A DYNAMIC EVOLUTION MODEL FOR PERFECTLY PLASTIC PLATES

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Abstract. We consider the dynamic evolution of a linearly elastic-perfectly plastic thin plate subject to a purely vertical body load. As the thickness of the plate goes to zero, we prove that the three-dimensional evolutions converge to a solution of a certain reduced model. In the limiting model admissible displacements are of Kirchhoff-Love type. Moreover, the motion of the body is governed by an equilibrium equation for the stretching stress, a hyperbolic equation involving the vertical displacement and the bending stress, and a rate-independent plastic flow rule. Some further properties of the reduced model are also discussed.

1. Introduction

Thin structures, such as beams, plates, or shells, are ubiquitous in the real world. A precise understanding of the laws governing their motion is therefore crucial in a large number of applications in mechanics and in civil engineering. From a mathematical point of view, the rigorous derivation of lower dimensional models for thin structures can be performed starting from their three-dimensional counterparts by using Γ-convergence techniques. This approach has been successfully applied to the stationary case: for instance, in the framework of nonlinear elasticity, to plates [18, 19, 23], beams [3, 29, 30, 32, 33], and shells [17, 24, 25]. We refer to [8] for the classical results in the framework of linearised elasticity. More recently, the Γ-convergence approach has been adapted to the evolutionary setting, as well: in nonlinear elastodynamics [1, 2], crack evolution [6, 16], plasticity [11, 26, 27], and delamination problems [28].

The subject of this paper is the rigorous derivation of a dynamic evolution model for a thin plate in perfect plasticity. The framework is that of small strain plasticity with additive decomposition for the strain field. The quasistatic case was treated in [11].

Let $\omega \subset \mathbb{R}^2$ be a domain with a C^2 boundary and let $h > 0$. We consider a plate, whose reference configuration is given by the set

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2}).$$

Here ω represents the mid-surface of the plate, while the parameter h denotes its thickness. The plate is assumed to be made of a homogeneous and isotropic material, whose elastic behaviour is linear and whose plastic response follows the Prandtl-Reuss law of perfect plasticity.

In this framework the dynamic evolution problem can be formulated as follows. Let $u_h(s)$ be the displacement field at time s and let $Eu_h(s)$ be the symmetric gradient of $u_h(s)$. The linearised strain $Eu_h(s)$ is decomposed as the sum of two symmetric matrices: the elastic strain $e_h(s)$ and the plastic strain $p_h(s)$. In the modelling of plastic behaviour of metals plastic deformation is usually assumed to be volume preserving: for this reason, we assume $p_h(s, x)$ to be a deviatoric matrix for every $x \in \Omega_h$ and every time s. We further suppose that the evolution is driven by a purely vertical time-dependent body load $f_h(s)$ and by a time-dependent boundary displacement $w_h(s)$ prescribed on a portion $\Gamma_{d,h} := \gamma_d \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ of the lateral boundary of the plate. The dynamic evolution problem consists in finding a triplet (u_h, e_h, p_h) such that the following conditions hold for every $s \geq 0$:

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(d1) *kinematic admissibility:* $Eu_h(s) = e_h(s) + p_h(s)$ in Ω_h and $u_h(s) = w_h(s)$ on $\Gamma_{d,h}$;

(d2) *constitutive equation:* $\sigma_h(s) := \mathbb{C}e_h(s)$ in Ω_h, where $\sigma_h(s)$ is the stress field at time s and $\mathbb{C}$ is the elasticity tensor;

(d3) *equation of motion:* $\ddot{u}_h(s) - \operatorname{div}\sigma_h(s) = f_h(s)e_3$ in Ω_h and $\sigma_h(s)\nu_{\partial\Omega_h} = 0$ on $\partial\Omega_h \setminus \Gamma_{d,h}$, where $\nu_{\partial\Omega_h}$ is the outer unit normal to $\partial\Omega_h$;

(d4) *stress constraint:* $(\sigma_h)_D(s) \in K$ in Ω_h, where $(\sigma_h)_D$ is the deviatoric part of σ_h and K is a convex and compact set in the space of deviatoric matrices $\mathbb{M}^{3\times 3}_D$;

(d5) *flow rule:* $\dot{p}_h(s,x)$ belongs to the normal cone to K at $(\sigma_h)_D(s,x)$ for every $x \in \Omega_h$.

Under suitable assumptions on the data, existence and uniqueness of solutions to system (d1)–(d5) has been proved in [5] and recently revisited in [7]. The natural setting for solutions is the space $BD(\Omega_h)$ of functions with bounded deformation on Ω_h for the displacement u_h, the space $L^2(\Omega_h;\mathbb{M}^{3\times 3}_{sym})$ for the elastic strain e_h, and the space $M_b(\Omega_h\cup\Gamma_{d,h};\mathbb{M}^{3\times 3}_D)$ of $\mathbb{M}^{3\times 3}_D$-valued bounded Radon measures on $\Omega_h \cup \Gamma_{d,h}$ for the plastic strain p_h. From a mechanical point of view this formulation is consistent with the well known fact that displacements in perfect plasticity can develop jump discontinuities along so-called slip-surfaces, on which plastic strain concentrates. Furthermore, the Dirichlet boundary condition on $\Gamma_{d,h}$ has to be relaxed and takes the form

$$p_h(s) = (w_h(s) - u_h(s)) \odot \nu_{\partial\Omega_h}\mathcal{H}^2 \qquad \text{on } \Gamma_{d,h},$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure and $\odot$ is the symmetrised tensor product. The mechanical interpretation of this condition is the following: if the prescribed boundary displacement is not attained at time s, a plastic slip develops at the boundary with a strength proportional to $w_h(s) - u_h(s)$.

Because of the weak regularity of p_h (p_h and $\dot{p}_h$ are only measures in the space variable), the meaning of condition (d5) has to be clarified. In [5] this issue is overcome by expressing (d5) as a variational inequality involving only the stress variable σ_h and the velocity $\dot{u}_h$. In [7] the authors replace condition (d5) by its equivalent form

(d5)$'$ *maximum dissipation principle:* $H(\dot{p}_h(s)) = (\sigma_h)_D(s) : \dot{p}_h(s)$ in Ω_h,

where $H(\xi) := \sup_{\eta\in K} \xi : \eta$ is the support function of K. The advantage of condition (d5)$'$, compared to (d5), is that one can give a meaning to the equality in (d5)$'$ in a measure setting. This relies on a notion of duality beween stresses and plastic strains that was introduced in [22] and further developed in [9, 15]. However, the definition of the duality requires some regularity of $\partial\Omega_h$ and of the relative boundary of $\Gamma_{d,h}$ in $\partial\Omega_h$. Since in our framework $\partial\Omega_h$ has only Lipschitz regularity, we prefer not to dwell on duality and we formulate (d5) as an energy inequality:

(d5)$''$ *energy inequality:* for every $0 \le t_1 \le t_2$

$$\begin{aligned} &\mathcal{Q}_h(e_h(t_2)) + \frac{1}{2}\|\dot{u}_h(t_2)\|^2_{L^2} + \int_{t_1}^{t_2} \mathcal{H}_h(\dot{p}_h(s))\,ds \\ &\le \mathcal{Q}_h(e_h(t_1)) + \frac{1}{2}\|\dot{u}_h(t_1)\|^2_{L^2} + \int_{t_1}^{t_2}\int_{\Omega_h} (\sigma_h(s) : E\dot{w}_h(s) + \ddot{u}_h(s)\cdot\dot{w}_h(s))\,dx\,ds \\ &\quad + \int_{t_1}^{t_2}\int_{\Omega_h} f_h(s)e_3\cdot(\dot{u}_h(s) - \dot{w}_h(s))\,dx\,ds, \end{aligned}$$

where

$$\mathcal{Q}_h(e_h(s)) := \frac{1}{2}\int_{\Omega_h} \mathbb{C}e_h(s,x) : e_h(s,x)\,dx$$

is the stored elastic energy at time s, while $\mathcal{H}_h(\dot{p}_h(s))$ is the plastic dissipation potential at time s, defined according to the theory of convex functions of measure (see Section 2.2). When the stress-strain duality is defined and (d1)–(d4) are satisfied, one can prove that conditions (d5)$'$ and (d5)$''$ are in fact equivalent. For the reader's convenience the proof of the existence for system (d1)–(d4) and (d5)$''$ is sketched in Section 3. In view of the subsequent analysis, a particular attention is paid to the dependence of the involved quantities on the thickness parameter h.

Existence of a dynamic evolution (u_h, e_h, p_h) in Ω_h is therefore established for every $h > 0$. Our main goal is to study the asymptotic behaviour of (u_h, e_h, p_h), as h tends to 0, and characterise its limit as a solution of a suitable limiting problem. This is the object of Section 4.

To discuss the limiting behaviour of (u_h, e_h, p_h) it is convenient to rescale Ω_h to a domain Ω independent of h and to rescale time by setting $t := hs$. According to this change of variables, we define the rescaled displacement u^h on $[0, +\infty) \times \Omega$ as

$$u^h(t,x) := \big(u_h(\tfrac{t}{h}, (x', hx_3)) \cdot e_\alpha \,,\, hu_h(\tfrac{t}{h}, (x', hx_3)) \cdot e_3\big) \qquad \text{for } x = (x', x_3),\ \alpha = 1, 2. \quad (1.1)$$

The spatial scaling of u_h is consistent with that of dimension reduction problems in linearised elasticity. In particular, the ratio of order h between the vertical and the tangential displacements can be rigorously justified starting from nonlinear elasticity, under the small strain assumption [19]. Note, however, that in linearised elasticity the problem is invariant under further scalings of u^h, while this is not the case in plasticity, because of the different homogeneity of the elastic energy and the dissipation potential. The scaling (1.1) is the correct one to see both elastic and plastic contributions in the limit as $h \to 0$ (see also [11]).

The time scaling of u_h is also consistent with the results in the context of elasticity (see, e.g., [1]): oscillations in Ω_h occur at a slow time scale, so that a time scaling is needed to observe oscillations in the limit as $h \to 0$.

The scaling for e_h and p_h is chosen in such a way that the rescaled triplet $(u^h(t), e^h(t), p^h(t))$ still satisfies the additive decomposition $Eu^h(t) = e^h(t) + p^h(t)$ in Ω for every t. Finally, we perform the same scaling as in (1.1) on the boundary datum w_h, while for the body load we set

$$f^h(t,x) := \tfrac{1}{h} f_h(\tfrac{t}{h}, (x', hx_3)).$$

In Theorem 4.1 we prove that, under suitable assumptions on the initial data and on the rescaled boundary condition and body load, the rescaled triplets $(u^h(t), e^h(t), p^h(t))$ converge, up to subsequences, to a limiting triplet $(u(t), e(t), p(t))$ for every time $t \geq 0$.

We now describe the conditions satisfied by the limiting triplet. For every $t \geq 0$ we have

(d1)$_r$ *reduced kinematic admissibility:* $u(t)$ is a Kirchhoff-Love displacement, that is,

$$u(t,x) = (\bar{u}_\alpha(t,x') - x_3 \partial_\alpha u_3(t,x'), u_3(t,x')) \qquad \text{for } x = (x', x_3),\ \alpha = 1, 2,$$

where $\bar{u}(t) \in BD(\omega)$ and $u_3(t) \in BH(\omega)$, the space of functions with bounded Hessian. The strains $e(t)$ and $p(t)$ satisfy

$$Eu(t) = e(t) + p(t) \quad \text{in } \Omega, \qquad p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d,$$
$$e_{i3}(t) = 0 \quad \text{in } \Omega, \qquad p_{i3}(t) = 0 \quad \text{in } \Omega \cup \Gamma_d, \qquad i = 1, 2, 3.$$

We note that the averaged tangential displacement $\bar{u}(t)$ may have jump discontinuities, while, because of the embedding of $BH(\omega)$ into $C(\overline{\omega})$, the normal displacement $u_3(t)$ is continuous, but its gradient may have jump discontinuities. In particular, the discontinuity sets of $u(t)$, that is, the limiting slip surfaces, are vertical surfaces. Condition (d1)$_r$ does not imply, in general, that $e(t)$ and $p(t)$ are affine with respect to x_3. However, they admit the following decomposition:

$$e(t) = \bar{e}(t) + x_3 \hat{e}(t) + e_\perp(t), \qquad p(t) = \bar{p}(t) \otimes \mathcal{L}^1 + \hat{p}(t) \otimes x_3 \mathcal{L}^1 - e_\perp(t),$$

where $\bar{e}(t), \hat{e}(t) \in L^2(\omega; \mathbb{M}^{2\times 2}_{sym})$, $e_\perp(t) \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $\bar{p}(t), \hat{p}(t) \in M_b(\omega \cup \gamma_d; \mathbb{M}^{2\times 2}_{sym})$ satisfy

$$E\bar{u}(t) = \bar{e}(t) + \bar{p}(t) \quad \text{in } \omega, \qquad \bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega} \mathcal{H}^1 \quad \text{on } \gamma_d,$$

and

$$-D^2 u_3(t) = \hat{e}(t) + \hat{p}(t) \quad \text{in } \omega, \qquad \hat{p}(t) = (\nabla u_3(t) - \nabla w_3(t)) \odot \nu_{\partial\omega} \mathcal{H}^1 \quad \text{on } \gamma_d.$$

Moreover, the vertical displacement $u_3(t)$ attains the boundary conditions $u_3(t) = w_3(t)$ on γ_d. Here, $\bar{w}(t)$ and $w_3(t)$ are the Kirchhoff-Love components of the limiting displacement $w(t)$.

Since the component $e_\perp(t)$ has in general a non trivial dependence on the variable x_3, the limiting problem has a genuinely three-dimensional nature and cannot be reduced to a purely two-dimensional setting. This feature was already observed in the quasistatic case [11] and is in contrast with the purely elastic case, see [31].

In addition, the limiting triplet $(u(t), e(t), p(t))$ satisfies the following conditions for every $t \geq 0$:

(d2)$_r$ *reduced constitutive equation:* $\sigma(t) := \mathbb{C}_r e(t)$ in Ω, where $\mathbb{C}_r$ is the reduced elasticity tensor, which is defined through a suitable minimisation formula (see (2.10));

(d3)$_r$ *equations of motion:* setting

$$\bar{f}(t, x') = \int_{-1/2}^{1/2} f(t, x)\, dx_3,$$

we have

$$\operatorname{div} \bar{\sigma}(t) = 0 \quad \text{in } \omega, \qquad \ddot{u}_3(t) - \tfrac{1}{12} \operatorname{div} \operatorname{div} \hat{\sigma}(t) = \bar{f}(t) \quad \text{in } \omega,$$

with corresponding Neumann boundary conditions on $\partial\omega \setminus \gamma_d$;

(d4)$_r$ *reduced stress constraint:* $\sigma(t) \in K_r$ in Ω, where $K_r := \partial H_r(0)$ is the subdifferential of the reduced dissipation potential H_r (whose expression is given in (2.12) through a minimisation formula) at 0;

(d5)$_r$ *reduced maximum dissipation principle:* $H_r(\dot{p}(t)) = \sigma(t) : \dot{p}(t)$ in Ω.

In (d3)$_r$ we denoted the limiting vertical body load by f.

Condition (d5)$_r$ has to be interpreted in a weak sense, as it is the case for condition (d5)$'$ in the three-dimensional problem. More precisely, we write (d5)$_r$ as

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle_r\,,$$

where the left-hand side is defined using the theory of convex functions of measures, while the right-hand side involves an ad-hoc notion of "reduced" stress-strain duality, introduced in [11, Section 7] for the study of the quasistatic case. We refer to Section 2 for the definition of the duality, as well as for a precise kinematic description of the reduced model.

We note that the stretching component $\bar{\sigma}(t)$ and the bending component $\hat{\sigma}(t)$ of the stress decouple in the equations of motion (d3)$_r$, while the whole stress $\sigma(t)$ is involved in the stress constraint (d4)$_r$ and in the maximum dissipation principle (d5)$_r$. Thus, the component $\sigma_\perp(t)$ will in general play a role in satisfying these two conditions, leading to a non trivial dependence of the solutions on the thickness variable x_3. As mentioned earlier, this behaviour is not peculiar of the dynamic case, but was already observed in the quasistatic case. Indeed, an explicit example in [12, Section 5] shows that the yielding threshold may be reached at different times along the vertical fibers of the plate, thus giving rise to a solution with $\sigma_\perp \neq 0$. The emergence of this multiyield behaviour was also observed in [21], where a formal asymptotic expansion of small strain oscillations in an elastoplastic plate with hardening was considered.

The proof of Theorem 4.1 is based on two main steps: first we deduce suitable compactness estimates for the three-dimensional evolutions, and then we pass to the limit in the equations via Γ-convergence arguments. Compactness estimates are obtained from the energy inequality (d5)$''$ and from some a posteriori regularity estimates for the three-dimensional problem (see (3.8) and (3.9)). Clearly the dependence of these inequalities on h is crucial in order to obtain meaningful bounds. While the behaviour of the energy inequality under scaling is relatively straightforward, dealing with the a posteriori estimate is more delicate. At this stage it is essential to have a purely vertical body load. Once these bounds are established, compactness is granted via Ascoli-Arzelà Theorem.

To pass to the limit in the equations, we cannot rely directly on Γ-convergence techniques, because of the inertial terms. However, the key ideas of the proof are borrowed by this theory. More precisely, to deduce the limiting equations of motion we construct suitable sequences of test functions for the three-dimensional problems. This is reminiscent of the recovery sequence construction in Γ-convergence. To pass to the limit in (d5)$''$ we apply the Γ-liminf inequality satisfied by $\mathcal{Q}_r$ and $\mathcal{H}_r$. Once we have a limiting energy inequality, condition (d5)$_r$ follows by using the reduced stress-strain duality and its properties.

The last section of the paper is devoted to the study of some properties of solutions to the limiting system (d1)$_r$–(d5)$_r$. In Proposition 5.1 we prove uniqueness of the normal displacement and of the elastic strain. This does not ensures uniqueness of the solution to the limiting problem. Indeed, in Proposition 5.2 we show that for "tangential" initial and boundary data system (d1)$_r$–(d5)$_r$ reduces to a two-dimensional quasistatic evolution, whose solutions are not unique (see, e.g., [34]).

2. Preliminaries

2.1. **Mathematical preliminaries.**

Measures. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space X, $M_b(B;X)$ denotes the space of bounded Borel measures on B with values in X, endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in M_b(B;\mathbb{R})$ is the variation of the measure μ. For every $\mu \in M_b(B;X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and μ^s is singular with respect to $\mathcal{L}^n$. If $\mu^s = 0$, we always identify μ with its density with respect to $\mathcal{L}^n$. If the relative topology of B is locally compact, by Riesz Representation Theorem $M_b(B;X)$ can be identified with the dual of $C_0(B;X)$, which is the space of continuous functions $\varphi : B \to X$ such that the set $\{\varphi \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $M_b(B;X)$ is defined using this duality. The duality between measures and continuous functions, as well as between other pairs of spaces, according to the context, is denoted by $\langle \cdot, \cdot \rangle$.

Matrices. The space of $n \times n$ symmetric matrices is denoted by $\mathbb{M}^{n\times n}_{sym}$ and is endowed with the euclidean scalar product $\xi : \zeta := \sum_{i,j} \xi_{ij}\zeta_{ij}$. The orthogonal complement of the subspace $\mathbb{R}I_{n\times n}$ spanned by the identity matrix $I_{n\times n}$ is the subspace $\mathbb{M}^{n\times n}_D$ of all symmetric matrices with zero trace. For every $\xi \in \mathbb{M}^{n\times n}_{sym}$, we obtain the orthogonal decomposition

$$\xi = \xi_D + \frac{1}{n}(\operatorname{tr}\xi)I_{n\times n},$$

where $\xi_D \in \mathbb{M}^{n\times n}_D$ is the deviatoric part of ξ. The symmetrised tensor product $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a \odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$.

Functions with bounded deformation. Let $U \subset \mathbb{R}^n$ be an open set. The space $BD(U)$ of functions with bounded deformation is the space of all $u \in L^1(U;\mathbb{R}^n)$, whose symmetric gradient (in the sense of distributions) $Eu := \frac{1}{2}(Du + Du^T)$ belongs to the space $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm

$$\|u\|_{BD} := \|u\|_{L^1} + \|Eu\|_{M_b}.$$

We say that a sequence $(u_k)_k$ converges to u weakly* in $BD(U)$ if $u_k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. Moreover, if U is bounded and has a Lipschitz boundary, then $BD(U)$ can be continuously embedded in $L^{n/(n-1)}(U;\mathbb{R}^n)$ and compactly embedded in $L^p(U;\mathbb{R}^n)$ for every $p < n/(n-1)$. Moreover, every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^n)$, and if Γ is a nonempty open subset of ∂U, there exists a constant $C > 0$, depending on U and Γ, such that

$$\|u\|_{BD} \leq C(\|u\|_{L^1(\Gamma)} + \|Eu\|_{M_b}).$$

For the general properties of $BD(U)$ we refer to [35].

Functions with bounded Hessian. The space $BH(U)$ of functions with bounded Hessian is the space of all functions $u \in W^{1,1}(U)$, whose Hessian D^2u (in the sense of distributions) belongs to $M_b(U; \mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BH(U)$ is a Banach space endowed with the norm

$$\|u\|_{BH} := \|u\|_{W^{1,1}} + \|D^2u\|_{M_b}.$$

If U has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $M_b(U; \mathbb{M}^{n\times n}_{sym})$. If U is bounded and has a Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,n/(n-1)}(U)$. If U is bounded and has a C^2 boundary, then for every function $u \in BH(U)$ one can define the traces of u and ∇u, still denoted by u and ∇u: they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U; \mathbb{R}^n)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau \in L^1(\partial U)$ for every τ tangent vector to ∂U. If in addition $n = 2$, then $BH(U)$ embeds into $C(\overline{U})$, which is the space of continuous functions on $\overline{U}$. For the general properties of $BH(U)$ we refer to [13].

Lipschitz functions with values in a Banach space. Let $T > 0$ and let X be the dual of a separable Banach space. We denote by $Lip([0, T]; X)$ the space of Lipschitz functions on $[0, T]$ with values in X. If $f \in Lip([0, T]; X)$, then the weak* limit

$$\dot{f}(t) := w^*\text{-}\lim_{s\to t} \frac{f(s) - f(t)}{s - t} \tag{2.1}$$

exists for a.e. $t \in [0, T]$ (see, e.g., [9, Theorem 7.1]). If in addition X is separable, then for every $f \in Lip([0, T]; X)$ the limit in (2.1) is actually in the strong topology of X, the map $t \mapsto \dot{f}(t)$ is measurable by Pettis Theorem, and

$$Lip([0, T]; X) = W^{1,\infty}([0, T]; X).$$

2.2. **Mechanical preliminaries: the three-dimensional problem.** In this section we describe the setting of the three-dimensional problem.

The reference configuration. Let $h > 0$ and let $\omega \subset \mathbb{R}^2$ be a domain (that is, an open, connected, and bounded set) with a C^2 boundary. We consider a thin plate whose reference configuration is given by

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2}).$$

We set $\Omega := \Omega_1$ and for $x \in \Omega$ we write $x = (x', x_3)$, where $x' \in \omega$ and $x_3 \in (-1/2, 1/2)$. We suppose that the boundary of ω is partitioned into two disjoint open sets γ_d, γ_n (which are the Dirichlet and the Neumann part of $\partial\omega$, respectively) and their common boundary $\partial_{|\partial\omega}\gamma_d$, that is,

$$\partial\omega = \gamma_d \cup \gamma_n \cup \partial_{|\partial\omega}\gamma_d.$$

We assume that $\partial_{|\partial\omega}\gamma_d = \{P_1, P_2\}$, where P_1 and P_2 are two points of $\partial\omega$. Moreover, we define $\Gamma_{d,h} := \gamma_d \times (-\frac{h}{2}, \frac{h}{2})$ and $\Gamma_{n,h} := \partial\Omega_h \setminus \overline{\Gamma}_{d,h}$. We also set $\Gamma_d := \Gamma_{d,1}$ and $\Gamma_n := \Gamma_{n,1}$. We will denote the outer unit normal to $\partial\Omega_h$ and to $\partial\omega$ by $\nu_{\partial\Omega_h}$ and by $\nu_{\partial\omega}$, respectively.

The stored elastic energy. Let $\mathbb{C}$ be the three-dimensional elasticity tensor, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$, and let $Q : \mathbb{M}^{3\times3}_{sym} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}.$$

It turns out that there exists two positive constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $\alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}}$, such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{2.2}$$

These inequalities imply that

$$|\mathbb{C}\xi| \leq 2\beta_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{2.3}$$

It is convenient to introduce the quadratic form $\mathcal{Q}_h : L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym}) \to [0, +\infty)$ given by

$$\mathcal{Q}_h(e) := \int_{\Omega_h} Q(e(x))\, dx$$

for every $e \in L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym})$. It describes the stored elastic energy of a configuration of Ω_h, whose elastic strain is e. Since $\mathcal{Q}_h$ is a convex functional, it is lower semicontinuous with respect to the weak convergence of $L^2(\Omega_h; \mathbb{M}^{3\times3}_{sym})$. We set $\mathcal{Q} := \mathcal{Q}_1$.

The plastic dissipation. Let K be a convex and compact set in $\mathbb{M}^{3\times3}_D$, whose boundary ∂K is interpreted as the yield surface. We assume that there exist two positive constants r_K and R_K, with $r_K \leq R_K$, such that

$$B(0, r_K) \subset K \subset B(0, R_K), \tag{2.4}$$

where $B(0,r) := \{\xi \in \mathbb{M}^{3\times3}_D : |\xi| \leq r\}$. The support function of K, which represents the three-dimensional plastic dissipation potential, is the function $H : \mathbb{M}^{3\times3}_D \to \mathbb{R}$ given by

$$H(\xi) := \sup_{\tau \in K} \xi : \tau \qquad \text{for every } \xi \in \mathbb{M}^{3\times3}_D.$$

It is easy to see that H is convex, positively 1-homogeneous, and satisfies the triangle inequality. Moreover, by (2.4) one deduces that

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \qquad \text{for every } \xi \in \mathbb{M}^{3\times3}_D. \tag{2.5}$$

From standard convex analysis we also have that the set K coincides with the subdifferential $\partial H(0)$ of H at 0.

Let $\mu \in M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$ and let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$. According to the theory of convex functions of measures (see [20]), we define the nonnegative Radon measure $H_h(\mu)$ as

$$H_h(\mu)(B) := \int_B H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $B \subset \Omega_h \cup \Gamma_{d,h}$. We also consider the functional

$$\mathcal{H}_h : M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D) \to [0, +\infty)$$

defined by

$$\mathcal{H}_h(\mu) := H_h(\mu)(\Omega_h \cup \Gamma_{d,h}).$$

One can prove (see, e.g., [35, Chapter II, Section 4]) that

$$\mathcal{H}_h(\mu) = \sup\Big\{ \int_{\Omega_h \cup \Gamma_{d,h}} \tau : d\mu \ : \ \tau \in C_0(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D),\ \tau(x) \in K \text{ for a.e. } x \in \Omega_h \cup \Gamma_{d,h} \Big\}.$$

From this characterisation it is clear that $\mathcal{H}_h$ is lower semicontinuous with respect to the weak* convergence of $M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$.

We also define the total variation of a function $\mu : [0,T] \to M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$ in an interval $[a,b] \subset [0,T]$ as

$$\mathcal{V}_h(\mu; a, b) := \sup\Big\{ \sum_{j=1}^N \|\mu(s_j) - \mu(s_{j-1})\|_{M_b} :\ a = s_0 \leq s_1 \leq \cdots \leq s_N = b,\ N \in \mathbb{N} \Big\},$$

and the dissipation of μ in $[a,b]$ as

$$\mathcal{D}_h(\mu; a, b) := \sup\Big\{ \sum_{j=1}^N \mathcal{H}_h(\mu(s_j) - \mu(s_{j-1})) :\ a = s_0 \leq s_1 \leq \cdots \leq s_N = b,\ N \in \mathbb{N} \Big\}.$$

It follows from (2.5) that

$$r_K \mathcal{V}_h(\mu; a, b) \leq \mathcal{D}_h(\mu; a, b) \leq R_K \mathcal{V}_h(\mu; a, b).$$

Moreover, if μ is absolutely continuous on $[a,b]$ with values in $M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$, then one has

$$\mathcal{D}_h(\mu; a, b) = \int_a^b \mathcal{H}_h(\dot{\mu}(s))\, ds \tag{2.6}$$

(see [9, Theorem 7.1]). We set $\mathcal{H} := \mathcal{H}_1$, $\mathcal{V} := \mathcal{V}_1$, and $\mathcal{D} := \mathcal{D}_1$.

Kinematic admissibility. Given a boundary datum $w \in H^1(\Omega_h;\mathbb{R}^3)$, we define the class $\mathcal{A}_h(w)$ of admissible displacements and strains, as the set of all triplets $(u,e,p) \in BD(\Omega_h) \times L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega_h \cup \Gamma_{d,h};\mathbb{M}^{3\times3}_D)$ such that

$$Eu = e + p \quad \text{in } \Omega_h, \qquad p = (w-u)\odot \nu_{\partial\Omega_h}\mathcal{H}^2 \quad \text{on } \Gamma_{d,h}.$$

We set $\mathcal{A}(w) := \mathcal{A}_1(w)$ for every $w \in H^1(\Omega;\mathbb{R}^3)$.

The trace of stresses. We recall that, if $\sigma \in L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym})$ with $\operatorname{div}\sigma \in L^2(\Omega_h;\mathbb{R}^3)$, we can define the trace $[\sigma\nu_{\partial\Omega_h}] \in H^{-1/2}(\partial\Omega_h;\mathbb{R}^3)$ of its normal component through the formula

$$\langle[\sigma\nu_{\partial\Omega_h}],\varphi\rangle := \int_{\Omega_h} \sigma : E\varphi\,dx + \int_{\Omega_h} \operatorname{div}\sigma\cdot\varphi\,dx$$

for every $\varphi \in H^1(\Omega_h;\mathbb{R}^3)$. In the following we say that $[\sigma\nu_{\partial\Omega_h}] = 0$ on $\Gamma_{n,h}$ if $\langle[\sigma\nu_{\partial\Omega_h}],\varphi\rangle = 0$ for every $\varphi \in H^1(\Omega_h;\mathbb{R}^3)$ with $\varphi = 0$ on $\Gamma_{d,h}$.

2.3. **Mechanical preliminaries: the reduced problem.** In this section we introduce the setting of the limiting problem.

The reduced stored elastic energy. Let $\mathbb{M} : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ be the operator given by

$$\mathbb{M}\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1(\xi) \\ \xi_{12} & \xi_{22} & \lambda_2(\xi) \\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}, \tag{2.7}$$

where the triplet $(\lambda_1(\xi),\lambda_2(\xi),\lambda_3(\xi))$ is the unique solution of the minimum problem

$$\min_{\lambda_i\in\mathbb{R}} Q\begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We observe that $(\lambda_1(\xi),\lambda_2(\xi),\lambda_3(\xi))$ can be characterised as the unique solution of the linear system

$$\mathbb{C}\mathbb{M}\xi : \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} = 0 \tag{2.8}$$

for every $\zeta_i \in \mathbb{R}$, $i = 1,2,3$. This implies that $\mathbb{M}$ is a linear map and

$$(\mathbb{C}\mathbb{M}\xi)_{i3} = (\mathbb{C}\mathbb{M}\xi)_{3i} = 0 \quad \text{for every } i = 1,2,3. \tag{2.9}$$

Let $Q_r : \mathbb{M}^{2\times2}_{sym} \to \mathbb{R}$ be the quadratic form given by

$$Q_r(\xi) := Q(\mathbb{M}\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}.$$

It follows from (2.2) that

$$\alpha_{\mathbb{C}}|\xi|^2 \le Q_r(\xi) \le \beta_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}.$$

We define the reduced elasticity tensor as the linear operator $\mathbb{C}_r : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ given by

$$\mathbb{C}_r\xi := \mathbb{C}\mathbb{M}\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{2.10}$$

Note that we can always identify $\mathbb{C}_r\xi$ with an element of $\mathbb{M}^{2\times2}_{sym}$ in view of (2.9). Moreover, by (2.8) we have

$$\mathbb{C}_r\xi : \zeta = \mathbb{C}_r\xi : \begin{pmatrix} \zeta_{11} & \zeta_{12} & 0 \\ \zeta_{12} & \zeta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym},\ \zeta \in \mathbb{M}^{3\times3}_{sym}. \tag{2.11}$$

This implies that

$$Q_r(\xi) = \frac{1}{2}\mathbb{C}_r\xi : \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}.$$

Finally, we introduce the functional $\mathcal{Q}_r : L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \to [0,+\infty)$, defined as

$$\mathcal{Q}_r(e) := \int_\Omega Q_r(e(x))\,dx$$

for every $e \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$. It describes the reduced elastic energy of a configuration, whose elastic strain is e.

The reduced plastic dissipation. In the reduced problem the plastic dissipation potential is given by the function $H_r : \mathbb{M}^{2\times 2}_{sym} \to [0,+\infty)$, defined as

$$H_r(\xi) := \min_{\lambda_i\in\mathbb{R}} H\begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11}+\xi_{22}) \end{pmatrix} \tag{2.12}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$. From the properties of H it follows that H_r is convex, positively 1-homogeneous, and satisfies

$$r_K|\xi| \le H_r(\xi) \le \sqrt{3}R_K|\xi| \text{ for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

The set $K_r := \partial H_r(0)$ represents the set of admissible stresses in the reduced problem and can be characterised as follows:

$$\xi \in K_r \quad \Leftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi)I_{3\times 3} \in K, \tag{2.13}$$

(see [11, Section 3.2]).

For every $\mu \in M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ we define the functional

$$\mathcal{H}_r(\mu) := \int_{\Omega\cup\Gamma_d} H_r\Big(\frac{d\mu}{d|\mu|}\Big)\,d|\mu|.$$

We also define the reduced dissipation of a function $\mu : [0,T] \to M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ in an interval $[a,b] \subset [0,T]$ as

$$\mathcal{D}_r(\mu;a,b) := \sup\Big\{\sum_{j=1}^N \mathcal{H}_r(\mu(s_j)-\mu(s_{j-1})) :\ a = s_0 \le s_1 \le \cdots \le s_N = b,\ N\in\mathbb{N}\Big\}.$$

If μ is absolutely continuous on $[a,b]$ with values in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$, then

$$\mathcal{D}_r(\mu;a,b) = \int_a^b \mathcal{H}_r(\dot\mu(s))\,ds. \tag{2.14}$$

Reduced kinematic admissibility. We introduce the set $KL(\Omega)$ of Kirchhoff-Love displacements, defined as

$$KL(\Omega) := \{u \in BD(\Omega) :\ (Eu)_{i3} = 0,\ i = 1,2,3\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar u \in BD(\omega)$ such that

$$u_\alpha(x) = \bar u_\alpha(x') - x_3\partial_\alpha u_3(x') \quad \text{for } x = (x',x_3)\in\Omega,\ \alpha = 1,2.$$

We call $\bar u$, u_3 the *Kirchhoff-Love components* of u.

Given a prescribed displacement $w \in H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$, we introduce the set $\mathcal{A}_{KL}(w)$ of *Kirchhoff-Love admissible triplets*, defined as the class of all triplets

$$(u,e,p) \in KL(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})\times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times 3}_{sym})$$

such that

$$Eu = e+p \quad \text{in } \Omega, \qquad p = (w-u)\odot\nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d,$$

$$e_{i3} = 0 \quad \text{in } \Omega, \qquad p_{i3} = 0 \quad \text{in } \Omega\cup\Gamma_d, \qquad i = 1,2,3.$$

The linear space $\{\xi \in \mathbb{M}^{3\times 3}_{sym} :\ \xi_{i3} = 0,\ i = 1,2,3\}$ is isomorphic to $\mathbb{M}^{2\times 2}_{sym}$. Thus, in the following, given $(u,e,p)\in\mathcal{A}_{KL}(w)$, we will always identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and p with a measure in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$.

The following closure property holds.

Lemma 2.1. *Let $(w_n)_n$ be a sequence in $H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u_n,e_n,p_n)\in\mathcal{A}_{KL}(w_n)$ be a sequence of admissible triplets. Assume that $u_n \rightharpoonup u$ weakly* in $BD(\Omega)$, $e_n \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $p_n \rightharpoonup p$ weakly* in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym})$, and $w_n \rightharpoonup w$ weakly in $H^1(\Omega;\mathbb{R}^3)$. Then $(u,e,p)\in\mathcal{A}_{KL}(w)$.*

Proof. The result easily follows by adapting the proof of [9, Lemma 2.1]. □

We now give a characterisation of the class of Kirchhoff-Love admissible triplets. To this purpose, we introduce the following definitions.

Definition 2.2. Let $f\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar f$, $\hat f\in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$ and by $f_\perp\in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$) of f:

$$\bar f(x') := \int_{-\frac12}^{\frac12} f(x',x_3)\,dx_3, \qquad \hat f(x') := 12\int_{-\frac12}^{\frac12} x_3 f(x',x_3)\,dx_3$$

for a.e. $x'\in\omega$, and

$$f_\perp(x) := f(x) - \bar f(x') - x_3\hat f(x')$$

for a.e. $x\in\Omega$. We call $\bar f$ the *zeroth order moment* of f and $\hat f$ the *first order moment* of f.

Definition 2.3. Let $q\in M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym})$. We denote by $\bar q$, $\hat q\in M_b(\omega\cup\gamma_d;\mathbb{M}^{2\times2}_{sym})$ and by $q_\perp\in M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym})$ the following measures:

$$\int_{\omega\cup\gamma_d}\varphi : d\bar q := \int_{\Omega\cup\Gamma_d}\varphi : dq, \qquad \int_{\omega\cup\gamma_d}\varphi : d\hat q := 12\int_{\Omega\cup\Gamma_d} x_3\varphi : dq$$

for every $\varphi\in C_0(\omega\cup\gamma_d;\mathbb{M}^{2\times2}_{sym})$, and

$$q_\perp := q - \bar q\otimes\mathcal{L}^1 - \hat q\otimes x_3\mathcal{L}^1,$$

where $\otimes$ denotes the usual product of measures. We call $\bar q$ the *zeroth order moment* of q and $\hat q$ the *first order moment* of q.

With these definitions at hand one can easily prove the following characterisation of the class $\mathcal{A}_{KL}(w)$.

Proposition 2.4. *Let $w\in H^1(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u,e,p)\in KL(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times2}_{sym})\times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym})$. Then $(u,e,p)\in\mathcal{A}_{KL}(w)$ if and only if the following three conditions are satisfied:*

(i) *$E\bar u = \bar e+\bar p$ in ω and $\bar p = (\bar w-\bar u)\odot\nu_{\partial\omega}\mathcal{H}^1$ on γ_d;*

(ii) *$D^2u_3 = -(\hat e+\hat p)$ in ω, $u_3=w_3$ on γ_d, and $\hat p = (\nabla u_3-\nabla w_3)\odot\nu_{\partial\omega}\mathcal{H}^1$ on γ_d;*

(iii) *$p_\perp = -e_\perp$ in Ω and $p_\perp = 0$ on Γ_d.*

Proof. The statement easily follows from the definition of moments and from the formula $(Eu)_{\alpha\beta} = (E\bar u)_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3$ for $\alpha,\beta=1,2$. □

Stress-strain duality. In the reduced model, we shall consider the set $\Sigma(\Omega)$ of *admissible stresses*, defined as

$$\Sigma(\Omega) := \{\sigma\in L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym}):\ \operatorname{div}\bar\sigma\in L^2(\omega;\mathbb{R}^2),\ \operatorname{div}\operatorname{div}\hat\sigma\in L^2(\omega)\}.$$

For every $\sigma\in\Sigma(\Omega)$ we can define the trace $[\bar\sigma\nu_{\partial\omega}]\in L^\infty(\partial\omega;\mathbb{R}^2)$ of its zeroth order moment normal component as

$$\langle[\bar\sigma\nu_{\partial\omega}],\psi\rangle := \int_\omega \bar\sigma : E\psi\,dx' + \int_\omega \operatorname{div}\bar\sigma\cdot\psi\,dx' \tag{2.15}$$

for every $\psi\in W^{1,1}(\omega;\mathbb{R}^2)$. Note that, since $\bar\sigma\in L^\infty(\omega;\mathbb{M}^{2\times2}_{sym})$ and $W^{1,1}(\omega;\mathbb{R}^2)$ embeds into $L^2(\omega;\mathbb{R}^2)$, all terms at the right-hand side of (2.15) are well defined.

Let $T(W^{2,1}(\omega))$ be the space of all traces of functions in $W^{2,1}(\omega)$ and let $(T(W^{2,1}(\omega)))'$ be its dual space. For every $\sigma \in \Sigma(\Omega)$ we can define the traces $b_0(\hat{\sigma}) \in (T(W^{2,1}(\omega)))'$ and $b_1(\hat{\sigma}) \in L^\infty(\partial\omega)$ of its first order moment as

$$-\langle b_0(\hat{\sigma}), \psi\rangle + \big\langle b_1(\hat{\sigma}), \frac{\partial\psi}{\partial\nu_{\partial\omega}}\big\rangle := \int_\omega \hat{\sigma} : D^2\psi\, dx' - \int_\omega \psi \operatorname{div}\operatorname{div}\hat{\sigma}\, dx' \tag{2.16}$$

for every $\psi \in W^{2,1}(\omega)$. Note that the right-hand side of (2.16) is well defined since $\hat{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times2}_{sym})$.

If $\hat{\sigma} \in C^2(\overline{\omega}, \mathbb{M}^{2\times2}_{sym})$, one can prove that

$$\begin{aligned} b_0(\hat{\sigma}) &= \operatorname{div}\hat{\sigma}\cdot\nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}\left(\hat{\sigma}\tau_{\partial\omega}\cdot\nu_{\partial\omega}\right), \\ b_1(\hat{\sigma}) &= \hat{\sigma}\nu_{\partial\omega}\cdot\nu_{\partial\omega}, \end{aligned}$$

where $\tau_{\partial\omega}$ is the tangent vector to $\partial\omega$.

Since $[\bar{\sigma}\nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$, the expression $[\bar{\sigma}\nu_{\partial\omega}] = 0$ on γ_n has a clear meaning. The same applies to $b_1(\hat{\sigma})$. As for $b_0(\hat{\sigma})$, in the following we say that $b_0(\hat{\sigma}) = 0$ on γ_n if $\langle b_0(\hat{\sigma}), \psi\rangle = 0$ for every $\psi \in W^{2,1}(\omega)$ with $\psi = 0$ on γ_d.

We also consider the space of *admissible plastic strains* $\Pi_{\Gamma_d}(\Omega)$, which is the set of all measures $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times2}_{sym})$ for which there exists $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) \times (H^1(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$.

For every $\sigma \in \Sigma(\Omega)$ and $\xi \in BD(\omega)$ we define the distribution $[\bar{\sigma} : E\xi]$ on ω as

$$\langle[\bar{\sigma} : E\xi], \varphi\rangle := -\int_\omega \varphi \operatorname{div}\bar{\sigma}\cdot\xi\, dx' - \int_\omega \bar{\sigma} : (\nabla\varphi \odot \xi)\, dx'$$

for every $\varphi \in C^\infty_c(\omega)$. It follows from [22, Theorem 3.2] that $[\bar{\sigma} : E\xi] \in M_b(\omega)$ and its variation satisfies

$$|[\bar{\sigma} : E\xi]| \le \|\bar{\sigma}\|_{L^\infty}|E\xi| \quad \text{ in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$, we define the measure $[\bar{\sigma} : \bar{p}] \in M_b(\omega \cup \gamma_d)$ as

$$[\bar{\sigma} : \bar{p}] := \begin{cases} [\bar{\sigma} : E\bar{u}] - \bar{\sigma} : \bar{e} & \text{in } \omega, \\ [\bar{\sigma}\nu_{\partial\omega}]\cdot(\bar{w} - \bar{u})\,\mathcal{H}^1 & \text{on } \gamma_d. \end{cases}$$

For every $\sigma \in \Sigma(\Omega)$ and $v \in BH(\omega)$ we define the distribution $[\hat{\sigma} : D^2 v]$ on ω as

$$\langle[\hat{\sigma} : D^2 v], \psi\rangle := \int_\omega \psi v \operatorname{div}\operatorname{div}\hat{\sigma}\, dx' - 2\int_\omega \hat{\sigma} : (\nabla v \odot \nabla\psi)\, dx' - \int_\omega v\hat{\sigma} : D^2\psi\, dx'$$

for every $\psi \in C^\infty_c(\omega)$. From [14, Proposition 2.1] it follows that $[\hat{\sigma} : D^2 v] \in M_b(\omega)$ and its variation satisfies

$$|[\hat{\sigma} : D^2 v]| \le \|\hat{\sigma}\|_{L^\infty}|D^2 v| \quad \text{ in } \omega.$$

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$, we define the measure $[\hat{\sigma} : \hat{p}] \in M_b(\omega \cup \gamma_d)$ as

$$[\hat{\sigma} : \hat{p}] := \begin{cases} -[\hat{\sigma} : D^2 u_3] - \hat{\sigma} : \hat{e} & \text{in } \omega, \\ b_1(\hat{\sigma})\dfrac{\partial(u_3 - w_3)}{\partial\nu_{\partial\omega}}\,\mathcal{H}^1 & \text{on } \gamma_d. \end{cases}$$

We are now in a position to introduce a duality between $\Sigma(\Omega)$ and $\Pi_{\Gamma_d}(\Omega)$. For every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$ we define the measure $[\sigma : p]_r \in M_b(\Omega \cup \Gamma_d)$ as

$$[\sigma : p]_r := [\bar{\sigma} : \bar{p}] \otimes \mathcal{L}^1 + \frac{1}{12}[\hat{\sigma} : \hat{p}] \otimes \mathcal{L}^1 - \sigma_\perp : e_\perp.$$

We also introduce the duality pairings

$$\langle\bar{\sigma}, \bar{p}\rangle := [\bar{\sigma} : \bar{p}](\omega \cup \gamma_d), \qquad \langle\hat{\sigma}, \hat{p}\rangle := [\hat{\sigma} : \hat{p}](\omega \cup \gamma_d)$$

and

$$\langle\sigma, p\rangle_r := [\sigma : p]_r(\Omega \cup \Gamma_d) = \langle\bar{\sigma}, \bar{p}\rangle + \frac{1}{12}\langle\hat{\sigma}, \hat{p}\rangle - \int_\Omega \sigma_\perp : e_\perp\, dx. \tag{2.17}$$

One can show (see [11, Proposition 7.8]) that

$$\mathcal{H}_r(p) = \sup\big\{\langle \sigma, p\rangle_r :\ \sigma \in \Sigma(\Omega),\ \sigma(x) \in K_r \text{ for a.e. } x \in \Omega\big\}. \tag{2.18}$$

Finally, the following integration by parts formula holds (see [12, Proposition 3.5]).

Proposition 2.5. *Let $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then*

$$\begin{aligned}
&\int_{\Omega\cup\Gamma_d} \varphi\, d[\sigma : p]_r + \int_\Omega \varphi\sigma : (e - Ew)\, dx \\
&= \ -\int_\omega \bar\sigma : (\nabla\varphi \odot (\bar u - \bar w))\, dx' - \int_\omega \operatorname{div}\bar\sigma \cdot \varphi(\bar u - \bar w)\, dx' \\
&\quad + \int_{\gamma_n} [\bar\sigma\nu_{\partial\omega}] \cdot \varphi(\bar u - \bar w)\, d\mathcal{H}^1 + \frac{1}{12}\int_\omega \hat\sigma : (u_3 - w_3) D^2\varphi\, dx' \\
&\quad + \frac{1}{6}\int_\omega \hat\sigma : (\nabla\varphi \odot (\nabla u_3 - \nabla w_3))\, dx' - \frac{1}{12}\int_\omega \varphi(u_3 - w_3)\operatorname{div}\operatorname{div}\hat\sigma\, dx' \\
&\quad + \frac{1}{12}\langle b_0(\hat\sigma), \varphi(u_3 - w_3)\rangle - \frac{1}{12}\int_{\gamma_n} b_1(\hat\sigma)\frac{\partial(\varphi(u_3 - w_3))}{\partial\nu_{\partial\omega}}\, d\mathcal{H}^1
\end{aligned}$$

for every $\varphi \in C^2(\overline{\omega})$.

3. Existence of three-dimensional dynamic evolutions

In this section we adapt the existence result [5, Theorem 1.3] of a dynamic evolution for perfectly plastic bodies to the context of a thin plate. Indeed, in view of the dimension reduction analysis of the next section, it is crucial to understand the dependence of all the involved quantities on the thickness parameter h.

We start by describing the assumptions on the data of the problem.

Forces. We assume the applied body loads to be purely vertical and with the following regularity:

$$f_h \in W^{1,1}_{loc}([0,+\infty); L^2(\Omega_h)). \tag{3.1}$$

We assume there are no traction forces on the Neumann part of the boundary $\Gamma_{n,h}$.

Boundary displacement. On $\Gamma_{d,h}$ we prescribe a boundary displacement

$$w_h \in H^2_{loc}([0,+\infty); H^1(\Omega_h;\mathbb{R}^3)) \cap W^{3,1}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{R}^3)). \tag{3.2}$$

Initial data. Let

$$\begin{gathered}
(u_{0,h}, e_{0,h}, p_{0,h}) \in \mathcal{A}_h(w_h(0)) \cap (H^1(\Omega_h;\mathbb{R}^3) \times L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym}) \times L^2(\Omega_h;\mathbb{M}^{3\times3}_D)),\\
v_{0,h} \in H^1(\Omega_h;\mathbb{R}^3),
\end{gathered} \tag{3.3}$$

be the initial data. Setting $\sigma_{0,h} := \mathbb{C}e_{0,h}$, we assume that

$$-\operatorname{div}\sigma_{0,h} = f_h(0)e_3 \text{ in } \Omega_h, \quad [\sigma_{0,h}\nu_{\partial\Omega_h}] = 0 \text{ on } \Gamma_{n,h}, \quad (\sigma_{0,h})_D \in K \text{ a.e. in } \Omega_h, \tag{3.4}$$

and

$$v_{0,h} = \dot w_h(0) \text{ on } \Gamma_{d,h}. \tag{3.5}$$

Theorem 3.1. *Assume* (3.1)–(3.5). *Then there exists a triplet (u_h, e_h, p_h), with*

$$\begin{gathered}
u_h \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{R}^3)) \cap Lip_{loc}([0,+\infty); BD(\Omega_h)),\\
e_h \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega_h;\mathbb{M}^{3\times3}_{sym})), \qquad p_h \in Lip_{loc}([0,+\infty); M_b(\Omega_h \cup \Gamma_{d,h};\mathbb{M}^{3\times3}_D)),
\end{gathered}$$

satisfying the following system of equations:

(i) kinematic admissibility: $(u_h(t), e_h(t), p_h(t)) \in \mathcal{A}_h(w_h(t))$ *for every* $t \geq 0$*;*

(ii) initial conditions: $(u_h(0), e_h(0), p_h(0)) = (u_{0,h}, e_{0,h}, p_{0,h})$ *and* $\dot u_h(0) = v_{0,h}$*;*

(iii) stress constraint: $(\sigma_h)_D(t) \in K$ *a.e. in* Ω_h *for every* $t \geq 0$*, where* $\sigma_h(t) := \mathbb{C}e_h(t)$*;*

(iv) equation of motion: *for a.e.* $t \geq 0$

$$\begin{cases} \ddot{u}_h(t) - \operatorname{div} \sigma_h(t) = f_h(t) e_3 \ \textit{in } \Omega_h, \\ [\sigma_h(t) \nu_{\partial\Omega_h}] = 0 \ \textit{on } \Gamma_{n,h}; \end{cases} \tag{3.6}$$

(v) energy inequality: *for every* $0 \leq t_1 \leq t_2$

$$\mathcal{Q}_h(e_h(t_2)) + \frac{1}{2}\|\dot{u}_h(t_2)\|_{L^2}^2 + \int_{t_1}^{t_2} \mathcal{H}_h(\dot{p}_h(s))\, ds \leq \mathcal{Q}_h(e_h(t_1)) + \frac{1}{2}\|\dot{u}_h(t_1)\|_{L^2}^2$$
$$+ \int_{t_1}^{t_2} \int_{\Omega_h} (\sigma_h(s) : E\dot{w}_h(s) + \ddot{u}_h(s) \cdot \dot{w}_h(s))\, dx\, ds + \int_{t_1}^{t_2} \int_{\Omega_h} f_h(s)\, ((\dot{u}_h)_3(s) - (\dot{w}_h)_3(s))\, dx\, ds. \tag{3.7}$$

Moreover, the following estimates hold:

- *there exists a constant* $C > 0$*, independent of* h*, such that*

$$\|\ddot{u}_h\|_{L^\infty([0,t];L^2)} + \|\dot{e}_h\|_{L^\infty([0,t];L^2)} \leq C(\|Ev_{0,h}\|_{L^2} + \|\dot{f}_h\|_{L^1([0,t];L^2)}$$
$$+ \|\dddot{w}_h\|_{L^1([0,t];L^2)} + \|\ddot{w}_h\|_{L^\infty([0,t];L^2)} + \sqrt{t}\, \|E\ddot{w}_h\|_{L^2([0,t];L^2)}) \tag{3.8}$$

  for every $t > 0$*;*

- *there exists a constant* $C' > 0$*, independent of* h*, such that*

$$\|p_h(t_2) - p_h(t_1)\|_{M_b} \leq C' \Big(\|e_h\|_{L^\infty([0,T];L^2)} \|e_h(t_2) - e_h(t_1)\|_{L^2}$$
$$+ \|\dot{u}_h\|_{L^\infty([0,T];L^2)} \|\dot{u}_h(t_2) - \dot{u}_h(t_1)\|_{L^2} + \|e_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|E\dot{w}_h(t)\|_{L^2}\, dt$$
$$+ \|\ddot{u}_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|\dot{w}_h(t)\|_{L^2}\, dt + \|\dot{u}_h - \dot{w}_h\|_{L^\infty([0,T];L^2)} \int_{t_1}^{t_2} \|f_h(t)\|_{L^2}\, dt \Big) \tag{3.9}$$

  for every $T > 0$ *and every* $t_1, t_2 \in [0, T]$*.*

Remark 3.2. The energy inequality (v) *formally* corresponds to the inequality

$$\int_{\Omega_h \cup \Gamma_{d,h}} (\sigma_h)_D(t) : \dot{p}_h(t)\, dx \geq \mathcal{H}_h(\dot{p}_h(t)) \tag{3.10}$$

for a.e. $t \geq 0$. Indeed, it is enough to choose $t_1 = t$ and $t_2 = t + \delta$ in (v), divide the inequality by δ, and pass to the limit as δ tends to zero. Using the kinematic admissibility $\dot{e}_h(t) = E\dot{u}_h(t) - \dot{p}_h(t)$, integration by parts, and (3.6), eventually yield (3.10). Note, however, that the left-handside of (3.10) is in general not well defined, since $(\sigma_h)_D(t) \in L^\infty(\Omega_h; \mathbb{M}^{3\times3}_D)$ and $\dot{p}_h(t) \in M_b(\Omega_h \cup \Gamma_{d,h}; \mathbb{M}^{3\times3}_D)$.

The formal equivalence of (v) and (3.10) suggests that (v) contains all the relevant information stored in the Prandtl-Reuss flow rule (see equation (d5)$'$ in the introduction). Indeed, the converse inequality

$$(\sigma_h)_D(t) : \dot{p}_h(t) \leq H_h(\dot{p}_h(t)) \quad \text{ in } \Omega_h \cup \Gamma_{d,h}$$

is an immediate consequence of (iii) and of the definition of H_h (if the left-handside is well defined). Moreover, as we will see in the proof of Theorem 4.1, condition (v) is enough to recover the limiting flow rule in the dimension reduction analysis of next section.

If the stress-strain duality in the sense of [22, 9, 15] is defined, the formal arguments above can be rigorously justified; thus, one can show that any solution to (i)–(v) satisfies condition (3.7) with an equality and this energy equality is equivalent to the Prandtl-Reuss flow rule (see, e.g., [7]). In this case uniqueness of solutions for the system (i)–(v) can also be proved by standard methods.

Proof of Theorem 3.1. We give here only a sketch of the proof (all the details can be found in [5, Theorem 1.3] or in [7, Theorem 4.1] in a slightly different setting). In order to simplify the notation we omit the dependence of the fields on h. Moreover, we denote the space $\{u \in H^1(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \Gamma_d\}$ by $H^1_{\Gamma_d}(\Omega;\mathbb{R}^3)$ and its dual by $H^{-1}_{\Gamma_d}(\Omega;\mathbb{R}^3)$.

We first prove existence for a visco-elastic regularisation of the problem. We start by regularising the initial velocities. More precisely, let $\varepsilon > 0$ and let $v_0^\varepsilon \in H^1(\Omega;\mathbb{R}^3)$ be the solution of the boundary value problem

$$\begin{cases} -\varepsilon \operatorname{div} E v_0^\varepsilon + v_0^\varepsilon = v_0 \text{ in } \Omega, \\ v_0^\varepsilon = \dot w(0) \text{ on } \Gamma_d, \\ [E v_0^\varepsilon \nu_{\partial\Omega}] = 0 \text{ on } \Gamma_n. \end{cases} \tag{3.11}$$

The standard theory of linear elliptic equations gives

$$v_0^\varepsilon \to v_0 \quad \text{in } H^1(\Omega;\mathbb{R}^3)$$

and

$$\varepsilon \operatorname{div} E v_0^\varepsilon \to 0 \quad \text{in } L^2(\Omega;\mathbb{R}^3).$$

Using a time-discretisation procedure and arguing exactly as in [7, Theorem 3.1], one can prove the existence and uniqueness of a triplet $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$, with

$$u_\varepsilon \in H^1_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \cap H^2_{loc}([0,+\infty); H^{-1}_{\Gamma_d}(\Omega;\mathbb{R}^3)),$$
$$e_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym})), \qquad p_\varepsilon \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_D)),$$

satisfying the following conditions:

- *kinematic admissibility:* $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w(t))$ for every $t \geq 0$;
- *initial conditions:* $(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u_0, e_0, p_0)$ and $\dot u_\varepsilon(0) = v_0^\varepsilon$;
- *stress constraint:* $(\sigma_\varepsilon)_D(t) \in K$ a.e. in Ω for every $t \geq 0$, where $\sigma_\varepsilon(t) := \mathbb{C} e_\varepsilon(t)$;
- *equation of motion:* for every $t \geq 0$ and every $\varphi \in L^2(0,t; H^1_{\Gamma_d}(\Omega;\mathbb{R}^3))$

$$\int_0^t \langle \ddot u_\varepsilon(s), \varphi(s)\rangle\, ds + \int_0^t \int_\Omega (\sigma_\varepsilon(s) + \varepsilon E \dot u_\varepsilon(s)) : E\varphi(s)\, dx\, ds = \int_0^t \int_\Omega f(s)\varphi_3(s)\, dx\, ds; \tag{3.12}$$

- *flow rule:* for a.e. $t \geq 0$

$$H(\dot p_\varepsilon(t)) = (\sigma_\varepsilon)_D(t) : \dot p_\varepsilon(t) \quad \text{a.e. in } \Omega. \tag{3.13}$$

We now prove the following bound: for every $t > 0$

$$\begin{aligned} \|\ddot u_\varepsilon\|^2_{L^\infty([0,t];L^2)} + \|\dot e_\varepsilon\|^2_{L^\infty([0,t];L^2)} + \varepsilon \|E\ddot u_\varepsilon\|^2_{L^2([0,t];L^2)} \leq C(\varepsilon^2 \|\operatorname{div} E v_0^\varepsilon\|^2_{L^2} + \|E v_0^\varepsilon\|^2_{L^2} \\ + \|\dot f\|^2_{L^1([0,t];L^2)} + \|\dddot w\|^2_{L^1([0,t];L^2)} + \|\ddot w\|^2_{L^\infty([0,t];L^2)} + (\varepsilon + t)\|E\ddot w\|^2_{L^2([0,t];L^2)}), \end{aligned} \tag{3.14}$$

where $C > 0$ is a constant independent of t and h.

To prove (3.14), we extend continuously the fields involved by setting for $s < 0$

$$u_\varepsilon(s) = u_0 + s v_0^\varepsilon, \quad w(s) = w(0) + s\dot w(0), \quad e_\varepsilon(s) = e_0, \quad p_\varepsilon(s) = p_0, \quad f(s) = f(0).$$

We introduce the time incremental quotient

$$D^\delta a(t) := \frac{a(t) - a(t-\delta)}{\delta}.$$

Let $T > 0$, $t \in (0,T]$, and $\delta > 0$. Using the equation of motion, for every test function $\varphi \in L^2([0,t+\delta]; H^1_{\Gamma_d}(\Omega;\mathbb{R}^3))$ we have (3.12) and

$$\begin{aligned} \int_\delta^{t+\delta} \langle \ddot u_\varepsilon(s-\delta), \varphi(s)\rangle\, ds + \int_\delta^{t+\delta} \int_\Omega (\sigma_\varepsilon(s-\delta) + \varepsilon E\dot u_\varepsilon(s-\delta)) : E\varphi(s)\, dx\, ds \\ = \int_\delta^{t+\delta} \int_\Omega f(s-\delta)\varphi_3(s)\, dx\, ds. \end{aligned}$$

Subtracting (3.12) from the previous equation and choosing $\varphi = \frac{1}{\delta}\chi_{(0,t)}D^\delta(\dot{u}_\varepsilon - \dot{w})$ yield

$$\begin{aligned}
&\int_0^t \langle D^\delta(\ddot{u}_\varepsilon - \ddot{w})(s), D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\rangle\, ds + \int_0^t\int_\Omega D^\delta(\sigma_\varepsilon + \varepsilon E\dot{u}_\varepsilon)(s) : D^\delta E(\dot{u}_\varepsilon - \dot{w})(s)\, dx\, ds\\
&\qquad + \int_0^t\int_\Omega D^\delta \ddot{w}(s)\cdot D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\, dx\, ds - \int_0^t\int_\Omega D^\delta f(s) D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s)\, dx\, ds\\
&\quad = \frac{1}{\delta}\int_0^\delta\int_\Omega f(0)D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s)\, dx\, ds\\
&\qquad -\frac{1}{\delta}\int_0^\delta\int_\Omega (\sigma_0(s) + \varepsilon E v_0^\varepsilon(s)) : D^\delta E(\dot{u}_\varepsilon - \dot{w})(s)\, dx\, ds.
\end{aligned}$$

Integrating by parts, the right-hand side can be rewritten as

$$\begin{aligned}
&\frac{1}{\delta}\int_0^\delta\int_\Omega f(0)D^\delta((\dot{u}_\varepsilon)_3 - \dot{w}_3)(s)\, dx\, ds + \frac{1}{\delta}\int_0^\delta\int_\Omega \operatorname{div}(\sigma_0(s) + \varepsilon E v_0^\varepsilon(s))\cdot D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\, dx\, ds\\
&\qquad -\frac{1}{\delta}\int_0^\delta \langle[(\sigma_0(s) + \varepsilon E v_0^\varepsilon(s))\nu_{\partial\Omega}], D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\rangle\, ds\\
&\quad = \frac{\varepsilon}{\delta}\int_0^\delta\int_\Omega \operatorname{div} E v_0^\varepsilon(s)\cdot D^\delta(\dot{u}_\varepsilon - \dot{w})(s)\, dx\, ds \;\leq\; \varepsilon\,\|\operatorname{div} E v_0^\varepsilon\|_{L^2}\|D^\delta(\dot{u}_\varepsilon - \dot{w})\|_{L^\infty([0,T];L^2)},
\end{aligned}$$

where the equality follows from (3.4) and (3.11). We now focus on the term

$$\int_0^t\int_\Omega D^\delta\sigma_\varepsilon(s) : D^\delta E\dot{u}_\varepsilon(s)\, dx\, ds.$$

Using the kinematic admissibility $E\dot{u}_\varepsilon = \dot{e}_\varepsilon + \dot{p}_\varepsilon$ a.e. in $[0,+\infty)\times\Omega$, we have that for every $\tau \in L^2([0,t+\delta]; L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}))$

$$\int_0^t\int_\Omega E\dot{u}_\varepsilon(s) : \tau(s)\, dx\, ds = \int_0^t\int_\Omega \dot{e}_\varepsilon(s) : \tau(s)\, dx\, ds + \int_0^t\int_\Omega \dot{p}_\varepsilon(s) : \tau(s)\, dx\, ds$$

and

$$\begin{aligned}
\int_\delta^{t+\delta}\int_\Omega E\dot{u}_\varepsilon(s-\delta) : \tau(s)\, dx\, ds &= \int_\delta^{t+\delta}\int_\Omega \dot{e}_\varepsilon(s-\delta) : \tau(s)\, dx\, ds\\
&\qquad + \int_\delta^{t+\delta}\int_\Omega \dot{p}_\varepsilon(s-\delta) : \tau(s)\, dx\, ds.
\end{aligned}$$

Testing the difference of the two previous equations by $\tau = \frac{1}{\delta}\chi_{(0,t)}D^\delta\sigma_\varepsilon$, we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega D^\delta E\dot{u}_\varepsilon(s) : D^\delta\sigma_\varepsilon(s)\, dx\, ds\\
&\quad = \int_0^t\int_\Omega D^\delta\dot{e}_\varepsilon(s) : D^\delta\sigma_\varepsilon(s)\, dx\, ds + \int_0^t\int_\Omega D^\delta\dot{p}_\varepsilon(s) : D^\delta(\sigma_\varepsilon)_D(s)\, dx\, ds\\
&\qquad -\frac{1}{\delta}\int_0^\delta\int_\Omega E v_0^\varepsilon : D^\delta\sigma_\varepsilon(s)\, dx\, ds\\
&\quad \geq \mathcal{Q}(D^\delta e_\varepsilon(t)) - \frac{1}{\delta}\int_0^\delta\int_\Omega E v_0^\varepsilon : D^\delta\sigma_\varepsilon(s)\, dx\, ds,
\end{aligned}$$

where we used that $D^\delta e_\varepsilon(0) = 0$ and

$$\int_0^t\int_\Omega D^\delta\dot{p}_\varepsilon(s) : D^\delta(\sigma_\varepsilon)_D(s)\, dx\, ds \geq 0$$

as a consequence of the stress constraint and of the flow rule (3.13). Since $D^\delta(\dot u_\varepsilon - \dot w)(0) = 0$, applying the Holder inequality we deduce

$$\begin{aligned}
&\frac{1}{2}\|D^\delta(\dot u_\varepsilon - \dot w)(t)\|^2_{L^2} + \mathcal{Q}(D^\delta e_\varepsilon(t)) + \varepsilon\|D^\delta E\dot u_\varepsilon\|^2_{L^2([0,t];L^2)}\\
&\quad\leq\ \varepsilon\,\|\mathrm{div}\, Ev_0^\varepsilon\|_{L^2}\|D^\delta(\dot u_\varepsilon - \dot w)\|_{L^\infty([0,T];L^2)} + 2\beta_{\mathbb{C}}\|Ev_0^\varepsilon\|_{L^2}\|D^\delta e_\varepsilon\|_{L^\infty([0,T];L^2)}\\
&\qquad + \|D^\delta f\|_{L^1([0,T];L^2)}\|D^\delta((\dot u_\varepsilon)_3 - \dot w_3)\|_{L^\infty([0,T];L^2)}\\
&\qquad + \big(2\beta_{\mathbb{C}}\sqrt{t}\|D^\delta e_\varepsilon\|_{L^\infty([0,T];L^2)} + \varepsilon\|D^\delta E\dot u_\varepsilon\|_{L^2([0,T];L^2)}\big)\|D^\delta E\dot w\|_{L^2([0,T];L^2)}\\
&\qquad + \|D^\delta(\dot u_\varepsilon - \dot w)\|_{L^\infty([0,T];L^2)}\|D^\delta \ddot w\|_{L^1([0,T];L^2)}
\end{aligned}$$

for every $T > 0$ and $t \in [0, T]$. By Young inequality and passing to the limit as δ tends to 0, we obtain (3.14).

As a consequence of (3.14), we deduce, in particular, that $u_\varepsilon \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3))$, so that the equation of motion (3.12) can be written in the strong formulation

$$\begin{cases} \ddot u_\varepsilon(t) - \mathrm{div}(\sigma_\varepsilon(t) + \varepsilon E\dot u_\varepsilon(t)) = f(t)e_3 \text{ in } \Omega,\\ [(\sigma_\varepsilon(t) + \varepsilon E\dot u_\varepsilon(t))\nu_{\partial\Omega}] = 0 \text{ on } \Gamma_n \end{cases} \tag{3.15}$$

for a.e. $t \geq 0$.

We now discuss how to pass to the limit, as $\varepsilon \to 0$. Arguing as in [7, Proposition 4.3], from the equation of motion and the flow rule we obtain the following energy balance:

$$\begin{aligned}
&\mathcal{Q}(e_\varepsilon(t)) + \frac{1}{2}\|\dot u_\varepsilon(t)\|^2_{L^2} + \int_0^t \mathcal{H}(\dot p_\varepsilon(s))\,ds + \varepsilon\int_0^t\int_\Omega |E\dot u_\varepsilon(s)|^2\,dx\,ds = \mathcal{Q}(e_0) + \frac{1}{2}\|v_0^\varepsilon\|^2_{L^2}\\
&+\int_0^t\int_\Omega ((\sigma_\varepsilon(s)+\varepsilon E\dot u_\varepsilon(s)) : E\dot w(s) + \ddot u_\varepsilon(s)\cdot\dot w(s))\,dx\,ds + \int_0^t\int_\Omega f(s)\,((\dot u_\varepsilon)_3(s) - \dot w_3(s))\,dx\,ds
\end{aligned} \tag{3.16}$$

for every $\varepsilon > 0$ and every $t > 0$. Combining this inequality with (3.14) and using Ascoli-Arzelà and Helly Theorem, we deduce the existence of

$$u \in W^{2,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \cap BV_{loc}([0,+\infty); BD(\Omega)),$$
$$e \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym})), \qquad p \in BV_{loc}([0,+\infty); M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D))$$

such that, up to subsequences,

$$\begin{aligned}
&u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BD(\Omega), \qquad \dot u_\varepsilon(t) \rightharpoonup \dot u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3),\\
&e_\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \qquad p_\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)
\end{aligned} \tag{3.17}$$

for every $t \in [0, T]$. From these convergences we immediately deduce that (u, e, p) satisfies conditions (i)–(iii). By (3.16) we have that $\varepsilon E\dot u_\varepsilon \to 0$ strongly in $L^2_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym}))$. Since $\ddot u_\varepsilon \rightharpoonup \ddot u$ weakly* in $L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3))$, we can pass to the limit in the weak formulation of (3.15) and, thus, deduce condition (iv).

Taking the difference of the equations of motion (3.15) and (3.6) and testing by $\dot u_\varepsilon - \dot w$ on $[0, t]\times\Omega$, one can prove (see [7, Lemma 4.5]) that

$$\begin{aligned}
&\dot u_\varepsilon \to \dot u \quad \text{strongly in } L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)),\\
&e_\varepsilon \to e \quad \text{strongly in } L^\infty_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)).
\end{aligned} \tag{3.18}$$

We now write the energy balance (3.16) between two times $t_1 \leq t_2$ and using the previous convergences and the lower semicontinuity of the elastic energy and of the dissipation, we obtain

$$\begin{aligned}
&\mathcal{Q}(e(t_2)) + \frac{1}{2}\|\dot u(t_2)\|^2_{L^2} + \mathcal{D}(p; t_1, t_2) \leq \mathcal{Q}(e(t_1)) + \frac{1}{2}\|\dot u(t_1)\|^2_{L^2}\\
&\quad + \int_{t_1}^{t_2}\int_\Omega (\sigma(s) : E\dot w(s) + \ddot u(s)\cdot\dot w(s))\,dx\,ds + \int_{t_1}^{t_2}\int_\Omega f(s)\,(\dot u_3(s) - \dot w_3(s))\,dx\,ds.
\end{aligned} \tag{3.19}$$

Let $T > 0$. Using the inequality

$$r_K \|p(t_2) - p(t_1)\|_{M_b} \leq \mathcal{D}(p; t_1, t_2)$$

in (3.19), we deduce that

$$\begin{aligned} r_K\|p(t_2)-p(t_1)\|_{M_b} &\leq 2\beta_{\mathbb{C}}\|e\|_{L^\infty([0,T];L^2)}\|e(t_2)-e(t_1)\|_{L^2}+\|\dot u\|_{L^\infty([0,T];L^2)}\|\dot u(t_2)-\dot u(t_1)\|_{L^2} \\ &+ 2\beta_{\mathbb{C}}\|e\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|E\dot w(t)\|_{L^2}\,dt + \|\ddot u\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|\dot w(t)\|_{L^2}\,dt \\ &+ \|\dot u_3 - \dot w_3\|_{L^\infty([0,T];L^2)}\int_{t_1}^{t_2}\|f(t)\|_{L^2}\,dt \end{aligned} \tag{3.20}$$

for every $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Hence, p is locally Lipschitz continuous on $[0,+\infty)$ with values in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)$, inequality (3.9) is satisfied, and (3.19) gives condition (v). Using the kinematic admissibility, one can prove that u is locally Lipschitz continuous on $[0,+\infty)$ with values in $BD(\Omega)$. Finally, inequality (3.8) easily follows from (3.14). □

4. Convergence of dynamic evolutions

In this section we discuss the convergence of three-dimensional dynamic evolutions, when the parameter h tends to 0. As it is usual in dimension reduction problems, we perform a change of variable in order to set the problem on a fixed domain Ω. We also perform a rescaling of the time variable (as done, e.g., in [1] in the context of nonlinear elasticity). We thus consider the change of variable $\phi_h : \overline{\Omega} \to \overline{\Omega}_h$ given by

$$\phi_h(x) := (x', hx_3)$$

for every $x = (x', x_3) \in \overline{\Omega}$. We define the linear operator $\Lambda_h : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ as

$$\Lambda_h\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{h}\xi_{13} \\ \xi_{12} & \xi_{22} & \frac{1}{h}\xi_{23} \\ \frac{1}{h}\xi_{13} & \frac{1}{h}\xi_{23} & \frac{1}{h^2}\xi_{33} \end{pmatrix}$$

for every $\xi \in \mathbb{M}^{3\times3}_{sym}$.

Let $t \mapsto (u_h(t), e_h(t), p_h(t))$ be a dynamic evolution in Ω_h with boundary datum w_h, force term f_h, and initial conditions $(u_{0,h}, e_{0,h}, p_{0,h})$ and $v_{0,h}$, as in Theorem 3.1. We associate with it an h-rescaled dynamic evolution in Ω, defined as follows:

$$t \mapsto (u^h(t), e^h(t), p^h(t)) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym}),$$

where for every $t \geq 0$ and a.e. $x \in \Omega$

$$u^h_\alpha(t,x) := (u_h)_\alpha\big(\tfrac{t}{h}, \phi_h(x)\big) \quad \alpha = 1,2, \qquad u^h_3(t,x) := h(u_h)_3\big(\tfrac{t}{h}, \phi_h(x)\big),$$
$$e^h(t,x) := \Lambda_h^{-1} e_h\big(\tfrac{t}{h}, \phi_h(x)\big),$$

and for every $t \geq 0$

$$p^h(t) := \tfrac{1}{h}\Lambda_h^{-1}\phi_h^\# p_h\big(\tfrac{t}{h}\big).$$

Here $\phi_h^\# q \in M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)$ denotes the pull-back measure of q, defined as

$$\int_{\Omega\cup\Gamma_d} \psi : d\phi_h^\# q := \int_{\Omega_h\cup\Gamma_{d,h}} \psi\circ\phi_h^{-1} : dq$$

for every $\psi \in C_0(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)$. Finally, we rescale the boundary datum w_h as

$$w^h_\alpha(t,x) := (w_h)_\alpha\big(\tfrac{t}{h}, \phi_h(x)\big) \quad \alpha = 1,2, \qquad w^h_3(t,x) := h(w_h)_3\big(\tfrac{t}{h}, \phi_h(x)\big)$$

for every $t \geq 0$ and a.e. $x \in \Omega$, and the vertical force f_h as

$$f^h(t,x) := \tfrac{1}{h} f_h\big(\tfrac{t}{h}, \phi_h(x)\big)$$

for every $t \geq 0$ and a.e. $x \in \Omega$.

The rescaled triplet satisfies the following conditions:

- *kinematic admissibility:* for every $t \geq 0$ we have

$$\begin{gathered} Eu^h(t) = e^h(t) + p^h(t) \text{ in } \Omega, \qquad p^h(t) = (w^h(t) - u^h(t)) \odot \nu_{\partial\Omega}\mathcal{H}^2 \text{ on } \Gamma_d, \\ p^h_{11}(t) + p^h_{22}(t) + \tfrac{1}{h^2}p^h_{33}(t) = 0 \text{ in } \Omega \cup \Gamma_d; \end{gathered} \tag{4.1}$$

- *stress constraint:* $\sigma^h_D(t) \in K$ a.e. in Ω for every $t \geq 0$, where $\sigma^h(t) := \mathbb{C}\Lambda_h e^h(t)$;
- *equation of motion:* for a.e. $t \geq 0$

$$\begin{cases} \begin{pmatrix} h^2\ddot{u}^h_\alpha(t) \\ \ddot{u}^h_3(t) \end{pmatrix} - \operatorname{div} \Lambda_h\sigma^h(t) = f^h(t)e_3 \text{ in } \Omega, \\ [\Lambda_h\sigma^h(t)\nu_{\partial\Omega}] = 0 \text{ on } \Gamma_n; \end{cases} \tag{4.2}$$

- *energy inequality:* for every $0 \leq t_1 \leq t_2$

$$\begin{aligned} \mathcal{Q}(\Lambda_h e^h(t_2)) + \frac{1}{2}\left\| \begin{pmatrix} h\dot{u}^h_\alpha(t_2) \\ \dot{u}^h_3(t_2) \end{pmatrix} \right\|^2_{L^2} + \int_{t_1}^{t_2} \mathcal{H}(\Lambda_h\dot{p}^h(s))\,ds \leq \mathcal{Q}(\Lambda_h e^h(t_1)) + \frac{1}{2}\left\| \begin{pmatrix} h\dot{u}^h_\alpha(t_1) \\ \dot{u}^h_3(t_1) \end{pmatrix} \right\|^2_{L^2} \\ + \int_{t_1}^{t_2}\int_\Omega \left(\sigma^h(s) : \Lambda_h E\dot{w}^h(s) + \begin{pmatrix} h\ddot{u}^h_\alpha(s) \\ \ddot{u}^h_3(s) \end{pmatrix} \cdot \begin{pmatrix} h\dot{w}^h_\alpha(s) \\ \dot{w}^h_3(s) \end{pmatrix} \right) dx\,ds \\ + \int_{t_1}^{t_2}\int_\Omega f^h(s)\,(\dot{u}^h_3(s) - \dot{w}^h_3(s))\,dx\,ds. \end{aligned} \tag{4.3}$$

We now state the assumptions on the rescaled data of the problem.

Forces. We consider a sequence of vertical loads $(f^h) \subset W^{1,1}_{loc}([0,+\infty); L^2(\Omega))$ such that for every $T > 0$ there exists a constant $C(T) > 0$ for which

$$\|f^h\|_{W^{1,1}([0,T];L^2)} \leq C(T) \tag{4.4}$$

for every $h > 0$. We also assume that there exists $f \in L^\infty_{loc}([0,+\infty); L^2(\Omega))$ such that

$$f^h(t) \to f(t) \quad \text{strongly in } L^2(\Omega) \tag{4.5}$$

for every $t \geq 0$.

Boundary displacements. We consider a sequence of boundary displacements

$$(w^h) \subset H^2_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3)) \cap W^{3,1}_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \tag{4.6}$$

such that for every $T > 0$ there exists a constant $C(T) > 0$ for which

$$\left\| \begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \right\|_{W^{2,1}([0,T];L^2)} + \|\Lambda_h E w^h\|_{H^2([0,T];L^2)} \leq C(T) \tag{4.7}$$

for every $h > 0$. We assume that for every $t \geq 0$

$$w^h(t) \rightharpoonup w(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3), \tag{4.8}$$

$$\Lambda_h E\dot{w}^h(t) \to \eta(t) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{4.9}$$

and

$$\begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \to \dot{w}_3 e_3 \quad \text{strongly in } L^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{R}^3)) \tag{4.10}$$

for some $w \in H^2_{loc}([0,+\infty); H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ and some $\eta \in H^1_{loc}([0,+\infty); L^2(\Omega;\mathbb{M}^{3\times3}_{sym}))$.

Initial data. We fix a triplet $(u_0^h, e_0^h, p_0^h) \in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ satisfying the kinematic admissibility conditions (4.1) and an initial velocity $v_0^h \in H^1(\Omega;\mathbb{R}^3)$ such that, setting $\sigma_0^h := \mathbb{C}\Lambda_h e_0^h$, we have

$$-\mathrm{div}\,\Lambda_h\sigma_0^h = f^h(0)e_3 \text{ in } \Omega, \quad [\Lambda_h\sigma_0^h\nu_{\partial\Omega}] = 0 \text{ on } \Gamma_n, \quad (\sigma_0^h)_D \in K \text{ a.e. in } \Omega, \tag{4.11}$$

and

$$v_0^h = \dot{w}^h(0) \text{ on } \Gamma_d \tag{4.12}$$

for every $h > 0$. Moreover, we suppose that

$$\begin{pmatrix} h(v_0^h)_\alpha \\ (v_0^h)_3 \end{pmatrix} \to v_0 e_3 \text{ strongly in } L^2(\Omega;\mathbb{R}^3), \tag{4.13}$$

$$\Lambda_h e_0^h \to \tilde{e}_0 \text{ strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{4.14}$$

$$\|\Lambda_h E v_0^h\|_{L^2} + \|\Lambda_h p_0^h\|_{M_b} \le C \tag{4.15}$$

for some $v_0 \in H^1(\Omega;\mathbb{R}^3)$, $\tilde{e}_0 \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, and some constant C independent of h.

We are now in a position to state the main result of this paper.

Theorem 4.1. *Assume* (4.4)–(4.15) *and let (u^h, e^h, p^h) be an h-rescaled dynamic evolution for the boundary datum w^h, the force term f^h, and the initial data (u_0^h, e_0^h, p_0^h) and v_0^h. Then there exists a map $t \mapsto (u(t), e(t), p(t))$ of class*

$$Lip_{loc}([0,+\infty); KL(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym}))$$

with $u_3 \in W^{2,\infty}_{loc}([0,+\infty); L^2(\omega))$, such that, up to subsequences,

$$u^h(t) \rightharpoonup u(t) \quad \textit{weakly}^* \textit{ in } BD(\Omega), \tag{4.16}$$

$$\dot{u}_3^h(t) \to \dot{u}_3(t) \quad \textit{strongly in } L^2(\Omega), \tag{4.17}$$

$$e^h(t) \to e(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{4.18}$$

$$\Lambda_h e^h(t) \to \mathbb{M}e(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{4.19}$$

$$p^h(t) \rightharpoonup p(t) \quad \textit{weakly}^* \textit{ in } M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym}) \tag{4.20}$$

for every $t \ge 0$. The map $t \mapsto (u(t), e(t), p(t))$ satisfies the following system of equations:

(i) kinematic admissibility: $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ *for every $t \ge 0$;*

(ii) initial conditions: $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$ *and $\dot{u}_3(0) = (v_0)_3$, where $u_0^h \rightharpoonup u_0$ weakly* in $BD(\Omega)$, $e_0^h \to e_0$ strongly in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, and $p_0^h \rightharpoonup p_0$ weakly* in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$ (these limits exist, up to subsequences);*

(iii) stress constraint: $\sigma(t) \in K_r$ *a.e. in Ω for every $t \ge 0$, where $\sigma(t) := \mathbb{C}_r e(t)$;*

(iv) equations of motion: *for every $t \ge 0$*

$$\begin{cases} \mathrm{div}\,\bar{\sigma}(t) = 0 & \textit{in } \omega, \\ [\bar{\sigma}(t)\nu_{\partial\omega}] = 0 & \textit{on } \gamma_n, \end{cases} \tag{4.21}$$

and for a.e. $t \ge 0$

$$\begin{cases} \ddot{u}_3(t) - \frac{1}{12}\mathrm{div}\,\mathrm{div}\,\hat{\sigma}(t) = \bar{f}(t) & \textit{in } \omega, \\ b_0(\hat{\sigma}(t)) = b_1(\hat{\sigma}(t)) = 0 & \textit{on } \gamma_n, \end{cases} \tag{4.22}$$

where

$$\bar{f}(x') := \int_{-1/2}^{1/2} f(x', x_3)\,dx_3 \quad \textit{for a.e. } x' \in \omega;$$

(v) flow rule: *for a.e. $t \ge 0$*

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t)\rangle_r\,. \tag{4.23}$$

Proof. The proof of Theorem 4.1 is subdivided into six steps.

Step 1: Compactness estimates. We first deduce some a priori estimates.

Writing the estimate (3.8) on $[0, t/h]$ and performing the scaling, we obtain

$$\Big\| \begin{pmatrix} h\ddot{u}^h_\alpha \\ \ddot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} + \|\Lambda_h \dot{e}^h\|_{L^\infty([0,t];L^2)} \le C\Big(\|\Lambda_h E v^h_0\|_{L^2} + \|\dot{f}^h\|_{L^1([0,t];L^2)} + \Big\| \begin{pmatrix} h\dddot{w}^h_\alpha \\ \dddot{w}^h_3 \end{pmatrix} \Big\|_{L^1([0,t];L^2)} + \Big\| \begin{pmatrix} h\ddot{w}^h_\alpha \\ \ddot{w}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} + \sqrt{t}\, \|\Lambda_h E\ddot{w}^h\|_{L^2([0,t];L^2)} \Big)$$

for every $t > 0$. By the assumptions on the data we deduce that for every $T > 0$ there exists a constant $C(T) > 0$, depending on T but independent of h, such that

$$\Big\| \begin{pmatrix} h\ddot{u}^h_\alpha \\ \ddot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,T];L^2)} + \|\Lambda_h \dot{e}^h\|_{L^\infty([0,T];L^2)} \le C(T). \tag{4.24}$$

We now write the rescaled energy inequality (4.3) with $t_1 = 0$ and $t_2 \in [0,t]$. By (2.2) and (2.3) we have

$$\alpha_{\mathbb{C}} \|\Lambda_h e^h\|^2_{L^\infty([0,t];L^2)} + \frac{1}{2} \Big\| \begin{pmatrix} h\dot{u}^h_\alpha \\ \dot{u}^h_3 \end{pmatrix} \Big\|^2_{L^\infty([0,t];L^2)} \le \beta_{\mathbb{C}} \|\Lambda_h e^h_0\|^2_{L^2} + \frac{1}{2} \Big\| \begin{pmatrix} h(v^h_0)_\alpha \\ (v^h_0)_3 \end{pmatrix} \Big\|^2_{L^2}$$
$$+ 2\beta_{\mathbb{C}} \|\Lambda_h e^h\|_{L^\infty([0,t];L^2)} \int_0^t \|\Lambda_h E\dot{w}^h(s)\|_{L^2}\, ds + \Big\| \begin{pmatrix} h\ddot{u}^h_\alpha \\ \ddot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,t];L^2)} \int_0^t \Big\| \begin{pmatrix} h\dot{w}^h_\alpha \\ \dot{w}^h_3 \end{pmatrix} \Big\|_{L^2} ds$$
$$+ (\|\dot{u}^h_3\|_{L^\infty([0,t];L^2)} + \|\dot{w}^h_3\|_{L^\infty([0,t];L^2)}) \int_0^t \|f^h(s)\|_{L^2}\, ds.$$

By the Cauchy inequality, the assumptions on the data and (4.24), we deduce that for every $T > 0$ there exists a constant $C(T) > 0$, depending on T but independent of h, such that

$$\|\Lambda_h e^h\|_{L^\infty([0,T];L^2)} + \Big\| \begin{pmatrix} h\dot{u}^h_\alpha \\ \dot{u}^h_3 \end{pmatrix} \Big\|_{L^\infty([0,T];L^2)} \le C(T). \tag{4.25}$$

Finally, we perform the scaling in (3.9) and by (4.24) and (4.25) we get

$$\|\Lambda_h p^h(t_2) - \Lambda_h p^h(t_1)\|_{M_b} \le C(T)\Big(\|\Lambda_h e^h(t_2) - \Lambda_h e^h(t_1)\|_{L^2} + \Big\| \begin{pmatrix} h\dot{u}^h_\alpha(t_2) - h\dot{u}^h_\alpha(t_1) \\ \dot{u}^h_3(t_2) - \dot{u}^h_3(t_1) \end{pmatrix} \Big\|_{L^2} + \int_{t_1}^{t_2} \|\Lambda_h E\dot{w}^h(t)\|_{L^2}\, dt + \int_{t_1}^{t_2} \Big\| \begin{pmatrix} h\dot{w}^h_\alpha(t) \\ \dot{w}^h_3(t) \end{pmatrix} \Big\|_{L^2} dt + \int_{t_1}^{t_2} \|f^h(t)\|_{L^2}\, dt \Big) \tag{4.26}$$

for every $T > 0$ and every $t_1, t_2 \in [0,T]$.

We now deduce some compactness properties for the triplets (u^h, e^h, p^h), as $h \to 0$. By (4.24), (4.25), and the Ascoli-Arzelà Theorem we infer the existence of

$$e, \tilde{e} \in W^{1,\infty}_{loc}([0,+\infty); L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$$

with $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$ for $\alpha, \beta = 1,2$ and $e_{i3} = 0$ for $i = 1,2,3$, such that, up to subsequences,

$$e^h(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{4.27}$$
$$\Lambda_h e^h(t) \rightharpoonup \tilde{e}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \tag{4.28}$$

for every $t \ge 0$. Moreover, by (4.24) and (4.26) the functions $\Lambda_h p^h$ are equi-Lipschitz continuous in time with values in $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D)$. Therefore, again by the Ascoli-Arzelà Theorem and by (4.15) there exist

$$p \in Lip_{loc}([0,+\infty); M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})), \quad \tilde{p} \in Lip_{loc}([0,+\infty); M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D)),$$

with $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$ for $\alpha, \beta = 1,2$ and $p_{i3} = 0$ for $i = 1,2,3$, such that, up to subsequences,

$$p^h(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym}), \tag{4.29}$$
$$\Lambda_h p^h(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D) \tag{4.30}$$

for every $t \ge 0$.

We now prove the weak* compactness in $BD(\Omega)$ of the sequence of displacements (u^h). Since γ_d is open in $\partial\omega$, there exists an open set $A \subset \mathbb{R}^2$ such that $\gamma_d = A \cap \partial\omega$. Let $\Omega' := (\omega \cup A) \times (-\frac{1}{2}, \frac{1}{2})$. By (4.7) and (4.8) we have that

$$Ew^h(t) \rightharpoonup Ew(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \tag{4.31}$$

for every $t \geq 0$. Thus, for every $t \geq 0$ we can extend $w^h(t)$ and $w(t)$ to Ω' in such a way that $w^h(t) \rightharpoonup w(t)$ weakly in $L^2(\Omega'; \mathbb{R}^3)$ and $Ew^h(t) \rightharpoonup Ew(t)$ weakly in $L^2(\Omega'; \mathbb{M}^{3\times3}_{sym})$ for every $t \geq 0$.

We now extend the triplets (u^h, e^h, p^h) to Ω' by setting

$$u^h(t) := w^h(t) \text{ in } \Omega' \setminus \Omega, \qquad e^h(t) := Ew^h(t) \text{ in } \Omega' \setminus \Omega, \qquad p^h(t) := 0 \text{ in } \Omega' \setminus (\Omega \cup \Gamma_d)$$

and we note that $Eu^h(t) = e^h(t) + p^h(t)$ in Ω'. Similarly, we set

$$e(t) := Ew(t) \text{ in } \Omega' \setminus \Omega, \qquad p(t) := 0 \text{ in } \Omega' \setminus (\Omega \cup \Gamma_d).$$

By (4.27) and (4.29) we deduce that $e^h(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega'; \mathbb{M}^{3\times3}_{sym})$ and $p^h(t) \rightharpoonup p(t)$ weakly* in $M_b(\Omega'; \mathbb{M}^{3\times3}_{sym})$, for every $t \geq 0$. Thus,

$$Eu^h(t) = e^h(t) + p^h(t) \rightharpoonup e(t) + p(t) \quad \text{weakly* in } M_b(\Omega'; \mathbb{M}^{3\times3}_{sym}).$$

Since $u^h(t) = w^h(t)$ in $\Omega' \setminus \Omega$ and $w^h(t)$ is bounded in $L^2(\Omega'; \mathbb{R}^3)$, the Korn-Poincaré inequality implies that the sequence $(u^h(t))$ is uniformly bounded in $BD(\Omega')$. Consequently, there exist $u(t) \in BD(\Omega')$ and a subsequence $u^{h_j}(t)$ such that $u^{h_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega')$. Since

$$u(t) = w(t) \text{ in } \Omega' \setminus \Omega \quad \text{ and } \quad Eu(t) = e(t) + p(t) \text{ in } \Omega',$$

the Korn-Poincaré inequality ensures that the limit $u(t)$ is uniquely determined. Therefore, the whole sequence converges in Ω' and in particular,

$$u^h(t) \rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega) \tag{4.32}$$

for every $t \geq 0$.

Since $e_{i3}(t) = p_{i3}(t) = 0$, it is easy to see that

$$(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$$

for every $t \geq 0$. Moreover, $u \in Lip_{loc}([0, +\infty); KL(\Omega))$, owing to the time regularity of e, p, and w, and as a consequence of Lemma 2.1,

$$(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t)) \tag{4.33}$$

for a.e. $t \geq 0$.

Finally, combining (4.24), (4.25), (4.32), together with the Ascoli-Arzelà Theorem, we conclude that

$$u_3 \in W^{2,\infty}_{loc}([0, +\infty); L^2(\Omega))$$

and

$$h\dot{u}^h_\alpha(t) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega) \quad \text{ for } \alpha = 1, 2, \tag{4.34}$$

$$\dot{u}^h_3(t) \rightharpoonup \dot{u}_3(t) \quad \text{weakly in } L^2(\Omega) \tag{4.35}$$

for every $t \geq 0$. Moreover, we also have that

$$h\dot{u}^h_\alpha \rightharpoonup 0 \quad \text{weakly* in } W^{1,\infty}([0,T]; L^2(\Omega)) \quad \text{ for } \alpha = 1, 2, \tag{4.36}$$

$$\dot{u}^h_3 \rightharpoonup \dot{u}_3 \quad \text{weakly* in } W^{1,\infty}([0,T]; L^2(\Omega)) \tag{4.37}$$

for every $T > 0$.

The previous arguments also prove that, up to subsequences, $u^h_0 \rightharpoonup u_0$ weakly* in $BD(\Omega)$, $e^h_0 \to e_0$ strongly in $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, and $p^h_0 \rightharpoonup p_0$ weakly* in $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, for some $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$, and the initial conditions are satisfied.

Step 2: Identification of the limiting elastic strain. We claim that

$$\tilde{e}(t) = \mathbb{M}e(t) \tag{4.38}$$

for every $t \geq 0$, where $\tilde{e}$ satisfies (4.28) and $\mathbb{M}$ is the operator defined in (2.7).

We first show that (4.38) holds for a.e. $t \geq 0$. Owing to (2.8), this is equivalent to prove that for a.e. $t \geq 0$

$$\mathbb{C}\tilde{e}(t,x) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0$$

for every $\lambda_i \in \mathbb{R}$ and a.e. $x \in \Omega$. Let $(a,b) \subset (-\frac{1}{2}, \frac{1}{2})$ and let $U \subset \omega$ be an open set. Let $(\ell_n) \subset C^1([-\frac{1}{2}, \frac{1}{2}])$ and $(\lambda_n^i) \subset C_c^1(\omega)$ be two sequences such that $\ell_n' \to \chi_{(a,b)}$ strongly in $L^4(-\frac{1}{2}, \frac{1}{2})$ and $\lambda_n^i \to \lambda_i \chi_U$ strongly in $L^4(\omega)$ for every $i = 1, 2, 3$, as $n \to \infty$.

We define

$$\phi_n^h(t,x) := \psi(t) \begin{pmatrix} 2h\lambda_n^1(x')\ell_n(x_3) \\ 2h\lambda_n^2(x')\ell_n(x_3) \\ h^2\lambda_n^3(x')\ell_n(x_3) \end{pmatrix},$$

where $\psi \in L^2(0, +\infty)$. Testing (4.2) by ϕ_n^h yields

$$\int_0^{+\infty}\!\!\int_\Omega \begin{pmatrix} h^2\ddot{u}_\alpha^h(t) \\ \ddot{u}_3^h(t) \end{pmatrix} \cdot \phi_n^h(t)\,dx\,dt + \int_0^{+\infty}\!\!\int_\Omega \mathbb{C}\Lambda_h e^h(t) : \Lambda_h E\phi_n^h(t)\,dx\,dt$$
$$= \int_0^{+\infty}\!\!\int_\Omega f^h(t)(\phi_n^h)_3(t)\,dx\,dt.$$

Owing to (4.25), (4.28), (4.36), and (4.37), we can pass to the limit as $h \to 0$ and then, as $n \to +\infty$. This yields

$$\int_0^{+\infty}\!\!\int_{U\times(a,b)} \psi(t)\,\mathbb{C}\tilde{e}(t,x) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} dx\,dt = 0.$$

Since the sets (a,b), U and the function ψ are arbitrary, we deduce that for a.e. $t \geq 0$ $\tilde{e}(t) = \mathbb{M}e(t)$ a.e. in Ω. Since $\tilde{e}$ and $\mathbb{M}e$ are continuous functions of time, this implies (4.38).

This argument also proves that $\tilde{e}_0 = \mathbb{M}e_0$, where $\tilde{e}_0$ is the limit in (4.14).

Step 3: Equations of motions. Let $T > 0$. Let $\varphi \in L^2([0,T]; KL(\Omega) \cap H^1(\Omega;\mathbb{R}^3))$ with $\varphi = 0$ on Γ_d. We test the rescaled equation of motion (4.2) by φ on $[0,T] \times \Omega$. This yields

$$\int_0^T\!\!\int_\Omega \begin{pmatrix} h^2\ddot{u}_\alpha^h(t) \\ \ddot{u}_3^h(t) \end{pmatrix} \cdot \varphi(t)\,dx\,dt + \int_0^T\!\!\int_\Omega \mathbb{C}\Lambda_h e^h(t) : E\varphi(t)\,dx\,dt = \int_0^T\!\!\int_\Omega f^h(t)\varphi_3(t)\,dx\,dt,$$

where we used that $\Lambda_h E\varphi(t) = E\varphi(t)$ since $\varphi(t) \in KL(\Omega)$. As a consequence of (4.25), (4.28), (4.36), and (4.37), we can pass to the limit in the previous equation and obtain

$$\int_0^T\!\!\int_\omega \ddot{u}_3(t) \cdot \varphi_3(t)\,dx'\,dt + \int_0^T\!\!\int_\Omega \sigma(t) : \begin{pmatrix} E\bar{\varphi}(t) - x_3 D^2\varphi_3(t) & 0 \\ 0 & 0 \end{pmatrix} dx\,dt$$
$$= \int_0^T\!\!\int_\omega \bar{f}(t)\varphi_3(t)\,dx'\,dt, \tag{4.39}$$

where $\sigma(t) := \mathbb{C}_r e(t) = \mathbb{C}\mathbb{M}e(t)$.

By choosing $\varphi = (\bar{\varphi}, 0)$ with $\bar{\varphi} \in L^2([0,T]; H^1(\omega;\mathbb{R}^2))$, $\bar{\varphi}(t) = 0$ on γ_d, in (4.39) we deduce that

$$\int_0^T\!\!\int_\omega \bar{\sigma}(t) : E\bar{\varphi}(t)\,dx'\,dt = 0.$$

This implies that for a.e. $t \geq 0$

$$\int_\omega \bar{\sigma}(t) : E\bar{\varphi}\,dx' = 0$$

for every $\bar\varphi \in H^1(\omega;\mathbb{R}^2)$, $\bar\varphi = 0$ on γ_d. The continuity of $\bar\sigma$ with respect to time implies that the above equation is actually satisfied for every $t \geq 0$. By [11, Lemma 7.10-(i)] we conclude that

$$\operatorname{div}\bar\sigma(t) = 0 \text{ in } \omega, \quad [\bar\sigma(t)\nu_{\partial\omega}] = 0 \text{ on } \gamma_n$$

for every $t \geq 0$.

We now choose φ in (4.39) of the form $\varphi = \varphi_3 e_3$, with $\varphi_3 \in L^2([0,T];H^2(\omega))$, $\varphi_3(t) = 0$ and $\nabla\varphi_3(t) = 0$ on γ_d and obtain

$$\int_0^T\int_\omega \ddot u_3(t)\varphi_3(t)\,dx'\,dt - \frac{1}{12}\int_0^T\int_\omega \hat\sigma(t) : D^2\varphi_3(t)\,dx'\,dt = \int_0^T\int_\omega \bar f(t)\varphi_3(t)\,dx'\,dt.$$

By [11, Lemma 7.10-(ii)] this implies that

$$\ddot u_3(t) - \frac{1}{12}\operatorname{div}\operatorname{div}\hat\sigma(t) = \bar f(t) \quad \text{ in } [0,+\infty)\times\omega,$$

together with the corresponding Neumann boundary conditions.

Step 4: Stress constraint. We recall that $(\mathbb{C}\Lambda_h e^h)_D(t) \in K$ a.e. in Ω for every $t \geq 0$ and every h. Since $\mathbb{C}\Lambda_h e^h(t) \rightharpoonup \sigma(t)$ weakly in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ for every $t \geq 0$ and K is a closed and convex set, we have that $\sigma_D(t) \in K$ a.e. in Ω. By (2.13) this is equivalent to saying that $\sigma(t) \in K_r$ a.e. in Ω for every $t \geq 0$.

Step 5: Flow rule. We first observe that

$$\mathcal{H}_r(\dot p(t)) \geq \langle\sigma(t),\dot p(t)\rangle_r \tag{4.40}$$

for a.e. $t \geq 0$. This follows from (2.18) combined with the fact that $\sigma(t) \in K_r$ a.e. in Ω for every $t \geq 0$. Moreover, as a consequence of Proposition 2.5, (4.21), (4.22), and (4.33), we have that

$$\begin{aligned}\langle\sigma(t),\dot p(t)\rangle_r &= \int_\Omega \sigma(t) : (E\dot w(t) - \dot e(t))\,dx - \frac{1}{12}\int_\omega \operatorname{div}\operatorname{div}\hat\sigma(t)(\dot u_3(t) - \dot w_3(t))\,dx' \\ &= \int_\Omega \sigma(t) : (E\dot w(t) - \dot e(t))\,dx + \int_\omega (\bar f(t) - \ddot u_3(t))(\dot u_3(t) - \dot w_3(t))\,dx' \end{aligned}\tag{4.41}$$

for a.e. $t \geq 0$.

On the other hand, we can pass to the limit in the rescaled energy inequality arguing as follows. By (4.30), the lower semicontinuity of the dissipation and the definition of $\mathcal{D}_r$, it turns out that

$$\mathcal{D}_r(p;0,T) \leq \liminf_{h\to0}\mathcal{D}(\Lambda_h p^h;0,T)$$

for every $T > 0$. Combining this inequality with the regularity of p, (2.6), and (2.14), we have that

$$\int_0^T \mathcal{H}_r(\dot p(t))\,dt \leq \liminf_{h\to0}\int_0^T \mathcal{H}(\Lambda_h\dot p^h(t))\,dt \tag{4.42}$$

for every $T > 0$. We now write the rescaled energy inequality (4.3) with $t_1 = 0$ and $t_2 = T$. Using the lower semicontinuity of $\mathcal{Q}$, the definition of $\mathcal{Q}_r$, and the assumptions on the data (4.7) and (4.9)–(4.14), we deduce that

$$\begin{aligned}&\mathcal{Q}_r(e(T)) + \frac12\|\dot u_3(T)\|^2_{L^2} + \int_0^T \mathcal{H}_r(\dot p(t))\,dt \leq \mathcal{Q}_r(e(0)) + \frac12\|\dot u_3(0)\|^2_{L^2} \\ &\quad + \int_0^T\int_\Omega \big(\mathbb{C}_r e(t) : E\dot w(t) + \ddot u_3(t)\dot w_3(t)\big)\,dx\,dt + \int_0^T\int_\omega \bar f(t)\,(\dot u_3(t) - \dot w_3(t))\,dx'\,dt\end{aligned}\tag{4.43}$$

for every $T > 0$. Here we used that $\eta_{\alpha\beta}(t) = Ew_{\alpha\beta}(t)$ by (4.9) and (4.31). By the time regularity of e and u, inequality (4.43) can be rewritten as

$$\int_0^T \mathcal{H}_r(\dot p(t))\,dt \leq \int_0^T\int_\Omega \sigma(t) : (E\dot w(t) - \dot e(t))\,dx\,dt + \int_0^T\int_\omega (\bar f(t) - \ddot u_3(t))(\dot u_3(t) - \dot w_3(t))\,dx'\,dt.$$

Hence, by (4.41)

$$\int_0^T \mathcal{H}_r(\dot{p}(t))\,dt \le \int_0^T \langle \sigma(t), \dot{p}(t)\rangle_r\,dt. \tag{4.44}$$

Combining the above inequality with (4.40), we deduce the flow rule (4.23).

We also note, for future references, that the flow rule implies that the inequality in (4.44) is actually an equality. Therefore, by (4.41) inequality (4.43) is an equality, as well. In other words, the following energy balance holds:

$$\begin{aligned}\mathcal{Q}_r(e(T)) + \frac{1}{2}\|\dot{u}_3(T)\|_{L^2}^2 + \int_0^T \mathcal{H}_r(\dot{p}(t))\,dt &= \mathcal{Q}_r(e(0)) + \frac{1}{2}\|\dot{u}_3(0)\|_{L^2}^2 \\ + \int_0^T\int_\Omega \mathbb{C}_r e(t) : E\dot{w}(t)\,dx\,dt &+ \int_0^T\int_\omega \big(\ddot{u}_3(t)\dot{w}_3(t) + \bar{f}(t)\,(\dot{u}_3(t) - \dot{w}_3(t))\big)\,dx'\,dt \end{aligned}\tag{4.45}$$

for every $T > 0$.

Step 6: Strong convergence of the stress and the velocity. We conclude the proof by showing the strong convergence of the sequences $(\dot{u}_3^h(t))$, $(e^h(t))$, and $(\Lambda_h e^h(t))$.

By (4.3), (4.45), and the assumptions on the data we have

$$\begin{aligned}&\limsup_{h\to 0}\Big\{\mathcal{Q}(\Lambda_h e^h(T)) + \frac{1}{2}\Big\|\begin{pmatrix} h\dot{u}_\alpha^h(T) \\ \dot{u}_3^h(T)\end{pmatrix}\Big\|_{L^2}^2 + \int_0^T \mathcal{H}(\Lambda_h \dot{p}^h(t))\,dt\Big\} \\ \le\;& \mathcal{Q}_r(e(0)) + \frac{1}{2}\|\dot{u}_3(0)\|_{L^2}^2 + \int_0^T\int_\Omega \sigma(t) : E\dot{w}(t)\,dx\,dt \\ &+ \int_0^t\int_\omega (\ddot{u}_3(t)\dot{w}_3(t) + \bar{f}(t)(\dot{u}_3(t) - \dot{w}_3(t))\,dx'\,dt \\ =\;& \mathcal{Q}_r(e(T)) + \frac{1}{2}\|\dot{u}_3(T)\|_{L^2}^2 + \int_0^T \mathcal{H}_r(\dot{p}(t))\,dx\,dt.\end{aligned}$$

Recalling (4.42) and

$$\mathcal{Q}_r(e(T)) \le \liminf_{h\to 0} \mathcal{Q}(\Lambda_h e^h(T)), \qquad \|\dot{u}_3(T)\|_{L^2}^2 \le \liminf_{h\to 0}\|\dot{u}_3^h(T)\|_{L^2}^2,$$

the inequality above implies that $\dot{u}_3^h(t) \to \dot{u}_3(t)$ strongly in $L^2(\Omega)$ and

$$\mathcal{Q}(\Lambda_h e^h(t)) \to \mathcal{Q}_r(e(t)) = \mathcal{Q}(\mathbb{M}e(t))$$

for every $t \ge 0$. Since

$$\mathcal{Q}(\Lambda_h e^h(t) - \mathbb{M}e(t)) = \mathcal{Q}(\Lambda_h e^h(t)) + \mathcal{Q}(\mathbb{M}e(t)) - \int_\Omega \mathbb{C}\Lambda_h e^h(t) : \mathbb{M}e(t)\,dx,$$

equations (4.28) and (4.38) imply that

$$\lim_{h\to 0}\mathcal{Q}(\Lambda_h e^h(t) - \mathbb{M}e(t)) = 0$$

for every $t \ge 0$. Hence, by (2.2) we conclude that $\Lambda_h e^h(t) \to \mathbb{M}e(t)$ strongly in $L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$ for every $t \ge 0$. As an immediate consequence, we deduce that $e^h(t) \to e(t)$ strongly in $L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$ for every $t \ge 0$.

This concludes the proof of Theorem 4.1. □

5. Some properties of the reduced model

In this section we collect some results about uniqueness for the reduced dynamic model, that has been derived in the previous section. We first prove uniqueness of the vertical displacement, of the elastic strain, and of some components of the plastic strain.

Proposition 5.1. *Let $t \mapsto (u(t), e(t), p(t))$ be a reduced dynamic evolution, that is, a solution to the system (i)-(v) in Theorem 4.1. Then the vertical displacement u_3, the elastic strain e, and the plastic strain components $\hat{p}$ and $p_\perp$ are unique.*

Proof. Let (u, e, p) and (v, η, q) be two solutions. Let $\sigma(t) := \mathbb{C}_r e(t)$ and $\tau(t) := \mathbb{C}_r \eta(t)$. Subtracting the two equations of motion for u_3 and v_3 leads to

$$\ddot{u}_3(t) - \ddot{v}_3(t) - \frac{1}{12}\operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t)) = 0 \quad \text{in } \omega$$

for a.e. $t \geq 0$. Multiplying this equation by $\dot{u}_3(t) - \dot{v}_3(t)$ and integrating on $[0, T] \times \omega$ yields

$$\int_0^T \int_\omega (\ddot{u}_3(t) - \ddot{v}_3(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt \\ - \frac{1}{12}\int_0^T \int_\omega \operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt = 0. \quad (5.1)$$

Since $\dot{u}_3(0) = \dot{v}_3(0)$, we have

$$\int_0^T \int_\omega (\ddot{u}_3(t) - \ddot{v}_3(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt = \frac{1}{2}\|\dot{u}_3(T) - \dot{v}_3(T)\|_{L^2}^2. \quad (5.2)$$

On the other hand, by Proposition 2.5, (4.21), and (4.22), we obtain

$$- \frac{1}{12}\int_0^T \int_\omega \operatorname{div}\operatorname{div}(\hat{\sigma}(t) - \hat{\tau}(t))(\dot{u}_3(t) - \dot{v}_3(t))\, dx'\, dt \\ = \int_0^T \int_\Omega (\sigma(t) - \tau(t)) : (\dot{e}(t) - \dot{\eta}(t))\, dx\, dt + \int_0^T \langle \sigma(t) - \tau(t), \dot{p}(t) - \dot{q}(t)\rangle_r\, dt, \quad (5.3)$$

where we have also used that $(\dot{u}(t) - \dot{v}(t), \dot{e}(t) - \dot{\eta}(t), \dot{p}(t) - \dot{q}(t)) \in \mathcal{A}_{KL}(0)$ for a.e. $t \geq 0$. Since $e(0) = \eta(0)$, we have

$$\int_0^T \int_\Omega (\sigma(t) - \tau(t)) : (\dot{e}(t) - \dot{\eta}(t))\, dx\, dt = \mathcal{Q}_r(e(T) - \eta(T)). \quad (5.4)$$

Moreover, using the flow rule, (2.18), and the fact that $\tau(t) \in K_r$ a.e. in Ω, we infer that

$$\langle \sigma(t) - \tau(t), \dot{p}(t)\rangle_r \geq 0$$

for a.e. $t \geq 0$. Similarly,

$$\langle \tau(t) - \sigma(t), \dot{q}(t)\rangle_r \geq 0$$

for a.e. $t \geq 0$. Summing up the previous inequalities and integrating in time yields

$$\int_0^T \langle \sigma(t) - \tau(t), \dot{p}(t) - \dot{q}(t)\rangle_r\, dt \geq 0. \quad (5.5)$$

Gathering (5.1)–(5.5) we deduce that

$$\frac{1}{2}\|\dot{u}_3(T) - \dot{v}_3(T)\|_{L^2}^2 + \mathcal{Q}_r(e(T) - \eta(T)) \leq 0.$$

By (2.2) we conclude that $\dot{u}_3 = \dot{v}_3$, hence $u_3 = v_3$, and that $e = \eta$. Finally, by Proposition 2.4 we deduce that $\hat{p} = \hat{q}$ and $p_\perp = q_\perp$. □

The following proposition gives a two-dimensional characterisation of the reduced dynamic evolution model for a specific choice of the data.

Proposition 5.2. *For every $t \geq 0$ let*

$$w(t, x) = \begin{pmatrix} \bar{w}(t, x') \\ 0 \end{pmatrix} \qquad \text{for a.e. } x \in \Omega,$$

where $\bar{w} \in H^2_{loc}([0, +\infty); H^1(\omega; \mathbb{R}^2))$. Let $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ be of the form

$$u_0(x) = \begin{pmatrix} \bar{u}_0(x') \\ 0 \end{pmatrix}, \qquad e_0(x) = \bar{e}_0(x') \quad \text{for a.e. } x \in \Omega, \qquad p_0 = \bar{p}_0 \otimes \mathcal{L}^1.$$

Then a map $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution, that is, a solution to (i)-(v) in Theorem 4.1, with boundary datum w, force term $\bar{f} = 0$, and initial conditions $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$ and $\dot{u}_3(0) = 0$, if and only if

$$u(t,x) = \begin{pmatrix} \bar{u}(t,x') \\ 0 \end{pmatrix}, \qquad e(t,x) = \bar{e}(t,x') \quad \text{for a.e. } x \in \Omega, \qquad p(t) = \bar{p}(t) \otimes \mathcal{L}^1 \tag{5.6}$$

for every $t \geq 0$, where

$$t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t)) \in BD(\omega) \times L^2(\omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\omega \cup \gamma_d; \mathbb{M}^{2\times 2}_{sym})$$

satisfies the following conditions:

(a) *$E\bar{u}(t) = \bar{e}(t) + \bar{p}(t)$ in ω, $\bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega}\mathcal{H}^1$ on γ_d for every $t \geq 0$;*

(b) *$(\overline{u}(0), \overline{e}(0), \overline{p}(0)) = (\overline{u}_0, \overline{e}_0, \overline{p}_0)$;*

(c) *$\bar{\sigma}(t) \in K_r$ a.e. in ω for every $t \geq 0$;*

(d) *for every $t \geq 0$*

$$\begin{cases} \operatorname{div} \bar{\sigma}(t) = 0 & \text{in } \omega, \\ [\bar{\sigma}(t)\nu_{\partial\omega}] = 0 & \text{on } \gamma_n; \end{cases}$$

(e) *$\mathcal{H}_r(\dot{\bar{p}}(t)) = \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle$ for a.e. $t \geq 0$.*

Proof. Assume that $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution with the given data. We have to prove that (5.6) and (a)–(e) are satisfied. To do this we argue as in [11, Proposition 7.16]. The theory of convex functions of measure ensures that

$$\mathcal{H}_r(\dot{p}(t)) = \mathcal{H}_r(\dot{p}^a(t)) + \mathcal{H}_r(\dot{p}^s(t)). \tag{5.7}$$

By the Fubini-Tonelli Theorem and the Jensen inequality we have

$$\begin{aligned} \mathcal{H}_r(\dot{p}^a(t)) &= \int_{\omega\cup\gamma_d} \int_{-\frac{1}{2}}^{\frac{1}{2}} H_r(\dot{\bar{p}}^a(t) + x_3\dot{\hat{p}}^a(t) - \dot{e}_\perp(t))\, dx_3\, dx' \\ &\geq \int_{\omega\cup\gamma_d} H_r\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} (\dot{\bar{p}}^a(t) + x_3\dot{\hat{p}}^a(t) - \dot{e}_\perp(t))\, dx_3\Big)\, dx' \\ &= \mathcal{H}_r(\dot{\bar{p}}^a(t)) \end{aligned} \tag{5.8}$$

for a.e. $t \geq 0$. Let $\lambda(t) := |\dot{\bar{p}}^s(t)| + |\dot{\hat{p}}^s(t)|$ for a.e. $t \geq 0$. Then the measure $\dot{\bar{p}}^s(t) + x_3\dot{\hat{p}}^s(t)$ is absolutely continuous with respect to $\lambda(t)$ for every $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. Thus, by the Radon-Nikodým Theorem we can write

$$\dot{p}^s(t) = \Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\lambda(t) \overset{gen.}{\otimes} \mathcal{L}^1,$$

where $\overset{gen.}{\otimes}$ denotes the generalised product of measures (see, e.g., [4, Definition 2.27]). By the Fubini-Tonelli Theorem and the Jensen inequality, we obtain

$$\begin{aligned} \mathcal{H}_r(\dot{p}^s(t)) &= \int_{\omega\cup\gamma_d} \int_{-\frac{1}{2}}^{\frac{1}{2}} H_r\Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\, dx_3\, d\lambda(t) \\ &\geq \int_{\omega\cup\gamma_d} H_r\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{d\dot{\bar{p}}^s(t)}{d\lambda(t)} + x_3\frac{d\dot{\hat{p}}^s(t)}{d\lambda(t)}\Big)\, dx_3\Big)\, d\lambda(t) \\ &= \mathcal{H}_r(\dot{\bar{p}}^s(t)) \end{aligned} \tag{5.9}$$

for a.e. $t \geq 0$. Combining (5.7)–(5.9), we conclude that

$$\mathcal{H}_r(\dot{p}(t)) \geq \mathcal{H}_r(\dot{\bar{p}}^a(t)) + \mathcal{H}_r(\dot{\bar{p}}^s(t)) = \mathcal{H}_r(\dot{\bar{p}}(t)) \tag{5.10}$$

for a.e. $t \geq 0$.

On the other hand, by (2.17), (2.18), (4.22), (4.23), and Proposition 2.5, we deduce that

$$\begin{aligned}\mathcal{H}_r(\dot p(t)) = \langle \sigma(t), \dot p(t)\rangle_r &= \langle \bar\sigma(t), \dot{\bar p}(t)\rangle + \frac{1}{12}\langle \hat\sigma(t), \dot{\hat p}(t)\rangle - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\,dx \\ &\leq \mathcal{H}_r(\dot{\bar p}(t)) - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\,dx - \frac{1}{12}\int_\omega \hat\sigma(t) : \dot{\hat e}(t)\,dx' - \int_\omega \dot u_3(t)\ddot u_3(t)\,dx'.\end{aligned} \tag{5.11}$$

Here we used that $w_3(t) = 0$ and $\bar f(t) = 0$ for every $t \geq 0$.

Therefore, by (5.10) we have

$$\begin{aligned}\int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\,dx + \frac{1}{12}\int_\omega \hat\sigma(t) : \dot{\hat e}(t)\,dx' + \int_\omega \dot u_3(t)\ddot u_3(t)\,dx' \\ = \frac{d}{dt}\big(\mathcal{Q}_r(e_\perp(t)) + \frac{1}{12}\mathcal{Q}_r(\hat e(t)) + \frac{1}{2}\|\dot u_3(t)\|^2_{L^2}\big) \leq 0\end{aligned}$$

for a.e. $t \geq 0$. Integrating with respect to time, this inequality yields

$$\mathcal{Q}_r(e_\perp(t)) + \frac{1}{12}\mathcal{Q}_r(\hat e(t)) + \frac{1}{2}\|\dot u_3(t)\|^2_{L^2} \leq \mathcal{Q}_r(e_\perp(0)) + \frac{1}{12}\mathcal{Q}_r(\hat e(0)) + \frac{1}{2}\|\dot u_3(0)\|^2_{L^2} = 0.$$

Since $u_3(0) = 0$, this implies that $u_3 = 0$ and $\hat e = e_\perp = 0$. By Proposition 2.4 we deduce that $\hat p = p_\perp = 0$. In other words, (5.6) is satisfied.

Condition (a) follows immediately from Proposition 2.4, (b) is straightforward, and (d) follows from (4.21). Since $\sigma(t) \in K_r$ a.e. in Ω, it is easy to check that $\bar\sigma(t) \in K_r$ a.e. in ω, that is, (c) holds. Finally, (5.11) and (5.10) yield (e).

Conversely, if $t \mapsto (u(t), e(t), p(t))$ is of the form (5.6) and conditions (a)–(e) are satisfied, it is trivial to check that $t \mapsto (u(t), e(t), p(t))$ is a reduced dynamic evolution. □

Remark 5.3. The previous proposition suggests that, in general, one cannot expect uniqueness for the components $\bar u$ and $\bar p$ of a reduced dynamic evolution. Indeed, Proposition 5.2 shows that for some specific choice of the data the reduced dynamic evolution coincides with a two-dimensional quasistatic model, for which uniqueness of displacement and plastic strain in general fails (see, e.g., [34]).

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