LOWER SEMICONTINUITY OF A CLASS OF INTEGRAL FUNCTIONALS ON THE SPACE OF FUNCTIONS OF BOUNDED DEFORMATION

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Abstract. We study the lower semicontinuity of some free discontinuity functionals with linear growth defined on the space of functions with bounded deformation. The volume term is convex and depends only on the Euclidean norm of the symmetrized gradient. We introduce a suitable class of surface terms, which make the functional lower semicontinuous with respect to L^1 convergence.

Keywords: free discontinuity problems, lower semicontinuity, functions of bounded deformation

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1. Introduction

Free discontinuity functionals, depending on a function u and on its jump set J_u, have an increasing importance for several problems in the Calculus of Variations, with applications to image processing, liquid crystals, and fracture mechanics (see [16, 22, 20, 3, 10]). A large literature is devoted to the study of the lower semicontinuity properties of free discontinuity functionals (see [1, 6, 12, 13, 9, 5, 8, 14, 17]). The techniques used in the proofs of the lower semicontinuity results strongly depend on the assumptions on the volume term of the functional, which typically involves the gradient ∇u. In the case of superlinear growth in ∇u, the volume and the surface term do not interact, while in the case of linear growth, which occurs in connection with elasto-plasticity problems, there is a strong interaction, which requires a different treatment.

The prototype of free discontinuity functionals with linear growth is

$$\int_\Omega f(|\nabla u|)\,\mathrm{d}x + C|D^c u|(\Omega) + \int_{J_u} g([u])\,\mathrm{d}\mathcal{H}^{n-1}, \tag{1.1}$$

where $u \in BV(\Omega)$, the space of functions of bounded variation, ∇u is the density of the absolutely continuous part of the distributional gradient of u, $D^c u$ is its Cantor part, J_u is the jump set of u, $[u]$ is the difference between the traces of u on both sides of J_u, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We refer to the book [3] for the definition and properties of BV functions. The lower semicontinuity of (1.1) with respect to the weak* convergence in BV has been proved in [6] under the following assumptions: $f\colon [0,+\infty) \to [0,+\infty)$ is convex and nondecreasing, $g\colon \mathbb{R} \to [0,+\infty)$ is even and subadditive, and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} = C \in (0,+\infty)\,. \tag{1.2}$$

In Remark 2.3 we shall see that an even subadditive function satisfying (1.2) is continuous. Note that all terms in (1.1) interact and equality (1.2) is also necessary for lower semicontinuity.

The purpose of this paper is to extend this result to functionals defined in the space $BD(\Omega)$ of functions of bounded deformation. We refer to the book [21] for the general properties of this space and to [2] for the fine properties of BD functions.

The natural extension of (1.1) to $BD(\Omega)$ is the functional

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{1.3}$$

where $Eu = \frac{1}{2}(Du + Du^T)$ is the symmetric part of the distributional gradient of u, $\mathcal{E}u$ is the density of the absolutely continuous part of Eu, while $E^c u$ is the Cantor part of Eu (defined in [2, Definition 4.1]). Here and in the rest of the paper, the space $\mathbb{M}^{n\times n}_{\rm sym}$ of $n\times n$ symmetric matrices is endowed with the Euclidean (or Frobenius) norm defined by

$$|A| = \Big(\sum_{i,j=1}^{n} A_{ij}^2\Big)^{\frac{1}{2}},$$

and the variation of the measure $|E^c u|$ is defined accordingly.

We shall see in Proposition 3.1 that the functional (1.3) cannot be lower semicontinuous. The main reason is that the last term in (1.3) does not take into account the orientation of the jump set J_u.

A possible way to overcome this drawback is to consider the restriction to J_u of the measure Eu which is given by

$$Eu \llcorner J_u = [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u\,,$$

where ν_u is the approximate unit normal to J_u and, for every pair of vectors $a, b \in \mathbb{R}^n$, $a \odot b$ is the matrix whose components are $\frac{1}{2}(a_i b_j + a_j b_i)$. The matrix $[u] \odot \nu_u$ encodes the behavior of the jump of u, taking into account also the orientation of the jump set. This suggests that a natural extension of (1.1) to $BD(\Omega)$ is

$$\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u] \odot \nu_u|)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.4}$$

We shall see in Proposition 3.2 that, if $g(t) = \min\{C|t|, 1\}$, then the functional given by (1.4) is not lower semicontinuous because the 1-homogeneous extension of the function $\nu \mapsto g(|z \odot \nu|)$ is not convex on $\mathbb{R}^n$.

The functional we propose as extension of (1.1) to $BD(\Omega)$ has the form

$$\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} G([u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{1.5}$$

where the function $G(z,\nu)$ has a specific structure and, in general, does not depend only on $|z \odot \nu|$. To explain the hypothesis we will consider on G, it is convenient to anticipate the technique of the proof. This will be based on a slicing argument which relies on the following well-known formula for the Euclidean norm of a symmetric $n\times n$ matrix A:

$$|A|^2 = \sup_{(\xi^1,\dots,\xi^n)} \sum_{i=1}^{n} |A\xi^i \cdot \xi^i|^2, \tag{1.6}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \dots, \xi^n)$ of $\mathbb{R}^n$ (see Proposition 2.1). This method suggests that the lower semicontinuity of $\mathcal{F}$ can be proved when G satisfies the following condition: there exists a function $g\colon \mathbb{R} \mapsto [0, +\infty)$ such that

$$G(z,\nu) = \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^{n} g(z \cdot \xi^i)^2 |\nu \cdot \xi^i|^2\Big)^{\frac{1}{2}}, \tag{1.7}$$

where the supremum is taken over all orthonormal bases $(\xi^1, \dots, \xi^n)$ of $\mathbb{R}^n$. Note that, in the particular case $g(t) = C|t|$, the function in (1.7) takes the form $G(z,\nu) = C|z \odot \nu|$.

The main result of the paper (Theorem 4.1) is that the functional $\mathcal{F}$ is L^1-lower semicontinuous on $BD(\Omega)$ under the following assumptions: f is convex and nondecreasing, G is given by (1.7), with g even and subadditive, and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} = C \in (0, +\infty)\,, \quad \liminf_{t\to+\infty} g(t) > 0\,. \tag{1.8}$$

The last assumption on g in (1.8) is used only to prove the semicontinuity in L^1 rather than in the weak* topology of BD.

The function $G(z,\nu)$ defined in (1.7) is computed explicitly in Section 6 when $g(t) = \min\{|t|, 1\}$ and $n = 2$. In particular, we find that in this case $G(z,\nu) = |z \odot \nu|$ if $|z| \le 1$, $G(z,\nu) = 1$ if $|z| \ge \sqrt{2}$, while there is a region in the annulus $1 < |z| < \sqrt{2}$ where $G(z,\nu) < \min\{|z \odot \nu|, 1\}$.

The results in Section 4 are used to prove a relaxation theorem for functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \psi([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

assuming that f is convex and nondecreasing and that there exists an even and subadditive function $g\colon \mathbb{R} \mapsto [0,+\infty)$ satisfying (1.8) such that

$$\psi(z,\nu) \geq \Big(\sum_{i=1}^n g(z\cdot\xi^i)^2|\nu\cdot\xi^i|^2\Big)^{\frac{1}{2}},$$

for every orthonormal basis $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$. In this case, the lower semicontinuous envelope $\mathrm{sc}^-\mathcal{F}$ takes the form

$$\mathrm{sc}^-\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \overline{\psi}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1},$$

for a suitable function $\overline{\psi}$. Note that the function f and the constant C do not change in the relaxation process.

2. Notation and preliminary results

Throughout the paper Ω is a bounded open set in $\mathbb{R}^n$, $\mathscr{A}(\Omega)$ is the class of open sets contained in Ω, and $\mathscr{B}(\Omega)$ is the class of Borel sets contained in Ω. The notation $A \Subset B$ means that A is relatively compact in B. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The set $\mathbb{S}^{n-1}$ is the unit sphere $\{x\in\mathbb{R}^n : |x|=1\}$, while $\mathrm{O}(n)$ stands for the group of $n{\times}n$ orthogonal matrices.

For every $n{\times}n$ matrix A, the Euclidean norm of A is defined by

$$|A| := \Big(\sum_{i,j=1}^n A_{ij}^2\Big)^{\frac{1}{2}}.$$

The following property will be used in the proof of the lower semicontinuity result. It is a well-known formula, but we include a proof for completeness of the presentation. The proof is based on the fact that the Euclidean norm is unitary invariant (see, e.g., [18]).

Proposition 2.1. *For every $n{\times}n$ symmetric matrix A we have*

$$|A| = \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n |A\xi^i\cdot\xi^i|^2\Big)^{\frac{1}{2}}, \tag{2.1}$$

where the supremum is taken over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$.

Proof. Let A be a symmetric matrix, let $(\xi^1,\dots,\xi^n)$ be an orthonormal basis and let $R\in\mathrm{O}(n)$ be a rotation such that $\xi^i = Re_i$, where $(e_1,\dots,e_n)$ is the standard basis in $\mathbb{R}^n$. Then

$$\sum_{i=1}^n |A\xi^i\cdot\xi^i|^2 = \sum_{i=1}^n |ARe_i\cdot Re_i|^2 = \sum_{i=1}^n |R^TARe_i\cdot e_i|^2 \leq |R^TAR|^2 = |A|^2.$$

To show that the supremum in (2.1) is attained, let $S\in\mathrm{O}(n)$ be a rotation such that S^TAS is a diagonal matrix with entries $\lambda_1,\dots,\lambda_n$, and let $\zeta^i := Se_i$ for $i=1,\dots,n$. Then $(\zeta^1,\dots,\zeta^n)$ is an orthonormal basis of $\mathbb{R}^n$ and we have that

$$\sum_{i=1}^n |A\zeta^i\cdot\zeta^i|^2 = \sum_{i=1}^n |ASe_i\cdot Se_i|^2 = \sum_{i=1}^n |S^TASe_i\cdot e_i|^2 = \sum_{i=1}^n \lambda_i^2 = |S^TAS|^2 = |A|^2.$$

This concludes the proof. ☐

Remark 2.2. We note that taking the supremum over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$ is equivalent to taking the supremum over the columns of all rotations $R\in\mathrm{O}(n)$. Therefore the supremum in (2.1) does not change if we consider only a countable dense family in $\mathrm{O}(n)$.

We recall that a function $g\colon\mathbb{R}\to[0,+\infty)$ is subadditive if

$$g(s+t)\leq g(s)+g(t) \quad \text{for every } s,t\in\mathbb{R}.$$

Remark 2.3. It is known (see, for instance, [19, Theorem 16.3.3]) that for a subadditive function we have

$$\lim_{t\to 0^+}\frac{g(t)}{t}=\sup_{t>0}\frac{g(t)}{t}$$

provided the right-hand side is finite. Moreover, if g is even, subadditive, and the right-hand side of the previous formula is finite, then $g(0)=0$ and g is continuous at 0, hence at every point of $\mathbb{R}$ (see, for instance, [19, Theorem 16.2.1]).

Remark 2.4. If g is a subadditive function satisfying (1.8) and $0<c<C$, then there exists a constant $b>0$ such that

$$g(t)\geq \min\{ct,b\}\quad \text{for every } t\geq 0. \tag{2.2}$$

Indeed, the first assumption in (1.8) implies that there exists $\delta>0$ such that $g(t)\geq ct$, for every $t\in[0,\delta]$. The second assumption in (1.8) implies that there exist $\eta>0$ and $M>\delta$ such that $g(t)\geq\eta$ for every $t\geq M$. We claim that

$$\inf_{t\in[\delta,M]} g(t)>0\,. \tag{2.3}$$

To prove the claim, we fix an integer $n\geq\frac{M}{\delta}$ and a constant $\varepsilon>0$ such that $n\varepsilon<\eta$. If (2.3) does not hold, then there exists $t\in[\delta,M]$ such that $g(t)<\varepsilon$. By subadditivity we have $g(nt)\leq n\varepsilon<\eta$. On the other hand $nt\geq n\delta\geq M$, hence $g(nt)\geq\eta$. This contradiction proves (2.3). To obtain (2.2) it is enough to take a constant b less than the infimum in (2.3) and with $0<b<\min\{c\delta,\eta\}$.

Remark 2.5. Let g be an even subadditive function satisfying (1.8) and let $a\in[0,C)$. Let us define the function

$$g^a(t):=\inf_{s\in\mathbb{R}}[\,g(s)+a|t-s|\,]\,.$$

It is easy to see that the function g^a is even, subadditive, and that $g^a\nearrow g$ as $a\nearrow C$. Moreover, using Remark 2.4, we can prove that there exists $\delta_a>0$ such that $g^a(t)=at$ for every $t\in[0,\delta_a]$.

Given a metric space X and a functional $F\colon X\to[0,+\infty]$, we recall that F is lower semicontinuous if for every sequence $x_k\in X$ such that $x_k\to x$ we have

$$F(x)\leq\liminf_{k\to+\infty}F(x_k)\,.$$

The following localization lemma is a classical tool useful to prove the lower semicontinuity of integral functionals. Its proof can be found, e.g., in [11, Lemma 15.2].

Lemma 2.6. *Let Λ be a function defined on the family of open subsets of Ω, which is superadditive on open sets with disjoint compact closure. Let λ be a positive measure on Ω, let φ_j, $j\in\mathbb{N}$, be nonnegative Borel functions such that*

$$\int_K\varphi_j\,\mathrm{d}\lambda\leq\Lambda(A)$$

for every open set $A\subset\Omega$, for every compact set $K\subset A$, and for every $j\in\mathbb{N}$. Then

$$\int_K\sup_j\varphi_j\,\mathrm{d}\lambda\leq\Lambda(A)$$

for every open set $A\subset\Omega$ and for every compact set $K\subset A$. Moreover, if $A\subset\Omega$ is an open set with $\Lambda(A)<+\infty$, then

$$\int_K\sup_j\varphi_j\,\mathrm{d}\lambda=\sup\Big\{\sum_{j=1}^r\int_{K^j}\varphi_j\,\mathrm{d}\lambda : (K^j)_{j=1}^r \textit{ disjoint compact subsets of } K,\ r\in\mathbb{N}\Big\}$$

for every compact set $K\subset A$.

We now recall some lower semicontinuity results about one dimensional functionals defined on the space $BV(U)$, where U is a bounded open subset of $\mathbb{R}$. Let us consider the functional $\Phi\colon BV(U)\times\mathscr{A}(U)\to[0,+\infty)$ defined by

$$\Phi(u,A):=\int_A f(|\nabla u|)\,\mathrm{d}t+C|D^cu|(A)+\sum_{t\in J_u\cap A}g([u](t))\,, \tag{2.4}$$

for every $u \in BV(U)$ and for every open set A contained in U. Even in dimension one we keep the notation ∇u for the absolutely continuous part of the distributional derivative Du of u. It is well known that the functional Φ is lower semicontinuous with respect to the weak* topology in $BV(U)$ under the following assumptions: f is convex and nondecreasing, g is even and subadditive, and (1.2) holds. For a proof of this result we refer to [7] and [3, Theorem 5.2]. If, in addition, g satisfies also the last condition in (1.8), the functional Φ is also lower semicontinuous with respect to the L^1 topology, which is weaker than the weak* topology in $BV(U)$. When g is nondecreasing on $[0,+\infty)$, this result can be obtained easily by a truncation argument. For the reader's convenience we give here a complete proof in the general case.

Proposition 2.7. *Let $f\colon [0,+\infty) \to [0,+\infty)$ be convex and nondecreasing, let $g\colon \mathbb{R} \to [0,+\infty)$ be even and subadditive, and assume that* (1.8) *holds. Then the functional $\Phi(\cdot, U)$ defined in* (2.4) *is L^1-lower semicontinuous on $BV(U)$, i.e.,*

$$\Phi(u,U) \le \liminf_{k\to+\infty} \Phi(u_k, U)\,, \tag{2.5}$$

for every $u_k, u \in BV(U)$ such that $u_k \to u$ in $L^1(U)$.

Proof. It is enough to prove the result when U is a bounded open interval, denoted by I. Let us fix $u_k, u \in BV(I)$ such that $u_k \to u$ in $L^1(I)$. Up to extracting a subsequence, we can assume that $u_k \to u$ a.e. in Ω, that the liminf in (2.5) is finite, and that it is actually a limit. Therefore

$$\Phi(u_k, I) \le M \tag{2.6}$$

for some positive constant M.

We start by proving that the number of large jumps of the functions u_k is equibounded. By Remark 2.4, there exist a constant $c > 0$ such that $g(t) \ge c\min\{t,1\}$. By (2.6), this implies that there exists a constant $M' > 0$ such that

$$|Du_k|(I \setminus J^1_{u_k}) + \mathcal{H}^0(J^1_{u_k}) \le M'\,, \tag{2.7}$$

where $J^1_{u_k} := \{t \in J_{u_k} : |[u_k](t)| \ge 1\}$ and $\mathcal{H}^0$ is the counting measure. Hence, up to a subsequence, we can assume that there exists an integer $m \ge 1$ such that $J^1_{u_k} = \{t^k_1, \dots, t^k_m\}$, with $t^k_1 < \cdots < t^k_m$. We can also assume that $t^k_i \to t_i$ as $k \to +\infty$, for $i = 1, \dots, m$, where $t_1 \le \cdots \le t_m$. Let us consider $s_1 < \cdots < s_\ell$ such that $\{s_1, \dots, s_\ell\} = \{t_1, \dots, t_m\}$. Let us fix $\delta > 0$ such that the following properties are satisfied:

- the intervals $[s_i - \delta, s_i + \delta]$, $i = 1, \dots, \ell$, are pairwise disjoint;
- $s_i - \delta$ and $s_i + \delta$ do not belong to $\bigcup_k J_{u_k} \cup J_u$;
- $u_k(s_i - \delta) \to u(s_i - \delta)$ and $u_k(s_i + \delta) \to u(s_i + \delta)$ as $k \to +\infty$, for $i = 1, \dots, \ell$.

Let us consider the open set $A_\delta := I \setminus \bigcup_{i=1}^{\ell} [s_i - \delta, s_i + \delta]$. First of all, we notice that, for k large enough, we have that $|[u_k](t)| < 1$ for all $t \in J_{u_k} \cap A_\delta$, i.e., $J^1_{u_k} \cap A_\delta = \emptyset$. Hence, by (2.7), we have that $|Du_k|(A_\delta) \le M'$ for all k. This implies that $u_k \overset{*}{\rightharpoonup} u$ in $BV(A_\delta)$, and by the lower semicontinuity of $\Phi(\cdot, A_\delta)$ with respect to the weak* convergence of $BV(A_\delta)$, we deduce that

$$\Phi(u, A_\delta) \le \liminf_{k\to+\infty} \Phi(u_k, A_\delta)\,. \tag{2.8}$$

Let us now fix $i \in \{1, \dots, \ell\}$ and let $I^i_\delta := (s_i - \delta, s_i + \delta)$. By (1.2), given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \ge t_\varepsilon$ we have that

$$f(t) \ge (C - \varepsilon)t\,. \tag{2.9}$$

We claim that

$$g(u_k(s_i+\delta) - u_k(s_i - \delta)) \le \frac{C}{C-\varepsilon}\Phi(u_k, I^i_\delta) + 2C\delta t_\varepsilon\,. \tag{2.10}$$

First of all we observe that

$$u_k(s_i+\delta) - u_k(s_i-\delta) = Du_k(I^i_\delta) = \int_{I^i_\delta} \nabla u_k \,\mathrm{d}t + D^c u_k(I^i_\delta) + \sum_{t \in J_{u_k} \cap I^i_\delta} [u_k](t)\,.$$

By the subadditivity and the continuity of g and by the inequality $g(t) \leq C|t|$, we have that

$$
\begin{aligned}
g(u_k(s_i+\delta)-u_k(s_i-\delta)) &\leq g\Big(\int_{I_\delta^i} \nabla u_k \,\mathrm{d}t\Big) + g(D^c u_k(I_\delta^i)) + \sum_{t\in J_{u_k}\cap I_\delta^i} g([u_k](t)) \\
&\leq C\int_{I_\delta^i} |\nabla u_k|\,\mathrm{d}t + C|D^c u_k|(I_\delta^i) + \sum_{t\in J_{u_k}\cap I_\delta^i} g([u_k](t))\,.
\end{aligned} \tag{2.11}
$$

By (2.11) and (2.9) we get

$$
\begin{aligned}
\frac{C}{C-\varepsilon}\Phi(u_k, I_\delta^i) &\geq \frac{C}{C-\varepsilon}\int_{I_\delta^i} f(|\nabla u_k|)\,\mathrm{d}t + C|D^c u_k|(I_\delta^i) + \sum_{t\in J_{u_k}\cap I_\delta^i} g([u_k](t)) \\
&\geq \frac{C}{C-\varepsilon}\int_{I_\delta^i} f(|\nabla u_k|)\,\mathrm{d}t + g(u_k(s_i+\delta)-u_k(s_i-\delta)) - C\int_{I_\delta^i}|\nabla u_k|\,\mathrm{d}t \\
&\geq \frac{C}{C-\varepsilon}\int_{\{|\nabla u_k|\geq t_\varepsilon\}\cap I_\delta^i} f(|\nabla u_k|)\,\mathrm{d}t + g(u_k(s_i+\delta)-u_k(s_i-\delta)) \\
&\qquad - C\int_{\{|\nabla u_k|\geq t_\varepsilon\}\cap I_\delta^i}|\nabla u_k|\,\mathrm{d}t - C\int_{\{|\nabla u_k|< t_\varepsilon\}\cap I_\delta^i}|\nabla u_k|\,\mathrm{d}t \\
&\geq g(u_k(s_i+\delta)-u_k(s_i-\delta)) - C\mathcal{L}^1(I_\delta^i)t_\varepsilon\,.
\end{aligned}
$$

Since $\mathcal{L}^1(I_\delta^i)=2\delta$, this proves (2.10).

Letting $k\to+\infty$ in (2.10) we obtain

$$
g(u(s_i+\delta)-u(s_i-\delta)) \leq \liminf_{k\to+\infty}\frac{C}{C-\varepsilon}\Phi(u_k, I_\delta^i) + 2C\delta t_\varepsilon\,. \tag{2.12}
$$

Summing (2.8) and (2.12) for $i=1,\dots,\ell$, it follows that

$$
\Phi(u, A_\delta) + \sum_{i=1}^{\ell} g(u(s_i+\delta)-u(s_i-\delta)) \leq \frac{C}{C-\varepsilon}\liminf_{k\to+\infty}\Phi(u_k, I) + 2\ell C\delta t_\varepsilon\,.
$$

Letting $\delta\to 0$ and then $\varepsilon\to 0$, we conclude the proof of (2.5). □

3. Some examples

In this section we show that, in general, the functionals defined in (1.3) and in (1.4) are not L^1-lower semicontinuous on $BD(\Omega)$.

We start by studying the functional $\mathcal{G}_1\colon BD(\Omega)\to[0,+\infty)$ defined by

$$
\mathcal{G}_1(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]|)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.1}
$$

where $f\colon[0,+\infty)\to[0,+\infty)$ is a convex nondecreasing function such that

$$
0 < \lim_{t\to+\infty}\frac{f(t)}{t} < +\infty\,, \tag{3.2}
$$

$C\in(0,+\infty)$, and $g\colon\mathbb{R}\to[0,+\infty)$ is a Borel function. As we shall see in the following proposition, the reason why the functional $\mathcal{G}_1$ fails to be lower semicontinuous is the fact that the surface density only depends on $|[u]|$.

Proposition 3.1. *The functional $\mathcal{G}_1$ defined in* (3.1) *is not L^1-lower semicontinuous on $BD(\Omega)$.*

Proof. For the sake of simplicity, we give the proof only when Ω is the unit cube in $\mathbb{R}^n$ centered at the origin, i.e., $\Omega=(-\frac12,\frac12)^n$. Let Q' be the unit cube in $\mathbb{R}^{n-1}$, i.e., $Q'=(-\frac12,\frac12)^{n-1}$. For every $x\in\Omega$, let x' be the vector in Q' with components $(x_1,\dots,x_{n-1})$. Let us assume, by contradiction, that the functional $\mathcal{G}_1$ is L^1-lower semicontinuous on $BD(\Omega)$.

Let us start by proving that

$$
\liminf_{s\to0^+}\frac{g(s)}{s} \geq \lim_{t\to+\infty}\frac{f(t)}{t}\,. \tag{3.3}
$$

Let us fix $t>2$, $z\in\mathbb{R}^n$, $z\neq 0$, and let us define the function u which connects linearly the vector 0 and the vector z in the rectangle $Q'\times[0,\frac{1}{t}]$:

$$u(x',x_n):=\begin{cases} z & \text{if } \frac{1}{t}\le x_n<\frac{1}{2}\,,\\ tzx_n & \text{if } 0<x_n<\frac{1}{t}\,,\\ 0 & \text{if } -\frac{1}{2}<x_n\le 0\,. \end{cases}$$

We now define a sequence of pure jump functions u_k which approximate in L^1 the function u. Let $t_k\to 0^+$ be a sequence such that the liminf in (3.3) is equal to $\lim_k \frac{g(t_k|z|)}{t_k|z|}$. For every $k\in\mathbb{N}$ let $h_k\in\mathbb{N}$ be such that $h_k\le\frac{1}{t_k}<h_k+1$. We define the function

$$\alpha_k(s):=\sum_{j=1}^{h_k}(j-1)t_k\mathbb{1}_{\left(\frac{j-1}{h_kt},\frac{j}{h_kt}\right)}(s),\qquad s\in(0,\tfrac{1}{t})\,,$$

where $\mathbb{1}_I$ is the indicator function of the interval I. Let

$$u_k(x',x_n):=\begin{cases} z & \text{if } \frac{1}{t}\le x_n<\frac{1}{2}\,,\\ z\alpha_k(x_n) & \text{if } 0<x_n<\frac{1}{t}\,,\\ 0 & \text{if } -\frac{1}{2}<x_n\le 0\,. \end{cases}$$

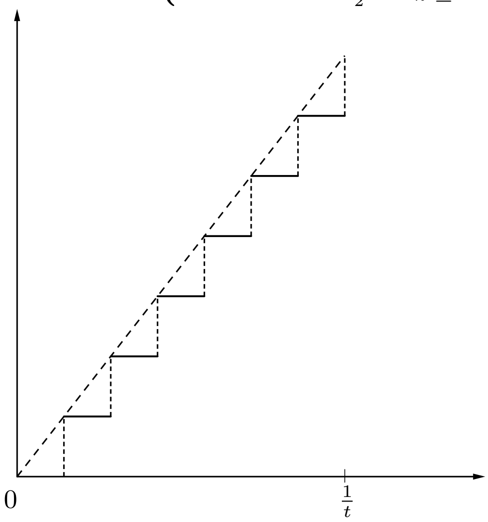

Figure 1. Graph of the function α_k.

It is easy to see that $u_k\to u$ in $L^1(\Omega;\mathbb{R}^n)$. Therefore, by the lower semicontinuity of $\mathcal{G}_1$ we get

$$\frac{f(t|z\odot e_n|)}{t}=\int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x=\mathcal{G}_1(u)\le\liminf_{k\to+\infty}\mathcal{G}_1(u_k)=\liminf_{k\to+\infty}\int_{J_{u_k}} g(|[u_k]|)\,\mathrm{d}\mathcal{H}^{n-1}$$

$$=\liminf_{k\to+\infty}h_kg(t_k|z|)=\lim_{k\to+\infty}\frac{g(t_k|z|)}{t_k}=|z|\liminf_{s\to 0^+}\frac{g(s)}{s}\,.$$

Letting $t\to+\infty$ in the inequality above, by (3.2) we get

$$|z\odot e_n|\lim_{t\to+\infty}\frac{f(t)}{t}\le|z|\liminf_{s\to 0^+}\frac{g(s)}{s}\,.$$

If we choose $z=e_n$, this proves (3.3).

Let us now prove that

$$\limsup_{s\to 0^+}\frac{g(s)}{s}\le\frac{1}{\sqrt{2}}\lim_{t\to+\infty}\frac{f(t)}{t}\,. \tag{3.4}$$

Taking (3.2) into account, this contradicts (3.3). To prove (3.4) we fix $z\in\mathbb{R}^n$, with $z\neq 0$, and we consider the pure jump function v defined by

$$v(x',x_n)=\begin{cases} z & \text{if } 0<x_n<\frac{1}{2}\,,\\ 0 & \text{if } -\frac{1}{2}<x_n\le 0\,. \end{cases}$$

We now construct a sequence of piecewise affine functions v_k which approximate v in L^1. For every $k \in \mathbb{N}$, $k \geq 2$, let

$$v_k(x', x_n) := \begin{cases} z & \text{if } \frac{1}{k} \leq x_n < \frac{1}{2}\,, \\ kzx_n & \text{if } 0 < x_n < \frac{1}{k}\,, \\ 0 & \text{if } -\frac{1}{2} < x_n \leq 0\,. \end{cases}$$

By the lower semicontinuity of $\mathcal{G}_1$ and by (3.2), we have that

$$\begin{aligned} g(|z|) &= \int_{J_v} g(|[v]|)\,\mathrm{d}\mathcal{H}^{n-1} = \mathcal{G}_1(v) \leq \liminf_{k\to+\infty} \mathcal{G}_1(v_k) = \liminf_{k\to+\infty} \int_\Omega f(|\mathcal{E}v_k|)\,\mathrm{d}x \\ &= \lim_{k\to+\infty} \frac{f(k|z\odot e_n|)}{k} = |z\odot e_n| \lim_{t\to+\infty} \frac{f(t)}{t}\,. \end{aligned}$$

By choosing z of the form $z = \delta e_1 = (\delta, 0, \dots, 0)$ we get

$$\frac{g(|\delta e_1|)}{\delta} \leq |e_1 \odot e_n| \lim_{t\to+\infty} \frac{f(t)}{t} = \frac{1}{\sqrt{2}} \lim_{t\to+\infty} \frac{f(t)}{t}\,,$$

and therefore, by letting $\delta \to 0^+$, we obtain (3.4). This concludes the proof. □

Let us now consider the functional $\mathcal{G}_2 \colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{G}_2(u) := \int_\Omega f(|\mathcal{E}u|)\,\mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} g(|[u]\odot\nu_u|)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.5}$$

where $f \colon [0, +\infty) \to [0, +\infty)$ is a convex nondecreasing function such that

$$\lim_{t\to+\infty} \frac{f(t)}{t} = C\,, \tag{3.6}$$

with $0 < C < +\infty$, and $g(t) = \min\{C|t|, 1\}$. In the next proposition, we prove that $\mathcal{G}_2$ is not lower semicontinuous. In this case, the main issue is the fact that the surface density does not satisfy a necessary condition for the lower semicontinuity of the functional. Indeed, the function

$$\psi(z, \nu) := g(|z \odot \tfrac{\nu}{|\nu|}|)|\nu|$$

is not convex in the variable ν.

Proposition 3.2. *The functional $\mathcal{G}_2$ defined in* (3.5) *is not L^1-lower semicontinuous on $BD(\Omega)$.*

Proof. For simplicity, we assume $C = 1$. Let us show that ψ is not convex with respect to the variable ν, i.e., there exist $z, \nu_0, \nu_1, \nu_2 \in \mathbb{R}^n$ such that $\nu_0 = \lambda\nu_1 + (1-\lambda)\nu_2$ for some $0 < \lambda < 1$ and

$$\psi(z, \nu_0) > \lambda\psi(z, \nu_1) + (1-\lambda)\psi(z, \nu_2)\,. \tag{3.7}$$

Indeed, let $z = \rho e_1 = (\rho, 0, \dots, 0)$, with $\rho > 0$. Then, if $\nu = (a_1, a_2, 0, \dots, 0)$, we have that

$$\psi(z, \nu)^2 = \min\{\rho^2(a_1^2 + \tfrac{1}{2}a_2^2), a_1^2 + a_2^2\}.$$

For $1 < \rho < \sqrt{2}$, the set $\psi(z, \nu) \leq 1$ is not convex, and it is possible to find $\nu_0, \nu_1, \nu_2 \in \mathbb{R}^n$ such that $\nu_0 = \lambda\nu_1 + (1-\lambda)\nu_2$ for some $0 < \lambda < 1$, $\psi(z, \nu_1) = \psi(z, \nu_2) = 1$ and $\psi(z, \nu_0) > 1$ (see Figure 2). This concludes the proof of (3.7).

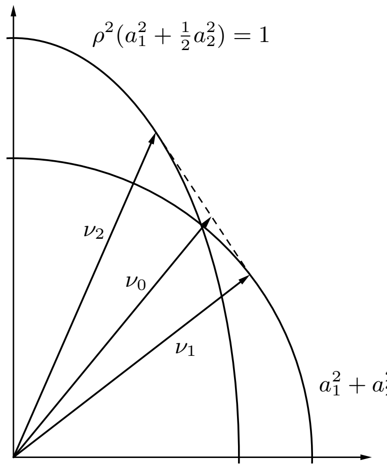

Figure 2. Construction of the vectors ν_0, ν_1, ν_2.

Let us consider the functional Φ defined on the collection of sets of finite perimeter in Ω by

$$\Phi(E) := \mathcal{G}_2(\mathbb{1}_E z) = \int_{\partial^* E} \psi(z, \nu_E)\, \mathrm{d}\mathcal{H}^{n-1},$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the approximate unit normal to E. Since $\psi(z, \cdot)$ is not convex, the functional Φ is not L^1-lower semicontinuous (see [3, Theorem 5.11]), and therefore $\mathcal{G}_2$ is not L^1-lower semicontinuous. $\square$

4. Semicontinuity by slicing

In this section we prove the lower semicontinuity of the functional $\mathcal{F}\colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} G([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.1}$$

under the following assumptions:

(H1) $f\colon [0, +\infty) \to [0, +\infty)$ is a convex nondecreasing function;

(H2) there exists an even subadditive function $g\colon \mathbb{R} \to [0, +\infty)$ such that the function $G\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty)$ can be written as

$$G(z, \nu) = \sup_{(\xi^1, \ldots, \xi^n)} \Big(\sum_{i=1}^n g(z \cdot \xi^i)^2 |\nu \cdot \xi^i|^2 \Big)^{\frac{1}{2}} \quad \text{for every } z \in \mathbb{R}^n, \nu \in \mathbb{S}^{n-1},$$

where the supremum is taken over all orthonormal bases $(\xi^1, \ldots, \xi^n)$ of $\mathbb{R}^n$;

(H3) $0 < C < +\infty$ and

$$\lim_{t \to +\infty} \frac{f(t)}{t} = \lim_{t \to 0^+} \frac{g(t)}{t} = C, \quad \liminf_{t \to +\infty} g(t) > 0.$$

The following theorem is the main result of the paper.

Theorem 4.1. *Under assumptions* (H1)–(H3)*, the functional* $\mathcal{F}$ *defined in* (4.1) *is* L^1*-lower semicontinuous on* $BD(\Omega)$*, i.e.,*

$$\mathcal{F}(u) \le \liminf_{k \to +\infty} \mathcal{F}(u_k)$$

for every sequence $u_k \in BD(\Omega)$ *and* $u \in BD(\Omega)$ *such that* $u_k \to u$ *in* $L^1(\Omega; \mathbb{R}^n)$.

In the proof of Theorem 4.1 it is convenient to consider the functional $\mathcal{F}\colon BD(\Omega) \times \mathscr{A}(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{F}(u, A) := \int_A f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(A) + \int_{J_u \cap A} G([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \tag{4.2}$$

for every $u \in BD(\Omega)$ and $A \in \mathscr{A}(\Omega)$. Clearly we have $\mathcal{F}(u) = \mathcal{F}(u, \Omega)$.

The proof is based on a slicing argument, which allows us to reduce the problem to the one-dimensional setting. We recall here some notation and some preliminary results about slicing. For every $\xi \in \mathbb{S}^{n-1}$ and for every set $B \subset \mathbb{R}^n$, we define

$$\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\} \quad \text{and} \quad B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\} \quad \text{for every } y \in \Pi^\xi .$$

If $w\colon \Omega \to \mathbb{R}$ is a scalar function and $v\colon \Omega \to \mathbb{R}^n$ is a vector function, we define their slices $w^\xi_y\colon \Omega^\xi_y \to \mathbb{R}$ and $\hat{v}^\xi_y\colon \Omega^\xi_y \to \mathbb{R}$ by

$$w^\xi_y(t) := w(y + t\xi) \quad \text{and} \quad \hat{v}^\xi_y := (v \cdot \xi)^\xi_y ,$$

respectively. If u_k is a sequence in $L^1(\Omega;\mathbb{R}^n)$ such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$, using Fubini Theorem we can prove that for every $\xi \in \mathbb{S}^{n-1}$ there exists a subsequence u_{k_j} such that $(\hat{u}_{k_j})^\xi_y \to \hat{u}^\xi_y$ in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

We recall that a function $u \in L^1(\Omega;\mathbb{R}^n)$ belongs to $BD(\Omega)$ if and only if we have

$$\hat{u}^\xi_y \in BV(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Pi^\xi} |Du^\xi_y|(\Omega^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty,$$

for every direction $\xi \in \mathbb{S}^{n-1}$. Moreover, we have that $(\mathcal{E}u\xi \cdot \xi)^\xi_y$ coincides $\mathcal{L}^1$-a.e. in Ω^ξ_y with the density $\nabla\hat{u}^\xi_y$ of the absolutely continuous part of the distributional derivative of $\hat{u}^\xi_y$; as for the Cantor part, we have

$$E^c u(B)\xi \cdot \xi = \int_{\Pi^\xi} D^c \hat{u}^\xi_y(B^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}$$

for every Borel set $B \subset \Omega$; finally, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have that $(J^\xi_u)^\xi_y = J_{\hat{u}^\xi_y}$ and $[u](y+t\xi)\cdot \xi = [\hat{u}^\xi_y](t)$, where $J^\xi_u = \{x \in J_u : [u](x) \cdot \xi \neq 0\}$. For more details about slicing of BD functions, we refer to [2].

The first step of the proof of Theorem 4.1 is a result about the lower semicontinuity of the functional $\mathcal{F}_\xi\colon BD(\Omega)\times\mathscr{A}(\Omega) \to [0,+\infty)$ defined for every $\xi \in \mathbb{S}^{n-1}$ by

$$\mathcal{F}_\xi(u, A) := \int_A f(|\mathcal{E}u\xi \cdot \xi|)\, \mathrm{d}x + C|E^c u\xi \cdot \xi|(A) + \int_{J_u \cap A} g([u] \cdot \xi)|\nu_u \cdot \xi|\, \mathrm{d}\mathcal{H}^{n-1}$$

for every $u \in BD(\Omega)$ and for every open set $A \subset \Omega$. In the previous formula, $|E^c u\xi \cdot \xi|$ is the total variation of the scalar measure $E^c u\xi \cdot \xi$ defined by $(E^c u\xi \cdot \xi)(B) := E^c u(B)\xi \cdot \xi$ for every Borel set $B \subset \Omega$.

Remark 4.2. Let us consider, for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, the one dimensional functional $\Phi_{\xi,y}\colon BV(\Omega^\xi_y)\times\mathscr{A}(\Omega^\xi_y) \to [0,+\infty)$ defined by

$$\Phi_{\xi,y}(v, U) := \int_U f(|\nabla v|)\, \mathrm{d}t + C|D^c v|(U) + \sum_{t \in J_v \cap U} g([v](t)) \tag{4.3}$$

for every $v \in BV(\Omega^\xi_y)$ and for every open set $U \subset \Omega^\xi_y$. Using the Coarea Formula and the slicing properties mentioned above, it is easy to see that

$$\mathcal{F}_\xi(u, A) = \int_{\Pi^\xi} \Phi_{\xi,y}(\hat{u}^\xi_y, A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \tag{4.4}$$

for every $u \in BD(\Omega)$ and for every open set $A \subset \Omega$.

Under assumptions (H1)–(H3), the lower semicontinuity of the functional $\mathcal{F}_\xi$ is easily obtained from the lower semicontinuity of the one dimensional functionals $\Phi_{\xi,y}$ proved in Proposition 2.7 and from formula (4.4), as the following lemma shows.

Lemma 4.3. *Let $\xi \in \mathbb{S}^{n-1}$ and let $u_k, u \in BD(\Omega)$ be such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$. Assume that (H1)–(H3) hold. Then*

$$\mathcal{F}_\xi(u, A) \le \liminf_{k\to+\infty} \mathcal{F}_\xi(u_k, A) \tag{4.5}$$

for every open set $A \subset \Omega$.

Proof. Let A be an open set contained in Ω. Up to a subsequence, we can assume that the liminf in (4.5) is actually a limit and that $(\hat{u}_k)^\xi_y \to \hat{u}^\xi_y$ in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Since by Proposition 2.7 the one dimensional functional $\Phi_{\xi,y}$ defined in (4.3) is L^1-lower semicontinuous, we obtain that

$$\Phi_{\xi,y}(\hat{u}^\xi_y, A^\xi_y) \le \liminf_{k\to+\infty} \Phi_{\xi,y}((\hat{u}_k)^\xi_y, A^\xi_y) \tag{4.6}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Integrating (4.6) with respect to y, by the Fatou Lemma we deduce that

$$\int_{\Pi^\xi} \Phi_{\xi,y}(\hat{u}^\xi_y, A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \le \liminf_{k\to+\infty} \int_{\Pi^\xi} \Phi_{\xi,y}((\hat{u}_k)^\xi_y, A^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y)\,. \tag{4.7}$$

Inequality (4.5) simply follows from the inequality above and from (4.4). $\square$

We prove now a lower semicontinuity result for functionals which are less than or equal to the original functional $\mathcal{F}$, but which have a much simpler structure. For every $a \in [0, C]$, we consider the function

$$g^a(t) := \inf_{s\in\mathbb{R}} [\, g(s) + a|t-s|\,]$$

and we define the function $G^a \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty)$ by

$$G^a(y,\nu) := \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n g^a(y\cdot\xi^i)^2 |\nu\cdot\xi^i|^2 \Big)^{\frac{1}{2}}, \tag{4.8}$$

where the supremum is taken over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$. In the following two lemmas, we shall study the functional

$$a\int_A |\mathcal{E}u|\, \mathrm{d}x + a|E^c u|(A) + \int_{J_u\cap A} G^a([u],\nu_u)\, \mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.9}$$

Note that, if $b \ge 0$ is such that $at - b \le f(t)$ for every $t \in [0,+\infty)$, then

$$\int_A (a|\mathcal{E}u| - b)\, \mathrm{d}x + a|E^c u|(A) + \int_{J_u\cap A} G^a([u],\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \le \mathcal{F}(u,A)$$

for every open set A contained in Ω. We will deduce the lower semicontinuity of $\mathcal{F}$ from the lower semicontinuity of the functional defined in (4.9) by passing to the supremum among all possible $a, b \ge 0$ such that $at - b \le f(t)$.

We start with a technical lemma.

Lemma 4.4. *Let $a \in [0,C]$, let $(\xi^1,\dots,\xi^n)$ be an orthonormal basis of $\mathbb{R}^n$, and let $u \in BD(\Omega)$. Then, for every open set $A \subset \Omega$,*

$$a\Big(\sum_{i=1}^n \Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\, \mathrm{d}x \Big)^2 \Big)^{\frac{1}{2}} \le a\int_A |\mathcal{E}u|\, \mathrm{d}x\,, \tag{4.10}$$

$$a\Big(\sum_{i=1}^n |E^c u\xi^i\cdot\xi^i|(A)^2 \Big)^{\frac{1}{2}} \le a|E^c u|(A)\,, \tag{4.11}$$

$$\Big(\sum_{i=1}^n \Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\, \mathrm{d}\mathcal{H}^{n-1} \Big)^2 \Big)^{\frac{1}{2}} \le \int_{J_u\cap A} G^a([u],\nu_u)\, \mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.12}$$

Proof. Let us prove (4.10). By the Hölder inequality with respect to the measure $|\mathcal{E}u|\mathcal{L}^n$, we get

$$\Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\, \mathrm{d}x \Big)^2 = \Big(\int_A \Big| \frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i \Big| |\mathcal{E}u|\, \mathrm{d}x \Big)^2 \le \int_A \Big| \frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i \Big|^2 |\mathcal{E}u|\, \mathrm{d}x \int_A |\mathcal{E}u|\, \mathrm{d}x\,.$$

Summing with respect to i, from (2.1) it follows that

$$\Big(\sum_{i=1}^n \Big(\int_A |\mathcal{E}u\xi^i\cdot\xi^i|\, \mathrm{d}x \Big)^2 \Big)^{\frac{1}{2}} \le \Big(\int_A \sum_{i=1}^n \Big| \frac{\mathcal{E}u}{|\mathcal{E}u|}\xi^i\cdot\xi^i \Big|^2 |\mathcal{E}u|\, \mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_A |\mathcal{E}u|\, \mathrm{d}x \Big)^{\frac{1}{2}} \le \int_A |\mathcal{E}u|\, \mathrm{d}x\,,$$

which proves (4.10).

To prove (4.11), we use the Hölder inequality with respect to the measure $|E^c u|$ and we obtain

$$|E^c u \xi^i \cdot \xi^i|(A)^2 = \Big(\int_A \Big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\xi^i \cdot \xi^i\Big| \,\mathrm{d}|E^c u|\Big)^2 \le \Big(\int_A \Big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\xi^i \cdot \xi^i\Big|^2 \,\mathrm{d}|E^c u|\Big)|E^c u|(A)\,.$$

Therefore, by (2.1),

$$\Big(\sum_{i=1}^n |E^c u \xi^i \cdot \xi^i|(A)^2\Big)^{\frac12} \le \Big(\int_A \sum_{i=1}^n \Big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\xi^i \cdot \xi^i\Big|^2 \,\mathrm{d}|E^c u|\Big)^{\frac12} |E^c u|(A)^{\frac12} \le |E^c u|(A)\,,$$

since $\big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\big| = 1$ $|E^c u|$-a.e. in Ω.

The strategy to prove (4.12) is the same. By the Hölder inequality with respect to the measure $G^a([u],\nu_u)\mathcal{H}^{n-1}$ we have

$$\begin{aligned}\Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i| \,\mathrm{d}\mathcal{H}^{n-1}\Big)^2 &= \Big(\int_{J_u\cap A} \frac{g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|}{G^a([u],\nu_u)} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2 \\ &\le \int_{J_u\cap A} \frac{g^a([u]\cdot\xi^i)^2|\nu_u\cdot\xi^i|^2}{G^a([u],\nu_u)^2} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \int_{J_u\cap A} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

and hence, by (4.8) we obtain

$$\begin{aligned}&\Big(\sum_{i=1}^n \Big(\int_{J_u\cap A} g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i| \,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac12} \\ &\le \Big(\int_{J_u\cap A} \sum_{i=1}^n \frac{g^a([u]\cdot\xi^i)^2|\nu_u\cdot\xi^i|^2}{G^a([u],\nu_u)^2} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^{\frac12} \Big(\int_{J_u\cap A} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\Big)^{\frac12} \\ &\le \int_{J_u\cap A} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

This concludes the proof. □

In the following lemma, we prove a preliminary result which is strongly connected with the lower semicontinuity of the functional defined in (4.9). The main idea of the proof is based on the following remark: by Proposition 2.1 we have that

$$|\mathcal{E}u| = \sup_j \Big(\sum_{i=1}^n |\mathcal{E}u\xi^i_j \cdot \xi^i_j|^2\Big)^{\frac12},$$

where $\{(\xi^1_j,\dots,\xi^n_j) : j\in\mathbb{N}\}$ is a suitable countable collection of orthonormal bases of $\mathbb{R}^n$. Therefore we can apply a localization argument based on Lemma 2.6 which leads to

$$\int_K |\mathcal{E}u|\,\mathrm{d}x = \sup\Big\{\sum_{j=1}^r \int_{K^j} \Big(\sum_{i=1}^n |\mathcal{E}u\xi^i_j\cdot\xi^i_j|^2\Big)^{\frac12}\mathrm{d}x : (K^j)_{j=1}^r \text{ disjoint compact subsets of } K,\ r\in\mathbb{N}\Big\}.$$

This will allow us to use the semicontinuity result already proved for $\mathcal{F}^a_{\xi^i_j}$ defined in (4.17) below.

Lemma 4.5. *Let $a\in[0,C]$, let $u_k, u\in BD(\Omega)$ be such that $u_k\to u$ in $L^1(\Omega;\mathbb{R}^n)$, and let*

$$\Lambda(A) := \liminf_{k\to+\infty}\Big[a\int_A |\mathcal{E}u_k|\,\mathrm{d}x + a|E^c u_k|(A) + \int_{J_{u_k}\cap A} G^a([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\Big] \tag{4.13}$$

for every open set $A\subset\Omega$. Then

$$a\int_K |\mathcal{E}u|\,\mathrm{d}x \le \Lambda(A)\,, \tag{4.14}$$

$$a|E^c u|(K) \le \Lambda(A)\,, \tag{4.15}$$

$$\int_{J_u\cap K} G^a([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \Lambda(A) \tag{4.16}$$

for every compact set K and for every open set A such that $K\subset A\subset\Omega$.

Proof. Let us fix an orthonormal basis $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$. For every $i=1,\dots,n$, let us consider the functional $\mathcal{F}^a_{\xi^i}\colon BD(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty)$ defined by

$$\mathcal{F}^a_{\xi^i}(u,A):=a\int_A|\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x+a|E^cu\xi^i\cdot\xi^i|(A)+\int_{J_u\cap A}g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}, \tag{4.17}$$

for every $u\in BD(\Omega)$ and for every open set $A\subset\Omega$. Since $\mathcal{F}^a_{\xi^i}$ satisfies the hypotheses of Lemma 4.3 (see Remark 2.5), we have that

$$\mathcal{F}^a_{\xi^i}(u,A)\le\liminf_{k\to+\infty}\mathcal{F}^a_{\xi^i}(u_k,A)\,, \tag{4.18}$$

for every $i=1,\dots,n$ and for every open set $A\subset\Omega$.

In order to prove (4.14), we observe that by (4.18)

$$a\int_K|\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\le\mathcal{F}^a_{\xi^i}(u,A)\le\liminf_{k\to+\infty}\mathcal{F}^a_{\xi^i}(u_k,A)$$

for every compact set $K\subset A$. From this inequality and from the superadditivity of the liminf, it follows that

$$a\Big(\sum_{i=1}^n\Big(\int_K|\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac12}\le\liminf_{k\to+\infty}\Big(\sum_{i=1}^n\mathcal{F}^a_{\xi^i}(u_k,A)^2\Big)^{\frac12}\,. \tag{4.19}$$

By the triangle inequality of the Euclidean norm in $\mathbb{R}^n$ and by Lemma 4.4, we obtain

$$\begin{aligned}\Big(\sum_{i=1}^n\mathcal{F}^a_{\xi^i}(u_k,A)^2\Big)^{\frac12}&\le a\Big(\sum_{i=1}^n\Big(\int_A|\mathcal{E}u_k\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac12}+a\Big(\sum_{i=1}^n|E^cu_k\xi^i\cdot\xi^i|(A)^2\Big)^{\frac12}\\&\quad+\Big(\sum_{i=1}^n\Big(\int_{J_{u_k}\cap A}g^a([u_k]\cdot\xi^i)|\nu_{u_k}\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac12}\\&\le a\int_A|\mathcal{E}u_k|\,\mathrm{d}x+a|E^cu_k|(A)+\int_{J_{u_k}\cap A}G^a([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned} \tag{4.20}$$

Hence, by (4.19) and (4.20) it follows that

$$a\Big(\sum_{i=1}^n\Big(\int_K|\mathcal{E}u\xi^i\cdot\xi^i|\,\mathrm{d}x\Big)^2\Big)^{\frac12}\le\Lambda(A)$$

for every compact set K, for every open set A such that $K\subset A\subset\Omega$, and for every orthonormal basis $(\xi^1,\dots,\xi^n)$.

Let us fix a sequence R_j dense in $\mathrm{O}(n)$ and let $\xi^1_j,\dots,\xi^n_j$ be the column vectors of R_j. Let us define the vector functions $\varphi^j=(\varphi^j_1,\dots,\varphi^j_n)$ with components given by $\varphi^j_i=|\mathcal{E}u\xi^i_j\cdot\xi^i_j|$, $i=1,\dots,n$. By the previous inequality, under the same assumptions on K and A, we have

$$a\Big|\int_K\varphi^j\,\mathrm{d}x\Big|\le\Lambda(A)$$

for every j. Since Λ is superadditive, we obtain

$$\begin{aligned}a\int_K|\varphi^j|\,\mathrm{d}x&=\sup\Big\{\sum_{h=1}^r a\Big|\int_{K^h}\varphi^j\,\mathrm{d}x\Big|:(K^h)_{h=1}^r\text{ disjoint compact subsets of }K,\ r\in\mathbb{N}\Big\}\\&\le\sup\Big\{\sum_{h=1}^r\Lambda(A^h):(A^h)_{h=1}^r\text{ disjoint open subsets of }A,\ r\in\mathbb{N}\Big\}\le\Lambda(A)\end{aligned} \tag{4.21}$$

for every compact set K and for every open set A such that $K\subset A\subset\Omega$. By Lemma 2.6 we deduce that

$$a\int_K\sup_j|\varphi^j|\,\mathrm{d}x\le\Lambda(A)\,.$$

On the other hand, by Proposition 2.1 and by Remark 2.2, we have that $\sup_j|\varphi^j|=|\mathcal{E}u|$. Together with the previous inequality, this concludes the proof of (4.14).

Let us now prove (4.15). Arguing as in the first part of the proof, we obtain that

$$a\Big(\sum_{i=1}^{n}|E^c u\xi^i\cdot\xi^i|(K)^2\Big)^{\frac{1}{2}}\leq\Lambda(A)$$

for every compact set K, for every open set A such that $K\subset A\subset\Omega$, and for every orthonormal basis $(\xi^1,\dots,\xi^n)$. Let $(\xi^1_j,\dots,\xi^n_j)$ be the sequence of orthonormal bases introduced above. We now define a sequence of vector functions $\varphi^j=(\varphi^j_1,\dots,\varphi^j_n)$ with components given by

$$\varphi^j_i=\Big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\xi^i_j\cdot\xi^i_j\Big|.$$

The inequality above gives, under the same assumptions on K and A,

$$a\Big|\int_K\varphi^j\,\mathrm{d}|E^c u|\Big|\leq\Lambda(A)$$

for every j. As in (4.21), we obtain that

$$a\int_K|\varphi^j|\,\mathrm{d}|E^c u|\leq\Lambda(A)\,,$$

hence, by Lemma 2.6, we deduce that

$$a\int_K\sup_j|\varphi^j|\,\mathrm{d}|E^c u|\leq\Lambda(A)$$

for every compact set K and for every open set A such that $K\subset A\subset\Omega$. On the other hand, since $\big|\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\big|=1$ $|E^c u|$-a.e. in Ω, we have $\sup_j|\varphi^j|=1$ $|E^c u|$-a.e. in Ω, by Proposition 2.1 and by Remark 2.2. Together with the previous inequality, this concludes the proof of (4.15).

The proof of (4.16) follows the same steps. Arguing as in the first part of the proof we obtain that

$$\Big(\sum_{i=1}^{n}\Big(\int_{J_u\cap K}g^a([u]\cdot\xi^i)|\nu_u\cdot\xi^i|\,\mathrm{d}\mathcal{H}^{n-1}\Big)^2\Big)^{\frac{1}{2}}\leq\Lambda(A)$$

for every compact set K, for every open set A such that $K\subset A\subset\Omega$, and for every orthonormal basis $(\xi^1,\dots,\xi^n)$. We now continue as in the previous step, replacing the measure $|E^c u|$ by $\mathcal{H}^{n-1}\llcorner J_u$ and defining $\varphi^j_i:=g^a([u]\cdot\xi^i_j)|\nu_u\cdot\xi^i_j|$. Since now $\sup_j|\varphi^j|=G^a([u],\nu_u)$, we finally obtain (4.16). $\square$

Remark 4.6. In order to treat separately the three terms of the functional $\mathcal{F}$, given $u\in BD(\Omega)$, it is useful to consider a partition of Ω into three Borel sets B_1,B_2,B_3 such that

$$\mathcal{L}^n(\Omega\setminus B_1)=0\,,\tag{4.22}$$

$$|E^c u|(\Omega\setminus B_2)=0\,,\tag{4.23}$$

$$\mathcal{H}^{n-1}((J_u\setminus B_3)\cup(B_3\setminus J_u))=0\,.\tag{4.24}$$

Let $\varepsilon>0$ and let K_1,K_2,K_3 be three pairwise disjoint compact sets such that $K_h\subset B_h$ for $h=1,2,3$ and

$$\int_{B_1\setminus K_1}f(|\mathcal{E}u|)\,\mathrm{d}x<\varepsilon\,,\tag{4.25}$$

$$C|E^c u|(B_2\setminus K_2)<\varepsilon\,,\tag{4.26}$$

$$\int_{B_3\setminus K_3}G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}<\varepsilon\,.\tag{4.27}$$

Finally, let A_1,A_2,A_3 be three pairwise disjoint open subsets of Ω such that $K_h\subset A_h$ for $h=1,2,3$.

We now exploit the property that a convex function can be written as the supremum of affine functions, which combined with Lemma 4.5 gives the following result.

Lemma 4.7. *Let $u_k, u \in BD(\Omega)$ be such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$ and let K_1, K_2, K_3 and A_1, A_2, A_3 be as in Remark 4.6. Then*

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_1)\,, \tag{4.28}$$

$$C|E^c u|(K_2) \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_2)\,, \tag{4.29}$$

$$\int_{K_3} G([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_3)\,. \tag{4.30}$$

Proof. Without loss of generality, we can assume $f(0) = 0$. Since f is convex and nonnegative, there exists a sequence of functions $f_j(t) = (a_j t - b_j)^+ = \max\{a_j t - b_j, 0\}$, with $a_j, b_j \ge 0$, such that $f_j \le f$ and $f(t) = \sup_j f_j(t)$, for every $t \in [0,+\infty)$. Hence, by Lemma 2.6, given $\delta > 0$ there exists a finite family of disjoint compact sets $K_1^1, \dots, K_1^r$ contained in K_1 such that

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x \le \sum_{j=1}^{r} \int_{K_1^j} (a_j|\mathcal{E}u| - b_j)^+\,\mathrm{d}x + \delta\,. \tag{4.31}$$

For every $j = 1, \dots, r$, let us fix a compact set $\widetilde{K}_1^j \subset K_1^j \cap \{a_j|\mathcal{E}u| - b_j \ge 0\}$ such that

$$\int_{K_1^j} (a_j|\mathcal{E}u| - b_j)^+\,\mathrm{d}x \le \int_{\widetilde{K}_1^j} (a_j|\mathcal{E}u| - b_j)\,\mathrm{d}x + \frac{\delta}{r}\,.$$

Let us consider a family of pairwise disjoint open sets $A_1^1, \dots, A_1^r$ such that $\widetilde{K}_1^j \subset A_1^j \subset A_1$ and $b_j \mathcal{L}^n(A_1^j \setminus \widetilde{K}_1^j) \le \delta/r$.

Note that by (H3) we have that $a_j \le C$, since $a_j t - b_j \le f(t)$. Therefore, we can apply (4.14) to $\widetilde{K}_1^j$ and A_1^j to obtain

$$a_j \int_{\widetilde{K}_1^j} |\mathcal{E}u|\,\mathrm{d}x \le \liminf_{k\to+\infty} \Big[a_j \int_{A_1^j} |\mathcal{E}u_k|\,\mathrm{d}x + a_j|E^c u_k|(A_1^j) + \int_{J_{u_k}\cap A_1^j} G^{a_j}([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\Big]$$

for every $j = 1, \dots, r$, and therefore

$$\int_{K_1^j} (a_j|\mathcal{E}u| - b_j)^+\,\mathrm{d}x \le \int_{\widetilde{K}_1^j} (a_j|\mathcal{E}u| - b_j)\,\mathrm{d}x + \frac{\delta}{r} \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_1^j) + \frac{2\delta}{r}\,,$$

where we have used the inequality $b_j \mathcal{L}^n(A_1^j \setminus \widetilde{K}_1^j) \le \delta/r$. This implies, by (4.31) and by the superadditivity of the liminf, that

$$\int_{K_1} f(|\mathcal{E}u|)\,\mathrm{d}x \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_1) + 3\delta\,.$$

Letting $\delta \to 0$, we conclude the proof of (4.28).

To prove (4.29) and (4.30), first of all we note that $\sup_j a_j = C$. Indeed, given $\delta > 0$, by (H3) there exists $T > 0$ such that $C < f(T)/T + \delta$. Since $f(T) = \sup_j f_j(T)$, there exists a j such that $f(T) < f_j(T) + \delta T \le a_j T + \delta T$. Therefore $C < a_j + 2\delta$. Letting $\delta \to 0$, we conclude that $\sup_j a_j = C$.

Since $\mathcal{L}^n(K_2) = 0$, for every $\delta > 0$ and for every j there exists an open set A_2^j such that $K_2 \subset A_2^j \subset A_2$ and $b_j \mathcal{L}^n(A_2^j) \le \delta$. Applying (4.15) we have that

$$\begin{aligned} a_j|E^c u|(K_2) &\le \liminf_{k\to+\infty} \Big[a_j \int_{A_2^j} |\mathcal{E}u_k|\,\mathrm{d}x + a_j|E^c u_k|(A_2^j) + \int_{J_{u_k}\cap A_2^j} G^{a_j}([u_k],\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}\Big] \\ &\le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_2^j) + b_j \mathcal{L}^n(A_2^j) \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_2) + \delta\,. \end{aligned}$$

Recalling that $\sup_j a_j = C$, we obtain

$$C|E^c u|(K_2) \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A_2) + \delta$$

and by the arbitrariness of δ, we deduce (4.29).

Finally, let us prove (4.30). Arguing as in the previous step, for every $\delta > 0$ and for every j we have that

$$\int_{K_3} G^{a_j}([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to+\infty} \mathcal{F}(u_k, A_3) + \delta\,.$$

Letting $\delta \to 0$, we obtain

$$\int_{K_3} G^{a_j}([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to+\infty} \mathcal{F}(u_k, A_3)\,. \tag{4.32}$$

Since $\sup_j a_j = C$, either there exists j_0 such that $a_{j_0} = C$ or there exists a strictly increasing subsequence a_{j_h} converging to C. In the first case, we have $g^{a_{j_0}} = g$, hence $G^{a_{j_0}} = G$, and (4.32) with j_0 coincides with (4.30). In the other case, $g^{a_{j_h}}$ is an increasing sequence and converges to g (see Remark 2.5). Consequently, $G^{a_{j_h}}$ is an increasing sequence and converges to G. Therefore, we can pass to the limit in (4.32) along the sequence j_h using the monotone convergence theorem, and we obtain (4.30). □

Theorem 4.1 is now a simple consequence of Lemma 4.7, thanks to the choice of K_1, K_2, K_3 made in Remark 4.6.

Proof of Theorem 4.1. Let us fix $u_k, u \in BD(\Omega)$ such that $u_k \to u$ in $L^1(\Omega;\mathbb{R}^n)$. Let us consider the three disjoint Borel sets B_1, B_2, B_3, the three disjoint compact sets K_1, K_2, K_3, and the three disjoint open sets A_1, A_2, A_3 as in Remark 4.6. By Lemma 4.7 and by the superadditivity of the liminf, we have

$$\int_{K_1} f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(K_2) + \int_{K_3} G([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to+\infty} \mathcal{F}(u_k, \Omega)\,.$$

From this inequality and from (4.22)–(4.27) we obtain

$$\begin{aligned} \mathcal{F}(u, \Omega) &= \int_{B_1} f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(B_2) + \int_{B_3} G([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \int_{K_1} f(|\mathcal{E}u|)\, \mathrm{d}x + \varepsilon + C|E^c u|(K_2) + \varepsilon + \int_{K_3} G([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \varepsilon \\ &\leq \liminf_{k\to+\infty} \mathcal{F}(u_k, \Omega) + 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we conclude the proof of Theorem 4.1. □

5. A relaxation result for functionals defined on $BD(\Omega)$

The aim of this section is to obtain an integral representation for the relaxation of the functional $\mathcal{F}\colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{F}(u) := \int_\Omega f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \psi([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.1}$$

for every $u \in BD(\Omega)$. We assume that:

(H1′) $f\colon [0, +\infty) \to [0, +\infty)$ is a convex nondecreasing function;

(H2′) $\psi\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty)$ is a Borel function and there exist a constant $c_1 > 0$ and an even subadditive function $g\colon \mathbb{R} \to [0, +\infty)$ such that for every orthonormal basis $(\xi^1, \dots, \xi^n)$

$$\Big(\sum_{i=1}^n g(z\cdot\xi^i)^2 |\nu\cdot\xi^i|^2\Big)^{\frac12} \leq \psi(z, \nu) \leq c_1 \min\{|z|, 1\}\,,$$

for every $z \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$;

(H3′) $0 < C < +\infty$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to0^+} \frac{g(t)}{t} = C\,, \quad \liminf_{t\to+\infty} g(t) > 0\,.$$

Remark 5.1. Note that, by assumption (H3′), there exist two constants $\alpha, \beta > 0$ such that

$$\alpha t - \beta \leq f(t) \leq \beta(t+1) \tag{5.2}$$

for every $t \geq 0$. Moreover, we claim that there exists a constant $c_2 > 0$ such that

$$c_2 \min\{|z|, 1\} \leq \psi(z, \nu)\,.$$

Indeed, let $c > 0$ be such that $g(t) \geq c \min\{|t|, 1\}$ (see Remark 2.4). Let ξ^1 be the unit vector lying on the plane spanned by z and ν with the direction of the bisector of the angle in $[0, \frac{\pi}{2}]$ between the directions $\pm\nu$ and $\frac{z}{|z|}$. Note that $|z \cdot \xi^1| \geq \frac{\sqrt{2}}{2}|z|$ and $|\nu \cdot \xi^1| \geq \frac{\sqrt{2}}{2}$. Let $\xi^2, \ldots, \xi^n \in \mathbb{S}^{n-1}$ be such that $(\xi^1, \ldots, \xi^n)$ is an orthonormal basis of $\mathbb{R}^n$. Then, by (H2′), we have

$$\begin{aligned}\psi(z, \nu) &\geq \Big(\sum_{i=1}^{n} g(z \cdot \xi^i)^2 |\nu \cdot \xi^i|^2\Big)^{\frac{1}{2}} \geq c \min\{|z \cdot \xi^1|, 1\} |\nu \cdot \xi^1| \\ &\geq c \min\{\tfrac{\sqrt{2}}{2}|z|, 1\} \tfrac{\sqrt{2}}{2} \geq c_2 \min\{|z|, 1\}\,,\end{aligned}$$

for a suitable constant $c_2 > 0$.

The L^1-lower semicontinuous envelope of $\mathcal{F}$ is the functional $\mathrm{sc}^-\mathcal{F}\colon BD(\Omega) \to [0, +\infty)$ defined by

$$\mathrm{sc}^-\mathcal{F}(u) := \inf\{\liminf_{k\to+\infty} \mathcal{F}(u_k) : \ u_k \in BD(\Omega),\ u_k \to u \text{ in } L^1(\Omega; \mathbb{R}^n)\}\,.$$

We are now in a position to state our relaxation result.

Theorem 5.2. *Under assumptions* (H1′)–(H3′)*, there exists a Borel function* $\overline{\psi}\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty)$ *such that*

$$\mathrm{sc}^-\mathcal{F}(u) = \int_\Omega f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(\Omega) + \int_{J_u} \overline{\psi}([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.3}$$

for every $u \in BD(\Omega)$.

To prove the theorem, we localize the problem and consider the functional $\mathcal{F}\colon L^1(\Omega; \mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}(u, A) := \int_A f(|\mathcal{E}u|)\, \mathrm{d}x + C|E^c u|(A) + \int_{J_u \cap A} \psi([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1},$$

if $u|_A \in BD(A)$, and $\mathcal{F}(u, A) := +\infty$ otherwise in $L^1(\Omega; \mathbb{R}^n)$. Its L^1-lower semicontinuous envelope is the functional $\mathrm{sc}^-\mathcal{F}\colon L^1(\Omega; \mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathrm{sc}^-\mathcal{F}(u, A) := \inf\{\liminf_{k\to+\infty} \mathcal{F}(u_k, A) : \ u_k \to u \text{ in } L^1(\Omega; \mathbb{R}^n)\}\,,$$

for every $u \in L^1(\Omega; \mathbb{R}^n)$ and $A \in \mathscr{A}(\Omega)$. Let us notice that $\mathrm{sc}^-\mathcal{F}(u, \Omega) = \mathrm{sc}^-\mathcal{F}(u)$ for every $u \in BD(\Omega)$.

One of the main tools used to prove Theorem 5.2 is the following integral representation result for the surface part of functionals defined on the space $BD(\Omega)$. We recall that a functional $\mathcal{G}\colon BD(\Omega) \times \mathscr{A}(\Omega) \to [0, +\infty)$ is said to be local if for every $A \in \mathscr{A}(\Omega)$ and for every $u, v \in BD(\Omega)$ such that $u = v$ a.e. in A, we have $\mathcal{G}(u, A) = \mathcal{G}(v, A)$. In the following statement, Q^ν_ε denotes a cube of side ε centered at the origin with two faces orthogonal to ν and u^ν_z is the pure jump function defined by

$$u^\nu_z(x) := \begin{cases} z & \text{if } x \cdot \nu > 0\,, \\ 0 & \text{if } x \cdot \nu < 0\,. \end{cases}$$

Theorem 5.3. *Assume that* $\mathcal{G}\colon BD(\Omega) \times \mathscr{A}(\Omega) \to [0, +\infty)$ *satisfies the following properties:*

- *(a)* $\mathcal{G}$ *is local;*
- *(b)* $\mathcal{G}(\cdot, A)$ *is* L^1*-lower semicontinuous, for every* $A \in \mathscr{A}(\Omega)$*;*
- *(c) there exists a constant* $c > 0$ *such that* $\frac{1}{c}|Eu|(A) \leq \mathcal{G}(u, A) \leq c(|Eu|(A) + \mathcal{L}^n(A))$*, for every* $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*;*
- *(d) for every* $u \in BD(\Omega)$*,* $\mathcal{G}(u, \cdot)$ *is the restriction to* $\mathscr{A}(\Omega)$ *of a Radon measure;*
- *(e)* $\mathcal{G}(u(\cdot - x_0) + b, x_0 + A) = \mathcal{G}(u, A)$ *for all* $b \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$ *such that* $x_0 + A \subset \Omega$*.*

Then, for every $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*, we have that*

$$\mathcal{G}(u, J_u \cap A) = \int_{J_u \cap A} \overline{\psi}([u], \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}\,, \tag{5.4}$$

where

$$\overline{\psi}(z,\nu) = \limsup_{\varepsilon \to 0} \Big[\frac{1}{\varepsilon^{n-1}} \inf\{\mathcal{G}(v, Q^\nu_\varepsilon) : v \in BD(Q^\nu_\varepsilon),\ v(x) = u^\nu_z(x) \ \text{on}\ \partial Q^\nu_\varepsilon\}\Big]\,. \tag{5.5}$$

Proof. The proof can be obtained by adapting the arguments of the proof of [4, Proposition 5.1], which is originally stated for the relaxation of a functional defined on $W^{1,1}(\Omega;\mathbb{R}^n)$. A careful inspection of the proof shows that the integral representation result still holds for a functional satisfying properties (a)–(e). In particular [4, Lemma 3.10], only uses assumptions (a), (c), and (d). Under assumptions (a)–(d) it is easy to check that [4, Lemma 3.11] holds for $\mathcal{G}$. Using these results, the proof of [4, Proposition 5.1] for $\mathcal{G}$ can be easily extended from the case $u \in SBD(\Omega)$ to the case $u \in BD(\Omega)$. As for (5.5), it is a consequence of the formula for the integrand given in [4, Proposition 5.1] and of the invariance properties due to (e). □

Another tool used in the proof of Theorem 4.1 regards the jump set of the sum of BD functions.

Proposition 5.4. *For every* $u, v \in BD(\Omega)$ *we have that* $\mathcal{H}^{n-1}(J_{u+v} \setminus (J_u \cup J_v)) = 0$*.*

Proof. The difficulty in the proof of this result is due to the fact that, for $u \in BD(\Omega)$, the Lebesgue discontinuity set S_u and the jump set J_u may be different (see [2, page 206] for the definitions). While it is easy to see that $J_u \subset S_u$, it is not known whether $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. However, it is proved in [2, Remark 6.3] that $S_u \setminus J_u$ is purely $(\mathcal{H}^{n-1}, n-1)$-unrectifiable, i.e., $\mathcal{H}^{n-1}((S_u \setminus J_u) \cap M) = 0$ for every $(\mathcal{H}^{n-1}, n-1)$-rectifiable set M.

Since $J_{u+v} \subset S_{u+v} \subset S_u \cup S_v$, we have that $J_{u+v} \setminus (J_u \cup J_v) \subset (S_u \setminus J_u) \cup (S_v \setminus J_v)$. Since $J_{u+v} \setminus (J_u \cup J_v)$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, by the pure $(\mathcal{H}^{n-1}, n-1)$-unrectifiability of $S_u \setminus J_u$ and $S_v \setminus J_v$ we conclude that $\mathcal{H}^{n-1}(J_{u+v} \setminus (J_u \cup J_v)) = 0$. □

Corollary 5.5. *Let* $u, v \in BD(\Omega)$*, let* $\varphi \in \mathcal{C}^\infty(\Omega)$ *with* $0 \le \varphi \le 1$*, and let* $w := \varphi u + (1-\varphi) v$*. Then*

$$\int_{J_w} \min\{|[w]|, 1\}\, \mathrm{d}\mathcal{H}^{n-1} \le \int_{J_u} \min\{|[u]|, 1\}\, \mathrm{d}\mathcal{H}^{n-1} + \int_{J_v} \min\{|[v]|, 1\}\, \mathrm{d}\mathcal{H}^{n-1}.$$

Proof. Let us define $[u]^* \colon J_w \to \mathbb{R}^n$ by $[u]^* = [u]$ on $J_w \cap J_u$ and $[u]^* = 0$ on $J_w \setminus J_u$. Similarly we define $[v]^* \colon J_w \to \mathbb{R}^n$ by $[v]^* = [v]$ on $J_w \cap J_v$ and $[v]^* = 0$ on $J_w \setminus J_v$. Using the sets S_u and S_v considered in the proof of Proposition 5.4 and taking into account the pure $(\mathcal{H}^{n-1}, n-1)$-unrectifiability of $S_u \setminus J_u$ and $S_v \setminus J_v$, we obtain that $\mathcal{H}^{n-1}(J_w \cap (S_u \setminus J_u)) = \mathcal{H}^{n-1}(J_w \cap (S_v \setminus J_v)) = 0$. This implies that $[w] = \varphi[u]^* + (1-\varphi)[v]^*$ $\mathcal{H}^{n-1}$-a.e. on J_w. The conclusion follows easily. □

We will show that the lower semicontinuous envelope of $\mathcal{F}$ satisfies the assumptions of Theorem 5.3. We start by proving a weak form of the subadditivity condition for $\mathrm{sc}^-\mathcal{F}$.

Lemma 5.6. *Let* $A, B \in \mathscr{A}(\Omega)$ *and* $A' \in \mathscr{A}(\Omega)$ *such that* $A' \Subset A$*. Then*

$$\mathrm{sc}^-\mathcal{F}(u, A' \cup B) \le \mathrm{sc}^-\mathcal{F}(u, A) + \mathrm{sc}^-\mathcal{F}(u, B)\,, \tag{5.6}$$

for every $u \in L^1(\Omega;\mathbb{R}^n)$*.*

Proof. Let us fix $u \in L^1(\Omega;\mathbb{R}^n)$ and let us consider two sequences u^A_k and u^B_k in $L^1(\Omega;\mathbb{R}^n)$ converging in $L^1(\Omega;\mathbb{R}^n)$ to the function u such that

$$\mathrm{sc}^-\mathcal{F}(u, A) = \lim_{k \to +\infty} \mathcal{F}(u^A_k, A) \quad \text{and} \quad \mathrm{sc}^-\mathcal{F}(u, B) = \lim_{k \to +\infty} \mathcal{F}(u^B_k, B)\,. \tag{5.7}$$

It suffices to prove (5.6) when the right hand side is finite. Hence we can assume that $\mathcal{F}(u^A_k, A)$ and $\mathcal{F}(u^B_k, B)$ are equibounded sequences. In particular, we have that $u^A_k \in BD(A)$ and $u^B_k \in BD(B)$.

We now use the De Giorgi slicing and averaging argument to construct a suitable sequence $u_k \in BD(A \cup B)$ converging to u in $L^1(\Omega;\mathbb{R}^n)$. Let $d := \mathrm{dist}(A', \partial A) > 0$ and let $h \in \mathbb{N}$. Let $A_0 := A'$ and $A_{h+1} := A$. We consider a chain of open sets $A_1, \dots, A_h$ such that $A_i \Subset A_{i+1}$ and $\mathrm{dist}(A_i, \partial A_{i+1}) \ge d/(h+1)$ for every $0 \le i \le h$. Let $\varphi_i \in \mathcal{C}^1_c(\Omega)$ be such that $0 \le \varphi_i \le 1$,

$\mathrm{supp}(\varphi_i) \subset A_{i+1}$, and $\varphi_i = 1$ in a neighborhood of $\overline{A}_i$. We assume in addition that $||\nabla\varphi_i||_{L^\infty(\Omega)} \leq 2(h+1)/d$. We set

$$u_k^i := \varphi_i u_k^A + (1-\varphi_i)u_k^B \in BD(A \cup B)\,, \tag{5.8}$$

for $i = 0, \ldots, h$. By the locality of $\mathcal{F}$, we obtain

$$\begin{aligned} \mathcal{F}(u_k^i, A' \cup B) &\leq \mathcal{F}(u_k^i, A_i) + \mathcal{F}(u_k^i, B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^i, B \setminus A_{i+1}) \\ &= \mathcal{F}(u_k^A, A_i) + \mathcal{F}(u_k^i, B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^B, B \setminus A_{i+1}) \\ &\leq \mathcal{F}(u_k^A, A) + \mathcal{F}(u_k^i, B \cap (A_{i+1} \setminus A_i)) + \mathcal{F}(u_k^B, B)\,, \end{aligned} \tag{5.9}$$

since $u_k^i = u_k^A$ on A_i and $u_k^i = u_k^B$ in a neighborhood of $\mathbb{R}^n \setminus A_{i+1}$. Let $S_i := B \cap (A_{i+1} \setminus A_i)$ and let us estimate $\mathcal{F}(u_k^i, S_i)$. From (5.8) we deduce that

$$Eu_k^i = \varphi_i Eu_k^A + (1-\varphi_i)Eu_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)$$

and therefore

$$\begin{aligned} \mathcal{F}(u_k^i, S_i) = &\int_{S_i} f(|\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|)\,\mathrm{d}x + \\ &+ C|\varphi_i E^c u_k^A + (1-\varphi_i)E^c u_k^B|(S_i) + \int_{J_{u_k^i} \cap S_i} \psi([u_k^i], \nu_{u_k^i})\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{5.10}$$

By (5.2) we have that

$$\begin{aligned} &\int_{S_i} f(|\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|)\,\mathrm{d}x \\ &\leq \beta \int_{S_i} |\varphi_i \mathcal{E}u_k^A + (1-\varphi_i)\mathcal{E}u_k^B + \nabla\varphi_i \odot (u_k^A - u_k^B)|\,\mathrm{d}x + \beta\mathcal{L}^n(S_i) \\ &\leq \beta\Big[\int_{S_i} |\mathcal{E}u_k^A|\,\mathrm{d}x + \int_{S_i} |\mathcal{E}u_k^B|\,\mathrm{d}x + \int_{S_i} |\nabla\varphi_i \odot (u_k^A - u_k^B)|\,\mathrm{d}x + \mathcal{L}^n(S_i)\Big] \\ &\leq c\Big[\int_{S_i} f(|\mathcal{E}u_k^A|)\,\mathrm{d}x + \int_{S_i} f(|\mathcal{E}u_k^B|)\,\mathrm{d}x + (h+1)\int_{S_i} |u_k^A - u_k^B|\,\mathrm{d}x + \mathcal{L}^n(S_i)\Big]\,, \end{aligned} \tag{5.11}$$

where $c > 0$ is a suitable constant. Moreover

$$C|\varphi_i E^c u_k^A + (1-\varphi_i)E^c u_k^B|(S_i) \leq C|E^c u_k^A|(S_i) + C|E^c u_k^B|(S_i)\,. \tag{5.12}$$

Finally, using the bounds $c_2 \min\{|z|, 1\} \leq \psi(z, \nu) \leq c_1 \min\{|z|, 1\}$, from (5.8) and Corollary 5.5, we deduce that

$$\begin{aligned} \int_{J_{u_k^i} \cap S_i} \psi([u_k^i], \nu_{u_k^i})\,\mathrm{d}\mathcal{H}^{n-1} &\leq c_1 \int_{J_{u_k^i} \cap S_i} \min\{|[u_k^i]|, 1\}\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq c_1\Big[\int_{J_{u_k^A} \cap S_i} \min\{|[u_k^A]|, 1\}\,\mathrm{d}\mathcal{H}^{n-1} + \int_{J_{u_k^B} \cap S_i} \min\{|[u_k^B]|, 1\}\,\mathrm{d}\mathcal{H}^{n-1}\Big] \\ &\leq c\Big[\int_{J_{u_k^A} \cap S_i} \psi([u_k^A], \nu_{u_k^A})\,\mathrm{d}\mathcal{H}^{n-1} + \int_{J_{u_k^B} \cap S_i} \psi([u_k^B], \nu_{u_k^B})\,\mathrm{d}\mathcal{H}^{n-1}\Big]\,, \end{aligned} \tag{5.13}$$

where $c > 0$ is a suitable constant. Summing (5.11)–(5.13), by (5.9) and (5.10) we get

$$\begin{aligned} \mathcal{F}(u_k^i, A' \cup B) \leq\; &\mathcal{F}(u_k^A, A) + \mathcal{F}(u_k^B, B) + c[\mathcal{F}(u_k^A, S_i) + \mathcal{F}(u_k^B, S_i) + \mathcal{L}^n(S_i)] \\ &+ c(h+1)\int_{S_i} |u_k^A - u_k^B|\,\mathrm{d}x\,. \end{aligned}$$

Summing the inequality above for $i = 0, \dots, h$ and taking the average with respect to i, we obtain that there exists an index i_k such that

$$\begin{aligned}
&\mathcal{F}(u_k^{i_k}, A' \cup B) \leq \mathcal{F}(u_k^A, A) + \mathcal{F}(u_k^B, B) \\
&\quad + \frac{c}{h+1}[\mathcal{F}(u_k^A, B \cap (A \setminus A')) + \mathcal{F}(u_k^B, B \cap (A \setminus A')) + \mathcal{L}^n(B \cap (A \setminus A'))] + c \int_{B \cap (A \setminus A')} |u_k^A - u_k^B| \,\mathrm{d}x \\
&\leq \mathcal{F}(u_k^A, A) + \mathcal{F}(u_k^B, B) + \frac{c}{h+1}[\mathcal{F}(u_k^A, A) + \mathcal{F}(u_k^B, B) + \mathcal{L}^n(\Omega)] + c \int_{B \cap (A \setminus A')} |u_k^A - u_k^B| \,\mathrm{d}x \,.
\end{aligned}$$

Let us define $u_k := u_k^{i_k}$. Letting $k \to +\infty$ and $h \to +\infty$ in the inequality above, we conclude that

$$\mathrm{sc}^-\mathcal{F}(u, A' \cup B) \leq \liminf_{k \to +\infty} \mathcal{F}(u_k, A' \cup B) \leq \lim_{k \to +\infty} \mathcal{F}(u_k^A, A) + \lim_{k \to +\infty} \mathcal{F}(u_k^B, B) \,.$$

By (5.7), this gives (5.6). □

We are now able to prove that the functional $\mathrm{sc}^-\mathcal{F}$ satisfies all the hypotheses of Theorem 5.3.

Lemma 5.7. *The functional* $\mathrm{sc}^-\mathcal{F}$ *satisfies the following properties:*

- $\mathrm{sc}^-\mathcal{F}$ *is local;*
- $\mathrm{sc}^-\mathcal{F}(\cdot, A)$ *is* L^1*-lower semicontinuous, for every* $A \in \mathscr{A}(\Omega)$*;*
- $\mathrm{sc}^-\mathcal{F}(u, A) \leq C|Eu|(A) + f(0)\mathcal{L}^n(A)$*, for every* $u \in BD(\Omega)$ *and for every* $A \in \mathscr{A}(\Omega)$*;*
- *for every* $u \in BD(\Omega)$*,* $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ *is the restriction to* $\mathscr{A}(\Omega)$ *of a Radon measure;*
- $\mathrm{sc}^-\mathcal{F}(u(\cdot - x_0) + b, x_0 + A) = \mathrm{sc}^-\mathcal{F}(u, A)$ *for all* $b \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$ *such that* $x_0 + A \subset \Omega$*.*

Proof. The proofs of the lower semicontinuity, of the upper bound, and of the translation invariance are immediate. The functional $\mathrm{sc}^-\mathcal{F}$ is local by [15, Proposition 16.15].

In order to prove that $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ is a measure, it is convenient to introduce the inner regular envelope of $\mathrm{sc}^-\mathcal{F}$, i.e., the functional $\overline{\mathcal{F}} \colon L^1(\Omega; \mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty)$ defined by

$$\overline{\mathcal{F}}(u, A) := \sup\{(\mathrm{sc}^-\mathcal{F})(u, U) : U \in \mathscr{A}(\Omega),\ U \Subset A\} \,,$$

for every $u \in L^1(\Omega; \mathbb{R}^n)$ and $A \in \mathscr{A}(\Omega)$. Note that $\overline{\mathcal{F}}(u, A) \leq \mathrm{sc}^-\mathcal{F}(u, A)$. Let us fix $u \in BD(\Omega)$. We claim that

$$\overline{\mathcal{F}}(u, A) = \mathrm{sc}^-\mathcal{F}(u, A) \,, \tag{5.14}$$

for every $A \in \mathscr{A}(\Omega)$. Indeed, let $\varepsilon > 0$. There exists a compact set $K \subset A$ such that $\mathcal{F}(u, A \setminus K) < \varepsilon$. Let us fix $A', A'' \in \mathscr{A}(\Omega)$ such that $K \subset A' \Subset A'' \Subset A$. Then, by Lemma 5.6, we have that

$$\begin{aligned}
\mathrm{sc}^-\mathcal{F}(u, A) &= \mathrm{sc}^-\mathcal{F}(u, A' \cup (A \setminus K)) \leq \mathrm{sc}^-\mathcal{F}(u, A'') + \mathrm{sc}^-\mathcal{F}(u, A \setminus K) \\
&\leq \overline{\mathcal{F}}(u, A) + \mathcal{F}(u, A \setminus K) \leq \overline{\mathcal{F}}(u, A) + \varepsilon \,.
\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude the proof of (5.14). This implies that $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ is inner regular on $\mathscr{A}(\Omega)$, for every $u \in BD(\Omega)$. By [15, Proposition 16.12], $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ is superadditive. The subadditivity of $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ follows from Lemma 5.6 and from the inner regularity. Hence, we can apply [15, Theorem 14.23] to extend $\mathrm{sc}^-\mathcal{F}(u, \cdot)$ to a Borel measure. Actually, it is a bounded measure thanks to the upper bound $\mathrm{sc}^-\mathcal{F}(u, A) \leq C|Eu|(A) + f(0)\mathcal{L}^n(A)$. □

In order to prove Theorem 5.2, it is useful to bound from below the functional $\mathcal{F}$ with the functional $\mathcal{F}_G \colon L^1(\Omega; \mathbb{R}^n) \times \mathscr{A}(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}_G(u, A) := \int_A f(|\mathcal{E}u|) \,\mathrm{d}x + C|E^c u|(A) + \int_{J_u \cap A} G([u], \nu_u) \,\mathrm{d}\mathcal{H}^{n-1} \tag{5.15}$$

if $u \in BD(A)$, and $\mathcal{F}_G(u, A) := +\infty$ otherwise, where G is the function introduced in (1.7). Note that, by Theorem 4.1, for every $A \in \mathscr{A}(\Omega)$ the functional $\mathcal{F}_G(\cdot, A)$ is L^1-lower semicontinuous in $BD(A)$.

Proof of Theorem 5.2. Let us fix $u \in BD(\Omega)$. By (H2′), we have that $G \leq \psi$ and hence

$$\mathcal{F}_G(\cdot, A) \leq \mathcal{F}(\cdot, A) \,,$$

for every open set $A \subset \Omega$. Therefore, by the lower semicontinuity of $\mathcal{F}_G(\cdot, A)$,

$$\mathcal{F}_G(u, A) \leq \mathrm{sc}^-\mathcal{F}(u, A) \leq \mathcal{F}(u, A) \,,$$

for every open set $A \subset \Omega$. By Lemma 5.7, $\mathrm{sc}^{-}\mathcal{F}(u,\cdot)$ is the restriction to $\mathscr{A}(\Omega)$ of a Radon measure defined on $\mathscr{B}(\Omega)$, still denoted by $\mathrm{sc}^{-}\mathcal{F}(u,\cdot)$. Hence

$$\mathcal{F}_G(u,B) \le \mathrm{sc}^{-}\mathcal{F}(u,B) \le \mathcal{F}(u,B)\,,$$

for every Borel set $B \subset \Omega$. Let us consider the sets $B_1, B_2, B_3 \in \mathscr{B}(\Omega)$ as in Remark 4.6. We recall that B_1, B_2, B_3 are pairwise disjoint and $B_1 \cup B_2 \cup B_3 = \Omega$. Moreover, $\mathcal{L}^n$ is concentrated on B_1, $E^c u$ is concentrated on B_2, and $\mathcal{H}^{n-1}((J_u \setminus B_3) \cup (B_3 \setminus J_u)) = 0$. Then

$$\int_{B_1} f(|\mathcal{E}u|)\,\mathrm{d}x = \mathcal{F}_G(u,B_1) \le \mathrm{sc}^{-}\mathcal{F}(u,B_1) \le \mathcal{F}(u,B_1) = \int_{B_1} f(|\mathcal{E}u|)\,\mathrm{d}x\,,$$

$$C|E^c u|(B_2) = \mathcal{F}_G(u,B_2) \le \mathrm{sc}^{-}\mathcal{F}(u,B_2) \le \mathcal{F}(u,B_2) = C|E^c u|(B_2)$$

and therefore

$$\mathrm{sc}^{-}\mathcal{F}(u,B_1) = \int_{\Omega} f(|\mathcal{E}u|)\,\mathrm{d}x\,, \tag{5.16}$$

$$\mathrm{sc}^{-}\mathcal{F}(u,B_2) = C|E^c u|(\Omega)\,. \tag{5.17}$$

On the other hand, for every $\delta > 0$ we can apply Theorem 5.3 to the functional $\mathcal{G}_\delta(u;A) := \mathrm{sc}^{-}\mathcal{F}(u;A) + \delta|Eu|(A)$. Thus we have

$$\mathrm{sc}^{-}\mathcal{F}(u,B_3) + \delta|Eu|(B_3) = \mathrm{sc}^{-}\mathcal{F}(u,J_u) + \delta|Eu|(J_u) = \int_{J_u} \overline{\psi}_\delta([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where

$$\overline{\psi}_\delta(z,\nu) = \limsup_{\varepsilon\to 0}\Big[\frac{1}{\varepsilon^{n-1}}\inf\{\mathcal{G}_\delta(v,Q^\nu_\varepsilon) : v \in BD(Q^\nu_\varepsilon),\ v(x) = u^\nu_z(x) \text{ on } \partial Q^\nu_\varepsilon\}\Big]\,.$$

Defining $\overline{\psi}(z,\nu) := \lim_{\delta\searrow 0}\overline{\psi}_\delta(z,\nu)$, we infer

$$\mathrm{sc}^{-}\mathcal{F}(u;B_3) = \int_{J_u} \overline{\psi}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.18}$$

Summing (5.16), (5.17), and (5.18) we obtain (5.3). □

6. An example of surface density

In this section we provide the explicit expression of the surface density $G(z,\nu)$ defined in (1.7) when $g(t) = \min\{|t|,1\}$, i.e.,

$$G(z,\nu) := \sup_{(\xi^1,\dots,\xi^n)} \Big(\sum_{i=1}^n \min\{|z\cdot\xi^i|,1\}^2|\nu\cdot\xi^i|^2\Big)^{\frac12}, \tag{6.1}$$

where the supremum is taken over all orthonormal bases $(\xi^1,\dots,\xi^n)$ of $\mathbb{R}^n$.

First of all we prove that the function G in (6.1) is invariant under rotations.

Lemma 6.1. *Let G be the function defined in* (6.1)*. Then for every $z \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $R \in \mathrm{O}(n)$ we have that $G(z,\nu) = G(Rz,R\nu)$. Moreover $G(z,\nu) = G(-z,\nu) = G(z,-\nu)$.*

Proof. Let us fix $z \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $R \in \mathrm{O}(n)$. For every orthonormal basis $(\xi^1,\dots,\xi^n)$ we have that $(R^T\xi^1,\dots,R^T\xi^n)$ is an orthonormal basis. Moreover, for every orthonormal basis $(\zeta^1,\dots,\zeta^n)$ there exists an orthonormal basis $(\xi^1,\dots,\xi^n)$ such that $\zeta^i = R^T\xi^i$, $i = 1,\dots,n$. Therefore

$$\begin{aligned} G(Rz,R\nu) &= \sup_{(\xi^1,\dots,\xi^n)}\Big(\sum_{i=1}^n g(Rz\cdot\xi^i)^2|R\nu\cdot\xi^i|^2\Big)^{\frac12} = \sup_{(\xi^1,\dots,\xi^n)}\Big(\sum_{i=1}^n g(z\cdot R^T\xi^i)^2|\nu\cdot R^T\xi^i|^2\Big)^{\frac12} \\ &= \sup_{(\zeta^1,\dots,\zeta^n)}\Big(\sum_{i=1}^n g(z\cdot\zeta^i)^2|\nu\cdot\zeta^i|^2\Big)^{\frac12} = G(z,\nu). \end{aligned}$$

The symmetry of the function G stated in the lemma is a straightforward consequence of (6.1). □

We study the function G in the two dimensional case, i.e., when $n=2$. Let us fix $\nu\in\mathbb{S}^1$. Thanks to Lemma 6.1, we can reduce to the case $\nu=(1,0)$ by applying a suitable rotation. To study the function $z\mapsto G(z,\nu)$, it is convenient to express the vector $z\in\mathbb{R}^2$ in polar coordinates. Let ρ be the norm of z and let φ be the angle between ν and z. By Lemma 6.1, it is enough to study the case $\varphi\in[0,\frac{\pi}{2}]$. In this way $z=(\rho\cos\varphi,\rho\sin\varphi)$.

Proposition 6.2. *Let $\nu=(1,0)$. Let $\rho>0$, $\varphi\in[0,\frac{\pi}{2}]$, and $z=(\rho\cos\varphi,\rho\sin\varphi)$. Then*

$$G(z,\nu)=\begin{cases}\rho\Big(\cos^4(\frac{\varphi}{2})+\sin^4(\frac{\varphi}{2})\Big)^{\frac12} & \text{if } \rho\cos(\frac{\varphi}{2})\le 1\,,\\ 1 & \text{if } \rho\cos\varphi\ge 1 \text{ or } \rho\ge\sqrt{2}\,,\\ \Big(\cos^2(\overline{\theta})+(\rho^2-1)\sin^2(\overline{\theta})\Big)^{\frac12} & \text{otherwise}\,,\end{cases} \tag{6.2}$$

where $\overline{\theta}=\varphi-\arccos(\frac{1}{\rho})\in[0,\frac{\varphi}{2}]$.

Proof. By (6.1) we have that

$$G(z,\nu)^2=\sup_{(\xi^1,\xi^2)}\big(g(z\cdot\xi^1)^2|\nu\cdot\xi^1|^2+g(z\cdot\xi^2)^2|\nu\cdot\xi^2|^2\big)\,, \tag{6.3}$$

where the supremum is taken over all orthonormal bases of $\mathbb{R}^2$. If we write ξ^1 in polar coordinates, it is easy to see that (6.3) is equivalent to

$$G(z,\nu)^2=\sup_{\varphi-\frac{\pi}{2}<\theta\le\varphi}\gamma(\theta)\,. \tag{6.4}$$

where

$$\gamma(\theta):=\min\{\rho|\cos(\varphi-\theta)|,1\}^2\cos^2(\theta)+\min\{\rho|\sin(\varphi-\theta)|,1\}^2\sin^2(\theta)\,.$$

Indeed, it is sufficient to take the supremum in (6.3) for θ ranging in an interval of length $\frac{\pi}{2}$. Note that $G(z,\nu)\le 1$, because $\gamma(\theta)\le 1$ for $\varphi-\frac{\pi}{2}<\theta\le\varphi$.

Let us assume that $\rho\cos(\frac{\varphi}{2})\le 1$. Then, for $\varphi-\frac{\pi}{2}<\theta\le\varphi$, we have that

$$\begin{aligned}\gamma(\theta)&\le\rho^2\cos^2(\varphi-\theta)\cos^2(\theta)+\rho^2\sin^2(\varphi-\theta)^2\sin^2(\theta)\\&=\tfrac14\rho^2\cos(2\varphi-4\theta)+\rho^2\cos^4(\tfrac{\varphi}{2})+\rho^2\sin^4(\tfrac{\varphi}{2})-\tfrac14\rho^2\,.\end{aligned} \tag{6.5}$$

Since $-\pi\le-2\varphi\le 2\varphi-4\theta<2\pi-2\varphi\le 2\pi$, the function $\theta\mapsto\cos(2\varphi-4\theta)$ attains its maximum at $\theta=\frac{\varphi}{2}$. Since $\rho\cos(\frac{\varphi}{2})\le 1$ and $0\le\frac{\varphi}{2}\le\frac{\pi}{4}$, we also have that $\rho\sin(\frac{\varphi}{2})\le 1$. This implies that $\gamma(\theta)$ attains its maximum at $\theta=\frac{\varphi}{2}$ and therefore $G(z,\nu)^2=\rho^2\cos^4(\frac{\varphi}{2})+\rho^2\sin^4(\frac{\varphi}{2})$. This concludes the study of the case $\rho\cos(\frac{\varphi}{2})\le 1$.

We therefore suppose that $\rho\cos(\frac{\varphi}{2})>1$ in what follows.

Let us assume that $\rho\cos\varphi\ge 1$ first. We simply note that in this case the maximum of $\gamma(\theta)$ is attained at $\theta=0$ and $\gamma(0)=1$.

Hence, let $\rho\cos\varphi<1$ and $\rho\le\sqrt{2}$. We claim that the maximum in (6.4) is attained at $\theta=\overline{\theta}=\varphi-\arccos(\frac{1}{\rho})$. Note that $0<\overline{\theta}<\frac{\varphi}{2}$, since $\rho\cos\varphi<1$ and $\rho\cos(\frac{\varphi}{2})>1$. For $\overline{\theta}\le\theta\le\varphi$, we have that $\rho\cos(\varphi-\theta)\ge 1$, and therefore $\rho^2\sin^2(\varphi-\theta)\le\rho^2-1\le 1$. This implies that

$$\begin{aligned}\gamma(\theta)&\le\cos^2(\theta)+\rho^2\sin^2(\varphi-\theta)^2\sin^2(\theta)\le\cos^2(\theta)+(\rho^2-1)\sin^2(\theta)\\&=(\rho^2-1)+(2-\rho^2)\cos^2\theta\,.\end{aligned} \tag{6.6}$$

The function $\theta\mapsto(\rho^2-1)+(2-\rho^2)\cos^2\theta$ is nonincreasing for $\theta\in[\overline{\theta},\varphi]$, and therefore its maximum in the interval $[\overline{\theta},\varphi]$ is attained at $\theta=\overline{\theta}$. Since $\rho\cos(\varphi-\overline{\theta})=1$ and $\rho\sin(\varphi-\theta)\le 1$, we get that the maximum of the function $\gamma(\theta)$ in the interval $[\overline{\theta},\varphi]$ is attained at $\theta=\overline{\theta}$. For $\varphi-\frac{\pi}{2}<\theta\le\overline{\theta}$, the maximum of the function $\theta\mapsto\cos(2\varphi-4\theta)$ is attained at $\theta=\overline{\theta}$, since $0\le 2\varphi-4\overline{\theta}\le 2\varphi-4\theta<2\pi-2\varphi\le 2\pi$ and $\overline{\theta}>0$. Hence, by inequality (6.5), we have that the maximum of the function $\gamma(\theta)$ in the interval $(\varphi-\frac{\pi}{2},\overline{\theta}]$ is attained at $\theta=\overline{\theta}$. This concludes the study of the case $\rho\cos\varphi<1$ and $\rho\le\sqrt{2}$.

We conclude the proof by observing that if $\rho\cos\varphi<1$ and $\rho>\sqrt{2}$, then $\gamma(\overline{\theta})=1$. Indeed, $\rho\cos(\varphi-\overline{\theta})=1$ and $\rho^2\sin^2(\varphi-\overline{\theta})=\rho^2-\rho^2\cos^2(\varphi-\overline{\theta})=\rho^2-1\ge 1$. □

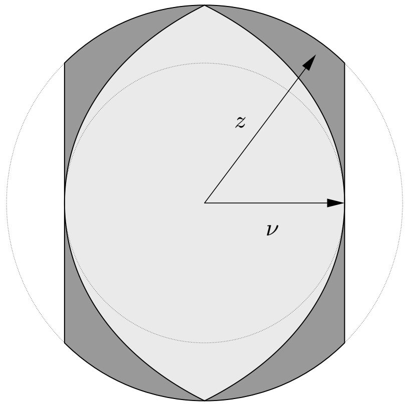

Figure 3. Behavior of the function $z \mapsto G(z,\nu)$.

Remark 6.3. Let us fix $\nu \in \mathbb{S}^{n-1}$. For every $z \in \mathbb{R}^n$ we have that

$$|z \odot \nu| = |z| \Big(\cos^4(\tfrac{\varphi}{2}) + \sin^4(\tfrac{\varphi}{2}) \Big)^{\frac{1}{2}} ,$$

where $\varphi \in [0, \frac{\pi}{2}]$ is the angle formed by the directions $\pm\nu$ and $\frac{z}{|z|}$. Hence, by formula (6.2), for $|z|\cos(\frac{\varphi}{2}) \leq 1$, i.e., in the region colored in light gray in Figure 3, we have that $G(z,\nu) = |z \odot \nu|$. For $|z| \geq \sqrt{2}$ or $|z|\cos(\varphi) \geq 1$, i.e., outside the colored regions in Figure 3, by (6.2) we obtain $G(z,\nu) = 1$. In the remaining part of $\mathbb{R}^2$, i.e., in the region colored in dark gray in Figure 3, the function $z \mapsto G(z,\nu)$ makes a transition between the function $z \mapsto |z \odot \nu|$ and the function with constant value 1.

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