

RELAXATION OF THE HENCKY MODEL IN PERFECT PLASTICITY

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ABSTRACT. In this paper we give a full proof of the relaxation of the Hencky model in perfect plasticity, under suitable assumptions for the domain and the Dirichlet boundary.

1. INTRODUCTION

The first complete mathematical treatment of the evolution problem in perfect plasticity is due to Suquet [13]. More recently, in [7] plastic evolution has been revisited in the framework of the variational theory for rate-independent processes (see, e.g., [11, 12]). In this variational approach existence of a quasi-static evolution is proved by approximation via time discretization and by solving a suitable incremental minimum problem at each discrete time.

More precisely, let $\Omega \subset \mathbb{R}^n$ denote the reference configuration of a plastic body. The elasto-plastic behaviour of Ω is described by three kinematic variables: the displacement $u : \Omega \rightarrow \mathbb{R}^n$, the elastic strain $e : \Omega \rightarrow \mathbb{M}_{sym}^{n \times n}$, and the plastic strain $p : \Omega \rightarrow \mathbb{M}_D^{n \times n}$. Here $\mathbb{M}_D^{n \times n}$ is the subspace of trace-free matrices in $\mathbb{M}_{sym}^{n \times n}$. Moreover, the strain $Eu := \text{sym } Du$ is related to e and p by the following kinematic admissibility condition:

$$Eu = e + p \quad \text{in } \Omega.$$

The requirement $p(x) \in \mathbb{M}_D^{n \times n}$ for every $x \in \Omega$ corresponds to the plastic incompressibility condition $\text{tr } p = 0$ in Ω , which is a usual requirement in the description of plastic behaviour in metals.

Let now $[0, T]$ be a time interval and assume for simplicity that the evolution is driven by a time-dependent boundary displacement $w : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ prescribed on a portion Γ_0 of $\partial\Omega$. Let t_i be a given discrete time and let $(u_{i-1}, e_{i-1}, p_{i-1})$ be the elasto-plastic configuration of the body at the previous discrete time t_{i-1} . Then, the configuration (u_i, e_i, p_i) at time t_i is found as a solution of the minimum problem

$$\min \left\{ \int_{\Omega} Q(e(x)) dx + \int_{\Omega} H(p(x) - p_{i-1}(x)) dx : \right. \\ \left. (u, e, p) \text{ such that } Eu = e + p \text{ in } \Omega, u = w(t_i) \text{ on } \Gamma_0 \right\}. \quad (1.1)$$

Here $Q : \mathbb{M}_{sym}^{n \times n} \rightarrow [0, +\infty)$ is a positive definite quadratic form, representing the elastic energy density. The function $H : \mathbb{M}_D^{n \times n} \rightarrow [0, +\infty)$ plays the role of a plastic dissipation potential and it is defined as the support function of a given convex and closed set $K \subset \mathbb{M}_D^{n \times n}$, that is,

$$H(\xi) := \sup_{\sigma \in K} \sigma : \xi \quad \text{for every } \xi \in \mathbb{M}_D^{n \times n}. \quad (1.2)$$

The set K represents the elasticity domain and its boundary ∂K defines the so-called yield surface. For $p_{i-1} = 0$ problem (1.1) is usually referred to as the Hencky model of perfect plasticity.

2010 *Mathematics Subject Classification.* 49J45 (74C05, 74G65).

Key words and phrases. Perfect plasticity, Hencky model, functions with bounded deformation, relaxation.

From the definition (1.2) it easily follows that the function H is convex and has linear growth. Thus, the natural domain of the functional in (1.1) is the class $\mathcal{A}_{reg}(w(t_i), \Gamma_0)$ of all triplets $(u, e, p) \in LD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times L^1(\Omega; \mathbb{M}_D^{n \times n})$ such that

$$Eu = e + p \quad \text{in } \Omega, \quad u = w(t_i) \quad \text{on } \Gamma_0.$$

We recall that $LD(\Omega)$ is the space of $L^1(\Omega; \mathbb{R}^n)$ functions whose symmetric gradient is in $L^1(\Omega; \mathbb{M}_{sym}^{n \times n})$.

However, since $LD(\Omega)$ and $L^1(\Omega; \mathbb{M}_D^{n \times n})$ are not reflexive spaces, problem (1.1) has in general no solution in the class $\mathcal{A}_{reg}(w(t_i), \Gamma_0)$. For this reason, in [7], as well as in the subsequent literature, problem (1.1) is replaced by the following weak formulation:

$$\min \left\{ \int_{\Omega} Q(e(x)) dx + \mathcal{H}_{\Omega \cup \Gamma_0}(p - p_{i-1}) : (u, e, p) \in \mathcal{A}(w(t_i), \Gamma_0) \right\}, \quad (1.3)$$

where $\mathcal{H}_{\Omega \cup \Gamma_0}(p - p_{i-1})$ is defined according to the theory of convex functions of measures (see Section 2 for more details) and the class $\mathcal{A}(w(t_i), \Gamma_0)$ is the set of all triplets $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ such that

$$Eu = e + p \quad \text{in } \Omega, \quad p = (w(t_i) - u) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1} \quad \text{on } \Gamma_0.$$

In other words, the plastic strain p is allowed to take values in a space of measures (and thus, the displacement u in the space $BD(\Omega)$ of functions with bounded deformation) and the boundary condition is relaxed (the boundary value may be not attained by u and in this case a discontinuity is developed along Γ_0).

It is easy to see that problem (1.3) is an extension of problem (1.1), meaning that any solution to (1.1) solves (1.3), as well. In the isotropic case with von Mises yield criterion it was shown in [2, Theorem 2.3] and in [14, Chapter II, Section 6.2] that (1.1) and (1.3) have the same infimum. In this article we prove that the relation between the two problems is much stronger: (1.3) is the relaxed problem of (1.1) in the sense of Γ -convergence, with respect to the natural topology, and as such, it is its natural extension (Theorem 4.3). In particular, by the abstract theory of Γ -convergence [6] this implies that not only the two problems have the same infimum, but also the minimisers of (1.3) coincide with all the limits of minimising sequences of (1.1).

A fundamental step in establishing Theorem 4.3 is to prove the density of the class $\mathcal{A}_{reg}(w(t_i), \Gamma_0)$ in the class $\mathcal{A}(w(t_i), \Gamma_0)$. Density is intended with respect to a topology that guarantees convergence of the energies, that is, strong- L^2 convergence for the elastic strains and strict convergence in the sense of measures for the plastic strains. This question is highly non trivial. Indeed, by the kinematic admissibility and the plastic incompressibility condition, displacements in $\mathcal{A}(w(t_i), \Gamma_0)$ belong to the space

$$U(\Omega) := \{u \in BD(\Omega) : \operatorname{div} u \in L^2(\Omega)\}.$$

For $n > 2$ this space is not local: if $u \in U(\Omega)$, then it is not true in general that $\varphi u \in U(\Omega)$ for every $\varphi \in C_c^\infty(\Omega)$. This fact prevents any naïve approximation based on local arguments and partitions of unity. Moreover, since displacements in $\mathcal{A}_{reg}(w(t_i), \Gamma_0)$ attain exactly the boundary condition on Γ_0 , one needs to correct the boundary value, again without leaving the space $U(\Omega)$, before any regularization procedure.

In Section 3 we prove two versions of this density result, under two different sets of assumptions for the domain Ω and the Dirichlet boundary Γ_0 . In Theorem 3.2 we assume $\partial\Omega$ to be of class $C^{2,1}$ and Γ_0 to be any open subset of $\partial\Omega$, while in Theorem 3.4 we consider the full Dirichlet case $\Gamma_0 = \partial\Omega$ with $\partial\Omega$ of Lipschitz regularity. We believe these density results to be of independent interest. For instance, Theorem 3.4 has been applied in the recent paper [5] to characterise the asymptotic behaviour of a certain family of quasistatic evolutions in a strain gradient plasticity model coupled with damage.

A crucial ingredient in the proof of both Theorems 3.2 and 3.4 is a result by Bogovsky [3, Theorem 1] (see also [4, Theorem 2.4]), stating the following: there exists a constant

$C > 0$ such that for every function $\psi \in L^p(\Omega)$ with null average on Ω , there exists a solution $v \in W_0^{1,n/(n-1)}(\Omega; \mathbb{R}^n)$ to the problem

$$\begin{cases} \operatorname{div} v = \psi & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

satisfying the estimate

$$\|v\|_{W^{1,n/(n-1)}} \leq C \|\psi\|_{L^{n/(n-1)}}.$$

By local mollifications we first construct smooth approximating triplets (u_k, e_k, p_k) with $\operatorname{div} u_k \in L^{n/(n-1)}(\Omega)$. Bogovski's result allows us to correct the displacements u_k in such a way to gain the L^2 -integrability of the divergence.

Concerning the boundary condition issue, in the Dirichlet case (Theorem 3.4) we extend u by the boundary datum w outside Ω and, before mollifying, we perform a local translation of the boundary towards the interior of Ω . If $\Gamma_0 \neq \partial\Omega$ (Theorem 3.2), we apply a clever construction by Anzellotti and Giaquinta [2], that requires higher regularity of the boundary. In Theorem 3.3 we provide a full proof of this construction, since the original proof in [2] contains several misprints and inaccuracies.

We finally mention that an approximation result close in spirit to Theorems 3.2 and 3.4 has been proved in [8, Theorem 4.7] for a model of perfectly plastic plates, but the proof is more conventional and does not require Bogovski's argument, since the model under consideration can be partially reduced to a two-dimensional setting.

This paper provides an answer to a basic question in the mathematical theory of perfect plasticity: what is the exact relation between the classical formulation (1.1) of the Hencky model and the weak formulation (1.3)? Although this is a very natural question, it has gone unregarded by the mathematical community and has been left open, up to the present contribution. The only established results available in the literature were those in [2] and in [14], concerning the isotropic case with von Mises yield condition and showing only equality of infima. However, we are much indebted to these works, since several arguments in the proofs of this article were inspired by ideas contained in [2] and in [14].

2. MATHEMATICAL PRELIMINARIES

In this section we introduce some notation and recall some notions that will be used throughout the article.

Distance function. Let U be a bounded open set of \mathbb{R}^n with a C^2 boundary and let $d(x) := \operatorname{dist}(x, \partial U)$ for every $x \in \bar{U}$. It is well known that there exists $a > 0$ such that, setting $U_a := \{x \in \bar{U} : d(x) < a\}$, then for every $x \in U_a$ there exists a unique projection $\pi(x) \in \partial U$ such that $d(x) = |x - \pi(x)|$; moreover, d is differentiable in U_a and $\nabla d(x) = -\nu_{\partial U}(\pi(x))$ for every $x \in U_a$, where $\nu_{\partial U}$ is the outward unit normal vector to ∂U . If $\partial U \in C^k$ with $k \geq 2$, then $d \in C^k(U_a)$. See, e.g., [9, Section 14.6].

Matrices. The space of symmetric $n \times n$ matrices is denoted by $\mathbb{M}_{sym}^{n \times n}$. It is endowed with the euclidean scalar product $\xi : \zeta = \sum_{i,j} \xi_{ij} \zeta_{ij}$ and the corresponding euclidean norm $|\xi|^2 = (\xi : \xi)^{1/2}$. The orthogonal complement of the subspace $\mathbb{R}I_{n \times n}$ spanned by the identity matrix $I_{n \times n}$ is the subspace $\mathbb{M}_D^{n \times n}$ of all matrices in $\mathbb{M}_{sym}^{n \times n}$ with trace zero. For every $\xi \in \mathbb{M}_{sym}^{n \times n}$ the orthogonal projection of ξ on $\mathbb{M}_D^{n \times n}$ is the deviator ξ_D of ξ , given by

$$\xi_D = \xi - \frac{1}{n}(\operatorname{tr} \xi)I_{n \times n}.$$

The symmetrised tensor product $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a \odot b)_{ij} = (a_i b_j + a_j b_i)/2$.

Measures. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space X , $M_b(B; X)$ denotes the space of all bounded Borel measures on B with values in X , endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in M_b(B; \mathbb{R})$ is the variation of the measure μ . If μ is

absolutely continuous with respect to the Lebesgue measure \mathcal{L}^n , we always identify μ with its density with respect to \mathcal{L}^n , which is a function in $L^1(B; X)$.

If the relative topology of B is locally compact, by the Riesz representation Theorem the space $M_b(B; X)$ can be identified with the dual of $C_0(B; X)$, which is the space of all continuous functions $\varphi : B \rightarrow X$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $M_b(B; X)$ is defined using this duality. Finally, we say that a sequence of measures μ_k converges to μ strictly in $M_b(B; X)$ if $\mu_k \rightharpoonup \mu$ weakly* in $M_b(B; X)$ and $|\mu_k|(B) \rightarrow |\mu|(B)$.

Convex functions of measures. Let U be an open set of \mathbb{R}^n . For every $\mu \in M_b(U; X)$ let $d\mu/d|\mu|$ be the Radon-Nikodým derivative of μ with respect to its variation $|\mu|$. Let $H : X \rightarrow [0, +\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H(\xi) \leq R|\xi| \quad \text{for every } \xi \in X,$$

where r and R are two constants, with $0 < r \leq R$. According to the theory of convex functions of measures, developed in [10], we introduce the nonnegative Radon measure $H(\mu) \in M_b(U)$ defined by

$$H(\mu)(A) := \int_A H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|$$

for every Borel set $A \subset U$. We also consider the functional $\mathcal{H}_U : M_b(U; X) \rightarrow [0, +\infty)$ defined by

$$\mathcal{H}_U(\mu) := H(\mu)(U) = \int_U H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|$$

for every $\mu \in M_b(U; X)$. One can prove that \mathcal{H}_U is lower semicontinuous on $M_b(U; X)$ with respect to weak* convergence (see, e.g., [1, Theorem 2.38]).

Functions with bounded deformation. Let U be an open set of \mathbb{R}^n . The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U; \mathbb{R}^n)$ whose symmetric gradient $Eu := \text{sym } Du$ (in the sense of distributions) belongs to $M_b(U; \mathbb{M}_{sym}^{n \times n})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{BD} := \|u\|_{L^1} + \|Eu\|_{M_b}.$$

We say that a sequence (u^k) converges to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U; \mathbb{R}^n)$ and $Eu^k \rightharpoonup Eu$ weakly* in $M_b(U; \mathbb{M}_{sym}^{n \times n})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, $BD(U)$ can be embedded into $L^{n/(n-1)}(U; \mathbb{R}^n)$ and every function $u \in BD(U)$ has a trace, still denoted by u , which belongs to $L^1(\partial U; \mathbb{R}^n)$. Moreover, if Γ is a nonempty open subset of ∂U , there exists a constant $C > 0$, depending on U and Γ , such that

$$\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Gamma)} + C\|Eu\|_{M_b} \quad (2.1)$$

(see [14, Chapter II, Proposition 2.4 and Remark 2.5]).

We will also use the space $LD(U)$ defined as the space of all functions $u \in L^1(U; \mathbb{R}^n)$ whose symmetric gradient $Eu := \text{sym } Du$ belongs to $L^1(U; \mathbb{M}_{sym}^{n \times n})$. The space $LD(U)$ is a Banach space endowed with the norm

$$\|u\|_{LD} := \|u\|_{L^1} + \|Eu\|_{L^1}.$$

For the general properties of the spaces $BD(U)$ and $LD(U)$ we refer to [14].

3. TWO DENSITY RESULTS

In this section we prove two density results in the class of admissible triplets. We first introduce some notation.

Definition 3.1. Let $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and let Γ_0 be an open subset of $\partial\Omega$ (in the relative topology). The class $\mathcal{A}_{reg}(w, \Gamma_0)$ of regular triplets with boundary datum w is defined as the set of all triplets $(u, e, p) \in LD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times L^1(\Omega; \mathbb{M}_D^{n \times n})$ such that

$$Eu = e + p \quad \text{a.e. in } \Omega, \quad (3.1)$$

$$u = w \quad \text{on } \Gamma_0. \quad (3.2)$$

The class $\mathcal{A}(w, \Gamma_0)$ of triplets with boundary datum w is defined as the set of all triplets $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ such that

$$Eu = e + p \quad \text{in } \Omega, \quad (3.3)$$

$$p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1} \quad \text{on } \Gamma_0. \quad (3.4)$$

The first result of this section is an approximation result for triplets in $\mathcal{A}(w, \Gamma_0)$ in terms of regular triplets in $\mathcal{A}_{reg}(w, \Gamma_0)$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with a $C^{2,1}$ boundary. Let $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and let $(u, e, p) \in \mathcal{A}(w, \Gamma_0)$. Then there exists a sequence of triplets (u_k, e_k, p_k) in $\mathcal{A}_{reg}(w, \Gamma_0)$ such that*

$$u_k \rightarrow u \quad \text{strongly in } L^{n/(n-1)}(\Omega; \mathbb{R}^n), \quad (3.5)$$

$$e_k \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{n \times n}), \quad (3.6)$$

$$p_k \rightharpoonup p \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n}), \quad (3.7)$$

and

$$\int_{\Omega} |p_k(x)| dx \rightarrow |p|(\Omega \cup \Gamma_0), \quad (3.8)$$

as $k \rightarrow \infty$.

The proof of Theorem 3.2 is based on the following auxiliary result, that was stated in [2]. We give here a complete proof of this result, since the proof proposed in [2] contains several inaccuracies and misprints.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with a $C^{2,1}$ boundary. Let $d(x) := \text{dist}(x, \partial\Omega)$ for every $x \in \bar{\Omega}$ and let $\Omega_a := \{x \in \bar{\Omega} : d(x) < a\}$ be such that $d \in C^2(\Omega_a)$. Finally, let $u \in L^1(\partial\Omega; \mathbb{R}^n)$ be such that $u \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$. Then there exists $v \in W^{1,1}(\Omega; \mathbb{R}^n) \cap L^2(\Omega; \mathbb{R}^n)$, with $\text{div } v \in L^2(\Omega)$, such that $\text{supp } v \subset \Omega_a$, $v = u$ on $\partial\Omega$, and*

$$v(x) \cdot \nabla d(x) = 0$$

for a.e. $x \in \Omega_a$.

Proof. We introduce the following notation:

$$Q := (-1, 1)^n, \quad Q^+ := (-1, 1)^{n-1} \times (0, 1), \quad Q_0 := (-1, 1)^{n-1} \times \{0\}.$$

The proof is subdivided into two steps.

Step 1. We first prove that, given $u \in L^1((-1, 1)^{n-1}; \mathbb{R}^{n-1})$, there exists a function $v \in W^{1,1}(Q^+; \mathbb{R}^n) \cap L^2(Q^+; \mathbb{R}^n)$, with $\partial_i v \in L^2(Q^+; \mathbb{R}^n)$ for $i = 1, \dots, n-1$, such that $v = (u, 0)$ on Q_0 and $v \cdot e_n = 0$ a.e. in Q^+ .

Let $u \in L^1((-1, 1)^{n-1}; \mathbb{R}^{n-1})$ be given. Let $\{\tau_j\}$ be a sequence of positive numbers decreasing to 0, as $j \rightarrow \infty$, and let $\{\theta_j\} \subset C_c^\infty((-1, 1)^{n-1}; \mathbb{R}^{n-1})$ be such that $\theta_0 \equiv 0$ and

$$\theta_j \rightarrow u \quad \text{strongly in } L^1((-1, 1)^{n-1}; \mathbb{R}^{n-1}), \quad (3.9)$$

as $j \rightarrow \infty$. We denote the coordinates in Q^+ by $(x', x_n) \in (-1, 1)^{n-1} \times (0, 1)$ and we define

$$v(x', x_n) := \begin{cases} 0 & \text{for } x_n \geq \tau_0, \\ \left(\theta_j(x') + \frac{x_n - \tau_j}{\tau_{j+1} - \tau_j} (\theta_{j+1}(x') - \theta_j(x')), 0 \right) & \text{for } \tau_{j+1} \leq x_n < \tau_j, \quad j \geq 0. \end{cases}$$

It is clear that $v \cdot e_n = 0$ in Q^+ . By straightforward computations we have that

$$\|v\|_{L^2}^2 \leq \sum_{j=0}^{\infty} (\tau_j - \tau_{j+1}) (\|\theta_j\|_{L^2}^2 + \|\theta_{j+1}\|_{L^2}^2),$$

and analogously,

$$\|\partial_i v\|_{L^2}^2 \leq \sum_{j=0}^{\infty} (\tau_j - \tau_{j+1}) (\|\partial_i \theta_j\|_{L^2}^2 + \|\partial_i \theta_{j+1}\|_{L^2}^2),$$

for $i = 1, \dots, n-1$, while

$$\|\partial_n v\|_{L^1} = \sum_{j=0}^{\infty} \|\theta_j - \theta_{j+1}\|_{L^1}.$$

Therefore, a suitable choice of the convergence rates of $\{\tau_j\}$ and $\{\theta_j\}$ guarantees that $v \in W^{1,1}(Q^+; \mathbb{R}^n) \cap L^2(Q^+; \mathbb{R}^n)$, with $\partial_i v \in L^2(Q^+; \mathbb{R}^n)$ for $i = 1, \dots, n-1$. Finally, in view of (3.9), the trace of v on Q_0 coincides with $(u, 0)$.

We also note that, if $\text{supp } u \subset \omega$ where ω is an open set compactly contained in $(-1, 1)^{n-1}$, then we can choose the sequence $\{\theta_j\}$ in such a way that $\text{supp } \theta_j \subset \omega$ for every j , so that $\text{supp } v \subset \omega \times [0, 1)$.

Step 2. We now prove the general statement. Let $u \in L^1(\partial\Omega; \mathbb{R}^n)$ be such that $u \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$. We can cover $\partial\Omega$ with a finite number of open sets A_j , $j = 1, \dots, N$, such that $\cup_j A_j \cap \Omega \subset \Omega_a$ and for every $j = 1, \dots, N$ there exists a $C^{1,1}$ diffeomorphism $\Phi_j : A_j \rightarrow Q$ satisfying $\Phi_j(A_j \cap \Omega) = Q^+$, $\Phi_j(A_j \cap \partial\Omega) = Q_0$, and $D\Phi_j(x) \nabla d(x) = e_n$ for every $x \in A_j \cap \Omega$.

Let $\{\varphi_j\}$ be a partition of unity subordinated to the covering $\{A_j\}$. For every $j = 1, \dots, N$ we set $u_j := \varphi_j u$ and we consider

$$\hat{u}_j(y') := (D\Phi_j^{-1}(y', 0))^T u_j(\Phi_j^{-1}(y', 0))$$

for a.e. $y' \in (-1, 1)^{n-1}$. Note that

$$\hat{u}_j \cdot e_n = u_j(\Phi_j^{-1}(y', 0)) \cdot D\Phi_j^{-1}(y', 0) e_n = -u_j(\Phi_j^{-1}(y', 0)) \cdot \nu_{\partial\Omega}(\Phi_j^{-1}(y', 0)) = 0,$$

where we used that $\nabla d = -\nu_{\partial\Omega} \circ \pi$. Therefore, by Step 1 there exists a function $v_j \in W^{1,1}(Q^+; \mathbb{R}^n) \cap L^2(Q^+; \mathbb{R}^n)$, with $\partial_i v_j \in L^2(Q^+; \mathbb{R}^n)$ for $i = 1, \dots, n-1$, such that $\text{supp } v_j$ is compactly contained in $A_j \cap \bar{\Omega}$, $v_j = \hat{u}_j$ on Q_0 and $v_j \cdot e_n = 0$ a.e. in Q^+ . We define

$$v := \sum_{j=1}^N (D\Phi_j)^T v_j \circ \Phi_j.$$

Since $D\Phi_j$ is $C^{0,1}$, we have that $v \in W^{1,1}(\Omega; \mathbb{R}^n) \cap L^2(\Omega; \mathbb{R}^n)$. Moreover, by construction it is clear that $\text{supp } v \subset \Omega_a$, $v = u$ on $\partial\Omega$, and $v \cdot \nabla d = 0$ a.e. in Ω_a .

It remains to check that $\text{div } v \in L^2(\Omega)$. Straightforward computations lead to

$$\text{div } v = \sum_{j=1}^N \Delta \Phi_j \cdot (v_j \circ \Phi_j) + \sum_{j=1}^N \text{tr}((D\Phi_j)^T (Dv_j \circ \Phi_j) D\Phi_j). \quad (3.10)$$

Let us focus on the second term on the right-hand side. For every $j = 1, \dots, n$, let $R_j \in C^1(A_k \cap \partial\Omega; \mathbb{M}^{n \times n})$ be such that $R_j(x) \in SO(n)$ and $R_j(x) e_n = -\nu_{\partial\Omega}(x)$ for every $x \in A_j \cap \partial\Omega$. Let $P_j := R_j \circ \pi$. Then,

$$\begin{aligned} \text{tr}((D\Phi_j)^T (Dv_j \circ \Phi_j) D\Phi_j) &= \sum_{i=1}^n (D\Phi_j)^T (Dv_j \circ \Phi_j) D\Phi_j P_j e_i \cdot P_j e_i \\ &= \sum_{i=1}^n (Dv_j \circ \Phi_j) D\Phi_j P_j e_i \cdot D\Phi_j P_j e_i. \end{aligned}$$

Since $D\Phi_j P_j e_n = D\Phi_j \nabla d = e_n$ and $v_j \cdot e_n = 0$, the previous expression reduces to

$$\operatorname{tr}((D\Phi_j)^T(Dv_j \circ \Phi_j)D\Phi_j) = \sum_{i=1}^{n-1} (Dv_j \circ \Phi_j)D\Phi_j P_j e_i \cdot D\Phi_j P_j e_i. \quad (3.11)$$

For every $i = 1, \dots, n-1$ and $x_0 \in A_j \cap \Omega$ the vector $P_j(x_0)e_i$ is orthogonal to $\nabla d(x_0)$, hence, it is a tangent vector to the level set $\{x \in \Omega : d(x) = d(x_0)\}$; the Jacobian $D\Phi_j$ maps such vectors into vectors orthogonal to e_n , thus, the expression in (3.11) depends only on $\partial_i v_j$ for $i = 1, \dots, n-1$, and as such, it belongs to $L^2(\Omega)$. Since v_j belongs to $L^2(\Omega; \mathbb{R}^n)$ as well, identity (3.10) implies that $\operatorname{div} v \in L^2(\Omega)$. \square

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Upon replacing the triplet (u, e, p) by $(u - w, e - Ew, p)$, we can assume that $w = 0$. The proof is subdivided into three steps.

Step 1. We first prove the statement assuming that $(u, e, p) \in \mathcal{A}(0, \Gamma_0)$ satisfies the additional condition $u = 0$ on Γ_0 (hence, $p = 0$ on Γ_0 by (3.4)). In this step we only need $\partial\Omega$ to be of Lipschitz regularity.

The proof follows closely that of [14, Theorem II-3.4]. Let $k \in \mathbb{N}$. Let $\{A_i\}_{i \in \mathbb{N}}$ be a locally finite covering of Ω and let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a partition of unity subordinated to it. Let $\{\varrho_\varepsilon\}$ be a family of mollifiers. For every i we can find ε_i such that

$$\|(\varphi_i u) * \varrho_{\varepsilon_i} - \varphi_i u\|_{L^{n/(n-1)}} \leq \frac{1}{k} \frac{1}{2^i}, \quad (3.12)$$

$$\|(\nabla \varphi_i \odot u) * \varrho_{\varepsilon_i} - \nabla \varphi_i \odot u\|_{L^{n/(n-1)}} \leq \frac{1}{k} \frac{1}{2^i}, \quad (3.13)$$

$$\|(\varphi_i e) * \varrho_{\varepsilon_i} - \varphi_i e\|_{L^2} \leq \frac{1}{k} \frac{1}{2^i}, \quad (3.14)$$

$$\left| \int_{\Omega} |(\varphi_i p) * \varrho_{\varepsilon_i}| dx - |\varphi_i p|(\Omega) \right| \leq \frac{1}{k} \frac{1}{2^i}. \quad (3.15)$$

We then define

$$\hat{u}_k := \sum_{i=0}^{\infty} (\varphi_i u) * \varrho_{\varepsilon_i}.$$

By (3.12)–(3.15) it is clear that $\hat{u}_k \in C^\infty(\Omega; \mathbb{R}^n) \cap LD(\Omega)$ and

$$\|\hat{u}_k - u\|_{L^{n/(n-1)}} \leq \frac{1}{k}. \quad (3.16)$$

Since $\operatorname{div} u = \operatorname{tr} e$ in Ω , (3.13) and (3.14) yield

$$\|\operatorname{div} \hat{u}_k - \operatorname{div} u\|_{L^{n/(n-1)}} \leq \frac{1}{k}. \quad (3.17)$$

Moreover, one can show (see [14, Theorem II-3.3]) that $\hat{u}_k = u$ on $\partial\Omega$, hence $\hat{u}_k = 0$ on Γ_0 . Since $\operatorname{div} u \in L^2(\Omega)$ and (3.17) holds, we can construct $\psi_k \in C_c^\infty(\Omega)$ such that

$$\|\psi_k - \operatorname{div} u\|_{L^2} \leq \frac{1}{k} \quad (3.18)$$

and

$$\int_{\Omega} \psi_k(x) dx = \int_{\Omega} \operatorname{div} \hat{u}_k(x) dx. \quad (3.19)$$

We denote by v_k a solution of the system

$$\begin{cases} \operatorname{div} v_k = \psi_k - \operatorname{div} \hat{u}_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

By [3, Theorem 1] (see also [4, Theorem 2.4]) condition (3.19) guarantee the existence of a solution $v_k \in W_0^{1, n/(n-1)}(\Omega; \mathbb{R}^n)$ such that

$$\|v_k\|_{W^{1, n/(n-1)}} \leq C \|\psi_k - \operatorname{div} \hat{u}_k\|_{L^{n/(n-1)}}, \quad (3.20)$$

where C is a constant independent of k .

We now set

$$\begin{aligned} u_k &:= \hat{u}_k + v_k, & e_k &:= \sum_{i=0}^{\infty} (\varphi_i e)_D * \varrho_{\varepsilon_i} + \frac{1}{n} \psi_k I_{n \times n}, \\ p_k &:= \sum_{i=0}^{\infty} (\varphi_i p) * \varrho_{\varepsilon_i} + \sum_{i=0}^{\infty} (\nabla \varphi_i \odot u)_D * \varrho_{\varepsilon_i} + (Ev_k)_D. \end{aligned}$$

It is easy to see that $(u_k, e_k, p_k) \in \mathcal{A}_{reg}(0, \Gamma_0)$ for every k . Moreover, by (3.16)–(3.18) and (3.20) it is clear that u_k converges to u strongly in $L^{n/(n-1)}(\Omega; \mathbb{R}^n)$ and by (3.14) and (3.18) that e_k converges to e strongly in $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, as $k \rightarrow \infty$. To conclude the proof, it is enough to show that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |p_k| dx \leq |p|(\Omega) = |p|(\Omega \cup \Gamma_0). \quad (3.21)$$

Indeed, if (3.21) holds, then $\{p_k\}$ is bounded in $M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$. In particular, there exists $q \in M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ such that, up to subsequences,

$$p_k \rightharpoonup q \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n}). \quad (3.22)$$

Since $Eu_k = e_k + p_k$ in Ω , the convergence of $\{u_k\}$ and $\{e_k\}$ imply that $Eu = e + q$ in Ω , that is, $q = p$ in Ω . On the other hand, by lower semicontinuity of the norm $\|\cdot\|_{M_b}$ with respect to weak* convergence and by (3.21), we obtain

$$|q|(\Omega \cup \Gamma_0) = |p|(\Omega) + |q|(\Gamma_0) \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |p_k| dx \leq \limsup_{k \rightarrow \infty} \int_{\Omega} |p_k| dx \leq |p|(\Omega).$$

Hence, $q = 0$ on Γ_0 , which implies that $q = p$ on $\Omega \cup \Gamma_0$. Thus, (3.22) gives (3.7) (note that in (3.22) the whole sequence converges, since the limit is uniquely determined) and the equality above gives (3.8).

We now prove (3.21). By (3.13) we have that

$$\left\| \sum_{i=0}^{\infty} (\nabla \varphi_i \odot u)_D * \varrho_{\varepsilon_i} \right\|_{L^{n/(n-1)}} \rightarrow 0,$$

as $k \rightarrow \infty$, and by (3.17), (3.18), and (3.20) we deduce that $(Ev_k)_D$ is converging to 0 in $L^{n/(n-1)}(\Omega; \mathbb{M}_D^{n \times n})$. Finally,

$$\sum_{i=0}^{\infty} \int_{\Omega} |(\varphi_i p) * \varrho_{\varepsilon_i}| dx \leq \sum_{i=0}^{\infty} \int_{\Omega} \varphi_i d|p| = |p|(\Omega).$$

Combining these observations together, we deduce (3.21).

Step 2. We now show that any triplet $(u, e, p) \in \mathcal{A}(0, \Gamma_0)$ can be approximated in the sense of (3.5)–(3.8) by a sequence of triplets (u_k, e_k, p_k) in $\mathcal{A}(0, \Gamma_0)$ such that $u_k = 0$ on Γ_0 (hence, $p_k = 0$ on Γ_0 by (3.4)) for every k . In this step we will use the $C^{2,1}$ regularity of $\partial\Omega$, since the construction will be based on Theorem 3.3.

Let $(u, e, p) \in \mathcal{A}(0, \Gamma_0)$ and let χ_{Γ_0} be the characteristic function of Γ_0 . By Theorem 3.3 there exists a function $v \in W^{1,1}(\Omega; \mathbb{R}^n)$, with $\operatorname{div} v \in L^2(\Omega)$, such that $\operatorname{supp} v \subset \Omega_a$, $v = -\chi_{\Gamma_0} u$ on $\partial\Omega$, and

$$v(x) \cdot \nabla d(x) = 0$$

for a.e. $x \in \Omega_a$. For $k \in \mathbb{N}$ large enough we define $\eta_k(x) := \max\{0, 1 - kd(x)\}$, $x \in \Omega$, and

$$\begin{aligned} u_k &:= u + \eta_k v, & e_k &:= e_D + \frac{1}{n} \eta_k \operatorname{div} v I_{n \times n}, \\ p_k &:= p \llcorner \Omega + \eta_k (Ev)_D + \nabla \eta_k \odot v. \end{aligned}$$

Note that $\nabla \eta_k = -k \nabla d$ a.e. in $\Omega_{1/k}$, so that $\nabla \eta_k(x) \odot v(x) \in \mathbb{M}_D^{n \times n}$ for a.e. $x \in \Omega$. Since $\operatorname{div} v \in L^2(\Omega)$, we have that $e_k \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$. Moreover, $u_k = 0$ on Γ_0 , since by

construction $v = -u$ on Γ_0 . Thus, $(u_k, e_k, p_k) \in \mathcal{A}(0, \Gamma_0)$ and satisfies the required additional condition on Γ_0 .

It is easy to see that $u_k \rightarrow u$ strongly in $L^{n/(n-1)}(\Omega; \mathbb{R}^n)$ and $e_k \rightarrow e$ strongly in $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, as $k \rightarrow \infty$. To conclude, it is enough to show that

$$\limsup_{k \rightarrow \infty} |p_k|(\Omega) \leq |p|(\Omega \cup \Gamma_0). \quad (3.23)$$

Indeed, if (3.23) holds, then $\{p_k\}$ is bounded in $M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$. In particular, there exists $q \in M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ such that

$$p_k \rightharpoonup q \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n}). \quad (3.24)$$

Let U be an open set of \mathbb{R}^n such that $U \cap \partial\Omega = \Gamma_0$ and let us consider the extensions

$$\tilde{u}_k := \begin{cases} u_k & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{e}_k := \begin{cases} e_k & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{p}_k := \begin{cases} p_k & \text{in } \Omega \cup \Gamma_0, \\ 0 & \text{in } U \setminus \bar{\Omega}, \end{cases}$$

and

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{e} := \begin{cases} e & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{q} := \begin{cases} q & \text{in } \Omega \cup \Gamma_0, \\ 0 & \text{in } U \setminus \bar{\Omega}. \end{cases}$$

Then, $\tilde{u}_k \rightarrow \tilde{u}$ strongly in $L^{n/(n-1)}(\Omega; \mathbb{R}^n)$, $\tilde{e}_k \rightarrow \tilde{e}$ strongly in $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, and $\tilde{p}_k \rightharpoonup \tilde{q}$ weakly* in $M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$. Since $E\tilde{u}_k = \tilde{e}_k + \tilde{p}_k$ in $\Omega \cup U$, we deduce that $E\tilde{u} = \tilde{e} + \tilde{q}$ in $\Omega \cup U$, hence $q = p$ in $\Omega \cup \Gamma_0$. Therefore, (3.24) and (3.23) yield (3.7) and (3.8).

We now prove (3.23). By definition of p_k we have that

$$|p_k|(\Omega) \leq |p|(\Omega) + \int_{\Omega} |\eta_k(Ev)_D| dx + \int_{\Omega} |\nabla \eta_k \odot v| dx.$$

It is immediate to see that the second term on the right-hand side converges to 0, as $k \rightarrow \infty$. Thus, to prove the claim (3.23), it suffices to prove that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla \eta_k \odot v| dx \leq |p|(\Gamma_0). \quad (3.25)$$

Using the definition of η_k we obtain

$$\int_{\Omega} |\nabla \eta_k \odot v| dx = k \int_{\Omega_{1/k}} |\nabla d \odot v| dx. \quad (3.26)$$

For k sufficiently small we can cover $\Omega_{1/k}$ with a finite number of open sets A_j , $j = 1, \dots, N$, such that for every $j = 1, \dots, N$ the map $\Psi_j : (-1, 1)^{n-1} \times (-\frac{2}{k}, \frac{2}{k}) \rightarrow A_j$ given by $\Psi_j(x', x_n) = (x', g_j(x')) - x_n \nu_{\partial\Omega}(x', g_j(x'))$ is a $C^{1,1}$ diffeomorphism. Here g_j is a $C^{2,1}$ function whose subgraph represents Ω in A_j . In other words, we may assume that $\Psi_j((-1, 1)^{n-1} \times (0, \frac{1}{k})) = A_j \cap \Omega$. Let $\{\varphi_j\}$ be a partition of unity subordinated to the covering $\{A_j\}$. Then

$$k \int_{\Omega_{1/k}} |\nabla d \odot v| dx = k \int_{\Omega_{1/k}} \left| \nabla d \odot \sum_{j=1}^N \varphi_j v \right| dx \leq k \sum_{j=1}^N \int_{\Omega_{1/k} \cap A_j} |\nabla d \odot \varphi_j v| dx. \quad (3.27)$$

By a change of variable we have

$$\begin{aligned} & k \int_{\Omega_{1/k} \cap A_j} |\nabla d \odot \varphi_j v| dx \\ &= k \int_0^{1/k} \int_{(-1,1)^{n-1}} |\nabla d(\Psi_j(x', x_n)) \odot (\varphi_j v)(\Psi_j(x', x_n))| \det D\Psi_j(x', x_n) dx' dx_n. \end{aligned}$$

By Fubini Theorem and the definition of trace we have that there exists a set M with $\mathcal{L}^1(M) = 0$ such that

$$(\varphi_j v)(\Psi_j(x', t)) \rightarrow (\varphi_j v)(\Psi_j(x', 0)) \quad \text{in } L^1((-1, 1)^{n-1}; \mathbb{R}^n),$$

as $t \rightarrow 0^+$, $t \notin M$, where the limit $(\varphi_j v)(\Psi_j(x', 0))$ is intended in the sense of traces. Thus, we conclude that

$$k \int_{\Omega_{1/k} \cap A_j} |\nabla d \odot \varphi_j v| dx \rightarrow \int_{(-1,1)^{n-1}} |\nu_{\partial\Omega}(\Psi_j(x', 0)) \odot w_j(x', 0)| \det D\Psi_j(x', 0) dx'.$$

By the area formula we have

$$\int_{(-1,1)^{n-1}} |\nu_{\partial\Omega}(\Psi_j(x', 0)) \odot w_j(x', 0)| \det D\Psi_j(x', 0) dx' = \int_{A_j \cap \partial\Omega} |\nu_{\partial\Omega} \odot \varphi_j v| d\mathcal{H}^{n-1}.$$

Combining the previous equations with (3.26) and (3.27), we deduce that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla \eta_k \odot v| dx &\leq \sum_{j=1}^N \int_{A_j \cap \partial\Omega} |\nu_{\partial\Omega} \odot \varphi_j v| d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} \sum_{j=1}^N \varphi_j |\nu_{\partial\Omega} \odot v| d\mathcal{H}^{n-1} \\ &= \int_{\Gamma_0} |-u \odot \nu_{\partial\Omega}| d\mathcal{H}^{n-1} = |p|(\Gamma_0), \end{aligned}$$

where we used that $v = -\chi_{\Gamma_0} u$ on $\partial\Omega$.

Step 3. To conclude, it is enough to apply a diagonal argument, together with the remark that bounded sets of $M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ are metrizable with respect to weak* convergence. \square

When the boundary condition is prescribed on the whole boundary, that is, $\Gamma_0 = \partial\Omega$, a different construction of the approximating sequence can be performed. This new construction requires only Lipschitz regularity of the boundary and leads to more regular approximating triplets. More precisely, we have the following theorem.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. Assume $\Gamma_0 = \partial\Omega$. Let $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and let $(u, e, p) \in \mathcal{A}(w, \Gamma_0)$. Then there exists a sequence of triplets (u_k, e_k, p_k) in $\mathcal{A}_{reg}(w, \Gamma_0)$ such that*

$$(u_k - w, e_k - Ew, p_k) \in C_c^\infty(\Omega; \mathbb{R}^n) \times C_c^\infty(\Omega; \mathbb{M}_{sym}^{n \times n}) \times C_c^\infty(\Omega; \mathbb{M}_D^{n \times n})$$

for every k , and

$$u_k \rightarrow u \quad \text{strongly in } L^{n/(n-1)}(\Omega; \mathbb{R}^n), \quad (3.28)$$

$$e_k \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{n \times n}), \quad (3.29)$$

$$p_k \rightharpoonup p \quad \text{weakly* in } M_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}), \quad (3.30)$$

and

$$\int_{\Omega} |p_k(x)| dx \rightarrow |p|(\overline{\Omega}), \quad (3.31)$$

as $k \rightarrow \infty$.

Proof. Upon replacing the triplet (u, e, p) by $(u - w, e - Ew, p)$, we can assume that $w = 0$.

We consider the extensions

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{e} := \begin{cases} e & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{p} := \begin{cases} p & \text{in } \overline{\Omega}, \\ 0 & \text{in } U \setminus \overline{\Omega}, \end{cases}$$

where U is an open and bounded set such that Ω is compactly contained in U . Note that $\tilde{u} \in BD(U)$, $\tilde{e} \in L^2(U; \mathbb{M}_{sym}^{n \times n})$, $\tilde{p} \in M_b(U; \mathbb{M}_D^{n \times n})$, and $E\tilde{u} = \tilde{e} + \tilde{p}$ in U . In particular, we have that $\operatorname{div} \tilde{u} \in L^2(U)$.

Since Ω is bounded and has a Lipschitz boundary, there exists a finite open cover $\{A_j\}$, $j = 1, \dots, N$, of $\partial\Omega$, made of open cubes centred at points on $\partial\Omega$, with a face orthogonal to some vector $\xi_j \in \mathbb{S}^1$ and such that the set $A_j \cap \Omega$ is a Lipschitz subgraph in the direction

ξ_j . We set $A_0 := \Omega$ and $\xi_0 := 0$. For every $j = 0, \dots, N$ and every $k \in \mathbb{N}$ we introduce the translation

$$\tau_{j,k}(x) := x + \frac{1}{k}\xi_j \quad \text{for } x \in \mathbb{R}^n.$$

Finally, let $\{\varphi_j\}$ be a partition of unity subordinated to $\{A_j\}$ and let $\{\varrho_\varepsilon\}$ be a family of mollifiers.

We define

$$\begin{aligned} \tilde{u}_k &:= \sum_{j=0}^N (\varphi_j \tilde{u}) \circ \tau_{j,k}, & \tilde{e}_k &:= \sum_{j=0}^N (\varphi_j \tilde{e})_D \circ \tau_{j,k} + \frac{1}{n} \operatorname{div} \tilde{u}_k I_{n \times n}, \\ \tilde{p}_k &:= \sum_{j=0}^N \tau_{j,k}^\# (\varphi_j \tilde{p}) + \sum_{j=0}^N (\nabla \varphi_j \odot \tilde{u})_D \circ \tau_{j,k}, \end{aligned}$$

where $\tau_{j,k}^\# (\varphi_j \tilde{p})$ denotes the pull-back measure of $\varphi_j \tilde{p}$. We observe that $\tilde{u}_k \in BD(\Omega)$, $(\tilde{e}_k)_D \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\operatorname{div} \tilde{u}_k = \operatorname{tr} \tilde{e}_k \in L^{n/(n-1)}(\Omega)$, $\tilde{p}_k \in M_b(\Omega; \mathbb{M}_D^{n \times n})$, and they all have compact support in Ω for k sufficiently small. Moreover, $E\tilde{u}_k = \tilde{e}_k + \tilde{p}_k$ in Ω and, as $k \rightarrow \infty$, we have

$$\tilde{u}_k \rightarrow u \quad \text{strongly in } L^{n/(n-1)}(\Omega; \mathbb{R}^n), \quad (3.32)$$

$$\operatorname{div} \tilde{u}_k \rightarrow \operatorname{div} u \quad \text{strongly in } L^{n/(n-1)}(\Omega), \quad (3.33)$$

$$(\tilde{e}_k)_D \rightarrow e_D \quad \text{strongly in } L^2(\Omega; \mathbb{M}_D^{n \times n}), \quad (3.34)$$

and

$$\sum_{j=0}^N (\nabla \varphi_j \odot \tilde{u})_D \circ \tau_{j,k} \rightarrow 0 \quad \text{strongly in } L^{n/(n-1)}(\Omega; \mathbb{M}_D^{n \times n}). \quad (3.35)$$

Since $\tau_{j,k}$ is an outward translation on $A_j \cap \Omega$, we have that

$$\sum_{j=0}^N |\tau_{j,k}^\# (\varphi_j \tilde{p})|(\Omega) \leq \sum_{j=0}^N |\varphi_j \tilde{p}|(U) = \sum_{j=0}^N \int_U \varphi_j d|\tilde{p}| = |p|(\bar{\Omega})$$

for every $k \in \mathbb{N}$, hence by (3.35) we deduce that

$$\limsup_{k \rightarrow \infty} |\tilde{p}_k|(\Omega) \leq |p|(\bar{\Omega}). \quad (3.36)$$

We now set

$$\hat{u}_k := \tilde{u}_k * \varrho_{\varepsilon_k}, \quad \hat{e}_k := \tilde{e}_k * \varrho_{\varepsilon_k}, \quad \hat{p}_k := \tilde{p}_k * \varrho_{\varepsilon_k},$$

where ε_k is chosen so that $\hat{u}_k \in C_c^\infty(\Omega; \mathbb{R}^n)$, $\hat{e}_k \in C_c^\infty(\Omega; \mathbb{M}_{sym}^{n \times n})$, $\hat{p}_k \in C_c^\infty(\Omega; \mathbb{M}_D^{n \times n})$, and

$$\|\hat{u}_k - \tilde{u}_k\|_{L^{n/(n-1)}} \leq \frac{1}{k}, \quad (3.37)$$

$$\|\operatorname{div} \hat{u}_k - \operatorname{div} \tilde{u}_k\|_{L^{n/(n-1)}} \leq \frac{1}{k}, \quad (3.38)$$

$$\|(\hat{e}_k)_D - (\tilde{e}_k)_D\|_{L^2} \leq \frac{1}{k}, \quad (3.39)$$

$$\left| \int_\Omega |\hat{p}_k| dx - |\tilde{p}_k|(\Omega) \right| \leq \frac{1}{k}, \quad (3.40)$$

where the last inequality follows from the fact that $|\tilde{p}_k|(\partial\Omega) = 0$, hence mollifications of \tilde{p}_k strictly converge to \tilde{p}_k in Ω . Clearly, we still have that $E\hat{u}_k = \hat{e}_k + \hat{p}_k$ in Ω .

We now introduce a correction for $\operatorname{div} \hat{u}_k$. Since $\operatorname{div} u \in L^2(\Omega)$ and $\operatorname{div} \hat{u}_k \rightarrow \operatorname{div} u$ in $L^{n/(n-1)}(\Omega)$ by (3.33) and (3.38), we can construct $\psi_k \in C_c^\infty(\Omega)$ such that

$$\|\psi_k - \operatorname{div} u\|_{L^2} \leq \frac{1}{k}, \quad (3.41)$$

and

$$\int_\Omega \psi_k(x) dx = \int_\Omega \operatorname{div} \hat{u}_k(x) dx. \quad (3.42)$$

We denote by v_k a solution of the system

$$\begin{cases} \operatorname{div} v_k = \psi_k - \operatorname{div} \hat{u}_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

By [3, Remark 4] (see also [4, Theorem 2.4]) condition (3.42) guarantees the existence of a solution $v_k \in C_c^\infty(\Omega; \mathbb{R}^n)$ such that

$$\|v_k\|_{W^{1,n/(n-1)}} \leq C \|\psi_k - \operatorname{div} \hat{u}_k\|_{L^{n/(n-1)}},$$

where C is a constant independent of k . Combining this inequality with (3.33), (3.38), and (3.41), we deduce that

$$v_k \rightarrow 0 \quad \text{strongly in } W^{1,n/(n-1)}(\Omega; \mathbb{R}^n). \quad (3.43)$$

We are now ready to define the approximating sequence. We set

$$u_k := \hat{u}_k + v_k, \quad e_k := \hat{e}_k + \frac{1}{n} \operatorname{div} v_k I_{n \times n}, \quad p_k := \hat{p}_k + (E v_k)_D.$$

It is immediate to see that $u_k \in C_c^\infty(\Omega; \mathbb{R}^n)$, $e_k \in C_c^\infty(\Omega; \mathbb{M}_{sym}^{n \times n})$, and $p_k \in C_c^\infty(\Omega; \mathbb{M}_D^{n \times n})$. Moreover, $E u_k = e_k + p_k$ in Ω , so that $(u_k, e_k, p_k) \in \mathcal{A}_{reg}(0, \Gamma_0)$ for every k . By (3.32), (3.37), and (3.43) we immediately deduce (3.28). Note that

$$e_k = (\hat{e}_k)_D + \frac{1}{n} \psi_k I_{n \times n},$$

hence, (3.34), (3.39), and (3.41) yield (3.29). Finally, by (3.36), (3.40), and (3.43) we deduce that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |p_k| dx \leq \limsup_{k \rightarrow \infty} \int_{\Omega} |\hat{p}_k| dx \leq \limsup_{k \rightarrow \infty} |\tilde{p}_k|(\Omega) \leq |p|(\bar{\Omega}). \quad (3.44)$$

This last inequality is enough to conclude. Indeed, extending to 0 the triplets (u_k, e_k, p_k) outside $\bar{\Omega}$, we have by (3.44) that $\{p_k\}$ is bounded in $M_b(U; \mathbb{M}_D^{n \times n})$, hence, up to subsequences,

$$p_k \rightharpoonup q \quad \text{weakly}^* \text{ in } M_b(U; \mathbb{M}_D^{n \times n}). \quad (3.45)$$

By (3.28) and (3.29) we deduce that $E \tilde{u} = \tilde{e} + q$ in U , hence, in particular, $q = p$ in $\bar{\Omega}$. This fact, together with (3.44) and (3.45), yields (3.30) and (3.31). This concludes the proof of the theorem. \square

4. THE RELAXATION RESULT

In this section we apply the density theorems of Section 3 to characterise the relaxation of the Hencky model. In the notation of the introduction we prove that (1.3) is the relaxed problem of (1.1) when $p_{i-1} = 0$.

For the sake of notation it is convenient to express the involved functionals in terms of the displacement u and of the elastic strain e , only. The plastic strain p can be always recovered a posteriori by the kinematic compatibility condition.

Definition 4.1. Let $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and let Γ_0 be an open subset of $\partial\Omega$. We define $\mathcal{B}_{reg}(w, \Gamma_0)$ as the class of all pairs $(u, e) \in LD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ such that

$$\operatorname{div} u = \operatorname{tr} e \quad \text{a.e. in } \Omega, \quad (4.1)$$

$$u = w \quad \text{on } \Gamma_0. \quad (4.2)$$

Definition 4.2. We define \mathcal{B} as the class of all pairs $(u, e) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ such that (4.1) holds.

Note that $(u, e) \in \mathcal{B}_{reg}(w, \Gamma_0)$ if and only if there exists $p \in L^1(\Omega; \mathbb{M}_D^{n \times n})$ such that $(u, e, p) \in \mathcal{A}_{reg}(w, \Gamma_0)$. Given any $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$ and any Γ_0 open subset of $\partial\Omega$, we have that $(u, e) \in \mathcal{B}$ if and only if there exists $p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ such that $(u, e, p) \in \mathcal{A}(w, \Gamma_0)$.

We are now in a position to state the main result of this section.

Theorem 4.3. *Let $Q : \mathbb{M}_{sym}^{n \times n} \rightarrow [0, +\infty)$ be a positive definite quadratic form and let $H : \mathbb{M}_D^{n \times n} \rightarrow [0, +\infty]$ a convex and positively one-homogeneous function such that*

$$r|\xi| \leq H(\xi) \leq R|\xi| \quad \text{for every } \xi \in \mathbb{M}_D^{n \times n}, \quad (4.3)$$

with $0 < r \leq R$. Assume one of the two following conditions: either

- (i) $\Omega \subset \mathbb{R}^n$ is an open and bounded set with a $C^{2,1}$ boundary and Γ_0 an open subset of $\partial\Omega$,

or

- (ii) $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary and $\Gamma_0 = \partial\Omega$.

Let $w \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$. Let $\mathcal{F} : BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \rightarrow [0, +\infty]$ given by

$$\mathcal{F}(u, e) := \begin{cases} \int_{\Omega} Q(e(x)) dx + \int_{\Omega} H(Eu(x) - e(x)) dx & \text{if } (u, e) \in \mathcal{B}_{reg}(w, \Gamma_0), \\ +\infty & \text{otherwise.} \end{cases}$$

The lower semicontinuous envelope of \mathcal{F} , with respect to the product of the $L^1(\Omega; \mathbb{R}^n)$ -strong topology and the $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ -weak topology, is the functional $\mathcal{G} : BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \rightarrow [0, +\infty]$ given by

$$\mathcal{G}(u, e) := \int_{\Omega} Q(e(x)) dx + \mathcal{H}_{\Omega}(Eu - e) + \int_{\Gamma_0} H((w - u) \odot \nu_{\partial\Omega}) d\mathcal{H}^{n-1}$$

if $(u, e) \in \mathcal{B}$, and $\mathcal{G}(u, e) := +\infty$ otherwise.

Proof. Since the functional \mathcal{F} is coercive in the elastic strain e with respect to the weak topology of $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, the lower semicontinuous envelope $\overline{\mathcal{F}}$ can be characterised sequentially as

$$\overline{\mathcal{F}}(u, e) = \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{F}(u_j, e_j) : \begin{aligned} &u_j \rightarrow u \text{ strongly in } L^1(\Omega; \mathbb{R}^n), \\ &e_j \rightarrow e \text{ weakly in } L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \end{aligned} \right\}.$$

We first prove that $\mathcal{G} \leq \overline{\mathcal{F}}$. Let $(u, e) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$ and let $u_j \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^n)$ and $e_j \rightarrow e$ weakly in $L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$. We want to show that

$$\liminf_{j \rightarrow \infty} \mathcal{F}(u_j, e_j) \geq \mathcal{G}(u, e).$$

Without loss of generality we can assume that

$$\liminf_{j \rightarrow \infty} \mathcal{F}(u_j, e_j) < +\infty$$

and, up to subsequences, that the above liminf is a limit. Thus, we deduce by (4.3) that there exists a constant $C > 0$ such that

$$\int_{\Omega} |Eu_j(x) - e_j(x)| dx \leq C$$

for every j . Up to extracting a further subsequence, we can thus assume that $p_j := Eu_j - e_j \rightarrow p$ weakly* in $M_b(\Omega; \mathbb{M}_D^{n \times n})$. Since $p = Eu - e$ in Ω , we have $(u, e) \in \mathcal{B}$.

Since Γ_0 is an open subset of $\partial\Omega$, there exists an open set U in \mathbb{R}^n such that $\Gamma_0 = \partial\Omega \cap U$. We define

$$\tilde{u}_j := \begin{cases} u_j & \text{in } \Omega, \\ w & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{e}_j := \begin{cases} e_j & \text{in } \Omega, \\ Ew & \text{in } U \setminus \Omega, \end{cases}$$

and

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ w & \text{in } U \setminus \Omega, \end{cases} \quad \tilde{e} := \begin{cases} e & \text{in } \Omega, \\ Ew & \text{in } U \setminus \Omega. \end{cases}$$

Clearly $\tilde{u}_j \rightharpoonup \tilde{u}$ weakly* in $BD(U)$ and $\tilde{e}_j \rightharpoonup \tilde{e}$ weakly in $L^2(U; \mathbb{M}_{sym}^{n \times n})$. By definition of the extensions and by lower semicontinuity we deduce that

$$\begin{aligned} \liminf_{j \rightarrow \infty} \mathcal{F}(u_j, e_j) &= \liminf_{j \rightarrow \infty} \int_{\Omega} Q(e_j(x)) dx + \mathcal{H}_U(E\tilde{u}_j - \tilde{e}_j) \\ &\geq \int_{\Omega} Q(e(x)) dx + \mathcal{H}_U(E\tilde{u} - \tilde{e}) = \mathcal{G}(u, e). \end{aligned}$$

We now prove that $\mathcal{G} \geq \overline{\mathcal{F}}$. Let $(u, e) \in \mathcal{B}$. Set

$$p := Eu - e \text{ in } \Omega, \quad p := (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1} \text{ on } \Gamma_0.$$

By Theorem 3.2 or Theorem 3.4 (according to the validity of assumption (i) or (ii), respectively) there exists a sequence $(u_j, e_j, p_j) \in \mathcal{A}_{reg}(w, \Gamma_0)$ such that

$$u_j \rightarrow u \quad \text{strongly in } L^{n/(n-1)}(\Omega; \mathbb{R}^n), \quad (4.4)$$

$$e_j \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{n \times n}), \quad (4.5)$$

$$p_j \rightarrow p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n}), \quad (4.6)$$

and

$$\int_{\Omega} |p_j(x)| dx \rightarrow |p|(\Omega \cup \Gamma_0). \quad (4.7)$$

We now extend p_j and p by 0 to an open subset U such that $U \cap \partial\Omega = \Gamma_0$, and we call \tilde{p}_j and \tilde{p} the extensions, respectively. By (4.6) and (4.7) we have that $\tilde{p}_j \rightarrow \tilde{p}$ strictly in $M_b(U; \mathbb{M}_D^{n \times n})$, hence the Reshetnyak Theorem (see [1, Theorem 2.39]) implies that

$$\int_{\Omega} H(p_j(x)) dx = \mathcal{H}_U(\tilde{p}_j) \rightarrow \mathcal{H}_U(\tilde{p}). \quad (4.8)$$

Note that

$$\mathcal{H}_U(\tilde{p}) = \mathcal{H}_{\Omega}(Eu - e) + \int_{\Gamma_0} H((w - u) \odot \nu_{\partial\Omega}) d\mathcal{H}^{n-1}.$$

Since $(u_j, e_j) \in \mathcal{B}_{reg}(w, \Gamma_0)$ and $p_j = Eu_j - e_j$, convergences (4.5) and (4.8) imply that

$$\lim_{j \rightarrow \infty} \mathcal{F}(u_j, e_j) = \mathcal{G}(e, u).$$

This concludes the proof. \square

Acknowledgements. The author acknowledges support by GNAMPA-INdAM and by the European Research Council under Grant No. 290888 ‘‘Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture’’.

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