm{div}X_t=\mathrm{div}\bigl(|\nabla u_t|^2X_t\bigr)\,,$$

integrating by parts in each connected component of $\Omega\setminus\Gamma_t$, and recalling that $X_t\cdot\nu_{\partial\Omega}=0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\biggl(\int_{U\setminus\Gamma_s}|\nabla u_s|^2\,\mathrm{d}x\biggr)_{|s=t}=\int_{\Gamma_t}\bigl(|\nabla u_t^-|^2-|\nabla u_t^+|^2\bigr)(X_t\cdot\nu_t)\,\mathrm{d}\mathcal{H}^{N-1}\,.$$

Finally we also notice that in the last expression, we can substitute the operator $\nabla$ with the operator $\nabla_{\Gamma_t}$, since $\partial_{\nu_t}u_t=0$.

For the second term, it is well known (see, *e.g.*, [63]) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(\mathcal{H}^{N-1}(\Gamma_s)\Bigr)_{|s=t}=\int_{\Gamma_t}\mathrm{div}_{\Gamma_t}X_t\,\mathrm{d}\mathcal{H}^{N-1}=\int_{\Gamma_t}H_t(X_t\cdot\nu_t)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma_t}X_t\cdot\eta_t\,\mathrm{d}\mathcal{H}^0\,.$$

Hence, defining the function $f_t$ on $\Gamma_t$ as $f_t:=|\nabla_{\Gamma_t}u_t^-|^2-|\nabla_{\Gamma_t}u_t^+|^2+H_t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{MS}\bigl((u_s,\Gamma_s);U\bigr)_{|s=t}=\int_{\Gamma_t}f_t(X_t\cdot\nu_t)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma_t}X_t\cdot\eta_t\,\mathrm{d}\mathcal{H}^0\,. \tag{3.7}$$

Notice that the functions $f_t$ are weel defined $C^1$ functions in a normal tubular neighborhood of $\Gamma_t$. In particular, for $t=0$, we deduce the following expression for the first variation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{MS}\bigl((u_t,\Gamma_t);U\bigr)_{|t=0}=\int_{\Gamma}\bigl(|\nabla_\Gamma u^+|^2-|\nabla_\Gamma u^-|^2+H\bigr)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma}X\cdot\eta\,\mathrm{d}\mathcal{H}^0\,.$$

*Computation of the second variation.* Now we want to compute

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{MS}\bigl((u_s,\Gamma_s);U\bigr)_{|s=t}\,,$$

for $s\in(-1,1)$. The derivative of the first term of (3.7) can be computed as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_{\Gamma_t}f_t(X_t\cdot\nu_t)\,\mathrm{d}\mathcal{H}^{N-1}\biggr)_{|t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_{\Gamma}(f_t\circ\Phi_t)(X_t\circ\Phi_t)\cdot(\nu_t\circ\Phi_t)J_{\Phi_t}\,\mathrm{d}\mathcal{H}^{N-1}\biggr)_{|t=0}\\ &= \int_\Gamma(\dot f+\nabla f\cdot X)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}+\int_\Gamma f\frac{\partial}{\partial t}\bigl(\dot\Phi_t\cdot(\nu_t\circ\Phi_t)J_{\Phi_t}\bigr)_{|t=0}\,\mathrm{d}\mathcal{H}^{N-1}\,.\end{aligned}$$

Now, using equality (7) of Lemma 3.8 to rewrite the second integral, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Gamma_t} f_t(X_t\cdot\nu_t)\,\mathrm{d}\mathcal{H}^{N-1}\right)_{|t=0} &= \int_\Gamma(\dot f+\nabla f\cdot X)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+\int_\Gamma f\big(Z\cdot\nu-2X^{||}\cdot\nabla_\Gamma(X\cdot\nu)+D\nu[X^{||},X^{||}]+\mathrm{div}_\Gamma\big((X\cdot\nu)X\big)\big)\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_\Gamma f\big(Z\cdot\nu-2X^{||}\cdot\nabla_\Gamma(X\cdot\nu)+D\nu[X^{||},X^{||}]\big)\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+\int_\Gamma(\dot f+\nabla f\cdot\nu(X\cdot\nu))(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}+\int_\Gamma\mathrm{div}_\Gamma\big(f(X\cdot\nu)X\big)\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_\Gamma(\dot f+\nabla f\cdot\nu(X\cdot\nu))(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma}f(X\cdot\nu)(X\cdot\eta)\mathrm{d}\mathcal{H}^0\\
&\quad+\int_\Gamma Hf(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^{N-1}+\int_\Gamma f\big(Z\cdot\nu-2X^{||}\cdot\nabla_\Gamma(X\cdot\nu)+D\nu[X^{||},X^{||}]\big)\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}
$$

where the last equality follows from integration by parts, while the previous one by writing $X=(X\cdot\nu)\nu+X^{||}$. Now, recalling that $f=|\nabla_\Gamma u^-|^2-|\nabla_\Gamma u^+|^2+H$, we have that

$$\nabla f=2\nabla_\Gamma u^+\,D^2u^--2\nabla_\Gamma u^-\,D^2u^++\nabla H\,,$$

$$\dot f=2\nabla_\Gamma u^+\cdot\nabla_\Gamma\dot u^--2\nabla_\Gamma u^-\cdot\nabla_\Gamma\dot u^++\dot H\,.$$

Using the above identities and (2), (4) and (5) of Lemma 3.8 we can write

$$
\begin{aligned}
\int_\Gamma(\nabla f\cdot\nu)(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^{N-1}&=\int_\Gamma(X\cdot\nu)^2\big[2D^2u^-[\nabla_\Gamma u^-,\nu]-2D^2u^+[\nabla_\Gamma u^+,\nu]+\partial_\nu H\big]\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_\Gamma(X\cdot\nu)^2\big[2D\nu[\nabla_\Gamma u^+,\nabla_\Gamma u^+]-2D\nu[\nabla_\Gamma u^-,\nabla_\Gamma u^-]-|D\nu|^2\big]\,\mathrm{d}\mathcal{H}^{N-1}\\
&=\int_\Gamma(H^2-2fH)(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}
$$

where the identity $D\nu[\tau,\tau]=H$ has been used in the last step.

Now we would like to treat the term $\int_\Gamma\dot f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}$. First of all we recall that $H=\mathrm{div}_\Gamma\nu$ and $\partial_\nu\dot\nu=0$ (since $|\nu_t|^2\equiv1$). Thus $\dot H=\mathrm{div}_\Gamma\dot\nu$, and hence

$$
\begin{aligned}
\int_\Gamma\dot H(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}&=\int_\Gamma(\mathrm{div}_\Gamma\dot\nu)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}\\
&=-\int_\Gamma\dot\nu\cdot\nabla_\Gamma(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma}(\dot\nu\cdot\eta)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^0\\
&=\int_\Gamma|\nabla_\Gamma(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma}(\dot\nu\cdot\eta)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^0,
\end{aligned}
$$

where in the last line we have used (6) of Lemma 3.8. Moreover

$$\int_\Gamma(\nabla_\Gamma u^\pm\cdot\nabla_\Gamma\dot u^\pm)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}=-\int_\Gamma\dot u^\pm\mathrm{div}_\Gamma\big(\nabla_\Gamma u^\pm(X\cdot\nu)\big)\,\mathrm{d}\mathcal{H}^{N-1}+2\int_{\partial\Gamma}\dot u^\pm(X\cdot\nu)(\nabla_\Gamma u^\pm\cdot\eta)\,\mathrm{d}\mathcal{H}^0,$$

Hence, recalling (3.4), we obtain

$$
\begin{aligned}
\int_\Gamma\dot f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^{N-1}=&-2\int_U|\nabla\dot u|^2\,\mathrm{d}x+\int_\Gamma|\nabla_\Gamma(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^{N-1}+\int_{\partial\Gamma}(\dot\nu\cdot\eta)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^0\\
&+2\int_{\partial\Gamma}\Big[\dot u^+(X\cdot\nu)(\nabla_\Gamma u^+\cdot\eta)-\dot u^-(X\cdot\nu)(\nabla_\Gamma u^-\cdot\eta)\Big]\,\mathrm{d}\mathcal{H}^0.
\end{aligned}
$$

Finaly, we have to compute the derivative of the second integral of (3.5). Using (*ii*) of Lemma 3.8, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\partial\Gamma_t} X_t\cdot\eta_t\,\mathrm{d}\mathcal{H}^0\right)_{|t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\partial\Gamma}(X_t\circ\Phi_t)\cdot(\eta_t\circ\Phi_t)\,\mathrm{d}\mathcal{H}^0\right)_{|t=0}\\ &= \int_{\partial\Gamma}\left(Z\cdot\eta + X\cdot\frac{\partial}{\partial t}(\eta_t\circ\Phi_t)_{|t=0}\right)\mathrm{d}\mathcal{H}^0\\ &= \int_{\partial\Gamma}\Big(Z\cdot\eta-(X\cdot\nu)(\dot\nu\cdot\eta)-H(X\cdot\nu)(X\cdot\eta)\Big)\,\mathrm{d}\mathcal{H}^0\,.\end{aligned}$$

We now observe that some integrals vanishes for regular admissible triple points. Indeed, by the Neumann conditions satisfied by $u$, we know that $\partial_\nu u^\pm=0$ on $\Gamma$ and that $\partial_{\nu_{\partial\Omega}}u^\pm=0$ on $\partial_N\Omega\cap\bar U$. The admissibility conditions we required on regular admissible triple points tell us that $\nu_\partial\Omega(x^i)$ and $\nu^i(x^i)$ are linear independent for every $i=1,2,3$, as well as $\nu_1(x_0)$ and $\nu_2(x_0)$. Using the fact that $\nabla u^\pm$ is continuous up to the closure of $\Gamma$, we can infer that $\nabla u^\pm(x^i)=0$ for each $i=0,1,2,3$.

Combining all the above identities, we obtain the desired formula for the second variation of our functional $\mathcal{MS}$ at a regular admissible triple point $(u,\Gamma)$. □

Remark 3.13. The above expression for the second variation can be also used to compute the second variation at a generic time $t\in(-1,1)$. Indeed, fix $t\in(-1,1)$, and consider the family of diffeomorphisms

$$\widetilde\Phi_s:=\Phi_{t+s}\circ\Phi_t^{-1}\,.$$

It is easy to see that this family is admissible for $(u,\Gamma)$ in $U$, and that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{MS}\big((u_s,\Gamma_s);U\big)_{|s=t}=\frac{\mathrm{d}^2}{\mathrm{d}h^2}\mathcal{MS}\big((u_{t+h},\widetilde\Phi_h(\Gamma_t));U\big)_{|h=0}\,.$$

Hence, *mutatis mutandis*, the same expression as in (3.6) holds true for the second variation at a generic time $t\in(-1,1)$.

The expression (3.5) of the first variation suggests the following definition.

Definition 3.14. Let $(u,\Gamma)$ be a regular admissible triple point and $U$ an admissible subdomain. We say that $(u,\Gamma)$ is *critical* if the following three conditions are satisfied:

- $H=|\nabla_\Gamma u^-|^2-|\nabla_\Gamma u^+|^2$ on $\Gamma$,
- the $\Gamma_i$'s meet in $x_0$ at $\frac{2}{3}\pi$,
- each $\Gamma_i$ meets $\partial\Omega$ orthogonally.

Remark 3.15. Notice that a critical triple point is such that $H_i=0$ on $\partial\Gamma_i$.

Now we want to rewrite the second variation in a critical triple point.

Proposition 3.16. *Let $(u,\Gamma)$ be a regular critical triple point. Then the second variation of $\mathcal{MS}$ at $(u,\Gamma)$ in $U$ can be written as follows:*

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{MS}\big((u_t,\Gamma_t);U\big)_{|t=0} &= -2\int_U|\nabla\dot u|^2\,\mathrm{d}x+\int_\Gamma|\nabla_\Gamma(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad+\int_\Gamma H^2(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^{N-1}-\sum_{i=1}^3(H_{\partial\Omega}(X\cdot\nu^i)^2)(x^i)\,.\end{aligned}$$

PROOF. We notice that for a regular admissible critical triple point $f=0$ on $\Gamma$, $\nabla u^{\pm}\equiv 0$ on $\partial\Gamma$, $X\cdot\eta = X\cdot\nu_{\partial\Omega}=0$ on $\bar\Gamma\cap\partial\Omega$ and, thanks to $(iii)$ of Lemma 3.8, that

$$Z\cdot\eta = -D\nu_{\partial\Omega}[X,X] = -(X\cdot\nu)^2 D\nu_{\partial\Omega}[\nu,\nu] = -H_{\partial\Omega}(X\cdot\nu)^2\,.$$

Recalling that the $\Gamma_i$'s meet in $x_0$ at $\frac{2}{3}\pi$, we also have that $\sum_{i=1}^3 Z\cdot\nu^i(x_0)=0$. This allows to conclude. □

The above result suggests to introduce the following definition.

DEFINITION 3.17. We introduce the space

$$\widetilde{H}^1(\Gamma) := \{\varphi:\Gamma\to\mathbb{R}\,:\,\varphi^i\in H^1(\Gamma_i)\,,\ (\varphi^1+\varphi^2+\varphi^3)(x_0)=0\}\,,$$

endowed with the norm given by:

$$\|\varphi\|_{\widetilde{H}^1(\Gamma)} := \sum_{i=1}^3 \|\varphi^i\|_{H^1(\Gamma_i)}\,.$$

Then, we define the quadratic form $\partial^2\mathcal{MS}\big((u,\Gamma);U\big):\widetilde{H}^1(\Gamma)\to\mathbb{R}$ as

$$\begin{aligned}\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi] \;:=\; & -2\int_U |\nabla v_\varphi|^2\,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma\varphi|^2\,\mathrm{d}\mathcal{H}^{N-1} + \int_\Gamma H^2\varphi^2\,\mathrm{d}\mathcal{H}^{N-1} \\ & -\sum_{i=1}^3 \big(\varphi_i^2 D\nu_{\partial\Omega}[\nu,\nu]\big)(x^i)\,,\end{aligned}$$

where $v_\varphi\in H^1_U(\Omega\backslash\Gamma)$ is the solution of

$$\int_\Omega \nabla v_\varphi\cdot\nabla z\,\mathrm{d}x = \langle \mathrm{div}_\Gamma\big(\varphi\nabla_\Gamma u^+\big), z^+\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} - \langle \mathrm{div}_\Gamma\big(\varphi\nabla_\Gamma u^-\big), z^-\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)}\,, \tag{3.8}$$

for every $z\in H^1_U(\Omega\backslash\Gamma)$.

The following lemma ensures that the right-hand side of (3.9) makes sense.

LEMMA 3.18. *Let $\varphi\in\widetilde{H}^1(\Gamma)$ and let $\Phi\in H^{\frac12}(\Gamma)\cap C^0(\Gamma)$. Then $\varphi\Phi\in H^{\frac12}(\Gamma)$.*

PROOF. We need to estimate the Gagliardo seminorm. So

$$\begin{aligned}[\varphi\Phi]^2_{H^{1/2}} &:= \int_\Gamma\int_\Gamma \frac{|\varphi(x)\Phi(x)-\varphi(y)\Phi(y)|^2}{|x-y|^2}\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\ &\le \int_\Gamma\int_\Gamma |\Phi(y)|^2\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2}\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\ &\quad + \int_\Gamma\int_\Gamma |\varphi(x)|^2\frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^2}\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\ &\le \|\Phi\|^2_{C^0}[\varphi]^2_{H^{1/2}} + \|\varphi\|^2_{L^\infty}[\Phi]^2_{H^{1/2}}\,.\end{aligned}$$

Using the Sobolev embedding $H^1(\Gamma)\subset H^{\frac12}(\Gamma)\cap L^\infty(\Gamma)$, we obtain that the above quantity is finite, and hence we conclude. □

REMARK 3.19. The above result holds just requiring $\Phi\in H^{\frac12}(\Gamma)$, but the proof is longer. Since in our case we already know that $\nabla_\Gamma u^{\pm}\in H^{\frac12}(\Gamma)\cap C^{0,\alpha}(\Gamma)$ for $\alpha\in(0,1/2)$, we prefer to give just this simplified version of the result.

REMARK 3.20. Notice that it is possible to write

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{MS}\big((u_t,\Gamma_t);U\big)_{|t=s} = \partial^2\mathcal{MS}\big((u_s,\Gamma_s);U\big)\big[(X\cdot\nu^1, X\cdot\nu^2, X\cdot\nu^3)\big] + R_s\,, \tag{3.9}$$

where $R_0$ vanishes whenever $(u,\Gamma)$ is a critical triple point.

We now introduce the space where we will prove the local minimimality result.

DEFINITION 3.21. Given $\delta>0$, we denote by the symbol $\mathcal{D}_\delta(\Omega,U)$ the space of all the diffeomorphisms $\Phi:\bar\Omega\to\bar\Omega$, with $\Phi=\mathrm{Id}$ in $(\Omega\backslash U)\cup\partial_D\Omega$ and $D\Phi(x_0)=\lambda\mathrm{Id}$ for some $\lambda\neq 0$, such that $\|\Phi-\mathrm{Id}\|_{W^{2,\infty}(\bar\Omega;\bar\Omega)}<\delta$.

As one would expect, the non negativity of the second variation is a necessary condition for local minimality, as shown in the following result. Since the proof is just technical, it will be postponed in the appendix.

PROPOSITION 3.22. *Let $(u,\Gamma)$ be a critical triple point such that there exists $\delta>0$ with the following property:*

$$\mathcal{MS}\big((u,\Gamma);U\big)\le\mathcal{MS}\big((v,\Gamma_\Phi);U\big)\,,$$

*for every $\|\Phi-\mathrm{Id}\|_{C^2(\bar\Omega;\bar\Omega)}<\delta$ with $\Phi=\mathrm{Id}$ on $\partial_D\Omega\cup(\Omega\backslash U)$, and every $v\in H^1(\Omega\backslash\Gamma_\Phi)$ such that $v=u$ in $(\Omega\backslash U)\cup\partial_D\Omega$. Then*

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi]\ge 0\,,\qquad \textit{for every } \varphi\in\widetilde H^1(\Gamma)\,.$$

The following strict stability condition will be shown to imply the local minimality result (see 3.24).

DEFINITION 3.23. We say that a critical triple point $(u,\Gamma)$ is *strictly stable* in an admissible subdomain $U$ if

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi]>0\qquad\text{for every } \varphi\in\widetilde H^1(\Gamma)\backslash\{0\}\,.$$

## 3.4. A local minimality result

The aim of this section is to prove the following result.

THEOREM 3.24. *Let $(u,\Gamma)$ be a strictly stable critical triple point. Then there exists $\bar\delta>0$ such that*

$$\mathcal{MS}\big((v,\Gamma_\Phi);U\big)\ge\mathcal{MS}\big((u,\Gamma);U\big)\,,$$

*for every $\Phi\in\mathcal{D}_{\bar\delta}(\Omega;U)$ and every $v\in H^1(\Omega\backslash\Gamma_\Phi)$ such that $v=u$ in $(\Omega\backslash U)\cup\partial_D\Omega$. Moreover equality holds true only when $\Gamma_\Phi=\Gamma$ and $v=u$.*

The rest of this section is devoted to the proof of the above result.

### 3.4.1. Construction of the family of diffeomorphisms.

The aim of this section is to construct the family of diffeomorphisms of the Step 1 described above.

PROPOSITION 3.25. *Let $(u,\Gamma)$ be a critical triple point and fix $\varepsilon>0$. Then it is possible to find a constant $\bar\delta_1=\bar\delta_1(\Gamma,\varepsilon)>0$ and constants $C_1>0$, $C_2>0$, depending only on $\Gamma$ and $\bar\delta_1$, with the following property:*

*let $\Phi\in C^3(\bar\Omega;\bar\Omega)$ be a diffeomorphism satisfying the following properties:*

- *there exist $\xi>0$ and $v\in\mathbb{R}^2$ such that $\Phi\big(\bar\Gamma\cap B_\xi(x_0)\big)=\bar\Gamma\cap B_\xi(x_0)+v$,*
- *$\Phi(x_0)\neq x_0$,*
- *$\|\Phi-\mathrm{Id}\|_{C^2(\bar\Omega;\bar\Omega)}<\bar\delta_1$.*

*Then it is possible to find an admissible family of diffeomorphisms* $(\Phi_t)_{t\in[0,1]}$ *with* $\|\Phi_t - \mathrm{Id}\|_{C^2(\bar\Omega;\bar\Omega)} < \varepsilon$*, such that*

$$\Phi_1(\Gamma) = \Phi(\Gamma)\,.$$

*Moreover the following estimates hold true for each time* $t\in[0,1]$*:*

$$\|X_t\cdot\tau_t\|_{L^2(\Gamma_t)} \le C_1\|X_t\cdot\nu_t\|_{L^2(\Gamma_t)}\,, \tag{3.10}$$

$$\|Z_t\cdot\nu_t\|_{L^1(\Gamma_t)} \le C_2\|X_t\cdot\nu_t\|_{L^2(\Gamma_t)}\,, \tag{3.11}$$

*where we recall that* $X_t := \dot\Phi_t\circ\Phi_t^{-1}$*,* $Z_t := \ddot\Phi_t\circ\Phi_t^{-1}$*, and* $\nu_t$ *and* $\tau_t$ *are the normal and the tangent vector field on* $\Gamma_t$ *respectively.*

*Heuristics.* We use two different strategies to construct the diffeomorphisms $\Phi_t$'s on $\bar\Gamma$, accordingly that we are closed to $x_0$ or not. Far from $x_0$ we just consider the flow of a suitable vector field that is (closed to) an extension of the normal vector field of $\bar\Gamma$. This part of the construction is easy. The difficult part is when we are closed to $x_0$. Our idea is the following: we first construct a vector field $Y$ on $\bar\Gamma\cap B_\mu(x_0)$, for some $\mu>0$, such that $Y$ has null tangential component on $\Gamma\cap B_\mu(x_0)\setminus B_{\mu/2}(x_0)$. This last condition will be used to glue together the two constructions. Then we define our diffeomorphisms $\Phi_t$'s as

$$\Phi_t(x) := x + tY(x)\,,$$

for $x\in\bar\Gamma\cap B_\mu(x_0)$. Thus, we need our vector field $Y$ to satisfy the following two conditions:

(i) $Y\in C^2(\bar\Gamma)$ (see Definition 3.35), with $\|Y\|_{C^2(\bar\Gamma)}$ sufficiently small,

(ii) $\|Y\cdot\nu\|_{L^2(\Gamma)} \ge M\|Y\cdot\tau\|_{L^2(\Gamma)}$, for some $M>0$.

The second condition suggests us to consider the sets

$$C^i := \left\{ v\in B_\mu \;:\; \left|\frac{v}{|v|}\cdot\tau^i(x_0)\right| \ge \frac{3}{5}\left|\frac{v}{|v|}\cdot\nu^i(x_0)\right| \right\},$$

and to distinguish whether $Y(x_0) := \Phi(x_0) - x_0 \in C^i$ for some $i=1,2,3$, or not. In the latter case we just let $Y$ to be a vector field such that $|Y\cdot\nu| \ge C|Y\cdot\tau|$. Then (ii) is clearly satisfied, while (i) follows from the assumption on $\Phi$ near $x_0$.

If, instead, $Y(x_0)\in C^i$ for some $i=1,2,3$, then we have to estimate the tangential part of our vector field on $\Gamma^i$ with its normal component on $\Gamma^j$, where $j\neq i$. Are we sure that we can do it? The really bad case is when the curve $\Phi(\Gamma^i)$ is completely over $\Gamma^i$, and $\Phi(\Gamma^j)$ is over $\Gamma^j$ out of a ball $B_r(x_0)$. We prove that $r$ can be at least of order $\sqrt{|Y(x_0)|}$, and that $|Y\cdot\nu^j| \ge 4|Y(x_0)|$ in a ball of radius of the same order of $r$. Thus, on $\Gamma^i$, we have to make our vector field $Y$ to have small tangential part out of a ball whose radius is comparable with $\sqrt{|Y(x_0)|}$. The idea is to look at $\Gamma^i$ and $\Phi(\Gamma^i)$ near $x_0$ as the graph, with respect to the axes given by $\tau^i(x_0)$ and $\nu^i(x_0)$, of functions $h^i$ and $\widetilde{h}^i$ respectively. Write $Y(x_0) = Y_1(x_0)\tau^i(x_0) + Y_2(x_0)\nu^i(x_0)$, and let $s$ be the coordinate with respect to $\tau^i(x_0)$. Notice that $|Y(x_0)|$ and $|Y_1(x_0)|$ are of the same order. Then we define our vector field $Y$ as follows

$$Y\big(s,h^i(s)\big) := \big(s - G(s), \widetilde{h}^i\big(G(s)\big) - h^i(s)\big)\,,$$

where $L>2$ is a fixed constant, and $G : [0, L\sqrt{Y_1(x_0)}] \to [Y_1(x_0), L\sqrt{Y_1(x_0)}]$ is a diffeomorphism with $\|G-\mathrm{Id}\|_{C^2}$ sufficiently small, and such that $G$ is equal to the identity in $[(L-1)\sqrt{Y_1(x_0)}, L\sqrt{Y_1(x_0)}]$. Thanks to this last condition we have that $|Y\cdot\nu^i| \ge C|Y\cdot\tau^i|$ in the last part of the interval. Then, as in the previous case, we can let the tangential part of $Y$ vanish on $Y^i$, thus having (ii) in force out of the ball $B$ centered at $x_0$ where $G$ is not the identity. Since the tangential component of $Y$ in $\Gamma^i\cap B$ is of order $|Y_1(x_0)|$, we can have estimate (ii) in force also in this case.

PROOF. The proof is divided in three parts: we first define our diffeomorphisms on $\bar{\Gamma}$, then we extend them to admissible ones defined in the whole $\bar{\Omega}$ and finally we will show that our construction is such that estimate (3.10) holds true. We start with some preliminaries.

*Preliminaries.* The constants $C > 0$ that will appear in the following computations may change from line to line, but we will keep the same notation. Fix $\mu > 0$ such that

- $\nu^i(x) \cdot \nu^i(x_0) \geq \frac{2}{3}$, for $x \in \Gamma_i \cap B_\mu(x_0)$,
- $B_{4\mu} \Subset \Omega$,
- $(\Gamma_i)_\mu$ is a tubular neighborhood of $\Gamma_i$,
- the sets $(\Gamma_i)_\mu \backslash B_{3\mu}$ are disjoint,
- $\Gamma^i \cap B_\mu(x_0)$ is a graph with respect to the axes given by $\tau^i(x_0)$ and $\nu^i(x_0)$.

We will take $\bar{\delta}_1 < \frac{\mu}{2}$. Moreover we will denote by $\chi : \mathbb{R} \to [0,1]$ a smooth cut-off function such that $\chi \equiv 0$ on $[1, +\infty)$ and $\chi \equiv 1$ on $(-\infty, \frac{1}{2}]$.

*Step 1: construction of the diffeomorphisms near $x_0$.* We define the diffeomorphisms $\Phi_t^O$ as

$$\Phi_t^O(x) := x + tN(x)\,,$$

for $x \in \bar{\Gamma} \cap B_\mu(x_0)$, where the vector field $N$ will be constructed as follows.

*Case 1: $x_0 \notin C^i$.* Write

$$\frac{\Phi(x) - x}{|\Phi(x) - x|} = a_i(x)\tau^i(x) + b_i(x)\nu^i(x)\,,$$

for some functions $a_i, b_i : \bar{\Gamma}^i \cap B_\mu(x_0) \to \mathbb{R}$. Up to take a smaller $\mu$, we can suppose $|b_i(x) - b_i(x_0)| < \frac{1}{4}$ and $|a_i(x) - a_i(x_0)| < \frac{1}{4}$ for $x \in \Gamma^i \cap B_\mu(x_0)$. Notice that, since $x_0 \notin C^i$, we have $|b_i(x_0)| \geq \frac{3}{4}$, $|a_i(x_0)| \leq \frac{\sqrt{5}}{4}$.

Consider the unitary vector field $Y^i$ on $\bar{\Gamma}^i \cap B_{3\mu}(x_0)$ given by

$$Y^i(x) := \frac{\widetilde{Y}^i}{|\widetilde{Y}^i|}\,.$$

where, if we define $\widetilde{\chi}(x) := \chi\Big(\frac{|x-x_0|^2}{\mu^2}\Big)$, we set

$$\widetilde{Y}^i := \widetilde{\chi}(x)\big(a_i(x)\tau^i(x) + b_i(x)\nu^i(x)\big) + (1 - \widetilde{\chi}(x))\nu^i(x)\,.$$

Then there exists a constant $C > 0$ such that $|Y^i \cdot \nu^i| \geq C|Y^i \cdot \tau^i|$ on $\Gamma^i \cap B_{3\mu}(x_0)$. Indeed, we have that

$$\begin{aligned} |\widetilde{\chi}(x)a_i(x)| &\leq |\widetilde{\chi}(x)(a_i(x) - a_i(x_0))| + |\widetilde{\chi}(x)a_i(x_0)| \leq \frac{1}{4} + |a_i(x_0)| \\ &\leq C \leq 1 - |b_i(x) - 1| \leq |1 + \widetilde{\chi}(x)(b_i(x) - 1)|\,, \end{aligned}$$

where in the second to last inequality we have used the fact that $|b_i(x) - 1| \leq \frac{1}{2}$. Moreover, it is possible to find a constant $C > 0$ independent of $a_i(x_0)$, $b_i(x_0)$, such that $\|Y^i\|_{C^3(\bar{\Gamma}^i \cap B_{3\mu}(x_0))} \leq C$. Since the vector $Y^i$ is constant in a neighborhood of $x_0$, it is possible to represent (a piece of) $\Phi(\bar{\Gamma}^i)$ as a graph of class $C^3$ over $\bar{\Gamma}^i$, with respect to the vector field $Y^i$. Namely, it is possible to find a function $\varphi^i \in C^3(\bar{\Gamma}^i \cap B_{3\mu}(x_0))$ such that

$$x \mapsto x + \varphi^i(x)Y^i(x)$$

is a diffeomorphisms of class $C^3$ from $\bar{\Gamma}^i \cap B_{3\mu}(x_0)$ to its image, that is contained in $\Phi(\bar{\Gamma}^i)$. Finally, or any $\xi > 0$ it is possible to find $\bar{\delta}_1 > 0$ such that if $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta}_1$, then $\|\varphi\|_{C^3(\bar{\Gamma}^i \cap B_{3\mu}(x_0))} \leq \xi$. Define $N := \varphi Y^i$.

*Case 2:* $x_0 \in C^i$. Consider the axes given by $\tau^i(x_0)$ and $\nu^i(x_0)$ centered at $x_0$, and denote by $s$ the coordinated with respect to $\tau^i(x_0)$. Our assumptions on $\mu$ allow us to write $\Gamma^i$ in a neighborhood of $x_0$ as a graph of a function $h_i$, with respect to the above axes. We can suppose $\bar{\delta}_1 > 0$ so small such that the same is true also for $\Phi(\Gamma^i)$, *i.e.*, we can represent $\Phi(\Gamma^i)$ in a neigborhood of $\Phi(x_0)$ as the graph of a function $\widetilde{h}_i$ with respect to the same axes.

Now write
$$\Phi(x_0) - x_0 = s_0 \tau^i(x_0) + t_0 \nu^i(x_0)$$
for some $s_0, t_0 \in \mathbb{R}$, where we can also suppose $s_0 < 1$, if $\bar{\delta}_1$ is sufficiently small. Since $x_0 \in C^i$, we have that $C_1 s_0 \leq |\Phi(x_0) - x_0| \leq C_2 s_0$, for some $C_1, C_2 > 0$.

Fix $L > 1$ and define the diffeomorphism $G_L : [0, (L+1)\sqrt{s_0}] \to [s_0, (L+1)\sqrt{s_0}]$ given by
$$G_L(s) := s + \chi\Big(\frac{s}{L\sqrt{s_0}}\Big) s_0 \,.$$
Notice that
$$|G_L'(s) - 1| \leq C\frac{\sqrt{s_0}}{L}\,, \qquad\qquad |G_L''(s)| \leq \frac{C}{L^2}\,. \tag{3.12}$$
Moreover $G_L$ is the identity in $[L\sqrt{s_0}, (L+1)\sqrt{s_0}]$.

Now define the vector field
$$S^i\big((s, h^i(s))\big) := \big(G_L(s) - s, \widetilde{h}_i(G_L(s)) - h_i(s)\big)\,.$$
Then, by a direct computation, we have
$$\begin{aligned}
\|S^i\|_{C^2} \;\leq\;& \Big(\Big(C\|h'\|_{C^0} + L^2(\|h'\|_{C^0} + \|\widetilde{h}'\|_{C^0})\Big)s_0 \\
+\;& \Big(\frac{C}{L} + \Big(1 + \frac{C}{L}\Big)\|\widetilde{h}_i'\|_{C^0} + \|h_i'\|_{C^0}\Big) \\
+\;& \Big(\frac{C}{L^2} + \Big(1 + \frac{C}{L}\Big)^2\|\widetilde{h}_i''\|_{C^0} + \frac{C}{L^2}\|\widetilde{h}_i'\|_{C^0} + \|h_i''\|_{C^0}\Big)\Big)\,.
\end{aligned}$$
Notice that $S^i\big((s, h(s))\big) = \lambda(s)\nu^i(x_0)$, for $\lambda(s) \in \mathbb{R}$ and $s \in [L\sqrt{s_0}, (L+1)\sqrt{s_0}]$.

We now want to extend the definition of the vector field $S^i$ to the whole $\bar{\Gamma}^i \cap B_\mu(x_0)$. Write
$$\nu^i(x_0) = a_i(x)\tau^i(x) + b_i(x)\nu^i(x)\,,$$
for some functions $a_i, b_i : \bar{\Gamma}^i \cap B_\mu(x_0) \to \mathbb{R}$. Let $\bar{x} \in \Gamma^i$ the point given by $((L+1)\sqrt{s_0}, h((L+1)\sqrt{s_0}))$, and let $r > 0$ such that the ball $B_r(x_0)$ intersect the curve $\Gamma^i$ in the point $\bar{x}$. Up to take a smaller $\mu$, we can suppose $|b_i(x) - b_i(\bar{x})| < \frac{1}{4}$ and $|a_i(x) - a_i(\bar{x})| < \frac{1}{4}$ for $x \in \Gamma^i \cap B_\mu(x_0)$. Up to decreasing the value of $\bar{\delta}_1$, we can also suppose $|b_i(\bar{x})| \geq \frac{3}{4}$, $|a_i(\bar{x})| \leq \frac{1}{4}$.

As done in the previous step, let us consider the vector
$$Y^i(x) := \frac{\widetilde{Y}^i}{|\widetilde{Y}^i|}\,.$$
where
$$\widetilde{Y}^i := \widetilde{\chi}(x)\big(a_i(x)\tau^i(x) + b_i(x)\nu^i(x)\big) + (1 - \widetilde{\chi}(x))\nu^i(x)\,.$$
Using the same computation of the previous step, we have that $|Y^i \cdot \nu^i| \geq C|Y^i \cdot \tau^i|$ on $\Gamma^i \cap B_{3\mu}(x_0) \setminus B_r(x_0)$, for some constant $C > 0$. Moreover, it is possible to represent (a piece of) $\Phi(\Gamma^i)$ as a graph of a function $\varphi$ of class $C^3$ over $\Gamma^i \cap B_{3\mu}(x_0) \setminus B_r(x_0)$, with

respect to the vector field $Y^i$. Notice that the vector field $Y^i$ turns out to be of clas $C^3$. Finally, for any $\xi > 0$ it is possible to find $\bar{\delta}_1 > 0$ such that if $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta}_1$, then $\|\varphi\|_{C^3(\bar{\Gamma}^i \cap B_{3\mu}(x_0) \setminus B_r(x_0))} \leq \xi$. Define

$$N := \begin{cases} S^i & \text{on } \bar{\Gamma}^i \cap B_r(x_0) \\ \varphi Y^i & \text{on } \bar{\Gamma}^i \cap B_{3\mu}(x_0) \setminus B_r(x_0) \end{cases}$$

Notice that $N$ turns out to be a well defined $C^3$ vector field.

*Step 2: construction of the diffeomorphisms far from $x_0$.* Let $R \in C^3(\bar{\Omega};\mathbb{R}^2)$ be a vector field with the following properties

- $|R| \leq 1$,
- $R(x + t\nu^i(x)) = \nu^i(x)$ for any $|t| < \mu$ and any $x \in (\Gamma^i)_\mu \setminus (B_\mu(x_0) \cup (\partial\Omega)_\mu)$,
- $|R \cdot \nu^i| \geq \frac{1}{2}$ on $\Gamma_i$,
- $R$ is tangential to $\partial\Omega$,
- $R \equiv 0$ on $\partial_D\Omega \cup (\Omega \setminus U)$.

Then, it is possible to find a function $\psi \in C^3((\Gamma^i)_\mu \setminus (B_\mu(x_0) \cup (\partial\Omega)_\mu)$ (extended in a constant way along the trajectories of $R$) such that, if we consider the flow $\Phi^B_t$ of the vector field $\psi R$, we have $\Phi^B_t((\Gamma^i)_\mu \setminus (B_\mu(x_0) \cup (\partial\Omega)_\mu) \in \Phi(\Gamma^i)$. Moreover, for any $\xi > 0$ it is possible to find $\bar{\delta}_1 > 0$ such that if $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta}_1$, then $\|\psi\|_{C^2} < \xi$.

*Step 3: definition of the diffeomorphisms in $\bar{\Gamma}$.* We define our family of diffeomorphisms $(\Phi_t)_{t\in[0,1]}$ as follows:

$$\Phi_t(x) := \chi\Big(\frac{|x|^2}{(3\mu)^2}\Big)\Phi^O_t(x) + \Big(1 - \chi\Big(\frac{|x|^2}{(3\mu)^2}\Big)\Big)\Phi^B_t(x)\,.$$

Notice that the two flows $\Phi^O_t$ and $\Phi^B_t$ are the same for points $x \in \Gamma \setminus (B_{2\mu}(x_0) \cup (\partial\Omega)_\mu)$. Moreover the above diffeomorphisms are of class $C^3$ and $\Phi_1(\bar{\Gamma}) = \Gamma_\Phi$.

We claim that it is possible to find $\bar{\delta}_1 > 0$ and $L > 1$ (where $L$ is the constant usend in the construction of the diffeomorphism $G_L$ in the previous step) such that if $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta}_1$, then (up to take a smaller $\mu$)

$$\|\Phi_t - \mathrm{Id}\|_{C^2} \leq \varepsilon\,. \tag{3.13}$$

Indeed, we first $L > 1$ such that $\frac{C}{L} < \frac{\varepsilon}{4}$ (where $C$ is the constant appearing in (3.12)), and then we choose $\bar{\delta}_1$ such that the desired estimate holds true.

*Step 4: extension of the diffeomorphisms.* First of all we extend our diffeomorphisms on $\partial\Omega$. For a point $x \in \partial\Omega$ we just consider the flow given by the vector field $\psi R$, where $\psi$ is the function found in Step 2. The fact that $R$ is tangential to $\partial\Omega$ ensures that if we start from a point $x \in \partial\Omega$, its evolution with respect to the above flow remains in $\partial\Omega$.

Now consider the function

$$f_t = \begin{cases} \Phi_t - \mathrm{Id} & \text{on } \bar{\Gamma} \cup \partial\Omega\,, \\ 0 & \text{in } \bar{\Omega} \setminus U\,. \end{cases}$$

It is easy to see that $f_t \in C^3\big(\bar{\Gamma} \cup \partial\Omega \cup (\bar{\Omega} \setminus U);\mathbb{R}^2\big)$ and that

$$\|f_t\|_{C^2\left(\bar{\Gamma} \cup \partial\Omega \cup (\bar{\Omega} \setminus U);\mathbb{R}^2\right)} \leq \|\Phi_t - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} \leq \varepsilon\,,$$

Without loss of generality, we can suppose $\varepsilon < \frac{1}{2}$. Use Whitney's extension theorem (see Section 3.6.2) to extend the functions $f_t$ to functions $u_t \in C^3(\mathbb{R}^2;\mathbb{R}^2)$ such that $\|u_t\|_{C^2(\mathbb{R}^2;\mathbb{R}^2)} < \varepsilon$

Define the functions (denoted with an abuse of notation) $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\Phi_t := \mathrm{Id} + u_t \,.$$

Notice that $\|\Phi_t - \mathrm{Id}\|_{C^2(\mathbb{R}^2;\mathbb{R}^2)} < \frac{1}{2}$, and hence $\Phi_t$ are diffeomorphisms of $\mathbb{R}^2$. Moreover, since $\Phi_t(\partial\Omega) = \partial\Omega$, we infer that $\Phi_t(\bar{\Omega}) = \bar{\Omega}$ and thus that $\Phi_t$ is a diffeomorphism of $\bar{\Omega}$ onto itself.

*Step 5: estimates.* First of all we prove estimate (3.10). By definition we have that

$$Z(x) = \Big(1 - \chi\Big(\frac{|x|^2}{(3\mu)^2}\Big)\Big) Z^B(x) \,,$$

where $Z^B(x) := \psi^2 DR[R]$ (where $\psi$ is the function given by Step 2). Since $|R \cdot \nu| \geq \frac{1}{2}$ in the region where we consider the flow of the vector field $\psi R$, we can take $\bar{\delta}_1$ so small such that $|R(\Phi^B_t(x)) \cdot \nu_t(\Phi^B_t(x))| \geq \frac{1}{4}$ for $x \in \Gamma \backslash B_\mu(x_0)$. Thus

$$\int_{\Gamma_t} |Z \cdot \nu_t| \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\Gamma_t} \psi^2 DR[R, \nu_t] \, \mathrm{d}\mathcal{H}^{N-1} \leq C \int_{\Gamma_t} \psi^2 |R \cdot \nu_t|^2 \, \mathrm{d}\mathcal{H}^{N-1} = C \int_{\Gamma_t} |X \cdot \nu_t|^2 \, \mathrm{d}\mathcal{H}^{N-1} \,.$$

To prove estimate (3.11) we first need to notice the following fact: let $\alpha_1, \alpha_2 > 0$ be small parameters, and take $a, b \in \mathbb{R}$ small such that $b \neq 0$ and $|b| \geq C|a|$ for some constant $C > 0$. Consider the two parabola given by

$$y = -\alpha_2 (x-a)^2 + b \,, \qquad , \text{ and } y = \alpha_1 x^2 \,.$$

Then the distance between these two parabola are greater than $\frac{1}{2} b$ if $x \in [0, C\sqrt{b}]$ (or $x \in [a, C\sqrt{b}]$ if $a < 0$), for some constant $C > 0$ depending on $\alpha_1$ and $\alpha_2$.

We use the above observation in this way: suppose $x_0 \notin C^i$ and represent the curves $\Gamma^i$ and $\Phi(\Gamma^i)$ in a neighborhood of $x_0$ as the graphs, with respect to the axes given by $\tau^i(x_0)$ and $\nu^i(x_0)$ centered at $x_0$, of $h_i$ and $\widetilde{h}_i$ respectively. Up to change $\nu^i(x_0)$ with $-\nu^i(x_0)$ we can suppose $\widetilde{h}_i \geq 0$. Thus, it is possible to find $\alpha_1, \alpha_2 > 0$ such that

$$h_i(s) \leq \alpha_1 s^2 \,, \qquad \widetilde{h}_i(s) \geq -\alpha_2 (s-a)^2 + b \,,$$

where we write $\Phi(x_0) - x_0 = a\tau^i(x_0) + b\nu^i(x_0)$, for some $a, b \in \mathbb{R}$ with $b \neq 0$ and $|b| \geq C|a|$. Set $d : |\Phi(x_0) - x_0| =$. Thanks to the above observation we can say that

$$|Y^i(x)| \geq \frac{1}{2} d \,,$$

for $x \in \Gamma^i \cap B_{D\sqrt{d}}(x_0)$, for some constant $D > 0$ depending on $\Gamma^i$ and $\bar{\delta}_1$.

We are now in the position to prove estimate (3.11). Suppose $x_0 \notin \cup_{i=1}^3 C^i$. Thanks to the definition of the vector field $N$ and the properies of $R$, we know that on $\Gamma$ it holds

$$|X \cdot \nu| \geq C |X \cdot \tau| \,, \tag{3.14}$$

for some constant $C > 0$. Thus, a similar inequaity holds on $\Gamma_t$ providing $\bar{\delta}_1$ sufficiently small. Hence the integral estimate follows directly.

If instead $x_0 \in C^i$ we have, for $j \neq i$, the following estimate in force

$$\int_{\Gamma^i \cap B_{r\sqrt{d}}(x_0)} |X \cdot \tau^i|^2 \, \mathrm{d}\mathcal{H}^{N-1} \leq C d^{\frac{3}{2}} \leq C \int_{\Gamma^j \cap B_{D\sqrt{d}}(x_0)} |X \cdot \nu^j|^2 \, \mathrm{d}\mathcal{H}^{N-1} \,. \tag{3.15}$$

For $\bar{\delta}_1$ sufficiently small, the same estimate continue to hold also for the curves $\Gamma^i_t$ and $\Gamma^j_t$ (with $\tau^i_t$ and $\nu^i_t$). Notice that in $\Gamma^i \cap B_{3\mu}(x_0) \backslash B_r(x_0)$ we have estimate (3.14) in force.

By using (3.14) and (3.15) we obtain estimate (3.11). $\square$

REMARK 3.26. From the above proof, it is easy to see that the following property holds: if $(\Phi_\varepsilon)_\varepsilon$ is a family of diffeomorphisms of class $C^2$ with the same properties as in the statement of the theorem, such that $\Phi_\varepsilon \to \Phi$ in the $C^1$ topology, where $\Phi$ is a diffeomorphism satisfying $\Phi(\Gamma) \neq \Gamma$, then there exists a constant $C > 0$ such that

$$\|X^\varepsilon \cdot \nu_t^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \geq C\,,$$

where $X^\varepsilon$ is the vector field associated to $\Phi_\varepsilon$, and $\Gamma_t^\varepsilon := \Phi_\varepsilon^t(\Gamma)$, where $\Phi_\varepsilon^t$ is the flow generated by $X^\varepsilon$.

**3.4.2. Uniform coercivity of the quadratic form.** The second technical result we prove is a sort of continuity of the quadratic form $\partial^2\mathcal{MS}\big((u,\Gamma);U\big)$ in a stable critical triple point $(u,\Gamma)$. This result is the fundamental estimate needed in order to prove Theorem 3.24.

PROPOSITION 3.27. *Let $(u,\Gamma)$ be a strictly stable critical triple point. Then there exists $\bar\delta_2 > 0$ and $\bar C > 0$ such that*

$$\partial^2\mathcal{MS}\big((u_\Phi,\Gamma_\Phi);U\big)[\varphi] \geq \bar C\|\varphi\|^2_{\widetilde H^1(\Gamma_\Phi)}\,,$$

*for each $\Phi \in \mathcal{D}_\delta(\Omega,U)$, where $\delta \in (0,\bar\delta_2)$, and each $\varphi \in \widetilde H^1(\Gamma_\Phi)$.*

In order to prove the above proposition, we first need to prove that if $(u,\Gamma)$ is strictly stable, then $\partial^2\mathcal{MS}\big((u,\Gamma);U\big)$ is coercive.

LEMMA 3.28. *Let $(u,\Gamma)$ be a strictly stable critical triple point. Then there exists $M > 0$ such that*

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi] \geq M\|\varphi\|^2_{\widetilde H^1(\Gamma)}\,, \qquad \forall \varphi \in \widetilde H^1(\Gamma)\,.$$

PROOF. It is sufficient to show that

$$M := \inf\{\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi] \,:\, \|\varphi\|_{\widetilde H^1(\Gamma)} = 1\} > 0\,.$$

Suppose for the sake of contradiction that $M = 0$, and let $(\varphi_n)_n$ be a minimizing sequence for $M$, *i.e.*, $\|\varphi_n\|_{\widetilde H^1(\Gamma)} = 1$ and $\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi_n] \to 0$. Then there exists $\varphi \in \widetilde H^1(\Gamma)$ such that, up to a not relabelled subsequence, $\varphi_n^i \rightharpoonup \varphi^i$ in $\widetilde H^1(\Gamma)$ and, by the Sobolev embeddings, $\varphi_n \to \varphi$ in $C^{0,\beta}(\bar\Gamma)$ for each $\beta \in (0,\frac12)$, and $\varphi_n \to \varphi$ in $H^{\frac12}(\Gamma)$. We claim that

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi] \leq \liminf_{n\to\infty} \partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi_n] = 0\,. \tag{3.16}$$

Indeed, it is easy to see that

$$\int_\Gamma |\nabla_\Gamma\varphi|^2 \,\mathrm{d}\mathcal{H}^{N-1} \leq \liminf_{n\to\infty}\int_\Gamma |\nabla_\Gamma\varphi_n|^2 \,\mathrm{d}\mathcal{H}^{N-1}\,,$$

$$\int_\Gamma H^2\varphi_n^2 \,\mathrm{d}\mathcal{H}^{N-1} \to \int_\Gamma H^2\varphi^2 \,\mathrm{d}\mathcal{H}^{N-1}\,,$$

and

$$\sum_{i=1}^3 \big(\varphi_i^2 D\nu_{\partial\Omega}[\nu,\nu]\big)(x^i) \to \sum_{i=1}^3 \big(\varphi_i^2 D\nu_{\partial\Omega}[\nu,\nu]\big)(x^i)\,.$$

Thus, we are left to prove that

$$\int_\Gamma z^\pm \mathrm{div}_\Gamma(\varphi_n\nabla_\Gamma u^\pm) \,\mathrm{d}\mathcal{H}^{N-1} \to \int_\Gamma z^\pm \mathrm{div}_\Gamma(\varphi\nabla_\Gamma u^\pm) \,\mathrm{d}\mathcal{H}^{N-1}\,,$$

for all $z \in H^1_U(\Omega\backslash\Gamma)$. Notice that $\varphi_n\nabla_\Gamma u^\pm \in H^{\frac{1}{2}}(\Gamma;\mathbb{R}^2)$ thanks to Lemma 3.18. To prove the above convergence we will show that $\varphi_n\nabla_\Gamma u^\pm \to \varphi\nabla_\Gamma u^\pm$ in $H^{\frac{1}{2}}(\Gamma;\mathbb{R}^2)$:

$$\int_\Gamma\int_\Gamma \frac{|(\varphi_n\nabla_\Gamma u^\pm - \varphi\nabla_\Gamma u^\pm)(x) - (\varphi_n\nabla_\Gamma u^\pm - \varphi\nabla_\Gamma u^\pm)(y)|}{|x-y|^2}\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y)$$
$$\leq \|\nabla_\Gamma u^\pm\|^2_{L^\infty(\bar\Gamma;\mathbb{R}^2)}\|\varphi_n-\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\varphi_n-\varphi\|^2_{L^\infty(\bar\Gamma)}\|\nabla_\Gamma u^\pm\|^2_{H^{\frac{1}{2}}(\Gamma;\mathbb{R}^2)}.$$

Now we have two cases: if $\varphi \neq 0$ then (3.16) gives the desired contradiction. On the other hand, if $\varphi = 0$, then $v_\varphi = 0$, and hence again by (3.16) we obtain that

$$\int_\Gamma |\nabla_\Gamma\varphi_n|^2\,\mathrm{d}\mathcal{H}^{N-1} \to 0\,,$$

and this contradicts the fact that $\|\varphi_n\|_{\widetilde{H}^1(\Gamma)} = 1$. □

Before proving Proposition 3.27 we need to observe the following fact, similar to [8, Lemma 5.1].

REMARK 3.29. Consider the function $u_\Phi$ (see Definition 3.5). We claim that, for every $\alpha < \frac{1}{2}$, the following convergence holds true:

$$\sup_{\Phi\in\mathcal{D}_\delta(\Omega,U)} \|\nabla_\Gamma(u^\pm_\Phi\circ\Phi) - \nabla_\Gamma u^\pm\|_{C^{0,\alpha}(\bar\Gamma;\mathbb{R}^2)} \to 0\,,$$

as $\delta \to 0^+$. First of all we notice that, what we are really claiming, is that, denoting by $A_1, A_2, A_3$ the three connected components of $\Omega\backslash\bar\Gamma$, and letting $u^i$ be the function $u$ restricted to $A_i$, we have that

$$\sup_{\Phi\in\mathcal{D}_\delta(\Omega,U)} \|\nabla_\Gamma(\widetilde{u}^i_\Phi\circ\Phi) - \nabla_\Gamma\widetilde{u}^i\|_{C^{0,\alpha}(\bar\Gamma\cap\partial A_i;\mathbb{R}^2)} \to 0\,,$$

as $\delta \to 0^+$, where $\widetilde{u}^i$ is the trace of $u^i$ on $\bar\Gamma\cap\partial A_i$. This can be proved by using the estimate of the $H^2$-norm of $\widetilde{u}^i_\Phi\circ\Phi$ in a neighborhood of $\Gamma$ (that turns out to be uniform for $\Phi\in\mathcal{D}_\delta(\Omega,U)$) and by the Ascoli-Arzelá theorem.

PROOF OF PROPOSITION 3.27. Suppose for the sake of contradiction that there exist a family of diffeomorphisms $\Phi_n : \bar\Omega \to \bar\Omega$ with $\Phi_n = \mathrm{Id}$ in $(\Omega\backslash U)\cup\partial_D\Omega$ such that $\Phi_n \to \mathrm{Id}$ in $C^2(\bar\Omega;\bar\Omega)$, and functions $\varphi_n \in \widetilde{H}^1(\Gamma_{\Phi_n})$ with $\|\varphi_n\|_{\widetilde{H}^1(\Gamma_{\Phi_n})} = 1$, such that

$$\partial^2\mathcal{MS}\big((u_{\Phi_n},\Gamma_{\Phi_n});U\big)[\varphi_n] \to 0\,. \tag{3.17}$$

Let $\widetilde{\varphi}_n := c_n\varphi_n\circ\Phi_n$, where $c_n := \|\varphi_n\circ\Phi_n\|^{-1}_{\widetilde{H}^1(\Gamma)} \to 1$. Then it is not difficult to prove that

$$\left|\int_{\Gamma_{\Phi_n}} H^2_{\Phi_n}\varphi_n^2\,\mathrm{d}\mathcal{H}^{N-1} - \int_\Gamma H^2\widetilde{\varphi}_n^2\,\mathrm{d}\mathcal{H}^{N-1}\right| \to 0\,,$$

$$\left|\int_{\Gamma_{\Phi_n}} |\nabla_{\Gamma_{\Phi_n}}\varphi_n|^2\,\mathrm{d}\mathcal{H}^{N-1} - \int_\Gamma |\nabla_\Gamma\widetilde{\varphi}_n|^2\,\mathrm{d}\mathcal{H}^{N-1}\right| \to 0\,,$$

and

$$\left|\sum_{i=1}^3\big((\varphi_n)_i^2 D\nu_{\partial\Omega}[\nu,\nu]\big)(\Phi_n(x^i)) - \sum_{i=1}^3\big((\widetilde{\varphi}_n)_i^2 D\nu_{\partial\Omega}[\nu,\nu]\big)(x^i)\right| \to 0\,.$$

We also claim that the following convergence holds

$$\int_U |\nabla v_{\widetilde{\varphi}_n} - \nabla(v_{\varphi_n}\circ\Phi_n)|^2\,\mathrm{d}x \to 0\,. \tag{3.18}$$

To prove it we proceed as in the proof of [9, Lemma 5.4] that, for the reader's convenience, we report here. Our argument only changes from the original one in the proof of the last convergence, where we take advantange of the fact that in dimension 2 functions in $H^1(\Gamma)$ are bounded in $L^\infty$. Otherwise we would have needed that $\nabla_\Gamma(u^\pm\circ\Phi_n)\to\nabla_\Gamma u^\pm$ in $C^{0,\alpha}(\bar\Gamma;\mathbb{R}^2)$ for some $\alpha>\frac{1}{2}$, while in our case, due to the singularity given by the triple point, we only have the above convergence for $\alpha<\frac{1}{2}$. So, setting $z_n:=v_{\widetilde\varphi_n}-v_{\varphi_n}\circ\Phi_n$, we obtain that $z_n$ solves the problem

$$\int_U A_n[\nabla z_n,\nabla z]\,\mathrm{d}x-\int_U(A_n-\mathrm{Id})[\nabla\widetilde\varphi_n,\nabla z]\,\mathrm{d}x+\int_\Gamma(h_n^+z^+-h_n^-z^-)\,\mathrm{d}\mathcal{H}^{N-1}=0\,,$$

for all $z\in H^1_U(\Omega\backslash\Gamma)$, where $h_n^\pm:=\mathrm{div}_\Gamma(\widetilde\varphi_n\nabla_\Gamma u^\pm)-\big(\mathrm{div}_{\Gamma_{\Phi_n}}(\varphi_n\nabla_{\Gamma_{\Phi_n}}u^\pm_{\Phi_n})\big)J_{\Phi_n}$ and $A_n:=(J_{\Phi_n}D^{-1}\Phi_nD^{-T}\Phi_n)\circ\Phi_n$. Since $A_n\to\mathrm{Id}$ in $C^1$ and the sequence $(v_{\widetilde\varphi_n})_n$ is bounded in $H^1(\Omega\backslash\Gamma)$, we have that $(A_n-\mathrm{Id})[\nabla\widetilde\varphi_n]\to0$ in $H^1(\Omega\backslash\Gamma)$. Thus (3.18) follows by showing that $h_n^\pm\to0$ in $H^{-\frac{1}{2}}(\Gamma)$. First of all we want to write the last term of $h_n$ in a divergence form. For let $\xi\in C_c^\infty(\Gamma)$ and write

$$\begin{aligned}
&\int_\Gamma\big(\mathrm{div}_{\Gamma_{\Phi_n}}(\varphi_n\nabla_{\Gamma_{\Phi_n}}u^\pm_{\Phi_n})\big)J_{\Phi_n}\xi\,\mathrm{d}\mathcal{H}^{N-1}=\int_{\Gamma_{\Phi_n}}\mathrm{div}_{\Gamma_{\Phi_n}}(\varphi_n\nabla_{\Gamma_{\Phi_n}}u^\pm_{\Phi_n})(\xi\circ\Phi_n^{-1})\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad=-\int_{\Gamma_{\Phi_n}}\varphi_n\nabla_{\Gamma_{\Phi_n}}u^\pm_{\Phi_n}\nabla_{\Gamma_{\Phi_n}}(\xi\circ\Phi_n^{-1})\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad=-\int_{\Gamma_{\Phi_n}}\varphi_n(D_\Gamma\Phi_n)^{-T}\circ\Phi_n^{-1}[\nabla_\Gamma(u^\pm_{\Phi_n}\circ\Phi_n)\circ\Phi_n^{-1}]\cdot(D_{\Gamma_{\Phi_n}}\Phi_n)^{-T}[(\nabla_\Gamma\xi)\circ\Phi_n^{-1}]\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad=-\int_\Gamma c_n^{-1}\widetilde\varphi_n(D_\Gamma\Phi_n)^{-1}(D_\Gamma\Phi_n)^{-T}[\nabla_\Gamma(u^\pm_{\Phi_n}\circ\Phi_n),\nabla_\Gamma\xi]J_{\Phi_n}\xi\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad=\int_\Gamma c_n^{-1}\mathrm{div}_\Gamma\big(\widetilde\varphi_n(D_\Gamma\Phi_n)^{-1}(D_\Gamma\Phi_n)^{-T})[\nabla_\Gamma(u^\pm_{\Phi_n}\circ\Phi_n)]J_{\Phi_n}\big)\xi\,\mathrm{d}\mathcal{H}^{N-1}
\end{aligned}$$

Thus we have that

$$h_n^\pm=\mathrm{div}_\Gamma\big(\widetilde\varphi_n\nabla_\Gamma u^\pm-c_n^{-1}\widetilde\varphi_n(D_\Gamma\Phi_n)^{-1}(D_\Gamma\Phi_n)^{-T}[\nabla_\Gamma(u^\pm_{\Phi_n}\circ\Phi_n)]J_{\Phi_n}\big)=:\mathrm{div}_\Gamma\Psi_n^\pm\,,$$

and hence, in order to prove that $h_n^\pm\to0$ in $H^{-\frac{1}{2}}(\Gamma)$ we will prove that $\Psi_n^\pm\to0$ in $H^{\frac{1}{2}}(\Gamma)$. In order to estimate the Gagliardo $H^{\frac{1}{2}}$-seminorm, we first simplify our notation by setting $\lambda_n:=c_n^{-1}(D_\Gamma\Phi_n)^{-1}(D_\Gamma\Phi_n)^{-T}J_{\Phi_n}$ and $u_n:=u^\pm_{\Phi_n}\circ\Phi_n$. Then we can proceed as follows:

$$\begin{aligned}
&(\widetilde\varphi_n\lambda_n\nabla_\Gamma u_n^\pm)(x)-(\widetilde\varphi_n\nabla_\Gamma u^\pm)(x)-(\widetilde\varphi_n\lambda_n\nabla_\Gamma u_n^\pm)(y)+(\widetilde\varphi_n\nabla_\Gamma u^\pm)(y)\\
&\qquad=[(\widetilde\varphi_n(\lambda_n-\mathrm{Id})\nabla_\Gamma u_n^\pm)(x)-(\widetilde\varphi_n(\lambda_n-\mathrm{Id})\nabla_\Gamma u_n^\pm)(x)]\\
&\qquad\quad+[(\widetilde\varphi_n(\nabla_\Gamma u_n^\pm-\nabla_\Gamma u^\pm))(x)-(\widetilde\varphi_n(\nabla_\Gamma u_n^\pm-\nabla_\Gamma u^\pm))(y)]\,.
\end{aligned}$$

The first term can be rewritten as folows

$$\begin{aligned}
&(\widetilde\varphi_n(\lambda_n-\mathrm{Id})\nabla_\Gamma u_n^\pm)(x)-(\widetilde\varphi_n(\lambda_n-\mathrm{Id})\nabla_\Gamma u_n^\pm)(x)\\
&\qquad=(\widetilde\varphi_n(\lambda_n-\mathrm{Id}))(x)[\nabla_\Gamma u_n^\pm)(x)-\nabla_\Gamma u_n^\pm)(y)]+\widetilde\varphi_n(x)[(\lambda_n-\mathrm{Id})(x)-(\lambda_n-\mathrm{Id})(y)]\\
&\qquad\quad+(\widetilde\varphi_n(x)-\widetilde\varphi_n(y))(\lambda_n-\mathrm{Id})(y)\,,
\end{aligned}$$

while the last one as

$$(\widetilde{\varphi}_n(\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm))(x) - (\widetilde{\varphi}_n(\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm))(y)$$
$$= \widetilde{\varphi}_n(x)[\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm)(x) - (\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm)(y))] + [\widetilde{\varphi}_n(x) - \widetilde{\varphi}_n(y)][\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm)(y)]\,.$$

Thus the Gagliardo $H^{\frac{1}{2}}$-seminorm of $\Phi_n$ can be estimated as follows:

$$\begin{aligned}\int_\Gamma\int_\Gamma & \frac{|\Psi_n(x)-\Psi_n(y)|^2}{|x-y|^2}\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\ & \le \|\widetilde{\varphi}_n\|^2_{C^0(\Gamma)}\|\lambda_n - \mathrm{Id}\|^2_{C^0(\bar\Omega;\mathbb{R}^{n^2})}\big[\|\nabla_\Gamma u_n^\pm\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\widetilde{\varphi}_n\|^2_{H^{\frac{1}{2}}(\Gamma)}\big] \\ & \quad + \mathcal{H}^1(\Gamma)\|\widetilde{\varphi}_n\|^2_{C^0(\Gamma)}\|\lambda_n - \mathrm{Id}\|^2_{C^0(\bar\Omega;\mathbb{R}^{n^2})} \\ & \quad + \|\widetilde{\varphi}_n\|^2_{C^0(\Gamma)}\|\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm\|^2_{H^{\frac{1}{2}}(\Gamma)} \\ & \quad + \|\widetilde{\varphi}_n\|^2_{H^{\frac{1}{2}}(\Gamma)}\|\nabla_\Gamma u_n^\pm - \nabla_\Gamma u^\pm\|^2_{C^0(\Gamma)}\,. \end{aligned} \tag{3.19}$$

To estimate the terms on the right-hand side we will use the following facts:

- $(\widetilde{\varphi}_n)_n$ is bounded in $H^{\frac{1}{2}}(\Gamma)$ and in $C^0(\Gamma)$,
- $\lambda_n \to \mathrm{Id}$ in $C^1(\Omega;\mathbb{R}^{n^2})$,
- $u_n \to u$ in $H^2\big((\Omega\setminus\Gamma)\cap V\big)$, where $V$ is a neighborhood of $\Gamma$ in $\Omega$ such that $V \cap \partial_D\Omega = \emptyset$.

Indeed the first fact follows directly from the Sobolev embeddings, since $(\widetilde{\varphi}_n)_n$ is bounded in $H^1(\Gamma)$, the second convergence is easy from the fact that $\Phi_n \to \mathrm{Id}$ in $C^2$, while the last claim is a consequence of the continuity property of elliptic boundary value problems: writing the equation satisfied by $u_n$ on $\Omega$ we notice that the coefficients of the elliptic operator converge to those of the laplacian. Thus, by Theorem 3.34 we get that $u_n \to u$ in $H^1(\Omega)$ and by the estimate 3.34 that the convergence is actually in $H^2((\Omega\setminus\Gamma)\cap V)$ (notice that we have to restrict ourselves to a neighborhood of $\Gamma$ in order to avoid the singularities of $u$ where the Neumann boundary condition transforms into a Dirichlet one).
Thus we conclude from (3.19) that $\Psi_n \to 0$ in $H^{\frac{1}{2}}(\Gamma)$.

Combining all the above convergence, one gets that

$$\big|\partial^2\mathcal{MS}\big((u_{\Phi_n},\Gamma_{\Phi_n});U\big)[\varphi_n] - \partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\widetilde{\varphi}_n]\big| \to 0\,,$$

and hence, by (3.17), that

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\widetilde{\varphi}_n] \to 0\,.$$

But this is in contradiction with the result of Lemma 3.28. □

### 3.4.3. Proof of Theorem 3.24.

We are now ready to prove Theorem 3.24.

Proof of Theorem 3.24. Let $\Phi$ a in the statement of the theorem.
*Step 1.* Suppose $\Phi$ satisfies the following additional hypothesis: $\Phi \in C^3(\bar\Omega;\bar\Omega)$, $\Phi(x_0) \neq x_0$ and $\Phi\big(\bar\Gamma \cap B_\xi(x_0)\big) = \bar\Gamma \cap B_\xi(x_0) + v$ for some $\xi > 0$ and $v \in \mathbb{R}^2$.

*First step.* Consider the diffeomorphisms $(\Phi_t)_t$ given by Proposition 3.25. Define the function $g(t) := \mathcal{MS}\big((u_t,\Gamma_t);U\big)$. Since $(u,\Gamma)$ is a critical point we have that $g'(0) = 0$. Hence we can write

$$\mathcal{MS}\big((u_\Phi,\Gamma_\Phi);U\big) - \mathcal{MS}\big((u,\Gamma);U\big) = g(1) - g(0) = \int_0^1 (1-t)g''(t)\,\mathrm{d}t\,.$$

We claim that there exists $\bar{\delta}_1 > 0$, and a constant $C > 0$ such that

$$g''(t) \geq C\|X \cdot \nu_t\|^2_{\widetilde{H}^1(\Gamma_t)}\,, \tag{3.20}$$

whenever $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta} < \bar{\delta}_1$. This allows us to conclude. Indeed the local minimality follows directly from (3.20), while the isolated local minimality can be deduce from the fact that $\mathcal{MS}\big((u,\Gamma);U\big) = \mathcal{MS}\big((v,\Gamma_\Phi);U\big)$ implies $g''(t) = 0$ for each $t \in [0,1]$. In particular $g''(0) = 0$, and this implies that $X \cdot \nu_t \equiv 0$ on $\Gamma_t$. Looking at the construction of the vector field $X$ (see Proposition 3.27) this implies that $X \equiv 0$ on $\Gamma$, that is $\Gamma_\Phi = \Gamma$. Since now the curve $\Gamma$ is fixed and we already know that $u$ minimizes the Dirichlet integral over $\Omega\backslash\Gamma$, we obtain the isolated local minimality of $(u,\Gamma)$ as wanted.

Let us now prove (3.20). First of all we notice that, by criticality of $(u,\Gamma)$, $\Gamma$ intersects $\partial\Omega$ orthogonally and $\nu^i(x_0), \nu^j(x_0)$ are linear independent for $i \neq j$. Thus it is possible to take $\bar{\delta}$ sufficiently small in order to have the that $\Gamma_t$ intersects $\partial\Omega$ in a non tangent way and that $\nu_t^i(x_0), \nu_t^j(x_0)$ are still linear independent for $i \neq j$. By the definition of $u_t$ we have that $\partial_{\nu_t} u_t^\pm = 0$ on $\Gamma_t$ and $\partial_{\nu_{\partial\Omega}} u^\pm = 0$ on $(\partial\Omega\backslash\partial_D\Omega) \cap \bar{U}$. Then

$$\nabla_\Gamma u^\pm(x_t^i) = 0 \qquad \text{for } i = 0,1,2,3\,.$$

In particular $f_t - H_t = 0$ on $\partial\Gamma_t$. Thus, by Remark 3.13, we can write

$$\begin{aligned} g''(t) &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{MS}\big((u_s,\Gamma_s);U\big)_{|s=t} = \partial^2\mathcal{MS}\big((u_t,\Gamma_t);U\big)[X\cdot\nu_t] \\ &\quad + \int_{\Gamma_t} f_t\big[Z\cdot\nu_t - 2X^{||}\cdot\nabla_{\Gamma_t}(X\cdot\nu_t) + D\nu_t[X^{||},X^{||}] - H_t(X\cdot\nu_t)^2\big]\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \sum_{i=1}^3 (X\cdot\nu_t)^2 D\nu_{\partial\Omega}[\nu_t,\nu_t](x_t^i) + \int_{\partial\Gamma_t} Z\cdot\eta_t\,\mathrm{d}\mathcal{H}^0, \end{aligned} \tag{3.21}$$

where we set $x_t^i := \Phi_t(x^i)$ for $i = 0,1,2,3$. Now we need to estimate each of the above terms. Fix $\zeta > 0$. For the first one we appeal to Proposition 3.27 to obtain

$$\partial^2\mathcal{MS}\big((u_t,\Gamma_t);U\big)[X\cdot\nu_t] \geq \bar{C}\|X\cdot\nu_t\|^2_{\widetilde{H}^1(\Gamma_t)}\,. \tag{3.22}$$

To estimate the second term of (3.21) we recall that Remark 3.29 and the continuity of the map $\Phi \mapsto H_\Phi$ assert that the map

$$\Phi \in \mathcal{D}_{\bar{\delta}_1}(\Omega;U) \mapsto \big\||\nabla_{\Gamma_\Phi}u_\Phi^+|^2 - |\nabla_{\Gamma_\Phi}u_\Phi^-|^2 + H_\Phi\big\|_{L^\infty(\Gamma_\Phi)}$$

is continuous with respect to the $C^2$-norm. Since by the criticality condition that quantity vanishes for $\Phi = \mathrm{Id}$, possibly reducing $\bar{\delta}_1$, it is possible to have

$$\big\||\nabla_{\Gamma_\Phi}u_\Phi^+|^2 - |\nabla_{\Gamma_\Phi}u_\Phi^-|^2 + H_\Phi\big\|_{L^\infty(\Gamma_\Phi)} \leq \zeta\,,$$

for each $\Phi \in \mathcal{D}_{\bar{\delta}_1}(\Omega;U)$. Hence

$$\begin{aligned} &\int_{\Gamma_t} f_t\big[Z\cdot\nu_t - 2X^{||}\cdot\nabla_{\Gamma_t}(X\cdot\nu_t) + D\nu_t[X^{||},X^{||}] - H_t(X\cdot\nu_t)^2\big]\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad \geq -\zeta\|Z\cdot\nu_t - 2X^{||}\cdot\nabla_{\Gamma_t}(X\cdot\nu_t) + D\nu_t[X^{||},X^{||}] - H_t(X\cdot\nu_t)^2\|_{L^1(\Gamma_t)} \\ &\quad \geq C\|X\cdot\nu_t\|^2_{\widetilde{H}^1(\Gamma_t)}\,, \end{aligned} \tag{3.23}$$

(3.24)

where in the last step we have used estimates (3.10) and (3.11) provided by Proposition 3.25.

To estimate the last term we recall that $Z \equiv 0$ in a neighborhood of $x_0$. Thus, we can rewrite the last term as

$$\begin{aligned}\sum_{i=1}^{3}(X\cdot\nu_t)^2 D\nu_{\partial\Omega}[\nu_t,\nu_t](x_t^i) + \int_{\partial\Gamma} Z\cdot\eta_t \,\mathrm{d}\mathcal{H}^0 &= \sum_{i=1}^{3}\big[(X\cdot\nu_t)^2 D\nu_{\partial\Omega}[\nu_t,\nu_t] + Z\cdot\eta_t\big](x_t^i)\\ &= \sum_{i=1}^{3}\big[(X\cdot\nu_t)^2 D\nu_{\partial\Omega}[\nu_t,\nu_t] + Z\cdot(\eta_t-\nu_{\partial\Omega}) + Z\cdot\nu_{\partial\Omega}\big](x_t^i)\\ &= \sum_{i=1}^{3}\big[-(X\cdot\eta_t)^2 D\nu_{\partial\Omega}[\eta_t,\eta_t] + Z\cdot(\eta_t-\nu_{\partial\Omega})\big](x_t^i)\,,\end{aligned}$$

where we have used equality *(iii)* of Lemma 3.8. We claim that it is possible to choose $\bar\delta_1$ in such a way that

$$|X\cdot\eta_t(x_t^i)|^2 \le \zeta|X\cdot\nu_t(x_t^i)|^2\,, \tag{3.25}$$

$$|\eta_t-\nu_{\partial\Omega}|(x_t^i)\le\zeta\,, \tag{3.26}$$

and

$$|Z(x_t^i)| \le C\|X\cdot\nu_t\|^2_{H^1(\Gamma_i)}\,, \tag{3.27}$$

for all $i=1,2,3$. Indeed, (3.26) follows easy by noticing that $\eta(x^i)=\nu_{\partial\Omega}(x^i)$ and by the identity

$$\eta_t = \frac{D\Phi_t[\eta]}{|D\Phi_t[\eta]|}\,.$$

To obtain (3.25) we notice that from $X=\nu$ on $\partial\Omega\cap\partial\Gamma$ we get $X\cdot\eta(x^i)=0$. Then we conclude thank to the continuity of the maps

$$G_i : \big\{(x,v,w)\in\big(\partial\Omega\cap B_{\bar\delta}(x^i)\big)\times\big(B_{\bar\delta}(\eta(x))\cap S^1\big)\times\big(B_{\bar\delta}(\nu(x))\cap S^1\big)\big\}\to\mathbb{R}\,,$$

given by

$$G_i(x,v,w) := \frac{|F(x)\cdot v|}{|F(x)\cdot w|}\,.$$

Finally, in order to obtain (3.27), we notice that, by construction of the vector field $X$, there exists a function $\Phi\in C^2((\Gamma)_{\bar\delta})$ that is constant along the trajectories of $F$, such that $X=\Phi F$ near $\partial\Omega$ (see Proposition 3.25). Hence

$$Z(x_t^i) = DX[X](x_t^i) = \Phi(x^i)^2 DF[F](x_t^i)\,.$$

Reasoning in a similar way as above, taking a $\bar\delta_1$ sufficiently small, we have that $|F\cdot\nu_t(x_t^i)|\ge\frac12$, and hence $\Phi(x^i)^2\le 2(X\cdot\nu_t)(x)^2$ for $x\in B_{\bar\delta}(x^i)$. Thus, we obtain the estimate

$$|Z(x_t^i)| \le C|X\cdot\nu_t|^2 \le C_2\|X\cdot\nu_t\|^2_{\widetilde H^1(\Gamma)}\,,$$

where $C>0$ depends only on $F$ and $\bar\delta$ and the last inequality follows by the Sobolev embedding. Using (3.25), (3.26) and (3.27) we obtain

$$\sum_{i=1}^{3}\big[-(X\cdot\eta_t)^2 D\nu_{\partial\Omega}[\eta_t,\eta_t] + Z\cdot(\eta_t-\nu_{\partial\Omega})\big](x_t^i) \ge -C_2\zeta\|X\cdot\nu_t\|^2_{\widetilde H^1(\Gamma)}\,. \tag{3.28}$$

Now, combining the estimates (3.22), (3.23) and (3.28), we get that:

$$\int_0^1 g''(t)\mathrm{d}t \ge \big(\bar C-(C_1+C_2)\zeta\big)\int_0^1 \|X\cdot\nu_t\|^2_{\widetilde H^1(\Gamma)}\mathrm{d}t\,.$$

Thus, by taking $\zeta$ sufficiently small, we finally have the claimed bound (3.20).

*Step 2.*It is easy to see that, given $\Phi$ as in Step 1, but with $\Phi(x_0) = x_0$, it is possible to construct a family of diffeomorphisms $\Psi_\varepsilon : \Phi(\bar{\Gamma}) \to \bar{\Omega}$ such that $\Psi_\varepsilon(x_0) \neq x_0$ and $\Psi_\varepsilon \to \mathrm{Id}$ in the $C^2$ norm, as $\varepsilon \to 0$. This implies that

$$\mathcal{MS}\big((u_{\Psi_\varepsilon}, \Psi_\varepsilon(\Phi(\Gamma)); U\big) \to \mathcal{MS}\big((u_\Phi, (\Phi(\Gamma); U\big)\,.$$

Thus the result follows by passing to the limit in the inequality proved in the previous case.

*Step 3.* We now drop the assumption $\Phi\big(\bar{\Gamma} \cap B_\xi(x_0)\big) = \bar{\Gamma} \cap B_\xi(x_0) + v$ for some $\xi > 0$ and $v \in \mathbb{R}^2$. It is easy to see that the condition $D\Phi(x_0) = \lambda \mathrm{Id}$, for some $\lambda \neq 0$, allows to construct a family of triple points $\Gamma_\varepsilon$ such that $\Gamma_\varepsilon$ is a traslation of $\Gamma$ in a ball $B_\varepsilon(x_0)$. Moreover there exist diffeomorphisms $\Psi_\varepsilon : \Phi(\Gamma) \to \Gamma_\varepsilon$ with $\Psi_\varepsilon \to \mathrm{Id}$ in the $C^2$ norm, as $\varepsilon \to \infty$. Roughly speaking, we define the curve $\Gamma^i_\varepsilon$ as follows: consider the parametrized curve $\Gamma_\varepsilon : [0,1] \to \mathbb{R}^2$ having curvature equal to those of $\Gamma^i$ in $[0, \varepsilon)$, and equal to those of $\Phi(\Gamma)$ in $[2\varepsilon, 1]$. In $[\varepsilon, 2\varepsilon]$ we just define the curvature as a linear function. The claim the follows directly.

Thus the result follows again by passing to the limit in the previous estimate.

*Step 4.* Since all the previous steps have been done just by usign the closeness of $\Phi$ to the identity in the $C^2$-norm, given $\Phi \in C^2(\bar{\Omega}; \bar{\Omega})$ such that $\|\Phi - \mathrm{Id}\|_{C^2(\bar{\Omega};\bar{\Omega})} < \bar{\delta}$, we can find $(\Phi_\varepsilon)_\varepsilon \subset C^3(\bar{\Omega}; \bar{\Omega})$ such that $\Phi_\varepsilon \to \Phi$ in $C^2(\bar{\Omega}; \bar{\Omega})$ and such that $\Phi_\varepsilon = \mathrm{Id}$ in $\partial_D\Omega \cup (\Omega \setminus U')$, where $U \subset U'$, with $U'$ an admissible subdomain. Using Remark 3.31, we know that, if $U'$ is close to $U$ in the Hausdorff sense, then $(u, \Gamma)$ is stable also in $U'$. Hence, the result follows by passing to the limit.

*Step 5.* Finally, the local minimality with respect to $W^{2,\infty}$-perturbations follows again by approximating an admissible diffeomorphism $\Phi \in W^{2,\infty}(\bar{\Omega}; \bar{\Omega})$ with a sequence of diffeomorphisms of class $C^2$ converging to $\Phi$ in the $W^{2,\infty}$-topology.

Thank to Remark 3.26 we also obtain the isolated local minimality result. □

## 3.5. Application

In this section we would like to give some examples of critical and strictly stable triple points.

**3.5.1. Local minimality in a tubular neighborhood.** Here we want to prove that, under an additional assumption (similar to those of [9] and [8]), every critical triple point is strictly stable in a suitable tubular neighborhood, and hence a local minimizer with respect to $W^{2,p}$-variations contained in that tubular neighborhood.

PROPOSITION 3.30. *Let $(u, \Gamma)$ be a critical triple point, and suppose that*

$$H_{\partial\Omega}(x^i) < 0\,, \tag{3.29}$$

*for each $i = 1, 2, 3$. Then there exists $\bar{\mu} > 0$ such that, for all $\mu < \bar{\mu}$, $(u, \Gamma)$ is strictly stable in $(\Gamma)_\mu$.*

PROOF. *Step 1.* First of all we prove that there exists a constant $C > 0$ such that:

$$\int_\Gamma |\nabla_\Gamma \varphi|^2 \,\mathrm{d}\mathcal{H}^{N-1} + \int_\Gamma H^2 \varphi^2 \,\mathrm{d}\mathcal{H}^{N-1} - \sum_{i=1}^{3} \varphi_i^2(x^i) H_{\partial\Omega}(x^i) \geq C \|\varphi\|^2_{\widetilde{H}^1(\Gamma)}\,,$$

for all $\varphi \in \widetilde{H}^1(\Gamma)$. Indeed, it is easy to see that

$$\int_{\Gamma_i} |\nabla_\Gamma \varphi_i|^2 \, d\mathcal{H}^{N-1} - \varphi_i^2(x^i) H_{\partial\Omega}(x^i) \geq C \int_{\Gamma_i} |\varphi_i|^2 \, d\mathcal{H}^{N-1} .$$

*Step 2.* The only thing we have to prove now is that

$$\lim_{\mu\to 0} \sup_{\substack{\varphi\in\widetilde{H}^1(\Gamma),\\ \|\varphi\|_{\widetilde{H}^1(\Gamma)}=1}} \int_{(\Gamma)_\mu} |\nabla v_\varphi^\mu|^2 \, dx = 0 , \tag{3.30}$$

where $v_\varphi^\mu \in H^1_{(\Gamma)_\mu}(\Omega\setminus\Gamma)$ is the solution of

$$\int_\Omega \nabla v_\varphi^\mu \cdot \nabla z \, dx = \langle \mathrm{div}_\Gamma\big(\varphi \nabla_\Gamma u^+\big), z^+\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} - \langle \mathrm{div}_\Gamma\big(\varphi\nabla_\Gamma u^-\big), z^-\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} , \tag{3.31}$$

for every $z \in H^1_{(\Gamma)_\mu}(\Omega\setminus\Gamma)$. For each $\mu > 0$, let $\bar\varphi^\mu \in \widetilde{H}^1(\Gamma)$, with $\|\bar\varphi^\mu\|_{\widetilde{H}^1(\Gamma)} = 1$, be such that

$$\int_{(\Gamma)_\mu} |\nabla v_{\bar\varphi^\mu}^\mu|^2 \, dx = \sup_{\substack{\varphi\in\widetilde{H}^1(\Gamma),\\ \|\varphi\|_{\widetilde{H}^1(\Gamma)}=1}} \int_{(\Gamma)_\mu} |\nabla v_\varphi^\mu|^2 \, dx .$$

Consider, for $\mu > 0$, the following minimum problem:

$$\min\{F^\mu(v) \, : \, v \in H^1_{(\Gamma)_\mu}(\Omega\setminus\Gamma)\} ,$$

where

$$F^\mu(v) := \frac12 \int_{(\Gamma)_\mu} |\nabla v|^2 \, dx - \langle \mathrm{div}_\Gamma\big(\bar\varphi^\mu \nabla_\Gamma u^+\big), v^+\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} + \langle \mathrm{div}_\Gamma\big(\bar\varphi^\mu\nabla_\Gamma u^-\big), v^-\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} .$$

Clearly, $v_{\bar\varphi^\mu}^\mu$ is the solution of the above minimum problem. We claim that

$$F^\mu(v) \geq \frac14 \int_{(\Gamma)_\mu} |\nabla v|^2 \, dx \, - C , \tag{3.32}$$

for a suitable constant $C > 0$. Taking (3.32) for grant, we conclude. Indeed, noticing that

$$\min\{F^\mu(v) \, : \, v \in H^1_{(\Gamma)_\mu}(\Omega\setminus\Gamma)\} = -\frac12 \int_{(\Gamma)_\mu} |\nabla v_{\bar\varphi^\mu}^\mu|^2 \, dx , \tag{3.33}$$

from (3.32) we get that

$$\sup_{\mu>0} \int_{(\Gamma)_\mu} |\nabla v_{\bar\varphi^\mu}^\mu|^2 \, dx \leq M ,$$

for some $M > 0$. So, up to a not relabelled subsequence, $v_{\bar\varphi^\mu}^\mu \rightharpoonup w$ weakly in $H^1(\Omega\setminus\Gamma)$, as $\mu \to 0$. It is easy to see that $w = 0$. Then, using equation (3.31) where we take as a test function $z$ itself, the uniform bound on $\|\mathrm{div}_\Gamma\big(\bar\varphi^\mu\nabla_\Gamma u^\pm\big)\|_{H^{-\frac12}(\Gamma)}$, and the compactenss of the trace operator, we finally get:

$$\int_{(\Gamma)_\mu} |\nabla v_{\bar\varphi^\mu}^\mu|^2 \, dx \to 0 , \qquad \text{as } \mu \to 0.$$

We are now left to prove estimate (3.32). Fix $\bar\mu > 0$ and let $\Phi_\mu := \bar\varphi^\mu \nabla_\Gamma u^+$. Then:

$$\begin{aligned}\langle \mathrm{div}_\Gamma\big(\bar\varphi^\mu \nabla_\Gamma u^\pm\big), v^+\rangle_{H^{-\frac12}(\Gamma)\times H^{\frac12}(\Gamma)} &\le \|\mathrm{div}_\Gamma \Phi_\mu\|^2_{H^{-\frac12}(\Gamma)}\|v^\pm\|^2_{H^{\frac12}(\Gamma)}\\ &\le C\frac{\varepsilon^2}{2}\|v\|_{H^1((\Gamma)_{\bar\mu})} + \frac{C}{2\varepsilon^2}\\ &\le C\frac{\varepsilon^2}{2}\|\nabla v\|_{L^2((\Gamma)_{\bar\mu})} + \frac{C}{2\varepsilon^2}\,.\end{aligned}$$

Thus, taking $\varepsilon > 0$ sufficiently small, we obtain the desired estimate. □

REMARK 3.31. In view of the result of Theorem 3.24, it is not restrictive to suppose an admissible set $U$ to be of class $C^\infty$, meeting $\partial\Omega$ orthogonally. Indeed, by (3.33), it follows that

$$-\int_{U_1} |\nabla v_\varphi|^2\,\mathrm{d}x \ge -\int_{U_2} |\nabla v_\varphi|^2\,\mathrm{d}x\,,$$

for every $\varphi \in \widetilde H^1(\Gamma)$, whenever $U_1$ and $U_2$ are admissible subdomains such that $U_1 \subset U_2$. Hence, given a regular critical and strictly stable triple point $(u,\Gamma)$ and generic admissible subdomain $U$, we can write $U = \bigcap_n U_n$, whith $U_n$ admissible subdomains where $(u,\Gamma)$ is strictly stable, that are of class $C^\infty$ meeting $\partial\Omega$ orthogonally.

## 3.6. Appendix

PROOF OF PROPOSITION 3.22. *Step 1.* Without loss of generality, we can suppose $\varphi^i \in C^\infty(\bar\Gamma_i)$. Indeed, if we take $\varphi \in \widetilde H^1(\Gamma)$, we can consider approximation by convolution $(\varphi^i_\varepsilon)_\varepsilon$ of each $\varphi_i$ (where we have previously extended each $\varphi^i$ to an $H^1$ function defined in a regular extension of $g_i$). Now let $h_\varepsilon := \big(\varphi^1_\varepsilon + \varphi^2_\varepsilon + \varphi^3_\varepsilon\big)(x_0)$ and define

$$\widetilde\varphi^3_\varepsilon = \varphi^3_\varepsilon - h_\varepsilon\,.$$

Then $\varphi_\varepsilon := (\varphi^1_\varepsilon, \varphi^2_\varepsilon, \widetilde\varphi^3_\varepsilon) \in \widetilde H^1(\Gamma)$, $\varphi^i_\varepsilon \to \varphi^i$ in $H^1(\Gamma_i)$ for $i = 1, 2$ and $\widetilde\varphi^3_\varepsilon \to \varphi^3$ in $H^1(\Gamma_3)$. By the continuity of the quadratic form $\partial^2\mathcal{MS}\big((u,\Gamma);U\big)$ with respect to the $H^1$ convergence, we obtain that

$$\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi] = \lim_{\varepsilon\to0}\partial^2\mathcal{MS}\big((u,\Gamma);U\big)[\varphi_\varepsilon]\,.$$

*Step 2.* Let $\varphi \in \widetilde h^1(\Gamma)$ such that $\varphi^i \in C^\infty(\bar\Gamma_i)$. For $x \in \bar\Gamma_i$, define

$$Y(x) := \varphi^i(x)\nu^i(x) + b^i(x)\tau^i(x)\,,$$

for some function $b^i \in C^1(\bar\Gamma_i)$ such that $b^i \equiv 0$ if $|x - x_0| \ge \delta_1$. The functions $b^i$ have to be choosen in such a way that the vector $Y$ is well defined in $x_0$ and that $DY(x_0)[\tau_1+\tau_2+\tau_3] = 0$. The first condition requires to impose that

$$\Big(\varphi^i + \frac12\varphi^{(i+1)mod3} - \frac{\sqrt3}{2}b^{(i+1)mod3}\Big)(x_0) = 0\,,$$

while the other one leads to

$$\sum_{i=1}^3\big[\nu^i D_{\Gamma_i}\varphi^i + \tau^i D_{\Gamma_i} b^i\big](x_0) = 0\,,$$

where we have use the fact that $H_i(x_0) = 0$, and hence $D_{\Gamma_i}\nu^i(x_0) = D_{\Gamma_i}\tau^i(x_0) = 0$. Now define

$$\bar Y := \begin{cases} Y(x) & \text{if } x \in \Gamma\,,\\ a(x)\tau_{\partial\Omega}(x) & \text{if } x \in \partial\Omega\,,\\ 0 & \text{in } \Omega\backslash U\,,\end{cases}$$

where $a \in C^1(\partial\Omega)$ is such that $a \equiv 1$ in a neighborhood of $\bar\Gamma \cap \partial\Omega$, $a \equiv 0$ in $\partial_D\Omega \cup (\bar U \cap \partial\Omega)$. Thanks to Lemma 3.37, $\bar Y \in C^1(\bar\Gamma \cup \partial\Omega \cup (\Omega \backslash U); \mathbb{R}^2)$. Then, use Whitney's extension theorem to extend it to a vector field $\widetilde{Y} \in C^1(\bar\Omega; \mathbb{R}^2)$. Using convolutions, we can approximate $\widetilde{Y}$ with vector fields $\widetilde{X}_\varepsilon \in C^\infty(\bar\Omega; \mathbb{R}^2)$ such that $\widetilde{X}_\varepsilon \to \widetilde{Y}$ in $C^1(\bar\Omega; \mathbb{R}^2)$. Notice that $\operatorname{supp} \widetilde{X}_\varepsilon \subset\subset U' \backslash \partial_D\Omega$, where $U' \supset U$ is an admissible subdomain. Now define

$$X^\varepsilon(x) := \begin{cases} \widetilde{X}_\varepsilon(x) - \chi\left(\frac{s^2}{\delta^2}\right)\left((\widetilde{X}_\varepsilon \cdot \nu_{\partial\Omega})\nu_{\partial\Omega}\right)(y) & \text{if } x = y + s\nu_{\partial\Omega}(y)\,,\ s < \delta, \\ \widetilde{X}_\varepsilon & \text{otherwise.} \end{cases}$$

In this way $X^\varepsilon \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$, and still $X^\varepsilon \to \widetilde{Y}$ in $C^1(\bar\Omega; \mathbb{R}^2)$. This allows to conclude. Indeed, call $(\Phi^\varepsilon)_t$ the flow generated by the vector field $X_\varepsilon$, with $\Phi^\varepsilon_0 = \mathrm{Id}$, and let $\Gamma^\varepsilon_t$ be the evolution of $\Gamma$ through this flow. Then, we have that

$$\partial^2 \mathcal{MS}\big((u,\Gamma); U\big)[\varphi] = \lim_{\varepsilon \to 0} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{MS}\big((u^\varepsilon, \Gamma^\varepsilon); U\big)_{|t=0} \geq 0\,,$$

where the last inequality follows from the local minimality of $(u, \Gamma)$. □

**3.6.1. Results on elliptic problems.** The following theorem collects some regularity results on elliptic problems in domains with corners we will need in the following. All these results can be found in the book of Grisvard (see [32]).

**Notation**. In this section we will consider operators $L$ written in the form

$$Lu = -\sum_{i,j=1}^{2} D_i(a_{ij} D_j u) + \sum_{i=1}^{2} a_i D_i u + a_0 u\,,$$

where $D_i$ denotes the partial derivatives with respec to the variable $x^i$.

Definition 3.32. We say that an open and bounded set $A \subset \mathbb{R}^2$ is a *curvilinear polygon* of class $C^{r,s}$, with $r \in \mathbb{N}$ and $s \in (0,1]$, if $\partial A$ is a simple and connected curve that can be written as

$$\partial A = \cup_{i=1}^{k} \bar\gamma_i\,,$$

where each $\gamma_i$ is a relatively open curve of class $C^{r,s}$, and $k \in \mathbb{N}$.

Moreover we will denote by $P_i$ the common boundary point of $\gamma_i$ and $\gamma_{i+1}$ (or $\gamma_k$ and $\gamma_1$), and by $\omega_i$ the angle in $P_i$ internal to $A$.

Theorem 3.33. *Let $A$ be a curvilinear polygon of class $C^{1,1}$, and let $L$ be an elliptic operator defined on $A$, with coefficients of class $C^{0,1}$. Then, the following a priori estimate holds true:*

$$\|u\|_{H^2(A)} \leq C_1\big(\|Lu\|_{L^2(A)} + \|\partial_\nu u\|_{H^{\frac12}(\partial A)} + \|u\|_{H^{\frac32}(\partial A)}\big) + C_2\|u\|_{H^1(A)}\,, \tag{3.34}$$

*for suitable constants $C_1, C_2 \geq 0$ and for all $u \in H^2(A)$.*

*Given $f \in L^2(A)$, let $u \in H^1(A)$ be a weak solution of the problem*

$$\begin{cases} Lu = f & \text{in } A\,, \\ \partial_\nu u = 0 & \text{on } \gamma_i\,,\ \text{for } i \in \mathcal{N}\,, \\ u = 0 & \text{on } \gamma_i\,,\ \text{for } i \in \mathcal{D}\,, \end{cases}$$

*where $\mathcal{N}, \mathcal{D}$ is a partition of $\{1, \dots, N\}$. Then $u$ can be written as*

$$u = u_{reg} + \sum_{i=1}^{N} u^i_{sing}\,,$$

*where* $u_{reg} \in H^2(A)$ *and* $u^i_{sing} \in H^1(A)$ *are such that* $u^i_{sing} \in H^2(V_i)$ *for each open set* $V_i$ *such that* $P_i \notin \bar{V}_i$.

*Finally, suppose that*

$$\omega_i \leq \begin{cases} \pi & \text{if } j, j+1 \in \mathcal{D}\,, \text{ or } j, j+1 \in \mathcal{N}\,, \\ \frac{\pi}{2} & \text{otherwise }\,. \end{cases}$$

*Then* $u \in H^2(A)$. *Moreover, if* $\mathcal{D}$ *is not empty,* (3.34) *holds with* $C_2 = 0$.

Finally, we need a continuity theorem for elliptic problems (see, *e.g.*, [41, Remark 2.2]).

THEOREM 3.34. *Let* $(L_s)_{s\in(-\delta,\delta)}$ *be a family of uniformly elliptic operators defined on a curvilinear polygon* $A$ *of class* $C^{1,1}$, *and let* $\mathcal{N}, \mathcal{D}$ *be a partition of* $\{1, \ldots, N\}$, *with* $\mathcal{D} \neq \emptyset$. *Suppose that, for* $s \in (-\delta, \delta)$, *the functions*

$$s \mapsto a_{ij}(\cdot, s) \qquad s \mapsto a_i(\cdot, s)\,, \qquad s \mapsto a_0(\cdot, s)\,,$$

*belong to* $L^\infty(A)$,

$$s \mapsto f_s \in H^{-1}(A)$$

*are continuous and that there exists a constant* $M > 0$ *such that*

$$|a_{ij}(x, s)| \leq M\,, \qquad |a_i(x, s)| \leq M\,, \qquad |a_0(x, s)| \leq M\,,$$

*for all* $x \in A$ *and for all* $s \in (-\delta, \delta)$. *Given* $v \in H^1(A)$ *let us consider the operator*

$$\begin{array}{rccc} T: & (-\delta, \delta) & \to & H^1 \\ & s & \mapsto & u_s \end{array}$$

*where* $u_s$ *is a weak solution of the problem*

$$\begin{cases} L_s u = f_s & \text{in } A\,, \\ \partial_\nu u = 0 & \text{on } \gamma_i\,, \text{ for } i \in \mathcal{N}\,, \\ u_s = v & \text{on } \gamma_i\,, \text{ for } i \in \mathcal{D}\,. \end{cases}$$

*Then* $T$ *is continuous.*

**3.6.2. Whitney's extension theorem.** Here we state the version of the Whitney's extension theorem needed in the chapter. But we first need to set some notation. Given $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^2$ and $v \in \mathbb{R}^2$, let

$$|\mathbf{k}| := k_1 + k_2\,, \qquad v^{\mathbf{k}} := v_1^{k_1} v_2^{k_2}\,.$$

If $f$ is a $|\mathbf{k}|$-times differentiable function, we set

$$D^{\mathbf{k}} f(x) := \frac{\partial^{|\mathbf{k}|} f}{\partial x^{\mathbf{k}}}(x) = \frac{\partial^{|\mathbf{k}|} f}{\partial x_1^{k_1} x_2^{k_2}}(x)\,,$$

where $D^{\mathbf{0}} = f$.

DEFINITION 3.35. Let $X$ be a compact subset of $\mathbb{R}^2$. We define the space $C^h(X)$ as the space of functions $f : X \to \mathbb{R}$ for which there exists a family $\mathcal{F} := \{F^{\mathbf{k}}\}_{|\mathbf{k}|\leq h}$ of continuous functions on $X$, with $F^{\mathbf{0}} = f$, such that, for every $|\mathbf{k}| \leq h$, it holds

$$\sup_{x,y\in X,\, 0<|x-y|<r} \left| F^{\mathbf{k}}(x) - F^{\mathbf{k}}(y) - \sum_{|\mathbf{j}|=1}^{h-|\mathbf{k}|} F^{\mathbf{j}}(x)(y-x)^{\mathbf{k}+\mathbf{j}} \right| = o(r^{h-|\mathbf{k}|})\,. \tag{3.35}$$

Moreovere, we define

$$\|\mathcal{F}\|_{C^h(X)} := \sum_{|\mathbf{k}|\le h} \|F^{\mathbf{k}}\|_{C^0(X)} \,.$$

THEOREM 3.36 (Whitney's extension theorem). *For every $h \ge 1$ and $L > 0$ there exists a constant $C_0 > 0$, depending on $h$ and $L$, with the following property: if $X \subset B_L$ is a compact set of $\mathbb{R}^2$ and $f \in C^h(X)$, then there exists a function $\widetilde{f} \in C^\infty(\mathbb{R}^2 \setminus X) \cap C^h(\mathbb{R}^2)$ such that*

$$D^{\boldsymbol{k}}\widetilde{f} = F^{\boldsymbol{k}} \quad \text{on } X\,, \quad \text{for every } |\boldsymbol{k}| \le h\,,$$

*and*

$$\|\widetilde{f}\|_{C^h(\mathbb{R}^2)} \le C_0 \|\mathcal{F}\|_{C^h(X)} \,.$$

We now prove a technical result we needed to use several times in this chapter.

LEMMA 3.37. *Let $\gamma \subset \Omega$ be a simple curve of class $C^1$ meeting $\partial\Omega$ orthogonally in a point $\bar{x}$. Let $X \in C^1(\gamma;\mathbb{R}^2)$ be such that $X(\bar{x}) = \tau_{\partial\Omega}(\bar{x})$. Then, the vector field defined as*

$$\widetilde{X} := \begin{cases} X & \text{on } \gamma\,, \\ \tau_{\partial\Omega} & \text{on } \partial\Omega\,, \end{cases}$$

*belongs to $C^1(\bar{\Gamma} \cup \partial\Omega;\mathbb{R}^2)$.*

PROOF. Denote by $\tau$ the tangent vector field on $\gamma$. Define

$$D\widetilde{X}(x)[\tau(x)] := DX(x)[\tau(x)]\,, \quad D\widetilde{X}(x)[\nu(x)] := \chi\Big(\frac{|x-\bar{x}|^2}{\varepsilon^2}\Big) D\tau_{\partial\Omega}(\bar{x})[\tau_{\partial\Omega}(\bar{x})]\,,$$

for $x \in \gamma$, and

$$D\widetilde{X}(x)[\tau_{\partial\Omega}(x)] := D\tau_{\partial\Omega}(x)[\tau_{\partial\Omega}(x)]\,, \quad D\widetilde{X}(x)[\nu_{\partial\Omega}(x)] := \chi\Big(\frac{|x-\bar{x}|^2}{\varepsilon^2}\Big) DX(\bar{x})[\tau(\bar{x})]\,,$$

for $x \in \partial\Omega$, for a constant $\varepsilon > 0$. Then condition (3.35) is easily satisfied if $x, y \in \gamma$ or $x, y \in \partial\Omega$. In the case $x \in \gamma$ and $y \in \partial\Omega$, we simply write $y - x = (y - \bar{x}) - (x - \bar{x})$ and we use the triangular inequality te get the desired estimate. $\square$

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