# Mean Field Games with Density Constraints

MSc Thesis

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# Abstract

The present thesis focuses on a recent theory called "Mean Field Games", introduced in 2006 by J.-M. Lasry and P.-L. Lions (see [18], [19], [20], [22]). Its goal is to study some first order Mean Field Games systems with density constraints. The density constraints are natural to assume, in the sense that agents usually do not want to go through places, where already there are many other agents. The model which is presented and studied is due to F. Santambrogio (see [30]) and it uses some ideas from recent works about crowd motion theory (see [23], [24], [25], [29]).

In the Introduction we briefly present the theory of Mean Field Games, using the original works by Lasry and Lions and the lecture notes of P. Cardaliaguet ([8]). Here is also presented a second order and first order system, the latter represents the main subject of study of the thesis.

In Chapter 1 we give some well-known definitions and results from the theory of optimal transportation. This theory seems to be very useful also in the study of MFG. The basic references of this topic are the two monographs by C. Villani (see [33] and [34]) and in this chapter we also use lecture notes of F. Santambrogio ([31]).

In Chapter 2 our aim is to present some well-known facts about optimal control theory and semi-concavity. Here the reference is mainly the book of P. Cannarsa and C. Sinestrari ([7]) which is a very important tool in our situation.

In Chapter 3 following the lecture notes by P. Cardaliaguet ([8]), which contain the lectures given by P.-L. Lions in Collège de France (years 2007 and 2008), we present a fixed point technique to obtain existence for a standard MFG system (without density constraints). We also present the assumptions (and the proof of the theorem) which are used to obtain uniqueness in this case. More or less we want to follow a similar way, using a fixed point scheme to study the case with density constraints.

In Chapter 4 we finally arrive to the model given by F. Santambrogio and try to construct a fixed point scheme to show the existence of a solution of our MFG system with density constraints. On this way of studying this system we realize that it is not so obvious to adapt the techniques from the usual case, hence we have to modify from time to time the ideas and at the end the problem is reduced to finding a fixed point of a multi-valued operator. Later in this chapter we give some possible further study possibilities in this topic and end the thesis with final conclusions. Basically this is the chapter which contains some original works and adaptations of other results (form the theory of crowd motion and from the theory of usual MFG theory) to our case.

# Contents

# Introduction

## 0.1 General facts and a second order MFG system

Mean Field Game theory was introduced recently by J.-M. Lasry and P.-L. Lions in a series of papers ([18],[19],[20]) and presented though several lectures of P.-L. Lions at the Collège de France. This theory is devoted to the analysis of differential games with a large number of players, who have very little influence on the overall system ("small" players). These type of models are derived from a "continuum limit" (in other words letting the number of players/agents go to infinity) which is somehow similar to the classical mean field approaches in Statistical Mechanics and Physics (as for example the derivation of Vlasov or Bolzmann equations in the kinetic theory of gases) or in Quantum Mechanics and Quantum Chemistry (density functional models, etc.). The applications of this theory cover a wide range of social life and economic problems.

In the present thesis we will use as basic references some very detailed lecture notes, such as the notes of P. Cardaliaguet ([8]), which covers a huge part of the lectures of P.-L. Lions at Collège de France (until 2008), as well as the notes by O. Guéant, P.-L. Lions and J.-M. Lasry ([10]), which was inspired from a "Cours Bachelier" held in 2009 and taught by J.-M. Lasry.

Now let us return to the theoretical introduction. The typical model for Mean Field Games (briefly MFG) is the following system

$$\begin{cases} (i) & -\partial_t u - \nu\Delta u + H(x,m,\nabla u) = F(x,m) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m - \nu\Delta m - \nabla\cdot(\nabla_p H(x,m,\nabla u)m) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0)=m_0, u(T,x)=G(x,m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{0.1.1}$$

In the above system, $\nu \geq 0$ is a parameter. The first equation (a Hamilton-Jacobi type equation) has to be understood backward in time and the second one (a Kolgomorov or Fokker-Planck type equation) forward in time. To study this system are crucial the following assumptions: the convexity of $H = H(x,m,p)$ w.r.t. the last variable. This condition will imply that the first equation is linked to an optimal control problem (in this case a stochastic control problem). The $u$ variable in the first equation will be the value function of a typical "small" player. The second crucial assumption is that $m_0$ (and therefore $m(t)$) is (the density of) a probability measure.

We can interpret the system heuristically as it follows. An arbitrary agent controls the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2\nu}B_t,$$

where $B_t$ is a standard Brownian motion. He aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T L(X_s, m(s), \alpha_s)ds + G(X_T, m(T))\right],$$

where $L$ is the Legendre-Flenchel conjugate of $H$ w.r.t. the $p$ variable. We remark that in this cost the evolution of the measure $m_s$ enters as a parameter.

The value function of our arbitrary player is given by the first (Hamilton-Jacobi) equation. His optimal control is (at least heuristically) given in feedback form by $\alpha^*(t,x) = -\nabla_p H(x, m, \nabla u)$. With a usual game theory argumentation, if all agents argue in this way, their distribution will move with a velocity which is due to the diffusion and to the drift term $-\nabla_p H(x, m, \nabla u)$. This idea will lead us to the second equation (Kolgomorov or Fokker-Planck) of our system.

As it is formulated also in [8], we can say that the so far developed MFG theory focuses on two main issues. First to study equation of the form (0.1.1) and give interpretations (in Economics for example) of such systems. Secondary to analyze differential games with a finite but large number of players and link their limiting behavior as the number of players goes to infinity and the equation (0.1.1).

So far the first problem is well understood and well documented. The original works by Lasry and Lions give a certain number of conditions under which equation (0.1.1) has a solution, discuss its uniqueness and its stability. Several papers also study the numerical approximation of the solution of (0.1.1): see for example Achdou and Capuzzo Dolcetta ([1]), Achdou, Camilli and Capuzzo Dolcetta ([2]), Gomes, Mohr and Souza ([9]), Lachapelle, Salomon and Turinici ([16]).

In [8] is pointed out also the fact that the MFG theory seem also particularly adapted to model problems in Economics: see Guéant ([11],[12]), Lachapelle ([17]), Lasry, Lions, Guéant ([21]), and the references therein. For the second part of the program, the limiting behavior of differential games when the number of players goes to infinity has been understood only for ergodic differential games. The general case remains largely open.

## 0.2 A first order (deterministic) MFG system

In our consideration the first order (deterministic) case of the system (0.1.1) plays a very important role, because it would be interesting to see whether a reasonable model including diffusion and density constraints makes sense . There are also some well-known results in this case (without the density constraints), regarding to existence, uniqueness, stability of the solution, which can be found in details in [8] and [22]. Our main aim is to prove some similar results, when we equip the system with density constraints. After our best knowledge these questions are still open and this kind of framework was proposed by F. Santambrogio in [30].

So a typical model to the first order MFG system is

$$\begin{cases} (i) & -\partial_t u(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m(t,x) - \nabla\cdot(\nabla u(t,x) m(t,x)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{0.2.1}$$

Let us say a few words about the heuristic interpretation of this system. The function $u$ in the first equation, as in the second order case, corresponds to the value function of a typical agent who controls his velocity $\alpha(t)$ and has to minimize his cost

$$\int_0^T \left(\frac{1}{2}|\alpha(t)|^2 + F(x(t), m(t))\right) dt + G(x(T), m(T)),$$

where $x(t) = x_0 + \int_0^t \alpha(s)ds$. The only knowledge of every agent on the overall world is the distribution of the other agents, represented by the density $m(t)$ of some probability measure. Then their "feedback strategy" (i.e., the way they ideally control at each time and at each point his velocity) is given by $\alpha(t, x) = -\nabla u(t, x)$. If all agents argue in this way, the density $m(t, x)$ of their distribution $m(t)$ over the space will evolve in time with the continuity equation (0.2.1)-(ii).

As in the case of the second order system, here we can work in the space $\mathcal{P}_1$, the space of Borel probability measures on $\mathbb{R}^d$ with finite first order moment, and we endow it with the Wasserstein (or Kantorovich-Rubinstein) distance $W_1$ (see the precise definition later on).

Moreover we present the basic assumptions from [8] which are that $F, G : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}$ have the following properties

- $F$ and $G$ are continuous over $\mathbb{R}^d \times \mathcal{P}_1$;
- There is a constant $C > 0$ s.t. for any $m \in \mathcal{P}_1$

  $$||F(\cdot, m)||_{C^2} \leq C, \quad ||G(\cdot, m)||_{C^2} \leq C,$$

  where $C^2$ is the space of functions with continuous second order derivatives endowed with the usual norm

  $$||f||_{C^2} = \sup_{x \in \mathbb{R}^d} \left(|f(x)| + |\nabla f(x)| + |\nabla^2 f(s)|\right);$$

- Here we suppose moreover that $m_0$ is absolutely continuous with respect to the $d$-dimensional Lebesgue measure, with a density still denoted by $m_0$ which is bounded and has compact support.

By a solution of (0.2.1) we mean a pair $(u, m) \in W^{1,\infty}_{loc}([0, T] \times \mathbb{R}^d) \times L^1((0, T) \times \mathbb{R}^d)$ such that $(i)$ is satisfied in the viscosity sense, while $(ii)$ is satisfied in the sense of distributions.

We formulate now the well-known results about this system (we will also present the sketch of the proofs later on).

**Theorem 0.2.1** ([8]). *Under the above assumptions, there is at least one solution of* (0.2.1).

**Theorem 0.2.2** ([8]). *Let us assume the following monotonicity properties, that are*

$$\int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2))\, d(m_1 - m_2) > 0, \forall m_1, m_2 \in \mathcal{P}_1, m_1 \neq m_2, \tag{0.2.2}$$

*and*

$$\int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))\, d(m_1 - m_2) \geq 0, \forall m_1, m_2 \in \mathcal{P}_1. \tag{0.2.3}$$

*Under these assumptions there is a unique classical solution of* (0.2.1).

*Remark* 0.2.3. The assumptions (0.2.2) and (0.2.3) ensure the uniqueness in the case of the system (0.1.1) too.

Later on we will use some elements, ideas from these proofs in the case of MFG systems with density constraints. We will also present the sketch of the proofs of these theorem in a further chapter.

## 0.3 MFG with density penalization and constraints

The idea to study MFG systems with density penalization is a natural procedure in this theory, however the idea of density constraints is due to F. Santambrogio (see [30]). Following his ideas we present here how can we construct first order MFG systems with these penalization and constraints.

We present a model from [30] as follows. We suppose that every agent follows the controlled trajectories

$$x'(t) = f(t, x(t), \alpha(t)),\ t \in [0, T],$$

where $\alpha : [0, T] \to \mathbb{R}^d$ is a control that each agent should choose, when they are maximizing the payoff function (previously in the introduction there was taken a cost function that agents wanted to minimize; without loss of generality, just by changing the corresponding signs, we can work with payoff functions)

$$-\int_t^T \left(\frac{1}{2}|\alpha(s)|^2 + g(\rho(s, y(s)))\right) ds + \Phi(y(T)),$$

where $g$ is a given increasing function. From this payoff function we can see that agents also pay (among others) for the densities of the regions they pass by. Later on we will see how can we arrive precisely to a MFG system from this formulation (i.e. how can we obtain the Hamilton-Jacobi and the continuity equations).

Now we can believe that we will arrive to a similar system to (0.2.1)

$$\begin{cases} (i) & \partial_t u(t, x) + \frac{1}{2}|\nabla u(t, x)|^2 = g(\rho(t, x)) & \text{in } (0, T) \times \mathbb{R}^d \\ (ii) & \partial_t \rho(t, x) + \nabla \cdot (\nabla u(t, x)\rho(t, x)) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ (iii) & \rho(0, x) = \rho_0(x), u(T, x) = \Phi(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{0.3.1}$$

It is also well-known fact that we can obtain the solution of the above system by minimization of a global functional, which in our case will be the following problem:

$$\min \int_0^T \int_\Omega \left(\frac{1}{2}|\alpha(t, x)|^2 \rho(t, x) + G(\rho(t, x))\right) dx dt - \int_\Omega \Phi(x)\rho(T, x) dx,$$

among solutions $(\rho, \alpha)$ of the continuity equation $\partial_t \rho + \nabla \cdot (\rho\alpha) = 0$ with the initial datum $\rho(0, x) = \rho_0(x)$. Here agents are moving in a domain $\Omega \subset \mathbb{R}^d$ instead of the whole space $\mathbb{R}^d$, but this is not a restriction neither. The function $G$ is chosen as the primitive of $g$, and in particular to be convex. This functional recall the functional studied by Benamou and Brenier in [5] to give a dynamical formulation of optimal transport.

Particulary if we choose for $g(\rho) = \rho^{m-1}(m > 1)$ (in this case $G(\rho) = \frac{1}{m}\rho^m$), then the above minimization problem as $m \to \infty$, tends to

$$\min \int_0^T \int_\Omega \frac{1}{2}|\alpha(t,x)|^2 \rho(t,x)dxdt - \int_\Omega \Phi(x)\rho(T,x)dx,$$

with the same assumptions as above and with the additional assumption that $\rho(t,x) \leq 1$ a.e.

We remark also that the energy will be infinite if we do not have for the initial density $\rho_0 \leq 1$ as well.

We can say that the above problem is equivalent, at least in the case of convex domains $\Omega$, with the following one: find $\rho_T \leq 1$ such that minimizes the quantity

$$TW_2^2(\rho_0, \rho_T) - \int_\Omega \Phi d\rho_T$$

and then choosing $\rho_t$ as a constant speed geodesic connecting $\rho_0$ and $\rho_T$. This can be deduced from the fact that the optimal curve connecting $\rho_0$ to $\rho_T$ and minimizing $\int\int \frac{1}{2}|\alpha|^2 \rho dxdt$ is precisely the geodesic between these two measure in the Wasserstein space, and the set $K$ of measure with density bounded by 1 is known to be geodesically convex.

Here it comes more or less in the picture the theory of optimal transportation, which seems to be a very useful tool in the study of this kind of problems (see also the chapter where the model for density constraint is presented). The above presented idea was the density penalization approach to MFG. However we will see later on the precise construction also for density constraints. We will also give a short presentation of the theory of optimal transportation and optimal control in the next chapters.

# Chapter 1

# Preliminaries on Optimal Transportation

We have seen in the introduction that the Fokker-Planck type equations of a MFG system are defined on the space of probability measures. A good and useful approach to understand these type of evolution equations is the framework of Optimal Transportation. So for the coherency of the present thesis we will give some basic definitions and well-known results from this theory. The main references in this field of Mathematics are (without any doubt) the two (monumental) books of C. Villani ([33],[34]). Despite this, we will use the notations and results from the graduate course given by F. Santambrogio at the University Paris-Sud in the second semester of the academic year 2011/2012. Here we will not present any proofs of the results, they can be found in a very detailed way in [31] (and of course in the textbooks [33],[34],[3]).

This theory has its roots in a more than 200 years old problem, proposed by Monge in 1781 ([28]). This problem is the following: given two densities of masses $f, g \geq 0$ on $\mathbb{R}^d$, with the property $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} g = 1$, find the map $T : \mathbb{R}^d \to \mathbb{R}^d$ pushing the first one onto the other, i.e. such that

$$\int_A g(x)dx = \int_{T^{-1}(A)} f(y)dy, \text{ for any Borel subset } A \subset \mathbb{R}^d \tag{1.0.1}$$

and minimizing the quantity

$$\int_{\mathbb{R}^d} |T(x) - x| f(x)dx$$

among all maps satisfying this condition.

In the following we will always define similarly to (1.0.1), the image measure of a measure $\mu$ on $X$ (measures will indeed replace the densities $f$ and $g$ in the most general formulation of the problem) through a measurable map $T : X \to Y$. This is the measure denoted by $T_\# \mu$ on $Y$ and characterized by

$$T_\# \mu(A) = \mu(T^{-1}(A)),$$

for every measurable subset $A$ of $Y$, or

$$\int_Y \varphi d(T_\# \mu) = \int_X \varphi \circ T d\mu,$$

for every measurable function $\varphi$.

This problem remained unsolved (does a minimizer exit? how to characterize it?, etc.) until the 1940s, when by the work of Kantorovich it has been inserted into a suitable framework, which gave the possibility to approach it, and later to show, that the problem actually has solutions, and one can characterize them. Since then the problem itself has been widely generalized, with very general cost functions $c(x, y)$ instead of the Euclidean distance $|x - y|$ and with more general measures and spaces (see the second book of C. Villani, [34] for a very general approach). We can say that in the past few decades it has grown up a huge theory from these two problems of Monge and Kantorovich, which is a very active field of research nowadays and it has many applications in geometry, PDEs, economical sciences, etc.

## 1.1 Primal and dual problems

In what follows we will focus on the transport problem from a measure $\mu$ on a space $X$ to $\nu$ on a space $Y$. $X$ and $Y$ could be complete and separable metric spaces (in general Polish spaces), but in the rest of this short insight we will consider the same (usually compact) subset $\Omega \subset \mathbb{R}^d$. We also will deal with general cost functions, $c : X \times Y \to [0, +\infty]$ which will be supposed to be continuous or semi-continuous (and very often symmetric).

The generalization of the problem of Monge, given by Kantorovich ([13]) is the following:

**Definition 1.1.1.** For two given probability measures $\mu$ and $\nu$ on $\Omega$, and a cost function $c : \Omega \times \Omega \to [0, +\infty]$ we consider the problem

$$(PK) \quad \min \left\{ K(\gamma) := \int_{\Omega \times \Omega} c d\gamma | \gamma \in \Pi(\mu, \nu) \right\}, \tag{1.1.1}$$

where $\Pi(\mu, \nu)$ is the set of so-called *transport plans*, i.e. $\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (\pi^x)_\# \gamma = \mu, (\pi^y)_\# \gamma = \nu\}$, where $\pi^x$ and $\pi^y$ are the two projections of $\Omega \times \Omega$ onto $\Omega$.

The solutions (minimizers) for this problem are called *optimal transport plans* between $\mu$ and $\nu$. If $\gamma$ is of the form $(id \times T)_\# \mu$ for a measurable map $T : \Omega \to \Omega$ (i.e. when no splitting of the mass occurs), then the map $T$ would be called *optimal transport map* from $\mu$ to $\nu$.

*Remark* 1.1.2. If $(id \times T)_\# \mu$ belongs to $\Pi(\mu, \nu)$ then $T$ pushes $\mu$ onto $\nu$ (i.e. $\nu(A) = \mu(T^{-1}(A))$ for any Borel set $A$) and our functional, we want to minimize, takes the form $\int c(x, T(x))\mu(dx)$, so this approach will be a generalization of Monge's problem.

*Remark* 1.1.3. We can also remark here that this formulation of the problem by Kantorovich is much better to work with, because in the case of Monge's problem we cannot ensure always, that there exists a map $T$, which satisfies the constraints (i.e. $T_\# \mu = \nu$), take for example the case when $\mu$ is a Dirac mass, and $\nu$ is not. For this case there exists no such $T$, but in contrary we have always $\mu \otimes \nu \in \Pi(\mu, \nu)$. But anyway there are links between the two formulations and we will see soon which are these.

An important tool will be duality theory and to introduce it we need in particular the notion of $c$-transform (a kind of generalization of the well-known Legendre-Flenchel transform).

**Definition 1.1.4.** Given a function $\chi : X \to \overline{\mathbb{R}}$ we define its $c-transform$ or $c$-conjugate $\chi^c : Y \to \overline{\mathbb{R}}$ by

$$\chi^c(y) = \inf_{x \in X} c(x,y) - \chi(x).$$

We also define the $\overline{c}$-transform of $\xi : Y \to \overline{\mathbb{R}}$ by

$$\xi^{\overline{c}}(x) = \inf_{y \in Y} c(c,y) - \xi(y).$$

Moreover, we say that a function $\psi$ defined on $Y$ is $\overline{c}$-concave if there exists $\chi$ such that $\psi = \chi^c$ (and analogously, a function $\phi$ over $X$ is said to be $c$-concave if there is $\xi$ such that $\phi = \xi^{\overline{c}}$ and we denote by $cC(X)$ and $\overline{c}C(Y)$ the sets of $c-$ and $\overline{c}-$ concave functions, respectively (when $X = Y$ and the cost $c$ is symmetric this distinction between $c-$ and $\overline{c}-$ concavity will play no more any role, and we will use just the notion of $c$-concavity).

We can see that for the cost function $c(x,y) = x \cdot y$, the notion of $c-$transform is exactly the well-know Legendre-Flenchel transform.

Now we can formulate the well known result, regarding to the duality of $(PK)$ (see [33] for example).

**Proposition 1.1.5.** *We have*

$$\min(PK) = \max_{\phi \in cC(X)} \int_\Omega \phi d\mu + \int_\Omega \phi^c d\nu. \tag{1.1.2}$$

*In particular the minimum value of $(PK)$ is a convex function of $(\mu,\nu)$ and it is the supremum of linear functionals.*

**Definition 1.1.6.** The functions $\phi$ which realize the maximum in (1.1.2) (they are not necessarily unique) are called *Kantorovich potentials* for the transport from $\mu$ to $\nu$.

We remark, that any $c-$concave function shares the same modulus of continuity of the cost $c$. In particular the cases when $c(x,y) = |x-y|^p$ for some $p \in [1,+\infty]$ have many interesting properties and are mainly well understood.

For example the case $c(x,y) = |x-y|$ shows a lot of interesting features, even if from the point of the existence of an optimal map $T$ it is one of the most difficult. A first interesting property is the following:

**Proposition 1.1.7.** *For any $1-$Lipschitz function $\phi$ we have $\phi^c = -\phi$. In particular the duality formula* (1.1.2) *may be rewritten as*

$$\min(PK) = \sup_{\phi \in Lip_1} \int_\Omega \phi d(\mu - \nu).$$

Another useful property of the $c-$transform is the following:

**Proposition 1.1.8.** *For any cost $c$ and any function $\phi : X \to \overline{\mathbb{R}}$ we have $\phi^{c\bar{c}} \geq \phi$ and the equality holds if and only if $\phi$ is c-concave.*

We summarize here some useful results for the case where the cost $c$ is of the form $c(x, y) = h(x - y)$, for a strictly convex function $h$.

**Theorem 1.1.9.** *Given $\mu$ and $\nu$ probability measures on a compact domain $\Omega \subset \mathbb{R}^d$, there exists an optimal transport plan $\pi$. It is unique and of the form $(id \times T)_{\#}\mu$, provided $\mu$ is absolutely continuous and $\partial\Omega$ is negligible. Moreover there exists also at least a Kantorovich potential $\phi$, and the optimal transport map $T$ and the potential $\phi$ are linked by*

$$T(x) = x - (\nabla h)^{-1}(\nabla\phi(x)).$$

*Moreover it holds $\phi(x) + \phi^c(T(x)) = c(x, T(x))$ for $\mu-$a.e. $x$. Conversely, every map $T$ which is of the form $T(x) = x - (\nabla h)^{-1}(\nabla\phi(x))$ for a function $\phi \in cC(\Omega)$ is an optimal transport plan from $\mu$ to $T_{\#}\mu$.*

*Remark* 1.1.10. The above theorem can be particularized in the case of the cost function $c(c, y) = \frac{|x-y|^2}{2}$. This is a well-known result of Brenier (see [6]), which says that there exits a unique optimal transport map, which send $\mu$ to $\nu$ and has it is a gradient of a convex function.

Naturally all the costs $c(x, y) = |x - y|^p$ with $p > 1$ fall under the above theorem. For the case $c(x, y) = |x - y|$ the results are a little bit weaker and are summarized below.

**Theorem 1.1.11.** *Given $\mu$ and $\nu$ probability measures on a domain $\Omega \subset \mathbb{R}^d$ there exists at least an optimal transport plan $\pi$. Moreover, one of such plans is of the form $(id\times T)_{\#}\mu$ provided $\mu$ is absolutely continuous. There exists also at least a Kantorovich potential $\phi$, and we have $\phi(x) - \phi(T(x)) = |x - T(x)|$ for $\mu-$a.e. $x$, for any choice of optimal $T$ and $\phi$.*

In the above result the absolute continuity assumption is crucial to have existence of an optimal transport map, in the sense that in general it cannot be replaced by weaker assumptions as in the strictly convex case. This can be seen from the following well-known example (see [34] and [31] for example).

**Example 1.1.12.** Set

$$\mu = \mathcal{H}^1\lfloor_A \text{ and } \nu = \frac{\mathcal{H}^1\lfloor_B + \mathcal{H}^1\lfloor_C}{2},$$

where $A, B$ and $C$ are three vertical parallel segments in $\mathbb{R}^2$ whose vertexes lie on the two lines $y = 0$ and $y = 1$ and the abscissas are $0, 1$ and $-1$, respectively, and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. It is clear that no transport plan may realize a cost better than 1 since, horizontally, every point needs to be displaced of a distance 1. Moreover, one can get a sequence of maps $T_n : A \to B \cup C$ by dividing $A$ into $2n$ equal segments $(A_i)_{i=1,\dots,2n}$ and $B$ and $C$ into $n$ segments each, $(B_i)_{i=1,\dots,n}$ and $(C_i)_{i=1,\dots,n}$ (all ordered downwards). Then define $T_n$ as a piecewise affine map which sends $A_{2i-1}$ onto $B_i$ and $A_{2i}$ onto $C_i$. In this way the cost of the map $T_n$ is less than $1 + 1/n$, which implies that the infimum of the Kantorovich problem is 1, as well as the infimum on transport maps only. Yet, no map $T$ may obtain a cost 1, as this would imply that all points are sent horizontally, but this cannot respect the push-forward constraint. On the other hand, the transport plan associated to $T_n$ weakly converge to the transport plan $\frac{1}{2}T^+_{\#}\mu + \frac{1}{2}T^-_{\#}\mu$, where $T^{\pm}(x) = x \pm e$ and $e = (1, 0)$. This transport plan turns out to be the only optimal transport plan and its cost is 1.

## 1.2 Wasserstein distances and spaces

Let $\Omega \subset \mathbb{R}^d$ and $p \geq 1$ and let us define

$$\mathcal{P}_p(\Omega) := \left\{ \mu \in \mathcal{P}(\Omega) : \int_\Omega |x|^p d\mu < +\infty \right\}.$$

The subset $\mathcal{P}_p(\Omega)$ of the set $\mathcal{P}(\Omega)$ of the probability Borel measure contains all the measures with finite $p^{th}$ order moment. Obviously, if $\Omega$ is bounded, then $\mathcal{P}_p(\Omega) = \mathcal{P}(\Omega)$. On these spaces we will define the Wasserstein distances as follows.

For $\mu, \nu \in \mathcal{P}_p(\Omega)$

$$W_p(\mu, \nu) = \inf \left\{ \int_{\Omega \times \Omega} |x - y|^p d\gamma : \gamma \in \Pi(\mu, \nu) \right\}^{\frac{1}{p}}.$$

We remark, that due to Jensen's inequality we have $W_p(\mu, \nu) \leq W_q(\mu, \nu)$ for all $p \leq q$. On the other hand, if $\Omega$ is bounded we have $W_p(\mu, \nu) \leq C W_1(\mu, \nu)^{\frac{1}{p}}$, for $p > 1$, where $C = diam(\Omega)^{\frac{p-1}{p}}$.

We also can define $W_\infty(\mu, \nu)$ (as a limit for $p \to \infty$).

We can formulate the following result:

**Theorem 1.2.1.** *For any $p \geq 1$ the functional $W_p$ is a distance over $\mathcal{P}_p(\Omega)$ and, given a measure $\mu$ and a sequence $(\mu_n)_{n \geq 1}$ in $\mathcal{P}_p(\Omega)$, the followings are equivalent:*

- $\mu_n \to \mu$ *w.r.t.* $W_p$;
- $\mu_n \rightharpoonup \mu$ *and* $\int_\Omega |x|^p \mu_n(dx) \to \int_\Omega |x|^p \mu(dx)$;
- $\int_\Omega \phi d\mu_n \to \int_\Omega \phi d\mu$ *for any* $\phi \in C(\Omega)$ *with a growth at most of order* $p$.

## 1.3 Curves, geodesics, continuity equation and displacement convexity

Now we continue our presentation studying some properties of curves in the Wasserstein spaces $\mathcal{P}_p$ endowed with the corresponding metrics $W_p$. For this theory the main reference is the book of Ambrosio, Gigli and Savaré (see [3]).

Before giving the main result we are interested in, we recall the definition of metric derivative, which is a concept that may be useful when studying curves which are valued in generic metric spaces.

**Definition 1.3.1.** Given a metric space $(X, d)$ and a curve $\omega : [0,1] \to X$ we define the *metric derivative* of the curve $\omega$ at time $t$ as the quantity

$$|\omega'|(t) = \lim_{h \to 0} \frac{d(w(t+h), w(t))}{|h|} \tag{1.3.1}$$

provided the limit exists.

As a consequence of Rademacher's theorem it can be seen that for any Lipschitz curve the metric derivative exists at almost every point $t \in [0,1]$. The following theorem guarantees the existence of the metric derivative for Lipschitz curves.

**Theorem 1.3.2.** *Suppose that $\omega : [0,1] \to X$ is Lipschitz continuous, then the metric derivative $|\omega'|(t)$ exists for a.e. $t \in [0,1]$. Moreover we have, for $t < s$,*

$$d(\omega(t), \omega(s)) \leq \int_t^s |\omega'|(\tau) d\tau.$$

Now the goal is to identify the Lipschitz curves in the space $\mathcal{P}_p(\Omega)$ endowed with the distance $W_p$ with the solution of the *continuity equation*

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0,$$

with $L^p$ vector fields $v_t$, and to connect the $L^p$ norm of $v_t$ with the metric derivative $|\mu'|(t)$.

We interpret this continuity equation as the equation which describes the evolution of the density $\mu_t$ of a family of particles initially distributed according to $\mu_0$ and each of them following the flow

$$\begin{cases} y_x'(t) = v(t, y_x(t)) \\ y_x(0) = x. \end{cases} \tag{1.3.2}$$

Now let us see how can we define solutions for this equation.

**Definition 1.3.3.** We say that a family of pairs measures/vector fields $(\mu_t, v_t)$ solves the continuity equation in the distributional sense if for any test function $\phi \in C_c^1((0,1) \times \overline{\Omega})$, compactly supported in time but not necessarily in space, we have

$$\int_0^1 \int_\Omega \partial_t \phi d\mu_t dt + \int_0^1 \int_\Omega \nabla \phi \cdot v_t d\mu_t dt = 0$$

Obviously this formulation includes Neumann boundary conditions on $\partial\Omega$ for $v_t$. If we want to impose the initial and final measures we can say that $(\mu_t, v_t)$ solves the same equation, in the sense of distributions, with initial and final data $\mu_0$ and $\mu_1$, respectively, if for any test function $\phi \in C^1([0,1] \times \overline{\Omega})$, we have

$$\int_0^1 \int_\Omega \partial_t \phi d\mu_t dt + \int_0^1 \int_\Omega \nabla \phi \cdot v_t d\mu_t dt = \int_\Omega \phi(1,x) d\mu_1(x) - \int_\Omega \phi(0,x) d\mu_0(x).$$

On the other hand we can define a weak solution of the continuity equation through the following condition: we say that $(\mu_t, v_t)$ solves the continuity equation in the weak sense if for any test function $\psi \in C^1([0,1]\times\Omega)$, the function $t \mapsto \int \psi d\mu_t$ is Lipschitz continuous and, for a.e. $t$, we have

$$\frac{d}{dt}\int_\Omega \psi d\mu_t = \int_\Omega \nabla\psi \cdot v_t d\mu_t.$$

Notice that in this case $t \mapsto \mu_t$ is automatically continuous for the weak convergence, and imposing the values of $\mu_0$ and $\mu_1$ may be done pointwisely.

We remark that the two notions are actually equivalent: every weak solution is actually a distributional solution and every distributional solution admits a representative (another family $\tilde{\mu}_t = \mu_t$ for a.e. $t$) which is weakly continuous and is a weak solution. The proof of this equivalence it is not so obvious at all, but it can be found in many references about optimal transportation. So due to this equivalence it is just a matter of taste, which notion of solution we will work with.

Now we formulate a characterization theorem of Lipschitz curves.

**Theorem 1.3.4.** *Let $(\mu_t)_{t\in[0,1]}$ be a Lipschitz curve for the distance $W_p$ ($p > 1$). Then for a.e. $t \in [0,1]$ there exists a vector field $v_t \in L^p(\mu_t;\mathbb{R}^d)$ such that*

- *the continuity equation $\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$ is satisfied in the weak sense (see above);*
- *for a.e. $t$ we have $||v_t||_{L^p(\mu_t)} \leq |\mu'|(t)$.*

*On the contrary, if $(\mu_t)_{t\in[0,1]}$ is a family of measures in $\mathcal{P}_p(\Omega)$ and for each $t$ we have a vector field $v_t \in L^p(\mu_t;\mathbb{R}^d)$ with the property $||v_t||_{L^p(\mu_t)} \leq C$, then $(\mu_t)_t$ is actually a Lipschitz curve for the $W_p$ distance and for a.e. $t$ we have $|\mu'|(t) \leq ||v_t||_{L^p(\mu_t)}$.*

Now on we want to give some characterizations to the geodesics in the Wasserstein spaces $\mathcal{P}_p(\Omega.)$ At first we give some definitions in arbitrary metric spaces $(X,d)$.

We can define the length of a curve in a metric space by the usual definition, moreover we know that for a Lipschitz curve $\omega : [0,1] \to X$, we have

$$\text{length}(\omega) = \int_0^1 |\omega'|(t)dt.$$

**Definition 1.3.5.** A curve $\omega : [0,1] \to X$ is said to be a *geodesic* between $x_0$ and $x_1 \in X$ if it minimizes the length among all curves such that $\omega(0) = x_0$ and $\omega(1) = x_1$.

A space $(X,d)$ is said to be a *length space* if it holds

$$d(x,y) = \inf\{\text{length}(\omega) : \omega \in \text{Lip}, \omega(0) = x, \omega(1) = y\}.$$

A space $(X,d)$ is said to be a *geodesic space* if it holds

$$d(x,y) = \min\{\text{length}(\omega) : \omega \in \text{Lip}, \omega(0) = x, \omega(1) = y\},$$

i.e. if it is a length space and there exit geodesics between two arbitrary points.

In a length space, a curve $\omega : [0,1] \to X$ is said to be a *constant speed geodesic* between $\omega(0)$ and $\omega(1) \in X$ if it satisfies

$$d(\omega(t),\omega(s)) = |t-s|d(\omega(0),\omega(1)), \forall t,s \in [0,1].$$

A curve with this property is automatically a geodesic.

**Proposition 1.3.6.** *Fix an exponent $p > 1$ and consider curves connecting $x_0$ and $x_1$. The following three facts are equivalent:*

- *$\omega$ is a constant speed geodesic;*
- *$|\omega'|(t) = d(\omega(0), \omega(1))$ a.e.;*
- *$\omega$ solves $\min \left\{ \int_0^1 |\omega'|(t)^p dt : \omega(0) = x_0, \omega(1) = x_1 \right\}$*

Now we give a link between the constant speed geodesics in $\mathcal{P}_p(\Omega)$ endowed with $W_p$ and the optimal transports.

**Theorem 1.3.7.** *Suppose that $\Omega$ is convex, take $\mu, \nu \in \mathcal{P}_p(\Omega)$ and $\gamma$ an optimal transport plan in $\Pi(\mu, \nu)$ for the cost $|x - y|^p$ $(p \geq 1)$. Define $\pi_t : \Omega \times \Omega \to \Omega$ through $\pi_t(x, y) = (1-t)x+ty$. Then the curve $\mu_t := (\pi_t)_\# \gamma$ is a constant speed geodesic in $\mathcal{P}_p(\Omega)$ connecting $\mu_0 = \mu$ and $\mu_1 = \nu$.*

*In the particular case where $\mu$ is absolutely continuous, or in general when $\gamma = \gamma_T$, this very curve is obtained as $((1 - t)id + tT)_\# \mu$.*

*As a consequence, the space $\mathcal{P}_p(\Omega)$ endowed with the Wasserstein distance $W_p$ is a geodesic space.*

Now let us recall the definitions of geodesic- and displacement convexity.

**Definition 1.3.8.** In an arbitrary metric space $(X, d)$ (actually, it is better if we are in a geodesic space) we can define $F : X \to \mathbb{R} \cup \{+\infty\}$ to be *geodesically convex* if for every two points $x_0, x_1 \in X$ there exists a constant speed geodesic $\omega$ connecting $\omega(0) = x_0$ and $\omega(1) = x_1$ such that $[0, 1] \ni t \mapsto F(\omega(t))$ is convex.

The notion of geodesic convexity in the space $\mathcal{P}_p(\Omega)$ for the distance $W_p$ has been introduced by McCann in [26] and it is particularly interesting since we know how to characterize the geodesics in such a space. This notion of convexity is usually referred to as *displacement convexity*. Notice, that it could a priori depend on the exponent $p$.

Now we present and study some functionals defined on Wasserstein spaces, which will be very important later on, because in many applied problems we face with minimization problems where the unknown to be determined is the distribution of a certain amount of mass, and there will be involved some functionals which we will present. So let us consider the following ones:

- The integral of a given function (potential energy)

$$\mathcal{V}(\mu) = \int V d\mu;$$

- The double integral of a function on $\Omega \times \Omega$ according to the tensor product $\mu \otimes \mu$ (interaction energy)

$$\mathcal{W}(\mu) = \int W(x, y) d\mu(x) d\mu(y);$$

- The Wasserstein distance (or a function of it) from a fixed measure

$$\mathcal{D}(\mu) = W_p(\mu, \nu);$$

- The norm in a dual function space: given a Banach space $X$ of functions on $\Omega$, define

$$||\mu||_{X'} = \sup_{\phi \in X, ||\phi|| \le 1} \int \phi d\mu = \sup_{\phi \in X \setminus \{0\}} \frac{\int \phi d\mu}{||\phi||_X};$$

- The integral of a function of the density

$$\mathcal{F}(\mu) = \begin{cases} \int f(\rho(x))dx & \text{if } \mu = \rho \cdot dx, \\ +\infty & \text{otherwise;} \end{cases}$$

- The sum of a function of the masses of the atomes

$$\mathcal{G}(\mu) = \begin{cases} \sum_i g(a_i) & \text{if } \mu = \sum_i a_i \delta_{x_i}, \\ +\infty & \text{if } \mu \text{ is not purely atomic.} \end{cases}$$

Now we can summarize some results about (semi)continuity and displacement convexity of these functionals.

**Theorem 1.3.9.**

- *If $V \in C_b(\Omega)$ then $\mathcal{V}$ is continuous for the weak convergence of probability measures. If $V$ is l.s.c. and bounded from below than $\mathcal{V}$ is l.s.c. Moreover, semi-continuity of $V$ (respectively, continuity) is necessary for the semi-continuity (continuity) of $\mathcal{V}$.*

- *If $W \in C_b(\Omega)$ then $\mathcal{W}$ is continuous for the weak convergence of probability measures. If $W$ is l.s.c. and bounded from below than $\mathcal{W}$ is semi-continuous.*

- *For any $1 \le p < +\infty$, the Wasserstein distance $W_p(\cdot, \nu)$ for any fixed measure $\nu \in \mathcal{P}(\Omega)$ is continuous w.r.t. weak convergence provided $\Omega$ is compact. If $\Omega$ is not compact and $\nu \in \mathcal{P}_p(\Omega)$, then $W_p(\cdot, \nu)$ is well-defined over $\mathcal{P}_p(\Omega)$ and it is only l.s.c.*

- *Let $X$ be a Banach space of functions over $\Omega$ such that $X \cap C_b(\Omega)$ is dense in $X$. Then*

$$\mu \mapsto ||\mu||_{X'} = \sup_{\phi \in X, ||\phi||_X \le 1} \int \phi d\mu = \sup_{\phi \in X \setminus \{0\}} \frac{\int \phi d\mu}{||\phi||_X}$$

*is l.s.c. for the weak convergence.*

**Proposition 1.3.10.** *Let $f : \mathbb{R} \to \mathbb{R}$ be convex function, with $f(0) = 0$, and set*

$$L := \lim_{t \to +\infty} \frac{f(t)}{t} = \sup_{t > 0} \frac{f(t)}{t} \in \mathbb{R} \cup \{+\infty\}.$$

*Let $\lambda$ be a fixed positive measure on $\Omega$. For every measure $\mu$ write $\mu = \rho \cdot \lambda + \mu_s$, where $\rho \cdot \lambda$ is the absolutely continuous part of $\mu$ and $\mu_s$ be the singular one (w.r.t. $\lambda$). Then, the functional defined through*

$$\mathcal{F}(\mu) = \int f(\rho(x)) d\lambda(x) + L\mu_s(\Omega)$$

*is l.s.c.*

**Lemma 1.3.11.** *Suppose that $g(0) = 0$, $g(t) \geq 0$, $g$ is sub-additive and l.s.c., moreover $\lim_{t\to+\infty} \frac{g(t)}{t} = +\infty$. Then $\mathcal{G}$ is l.s.c.*

**Proposition 1.3.12.**

- *The functional $\mathcal{V}$ is displacement convex if and only if $V$ is convex.*
- *The functional $\mathcal{W}$ is displacement convex if $W$ is convex.*
- *Suppose that $f$ is convex and superlinear, $f(0) = 0$ and $s \mapsto s^d f(s^{-d})$ is convex and non-increasing. Then $\mathcal{F}$ is displacement convex for the distance $W_2$.*

## 1.4 Gradient flows for $W_2$

Now we will give a short presentation of the gradient flow of a functional $F : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$. This functional is supposed to be l.s.c. for the weak convergence of probability measures and we also suppose that $\Omega$ is compact. The main reference of this theory is the textbook [3].

We will use an implicit Euler-like iterated minimization scheme to approach the gradient flows w.r.t. $W_2$. This method is borrowed of course from the Euclidean case, when $F$ is defined on $\mathbb{R}^d$.

So let us consider the discrete-time scheme

$$\rho^\tau_{k+1} \in \text{argmin} F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau},$$

for a fixed time step $\tau > 0$.

Each of these problems has a solution, by the compactness of $\mathcal{P}(\Omega)$ and semi-continuity of the r.h.s.

In order to have further investigations about these problems, we need to introduce a good notion of functional derivatives. Given a functional $G : \mathcal{P}(\Omega) \to \mathbb{R}$, we call $\frac{\delta G}{\delta \rho}(\rho)$ (if it exists), any function such that (they are unique up to additive constants)

$$\frac{d}{d\varepsilon} G(\rho + \varepsilon\chi)|_{\varepsilon=0} = \int \frac{\delta G}{\delta \rho}(\rho) d\chi,$$

for every perturbation $\chi$ s.t. at least for $\varepsilon$ sufficiently small $\rho + \varepsilon\chi \in \mathcal{P}(\Omega)$.

Now we will suppose that $\frac{\delta F}{\delta \rho}(\rho)$ is known, and we try to write the limit equation in terms of this operator. We also need the functional derivative of $\frac{1}{2} W_2^2(\cdot, \nu)$, for a fixed $\nu \in \mathcal{P}(\Omega)$.

**Lemma 1.4.1.** *Let $\nu$ be a fixed probability measure. Consider $\mu \in \mathcal{P}(\Omega)$ such that in the optimal transport from $\mu$ to $\nu$ for the cost $c(x, y) = \frac{1}{2}|x - y|^2$ the Kantorovich potential is unique (up to additive constants). This means that there is only one pair of $(\overline{\phi}, \overline{\psi})$ solving*

$$\max \int \phi d\mu + \psi d\nu : \phi(x_0) = 0, \phi(x) + \psi(y) \leq c(x, y)$$

*where $x_0$ is any fixed point in $\Omega$, so as to get rid of additive constants.*

*Then, for any $\mu_1 \in \mathcal{P}(\Omega)$, setting $\mu_\varepsilon := (1-\varepsilon)\mu + \varepsilon\mu_1$, we have*

$$\frac{d}{d\varepsilon}\left(\frac{1}{2}W_2^2(\mu_\varepsilon, \nu)\right)\Big|_{\varepsilon=0} = \int \overline{\phi} d(\mu_1 - \mu).$$

Now we want to construct perturbations for which we will calculate the the first variation of the functional, which is defined as the r.h.s. of our time discrete scheme. So for this take an optimal measure $\overline{\rho}$ for the minimization problem at step $k$ and construct some perturbations with the help of $\rho_\varepsilon = (1-\varepsilon)\overline{\rho} + \varepsilon\tilde{\rho}$, where $\tilde{\rho}$ is any other probability measure.

Now we choose a perturbation $\chi = \tilde{\rho} - \overline{\rho}$, which guarantees that, for $\varepsilon > 0$, the measure $\rho_\varepsilon$ is actually a probability over $\Omega$.

Let us compute now the first variation and, due to optimality, we have

$$0 \leq \frac{d}{d\varepsilon}\left(F(\overline{\rho} + \varepsilon\chi) + \frac{W_2^2(\overline{\rho} + \varepsilon\chi, \rho_k^\tau)}{2\tau}\right)\Bigg|_{\varepsilon=0} = \int \left(\frac{\delta F}{\delta\rho}(\overline{\rho}) + \frac{\phi}{\tau}\right) d\chi. \tag{1.4.1}$$

If we assume the fact that $\chi$ is an almost arbitrary measure with zero mass (which is not necessarily true, but when we deal with concrete examples we can work more carefully, so without loss of generality we can have this assumption), we deduce from the above condition the very important observation, namely that

$$\frac{\delta F}{\delta\rho}(\overline{\rho}) + \frac{\phi}{\tau}$$

must be constant.

But we know that $T(x) = x - \nabla\phi(x)$ for the optimal $T$, so taking in consideration the observation we made recently, we get

$$\frac{T(x) - x}{\tau} = -\frac{\nabla\phi(x)}{\tau} = \nabla\left(\frac{\delta F}{\delta\rho}(\rho)\right)(x). \tag{1.4.2}$$

We can view the ratio between the displacement and time as velocity, so let us denote $v := -\frac{T(x)-x}{\tau}$, the minus sign is because the displacement is associated to the transport from $\rho_{k+1}^\tau$ to $\rho_k^\tau$.

Now we arrived again to a very important observation, which is the fact that we have $v = -\nabla\left(\frac{\delta F}{\delta\rho}(\rho)\right)$, so this will suggest (and it can be analyzed rigourously) that at the limit $\tau \to 0$ we will find a solution of

$$\partial_t\rho - \nabla\cdot\left(\rho\nabla\left(\frac{\delta F}{\delta\rho}(\rho)\right)\right) = 0.$$

Let us present some well-known examples of this kind of equations. The three main classes of examples are some of the functionals presented in the previous section, and more precisely those considered by McCann in [26].

Consider the functionals $\mathcal{F}, \mathcal{V}, \mathcal{W}$, already presented previously. In the case, where the function $W$ is involved, we study only the simple case when it is depending on the difference $x-y$, and symmetric, i.e. $W(z) = W(-z)$. So in this case it is quite easy to realize that we have

$$\frac{\delta \mathcal{F}}{\delta \rho}(\rho) = f'(\rho), \quad \frac{\delta \mathcal{V}}{\delta \rho}(\rho) = V, \quad \frac{\delta \mathcal{W}}{\delta \rho}(\rho) = 2W * \rho.$$

A very important example is the case when $f(t) = t \ln t$. In such a case we obviously have $f'(t) = \ln t + 1$ and $\nabla(f'(\rho)) = \frac{\nabla \rho}{\rho}$ this means that the gradient flow equation associated to the functional $\mathcal{F}$ would be the *Heat Equation*

$$\partial_t \rho - \Delta \rho = 0,$$

and that for $\mathcal{F} + \mathcal{V}$ we would have the *Fokker-Planck Equation*

$$\partial_t \rho - \Delta \rho - \nabla \cdot (\rho \nabla V) = 0.$$

The case of the interaction functional $\mathcal{W}$ gives a not so elegant non-linear and non-local equation, which is

$$\partial_t \rho - \nabla \cdot (\rho(\nabla W * \rho)) = 0.$$

Finally we remark the fact that if we want uniqueness of the gradient flows, naturally we would need some kind of convexity notion for the functional which is involved in the minimization (this notion could be the geodesic or displacement convexity, or some $\lambda$-geodesic convexity). In PDE applications usually it is a very important step to to prove uniqueness for weak solutions of the continuity equation with the velocity field $v = -\nabla \left(\frac{\delta F}{\delta \rho}\right)$.

## 1.5 Gradient flows with density constraints

In this section we present some remarks about the gradient flow of a special functional in the Wasserstein space. This functional is used in the study of problem arising in the theory of crowd motion with density constraints and we will use some of these ideas in the study of MFG systems with density constraints. One can see the construction of the precise model later on, where it is explained as well where comes from the definition of the functional. We consider at first the set $K := \{\rho \in \mathcal{P}_2(\Omega) : \rho \leq 1\}$, which helps us in the formulation of the constraints.

Let us consider the functional

$$F(\rho) = \int_\Omega D d\rho + I_K(\rho),$$

where

$$I_K(\rho) = \begin{cases} 0, & \text{if } \rho \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here the function $D$ is sufficiently regular, in the special case of crowd motion theory (see for example [24]) the authors wanted to gain information about a motion, where the vector field $u := -\nabla D$ was used.

So for the discrete time scheme for gradient flow of the functional $F$ we fix a time step $\tau > 0$ and consider the minimization scheme as usual:

$$\rho_{k+1}^{\tau} \in \operatorname{argmin}_{\rho \in \mathcal{P}_2(\Omega)} F(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^{\tau}).$$

Now we want to say a few words about the optimality conditions (similarly to (1.4.1)) in terms of $\frac{\delta F}{\delta \rho}$. This is explained also in [32]. So let us denote by $\psi := \frac{\delta F}{\delta \rho} + \frac{\phi}{\tau}$, where $\phi$ is the Kantorovich potential. If we do not consider for the moment the constraint, $\rho \in K$, we have that the optimal $\overline{\rho}$ is concentrated on argmin $\psi = D + \frac{\phi}{\tau}$. This fact is also because $\overline{\rho}$ minimizes also the functional $\rho \mapsto \int \psi d\rho$.

But now, if we want to take in consideration the constraint, i.e. $\rho \in K$, and we know that $K$ is a convex set, we can take a variation of the form $\rho_\varepsilon := (1-\varepsilon)\overline{\rho} + \varepsilon \tilde{\rho}$, for any $\tilde{\rho} \in K$. Here we mention, that we cannot take an arbitrary probability measure for $\tilde{\rho}$.

In this case instead of (1.4.1) we will have:

$$\int \psi d\overline{\rho} \leq \int \psi d\tilde{\rho}, \ \forall \tilde{\rho} \in K.$$

We have here an important remark, namely that for a given function $\psi$ for the minimizers of $\rho \mapsto \int \psi d\rho$ it is sufficient to concentrate $\rho$ on a level set of $\psi$, and put the maximal possible density (which is 1) on it. This implies that there is a constant $l$ such that

$$\overline{\rho} = \begin{cases} 1, & \text{on } \psi < l, \\ 0, & \text{on } \psi > l, \\ \in [0,1], & \text{on } \psi = l. \end{cases}$$

We will see later on, that this also will motivate the introduction of a pressure $p$, and consider the constraint in the term of this pressure.

# Chapter 2

# Some Preliminaries on Optimal Control Theory and Semi-Concavity

We also have seen, that Optimal Control theory is also a very important tool in MFG theory. While the Optimal Transportation theory helps us in the understanding of the Fokker-Planck type equations in a MFG system, the theory of Optimal Control is an indispensable tool to study the Hamilton-Jacobi type equation of the system. We will also see later on, that some semi-concave estimates on the value function are also very important. That is why we spend some time giving some definitions and properties on these subjects. A very good reference, which is exactly what we need at this point is the monograph of P. Cannarsa and C. Sinestrari, [7]. So following this book we give some basic definitions and results on this topic.

## 2.1 Basic definitions and assumptions

**Definition 2.1.1.** [7] A *control system* consists of a pair of $(f, \mathcal{A})$, where $\mathcal{A} \in \mathbb{R}^m$ is a closed set and $f : \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d$ is a continuous function. The set $\mathcal{A}$ is called the *control set*, while $f$ is called the *dynamics* of the system. The *state equation* associated with the system is

$$\begin{cases} y'(t) = f(y(t), \alpha(t)), & t \in [t_0, +\infty) \ a.e., \\ y(t_0) = x, \end{cases} \tag{2.1.1}$$

where $t_0 \in \mathbb{R}, x \in \mathbb{R}^d$ and $\alpha \in L^1_{loc}([t_0, +\infty), \mathcal{A})$. The function $\alpha$ is called a *control strategy* or simply a *control*. We denote the solution of (2.1.1) by $y(\cdot, t_0, x, \alpha)$ and we call is the *trajectory* of the system corresponding to the initial condition $y(t_0) = x$ and to the control $\alpha$.

The word "control" sometimes is used to denote elements $v \in \mathcal{A}$ and sometimes for functions $\alpha : [t_0, +\infty) \to \mathcal{A}$; the meaning should be clear from the context.

We restrict ourselves to autonomous control systems, where $f(x, \alpha)$ is not depending explicitly on $t$, this is done for the sake of simplicity since the results we present can be easily extended to the non-autonomous case and in the studying of MFG systems we usually work with autonomous control systems.

We now list some basic assumptions on our control system which will be needed in most of the results we will present in this chapter. These formulations are borrowed from [7].

(H0) The control set $\mathcal{A}$ is compact.

(H1) There exists $K_1 > 0$ such that $|f(x_2, \alpha) - f(x_1, \alpha)| \leq K_1|x_2 - x_1|$, for all $x_1, x_2 \in \mathbb{R}^d$ and $\alpha \in \mathcal{A}$.

(H2) $\frac{\partial f}{\partial x}$ exists and it is continuous, in addition, there exists $K_2 > 0$ such that

$$\left|\left|\frac{\partial f}{\partial x}(x_2, \alpha) - \frac{\partial f}{\partial x}(x_1, \alpha)\right|\right| \leq K_2|x_2 - x_1|,$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $\alpha \in \mathcal{A}$.

It is well-known fact from the theory of ODEs that assumption $(H1)$ ensures the existence of a unique global solution to the state equation (2.1.1) for any choice of $t_0, x$ and $\alpha$.

An *optimal control problem* consists of choosing the control strategy $\alpha$ in the state equation (2.1.1) in order to minimize a given functional. We will introduce the so-called *Bolza problem*, which is exactly the type of problem, which will arrive in our consideration later on, studying MFG systems.

We are giving a control system $(f, \mathcal{A})$, a function $g \in C(\mathbb{R}^d)$ and a finite time $T > 0$, in addition, a function $L \in C(\mathbb{R}^d \times \mathcal{A})$ is assigned, called *running cost*. The function $g$ is called the *final cost*. For any $(t, x) \in [0, T] \times \mathbb{R}^d$ we consider the functional

$$J_{t,x}(\alpha) = \int_t^T L(y(s), \alpha(s))ds + g(y(T)), \tag{2.1.2}$$

where $y(\cdot) = y(\cdot, t, x, \alpha)$ obtained from the state equation (2.1.1), and we consider the control problem

$$(BP) \qquad \min_{\alpha:L^1([t,T],\mathcal{A})} J_{t,x}(\alpha).$$

We also give some assumption on $L$, which we will need later in some results.

(L1) For any $R > 0$ there exists $c_R > 0$ such that $|L(x_2, \alpha) - L(x_1, \alpha)| \leq c_R|x_2 - x_1|$, for all $x_1, x_2 \in B(0, R), \alpha \in \mathcal{A}$.

(L2) For any $x \in \mathbb{R}^d$ the following set is convex: $\mathcal{L}(x) := \{(\lambda, v) \in \mathbb{R}^{d+1} : \exists \alpha \in \mathcal{A} \text{ s.t. } v = f(x, \alpha), \lambda \geq L(x, \alpha)\}$.

(L3) For any $R > 0$ there exists $\lambda_R > 0$ s.t.

$$L(x, \alpha) + L(y, \alpha) - 2L\left(\frac{x+y}{2}, \alpha\right) \leq \lambda_R|x - y|^2, \forall x, y \in B(0, R), \alpha \in \mathcal{A}.$$

**Definition 2.1.2.** [7] A control $\alpha : [t, T] \to \mathcal{A}$ such that the infimum in $(BP)$ is attained is called *optimal* for problem $(BP)$ with initial point $(t, x)$. The corresponding solution $y(\cdot) = y(\cdot, t, x, \alpha)$ of (2.1.1) is called an *optimal trajectory* or a *minimizer*.

We now present the following result concerning the existence of optimal trajectories for $(BP)$. The main step is contained in the following theorem.

**Theorem 2.1.3.** *[7] Assume that (H0),(H1),(L1) and (L2) hold. Let $\{y_k\}$ be a set of trajectories in* (2.1.1) *in some given interval $[t_0, t_1]$, that is $y_k(\cdot) = y_k(\cdot, t_0, x_k, \alpha_k)$ for some $x_k \in \mathbb{R}^d$ and $\alpha_k : [t_0, t_1] \to \mathcal{A}$. If the trajectories $y_k$ are uniformly bounded, then there exists a subsequence, still denoted by $\{y_k\}$ converging uniformly to an arc $\overline{y}$ which is a trajectory of* (2.1.1) *associated with some control $\overline{\alpha}$ and satisfies*

$$\int_{t_0}^{t_1} L(\overline{y}(s), \overline{\alpha}(s))ds \leq \liminf_{k\to+\infty} \int_{t_0}^{t_1} L(y_k(s)\alpha_k(s))ds.$$

**Theorem 2.1.4.** *[7] Assume that (H0),(H1),(L1) and (L2) hold and that $g$ is continuous. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, there exists an optimal control for problem $(BP)$.*

Let us now consider the case when the control set $\mathcal{A}$ is not bounded (for instance is the whole space $\mathbb{R}^m$) and thus (H0) is not satisfied. In this case we can replace the assumption (H0) with the following ones:

$(H^*)$ There exists $K_0 > 0$ such that

$$|f(x, \alpha)| \leq K_0(1 + |x| + |\alpha|), \forall x \in \mathbb{R}^d, \alpha \in \mathcal{A}.$$

$(L^*)$ There exists $l_0 \geq 0$ and a function $l : [0, +\infty) \to [0, +\infty)$ with $\lim_{r\to\infty} \frac{l(r)}{r} = +\infty$ and such that

$$L(x, \alpha) \geq l(|\alpha|) - l_0, \forall x \in \mathbb{R}^d, \alpha \in \mathcal{A}.$$

Now we have the following result:

**Theorem 2.1.5.** *[7] Let $(f, \mathcal{A})$ be a control system satisfying $(H^*)$ and (H1). Let $T > 0$, let $g$ be locally Lipschitz and bounded from below and let $L \in C(\mathbb{R}^d \times \mathcal{A})$ satisfying $(L^*)$ and (L1). Then, for any $R > 0$ there exists $\mu_R > 0$ with the following property: given $(t, x) \in \mathbb{R}^d \times B(0, R)$, if we set $\mathcal{M}_R := \{\alpha : [t, T] \to \mathcal{A} : ||\alpha||_\infty \leq \mu_R\}$ we have*

$$\inf_{\alpha\in L^1([t,T])} J_{t,x}(\alpha) = \inf_{\alpha\in\mathcal{M}_R} J_{t,x}(\alpha).$$

The previous result shows that the control problem under consideration is equivalent to one with compact control space. Thus we can obtain the following result.

**Theorem 2.1.6.** *[7] Let the hypotheses of the previous theorem be satisfied and let (L2) also hold. Then there exists an optimal control for the problem $(BP)$ for any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$.*

## 2.2 The value function and a Hamilton-Jacobi equation

Now let us introduce the value function of the Bolza problem.

**Definition 2.2.1.** [7] Given $(t,x) \in [0,T] \times \mathbb{R}^d$, we define

$$V(t,x) = \inf_{\alpha \ measurable} J_{t,x}(\alpha).$$

The function $V$ is called the *value function* of the control problem $(BP)$.

One can prove that the value function satisfies the *dynamic programming principle*: for any given $(t,x) \in (0,T)\mathbb{R}^d$ and $s \in [t,T]$ we have

$$V(t,x) = \inf_{\alpha:[t,s]\to\mathcal{A}} V(s, y(s,t,x,\alpha)) + \int_t^s L(y(\tau,t,x,\alpha), \alpha(\tau))d\tau. \tag{2.2.1}$$

Now we can have some regularity properties for the value function, under some assumptions.

**Theorem 2.2.2.** *[7] Let our control system satisfy the assumptions (H0),(H1), (L1) and let $g \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$. Then $V \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{R}^d)$.*

**Theorem 2.2.3.** *[7] Let the assumptions (H0), (H1), (H2), (L1), (L3) be satisfied and let $g$ be locally semi-concave with linear modulus in $\mathbb{R}^d$. Then $V$ is locally semi-concave with linear modulus in $[0,T] \times \mathbb{R}^d$.*

Analogous results hold when the control space is not necessarily compact but the running cost is super-linear w.r.t. $\alpha$, more precisely

**Theorem 2.2.4.** *[7] Let $(f, \mathcal{A})$ be a control system satisfying ($H^*$) and (H1).*

*(i) Let $g$ be locally Lipschitz and bounded from below, and let $L \in C(\mathbb{R}^d \times \mathcal{A})$ satisfying (L1) and ($L^*$). Then the value function $V$ for $(BP)$ is locally Lipschitz.*

*(ii) Suppose, in addition, that $g$ is locally semi-concave with linear modulus, $f$ satisfies $(H2)$ and $L$ satisfies the following: for any $R > 0$, there exists $\lambda_R$ such that*

$$L(x,\alpha) + L(y,\alpha) - 2L\left(\frac{x+y}{2}, \alpha\right) \le \lambda_R |x-y|^2, \forall x,y \in B(0,R), \alpha \in \mathcal{A} \cap B(0,R).$$

*Then $V$ is locally semi-concave with linear modulus in $[0,T] \times \mathbb{R}^d$.*

The value function of a Bolza problem also satisfies a suitable Hamilton-Jacobi equation in viscosity sense. The Hamiltonian in this case is defined as

$$H(x,p) = \max_{\alpha\in\mathcal{A}} \left(-p \cdot f(x,\alpha) - L(x,\alpha)\right).$$

Then we have the following result.

**Theorem 2.2.5.** *[7] Under the hypotheses of Theorem 2.2.2 and Theorem 2.2.4(i), the value function $V$ is a viscosity solution of the problem*

$$\begin{cases} -\partial_t V(t,x) + H(x, \nabla V(t,x)) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ V(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.2.2}$$

Now we can formulate a very important theorem of the control theory w.r.t. the Bolza problem, namely the Pontryagin maximum principle formulated for a compact $\mathcal{A}$, but which can be extended, as we have seen already.

**Theorem 2.2.6.** *[7] Let $f$ and $L$ satisfy the hypotheses (H0), (H1), (L1) and let $g$ be continuous. Suppose in addition that $\frac{\partial f}{\partial x}$ and $\frac{\partial L}{\partial x}$ exist and are continuous w.r.t. $x$. Given $(t,x) \in [0,T] \times \mathbb{R}^d$, let $\alpha : [t,T] \to \mathcal{A}$ be an optimal control for problem $(BP)$ with initial point $(t,x)$ and let $y(\cdot) = y(\cdot, t, x, \alpha)$ be the corresponding optimal trajectory. For a given $q \in D^+ g(y(t))$ (the set of all super-differentials of $g$ in $y(T)$), let $p : [t,T] \to \mathbb{R}^d$ be a solution of the equation*

$$\begin{cases} p'(s) = -\frac{\partial f}{\partial x}^t (y(s), \alpha(s)) p(s) - \frac{\partial L}{\partial x}(y(s), \alpha(s)), & s \in [t,T] \text{ a.e.} \\ p(T) = q. \end{cases} \tag{2.2.3}$$

*Then, $p(s)$ satisfies, for $s \in [t,T]$ a.e.*

$$-f(y(s), \alpha(s)) \cdot p(s) - L(y(s), \alpha(s)) \geq -f(y(s), v) \cdot p(s) - L(y(s), v),$$

*for all $v \in \mathcal{A}$. In addition*

$$p(s) \in \nabla^+ V(s, y(s)), \forall s \in [t,T].$$

*Remark* 2.2.7. Here we denoted by $\frac{\partial f}{\partial x}^t$ the transpose of the Jacobian matrix $\frac{\partial f}{\partial x}$.

Throughout this chapter we used the usual definition of the semi-concavity, which we can formulate and characterize in the following way.

**Definition 2.2.8.** [7] Let $A \in \mathbb{R}^d$ be an open set. We say that a function $u : A \to \mathbb{R}$ is *semi-concave* with linear modulus if it is continuous in $A$ and there exists $C \geq 0$ such that

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2,$$

for all $x, h \in \mathbb{R}^d$ such that $[x-h, x+h] \subset A$. The constant $C$ is called the *semi-concavity* constant for $u$ in $A$.

**Proposition 2.2.9.** *[7] Given $u : A \to \mathbb{R}$, with $A \subset \mathbb{R}^d$ open, convex and given $C \geq 0$, the following properties are equivalent:*

*(a) $u$ is semi-concave with a linear modulus in $A$ and with semi-concavity constant $C$;*

*(b) $u$ satisfies*

$$\lambda u(x) + (1-\lambda) u(y) - u(\lambda x + (1-\lambda) y) \leq C \frac{\lambda(1-\lambda)}{2} |x-y|^2,$$

*for all $x, y$ such that $[x,y] \subset A$ and $\lambda \in [0,1]$;*

*(c)* *the function* $x \mapsto u(x) - \frac{C}{2}|x|^2$ *is concave in* $A$;

*(d)* *there exist two functions* $u_1, u_2 : A \to \mathbb{R}$ *such that* $u = u_1 + u_2$, *where* $u_1$ *is concave,* $u_2 \in C^2(A)$ *and satisfies* $||\nabla^2 u_2||_\infty \leq C$;

*(e)* *for any* $\nu \in S^{d-1}$ *we have* $\frac{\partial^2 u}{\partial \nu^2} \leq C$ *in* $A$ *in the sense of distributions, that is*

$$\int_A u(x) \frac{\partial^2 \phi}{\partial \nu^2}(x) dx \leq C \int_A \phi(x) dx, \quad \forall \phi \in C_0^\infty(A), \phi \geq 0;$$

*(f)* $u$ *can be represented as* $u(x) = \inf_{i \in I} u_i(x)$, *where* $\{u_i\}_{i \in I}$ *is a family of functions of* $C^2(A)$ *such that* $||\nabla^2 u_i||_\infty \leq C$ *for all* $i \in I$.

# Chapter 3

# Existence and Uniqueness for a Standard First Order MFG System

In this chapter we want to present briefly how can we obtain existence and uniqueness results for a standard first order MFG system (i.e. without any density constraints). Basically what we will do is to show the mains ideas which were used in the proofs of Theorem 0.2.1 and Theorem 0.2.2. These are contained in a very nice and detailed way in [8], so we borrow these ideas from this reference, moreover we recommend this reference for a better insight to this theory.

The basic idea behind all the machinery which is needed in the proof of the Theorem 0.2.1 (i.e. proving existence result in the case of the system (0.2.1)) is to split the system in two part. First to take the unknown density from the system as a given quantity and study (just) the Hamilton-Jacobi equation (show the existence and uniqueness of a solution). Secondary to prove existence and uniqueness of a solution of the continuity equation for a given value function. And finally to "glue" these two information together, in the following sense: define an abstract operator, which for example for an input density creates through the unique solution of the Hamilton-Jacobi equation (with this given input density), an output, which is the unique solution of the continuity equation (with the recently got value function as a given quantity). This operator will be well-defined, because of the existence and uniqueness results in the two little problems, hence it remains just to prove some continuity property of the operator and use a fixed point theorem (for example Schauder's) and we are done with the existence.

For the uniqueness result in the case of (0.2.1) (i.e. Theorem 0.2.2) we just have to use some monotonicity assumptions, which were mentioned in the Introduction. We will give some details later on.

## 3.1 Existence results

At fist let us consider the Hamilton-Jacobi equation of the system (0.2.1).

$$\begin{cases} -\partial_t u + \frac{1}{2}|\nabla_x u|^2 = f(t,x), & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = g(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{3.1.1}$$

We have seen a more or less detailed theory on this type of equations in the previous chapter (see also [7]) and as we have seen, the most fundamental regularity property of the solutions of this equation is semi-concavity.

So we formulate a result regarding the solution of this problem.

**Lemma 3.1.1** ([8])**.** *For any $C > 0$ there is a constant $C_1 = C_1(C)$ such that is $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and such that*

$$||f(t,\cdot)||_{C^2} \leq C, \forall t \in [0,T], \quad ||g||_{C^2} \leq C,$$

*then the equation* (3.1.1) *has a unique bounded continuous viscosity solution which is given by the formula:*

$$u(t,x) = \inf_{\alpha \in L^2([t,T],\mathbb{R}^d)} \int_t^T \frac{1}{2}|\alpha(s)|^2 + f(s,x(s))ds + g(x(T)), \tag{3.1.2}$$

*where $x(s) = x + \int_t^s \alpha(\tau)d\tau$. Moreover $u$ is Lipschitz continuous and satisfies*

$$||\nabla_{x,t}u||_\infty \leq C_1 \quad \text{and} \quad \nabla^2_{xx}u \leq C_1 Id,$$

*where the last inequality holds in the sense of distributions.*

For a point $(t,x) \in [0,T) \times \mathbb{R}^d$ as before in the previous chapter we denote by $\mathcal{A}(t,x)$ the set of optimal controls in the problem (3.1.2). It is a well-known result from optimal control theory that this set is nonempty, moreover if $(t_n, x_n) \to (t,x)$ and $\alpha_n \in \mathcal{A}(t_n,x_n)$ then up to a subsequence $(\alpha_n)$ weakly converges in $L^2$ to some $\alpha \in \mathcal{A}(t,x)$.

We can formulate now the next result.

**Lemma 3.1.2** ([8])**.** *Let $(t,x) \in [0,T] \times \mathbb{R}^d$, $\alpha \in \mathcal{A}(t,x)$ and let us set $x(s) = x + \int_t^s \alpha(\tau)d\tau$. Then the following statements are true:*

- *(i) For any $s \in (t,T]$, the restriction of $\alpha$ to $[s,T]$ is the unique element of $\mathcal{A}(s,x(s))$;*
- *(ii) $\nabla_x u(t,x,)$ exists if and only if $\mathcal{A}(t,x)$ is reduced to a singleton. In this case, $\nabla_x u(t,x) = -\alpha(t)$, where $\mathcal{A}(t,x) = \{\alpha\}$.*

We also can formulate a “reverse” result in the following lemma:

**Lemma 3.1.3** ([8])**.** *Let $(t,x) \in [0,T) \times \mathbb{R}^d$ and $x(\cdot)$ be an absolutely continuous solution to the differential equation*

$$\begin{cases} x'(s) = -\nabla_x u(s,x(s)), & \text{a.e. in } [t,T], \\ x(t) = x. \end{cases} \tag{3.1.3}$$

*Then the control $\alpha := x'$ is optimal for $u(t,x)$. Moreover if $u(t,\cdot)$ is differentiable at $x$, then the equation* (3.1.3) *has a unique solution, corresponding to the optimal trajectory.*

We remark here that the graph of $(t,x) \mapsto \mathcal{A}(t,x)$ is weakly closed in $L^2([0,T],\mathbb{R}^d)$. From this we can conclude that this map is measurable with nonempty closed values, so it has a Borel measurable selection $\overline{\alpha}$, more precisely $\overline{\alpha}(t,x) \in \mathcal{A}(t,x), \forall (t,x)$. This motivates the definition of the flow

$$\Phi(x,t,s) = x + \int_t^s \overline{\alpha}(t,x)(\tau)d\tau, \quad \forall s \in [t,T].$$

We have the following result for this recently defined flow:

**Lemma 3.1.4** ([8])**.** *The flow $\Phi$ has the semi-group property, i.e.*

$$\Phi(x,t,s') = \Phi(\Phi(x,t,s),s,s'), \quad \forall t \leq s \leq s' \leq T.$$

*Moreover it satisfies*

$$\partial_s \Phi(x,t,s) = -\nabla u(s,\Phi(x,t,s)), \quad \forall x \in \mathbb{R}^d, s \in (t,T),$$

*and*

$$|\Phi(x,t,s') - \Phi(s,t,s)| \leq ||\nabla u||_{L^\infty}|s'-s|, \quad \forall x \in \mathbb{R}^d, \forall t \leq s \leq s' \leq T.$$

Now our aim is to present some existence and uniqueness result for the solutions of the continuity equation. So if it is given the unique solution of (3.1.1), we study the continuity equation

$$\begin{cases} \partial_t \mu(s,x) - \nabla \cdot (\nabla_x u(s,x)\mu(s,x)) = 0, & \text{in } (0,T)\times\mathbb{R}^d \\ \mu(0,x) = m_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{3.1.4}$$

**Theorem 3.1.5** ([8])**.** *Given a solution $u$ to* (3.1.1) *and under the assumptions of Theorem 3.1.1 the problem* (3.1.4) *has a unique weak solution given by the formula*

$$s \mapsto \mu(s) := \Phi(\cdot,0,s)_{\#}m_0.$$

Now we present a stability result which will be needed in the construction of the fixed point scheme. Consider a sequence $(m_n)$ in $C([0,T],\mathcal{P}_1)$ which uniformly converges to $m \in C([0,T],\mathcal{P}_1)$ and let $u_n$ be the solution of

$$\begin{cases} -\partial_t u_n + \frac{1}{2}|\nabla_x u_n|^2 = F(x,m_n(t)), & \text{in } (0,T)\times\mathbb{R}^d, \\ u_n(T,x) = g(x,m_n(T)), & \text{in } \mathbb{R}^d, \end{cases} \tag{3.1.5}$$

and $u$ let be the solution of

$$\begin{cases} -\partial_t u + \frac{1}{2}|\nabla_x u|^2 = F(x,m(t)), & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = g(x,m(T)), & \text{in } \mathbb{R}^d. \end{cases} \tag{3.1.6}$$

Furthermore we denote by $\Phi_n$ the flow associated to $u_n$ and $\Phi$ the flow associated to $u$ and set $\mu_n(s) := \Phi_n(\cdot,0,s)_{\#}m_0$ and $\mu(s) := \Phi(\cdot,0,s)_{\#}m_0$. In this setting we have the following result:

**Lemma 3.1.6** ([8])**.** *The solution $u_n$ locally uniformly converges to $u$ in $[0,T]\times\mathbb{R}^d$ while $\mu_n$ converges to $\mu$ in $C([0,T],\mathcal{P}_1)$.*

Now we are able to obtain an existence result of the solution of the system (0.2.1), i.e. proving the Theorem 0.2.1.

*Proof of Theorem 0.2.1.* Let us consider $K$ the convex subset of maps $m \in C([0,T], \mathcal{P}_1)$ such that $m(0) = m_0$. Let us consider the following operator on $K$ : to every $m \in K$ let us associate the unique solution $u$ of the problem (3.1.6), and to this solution $u$ let us associate the unique solution of (3.1.4). Then $\mu \in K$, so the operator is well-defined and from the above mentioned stability result this is also continuous.

Moreover we also know from [8] that there exists a constant $R > 0$, independent of $m$, such that for any $s \in [0,T]$, $\mu(s)$ has support in $B(0,R)$ and satisfies

$$W_1(\mu(s), \mu(s')) \leq R|s - s'|, \quad \forall s, s' \in [0,T].$$

This fact will imply that the above defined operator $m \mapsto \mu$ is compact because $s \mapsto \mu(s)$ is uniformly Lipschitz continuous with values in the compact set of probability measures $\mathcal{P}_1(B(0,R))$.

So at this point we are able to use Schauder's fixed point theorem, which says that any continuous map defined on a convex, compact set of a Banach space has a fixed point. By this result we concluded the proof of the theorem, i.e. the existence of a solution of the first order MFG system. □

## 3.2 Uniqueness results

Now having in mind the assumptions of the Theorem 0.2.2, i.e.

$$\int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2))\, d(m_1 - m_2) > 0, \forall m_1, m_2 \in \mathcal{P}_1, m_1 \neq m_2, \tag{3.2.1}$$

and

$$\int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))\, d(m_1 - m_2) \geq 0, \forall m_1, m_2 \in \mathcal{P}_1. \tag{3.2.2}$$

we will prove the uniqueness of a solution of the standard firts order MFG system, (0.2.1).

*Proof of the Theorem 0.2.2.* We will work with a contradiction argument, so assume that there exist two solutions $(u_1, m_1)$ and $(u_2, m_2)$ of the system (0.2.1). Let us define $\overline{u} := u_1 - u_2$ and $\overline{m} = m_1 - m_2$. With the help of these notations we can write the Hamilton-Jacobi and the continuity equations satisfied by $\overline{u}$ and $\overline{m}$. These are:

$$-\partial_t \overline{u} + \frac{1}{2}\left(|\nabla_x u_1|^2 - |\nabla_x u_2|^2\right) - (F(x, m_1) - F(x, m_2)) = 0, \tag{3.2.3}$$

and

$$-\partial_t \overline{m} - \nabla \cdot (m_1 \nabla_x u_1 - m_2 \nabla_x u_2) = 0. \tag{3.2.4}$$

Now we write the weak formulations of these equations, starting with the second one. We use as test function $\overline{u}$ (which we can assume that it is of class $C^1$) in equation (3.2.4), hence we multiply the equation and integrate w.r.t. time and space. We have

$$-\int_{\mathbb{R}^d} \overline{m}(T)\overline{u}(T) + \int_{\mathbb{R}^d} \overline{m}(0)\overline{u}(0) + \int_0^T \int_{\mathbb{R}^d} (\partial_t \overline{um} - \nabla_x \overline{u} \cdot (m_1 \nabla_x u_1 - m_2 \nabla_x u_2)) = 0.$$

Now we multiply the equation (3.2.3) by $\overline{m}$ and integrate over time and space, so we have

$$\int_0^T \int_{\mathbb{R}^d} \left( -\partial_t \overline{um} + \frac{\overline{m}}{2} \left( |\nabla_x u_1|^2 - |\nabla_x u_2|^2 \right) - \overline{m}(F(x, m_1) - F(x, m_2)) \right) = 0.$$

Now we assume that $\overline{m}(0) = 0$ and add the two weak formulations. After simplifications we get

$$\begin{aligned} &-\int_{\mathbb{R}^d} \overline{m}(T)(G(x, m_1(T)) - G(x, m_2(T))) + \int_0^T \int_{\mathbb{R}^d} \left( \frac{\overline{m}}{2} \left( |\nabla_x u_1|^2 - |\nabla_x u_2|^2 \right) \right) \\ &- \int_0^T \int_{\mathbb{R}^d} (\overline{m}(F(x, m_1) - F(x, m_2)) + \nabla_x \overline{u} \cdot (m_1 \nabla_x u_1 - m_2 \nabla_x u_2)) = 0 \end{aligned}$$

We remark here that we have by assumption that

$$\int_{\mathbb{R}^d} \overline{m}(T)(G(x, m_1(T)) - G(x, m_2(T))) \geq 0,$$

moreover we have the formula

$$\frac{\overline{m}}{2} \left( |\nabla_x u_1|^2 - |\nabla_x u_2|^2 \right) - \nabla_x \overline{u} \cdot (m_1 \nabla_x u_1 - m_2 \nabla_x u_2) = -\frac{m_1 + m_2}{2} |\nabla_x u_1 - \nabla_x u_2|^2.$$

These statements will imply that

$$\int_0^t \int_{\mathbb{R}^d} \overline{m}(F(x, m_1) - F(x, m_2)) \leq 0,$$

which together with the assumption (3.2.1) will imply that $\overline{m} = 0$, so $m_1 = m_2$. But this will imply that $u_1 = u_2$, because they solve the equation for the same measure. So finally we have the uniqueness result we wanted. □

# Chapter 4

# An Approach of MFG with Density Constraints

In this chapter we actually give a model for Mean Field Games with density constraints in the deterministic case. This problem arises from the "congestion" case, where in the cost that agents optimize there is a penalization for passing through zones with high density of other agents. This is a natural problem in real life, because the agents do not want to go to places, where there is already a huge number of other agents.

This model, we will present is due to Filippo Santambrogio, and can be found in [30]. Although in this paper the model is presented in a very detailed way, for the completeness we will present it shortly.

The idea of this model comes from some recent works about crowd motion, where the density constraints were treated in terms of projections of the velocity field onto the set of admissible velocities (with a constraint on the divergence) and a pressure field was introduced. In the paper [30] it was written the corresponding system of PDEs and also proposed some open problems. In what it follows, we would like to understand this model and answer some of the open problems regarding mainly to the existence and uniqueness of the solutions.

Now we will present some results about the crowd motion with density constraints and present the construction of the MFG system with this constraints, i.e. the adaptation of the ideas from crowd motion.

## 4.1 Crowd motion with density constraints

In the usual models for pedestrian motion we can use the terminology of "soft congestion", which is the description of the natural fact that people slow down when the density of the crowd around them is too high. Here is also very natural to say that their speed is (or influenced by) a decreasing function of the density $\rho$.

Maury and Venel in [23] presented another model where the idea is that particles can move as they want as far as they are not too dense, but, if a density constraint is fulfilled, there velocity field $u$ will be modified into another field $v$, which is less concentrating and usually slower. Here comes the main assumption of the model, which we will somehow also use in our MFG model with density constraints, i.e. the new field $v$ is the projection of $u$ onto the cone of admissible velocities. These velocities are those that infinitesimally preserve the constraints. The authors in [23] were considering a discrete (microscopic) case, where individuals are represented by small disks. The density constraint is interpreted as a non-superposition constraint: particles cannot overlap, but as soon as they are not in touch their motion is unconstrained. When they touch, the set of admissible velocities restricts to those that increase the distance of every pair of particles in contact.

Later on, a continuous (macroscopic) model has also been established in [24] (see also [25] for the latest developments and a micro-macro comparison).

The studied model is the following:

- The population of the particles is described by a probability measure (here actually we can identify probability measures with their densities, because we are working only with absolutely continuous measures) $\rho \in \mathcal{P}(\Omega)$;
- The non-overlapping constraint is replaced by the condition $\rho \in K := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$;
- For every time $t$, we consider $u_t : \Omega \to \mathbb{R}^d$ a vector field, possibly depending on $\rho$. We remark here that this scheme is more general, than the one briefly discussed in the Section 1.5, because while there we took for the vector field the opposite of the gradient of a (semi-convex) function, here we take an arbitrary vector field, possibly depending on $\rho$;
- For every density $\rho$ we have a set of admissible velocities, characterized by the sign of the divergence on the saturated region $\{\rho = 1\}$, so the set is: $adm(\rho) := \left\{v : \Omega \to \mathbb{R}^d : \nabla \cdot v \geq 0 \text{ on } \{\rho = 1\}\right\}$;
- We consider the projection operator $P$, which is either the projection in $L^2(\mathcal{L}^d)$ or in $L^2(\rho)$ (this will turn out to be the same, since the only relevant zone is $\{\rho = 1\}$);
- Finally we solve the equation

$$\partial_t \rho_t + \nabla \cdot \left(\rho_t P_{adm(\rho_t)}[u_t]\right) = 0, \tag{4.1.1}$$

  in the weak sense, which definition was already given.

One can show the existence of a solution of this problem by a discrete scheme, taking a fixed positive time step $\tau > 0$ and consider the sequence $\rho_0^\tau := \rho_0$ and for $k \geq 0$ define $\tilde{\rho}_{k+1}^\tau = (id + \tau u)_{\#}\rho_k$ and finally $\rho_{k+1}^\tau = P_K[\tilde{\rho}_{k+1}^\tau]$. By this construction letting $\tau \to 0$ we will have existence of a solution in the described model.

But the main difficulty solving explicitly (4.1.1) even in the weak sense, is that the vector field $v_t = P_{adm(\rho_t)}[u_t]$ is nor regular at all, because it was obtained as an $L^2$ projection, and may only be expected $L^2$ a priori, neither depends regularly on $\rho$.

By duality arguments we can modify the set of admissible velocities, and due to this modification the problem will be more easier to handle.

$$adm(\rho) = \left\{v \in L^2(\rho) : \int v \cdot \nabla p \leq 0 \ \forall p \in H^1(\Omega) : p \geq 0, p(1-\rho) = 0\right\}.$$

In this sense we characterize $v = P_{adm(\rho)}[u]$ through

$$u = v + \nabla p, \; v \in adm(\rho), \; \int v \cdot \nabla p = 0,$$

$$p \in press(\rho) := \left\{ p \in H^1(\Omega) : p \geq 0, p(1-\rho) = 0 \right\},$$

where $press(\rho)$ is the space of functions $p$ used as test functions in the dual definition of $adm(\rho)$. They play the role of pressures in the movement. The two cones $\nabla press(\rho)$ and $adm(\rho)$ are orthogonal to each other and that is why this allows the orthogonal decomposition of $u = v + \nabla p$.

So we can write our continuity equation in the form:

$$\partial_t \rho + \nabla \cdot (\rho(u - \nabla p)) = 0, \; p \geq 0, \; p(1-\rho) = 0. \tag{4.1.2}$$

We remark here the fact that the orthogonality condition $\int v \cdot \nabla p = 0$ is no more necessary and is a consequence of the continuity equation. This may be informally seen in the following way. Fix a time $t_0$ and notice that

$$\int p_{t_0} d\rho_{t_0} = \int p_{t_0} \geq \int p_{t_0} d\rho_t,$$

which means that the function $t \mapsto \int p_{t_0} d\rho_t$ is maximal at $t = t_0$. By differentiating and using the definition of the weak solution of the continuity equation, we get

$$0 = \int \nabla p_{t_0} \cdot v_{t_0} d\rho_{t_0} = \int \nabla p_{t_0} \cdot v_{t_0},$$

(however this proof is only formal because nothing guarantees that $t \mapsto \int p_{t_0} d\rho_t$ is differentiable at $t = t_0$, but this can be fixed and it is possible to obtain $\int \nabla p_{t_0} \cdot v_t = 0$ for a.e. $t$).

Because of the lack of regularity of $v$ the main tools to prove at least existence of a solution of the continuity equation lie in the theory of Optimal Transportation and Wasserstein distances (and gradient flows for $W_2$).

An interesting point here is that the constraint $\rho \in K$ may be seen as a limit of $L^m$ penalization as $m \to +\infty$, however it is pointed out in [30], that this method of density penalization is hard to adapt to MFG systems with density constraints. Later on we will present some other alternative methods to handle this problem.

## 4.2 The MFG system with density constraints

In [30] we can see how is constructed a MFG system with density constraints using the approach with the pressure, presented in the previous section. So we recall the construction of this model.

Actually we will study a first order deterministic problem, similar to the one presented in the Introduction (0.2.1). Now we will give the expression of the effect of the density constraints through the existence of a non-zero pressure field. This of course requires changing the definition of the equilibrium.

The model is as follows:

- The situation is described through a pair $(\rho, \overline{\alpha})$, where $\rho_t$ stands for the density of the agents at time $t$ and $\overline{\alpha}(t,x)$ for the effort that the agent located at $x$ at time $t$ makes to control his movement. We require that $(\rho, \overline{\alpha})$ satisfies

$$\partial_t \rho_t + \nabla \cdot (\rho_t (P_{adm(\rho_t)}[\overline{\alpha}_t])) = 0$$

which means that $\rho$ is convected by the projection of this effort vector field.

- Considering the projection of $\overline{\alpha}_t$ onto $adm(\rho_t)$ a pressure $p_t$ appears such that

$$P_{adm(\rho_t)}[\overline{\alpha}_t] = \overline{\alpha}_t - \nabla p_t, \;\; p_t \in press(\rho_t), \;\; \int P_{adm(\rho_t)}[\overline{\alpha}_t] \cdot \nabla p_t = 0$$

(but again, the orthogonality condition in included in the continuity equation).

- Every agent wants to optimize his own control problem, where the state equation is influenced by the pressure:

$$\min \int_t^T \frac{|\alpha(s)|^2}{2} ds + G(y(T)) : \;\; y'(s) = \alpha(s) - \nabla p_s(y(s)), y(t) = x.$$

- The configuration $(\rho, \overline{\alpha})$ is said to be an equilibrium if the original effort field $\overline{\alpha}$ coincides with the optimal effort in this control problem and if the original densities $\rho_t$ coincide with the densities realized at time $t$ according to these optimal trajectories.

We compute the Hamiltonian of this control problem and after this we will write down the system which is associated to such an equilibrium.

$$H(t,x,\xi) = \sup_{\alpha} \left( -\xi \cdot (\alpha - \nabla_x p(t,x)) - \frac{|\alpha|^2}{2} \right) = \frac{|\xi|^2}{2} + \xi \cdot \nabla_x p(t,x),$$

the optimal $\alpha$ in this maximization being exactly $-\xi$. This means that the optimal effort at $(t,x)$ will be $-\nabla_x u(t,x)$ (the gradient of the value function) and that the vector field appearing in the continuity equation will be $-\nabla_x u - \nabla_x p$. We finally get our MFG system

$$\begin{cases} (i) & -\partial_t u + \frac{|\nabla_x u|^2}{2} - \nabla_x u \cdot \nabla_x p = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\ (ii) & \partial_t \rho - \nabla \cdot (\rho(\nabla_x u + \nabla_x p)) = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\ (iii) & p \geq 0, p(1-\rho) = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\ (iv) & u(T,x) = G(x), \;\; \rho(0,x) = \rho_0(x), & \text{in } \mathbb{R}^d, \end{cases} \tag{4.2.1}$$

where the first equation is understood in viscosity sense, while the second in distributional sense.

At this point it is very important to remark that the density only is not sufficient to describe the configuration, so the equilibrium may only be defined in terms of a pair $(\rho, \overline{\alpha})$. This is because of the fact that the pressure at time $t$ does not depend on the density $\rho_t$ only, but on the vector field that we project onto $adm(\rho_t)$. This is the main difference between this approach and the gradient flow framework presented for crowd motion in [24], where the vector field to be projected was itself a function of $\rho$.

## 4.3 Towards to existence results with density constraints

We mentioned already that in [30] there were proposed many open questions regarding to the given first order MFG system with density constraints, presented in the previous chapter. Actually by now, we do not know any results for example considering existence and uniqueness of a solution of the presented system.

Our main aim in this chapter is to give some positive answers to these questions. At first we will not follow the way described in [30], i.e. the method of density penalization, which guarantee that the constraints will be fulfilled, but we try to build in the constraints in a good constructed fixed point scheme (somehow similar to the scheme presented in [8]) which will ensure the existence of a solution, moreover the satisfaction of the constraints.

*Remark* 4.3.1. We mention here that for the simplicity we will denote later on by $\nabla f = \nabla_x f$, the gradient of a function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ with respect to the space variable.

### 4.3.1 A fixed point method for the existence

As we said it before, to show the existence of a solution of (4.2.1) we would like to use a fixed point argumentation. Let us describe the idea.

- As an initial step we take a pair $(\rho, \overline{\alpha})$, where $\rho$ satisfies the density constraint and $\overline{\alpha}$ is an optimal control for our control problem (which will exist under the correct assumptions). Now from this pair we will define a pressure (more precisely the gradient of the pressure) using the projection operator onto the space $adm(\rho)$.
- With the obtained pressure we solve the Hamilton-Jacobi equation with the final state condition ($(i)$ from (4.2.1)), and we get a (unique) value function $u$.
- Now we would like to show the existence and uniqueness of a pair $(\rho, p)$ which solves the continuity equation equipped with the initial condition, moreover where $\rho$ satisfies the density constraint defined with the help of the pressure $p$ ($(ii), (iii)$ from (4.2.1)).
- As next step we define an abstract operator, which associates to this pair $(\rho, p)$ the unique pair, what we get after another loop (now we take the obtained $p$ as the initial pressure when we solve the Hamilton-Jacobi equation).
- As last step we have to prove some properties of this operator (continuity, etc.) and of the space, on which it acts in order to use some fixed point theorem (for example Schauder's), which will give us the existence of the density $\rho$ saturating also the constraint. The existence of $u$ will be a consequence.

So this is the idea of proving the existence. Of course we have to prove each intermediate steps rigorously.

*Remark* 4.3.2. As we mentioned in Chapter 2, we use the same notation $\mathcal{A}$ for a function space, and also a subset of the Euclidean space of controls, but this will be clear in the context.

Now we want to do each step in the proof of the existence algorithm precisely. For this at first we study the Hamilton-Jacobi equation with the final state condition, i.e.

$$\begin{cases} -\partial_t u + \frac{1}{2}|\nabla u|^2 - \nabla p \cdot \nabla u = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\ u(T,x) = g(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{4.3.1}$$

As we have seen in Chapter 2, under some assumptions on the $p$ and $g$, we can gain existence, uniqueness and also some regularities on the value function $u$. So at first let us translate the hypotheses, which were used in the mentioned chapter, to our problem and then we formulate some properties.

So these hypotheses are:

$(H0')$ The set $\mathcal{A}$ is compact;

$(H1')$ There exists $K_1 > 0$ such that $|\nabla p(x_1) - \nabla p(x_2)| \leq K_1|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}^d$, i.e. $p \in C^{1,1}(\mathbb{R}^d)$. We remark that so far we just translated this hypothesis from [7] to our case, but as we will see it is to strong to assume this property on the pressure $p$. But this property is needed only to ensure the existence and uniqueness of the solution of the Cauchy problem, and it is well-known that in our case for this it is enough for $p$ to be semi-convex (which would be more natural assumption to have), hence we transform this hypothesis into the following one:

$$\exists K_1 > 0 : (\nabla p(x_1) - \nabla p(x_2)) \cdot (x_1 - x_2) \geq K_1|x_1 - x_2|^2, \forall x_1, x_2 \in \mathbb{R}^d.$$

$(H2')$ $\nabla^2 p$ exists and is continuous; in addition, there exists $K_2 > 0$ such that $||\nabla^2 p \cdot x_1 - \nabla^2 p \cdot x_2|| \leq K_2|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}^d$. Here we also have the remark, that this assumption is needed only to ensure the semi-concavity regularity for the value function, which we cannot use in our scheme, hence we will not need this regularity for the introduced pressure $p$.

The running cost in our control problem is $L(x, \alpha) := \frac{1}{2}|\alpha|^2$, so the assumptions (L1) and (L3) will hold trivially. But this is not the situation in the case of assumption (L2), so we reformulate this as follows.

$(L2')$ For any $x \in \mathbb{R}^d$, the following set is convex:

$$\mathcal{L}(x) := \left\{ (\lambda, v) \in \mathbb{R}^{d+1} : \exists \alpha \in \mathcal{A} \text{ s.t. } v = \alpha - \nabla p(x), \lambda \geq \frac{1}{2}|\alpha|^2 \right\}.$$

Here let us consider the case when the control set is unbounded and thus assumption $(H0')$ does not hold. In this case we replace again this assumption with the following one:

$(H^{*'})$ There exists $K_0$ such that

$$|\nabla p(x)| \leq K_0|x|, \forall x \in \mathbb{R}^d;$$

Because of the particularity of the running cost, the $(L^*)$ property is automatically satisfied (we can take $l_0 = 0$ and $l(r) = \frac{1}{2}r^2$).

Now we are dealing with the following dynamic programming problem and state equation for a given $p \in H^1(\mathbb{R}^d)$.

$$u(t,x) = \sup_{\alpha \in L^2([t,T],\mathbb{R}^d)} -\int \frac{1}{2}|\alpha(s)|^2 ds + g(y(T)) = \inf_{\alpha \in L^2([t,T],\mathbb{R}^d)} \int \frac{1}{2}|\alpha(s)|^2 ds + g(y(T)), \tag{4.3.2}$$

where $y$ solves the state equation

$$\begin{cases} y'(s) = \alpha(s) - \nabla p(s, y(s)), & s \in (t,T], \\ y(t) = x. \end{cases} \tag{4.3.3}$$

So we can formulate for our problem similar results as in Theorem 2.2.2, Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.5.

**Proposition 4.3.3.** *(i) Let our control system satisfy the assumptions $(H0')$,$(H1')$ and let $g \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$. Then $u \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{R}^d)$.*

*(ii) Let assume moreover the assumption $(H2')$ be satisfied and let $g$ be locally semi-concave with linear modulus in $\mathbb{R}^d$. Then $u$ is locally semi-concave with linear modulus in $[0,T] \times \mathbb{R}^d$.*

Analogous results hold when the control space is not necessarily compact.

**Proposition 4.3.4.** *Let our control system satisfying $(H^{*'})$ and $(H1')$.*

*(i) Let $g$ be locally Lipschitz and bounded from below. Then the value function $u$, the solution of* (4.3.1) *is locally Lipschitz.*

*(ii) Suppose, in addition, that $g$ is locally semi-concave with linear modulus and $(H2')$ holds. We see that $L(x,\alpha) = \frac{1}{2}|\alpha|^2$ satisfies immediately the following condition (which was used in [7]): for any $R > 0$, there exists $\lambda_R$ such that*

$$L(x,\alpha) + L(y,\alpha) - 2L\left(\frac{x+y}{2}, \alpha\right) \leq \lambda_R |x-y|^2, \forall x, y \in B(0,R), \alpha \in \mathcal{A} \cap B(0,R).$$

*Then $u$ is locally semi-concave with linear modulus in $[0,T] \times \mathbb{R}^d$.*

Of course we can formulate the existence and uniqueness of a viscosity solution of the problem (4.3.1), which is exactly given by the dynamic programming principle (4.3.2).

**Theorem 4.3.5.** *Under the hypotheses of Proposition 4.3.3(i) and Proposition 4.3.4(i), the value function $u$ (given by the formula* (4.3.2)*) is a unique viscosity solution of the problem* (4.3.1)*.*

Now we give the Euler-Lagrange optimality condition, satisfied by the optimal control $\alpha$ in (4.3.2).

**Lemma 4.3.6.** *If $\alpha$ is an optimal control in* (4.3.2) *(with a corresponding $\rho$) then it solves the ODE*

$$\begin{cases} \alpha'(s) = \nabla^2 p(s, y(s))\alpha(s), & \forall s \in [t,T), \\ \alpha(T) = -\nabla g(y(T)). \end{cases} \tag{4.3.4}$$

*Proof.* At first we mention, that if $\alpha$ solves (4.3.4), then we have the identity:

$$\alpha(s) = -\exp\left(-\int_s^T \nabla^2 p(\tau, y(\tau))d\tau\right)\nabla g(y(T)),\ \forall t < s \leq T.$$

Now let us show how we arrive to the equation (4.3.4). We can write the control problem (4.3.2) in the following way:

$$\begin{aligned} u(t,x) &= \inf_\alpha \int_t^T \frac{1}{2}|y'(s) + \nabla p(s, y(s))|^2 + \frac{d}{ds}\left(g(y(s))\right) ds + g(y(t)) \\ &= \inf_\alpha \int_t^T \frac{1}{2}|y'(s) + \nabla p(s, y(s))|^2 + \nabla g(y(s))y'(s)ds + g(x) \end{aligned}$$

Now we can define a Lagrangian, as the inner part of the integral, more precisely

$$L(y, y') := \frac{1}{2}|y' + \nabla p(\cdot, y)|^2 + \nabla g(y)y'.$$

By the Euler-Lagrange theorem we know, that if $y$ solves the minimization problem, then we have for this $y$ the Euler-Lagrange equation satisfied

$$\frac{d}{dt}\left(\nabla_{y'} L(y(t), y'(t))\right) = \nabla_y L(y(t), y'(t)).$$

So let us write down the Euler-Lagrange equation in our case. We have

$$\nabla_y L(y, y') = \nabla^2 p(\cdot, y)(y' + \nabla p(\cdot, y)) + \nabla^2 g(y)y' = \nabla^2 p(\cdot, y)\alpha + \nabla^2 g(y)y',$$

and

$$\nabla_{y'} L(y, y') = \alpha + \nabla g(y).$$

Moreover

$$\frac{d}{dt}\left(\nabla_{y'} L(y(s), y'(s))\right) = \alpha'(s) + \nabla^2 g(y(s))y'(s).$$

We also know that $\nabla_{y'} L(y(T), y'(T)) = 0$, so by the Euler-Lagrange equation we have obtained exactly (4.3.4). $\square$

Our next program point would be to prove the next step in the fixed point approach idea, i.e. the existence of a unique pair $(\rho, p)$ obtained from the unique value function $u$. But as we will see, we will not be able to show uniqueness and under the assumptions we gave more or less there is no hope neither to have uniqueness. Hence we need to modify the fix point scheme.

We can formulate the following theorems, which are somehow the translations of some similar theorems from [24] and [29] to our case.

**Theorem 4.3.7.** *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded convex domain, $w = -\nabla D$, where $D : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz function, moreover $\rho_0 \leq 1$ is an admissible initial density. Then $\exists\ \rho_t, v_t$ such that they solve*

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0 \\ \rho_t \leq 1,\ \rho_{t=0} = \rho_0 \\ \forall t\ v_t = P_{adm(\rho_t)}[w_t]. \end{cases} \tag{4.3.5}$$

*Under the assumptions, we can say equivalently that $\exists\ \rho_t, p_t$ such that they solve*

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t(w_t - \nabla p_t)) = 0 \\ p_t \geq 0,\ (1-\rho_t)p_t = 0 \\ \rho_{t=0} = \rho_0. \end{cases} \tag{4.3.6}$$

**Theorem 4.3.8.** *If in addition to the assumptions of the previous theorem $D$ is also $\lambda-$convex (semi-convex) in the space variable, then the pair $(\rho_t, v_t)$ or equivalently $(\rho_t, p_t)$ which solves* (4.3.5) *and* (4.3.6) *respectively, is unique.*

In the proof of these theorems we will adapt some techniques from [24] and [29]. For the first theorem our goal would be to show the existence of $(\rho_t, v_t)$ or equivalently $(\rho_t, p_t)$ which satisfy (4.3.5) or (4.3.6) respectively. For this it is natural to consider an implicit Euler (JKO) scheme for the gradient flow approach as follows.

Let us consider $K := \{\rho \in \mathcal{P}_2(\Omega) : \rho \leq 1\}$, the set of feasible densities and the indicator function $I_K$ as in the section "Gradient flows with density constraints". The scheme will be almost the same as the one presented in the mentioned section, with a little modification, because the function $D$ is depending also on time. So we consider a fixed time step $\tau > 0$, $\rho_0^\tau := \rho_0$ and for $k \in \mathbb{N}, k \geq 0$ we consider the scheme

$$\rho_{k+1}^\tau \in \operatorname{argmin}_{\rho \in \mathcal{P}_2(\Omega)} \left( \int_\Omega D(k\tau, x) d\rho(x) + I_K(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right) \tag{4.3.7}$$

Here we remark that there exists always a minimizer in the above scheme, because the argument is l.s.c., but we cannot expect to be unique (the argument which would ensure the uniqueness would be the semi-convexity of $D$).

Moreover we define the discrete velocity and momentum as $v_k^\tau = \frac{id - T_k^\tau}{\tau}$, where $T_k^\tau$ is the unique transport map from $\rho_k^\tau$ to $\rho_{k-1}^\tau$ and $E_k^\tau = \rho_k^\tau v_k^\tau$ respectively. We can interpolate these discrete values by

$$\begin{cases} \rho^\tau(t, \cdot) = \rho_k^\tau, \\ v^\tau(t, \cdot) = v_k^\tau, \qquad \text{if } t \in ](k-1)\tau, k\tau] \\ E^\tau(t, \cdot) = E_k^\tau. \end{cases}$$

Our goal would be now to prove that if $\tau \to 0$ then we will get a limit $(\rho_t, \rho_t v_t)$ for $(\rho^\tau, E^\tau)$ which solves the problem (4.3.5), fulfilling the constraint. This is similar to Theorem 2.1 form [24] and Theorem 2.2.12 from [29], the only difference is that $D$ is depending also on time and is not semi-convex.

Now let us prove some technical lemmas, which will be useful in the proof of the Theorem 4.3.7.

**Lemma 4.3.9.** *For the function $D$ in Theorem 4.3.7 and a measure $\overline{\rho} \in K$ there exists a $\tau^* > 0$ such that for all $\tau < \tau^*$ we have:*

*(i) The functional $\rho \mapsto \int_\Omega D(\overline{\tau}, x) d\rho(x) + I_K(\rho) + \frac{1}{2\tau} W_2^2(\rho, \overline{\rho})$ admits a minimizer, where $\overline{\tau}$ is an arbitrary fixed time moment, possibly depending on $\overline{\rho}$;*

*(ii) There exists a Kantorivich potential $\overline{\phi}$ from $\rho_m$ to $\overline{\rho}$, for all minimizers $\rho_m$ in the previous point, such that*

$$\int_\Omega \left( D(\overline{\tau}, x) + \frac{\overline{\phi}(x)}{\tau} \right) \rho \geq \int_\Omega \left( D(\overline{\tau}, x) + \frac{\overline{\phi}(x)}{\tau} \right) \rho_m,$$

*for all $\rho \leq 1$ a.e.*

*Proof.* The point $(i)$ can be easily shown, because the function is l.s.c., so taking a minimizing sequence we can obtain a minimizer.

$(ii)$ At first let us assume that $\overline{\rho} > 0$. Now if we take a minimizer $\rho_m$, there exists a unique Kantorovich potential $\overline{\phi}$ from $\rho_m$ to $\overline{\rho}$ such that $\overline{\phi}(x_0) = 0$ for an $x_0 \in \Omega$. We define a small perturbation of $\rho_m$ in $K$ by $\rho_\varepsilon := \rho_m + \varepsilon(\rho - \rho_m)$, where $\rho \leq 1$ is a probability density from $K$ and $\varepsilon > 0$. Let us use the notation $J(\rho) = \int_\Omega D(\overline{\tau}, x) d\rho(x)$. Because $\rho_m$ is a minimizer, we have that

$$J(\rho_\varepsilon) - J(\rho_m) + \frac{1}{2\tau} \left( W_2^2(\rho_\varepsilon, \overline{\rho}) - W_2^2(\rho_m, \overline{\rho}) \right) \geq 0.$$

We have explicitly

$$J(\rho_\varepsilon) - J(\rho_m) = \int_\Omega D(\overline{\tau}, x)(\rho_\varepsilon - \rho_m) dx = \varepsilon \int_\Omega D(\overline{\tau}, x)(\rho - \rho_m) dx.$$

Now we estimate the term with the Wasserstein distances. Let us denote by $\phi_\varepsilon$ a Kantorovich potential from $\rho_\varepsilon$ to $\rho$, so we have

$$\frac{1}{2} W_2^2(\rho_\varepsilon, \rho) = \int_\Omega \phi_\varepsilon(x) \rho_\varepsilon(x) dx + \int_\Omega \phi_\varepsilon^c(y) \rho(y) dy$$

and

$$\frac{1}{2} W_2^2(\rho_m, \rho) \geq \int_\Omega \phi_\varepsilon(x) \rho_m(x) dx + \int_\Omega \phi_\varepsilon^c(y) \rho(y) dy,$$

so we get

$$\frac{1}{2\tau} \left( W_2^2(\rho_\varepsilon, \overline{\rho}) - W_2^2(\rho_m, \overline{\rho}) \right) \leq \frac{\varepsilon}{\tau} \int_\Omega \phi_\varepsilon(x)(\rho(x) - \rho_m(x)) dx.$$

Hence

$$\int_\Omega D(\overline{\tau}, x)(\rho - \rho_m) dx + \frac{1}{\tau} \int_\Omega \phi_\varepsilon(x)(\rho(x) - \rho_m(x)) dx.$$

Now if we send $\varepsilon \to 0$, the potential $\phi_\varepsilon$ will tend to the unique potential $\overline{\phi}$, so we get the desired inequality when $\overline{\rho} > 0$.

To prove the general case, we will argue as in Remark 3.1 from [24]. Let $\overline{\rho}_\delta > 0$ from $K$ such that it converges to $\overline{\rho}$ as $\delta \to 0$. Now define $\rho_m^\delta$ as a minimizer of the functional

$$\rho \mapsto J(\rho) + I_K(\rho) + \frac{1}{2\tau} W_2^2(\rho, \overline{\rho}_\delta) + c_\delta W_2^2(\rho, \rho_m),$$

where $c_\delta \to 0$, as $\delta \to 0$ so that $\rho_m^\delta$ converges to $\rho_m$. Using the Kantorovich potential $\phi_\delta$ from $\rho_m^\delta$ to $\overline{\rho}$, which will tend to $\overline{\phi}$, in the limit as $\delta \to 0$ we will get the desired inequality. $\square$

Now we follow the same path as in [24] or [29], proving similar results as in the references for $D$ also depending on time and not semi-convex to achieve the proof of the Theorem 4.3.7.

**Lemma 4.3.10.** *We have for the velocity field $w = -\nabla D(k\tau, \cdot)$ and $\tau > 0$ the following decomposition*

$$w = v_k^\tau + \nabla p_k^\tau, \text{ where } p_k^\tau \in H^1(\Omega), p_k^\tau(1 - \rho_k^\tau) = 0, p_k^\tau \geq 0.$$

*Proof.* Having in mind the statement of the previous lemma and also the ideas for the section "Gradient flows with density constraints" we know that there exists a Kantorovich potential $\phi$ from $\rho_k^\tau$ to $\rho_{k-1}^\tau$ such that $\rho_k^\tau$ is a minimizer of the problem

$$\rho_k^\tau \in \operatorname{argmin}_{\rho \in K} \left\{ \int_\Omega D(k\tau, x)\rho(x)dx + \frac{1}{\tau}\int_\Omega \phi(x)\rho(x)dx \right\},$$

which imposes

$$\begin{cases} \rho_k^\tau = 1, & \text{on } \{F < l\}, \\ \rho_k^\tau < 1, & \text{on } \{F = l\}, \\ \rho_k^\tau = 0, & \text{on } \{F > l\}, \end{cases} \tag{4.3.8}$$

where $F : \Omega \to \mathbb{R},\ F(x) = D(k\tau, x) + \frac{\phi(x)}{\tau}$ and $l \in \mathbb{R}$ so that $\int_\Omega \rho_k^\tau(x)dx = 1$.

Now we can define a pressure by

$$p_k^\tau(x) = (l - F(x))_+ = \left( l - D(k\tau, x) - \frac{\phi(x)}{\tau} \right)_+.$$

This newly defined pressure satisfies that is non-negative, is in $H^1(\Omega)$ and is zero where $\rho_k^\tau < 1$. Furthermore on the region where $\rho_k^\tau > 0$ we have $\nabla p_k^\tau = -\nabla D(k\tau, x) - \frac{\nabla\phi(x)}{\tau}$, where $\rho_k^\tau = 0$ we can modify the velocity to hold true the formula. Since $v_k^\tau = \frac{id - T_k^\tau}{\tau} = \frac{\nabla\phi}{\tau}$, the decomposition is true, which is that $w = -\nabla D(k\tau, \cdot) = v_k^\tau + \nabla p_k^\tau$. □

Now we will construct densities through interpolation along geodesics of the discrete densities $\rho_k^\tau$ by

$$\tilde{\rho}^\tau(t, \cdot) := \left( \frac{t - (k-1)\tau}{\tau}(id - T_k^\tau) + T_k^\tau \right)_\# \rho_k^\tau,$$

furthermore we define the velocities $\tilde{v}^\tau(t, \cdot)$ as the unique velocity field in $\mathrm{Tan}_{\tilde{\rho}^\tau}$ which together with $\tilde{\rho}^\tau$ solves the continuity equation. Moreover we also define the momentums as $\tilde{E}^\tau = \tilde{\rho}^\tau \tilde{v}^\tau$.

As is [24] we can formulate some a priori bounds for these quantities defined above. But before these let us give some estimations on the curves.

By the optimality of $\rho_k^\tau$ with respect to $\rho_{k-1}^\tau$ we easily can write the estimate

$$W_2^2(\rho_{k-1}^\tau, \rho_k^\tau) \leq 2\tau \left( \int_\Omega D(k\tau, x)d\rho_{k-1}^\tau(x) - \int_\Omega D(k\tau, x)d\rho_k^\tau(x) \right),$$

and the integral terms on the r.h.s. are bounded, so we have

$$W_2^2(\rho_{k-1}^\tau, \rho_k^\tau) \leq C\tau,$$

for a $C > 0$ constant. This would be the discrete Hölder estimate of order $\frac{1}{2}$.

Summing up over $k$ the above inequality with the integrals, we get

$$\begin{aligned}
\frac{1}{2}\sum_k \tau \left(\frac{W_2^2(\rho_{k-1}^\tau, \rho_k^\tau)}{\tau}\right)^2 &\leq \int_\Omega D(\tau, x)\rho_0^\tau(x)dx \\
&+ \int_\Omega |D(2\tau, x) - D(\tau, x)|\rho_1^\tau(x)dx \\
&+ \int_\Omega |D(3\tau, x) - D(2\tau, x)|\rho_2^\tau(x)dx \\
&+ \dots \\
&+ \int_\Omega |D((N+1)\tau, x) - D(N\tau, x)|\rho_N^\tau(x)dx \\
&- \int_\Omega D((N+1)\tau, x)d\rho_{N+1}^\tau(x) \\
&\leq TC|\Omega| + \int_\Omega D(\tau, x)\rho_0^\tau(x)dx - \int_\Omega D((N+1)\tau, x)\rho_{N+1}^\tau(x)dx \\
&\leq TC|\Omega| + \int_\Omega |D(\tau, x)|dx + \int_\Omega |D(T, x)|dx \\
&\leq TC|\Omega| + \tilde{C}.
\end{aligned}$$

where $(N+1)\tau = T$ and $D$ is Lipschitz continuous in time with constant $C$. Here we also used the fact that $D(t, \cdot)$ is integrable w.r.t. space variable for all $t$, so the last sum of the two integral terms is bounded by $\tilde{C} > 0$. This implies that

$$\sum_k \tau \left(\frac{W_2^2(\rho_{k-1}^\tau, \rho_k^\tau)}{\tau}\right)^2 \leq 2(TC|\Omega| + \tilde{C}), \tag{4.3.9}$$

which we can interpret as a discrete version of $H^1$ estimate.

Since on the interval $[(k-1)\tau, k\tau]$ the quantity $\frac{W_2^2(\rho_{k-1}^\tau, \rho_k^\tau)}{\tau}$ is the velocity of $\tilde{\rho}^\tau$, we can calculate the $L^2$ norm of its velocity on $[0, T]$ by

$$\int_0^T |(\tilde{\rho}^\tau)'|(t)dt = \sum_k \frac{W_2^2(\rho_{k-1}^\tau, \rho_k^\tau)}{\tau},$$

which by (4.3.9) is uniformly bounded independent of $\tau$. This gives a compactness result of the curves $\tilde{\rho}^\tau$.

Now we can formulate the results for a priori bounds we mentioned before and another useful lemma. These are borrowed from [24].

**Lemma 4.3.11** (A priori estimates, [24])**.** *The followings are true:*

*(i)* $v^\tau$ *is* $\tau-$*uniformly bounded in* $L^2((0, T); L^2_{\rho^\tau}(\Omega))$;

*(ii)* $p^\tau$ *is* $\tau-$*uniformly bounded in* $L^2((0, T); H^1(\Omega))$;

*(iii) $E^\tau$ and $\tilde{E}^\tau$ are $\tau-$uniformly bounded measures.*

**Lemma 4.3.12** ([24])**.** *Assume that $\mu$ and $\nu$ are two absolutely continuous measure which densities are bounded by a same constant $C > 0$. Then for all functions $f \in H^1(\Omega)$ we have the following inequality:*

$$\int_\Omega f d(\mu - \nu) \leq \sqrt{C}||\nabla f||_{L^2} W_2(\mu, \nu). \tag{4.3.10}$$

The proof of the first lemma about the a priori estimates is more or less the same as in [24], using the estimations we obtained before. The proof of the second lemma is the same as in [24], so that is why we omit them at this point.

Now we have in our pocket all the necessary tools to prove the Theorem 4.3.7. We are following exactly the same way as in [24], but for the coherency we will try to do each step more or less in details, even though we have to borrow many arguments from the proof from [24].

*Proof of the Theorem 4.3.7.* We divide the proof of this theorem in three steps.

*Step 1: convergence of $(\tilde{\rho}^\tau, \tilde{E}^\tau)$ and $(\rho^\tau, E^\tau)$.*
The previous results show that $\tilde{\rho}^\tau$ and $\tilde{E}^\tau$ are $\tau-$uniformly bounded measures, so there exists $(\rho, E)$ such that the sequence $(\tilde{\rho}^\tau, \tilde{E}^\tau)$ narrowly converges to it. Let us prove that the other sequence $(\rho^\tau, E^\tau)$ converges to the same limit.

At first let us prove the result for $\rho^\tau$. By construction both measures $\tilde{\rho}^\tau$ and $\rho^\tau$ coincide on the discrete time moments of the form $k\tau$ and on each interval $](k-1)\tau, k\tau]$ the first one in Hölder continuous of order $\frac{1}{2}$ (as we have seen), while the second one is constant. Moreover $\tilde{\rho}^\tau$ is uniformly converging on $[0, T]$ w.r.t. $W_2$. We have that $W_2(\tilde{\rho}^\tau(t), \rho^\tau(t)) \leq C\tau^{\frac{1}{2}}$ for all $t \in [0, T]$. And so $\rho^\tau$ converges uniformly to the same limit as $\tilde{\rho}^\tau$.

Now let us focus to the other part, i.e. the convergence of $E^\tau$ to the same limit as $\tilde{E}^\tau$. For this let us take an arbitrary test function $f \in C_c^\infty([0, T] \times \Omega)$ and prove that the quantity $\int_0^T \int_\Omega f(\tilde{E}^\tau - E^\tau)$ tends to 0, when $\tau$ goes to 0. We know that $\tilde{\rho}^\tau(t, \cdot)$ was obtained by an interpolation, more precisely $\tilde{\rho}^\tau(t, \cdot) = (T_t)_\# \rho_k^\tau$, where

$$T_t = (t - (k-1)\tau)v_k^\tau + T_k^\tau.$$

Hence for a small $h$ we have

$$\tilde{\rho}^\tau(t+h, \cdot) = (T_t + hv_k^\tau)_\# \rho_k^\tau = (id + hv_k^\tau \circ T_t^{-1})T_{t\#}\rho_k^\tau = (id + hv_k^\tau \circ T_t^{-1})_\# \tilde{\rho}^\tau(t, \cdot).$$

This allows us to define a transport map between $\tilde{\rho}^\tau(t, \cdot)$ and $\tilde{\rho}^\tau(t+h, \cdot)$ as $\tilde{T} = id + hv_k^\tau \circ T_t^{-1}$. By this we can write explicitly the velocity $\tilde{v}^\tau$ as

$$\tilde{v}^\tau(t, \cdot) = \lim_{h \to 0} \frac{\tilde{T} - id}{h} = \lim_{h \to 0} \frac{hv^\tau \circ T_t^{-1}}{h} = v^\tau \circ T_t^{-1}.$$

We have then

$$\int_\Omega f(t, x)\tilde{\rho}^\tau(t, x)\tilde{v}^\tau(t, x)dx = \int_\Omega f(t, T_t(x))\rho^\tau(t, x)v^\tau(t, x)dx.$$

And finally we have

$$\int_0^T \int_\Omega f(\tilde{E}^\tau - E^\tau) \leq \sum_k \int_{k\tau}^{(k+1)\tau} \int_\Omega |f(t,x) - f(t,T_t(x))||v_k^\tau(x)|\rho_k^\tau(x)dxdt$$
$$\leq \sum_k \int_{k\tau}^{(k+1)\tau} \int_\Omega \mathrm{Lip}(f)|x - T_t(x)||v_k^\tau(x)|\rho_k^\tau(x)dxdt$$
$$\leq \sum_k \int_{k\tau}^{(k+1)\tau} \int_\Omega \mathrm{Lip}(f)\tau|v_k^\tau(x)|^2\rho_k^\tau(x)dxdt \leq \mathrm{Lip}(f)C\tau.$$

So we have the result we wanted.

*Step 2: existence of the limit velocity*

We want to show that the limit momentum $E$ is absolutely continuous w.r.t. the limit measure $\rho$. For this let us consider the functional $\Psi$ which is defined on measure pairs $(F,\theta)$, where $F$ is a vector valued measure and $\theta$ is a scalar measure, and its values are real numbers, more precisely

$$\Psi(F,\theta) = \begin{cases} \int_0^T \int_\Omega \frac{|F|^2}{\theta}, & \text{if } F \ll \theta \text{ for a.e. } t \in [0,T], \\ +\infty, & \text{otherwise.} \end{cases}$$

This functional is l.s.c. w.r.t. the weak-* convergence of measures. But from a previous lemma we know the uniform bound

$$\int_0^T \int_\Omega |v^\tau|^2 \rho^\tau = \int_0^T \int_\Omega \frac{|E^\tau|^2}{\rho^\tau} \leq C,$$

hence by lower semi-continuity of $\Psi$ we have $\Psi(E,\rho) < +\infty$ so $E$ is absolutely continuous w.r.t. $\rho$. Other hand this means that there exists a $v(t,\cdot) \in L^2(\rho(t,\cdot))$ such that $E = \rho v$, moreover $(\rho, \rho v)$ satisfies the continuity equation as a limit of $(\tilde{\rho}^\tau, \tilde{E}^\tau)$.

Now it is left to prove that $v_t$ is an admissible velocity, which is that $v_t \in adm(\rho_t)$. Let us prove this statement. Let $t_0 \in (0,T)$ and let $q \in H^1(\Omega)$ such that $q \geq 0$ and $q(1-\rho_{t_0}) = 0$. By the continuity equation we have for a positive small $h$ the following

$$\int_{t_0}^{t_0+h} \int_\Omega \nabla q(x) v_t(x) \rho_t(x) dxdt = \int_\Omega (\rho_{t_0}(x) - \rho_{t_0+h}(x))\, q(x)dx.$$

Other hand we know that $\rho_{t_0} = 1$, where $q > 0$ and $\rho_{t_0+h} \leq 1$, *a.e.*, hence we have for a.e. $t_0 \in (0,T)$ that

$$\int_\Omega (\rho_{t_0}(x) - \rho_{t_0+h}(x))\, q(x)dx \geq 0.$$

Using this inequality, we have the following limit as $h \to 0$

$$0 \leq \frac{1}{h} \int_{t_0}^{t_0+h} \int_\Omega \nabla q(x) v_t(x) \rho_t(x) dxdt \to \int_\Omega \nabla q(x) v_{t_0}(x) \rho_{t_0}(x) dx$$
$$= \int_\Omega \nabla q(x) v_{t_0}(x) dx.$$

We get the opposite inequality, if in the integration w.r.t. time we integrate from $t_0 - h$ to $t_0$. So finally we get, what we wanted, i.e. $v_t \in adm(\rho_t)$.

*Step 3: the limit velocity $v_t$ is the projection of $w_t$ onto $adm(\rho_t)$*
First of all let us prove that it holds the decomposition $w_t = v_t + \nabla p_t$. We already know that on an interval of type $](k-1)\tau, k\tau]$ we have that $w = -\nabla D(k\tau, \cdot) = v_k^\tau + \nabla p_k^\tau$. So it is a natural idea to define also an approximation of $-\nabla D$ in time as

$$-\nabla D^\tau(t, \cdot) = -\nabla D(k\tau, \cdot), \text{ if } t \in ](k-1)\tau, k\tau].$$

Of course this sequence of piecewise constant functions tends to $-\nabla D$ (remember that $D$ is a locally Lipschitz function, so its space gradient is still locally Lipschitz is time).

So we have that $E^\tau = \rho^\tau v^\tau = -\rho^\tau(\nabla D^\tau + \nabla p^\tau) = -\rho^\tau \nabla D^\tau - \nabla p^\tau$, because $p^\tau = 0$ on $\{\rho^\tau < 1\}$. Our aim now is to prove that $p^\tau$ converges to a $p \in H^1(\Omega)$, where $p \geq 0$ and $p(1-\rho) = 0$. Because $p^\tau \in L^2([0,T], H^1(\Omega))$, there exists a $p$ such that $p^\tau$ weakly converges to $p$ in $L^2([0,T], H^1(\Omega))$. We have of course that $p \geq 0$ a.e., so it remains to show that $p_t = 0$ where $\{\rho_t < 1\}$. To show this, as in [24], we consider the average functions

$$p_{a,b}^\tau = \frac{1}{b-a} \int_a^b p^\tau(t, \cdot)dt \quad \text{and} \quad p_{a,b} = \frac{1}{b-a} \int_a^b p(t, \cdot)dt.$$

Since $p^\tau = 0$ on $\{\rho^\tau < 1\}$, we have

$$0 = \int_a^b \int_\Omega p^\tau(t,x)(1-\rho^\tau(t,x))dxdt = \frac{1}{b-a} \int_a^b \int_\Omega p^\tau(t,x)(1-\rho^\tau(a,x))dxdt$$
$$+ \frac{1}{b-a} \int_a^b \int_\Omega p^\tau(t,x)(\rho^\tau(a,x) - \rho^\tau(t,x))dxdt$$

As $\tau \to 0$ we have that

$$\int_\Omega p_{a,b}^\tau(t,x)(1-\rho^\tau(a,x))dx \to \int_\Omega p_{a,b}(t,x)(1-\rho(a,x))dx.$$

This is because $p_{a,b}^\tau \rightharpoonup p_{a,b}$ in $H^1(\Omega)$ (therefore strongly in $L^2(\Omega)$) and $\rho^\tau(a,\cdot)$ converges weakly-* to $\rho(a,\cdot)$ in $L^\infty(\Omega)$. Furthermore for all Lebesgue points $a$ of $p(\cdot,x)$ we have that $p_{a,b} \to p(a,\cdot)$, when $b \to a$, so for all these Lebesgue points we have

$$\int_\Omega p_{a,b}(t,x)(1-\rho(a,x))dx \to \int_\Omega p(a,x)(t,x)(1-\rho(a,x))dx, \text{ as } b \to a.$$

For the second integral in the above sum we use the Lemma 4.3.12 and we have

$$\int_a^b \int_\Omega p^\tau(t,x)(\rho^\tau(a,x) - \rho^\tau(t,x))dxdt \leq \int_a^b ||\nabla p^\tau(t,\cdot)||_{L^2} W_2(\rho^\tau(a,\cdot), \rho^\tau(t,\cdot))dt$$
$$\leq C\sqrt{b-a} \left( \int_a^b ||\nabla p^\tau(t,\cdot)||_{L^2}^2 dt \right)^{\frac{1}{2}} \left( \int_a^b dt \right)^{\frac{1}{2}}$$
$$= C(b-a) \left( \int_a^b ||\nabla p^\tau(t,\cdot)||_{L^2}^2 dt \right)^{\frac{1}{2}}.$$

Moreover we know that $\int_0^T ||\nabla p^\tau(t,\cdot)||_{L^2}^2 dt$ is $\tau-$uniformly bounded, so $||\nabla p^\tau(t,\cdot)||_{L^2}^2$ converges weakly to a measure $\mu$. That is why for a.e. $a$ we have that

$$\lim_{\tau\to 0}\frac{1}{b-a}\int_a^b\int_\Omega p^\tau(t,x)(\rho^\tau(a,x)-\rho^\tau(t,x))dxdt \le C\sqrt{\mu([a,b])}\to 0,\ \text{as } b\to a.$$

So finally we obtained that

$$\int_\Omega p(a,x)(1-\rho(a,x))dx = 0$$

for a.e. $a\in[0,T]$.

We also know that $-\rho^\tau\nabla D^\tau$ converges to $-\rho\nabla D$, so we have that $E=-\rho\nabla D-\nabla p$, where $p=0$ on the set $\{\rho<1\}$, so $E=-\rho(\nabla D-\nabla p)$. Other hand $E=\rho v$, so we have the decomposition $v=-\nabla D-\nabla p$, which is $w=v+\nabla p$.

By the *Step 2* we also know that

$$\int_\Omega \nabla p(t,x)\cdot v(t,x)=0,\ \text{for a.e. } t\in[0,T],$$

and this ensures that $v_t=P_{adm(\rho_t)}[w_t]$.

$\square$

Now we will show that having the additional assumption on $D$, namely the semi-convexity in the space variable, we can gain uniqueness. But at first we present a lemma, which would be needed in the proof of Theorem 4.3.8.

**Lemma 4.3.13.** *Let $\Omega$ be a convex bounded domain of $\mathbb{R}^d$ and let $\rho,\tilde\rho\in\mathcal{P}_2(\Omega)$ two absolutely continuous measures such that $\rho\le 1$ and $\tilde\rho\le 1$. Take a Kantorovich potential $\phi$ from $\rho$ to $\tilde\rho$ and $p\in H^1(\Omega)$ such that $p\ge 0$ and $p(1-\rho)=0$.*

*Then*

$$\int_\Omega \nabla\phi\cdot\nabla p d\rho \ge 0.$$

*Proof of the lemma.* By the assumption on the pressure $p$, that it vanishes where the density is less than 1 (and so vanishes also its distributional gradient), the above integral is nonzero only on the set $\{x\in\Omega:\rho(x)=1\}$, and on this set we can do the integration with respect to the Lebesgue measure, because $\rho$ is absolutely continuous.

We would like to integrate by parts, since we know how to handle the sign of the Laplacian. Yet, this does not make sense because we cannot expect much regularity for the potential $\phi$, hence its Laplacian will be a measure, $p$ is also just an $H^1(\Omega)$ function. If $\rho$ and $\tilde\rho$ would be bounded from below away from zero, we would know that the potential is in $W^{2,1}(\Omega)$, so its Laplacian would be in $L^1(\Omega)$, so it would be enough for $p$ to be in $L^\infty(\Omega)$, and we know how to handle this situation. However this is not the case, the Laplacian of $\phi$ is just a measure, so for the pressure we would need to be continuous. That is why we have problems with the integration by parts.

But we can show that the sign of the integral is the same as if we could use some integration by parts formula. For this we will use some approximation arguments.

It is enough to prove the statement in the case when $p$ is bounded, because in the case when $p$ is unbounded, we can use the non-negativity of the integral for a sequence $p_n := \min\{p, n\}$, which approximates $p$ in $H^1(\Omega)$ and take its limit, so we will have the result also for unbounded pressures.

We remark here that without the loss of generality, using some extension arguments for functions in $H^1(\Omega)$, we extend $p$ to the whole $\mathbb{R}^d$, imposing compact support for the new function, and work with this new pressure, to avoid some technical issues on the boundary later on in the construction of the approximating sequence.

Now we will take a sequence $(p_\varepsilon)_{\varepsilon>0}$, which approximates $p$ in $H^1(\Omega)$, and which support is "almost" containing in the set $\{\rho = 1\}$. We explain what we mean exactly in a moment.

For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \{\rho < 1\}$ such that $|\Delta\phi|\left(\{\rho < 1\} \setminus K_\varepsilon\right) < \varepsilon$, and $O_\varepsilon := \mathbb{R}^d \setminus K_\varepsilon$ is an open set. Moreover $p = 0$ a.e. outside of $O_\varepsilon$, so it is true that $p \in H_0^1(O_\varepsilon)$, hence it can be approximated by a sequence $p_\varepsilon \in C_c^\infty(O_\varepsilon)$, such that $supp(p_\varepsilon) \subset O_\varepsilon$ and $||p - p_\varepsilon||_{H^1} < \varepsilon$.

Now we use the integration by parts formula for $p_\varepsilon$, more precisely

$$\int_\Omega \nabla\phi \cdot \nabla p_\varepsilon dx = \int_{\partial\Omega} \frac{\partial\phi}{\partial n} p_\varepsilon d\sigma(x) - \int_\Omega \Delta\phi p_\varepsilon dx.$$

Since the domain is convex, it is easy to see that the angle between $\nabla\phi$ and $n$, the outer normal vector is always less or equal than $\frac{\pi}{2}$, so $\frac{\partial\phi}{\partial n} = \nabla\phi \cdot n \geq 0$. Moreover $p_\varepsilon \geq 0$, that is why the boundary term in the above formula is always non-negative. We have that

$$\int_\Omega \nabla\phi \cdot \nabla p_\varepsilon dx \geq -\int_\Omega \Delta\phi p_\varepsilon dx = -\int_{O_\varepsilon} \Delta\phi p_\varepsilon dx. \tag{4.3.11}$$

We can decompose the last integral as follows:

$$-\int_{O_\varepsilon} \Delta\phi p_\varepsilon dx = -\int_{O_\varepsilon \cap \{\rho=1\}} \Delta\phi p_\varepsilon dx - \int_{O_\varepsilon \setminus \{\rho=1\}} \Delta\phi p_\varepsilon dx \tag{4.3.12}$$

It was important this decomposition, because we know that $\Delta\phi \leq 0$ on the set $\{\rho = 1\}$, and by this we can have further good estimations. But let us explain why this statement is true. First of all we know that the Kantorovich potential satisfies that $\psi(x) := \frac{|x|^2}{2} - \phi(x)$ is convex. Secondary we also know that $\psi$ satisfies a Monge-Ampère equation, namely

$$\frac{\rho}{\tilde{\rho}(\nabla\psi)} = \det(\nabla^2\psi).$$

We are interested in the region, where $\rho = 1$, and we also know that $\tilde{\rho} \leq 1$, so the left hand side is grater or equal than 1. So we have

$$1 \leq \det(I - \nabla^2\phi).$$

Moreover the matrix $I - \nabla^2\phi$ is positive definite, so all of its eigenvalues are positive, and we know that the determinant is the product of the eigenvalues. So that is why we can use the inequality between the arithmetic and geometric means, which will be a relation between the determinant and the trace of $I - \nabla^2\phi$. So we will have

$$1 \leq \det(I - \nabla^2\phi) \leq \left(\frac{1}{d}(d - \Delta\phi)\right)^d$$

which is equivalent to

$$1 \leq 1 - \frac{\Delta\phi}{d},$$

from where we can deduce that $\Delta\phi \leq 0$. To be more precise we remark here that in the proof of this inequality we actually used the absolutely continuous part of the measure $\Delta\phi$, while in the integration by parts formula we used the full measure. But this is not a problem, because we know that $\Delta\phi \leq \Delta_a\phi$, where we denoted by $\Delta_a\phi$ the absolutely continuous part.

Now let us return to the inequality (4.3.11). By the above argumentation and using also (4.3.12) we have

$$\int_\Omega \nabla\phi \cdot \nabla p_\varepsilon dx \geq -\int_{O_\varepsilon \setminus \{\rho=1\}} \Delta\phi p_\varepsilon dx.$$

The set $O_\varepsilon \setminus \{\rho = 1\}$ is exactly $\{\rho < 1\} \setminus K_\varepsilon$ and using the fact that the approximating sequence is also bounded (because it was enough to work originally with bounded pressures) we will have the following estimation

$$\int_\Omega \nabla\phi \cdot \nabla p_\varepsilon dx \geq -\int_{\{\rho<1\}\setminus K_\epsilon} \Delta\phi p_\varepsilon dx \geq -c|\Delta\phi|(\{\rho < 1\} \setminus K_\varepsilon) \geq -c\varepsilon.$$

The last inequality if due to the construction of $K_\varepsilon$, where $c > 0$ is a constant, which controls the (supremum) norm of the sequence.

So finally we send $\varepsilon$ to 0, then in the limit the left hand side tends to $\int_\Omega \nabla\phi \cdot \nabla p dx$ and the right hand side tends to 0 and we are done. $\square$

*Proof of the Theorem 4.3.8.* Let us take two curves $\rho_t$ and $\tilde{\rho}_t$ satisfying the continuity equation in (4.3.5) with the corresponding vector fields $v_t$ and $\tilde{v}_t$ (which are actually in the tangent spaces of the curves). Now the idea would be to calculate $\frac{d}{dt}\frac{1}{2}W_2^2(\rho_t, \tilde{\rho}_t)$ and estimate it.

From Theorem 8.4.7 from [3] we know that for an absolutely continuous curve $\mu_t$ in $\mathcal{P}_p(\Omega)$ and its tangent vector field $v_t$ we have the formula

$$\frac{d}{dt}\frac{1}{p}W_p^p(\mu_t, \nu) = \int_{\Omega\times\Omega} |x - y|^{p-2}(x - y) \cdot v_t(x) d\gamma, \forall\gamma \in \Pi(\mu_t, \nu),$$

where $\nu \in \mathcal{P}_p(\Omega)$ is a fixed measure.

Using this formula in our case for $\mu_t = \rho_t$ and $\nu = \tilde{\rho}_s$, for a fixed $s$, then changing the roles of the two measures we will obtain adding this two inequalities (using also the Lemma 4.3.4 from [3]) the following inequality

$$\frac{d}{dt}\frac{1}{2}W_2^2(\rho_t, \tilde{\rho}_t) \leq \int_{\Omega\times\Omega} (x - y) \cdot (v_t(x) - \tilde{v}_t(y)) d\gamma, \forall\gamma \in \Pi(\rho_t, \tilde{\rho}_t).$$

We also know that for absolutely continuous measures (assuming also that the border of $\Omega$ is negligible) there is an optimal transport map $T$ such that $T_{\#}\rho_t = \tilde{\rho}_t$ and for $\gamma \in \Pi(\rho_t, \tilde{\rho}_t)$ we have $\gamma = (id \times T)_{\#}\rho_t$. Using this, we can write the above formula in terms of transport maps, instead of plans, so we have

$$\frac{d}{dt}\frac{1}{2}W_2^2(\rho_t, \tilde{\rho}_t) \leq \int_\Omega (x - T(x)) \cdot (v_t(x) - \tilde{v}_t(T(x)))d\rho_t(x).$$

Now we know for the vector fields $v_t$ and $\tilde{v}_t$ their representation with the introduced pressure fields, i.e. $v_t = w_t - \nabla p_t$ and $\tilde{v}_t = w_t - \nabla \tilde{p}_t$. Using these we try to estimate the r.h.s. We also use the fact, that the vector field $w_t$ is the opposite of the gradient of a semi-convex function. By this fact we have

$$\begin{aligned}
\int_\Omega (x - T(x)) \cdot (v_t(x) - \tilde{v}_t(T(x)))d\rho_t(x) &= \int_\Omega (x - T(x)) \cdot (-\nabla D(t,x) + \nabla D(t, T(x)))d\rho_t(x) \\
&- \int_\Omega (x - T(x)) \cdot (\nabla p_t(x) - \nabla \tilde{p}_t(T(x)))d\rho_t(x) \\
&\leq -\lambda \int_\Omega |x - T(x)|^2 d\rho_t - \int_\Omega (x - T(x)) \cdot (\nabla p_t(x) - \nabla \tilde{p}_t(T(x)))d\rho_t(x) \\
&= -\lambda W_2^2(\rho_t, \tilde{\rho}_t) - \int_\Omega (x - T(x)) \cdot (\nabla p_t(x) - \nabla \tilde{p}_t(T(x)))d\rho_t(x)
\end{aligned}$$

Now we will prove that $\int_\Omega (x - T(x)) \cdot (\nabla p_t(x) - \nabla \tilde{p}_t(T(x)))d\rho_t(x)$ is a positive quantity. We know that $x - T(x) = \nabla\phi(x)$, for a Kantorovich potential $\phi$. So we can use simply the Lemma 4.3.13, which tells us that $\int_\Omega \nabla\phi(x) \cdot \nabla p_t(x) d\rho_t(x) \geq 0$ and similarly $-\int_\Omega \nabla\phi(x) \cdot \nabla\tilde{p}_t(T(x))d\rho_t(x) \geq 0$. Let us explain a little bit more detailed, why is this analogous. If we write again for $\nabla\phi(x) = x - T(x)$ and make a change of variable formula $x \mapsto T^{-1}(y)$ in the integral we will get

$$\int_\Omega (y - T^{-1}(y)) \cdot \nabla\tilde{p}_t(y)\rho_t(T^{-1}(y))\frac{1}{|JT|}dy,$$

where we denoted by $|JT|$ the absolute value of the Jacobian determinant of $T$. But we know that this will be

$$\int_\Omega \nabla\psi(y) \cdot \nabla\tilde{p}_t(y)\tilde{\rho}_t(y)dy,$$

where $\psi$ is the Kantorovich potential from $\tilde{\rho}$ to $\rho$. Now we can use again the Lemma 4.3.13 showing the non-negativity of this term.

So we will get

$$\frac{1}{2}\frac{d}{dt}W_2^2(\rho_t, \tilde{\rho}_t) \leq -\lambda W_2^2(\rho_t, \tilde{\rho}_t).$$

By Gronwall's lemma we have that

$$W_2^2(\rho_t, \tilde{\rho}_t) \leq e^{-2\lambda t}W_2^2(\rho_0, \tilde{\rho}_0),$$

but naturally we use the same initial measures for both curves, so we can deduce the uniqueness.

□

Unfortunately we see that in our case we would need to take $D$ as $u$, the value function from the Hamilton-Jacobi equation of our system, which under the hypotheses we had would be more likely semi-concave, not semi-convex what we would need for this theorem (anyway we are not able at this point to prove semi-convexity under some suitable assumptions), so there is no hope to guarantee the uniqueness of the solution $(\rho_t, p_t)$.

In the lack of uniqueness, our abstract operator would not be well-defined, so the scheme as we stated in the beginning of this section will not work in that form. We need some modifications in the idea and a natural modification would be to try to use some fixed point theorem for multi-valued maps. Let us come back to this issue in the next section.

## 4.4 Further ideas. Conclusions

### 4.4.1 A more general fixed point theorem for the existence

As we have seen previously at this moment unfortunately we are not able to show the existence of a solution of our first order MFG system with density constraints. But we have at least two more ideas to approach this issue. The first one would be not to drop the fixed point theories yet, but try to use a more general one.

Our first problem which arose in the fixed point scheme we wanted to use, is that the defined operator will be multi-valued and if we want to translate for example Schauder's fixed point theorem for multi-valued maps, we need the convexity of the image sets. But it seems that this property is hard to be proven, i.e. that all the solutions $(\rho, p)$ of (4.3.6) form a convex set. So if it is possible we want to get rid of this property and find a more general fixed point theorem, for which we do not need the convexity. Fortunately there is some hope, because we have found one, which assumes more topological properties instead of algebraic ones (for ex. convexity). This theorem is due to Begle (see [4]) and later on it was used by many other authors (see [27], [14]) to deal with similar issues.

Let us say a few words about the definitions, which we need for this theorem.

**Definition 4.4.1.** A space is said to be *contractible* if its characteristic function is homotopic to a constant (so a contractible space is always nonempty). A nonempty convex set is always contractible, but there are examples when the converse is not true.

A nonempty space is *acyclic* if its Čech homology (coefficients in a fixed field) is zero in dimensions greater than zero. With other words an acyclic space is a space which has the same homology group as does the space which is consisting of just one point. Every contractible space is acyclic, but we have also examples which show that the converse implication is not necessarily true.

And now we present the fixed point theorem which we were talking about and which is due to Begle (in 1950).

**Theorem 4.4.2** ([4],[27])**.** *Suppose $X$ is a compact acyclic lc space and $f : X \to X$ is a closed-graphic acyclic valued multifunction. Then $f$ has a fixed point, i.e. $\exists x \in X : x \in f(x)$.*

*Remark* 4.4.3. At this point we omit the definition of the lc (locally connected) space, because it requires many algebraic topological preparations, and to give these notions is not the goal of our thesis. But it can be found in the original work of Begle (see [4]).

So basically our next program point would be to check that under which conditions the assumptions of this fixed point theorem could be satisfied in our case, and if it is possible deduce the existence of a solution of our MFG system with density constraints. But it seems that this is not completely trivial, i.e. checking this algebraic topological notions in our case, but there is still some hope.

### 4.4.2 The vanishing viscosity method for existence

A next interesting idea would be to study the first order MFG system with density constraints as a limit of a certain regularized (second order) system. This is a well-known method (called as vanishing viscosity method) and it was used also in [8] to show the existence of a first order MFG system, which was obtained as the limit of second order systems.

Here at first it is necessary to build second order MFG systems with density constraints (using stochastic control) and after a big work on these systems maybe it will be easier to obtain existence results also in the first order case. Hence this method would be very useful.

Let us see what is happening mathematically. We can see in [8] that in the limit as $\sigma \to 0$ the solutions $(u_\sigma, m_\sigma)$ of

$$\begin{cases} (i) & -\partial_t u - \sigma\Delta u + \frac{1}{2}|\nabla u| = F(x,m) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & -\partial_t m - \sigma\Delta m - \nabla\cdot(\nabla u m) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x,m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{4.4.1}$$

converge to a solution $(u,m)$ of (0.2.1). For $(u_\sigma)$ the convergence is locally uniformly in $[0,T]\times\mathbb{R}^d$ and for $(m_\sigma)$ it is in $C([0,T],\mathcal{P}_1(\mathbb{R}^d))$.

Anyway we can say that after the study of first order MFG systems with density constraints it would be a natural further idea to study the second order case with these constraints.

### 4.4.3 MFG with density constraints on manifolds

It would be also nice to study MFG systems with density constraints on manifolds (Riemannian or Finslerian). There is a huge literature for optimal control and optimal transportation theory on manifolds, so it would be very natural and important to study also MFG systems on manifolds. At first it is important to study MFG systems with density constraints in the whole space $\mathbb{R}^d$, instead of a bounded (and convex) domain $\Omega \subset \mathbb{R}^d$.

We can also imagine examples, where the metric on a manifold is not symmetric and the agents are moving on this manifold. An idea would be that the agents are moving "between mountains" and going upwards on a mountain is more expensive than going downwards, so the cost or the payoff could depend on the (non-symmetric) metric.

There is a paper by Kristály et al. which deals with optimal placement of a deposit between markets on Finslerian manifolds (see [15]), and the authors use some techniques from Riemann-Finsler geometry and some computational techniques (evolutive algorithms) as well to study this problem. It would be an interesting approach to study this problem in the framework of optimal transportation an adapt some techniques to study MFG systems with density constraints on Finslerian manifolds.

Another interesting idea could be to study the location of Nash equilibria points (also on manifolds) when the number of players tends to infinity, i.e. to characterize the Nash equilibria of a MFG system (with and without density constraints). Here we also could adapt some ideas from the works of Kristály (see for example [14]).

### 4.4.4 Final conclusions and remarks

So far this thesis contains some possible approaches to study first order MFG systems with density constraints. The model is due to F. Santambrogio (see [30]) and it consists some ideas from the theory of crowd motion.

We wanted to construct a fixed point scheme to show the existence of a solution of such a system, but at the present moment we just have some conclusions and further ideas to develop, instead of full results. One of the main issues is that we cannot guarantee the semi-convexity of the value function and this fact leads us to the necessity of the use of some multi-valued fixed point theorem. But almost all the multi-valued fixed point theorems require the convexity of the image sets. And this fact seems hard to be shown, hence we looked for a more general fixed point theorem for multi-valued maps (Begle's fixed point theorem) and hopefully after some works we will be able to show the existence. But anyway we have some other ideas for this issue (for example the already mentioned vanishing viscosity method) which require further works.

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